

MODEL-CHECKING PROBLEMS AS A BASIS FOR PARAMETERIZED INTRACTABILITY

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ABSTRACT. Most parameterized complexity classes are defined in terms of a parameterized version of the Boolean satisfiability problem (the so-called *weighted satisfiability problem*). For example, Downey and Fellows’s W-hierarchy is of this form. But there are also classes such as the A-hierarchy, that are more naturally characterised in terms of *model-checking problems* for certain fragments of first-order logic.

Downey, Fellows, and Regan (1998) were the first to establish a connection between the two formalisms by giving a characterisation of the W-hierarchy in terms of first-order model-checking problems. We improve their result and then prove a similar correspondence between weighted satisfiability and model-checking problems for the A-hierarchy and the W*-hierarchy. Thus we obtain very uniform characterisations of many of the most important parameterized complexity classes in both formalisms.

Our results can be used to give new, simple proofs of some of the core results of structural parameterized complexity theory.

1. INTRODUCTION

Parameterized complexity theory allows a refined complexity analysis of problems whose input consists of several parts of different sizes. Such an analysis is particularly well-suited for a certain type of logic based algorithmic problems such as model-checking problems in automated verification or database query evaluation. In such problems one has to evaluate a formula of some logic in a finite structure. Typical examples are the evaluation of formulas of linear time temporal logic (LTL) in finite Kripke structures or formulas of first-order logic (FO; relational calculus in database terminology) in finite relational structures. Throughout this paper we adopt the term *model-checking problems* from verification when referring to problems of this general type. It has turned out that usually the complexity of these problems is quite high; for example, for both LTL and FO, it is PSPACE-complete [15, 17]. This high complexity of model-checking problems is usually caused by large and complicated formulas. However, in the practical situations in which model-checking problems occur one usually has to evaluate a small formula in a very large structure. In our examples from verification and database theory this is obvious. So an exponential time complexity may still

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be acceptable as long as the exponential term in the running time only involves the size of the input formula and not the much larger size of the input structure. Lichtenstein and Pnueli [14] argue along these lines to support their LTL-model-checking algorithm with a running time of $2^{O(k)} \cdot n$, where k is the size of the input formula and n the size of the input structure. While this argument just follows algorithmic common sense, parameterized complexity theory, or more precisely the theory of parameterized intractability, comes into play if one wants to argue that no algorithm with a comparable running time exists for FO-model-checking. Indeed, no algorithm for FO-model-checking with a running time better than the trivial $n^{O(k)}$ is known, but classical complexity theory does not provide the tools to show that no better algorithm exists.

So far we have argued that parameterized complexity theory is useful for analysing certain algorithmic problems from logic. But it turns out that the same logical problems are also very useful to lay a foundation for parameterized complexity theory, and this is what the present paper is about.

Before describing our results, let us briefly recall the basic notions of parameterized complexity theory. Instances of a parameterized problem consist of two parts, which we call *input* and *parameter*. The idea is that in the instances occurring in practice the parameter can be expected to be small, whereas the input may be very large. For example, an instance of a *parameterized model-checking problem* consists of a structure and a formula, and we take the formula to be the parameter. Let n denote the size of the input of a parameterized problem and k the size of the parameter. A parameterized problem is *fixed parameter-tractable* if it can be solved in time $f(k) \cdot p(n)$ for an arbitrary computable function f and a polynomial p . FPT denotes the class of all fixed-parameter tractable problems. Just as the Boolean satisfiability problem can be seen as the most basic intractable problem in the classical theory of NP-completeness, a natural parameterization of the satisfiability problem serves as a basis for the theory of parameterized intractability: The *weighted satisfiability problem* for a class of Boolean formulas asks whether a given formula has a satisfying assignment in which precisely k variables are set to TRUE; here k is treated as the parameter. Unfortunately, it turns out that the complexity of the weighted satisfiability problem is much less robust than that of the unweighted problem. For example, the weighted satisfiability problem for formulas in conjunctive normal form does not seem to have the same complexity as the weighted satisfiability problem for arbitrary formulas. So instead of getting just one class of intractable problems, we get a whole family of classes of intractable parameterized problems each having a complete weighted satisfiability problem. The most basic of these classes form the so-called *W-hierarchy*.

Downey, Fellows, and Regan [7] gave an alternative characterisation of the W-hierarchy, which resembles Fagin's [9] and Stockmeyer's [16] characterisation of the class NP and the polynomial hierarchy. They proved that for each level $W[t]$ of the W-hierarchy there is a family $\Sigma_{t,u}[\tau]$, for $u \geq 1$, of classes of first-order formulas of a certain vocabulary τ such that the model-checking problem for each $\Sigma_{t,u}[\tau]$ is in $W[t]$, and conversely each problem in $W[t]$ can be reduced to the model-checking problem for $\Sigma_{t,u}[\tau]$ for some $u \geq 1$. In [11] we improved this characterisation by showing that u can be taken to be 1 and τ any vocabulary, which is not unary. In other words, we showed that model-checking for $\Sigma_{t,1}[\tau]$ is $W[t]$ -complete for any vocabulary τ that is not binary. This result is the starting point for our present investigation. We further improve the result by showing that the vocabulary τ can be taken to be part of the input and does not have to be fixed in advance. This gives us a very robust characterisation of the W-hierarchy in terms of first-order model checking problems. To underline the significance of this characterisation, we show that some of the most important structural results on the W-hierarchy, the previously known proofs of which are very complicated (cf. Part II of Downey and Fellow's monograph [6]), can be derived as

easy corollaries of our results. Moreover, we derive a strengthening of the so-called monotone and antimonotone collapse.

The correspondence between weighted satisfiability problems and model-checking problems for first-order logic can be extended beyond the W-hierarchy. We establish such a correspondence for the W^* -hierarchy (introduced in [8]) and the A-hierarchy (introduced in [11]). For each of these hierarchies a characterisation either in terms of weighted satisfiability problems or in terms of model-checking problems was known before; and for each of them we provide the counterpart.

The W^* -hierarchy is a small variation of the W-hierarchy. As the classes of the W-hierarchy, the classes of the W^* -hierarchy are defined via the weighted satisfiability problem; we give a characterisation in terms of model-checking problems of first-order logic. It is an open problem whether the W-hierarchy and the W^* -hierarchy coincide. Downey, Fellows, and Taylor were able to prove that $W[1] = W^*[1]$ [8] and $W[2] = W^*[2]$ [5]. The latter result has a highly non-trivial proof; here we are able to derive $W[1] = W^*[1]$ and $W[2] = W^*[2]$ as simple corollaries of our characterisation of the W^* -hierarchy. This gives a very transparent proof of these results that also clearly shows why it only works for the first two levels.

The A-hierarchy, which may be viewed as the parameterized analogue of the polynomial hierarchy, is defined in terms of the parameterized halting problem for alternating Turing machines. In [11], we gave a characterisation of the hierarchy in terms of model-checking problems for fragments of first-order logic; in this characterisation the levels of the A-hierarchy correspond to levels of quantifier alternation in first-order formulas. Here we give a propositional characterisation in terms of the *alternating weighted satisfiability problem* (which may be viewed as the parameterized version of the satisfiability problem for quantified Boolean formulas). The overall picture that evolves is that in parameterized complexity theory we have two different sources of increasing complexity: the alternation of propositional connectives (leading to the W-hierarchy) and quantifier alternation (leading to the A-hierarchy). Thus we actually obtain a 2-dimensional family of parameterized classes which we call the *A-matrix* (see Figure 1 on page 30). Each class of this matrix has natural characterisations in terms of an alternating weighted satisfiability problem and a model-checking problem for a fragment of first-order logic. Let us remark that in classical complexity, only quantifier alternation is relevant, because the classes are closed under Boolean connectives. Thus there is only the (1-dimensional) polynomial hierarchy.

In a last section, we use certain normal forms established here and a known characterisation of the AW-hierarchy (introduced in [1]) by first-order model-checking to give a simple proof of the collapse of the AW-hierarchy to its first-level [1]. Actually, we slightly strengthen the result of [1]. An application of this stronger result can be found in [12].

On a more technical level, our main contribution is a new and greatly simplified proof technique for establishing the correspondence between weighted satisfiability problems and model-checking problems. This technique enables us to obtain all our results in a fairly uniform way. A major problem in structural parameterized complexity theory is the lacking robustness of most classes of intractable parameterized problems, leading to the abundance of classes and hierarchies of classes. Maybe the technically most difficult result of this paper is a normalisation lemma for the relevant fragments of first-order logic which shows that the vocabulary can be treated as part of the input of a model-checking problem.

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We would like to acknowledge that McCartin, Downey, and Fellows already conjectured the characterisation of the A-hierarchy by alternating weighted satisfiability problems that we prove here.

2. PRELIMINARIES

In this section we recall some definitions and fix our notations.

2.1. Fixed-Parameter Tractability. A *parameterized problem* is a set $Q \subseteq \Sigma^* \times \Pi^*$, where Σ and Π are finite alphabets. If $(x, y) \in \Sigma^* \times \Pi^*$ is an instance of a parameterized problem, we refer to x as the *input* and to y as the *parameter*.

To illustrate our notation, let us give one example of a parameterized problem, the *parameterized clique problem* p -CLIQUE:

<p>p-CLIQUE</p> <p style="margin-left: 2em;"><i>Input:</i> A graph \mathcal{G}.</p> <p style="margin-left: 2em;"><i>Parameter:</i> $k \in \mathbb{N}$ (say, in binary).</p> <p style="margin-left: 2em;"><i>Problem:</i> Decide if \mathcal{G} has a clique of size k.</p>

Definition 2.1. A parameterized problem $Q \subseteq \Sigma^* \times \Pi^*$ is *fixed-parameter tractable*, if there is a computable function $f : \mathbb{N} \rightarrow \mathbb{N}$, a polynomial p , and an algorithm that, given a pair $(x, y) \in \Sigma^* \times \Pi^*$, decides if $(x, y) \in Q$ in at most $f(|y|) \cdot p(|x|)$ steps.

FPT denotes the complexity class consisting of all fixed-parameter tractable parameterized problems.

Occasionally we use the term *fpt-algorithm* to refer to an algorithm that takes as input pairs $(x, y) \in \Sigma^* \times \Pi^*$ and has a running time bounded by $f(|y|) \cdot p(|x|)$ for some computable function $f : \mathbb{N} \rightarrow \mathbb{N}$ and polynomial p . Thus a parameterized problem is in FPT if it can be decided by an fpt-algorithm. However, we use the term fpt-algorithm mostly when referring to algorithms computing mappings.

Complementing the notion of fixed-parameter tractability, there is a theory of parameterized intractability. It is based on the following notion of parameterized reduction:

Definition 2.2. An *fpt-reduction* from the parameterized problem $Q \subseteq \Sigma^* \times \Pi^*$ to the parameterized problem $Q' \subseteq (\Sigma')^* \times (\Pi')^*$ is a mapping $R : \Sigma^* \times \Pi^* \rightarrow (\Sigma')^* \times (\Pi')^*$ such that:

- (1) For all $(x, y) \in \Sigma^* \times \Pi^*$: $(x, y) \in Q \iff R(x, y) \in Q'$.
- (2) There is a computable function $g : \mathbb{N} \rightarrow \mathbb{N}$ such that for all $(x, y) \in \Sigma^* \times \Pi^*$, say with $R(x, y) = (x', y')$, we have $|y'| \leq g(|y|)$.
- (3) R can be computed by an fpt-algorithm.

We write $Q \leq^{\text{fpt}} Q'$ or simply $Q \leq Q'$, if there is an fpt-reduction from Q to Q' and set

$$[Q]^{\text{fpt}} := \{Q' \mid Q' \leq^{\text{fpt}} Q\}.$$

For a class C of parameterized problems, we let

$$[C]^{\text{fpt}} := \bigcup_{Q \in C} [Q]^{\text{fpt}}.$$

2.2. Relational Structures and First-order Logic. A (*relational*) *vocabulary* τ is a finite set of relation symbols. Each relation symbol has an *arity*. The *arity* of τ is the maximum of the arities of the symbols in τ . A *structure* \mathcal{A} of vocabulary τ , or τ -*structure* (or, simply *structure*), consists of a set A called the *universe*, and an interpretation $R^{\mathcal{A}} \subseteq A^r$ of each r -ary relation symbol $R \in \tau$. We synonymously write $\bar{a} \in R^{\mathcal{A}}$ or $R^{\mathcal{A}}\bar{a}$ to denote that the tuple $\bar{a} \in A^r$ belongs to the relation $R^{\mathcal{A}}$. For example, we view a *directed graph* as a structure $\mathcal{G} = (G, E^{\mathcal{G}})$, whose vocabulary consists of one binary relation symbol E . \mathcal{G} is an (undirected) *graph*, if $E^{\mathcal{G}}$ is irreflexive and symmetric. We define the *size* of a τ -structure \mathcal{A} to be the number

$$\|\mathcal{A}\| := |A| + \sum_{R \in \tau} \text{arity}(R) \cdot (|R^{\mathcal{A}}| + 1).$$

$\|\mathcal{A}\|$ is the size of a reasonable encoding of \mathcal{A} (see [10] for details). For example, the size of a graph with n vertices and m edges is $O(n + m)$.

The class of all first-order formulas is denoted by FO. They are built up from atomic formulas using the usual boolean connectives and existential and universal quantification. Recall that *atomic formulas* are formulas of the form $x = y$ or $Rx_1 \dots x_r$, where x, y, x_1, \dots, x_r are variables and R is an r -ary relation symbol. For $t \geq 1$, let Σ_t denote the class of all FO-formulas of the form

$$\exists x_{11} \dots \exists x_{1k_1} \forall x_{21} \dots \forall x_{2k_2} \dots Qx_{t1} \dots Qx_{tk_t} \psi,$$

where $Q = \forall$ if t is even and $Q = \exists$ otherwise, and where ψ is quantifier-free. Π_t -formulas are defined analogously starting with a block of universal quantifiers. Let $t, u \geq 1$. A formula φ is $\Sigma_{t,u}$, if it is Σ_t and all quantifier blocks after the leading existential block have length $\leq u$. For example, a formula

$$\exists x_1 \dots \exists x_k \forall y \exists z_1 \exists z_2 \psi,$$

where ψ is quantifier-free, is in $\Sigma_{3,2}$ (for every $k \geq 1$).

If \mathcal{A} is a structure, a_1, \dots, a_n are elements of the universe A of \mathcal{A} , and $\varphi(x_1, \dots, x_n)$ is a first-order formula whose free variables are among x_1, \dots, x_n , then we write $\mathcal{A} \models \varphi(a_1, \dots, a_n)$ to denote that \mathcal{A} satisfies φ if the variables x_1, \dots, x_n are interpreted by a_1, \dots, a_n , respectively.

If Φ is a class of first-order formulas, then $\Phi[\tau]$ denotes the class of all formulas of vocabulary τ in Φ and $\Phi[r]$, for $r \in \mathbb{N}$, the class of all formulas in Φ whose vocabulary has arity $\leq r$.

If again Φ is a class of first-order formulas, then $p\text{-MC}(\Phi)$ denotes the (*parameterized*) *model-checking problem for formulas in Φ* , i.e., the parameterized problem

<p>$p\text{-MC}(\Phi)$</p> <p style="margin-left: 20px;"><i>Input:</i> A structure \mathcal{A}.</p> <p style="margin-left: 20px;"><i>Parameter:</i> A sentence φ in Φ</p> <p style="margin-left: 20px;"><i>Problem:</i> Decide if \mathcal{A} satisfies φ.</p>

Often, the natural formulation of a parameterized problem in first-order logic immediately gives an fpt-reduction to a model-checking problem, e.g.,

- $p\text{-CLIQUE} \leq p\text{-MC}(\Sigma_1[2])$, since the existence of a clique of size k is expressed by the Σ_1 -sentence

$$\exists x_1 \dots \exists x_k \bigwedge_{1 \leq i < j \leq k} Ex_i x_j.$$

- $p\text{-DOMINATING SET} \leq p\text{-MC}(\Sigma_{2,1}[2])$. Here, $p\text{-DOMINATING SET}$ is the problem that asks if a graph \mathcal{G} (the input) has a dominating set of size k (the parameter); so we want to

know if \mathcal{G} satisfies the $\Sigma_{2,1}$ -sentence

$$\exists x_1 \dots \exists x_k \forall y \left(\bigwedge_{1 \leq i < j \leq k} \neg x_i = x_j \wedge \bigvee_{1 \leq i \leq k} (y = x_i \vee E y x_i) \right).$$

2.3. Propositional logic. Formulas of propositional logic are important ingredients in the definitions of various complexity classes of intractable parameterized problems. We recall a few notions and fix our notations: Formulas of propositional logic are built up from *propositional variables* X_1, X_2, \dots by taking conjunctions, disjunctions, and negations. The negation of a formula α is denoted by $\neg\alpha$. We distinguish between *small conjunctions*, denoted by \wedge , which are just conjunctions of two formulas, and *big conjunctions*, denoted by \bigwedge , which are conjunctions of arbitrary finite sets of formulas. Analogously, we distinguish between *small disjunctions*, denoted by \vee , and *big disjunctions*, denoted by \bigvee . A formula is *small* if it neither contains big conjunctions nor big disjunctions. By $\alpha = \alpha(Z_1, \dots, Z_m)$ we indicate that the variables in α are among Z_1, \dots, Z_m .

Let V be a set of propositional variables. We identify each assignment

$$S : V \rightarrow \{\text{TRUE}, \text{FALSE}\}$$

with the set $\{X_i \in V \mid S(X_i) = \text{TRUE}\} \in 2^V$. The *weight* of an assignment $S \in 2^V$ is $|S|$, the number of variables set to TRUE. A propositional formula α is *k-satisfiable* (where $k \in \mathbb{N}$), if there is an assignment for the set of variables of α of weight k satisfying α .

For a set Γ of propositional formulas, the *weighted satisfiability problem* $\text{WSAT}(\Gamma)$ for formulas in Γ is the following parameterized problem:

<p>WSAT(Γ) <i>Input:</i> A propositional formula $\alpha \in \Gamma$. <i>Parameter:</i> $k \in \mathbb{N}$ <i>Problem:</i> Decide if α is k-satisfiable.</p>

The *depth* of a formula is the maximum number of nested (big and small) conjunctions and disjunctions appearing in this formula. The *weft* of a formula is the maximum number of nested big conjunctions and big disjunctions appearing in it. Hence, the weft of a formula always is less than or equal to its depth. For $t, d \in \mathbb{N}$ with $t \leq d$, we set

$$\Omega_{t,d} := \{\alpha \mid \text{propositional formula } \alpha \text{ has weft } \leq t \text{ and depth } \leq d\}.$$

For $t \geq 0$ and $d \geq 1$ define the sets $\Gamma_{t,d}$ and $\Delta_{t,d}$ by induction on t (here, by $(\lambda_1 \wedge \dots \wedge \lambda_r)$ we mean the iterated small conjunction $((\dots (\lambda_1 \wedge \lambda_2) \dots) \wedge \lambda_r)$):

$$\begin{aligned} \Gamma_{0,d} &:= \{(\lambda_1 \wedge \dots \wedge \lambda_r) \mid \lambda_1, \dots, \lambda_r \text{ literals and } r \leq d\}, \\ \Delta_{0,d} &:= \{(\lambda_1 \vee \dots \vee \lambda_r) \mid \lambda_1, \dots, \lambda_r \text{ literals and } r \leq d\}, \\ \Gamma_{t+1,d} &:= \{\bigwedge \Pi \mid \Pi \subseteq \Delta_{t,d}\}, \\ \Delta_{t+1,d} &:= \{\bigvee \Pi \mid \Pi \subseteq \Gamma_{t,d}\}. \end{aligned}$$

If in the definition of $\Gamma_{0,d}$ and $\Delta_{0,d}$ we require that all literals are positive (negative) we obtain the sets denoted by $\Gamma_{t,d}^+$ and $\Delta_{t,d}^+$ ($\Gamma_{t,d}^-$ and $\Delta_{t,d}^-$), respectively. Clearly, $\Gamma_{t,d} \subseteq \Omega_{t,t+d}$ and $\Delta_{t,d} \subseteq \Omega_{t,t+d}$.

3. NORMALISATION

We have introduced two logically defined families of parameterized problems, the first based on model-checking problems for classes of first-order sentences and the second based on the weighted satisfiability problem for classes of propositional formulas. The main results of this paper establish a tight correspondence between the two approaches; in fact, we present formalisms that allow to

translate from one family of parameterized problems into the other. To prove these results, it is convenient to first simplify each of the two sides separately.

3.1. Propositional Normalisation. The following lemma has been used by Downey, Fellows, and others as the first step in numerous fpt-reductions (cf. [6]).

Lemma 3.1 (Propositional Normalisation). *Let $d \geq t \geq 0$. Then there is a polynomial time algorithm that computes for every formula in $\Omega_{t,d}$ an equivalent formula in $\Delta_{t+1,2^d}$.*

Proof (sketch): We can restrict our attention to formulas in $\Omega_{t,d}$ with negation symbols only in front of atomic formulas. We proceed by induction on t : If $\alpha \in \Omega_{0,d}$, then α contains at most 2^d variables and we just compute an equivalent formula in disjunctive normal form. For $t \geq 1$, we use the distributive law:

$$\left(\bigvee_{i \in I} \alpha_i \wedge \bigvee_{j \in J} \beta_j \right) \text{ is equivalent to } \bigvee_{(i,j) \in I \times J} (\alpha_i \wedge \beta_j).$$

□

Note that the algorithm in Lemma 3.1 is polynomial, because the depth of the formulas is bounded by a fixed constant d . Obviously, no such normalisation is possible for formulas of arbitrary depth. Even if the depth of the formula is treated as a parameter, the reduction is not fixed parameter tractable: the formula $\alpha' \in \Delta_{t+1,2^d}$ equivalent to a formula $\alpha \in \Omega_{t,d}$ may have size $\Omega(|\alpha|^d)$. However, as we shall see in Section 5.1, if we treat the depth as parameter we can at least prove a weaker normalisation lemma (Lemma 5.2).

Corollary 3.2. *For all $d \geq t \geq 0$,*

$$\text{WSAT}(\Omega_{t,d}) \leq \text{WSAT}(\Delta_{t+1,2^d}).$$

Remark 3.3. Instead of propositional *formulas*, Downey and Fellows always work with Boolean *circuits* (cf. [6]). However, since we are only dealing with circuits and formulas of bounded depth, this does not really make a difference. We can always transform circuits into formulas in the most straightforward way. More precisely, if we define depth and weft of a circuit in the natural way and denote by $C_{t,d}$ the class of all circuits of weft t and depth d , then we get the following results:

- (1) Let $d \geq t \geq 0$. Then there is a polynomial time algorithm that computes for every circuit in $C_{t,d}$ an equivalent formula in $\Omega_{t,d}$.
- (2) Let $t \geq 0$. Then there is an fpt-algorithm that computes for every circuit in $C_{t,k}$ an equivalent formula in $\Omega_{t,k}$. Here k is treated as the parameter.

3.2. First-order normalisation. The normalisation results for first-order logic presented in this subsection are concerned with the vocabulary of the formulas in parameterized model-checking problems. Actually, we prove that it is irrelevant, whether we consider arbitrary formulas or we restrict ourselves to a fixed vocabulary, as long as it contains at least one binary relation symbol. This may not sound very surprising, but is not easy to prove and was left open in our earlier paper [11].

The main results of this section are summarised in the following First-Order Normalisation Lemma. To state the lemma we need two more definitions: For all $t, u \geq 1$, we call a $\Sigma_{t,u}$ -formula

$$\exists x_1 \dots \exists x_k \forall \bar{y}_1 \dots Q_t \bar{y}_t \varphi$$

strict if no atomic subformula of φ contains more than one of the variables x_1, \dots, x_k . We denote the class of all strict $\Sigma_{t,u}$ -formulas by $\text{strict-}\Sigma_{t,u}$. A Σ_t -formula is *simple*, if its quantifier-free part

is a conjunction of literals in case t is odd, and is a disjunction of literals in case t is even.¹ We denote the class of all simple Σ_t -formulas by $\text{simple-}\Sigma_t$.

Lemma 3.4 (First-Order Normalisation Lemma).

- (1) For $t \geq 2, u \geq 1$, $p\text{-MC}(\Sigma_{t,u}) \leq p\text{-MC}(\text{strict-}\Sigma_{t,1}[2])$.
- (2) For all $t \geq 1$, $p\text{-MC}(\Sigma_t) \leq p\text{-MC}(\text{simple-}\Sigma_t[2])$.
- (3) $p\text{-MC}(\text{FO}) \leq p\text{-MC}(\text{FO}[2])$ ([11]).

The First-Order Normalisation Lemma is the only result of this section used in the rest of the paper. Hence the reader not interested in its proof may pass to Section 4 directly.

It will be useful to first recall the proof of (3) (from [11]). We then point out the difficulties in proving (2) by the same simple technique and resolve these difficulties by Lemmas 3.5–3.7. The proof of (1) is also complicated and will be carried out in several steps in Lemmas 3.8–3.10.

Proof of Lemma 3.4(3): Let (\mathcal{A}, φ) be an instance of $p\text{-MC}(\text{FO})$. We construct a structure \mathcal{A}_b and a sentence $\varphi_b \in \text{FO}[2]$ such that $(\mathcal{A} \models \varphi \iff \mathcal{A}_b \models \varphi_b)$.

Let τ be the vocabulary of \mathcal{A} . We let \mathcal{A}_b be the *bipartite structure* or *incidence structure* associated with \mathcal{A} : Let τ_b be the vocabulary of arity 2 that contains a unary relation symbol P_R for every $R \in \tau$ and binary relation symbols E_1, \dots, E_s , where s is the arity of τ . The universe A_b of \mathcal{A}_b consists of A together with a new vertex $b_{R,\bar{a}}$ for all $R \in \tau$ and $\bar{a} \in R^A$. The relation $E_i^{\mathcal{A}_b}$ holds for all pairs $(a_i, b_{R,a_1\dots a_r})$, and $P_R^{\mathcal{A}_b} := \{b_{R,\bar{a}} \mid \bar{a} \in R^A\}$. Let φ_b be the FO-sentence equivalent to the τ' -formula obtained from φ by replacing every atomic formula $Rx_1 \dots x_r$ by (the simple Σ_1 -formula)

$$\exists y(P_R y \wedge E_1 x_1 y \wedge \dots \wedge E_r x_r y). \quad (3.1)$$

Then clearly $(\mathcal{A} \models \varphi \iff \mathcal{A}_b \models \varphi_b)$. To see that this construction yields an fpt-reduction, note that $\|\mathcal{A}_b\| = O(\|\mathcal{A}\|)$. \square

Why does the same construction not also work to get $p\text{-MC}(\Sigma_t) \leq p\text{-MC}(\Sigma_t[2])$? Because if, say, a Σ_1 formula contains a negated atom $\neg Rx_1 \dots x_r$, then it will be replaced by a formula equivalent to

$$\forall y(\neg P_R y \vee \neg E_1 x_1 y \vee \dots \vee \neg E_r x_r y) \quad (3.2)$$

and we obtain a formula that is no longer equivalent to a Σ_1 -formula. At first sight it seems that we can easily resolve this problem by just extending the bipartite structure \mathcal{A}_b by additional points $b_{\neg R, \bar{a}}$ for all $\bar{a} \notin R^A$ and relation symbols $P_{\neg R}$. Unfortunately, in general the size of the resulting structure is not polynomially bounded in the size of \mathcal{A} , since the vocabulary is not fixed in advance.

We denote by Σ_t^+ the class of all Σ_t -formulas without negation symbols and by Σ_t^- the class of all Σ_t -formulas in which there is a negation symbol in front of every atom and there are no other negation symbols. Using the transition $(\mathcal{A}, \varphi) \mapsto (\mathcal{A}_b, \varphi_b)$ we derive part (1) and (2) of the following lemma:

Lemma 3.5. (1) If $t \geq 1$ is odd, then

$$p\text{-MC}(\Sigma_t^+) \leq p\text{-MC}(\Sigma_t^+[2]) \quad \text{and} \quad p\text{-MC}(\text{simple-}\Sigma_t^+) \leq p\text{-MC}(\text{simple-}\Sigma_t^+[2]).$$

(2) If $t \geq 1$ is even, then

$$p\text{-MC}(\Sigma_t^-) \leq p\text{-MC}(\Sigma_t^-[2]) \quad \text{and} \quad p\text{-MC}(\text{simple-}\Sigma_t^-) \leq p\text{-MC}(\text{simple-}\Sigma_t^-[2]).$$

(3) If $t \geq 1$ and $r \geq 1$, then $p\text{-MC}(\text{simple-}\Sigma_t[r]) \leq p\text{-MC}(\text{simple-}\Sigma_t[2])$.

¹Simple Σ_1 -formulas are also called conjunctive queries with negation.

Proof. If t is odd and $\varphi \in \Sigma_t^+$ (is simple), then the last quantifier block in φ is existential (and the quantifier-free part is a conjunction of literals). Since φ only has positive literals, in φ_b this last existential block can absorb the quantifiers introduced by (3.1) (and only further conjunctions are added to the quantifier-free part). Similarly, if t is even and $\varphi \in \Sigma_t^-$ (is simple), then the last quantifier block in φ is universal (and the quantifier-free part is a disjunction of literals) and in φ_b this block can absorb the quantifiers introduced by (3.2) (and only further disjunctions are added to the quantifier-free part).

It remains to prove (3). Fix $r \geq 1$. Given any structure \mathcal{A} in a vocabulary τ of arity r , we obtain the structure \mathcal{A}' by adding the complement of the relations of \mathcal{A} , more precisely: We set $\tau' := \tau \cup \{R^c \mid R \in \tau\} \cup \{\neq\}$ and we obtain \mathcal{A}' from \mathcal{A} setting $(R^c)^{\mathcal{A}'} := A^{\text{arity}(R)} \setminus R^{\mathcal{A}}$ and $\neq^{\mathcal{A}'} := \{(a, b) \mid a, b \in A, a \neq b\}$. Thus, $\|\mathcal{A}'\| = O(\|\mathcal{A}\|^r)$. The transition from \mathcal{A} to \mathcal{A}' allows to replace in any formula positive by negative literals and vice versa, thus showing that

$$p\text{-MC}(\Sigma_t^+[r]) \equiv^{\text{fpt}} p\text{-MC}(\Sigma_t[r]) \equiv^{\text{fpt}} p\text{-MC}(\Sigma_t^-[r])$$

and

$$p\text{-MC}(\text{simple-}\Sigma_t^+[r]) \equiv^{\text{fpt}} p\text{-MC}(\text{simple-}\Sigma_t[r]) \equiv^{\text{fpt}} p\text{-MC}(\text{simple-}\Sigma_t^-[r]),$$

which yields part (3) by what we already have proven. \square

A reduction to the positive (resp. negative) fragment is accomplished by:

Lemma 3.6. (1) *If $t \geq 1$ is odd, then $p\text{-MC}(\Sigma_t) \leq p\text{-MC}(\Sigma_t^+)$.*
(2) *If $t \geq 1$ is even, then $p\text{-MC}(\Sigma_t) \leq p\text{-MC}(\Sigma_t^-)$.*

Proof. Let (\mathcal{A}, φ) be an instance of $p\text{-MC}(\Sigma_t)$. We may assume that all negation symbols in φ are in front of atomic subformulas. We give an fpt-reduction mapping (\mathcal{A}, φ) to a pair (\mathcal{A}', φ') with $(\mathcal{A} \models \varphi \iff \mathcal{A}' \models \varphi')$, where φ' is a Σ_t^+ -formula if t is odd and a Σ_t^- -formula if t is even.

Let τ be the vocabulary of \mathcal{A} . The τ' -structure \mathcal{A}' will be an expansion of \mathcal{A} . It has an ordering $<^{\mathcal{A}'}$ of its universe $A' = A$. If $R \in \tau$ is r -ary, in τ' we have r -ary relation symbols R_f and R_l , and a $2 \cdot r$ -ary relation symbol R_s . $R_f^{\mathcal{A}'}$ and $R_l^{\mathcal{A}'}$ are singletons consisting of the first and last tuple in $R^{\mathcal{A}}$, respectively, in the lexicographic ordering of r -tuples induced by $<^{\mathcal{A}'}$ (and are empty in case $R^{\mathcal{A}}$ is empty). The relation $R_s^{\mathcal{A}'}$ contains (\bar{a}, \bar{b}) iff $(R^{\mathcal{A}}\bar{a}, R^{\mathcal{A}}\bar{b}, \bar{a})$ is less than \bar{b} , and no tuple in $R^{\mathcal{A}}$ is between \bar{a} and \bar{b} in the lexicographic ordering of r -tuples). Let $\bar{y} <_r \bar{z}$ denote a quantifier-free formula of vocabulary τ' without the negation symbol expressing that \bar{y} is less than \bar{z} in the lexicographic ordering of r -tuples.

Now assume that t is odd. Then in φ we replace every negative occurrence $\neg R x_1 \dots x_r$ of R by

$$\exists y_1 \dots \exists y_r \exists z_1 \dots \exists z_r ((R_f \bar{y} \wedge \bar{x} <_r \bar{y}) \vee (R_s \bar{y} \bar{z} \wedge \bar{y} <_r \bar{x} \wedge \bar{x} <_r \bar{z}) \vee (R_l \bar{z} \wedge \bar{z} <_r \bar{x}))$$

and every negative occurrence $\neg x = y$ by $(x < y \vee y < x)$. The resulting formula is easily seen to be equivalent to a Σ_t^+ -formula φ' . If t is even, we replace every positive occurrence $R x_1 \dots x_r$ of R in φ by

$$\neg \exists y_1 \dots \exists y_r \exists z_1 \dots \exists z_r ((R_f \bar{y} \wedge \bar{x} <_r \bar{y}) \vee (R_s \bar{y} \bar{z} \wedge \bar{y} <_r \bar{x} \wedge \bar{x} <_r \bar{z}) \vee (R_l \bar{z} \wedge \bar{z} <_r \bar{x}))$$

and every positive occurrence $x = y$ by $(\neg x < y \wedge \neg y < x)$. We obtain a formula that is equivalent to a Σ_t^- -formula φ' . \square

The last gap in a proof of Lemma 3.4(2), namely the transition to the simple fragments, will be closed by the following result:

Lemma 3.7. *For $t \geq 1$,*

$$p\text{-MC}(\Sigma_t[2]) \leq p\text{-MC}(\text{simple-}\Sigma_t[3]).$$

Proof. To simplify the notation we fix the parity of t , say, t is even. Let (\mathcal{A}, φ) be an instance of $p\text{-MC}(\Sigma_t[2])$. Thus, the vocabulary τ of \mathcal{A} has arity ≤ 2 and we can assume that the quantifier-free part of the sentence φ is in conjunctive normal form,

$$\varphi = \exists \bar{y}_1 \forall \bar{y}_2 \exists \bar{y}_3 \dots \forall \bar{y}_t \bigwedge_{i \in I} \bigvee_{j \in J} \lambda_{ij}$$

with literals λ_{ij} . First we replace the conjunction $\bigwedge_{i \in I}$ in φ by a universal quantifier. For this purpose, we add to the vocabulary τ unary relation symbols R_i for $i \in I$ and consider an expansion $\mathcal{B} := (\mathcal{A}, (R_i^{\mathcal{B}})_{i \in I})$ of \mathcal{A} , where $(R_i^{\mathcal{B}})_{i \in I}$ is a partition of A into nonempty disjoint sets. Then,

$$\mathcal{A} \models \varphi \iff \mathcal{B} \models \exists \bar{y}_1 \forall \bar{y}_2 \exists \bar{y}_3 \dots \forall \bar{y}_t \forall y \bigvee_{i \in I} \bigvee_{j \in J} (R_i y \wedge \lambda_{ij}).$$

Since the arity of τ is ≤ 2 , every λ_{ij} contains at most two variables, say, $\lambda_{ij} = \lambda_{ij}(x_{ij}, y_{ij})$. We expand \mathcal{B} to a structure \mathcal{C} by adding, for all $i \in I$ and $j \in J$, a relation $T_{ij}^{\mathcal{C}}$ of arity 3 containing all triples (a, b, c) such that $R_i^{\mathcal{B}} a$ and $\mathcal{B} \models \lambda_{ij}(b, c)$. Then,

$$\mathcal{A} \models \varphi \iff \mathcal{C} \models \exists \bar{y}_1 \forall \bar{y}_2 \exists \bar{y}_3 \dots \forall \bar{y}_t \forall y \bigvee_{i \in I} \bigvee_{j \in J} T_{ij} y x_{ij} y_{ij}.$$

The formula on the right hand side is simple, so this equivalence yields the desired reduction. \square

Proof of Lemma 3.4(2): By applying Lemma 3.6, Lemma 3.5, Lemma 3.7, and Lemma 3.5 one by one, we obtain the following chain of reductions, say, for even t ,

$$p\text{-MC}(\Sigma_t) \leq p\text{-MC}(\Sigma_t^-) \leq p\text{-MC}(\Sigma_t[2]) \leq p\text{-MC}(\text{simple-}\Sigma_t[3]) \leq p\text{-MC}(\text{simple-}\Sigma_t[2]).$$

\square

When trying to prove Lemma 3.4(1) we are facing another difficulty: For example, consider the case $t = 3$. If we apply the reduction used to prove Lemma 3.4(2) to a formula in $\Sigma_{3,u}$, the resulting formula, even though equivalent to a formula in $\Sigma_3[2]$, is not necessarily equivalent to a formula in $\Sigma_{3,u}[2]$.

The crucial property we exploit in our proof of Lemma 3.4(1) is that in a $\Sigma_{t,u}$ -formula the number of variables not occurring in the first, existentially quantified, block of variables is bounded by $(t-1) \cdot u$. We proceed in three steps: We start with $p\text{-MC}(\Sigma_{t,u})$. In Lemma 3.8 we show how to pass from $\Sigma_{t,u}$ to $\Sigma_{t,u'}[r]$ for some u', r ; in Lemma 3.9 we see that we can choose $r = 2$. Finally, we get $u' = 1$ by Lemma 3.10.

Lemma 3.8. *For $t \geq 2$ and $u \geq 1$,*

$$p\text{-MC}(\Sigma_{t,u}) \leq p\text{-MC}(\Sigma_{t,u+1}[t \cdot u]).$$

Proof. Let (\mathcal{A}, φ) be an instance of $p\text{-MC}(\Sigma_{t,u})$. Say, $\varphi = \exists x_1 \dots \exists x_k \psi$, where ψ begins with a universal quantifier. Set $q := (t-1) \cdot u$ and let $\bar{y} = y_1, \dots, y_q$ contain the variables in φ distinct from x_1, \dots, x_k . We shall define a structure \mathcal{A}' and a $\Sigma_{t,u+1}[t \cdot u]$ -sentence φ' with $(\mathcal{A} \models \varphi \iff \mathcal{A}' \models \varphi')$.

Let Λ be the set of all atomic subformulas of φ . Here the notation $\lambda(x_{i_1}, \dots, x_{i_\ell}, \bar{y})$ indicates that $x_{i_1}, \dots, x_{i_\ell}$ are the variables from x_1, \dots, x_k in λ . The vocabulary τ' of \mathcal{A}' contains a unary relation symbol O (the “old element relation”), binary relation symbols E_1, \dots, E_k (the “component relations”) and for every $\lambda(x_{i_1}, \dots, x_{i_\ell}, \bar{y}) \in \Lambda$ a unary relation symbol W_λ and a $(1+q)$ -ary relation symbol R_λ . Thus the arity of τ' is at most $1+q \leq t \cdot u$. For every $\lambda \in \Lambda$ and $a_1, \dots, a_\ell \in A$ with

$$\mathcal{A} \models \exists \bar{y} \lambda(a_1, \dots, a_\ell, \bar{y}) \quad (3.3)$$

we have in A' a new element $w(\lambda, a_1, \dots, a_\ell)$, a “witness” for (3.3). We let

$$\begin{aligned} A' &:= A \cup \{w(\lambda, a_1, \dots, a_\ell) \mid \lambda(x_{i_1}, \dots, x_{i_\ell}, \bar{y}) \in \Lambda, \bar{a} = (a_1, \dots, a_\ell) \in A^\ell, \mathcal{A} \models \exists \bar{y} \lambda(\bar{a}, \bar{y})\}, \\ O^{A'} &:= A \\ E_i^{A'} &:= \{(a_i, w(\lambda, a_1, \dots, a_\ell)) \mid w(\lambda, a_1, \dots, a_\ell) \in A'\} \quad (\text{for } 1 \leq i \leq k). \end{aligned}$$

For every $\lambda \in \Lambda$ we let:

$$\begin{aligned} W_\lambda^{A'} &:= \{w(\lambda, a_1, \dots, a_\ell) \mid \bar{a} \in A^\ell \text{ and } \mathcal{A} \models \exists \bar{y} \lambda(\bar{a}, \bar{y})\}, \\ R_\lambda^{A'} &:= \{(w(\lambda, a_1, \dots, a_\ell), b_1, \dots, b_q) \mid \bar{a} \in A^\ell, \bar{b} \in A^q, \text{ and } \mathcal{A} \models \lambda(\bar{a}, \bar{b})\}. \end{aligned}$$

This completes the definition of \mathcal{A}' . Note that $\|\mathcal{A}'\| \leq O(\|\mathcal{A}\|^q \cdot |\varphi|)$.

For every $\lambda(x_{i_1}, \dots, x_{i_\ell}, \bar{y}) \in \Lambda$ let $\chi_\lambda(x_{i_1}, \dots, x_{i_\ell}, z_\lambda)$ be a formula expressing:

“Either $z_\lambda \in W_\lambda$ is the witness for $x_{i_1}, \dots, x_{i_\ell}$ or $z_\lambda \notin W_\lambda$ and there is no witness in W_λ for $x_{i_1}, \dots, x_{i_\ell}$.”

That is, we let

$$\chi_\lambda(x_{i_1}, \dots, x_{i_\ell}, z_\lambda) := (W_\lambda z_\lambda \wedge \bigwedge_{j=1}^{\ell} E_j x_{i_j} z_\lambda) \vee (\neg W_\lambda z_\lambda \wedge \forall y \neg (W_\lambda y \wedge \bigwedge_{j=1}^{\ell} E_j x_{i_j} y)).$$

Then, for $\bar{a} \in A^\ell, \bar{b} \in A^q$, and $c \in A'$, we have:

$$\text{If } \mathcal{A}' \models \chi_\lambda(\bar{a}, c) \text{ then } (\mathcal{A} \models \lambda(\bar{a}, \bar{b}) \iff \mathcal{A}' \models R_\lambda z_\lambda \bar{y}(c\bar{b})).$$

Let $\chi := \bigwedge_{\lambda \in \Lambda} \chi_\lambda$. Let ψ' be the formula obtained from ψ by replacing every atomic subformula $\lambda(x_{i_1}, \dots, x_{i_\ell}, \bar{y})$ by $R_\lambda z_\lambda \bar{y}$ and relativizing all quantifiers to O . Finally, we let

$$\varphi' := \exists x_1 \dots \exists x_k \exists (z_\lambda)_{\lambda \in \Lambda} (O x_1 \wedge \dots \wedge O x_k \wedge \psi' \wedge \chi).$$

Then

$$\mathcal{A} \models \varphi \iff \mathcal{A}' \models \varphi'.$$

Since χ is equivalent to a formula of the form $\forall z \chi'$ with quantifier-free χ' , the quantifier $\forall z$ can be added to the first block of ψ' (recall that $t \geq 2$). Thus, the formula φ' is equivalent to a formula in $\Sigma_{t, u+1}[t \cdot u]$. \square

Lemma 3.9. For $t \geq 2$ and $u, r \geq 1$,

$$p\text{-MC}(\Sigma_{t,u}[r]) \leq p\text{-MC}(\text{strict-}\Sigma_{t,u+1}[2]).$$

Proof. Let (\mathcal{A}, φ) be an instance of $p\text{-MC}(\Sigma_{t,u}[r])$. We shall define a structure \mathcal{A}' of vocabulary τ' of arity 2 and a strict- $\Sigma_{t,u+1}$ -sentence φ' such that $(\mathcal{A} \models \varphi \iff \mathcal{A}' \models \varphi')$.

For notational simplicity, let us assume that $t \geq 2$ is even. Suppose that

$$\varphi = \exists x_1 \dots \exists x_k \forall \bar{y}_1 \exists \bar{y}_2 \dots \forall \bar{y}_{t-1} \psi,$$

where ψ is quantifier-free and $\bar{y}_i = (y_{(i-1)u+1}, \dots, y_{iu})$. Let $\bar{y} = (y_1, \dots, y_{(t-1)u})$. Let Λ be the set of all atomic subformulas of φ , τ the vocabulary of \mathcal{A} , and $r_0 := \max\{r, (t-1) \cdot u\}$.

The vocabulary τ' contains the unary relations symbols T_1, \dots, T_{r_0} , the binary relation symbols E_1, \dots, E_{r_0} , and a binary relation symbol S_λ for every $\lambda \in \Lambda$.

The universe of the structure \mathcal{A}' is $A' := A \cup A^2 \cup \dots \cup A^{r_0}$. The relation symbols are interpreted as follows:

- For $1 \leq i \leq r_0$, $T_i^{\mathcal{A}'} = A^i$.
- For $1 \leq i \leq r_0$,

$$E_i^{\mathcal{A}'} := \{(a_i, (a_1, \dots, a_s)) \mid i \leq s \leq r_0, (a_1, \dots, a_s) \in A^s\}.$$

- For every $\lambda(x_{i_1}, \dots, x_{i_s}, \bar{y}) \in \Lambda$ we let

$$S_\lambda^{\mathcal{A}'} := \{(\bar{a}, \bar{b}) \mid \bar{a} \in A^s, \bar{b} \in A^{(t-1)u}, \text{ and } \mathcal{A} \models \lambda(\bar{a}, \bar{b})\}.$$

Note that the size of \mathcal{A}' is $O(|A|^{r_0(t-1)u})$ and thus polynomial in the size of \mathcal{A} .

To define the formula φ' , for every $\lambda(x_{i_1}, \dots, x_{i_s}, \bar{y}) \in \Lambda$ we introduce a new variable x_λ and let

$$\chi_\lambda(\bar{x}, x_\lambda) := T_s x_\lambda \wedge E_1 x_{i_1} x_\lambda \wedge \dots \wedge E_s x_{i_s} x_\lambda.$$

Furthermore, we let $\chi = \bigwedge_{\lambda \in \Lambda} \chi_\lambda$. We introduce another new variable y representing the whole tuple \bar{y} and let

$$\xi(\bar{y}, y) := T_{(t-1)u} y \wedge \bigwedge_{i=1}^{(t-1)u} E_i y_i y.$$

Finally, we let

$$\eta(v_1, \dots, v_u) := T_1 v_1 \wedge \dots \wedge T_1 v_u$$

and let φ'' be the formula

$$\begin{aligned} & \exists x_1 \dots \exists x_k \exists (x_\lambda)_{\lambda \in \Lambda} \left(\chi \wedge \right. \\ & \left. \forall \bar{y}_1 (\eta(\bar{y}_1) \rightarrow \exists \bar{y}_2 (\eta(\bar{y}_2) \wedge \dots \wedge \forall \bar{y}_{t-1} (\eta(\bar{y}_{t-1}) \rightarrow \forall y (\xi(\bar{y}, y) \rightarrow \psi'))) \dots) \right), \end{aligned}$$

where ψ' is the formula obtained from ψ by replacing each $\lambda \in \Lambda$ by the atom $S_\lambda x_\lambda y$. It is easy to see that $(\mathcal{A} \models \varphi \iff \mathcal{A}' \models \varphi'')$ and that φ'' is equivalent to a formula in $\Sigma_{t,u+1}[2]$.

However, it is not obvious how to translate φ'' to a formula in strict- $\Sigma_{t,u+1}[2]$. The problematic atoms are those of the form $E_i x_j x_\lambda$ in the formula χ . To resolve the problem, we introduce a new variable z and let, for $\lambda(x_{i_1}, \dots, x_{i_s}, \bar{y}) \in \Lambda$,

$$\chi'_\lambda(\bar{x}, x_\lambda) = T_s x_\lambda \wedge \forall z (z = x_\lambda \rightarrow E_1 x_{i_1} z \wedge \dots \wedge E_s x_{i_s} z).$$

We let $\chi' = \bigwedge_{\lambda \in \Lambda} \chi'_\lambda$ and φ''' the formula obtained from φ'' by replacing the subformula χ by χ' . It is easy to transform φ''' into a formula in strict- $\Sigma_{t,u+1}[2]$. \square

The following lemma, the last step of our proof, is a the “strict version” of a result of [11]. For the reader’s convenience, we sketch the simple proof:

Lemma 3.10. *For $t \geq 2$ and $u, r \geq 1$,*

$$p\text{-MC}(\text{strict-}\Sigma_{t,u}[r]) \leq p\text{-MC}(\text{strict-}\Sigma_{t,1}[r]).$$

Proof (sketch): For simplicity we let $t = 3$. Let (\mathcal{A}, φ) be an instance of $p\text{-MC}(\text{strict-}\Sigma_{3,u}[r])$. Let τ be the vocabulary of \mathcal{A} . The sentence φ has the form

$$\exists x_1 \dots \exists x_k \forall \bar{y} \exists \bar{z} \psi$$

with quantifier-free ψ and with $|\bar{y}| = |\bar{z}| = u$. We shall construct an equivalent instance (\mathcal{A}', φ') of $p\text{-MC}(\text{strict-}\Sigma_{t,1}[r])$.

We set $A' := A \cup A^u$. The new unary relation symbol T is interpreted in \mathcal{A}' by $T^{\mathcal{A}'} := A^u$. For every atomic subformula λ of φ , say $\lambda = Rx_2y_3z_6y_4x_2$, we introduce a new relation symbol R_λ and set

$$R_\lambda^{\mathcal{A}'} abc \text{ iff } a \in A, b, c \in A^u \text{ and } R^{\mathcal{A}} ab_3c_6b_4a, \text{ where } b_3, b_4, \text{ and } c_6 \text{ are}$$

the third and fourth component of b and the sixth component of c , respectively.

Finally, we set $\varphi' := \exists x_1 \dots \exists x_k \forall y \exists z (\bigwedge_{1 \leq i \leq k} \neg Tx_i \wedge (Ty \rightarrow (Tz \wedge \psi')))$, where ψ' is obtained from ψ by replacing any atomic subformula $\lambda = Rx_2y_3z_6y_4x_2$ by $R_\lambda x_2yz$. \square

Proof of Lemma 3.4(1): Combining Lemmas 3.8, 3.9, and 3.10, we obtain the following chain of reductions:

$$p\text{-MC}(\Sigma_{t,u}) \leq p\text{-MC}(\Sigma_{t,u+1}[t \cdot u]) \leq p\text{-MC}(\text{strict-}\Sigma_{t,u+2}[2]) \leq p\text{-MC}(\text{strict-}\Sigma_{t,1}[2]).$$

\square

Remark 3.11. The First-Order Normalisation Lemma shows that for the model-checking problems for the various classes of first-order formulas we are interested in, it suffices to consider binary vocabularies. However, these vocabularies may still contain arbitrarily many unary and binary relation symbols. We can further strengthen the results to vocabularies with just one binary relation symbol and also restrict the input structures in the model-checking problems to be (simple undirected) graphs.

For every class Φ of formulas we consider the following restriction of $p\text{-MC}(\Phi[2])$:

$p\text{-MC}(\Phi[\text{GRAPH}])$ <i>Input:</i> A graph \mathcal{G} . <i>Parameter:</i> A sentence φ in Φ <i>Problem:</i> Decide if \mathcal{G} satisfies φ .

The following strengthening of Lemma 3.4(3) is already proved in [11]:

$$(3') \ p\text{-MC}(\text{FO}) \leq p\text{-MC}(\text{FO}[\text{GRAPH}]).$$

Furthermore, it is proved in [11] that for every $t \geq 1$,

$$p\text{-MC}(\Sigma_t[2]) \leq p\text{-MC}(\Sigma_t[\text{GRAPH}]).$$

Together with Lemma 3.4(2) this yields

$$(2') \ \text{For all } t \geq 1, \ p\text{-MC}(\Sigma_t) \leq p\text{-MC}(\Sigma_t[\text{GRAPH}]).$$

The corresponding strengthening of (1) is not so obvious, and we still do not know a direct proof. Surprisingly, the result can be shown by taking a detour via propositional logic, as we will see in the next section (cf. Corollary 4.15).

4. BACK AND FORTH BETWEEN PROPOSITIONAL AND FIRST-ORDER LOGIC: THE BASIC MACHINERY

In their most basic form, the results of this section go back to Downey, Fellows, and Regan [7]. We have (slightly) improved these results in an earlier paper [11], and here we give another improvement. Moreover, we present a new proof, which we believe is significantly simpler than those known for the weaker versions of the theorems. The proofs of all results presented in the later sections of this paper are based on the ideas developed here.

As an application, we show some of the core results of Downey and Fellows structure theory for the W-hierarchy; in particular, the “Normalisation Theorem” and its sharpened version for monotone/anti-monotone formulas (cf. Chapter 12 of [6]) are easy corollaries of Theorem 4.13.

4.1. From propositional to first-order logic. In this subsection we show how to reduce weighted satisfiability problems for propositional formulas to model-checking problems for fragments of first-order logic. For this purpose we need a known algorithm computing minimal covers in hypergraphs. We recall the fact.

Let $\mathcal{H} = (H, E)$ be a *hypergraph*, i.e., H is a set, the set of *points* of \mathcal{H} , and E is a set of non-empty subsets of H , the set of *edges* of \mathcal{H} . A subset $X \subseteq H$ *covers* an edge $e \in E$, if $X \cap e \neq \emptyset$; X covers \mathcal{H} , if X covers all edges of \mathcal{H} . If X , but no proper subset of X , covers \mathcal{H} , then X is a *minimal cover* of \mathcal{H} . The *arity* of a hypergraph is the maximum cardinality of its edges.

Lemma 4.1. *There is an algorithm that, given a hypergraph \mathcal{H} of arity at most d , computes in time $O(k \cdot d^k \cdot \|\mathcal{H}\|)$ a list of all minimal covers of \mathcal{H} of size at most k .*

Proof. The algorithm is a straightforward generalisation of a standard algorithm (using the bounded search tree technique, cf. [6]) showing that the parameterized vertex cover problem is in FPT.

Let $\mathcal{H} = (H, E)$ be as in the statement of the lemma. Let e_1, \dots, e_m be an enumeration of E . The algorithm builds a labelled d -ary tree of depth $\leq k$. The labels of the nodes are pairs (X, i) , where $X \subseteq H$ with $|X| \leq k$ and $0 \leq i \leq m$. Label (X, i) gives the information that X covers the edges e_1, \dots, e_i , but not e_{i+1} (if $i + 1 \leq m$).

The construction of the tree is by induction: The label of the root is $(\emptyset, 0)$. Suppose that a node t is labelled by (X, i) . If $i = m$ or if the depth of t is k , then t has no child. Otherwise, let $e_{i+1} = \{h_1, \dots, h_s\}$. Node t has children t_1, \dots, t_s . For $1 \leq j \leq s$, the label of t_j is (X_j, i_j) , where $X_j = X \cup \{h_j\}$ and i_j is the maximum index such that X_j covers e_1, \dots, e_{i_j} .

One easily verifies that any set Y of at most k points is a cover of \mathcal{H} if and only if there is a leaf of the tree labelled by (X, m) with $X \subseteq Y$. Thus, to obtain the list of all minimal covers of size $\leq k$, the algorithm checks for every leaf labelled by (X, m) , whether X is a minimal cover. For this purpose, it simply tests for each of the at most k subsets obtained by removing a single element from X if it is a cover. □

In a first step (Lemma 4.2) we give the translation of formulas in $\Gamma_{1,d}$ to first-order logic. Recall that a set $\{X_1, \dots, X_k\}$ of propositional variables represents the assignment that sets X_1, \dots, X_k to TRUE and all other variables to FALSE.

Lemma 4.2. *For all $d, k \geq 1$ and for all formulas $\alpha(X_1, \dots, X_m) \in \Gamma_{1,d}$ there are*

- a structure $\mathcal{A} = \mathcal{A}_{\wedge, \alpha(X_1, \dots, X_m), d, k}$ with universe $A := \{1, \dots, m\}$,
- a quantifier-free formula $\psi = \psi_{\wedge, d, k}(x_1, \dots, x_k)$ that only depends on d and k , but not on α

such that the mapping $(\alpha, d, k) \mapsto (\mathcal{A}, \psi)$ is computable in time $k^{d+2} \cdot d^k \cdot p(|\alpha|)$ for some polynomial p and such that for pairwise distinct $m_1, \dots, m_k \in A$

$$\{X_{m_1}, \dots, X_{m_k}\} \text{ satisfies } \alpha \iff \mathcal{A} \models \psi(m_1, \dots, m_k).$$

Proof. Let $d \geq 1$, $\alpha(X_1, \dots, X_m) \in \Gamma_{1,d}$, and $k \in \mathbb{N}$ be given, say,

$$\alpha = \bigwedge_{i \in I} \delta_i, \quad \text{where each } \delta_i \text{ is the disjunction of } \leq d \text{ literals.}$$

We may assume that every δ_i has the form

$$\neg X_{i_1} \vee \dots \vee \neg X_{i_r} \vee X_{j_1} \vee \dots \vee X_{j_s} \tag{4.1}$$

with $0 \leq r, s$ and $1 \leq r + s \leq d$ and with pairwise distinct X_{i_1}, \dots, X_{j_s} .

We call $t := (r, s)$ the *type* of δ_i . The structure \mathcal{A} has universe $A := \{1, \dots, m\}$; for every type $t = (r, s)$, the structure \mathcal{A} contains the r -ary relation

$$V_t^{\mathcal{A}} := \{(i_1, \dots, i_r) \mid \text{there are } j_1, \dots, j_s \text{ such that clause (4.1) occurs in } \alpha\}.$$

The structure \mathcal{A} contains further relations that will be defined later.

The formula ψ will have the form $\bigwedge_{t \text{ type}} \psi_t$, where $\psi_t = \psi_t(x_1, \dots, x_k)$ will express in \mathcal{A} that X_{x_1}, \dots, X_{x_k} satisfies every clause of type t of α . If $t = (r, 0)$ set

$$\psi_t := \bigwedge_{1 \leq i_1, \dots, i_r \leq k} \neg V_t x_{i_1} \dots x_{i_r}.$$

Let $t = (r, s)$ with $s \neq 0$. Fix $(i_1, \dots, i_r) \in V_t^{\mathcal{A}}$. Then, for $1 \leq m_1, \dots, m_k \leq m$,

the assignment X_{m_1}, \dots, X_{m_k} satisfies all clauses $\neg X_{i_1} \vee \dots \vee \neg X_{i_r} \vee X_{j_1} \vee \dots \vee X_{j_s}$ in α

if and only if

X_{m_1}, \dots, X_{m_k} satisfies $\neg X_{i_1} \vee \dots \vee \neg X_{i_r}$ or $\{m_1, \dots, m_k\}$ is a cover of the hypergraph $\mathcal{H} (= \mathcal{H}_t(i_1, \dots, i_r)) := (H, E)$ with

$$H := \{1, \dots, m\} \text{ and}$$

$$E := \{\{j_1, \dots, j_s\} \mid \neg X_{i_1} \vee \dots \vee \neg X_{i_r} \vee X_{j_1} \vee \dots \vee X_{j_s} \text{ occurs in } \alpha\}.$$

Let C_1, \dots, C_{d^k} be an enumeration (with repetitions if necessary) of the minimal covers of \mathcal{H} of size $\leq k$. View every C_i as a sequence of length k (with repetitions if necessary). For $u = 1, \dots, d^k$ and $\ell = 1, \dots, k$ add to \mathcal{A} the $(r+1)$ -ary relations $L_{t,u,\ell}^{\mathcal{A}}$, where

$$L_{t,u,\ell}^{\mathcal{A}} := \{(i_1, \dots, i_r, v) \mid v \text{ is the } \ell\text{th element of the } u\text{th cover } C_u \text{ of } \mathcal{H}_t(i_1, \dots, i_r)\}$$

(if $\mathcal{H}_t(i_1, \dots, i_r)$ has no cover of size $\leq k$, then $L_{r,u,\ell}^{\mathcal{A}}$ contains no tuple of the form (i_1, \dots, i_r, v)).

Now the preceding equivalence shows that we can set

$$\psi_t := \bigwedge_{1 \leq i_1, \dots, i_r \leq k} (V_t x_{i_1} \dots x_{i_r} \rightarrow \bigvee_{1 \leq u \leq d^k} \bigwedge_{1 \leq \ell \leq k} \bigvee_{1 \leq j \leq k} L_{t,u,\ell} x_{i_1} \dots x_{i_r} x_j).$$

It is easy to see that \mathcal{A} and ψ can be computed from α , d , and k in time $k^{d+2} \cdot d^k \cdot p(\alpha)$ for some polynomial p ; the only nontrivial part is the computation of the list of minimal covers, which is taken care of by Lemma 4.1. \square

Corollary 4.3. For all $d, k \geq 1$ and for all formulas $\alpha(X_1, \dots, X_m) \in \Delta_{1,d}$ there are

- a structure $\mathcal{A} = \mathcal{A}_{\forall, \alpha(X_1, \dots, X_m), d, k}$ with universe $A := \{1, \dots, m\}$,
- a quantifier-free formula $\psi = \psi_{\forall, d, k}(x_1, \dots, x_k)$ that only depends on d and k , but not on α

such that the mapping $(\alpha, d, k) \mapsto (\mathcal{A}, \psi)$ is computable in time $k^{d+2} \cdot d^k \cdot p(|\alpha|)$ for some polynomial p and for pairwise distinct $m_1, \dots, m_k \in A$

$$\{X_{m_1}, \dots, X_{m_k}\} \text{ satisfies } \alpha \iff \mathcal{A} \models \psi(m_1, \dots, m_k).$$

Proof. Exploiting the fact that $\neg\alpha$ is equivalent to a formula α' in $\Gamma_{1,d}$, we let \mathcal{A} be the structure constructed in Lemma 4.2 for the formula α' and $\psi_{\forall, d, k} := \neg\psi_{\wedge, d, k}$. \square

Corollary 4.4. $\text{WSAT}(\Gamma_{1,d} \cup \Delta_{1,d}) \leq p\text{-MC}(\Sigma_1)$.

Proof. Given an instance (α, k) of $\text{WSAT}(\Gamma_{1,d} \cup \Delta_{1,d})$, compute (\mathcal{A}, ψ) as in Lemma 4.2 or Corollary 4.3. Let

$$\varphi = \exists x_1 \dots \exists x_k \left(\bigwedge_{1 \leq i < j \leq k} x_i \neq x_j \wedge \psi \right).$$

Then

$$\alpha \text{ is } k\text{-satisfiable} \iff \mathcal{A} \models \varphi,$$

which gives the desired reduction. \square

Lemma 4.2 and Corollary 4.3 show how to translate formulas in $\Gamma_{1,d} \cup \Delta_{1,d}$ to quantifier-free formulas. When translating propositional formulas of a weighted satisfiability problem into first-order formulas of a model-checking problem, every additional big conjunction and big disjunction leads to a universal and an existential quantifier, respectively. The following proposition is based on this observation.

Proposition 4.5. For all $d, t \geq 1$

$$\text{WSAT}(\Delta_{t+1,d}) \leq p\text{-MC}(\Sigma_{t,1}).$$

Proof. Fix $d, t \geq 1$. Let (α, k) be an instance of $\text{WSAT}(\Delta_{t+1,d})$. We shall construct a structure \mathcal{A} and a $\Sigma_{t,1}$ -sentence φ such that

$$\alpha \text{ is } k\text{-satisfiable} \iff \mathcal{A} \models \varphi. \quad (4.2)$$

Let the variables of α be X_1, \dots, X_m . We assume that t is even, the case “ t is odd” is handled analogously. Thus α is of the form

$$\bigvee_{i_1 \in I_1} \bigwedge_{i_2 \in I_{i_1}} \dots \bigwedge_{i_t \in I_{i_1 \dots i_{t-1}}} \delta_{(i_1, \dots, i_t)},$$

where the $\delta_{\vec{i}} \in \Delta_{1,d}$. A simple argument shows that we can pass to an equivalent formula α' with $|\alpha'| \leq |\alpha|^t$ of the form

$$\bigvee_{i_1 \in I_1} \bigwedge_{i_2 \in I_2} \dots \bigwedge_{i_t \in I_t} \delta_{(i_1, \dots, i_t)},$$

so we assume that α itself already has this form. Let $\bar{I} := I_1 \times \dots \times I_t$.

The structure \mathcal{A} consists of two parts: The first part is the tree \mathcal{T} of height t obtained from the “parse tree” of α by removing all nodes that correspond to small subformulas of α . The edge relation of this tree, directed from the root to the leaves (which by definition have height 0), is represented by the binary relation $E^{\mathcal{A}}$. Moreover, we add a unary relation symbol Root and let $\text{Root}^{\mathcal{A}}$ be the singleton containing the root of \mathcal{T} . Note that each leaf of \mathcal{T} corresponds to a subformula $\delta_{\vec{i}}$, for

some $\bar{i} \in \bar{I}$, of α . We denote the leaf corresponding to $\delta_{\bar{i}}$ by $\ell_{\bar{i}}$. Each node of \mathcal{T} of height s corresponds to a subformula contained in $\Gamma_{s+1,d}$ if s is odd or $\Delta_{s+1,d}$ if s is even.

The universe of the second part of \mathcal{A} is $\{1, \dots, m\}$ (the set of indices of the variables of α). For every $\bar{i} \in \bar{I}$, let $\mathcal{A}_{\bar{i}} = \mathcal{A}_{\forall, \delta_{\bar{i}}(X_1, \dots, X_m), d, k}$ be the structure defined in Corollary 4.3. Essentially, the second part of \mathcal{A} simply consists of all $\mathcal{A}_{\bar{i}}$ s. However, all $\mathcal{A}_{\bar{i}}$ s have the same universe $\{1, \dots, m\}$. To keep them apart, we “tag” the tuples belonging to a relation in $\mathcal{A}_{\bar{i}}$ with the leaf $\ell_{\bar{i}}$ of \mathcal{T} that corresponds to $\delta_{\bar{i}}$. More precisely, for each r -ary relation symbol R in the vocabulary of the $\mathcal{A}_{\bar{i}}$ s, the vocabulary of \mathcal{A} contains an $(r+1)$ -ary relation symbol R' . We let

$$(R')^{\mathcal{A}} := \{(\ell_{\bar{i}}, a_1, \dots, a_r) \mid \bar{i} \in \bar{I}, (a_1, \dots, a_r) \in R^{\mathcal{A}_{\bar{i}}}\}.$$

Finally, to be able to tell the two parts of \mathcal{A} apart, we add one unary relation symbol V and let $V^{\mathcal{A}} := \{1, \dots, m\}$. This completes the definition of \mathcal{A} .

We now define, by induction on $s \geq 0$, formulas $\psi_s(y, x_1, \dots, x_k)$ such that for every node b of \mathcal{T} of height s , corresponding to a subformula β of α , and all pairwise distinct $a_1, \dots, a_k \in \{1, \dots, m\}$ we have

$$\{X_{a_1}, \dots, X_{a_k}\} \text{ satisfies } \beta \iff \mathcal{A} \models \psi_s(b, a_1, \dots, a_k). \quad (4.3)$$

$\psi_0(y, x_1, \dots, x_k)$ is the formula obtained from the formula $\psi_{\forall, d, k}$ of Corollary 4.3 by replacing each atomic subformula $Rx_1 \dots x_r$ by $R'yx_1 \dots x_r$. Then for $s = 0$, (4.3) follows from our construction of \mathcal{A} and Corollary 4.3.

For even $s \geq 0$, we let

$$\psi_{s+1}(y, x_1, \dots, x_k) := \forall z (Eyz \rightarrow \psi_s(z, x_1, \dots, x_k)),$$

and (4.3) follows from the fact that all nodes of height $s+1$ correspond to conjunction of formulas corresponding to nodes of height s . Similarly, for odd $s \geq 0$ we let

$$\psi_{s+1}(y, x_1, \dots, x_k) := \exists z (Eyz \wedge \psi_s(z, x_1, \dots, x_k)).$$

Finally, we let

$$\varphi := \exists x_1 \dots \exists x_k \exists y \left(\bigwedge_{i=1}^k Vx_i \wedge \bigwedge_{\substack{i,j=1 \\ i \neq j}}^k x_i \neq x_j \wedge \text{Root } y \wedge \psi_t(y, x_1, \dots, x_k) \right).$$

It is easy to see that φ is equivalent to a formula in $\Sigma_{t,1}$. □

We consider a more general weighted satisfiability problem, in which the depth of the formula is not fixed but treated as a parameter. The preceding proof yields:

Corollary 4.6. $P \leq p\text{-MC}(\Sigma_{t,1})$, where P is the parameterized problem

Input: $k \in \mathbb{N}$ and $\alpha \in \Delta_{t+1,k}$.
Parameter: k .
Problem: Decide if α is k -satisfiable.

For later reference, let us state the following lemma, which is an immediate consequence of the preceding proof:

Lemma 4.7. *Let $t, d \geq 1$. Then for all $k \geq 1$ and for all formulas $\alpha(X_1, \dots, X_m) \in \Delta_{t+1,d}$ there are*

- a structure \mathcal{A} with a unary relation $V^{\mathcal{A}} = \{1, \dots, m\}$,

- a formula $\psi(x_1, \dots, x_k)$ of the form $\exists y \exists y_1 \forall y_2 \exists y_3 \dots Q_t y_t \psi'$, where $Q_t = \exists$ if t is odd and $Q_t = \forall$ if t is even and ψ' is quantifier free, and the formula ψ only depends on t, d, k , but not on α ,

such that the mapping $(\alpha, k) \mapsto (\mathcal{A}, \psi)$ is fixed-parameter tractable and for pairwise distinct $m_1, \dots, m_k \in V^{\mathcal{A}}$,

$$\{X_{m_1}, \dots, X_{m_k}\} \text{ satisfies } \alpha \iff \mathcal{A} \models \psi(m_1, \dots, m_k).$$

Together with Lemma 3.1, Proposition 4.5 immediately yields:

Corollary 4.8. *For all $d \geq t \geq 1$ we have*

$$\text{WSAT}(\Omega_{t,d}) \leq p\text{-MC}(\Sigma_{t,1}).$$

4.2. From first-order to propositional logic. We turn to a reduction from model-checking problems for the fragments $\Sigma_{t,u}$ to weighted satisfiability problems for propositional formulas. We shall see that single quantifiers (or blocks of quantifiers of bounded length) translate into big disjunctions and conjunctions; the leading unbounded block yields the propositional variables, and its length yields the parameter.

We start by collecting some simple facts.

Let A be a set and $k \geq 1$. For all $a \in A$ and $1 \leq i \leq k$, let $X_{i,a}$ be a propositional variable with $X_{i,a} \neq X_{j,b}$ for $(i, a) \neq (j, b)$. Let V be the set of all these propositional variables. Let us call an assignment $S \in 2^V$ *functional* if for each i there is exactly one a such that $X_{i,a}$ is TRUE. The proof of the following lemma is straightforward.

Lemma 4.9. *Let $V = \{X_{i,a} \mid 1 \leq i \leq k, a \in A\}$.*

(1) *For*

$$\chi^- := \bigwedge_{\substack{1 \leq i \leq k \\ a, b \in A, a \neq b}} (\neg X_{i,a} \vee \neg X_{i,b}) \quad \text{and} \quad \chi^+ := \bigwedge_{i=1}^k \bigvee_{b \in A} X_{i,b}$$

and for every assignment $S \subseteq V$ of weight $|S| = k$ we have

$$S \text{ satisfies } \chi^- \iff S \text{ is functional} \iff S \text{ satisfies } \chi^+.$$

Observe that $\chi^- \in \Gamma_{1,2}^-$ and $\chi^+ \in \Gamma_{2,1}^+$. In addition, we may as well consider χ^- as a formula in $\Gamma_{2,1}^-$.

(2) *Let \mathcal{A} be a structure with universe A , $\bar{b} \in A^s$, $1 \leq i \leq k$, and $\psi(x_i, \bar{y})$ a formula in the vocabulary of \mathcal{A} with $\bar{y} = y_1 \dots y_s$. For*

$$\xi^\vee(\mathcal{A}, \psi, \bar{b}) := \bigvee_{\substack{a \in A \\ \mathcal{A} \models \psi(a, \bar{b})}} X_{i,a} \quad \text{and} \quad \xi^\wedge(\mathcal{A}, \psi, \bar{b}) := \bigwedge_{\substack{a \in A \\ \mathcal{A} \not\models \psi(a, \bar{b})}} \neg X_{i,a}$$

and for every functional assignment $S \subseteq V$ with, say, $S(X_{i,a_0}) = \text{TRUE}$ we have

$$S \text{ satisfies } \xi^\vee(\mathcal{A}, \psi, \bar{b}) \iff \mathcal{A} \models \psi(a_0, \bar{b}) \iff S \text{ satisfies } \xi^\wedge(\mathcal{A}, \psi, \bar{b}).$$

Proposition 4.10. *Let $t \geq 2$.*

- (1) *If t is even then $p\text{-MC}(\Sigma_{t,1}) \leq \text{WSAT}(\Gamma_{t,1}^+)$.*
- (2) *If t is odd then $p\text{-MC}(\Sigma_{t,1}) \leq \text{WSAT}(\Gamma_{t,1}^-)$.*

Proof. Let $t \geq 2$ and (\mathcal{A}, φ) an instance of $p\text{-MC}(\Sigma_{t,1})$. By the First-Order Normalisation Lemma we may assume that $\varphi \in \text{strict-}\Sigma_{t,1}$. We shall define a propositional formula α of the desired syntactical form such that

$$\mathcal{A} \models \varphi \iff \alpha \text{ is } k\text{-satisfiable.} \quad (4.4)$$

Suppose that

$$\varphi = \exists x_1 \dots \exists x_k \forall y_1 \exists y_2 \dots Q_{t-1} y_{t-1} \psi,$$

where $Q_{t-1} = \exists$ if t is odd and $Q_{t-1} = \forall$ if t is even and ψ is quantifier-free. We shall make further assumptions on φ when we branch depending on t later. Let $\bar{y} := (y_1, \dots, y_{t-1})$. Without loss of generality we assume that ψ is in negation normal form. We let Λ denote the set of all literals occurring in ψ (deviating from our earlier proofs, where Λ denoted a set of atoms). Recall that, because φ is in $\text{strict-}\Sigma_{t,1}$, at most one of the variables x_1, \dots, x_k occurs in a literal $\lambda \in \Lambda$.

The formula α will have propositional variables $X_{i,a}$ for all $a \in A$ and $1 \leq i \leq k$. The intended meaning of $X_{i,a}$ is: ‘‘First-order variable x_i takes value a .’’ Let V be the set of all these propositional variables.

Now assume that t is even. Without loss of generality we may assume that $\psi = \bigwedge_{i=1}^{\ell} \bigvee_{j=1}^{m_i} \lambda_{ij}$ is in conjunctive normal form, i.e.,

$$\varphi = \exists x_1 \dots \exists x_k \forall y_1 \exists y_2 \dots \forall y_{t-1} \bigwedge_{i=1}^{\ell} \bigvee_{j=1}^{m_i} \lambda_{ij},$$

We use the formulas χ^+ and $\xi^{\vee}(\dots)$ of the preceding lemma and let

$$\alpha := \chi^+ \wedge \bigwedge_{b_1 \in A} \bigvee_{b_2 \in A} \dots \bigwedge_{b_{t-1} \in A} \bigwedge_{i=1}^{\ell} \bigvee_{j=1}^{m_i} \xi^{\vee}(\mathcal{A}, \lambda_{ij}, b_1, \dots, b_{t-1}).$$

Clearly, α satisfies (4.4) and can easily be transformed into an equivalent $\Gamma_{t,1}^+$ -formula.

For odd $t \geq 3$ we proceed similarly, except that we assume that ψ is in disjunctive normal form and that we replace χ^+ and $\xi^{\vee}(\dots)$ by χ^- and $\xi^{\wedge}(\dots)$, respectively. \square

Proposition 4.11.

$$p\text{-MC}(\Sigma_1) \leq \text{WSAT}(\Gamma_{1,2}^-).$$

It is straightforward to derive this proposition from the well known result that $p\text{-MC}(\Sigma_1[2])$ is reducible to the parameterized clique problem (cf. e.g. [13]). However, to keep this paper self-contained we give a direct proof.

Proof of Proposition 4.11: Let (\mathcal{A}, φ) be an instance of $p\text{-MC}(\Sigma_1)$. By the First-Order Normalisation Lemma we may assume that the vocabulary of φ is binary. We may further assume that φ is of the form

$$\exists x_1 \dots \exists x_k \bigvee_{p=1}^{\ell} \bigwedge_{q=1}^{m_p} \lambda_{pq},$$

where each λ_{pq} is a literal.

In a first step of the proof we shall define formulas $\alpha_1, \dots, \alpha_{\ell} \in \Gamma_{1,2}^-$ such that for $1 \leq p \leq \ell$,

$$\mathcal{A} \models \exists x_1 \dots \exists x_k \bigwedge_{q=1}^{m_p} \lambda_{pq} \iff \alpha_p \text{ is } k\text{-satisfiable.} \quad (4.5)$$

Thus

$$\mathcal{A} \models \varphi \iff \text{exists } p, 1 \leq p \leq \ell : \alpha_p \text{ is } k\text{-satisfiable.} \quad (4.6)$$

Let us fix p . We let V_p be the set of propositional variables $X_{i,a}^p$ for $1 \leq i \leq k$ and $a \in A$. Let χ_p^- be the corresponding formula χ^- according to Lemma 4.9(1) for $V = V_p$.

Similarly as $\chi^\wedge(\dots)$ in Lemma 4.9(2), we define, for $1 \leq q \leq m_p$,

$$\xi_{pq} := \bigwedge_{\substack{a_1, a_2 \in A \\ \mathcal{A} \not\models \lambda_{pq}(a_1, a_2)}} (\neg X_{i_1, a_1}^p \vee \neg X_{i_2, a_2}^p),$$

where we assume that the free variables of λ_{pq} are among x_{i_1}, x_{i_2} . Recall that the vocabulary of φ is binary, thus a literal never has more than two free variables. For every functional assignment $\{X_{1, a_1}^p, \dots, X_{k, a_k}^p\} \in 2^{V_p}$ we have

$$\{X_{1, a_1}^p, \dots, X_{k, a_k}^p\} \text{ satisfies } \xi_{pq} \iff \mathcal{A} \models \lambda_{pq}(a_{i_1}, a_{i_2}).$$

Thus

$$\alpha_p := \chi_p^- \wedge \bigwedge_{q=1}^{m_p} \xi_{pq}$$

satisfies (4.5).

By (4.6), it remains to define a formula α such that

$$\alpha \text{ is } k\text{-satisfiable} \iff \text{exists } p, 1 \leq p \leq \ell : \alpha_p \text{ is } k\text{-satisfiable.} \quad (4.7)$$

Let $V := V_1 \cup \dots \cup V_\ell$. We call an assignment $S \in 2^V$ *good* if there is a $p, 1 \leq p \leq \ell$ such that $S \subseteq V_p$. The formula

$$\chi := \bigwedge_{\substack{1 \leq i_1, i_2 \leq k \\ a_1, a_2 \in A \\ 1 \leq p_1 < p_2 \leq \ell}} (\neg X_{i_1, a_1}^{p_1} \vee \neg X_{i_2, a_2}^{p_2})$$

says that an assignment is good. Note that if $S \subseteq V_p$ then S satisfies $\alpha_{p'}$ for all $p' \neq p$, because variables only occur negatively in $\alpha_{p'}$. Thus S satisfies $\bigwedge_{r=1}^{\ell} \alpha_r$ if and only if S satisfies α_p . Therefore,

$$\alpha := \chi \wedge \bigwedge_{p=1}^{\ell} \alpha_p$$

satisfies (4.7). Altogether, $(\mathcal{A}, \varphi) \mapsto (\alpha, k)$ is an fpt-reduction. \square

4.3. The W-hierarchy. We apply the results of the preceding two sections to the W-hierarchy. By definition the t th class of this hierarchy consists of all parameterized problems fpt-reducible to the weighted satisfiability problem $\text{WSAT}(\Omega_{t,d})$ for some d :

Definition 4.12. For $t \geq 1$, $\text{W}[t] := [\{\text{WSAT}(\Omega_{t,d}) \mid d \geq t\}]^{\text{fpt}}$.

Putting all together, we get:

Theorem 4.13. For $t \geq 1$,

$$\text{W}[t] = [p\text{-MC}(\Sigma_{t,1}[2])]^{\text{fpt}} = [p\text{-MC}(\Sigma_{t,u}) \mid u \geq 1]^{\text{fpt}}.$$

Moreover,

- if t is even, $\text{W}[t] = [\text{WSAT}(\Gamma_{t,1}^+)]^{\text{fpt}}$;
- if $t \geq 3$ is odd, $\text{W}[t] = [\text{WSAT}(\Gamma_{t,1}^-)]^{\text{fpt}}$;
- $\text{W}[1] = [\text{WSAT}(\Gamma_{1,2}^-)]^{\text{fpt}}$.

Proof. All statements are immediate consequences of preceding results, e.g.:

$$\begin{aligned} \text{WSAT}(\Omega_{t,d}) &\leq \text{WSAT}(\Delta_{t,2^d}) \quad (\text{by Lemma 3.1}) \\ &\leq p\text{-MC}(\Sigma_{t,1}[2]) \quad (\text{by Proposition 4.5 and Lemma 3.4}) \\ &\leq \text{WSAT}(\Gamma_{t,2}) \quad (\text{by Proposition 4.10 and Proposition 4.11}). \end{aligned}$$

□

Now, Proposition 4.5 yields:

Corollary 4.14. *For all $t, d \geq 1$,*

$$\text{WSAT}(\Delta_{t+1,d}) \in \mathbb{W}[t].$$

If we identify $\mathbb{W}[0]$ with FPT, then the statement of the preceding corollary is true for $t = 0$, too; in fact, there is even a polynomial time algorithm deciding for given (α, k) with $\alpha \in \Delta_{1,d}$ and $k \in \mathbb{N}$ if α is k -satisfiable.

The following corollary fills the gap that was left open in Remark 3.11.

Corollary 4.15. *For all $t \geq 2, u \geq 1$,*

$$p\text{-MC}(\Sigma_{t,u}) \leq p\text{-MC}(\Sigma_{t,1}[\text{GRAPH}]).$$

Proof. Let $t \geq 2$, say $t = 3$. By Theorem 4.13, it suffices to show

$$\text{WSAT}(\Gamma_{t,1}^-) \leq p\text{-MC}(\Sigma_{t,1}[\text{GRAPH}]).$$

Fix $\alpha \in \Gamma_{t,1}^-$ and $k \in \mathbb{N}$. Let \mathcal{G} be the graph obtained from the tree of α by removing the leaves, identifying the nodes corresponding to negative literals with the same variable and adding two cycles of length 3 to its root r . We can assume that in \mathcal{G} all branches from the root to a leaf of \mathcal{G} have the same length, namely 3. We say that pairwise distinct w_1, w_2, w_3, w_4 with $Ew_1w_2, Ew_2w_3, Ew_3w_4$, with $w_1 = r$ and with $w_4 = x$ “witness that x is a leaf”. Then, as formula φ we can choose a $\Sigma_{3,1}$ -formula equivalent to

$$\begin{aligned} &\exists x_1 \dots \exists x_k \exists x \exists u_1 \exists u_2 \exists v_1 \exists v_2 \exists \bar{w}_1 \dots \exists \bar{w}_k (\text{“}x, u_1, u_2 \text{ and } x, v_1, v_2 \text{ are distinct cycles”} \\ &\quad \wedge \bigwedge_{1 \leq i < j \leq k} x_i \neq x_j \wedge \text{“}\bar{w}_1 \text{ witness that } x_1 \text{ is a leaf”} \wedge \dots \wedge \text{“}\bar{w}_k \text{ witness that } x_k \text{ is a leaf”} \\ &\quad \wedge \forall y ((Exy \rightarrow \exists z (Eyz \wedge z \neq x \wedge \neg Ezx_1 \wedge \dots \wedge \neg Ezx_k))). \end{aligned}$$

Clearly, $(\alpha, k) \in \text{WSAT}(\Gamma_{t,1}^-) \iff \mathcal{G} \models \varphi$. □

Theorem 4.13 shows that for the weighted satisfiability problem for $\Gamma_{t,d}$ the relevant class of formulas are the monotone ones in case t is even, and the antimonotone ones in case t is odd. The so-called monotone and antimonotone collapse theorem due to Downey and Fellows [3, 4] states that for all $t, d \geq 1$,

$$\text{WSAT}(\Gamma_{2 \cdot t, d}^-) \in \mathbb{W}[2 \cdot t - 1] \quad \text{and} \quad \text{WSAT}(\Gamma_{2 \cdot t + 1, d}^+) \in \mathbb{W}[2 \cdot t].$$

We get the following stronger result:

Theorem 4.16. *For all $t, d \geq 1$,*

$$\text{WSAT}(\Delta_{2 \cdot t + 1, d}^-) \in \mathbb{W}[2 \cdot t - 1] \quad \text{and} \quad \text{WSAT}(\Delta_{2 \cdot t + 2, d}^+) \in \mathbb{W}[2 \cdot t].$$

Proof. Fix $k \in \mathbb{N}$. First, consider a formula α in $\Gamma_{1,d}^+$ with variables X_1, \dots, X_m , say

$$\alpha = \bigwedge_{i \in I} (Y_{i1} \vee \dots \vee Y_{ir_i}).$$

Compute the minimal covers of size $\leq k$ of the hypergraph $\mathcal{H} = (H, E)$, where $H := \{1, \dots, m\}$ and

$$E := \{\{i_1, \dots, i_{r_i}\} \mid \text{for some } i \in I: X_{i_1} = Y_{i1}, \dots, X_{i_{r_i}} = Y_{ir_i}\}.$$

For every such cover C let γ_C be the conjunction of the variables X_j with $j \in C$. Then, γ_C is the conjunction of at most k variables. With respect to assignments of α of weight k , the formulas α and $\bigvee_{C \text{ cover}} \gamma_C$ are equivalent.

Now, let $\beta \in \Delta_{2,t+2,d}^+$ and $k \in \mathbb{N}$. We replace every subformula $\alpha \in \Gamma_{1,d}^+$ by the corresponding $\bigvee_{C \text{ cover}} \gamma_C$, thus obtaining a formula β^* in $\Delta_{2,t+1,k}^+$. Then, the result follows from Corollary 4.6. (In case β^* but not β has $e < k$ variables, we check if β^* is e -satisfiable.)

The proof for $\text{WSAT}(\Delta_{2,t+1,d}^-)$ is obtained by treating subformulas in $\Delta_{1,d}^-$ in the dual way. \square

5. BACK AND FORTH BETWEEN PROPOSITIONAL AND FIRST-ORDER LOGIC: THE EXTENSIONS

5.1. The W^* -hierarchy. In [8], Downey, Fellows, and Taylor introduced the W^* -hierarchy and showed that the first two levels of the W^* -hierarchy coincide with first the two levels of the W -hierarchy ([8], [5]). We first recall the definition of the W^* -hierarchy and then give complete model-checking problems for the classes of this hierarchy. This characterization allows simple proofs of $W^*[1] = W[1]$ and $W^*[2] = W[2]$.

The crucial difference between the W -hierarchy and the W^* -hierarchy is that instead of being fixed, in the definition of the W^* -hierarchy the depth is treated as a parameter.

For a set Γ of propositional formulas we let

$\text{WSAT}^*(\Gamma)$ <i>Input:</i> $k \in \mathbb{N}$ and $\alpha \in \Gamma$ such that the depth of α is at most k . <i>Parameter:</i> k . <i>Problem:</i> Decide if α is k -satisfiable.
--

For every $t \geq 0$ we let Ω_t denote the set of all propositional formulas of weft at most t .

Definition 5.1. For $t \geq 1$,

$$W^*[t] := [\text{WSAT}^*(\Omega_t)]^{\text{fpt}}.$$

Before we turn to the first-order characterisation of the W^* -hierarchy, we normalise the propositional formulas involved. For $k \geq 1$ we define two new families $\Gamma_{t,k}^*$ and $\Delta_{t,k}^*$ of propositional formulas. We use $\bigwedge_{i=1}^k \alpha_i$ as an abbreviation for the formula $(\dots((\alpha_1 \wedge \alpha_2) \wedge \alpha_3) \dots \wedge \alpha_k)$. Similarly, we use $\bigvee_{i=1}^k \alpha_i$.

- We let $\Gamma_{1,k}^* = \Gamma_{1,k}$ and $\Delta_{1,k}^* = \Delta_{1,k}$.
- For $t \geq 2$, we let $\Gamma_{t,k}^*$ be the class of all formulas of the form

$$\bigwedge_{i \in I} \bigvee_{j=1}^k \alpha_{ij}$$

where I is an arbitrary (finite) index set and $\alpha_{ij} \in \Gamma_{t-1,k}^* \cup \Delta_{t-1,k}^*$ for all $i \in I, 1 \leq j \leq k$. Similarly, we let $\Delta_{t,k}^*$ be the class of all formulas of the form

$$\bigvee_{i \in I} \bigwedge_{j=1}^k \alpha_{ij}$$

where I and the α_{ij} are as above.

Observe that $\Gamma_{t,k}^* \cup \Delta_{t,k}^* \subseteq \Omega_{t,t \cdot k}$.

The following lemma, which may be viewed as the starred analogon of Lemma 3.1, is essentially due to Downey, Fellows, and Taylor [8]. Denote by PROP the class of all propositional formulas.

Lemma 5.2. *Let $t \geq 1$. Then there is an fpt-algorithm that assigns to every instance (α, k) of $\text{WSAT}^*(\Omega_t)$ an instance (β, ℓ) of $\text{WSAT}^*(\text{PROP})$ with $\beta \in \Delta_{t+1,\ell}^*$.*

Proof. By induction on $t \geq 1$ we first prove that every formula α in $\Omega_{t,k}$ whose outermost connective is a big conjunction is equivalent to a formula in $\Gamma_{t,2^k}^*$ and simultaneously that every formula in $\Omega_{t,k}$ whose outermost connective is a big disjunction is equivalent to a formula in $\Delta_{t,2^k}^*$.

Suppose that $t \geq 1$ and $\alpha \in \Omega_{t,k}$ is of the form $\bigwedge_{i \in I} \beta_i$. By the induction hypothesis, we can assume that each β_i is a Boolean combination of at most 2^k formulas in $\Gamma_{t-1,2^k}^* \cup \Delta_{t-1,2^k}^*$ or, if $t = 1$, propositional variables. Transforming these Boolean combinations into conjunctive normal form, which can be achieved by an fpt-reduction since the number (at most 2^k) of formulas is bounded in terms of the parameter, and merging the outermost conjunctions we obtain a formula of the desired form. Formulas α whose outermost connective is a big disjunction can be treated analogously.

Now it easily follows that there is an fpt-algorithm that assigns to every instance (α, k) of $\text{WSAT}^*(\Omega_t)$ a formula $\alpha' \in \Delta_{t+1,2^k}^*$ such that $(\alpha \text{ is } k\text{-satisfiable}) \iff (\alpha' \text{ is } k\text{-satisfiable})$. Let $\alpha' = \bigvee_{i \in I} \bigwedge_{j=1}^{2^k} \alpha'_{ij}$ and let X_1, \dots, X_{2^k+1-k} be new propositional variables. We set $\alpha'_{i,2^k+1} := \bigwedge_{m=1}^{2^k+1-k} X_m$, for $i \in I$, and

$$\beta = \bigvee_{i \in I} \bigwedge_{j=1}^{2^k+1} \alpha'_{ij}.$$

Then, $\beta \in \Delta_{t+1,2^k+1}^*$ and $(\alpha' \text{ is } k\text{-satisfiable}) \iff (\beta \text{ is } 2^k + 1\text{-satisfiable})$. Therefore, $(\alpha, k) \mapsto (\beta, 2^k + 1)$ is the desired reduction. \square

We turn to the characterisation of $\mathbb{W}^*[t]$ in terms of a complete model-checking problem. To get the corresponding fragment of first-order logic, we first point out a closure property of the classes Σ_t not shared by the $\Sigma_{t,u}$. The closure of $\Sigma_{t,u}$ under this operation yields the desired fragment.

The formula

$$\exists \bar{x} (\forall y_1 \forall y_2 \exists z_1 \exists z_2 \psi \wedge \exists v_1 \exists v_2 \forall w_1 \forall w_2 \chi) \quad (5.1)$$

with quantifier-free $\psi(\bar{x}, \bar{y}, \bar{z})$ and $\chi(\bar{x}, \bar{v}, \bar{w})$ is an existential quantification of a Boolean combination of Σ_2 -formulas; it is logically equivalent to the Σ_3 -formula

$$\exists \bar{x} \exists v_1 \exists v_2 \forall y_1 \forall y_2 \forall w_1 \forall w_2 \exists z_1 \exists z_2 (\psi \wedge \chi);$$

more generally, every existential quantification of a Boolean combination of Σ_2 -formulas is equivalent to a Σ_3 -formula.

The class $\Sigma_{3,2}$ does not have this closure property, the formula in (5.1) is an existential quantification of a Boolean combination of $\Sigma_{2,2}$ -formulas with *all* blocks of length ≤ 2 , but, in general,

it is not logically equivalent to a formula in $\Sigma_{3,2}$. The class $\Sigma_{3,2}^*$ and (the classes $\Sigma_{t,u}^*$) are defined in such a way that they have this closure property.

For this purpose, first define the set $\Theta_{t,u}$ of first-order formulas by induction:

- $\Theta_{0,u} :=$ the set of quantifier-free formulas
- $\Theta_{t+1,u} :=$ Boolean combinations of formulas of the form $\exists y_1 \dots \exists y_u \psi$ with $\psi \in \Theta_{t,u}$

Now let $\Sigma_{t,u}^*$ be the set of formulas of the form

$$\exists x_1 \dots \exists x_k \psi$$

where $\psi \in \Theta_{t-1,u}$.

As for the un-starred version, a $\Sigma_{t,u}^*$ -formula is in strict- $\Sigma_{t,u}^*$ if each atomic subformula contains at most one variable of the first block of its prefix. We leave it to the reader to verify the following lemma, which is the analogon for $\Sigma_{t,u}^*$ of part (1) of the First-Order Normalisation Lemma.

Lemma 5.3. *For $t \geq 2, u \geq 1$, $p\text{-MC}(\Sigma_{t,u}^*) \leq p\text{-MC}(\text{strict-}\Sigma_{t,1}^*[2])$.*

The following lemma is a stronger version of Lemma 4.2:

Lemma 5.4. *For all $k \geq 1$ and for all formulas $\alpha := \bigwedge_{i=1}^k \alpha_i$, where $\alpha_1, \dots, \alpha_k \in \Gamma_{1,k} \cup \Delta_{1,k}$, there are*

- a structure $\mathcal{A} := \mathcal{A}_{\wedge, \alpha, k}$ with universe $A := \{1, \dots, m\}$, where the variables of α are among X_1, \dots, X_m ,
- a quantifier-free formula $\psi := \psi_{\wedge, k}$ depending only on k

such that the mapping $(\alpha, k) \mapsto (\mathcal{A}, \psi)$ is fixed-parameter tractable and for pairwise distinct $m_1, \dots, m_k \in A$,

$$\{X_{m_1}, \dots, X_{m_k}\} \text{ satisfies } \alpha \iff \mathcal{A} \models \psi(m_1, \dots, m_k).$$

Proof. For $1 \leq i \leq k$, if $\alpha_i \in \Gamma_{1,k}$ we let $\mathcal{A}_i := \mathcal{A}_{\wedge, \alpha_i(X_1, \dots, X_m), k, k}$ and $\psi_i := \psi_{\wedge, k, k}$ be the structure and sentence obtained from Lemma 4.2, and if $\alpha_i \in \Delta_{1,k}$ we let $\mathcal{A}_i := \mathcal{A}_{\vee, \alpha_i(X_1, \dots, X_m), k, k}$ and $\psi_i := \psi_{\vee, k, k}$ be the structure and sentence obtained from Corollary 4.3. We let τ_i be the vocabulary obtained from the vocabulary of \mathcal{A}_i and ψ_i by replacing each relation symbol R by a new symbol R_i of the same arity. We let \mathcal{A}'_i and ψ'_i be the τ_i -structure and sentence obtained from \mathcal{A}_i and ψ_i , respectively, by replacing each relation symbol R by R_i .

Note that the universe of $\mathcal{A}'_1, \dots, \mathcal{A}'_k$ is $\{1, \dots, m\} =: A$. Let $\tau := \bigcup_{i=1}^k \tau_i$, and let \mathcal{A}' be the τ -structure with universe A and $R_i^{\mathcal{A}'} := R_i^{\mathcal{A}'_i}$ for all relation symbols $R_i \in \tau_i$ and $1 \leq i \leq k$.

Note that for $1 \leq i \leq k$ and pairwise distinct $m_1, \dots, m_k \in A$

$$\mathcal{A} \models \psi'_i(m_1, \dots, m_k) \iff \mathcal{A}_i \models \psi_i(m_1, \dots, m_k).$$

Thus by Lemma 4.2 and Corollary 4.3, for pairwise distinct $m_1, \dots, m_k \in A$ we have

$$X_{m_1}, \dots, X_{m_k} \text{ satisfies } \alpha \iff \mathcal{A} \models \bigwedge_{i=1}^k \psi'_i(m_1, \dots, m_k).$$

The only remaining problem is that the formula $\bigwedge_{i=1}^k \psi'_i(x_1, \dots, x_k)$ depends on α . But actually it only depends on which of $\alpha_1, \dots, \alpha_k$ are in $\Gamma_{1,k}$ and which in $\Delta_{1,k}$. We introduce k new unary relation symbols C_1, \dots, C_k and let \mathcal{A} be the expansion of \mathcal{A}' with

$$C_i^{\mathcal{A}} := \begin{cases} A & \text{if } \alpha_i \in \Gamma_{1,k}, \\ \emptyset & \text{if } \alpha_i \in \Delta_{1,k}. \end{cases}$$

We let ψ_\wedge^i be the formula obtained from the formula $\psi_{\wedge,k,k}$ of Lemma 4.2 by replacing each relation symbol R by the corresponding R_i and define ψ_\vee^i accordingly. Thus ψ_i is either ψ_\wedge^i or ψ_\vee^i , depending on whether $\alpha_i \in \Gamma_{1,k}$ or $\alpha_i \in \Delta_{1,k}$. Finally, we let

$$\psi(x_1, \dots, x_k) := \bigwedge_{i=1}^k \left((C_i x_1 \rightarrow \psi_\wedge^i(x_1, \dots, x_k)) \wedge (\neg C_i x_1 \rightarrow \psi_\vee^i(x_1, \dots, x_k)) \right).$$

□

Corollary 5.5. *For all $k \geq 1$ and for all formulas $\alpha := \bigvee_{i=1}^k \alpha_i$, where $\alpha_1, \dots, \alpha_k \in \Gamma_{1,k} \cup \Delta_{1,k}$, there are*

- a structure \mathcal{A} with universe $A := \{1, \dots, m\}$, where the variables of α are among X_1, \dots, X_m ,
- a quantifier-free formula $\psi_{\vee,k}$ depending only on k

such that the mapping $(\alpha, k) \mapsto (\mathcal{A}, \psi_{\vee,k})$ is fixed-parameter tractable and for pairwise distinct $m_1, \dots, m_k \in A$

$$\{X_{m_1}, \dots, X_{m_k}\} \text{ satisfies } \alpha \iff \mathcal{A} \models \psi_{\vee,k}(m_1, \dots, m_k).$$

The following two propositions will yield the characterisation of the \mathbf{W}^* -hierarchy in terms of model-checking problems.

Proposition 5.6. *For $t \geq 1$,*

$$\text{WSAT}^*(\Omega_t) \leq p\text{-MC}(\Sigma_{t,2}^*).$$

Proof. Recall the proof of Proposition 4.5; we proceed very similarly here and mainly point out where the the proofs differ. Fix $t \geq 1$. Let (α, k) be an instance of $\text{WSAT}^*(\Omega_t)$. We shall construct a structure \mathcal{A} and a $\Sigma_{t,2}$ -sentence φ such that

$$\alpha \text{ is } k\text{-satisfiable} \iff \mathcal{A} \models \varphi. \quad (5.2)$$

By Lemma 5.2, we may assume that $\alpha \in \Delta_{t+1,k}^*$. Let the variables of α be X_1, \dots, X_m . As in the proof of Proposition 4.5, the structure \mathcal{A} consists of two parts: a tree representing the parse tree of the formula α and, attached to the leaves of the tree, a structure on the variables representing the innermost subformulas.

However, a formula in $\Delta_{t+1,k}^*$ is not as regular as a formula in $\Delta_{t+1,d}$, and therefore the definition of the tree is more involved. In particular, some of the nodes and edges of the tree carry additional information.

First, we let \mathcal{T} be the tree obtained from the “parse tree” of α by removing all nodes that correspond to subformulas of α in $\Gamma_{1,k} \cup \Delta_{1,k}$. Thus, the leaves correspond to subformulas of the form $\bigvee_{i=1}^k \beta_i$ or $\bigwedge_{i=1}^k \beta_i$, where $\beta_1, \dots, \beta_k \in \Gamma_{1,k} \cup \Delta_{1,k}$. In addition to the relation symbol E for the edge relation of this tree (directed from the root to the leaves), we have binary relation symbols $\{E_1, \dots, E_k\}$ and unary relation symbols K_1, \dots, K_k and Root whose interpretation in \mathcal{T} is fixed by the following clauses: Let u be a node of \mathcal{T} and β the subformula of α corresponding to the node u .

- If u is the root of the tree, then $\text{Root}^{\mathcal{T}} u$;
- If $\beta = \bigvee_{i=1}^k \beta_i$ or $\beta = \bigwedge_{i=1}^k \beta_i$, where $\beta_1, \dots, \beta_k \in \Gamma_{s,k}^* \cup \Delta_{s,k}^*$ for some $s \geq 2$, then, for $1 \leq j \leq k$, $E_j^{\mathcal{T}} u u_j$ where u_j is the child of u corresponding to β_j . Moreover, $K_j^{\mathcal{T}} u$ if $\beta_j \in \Gamma_{s,k}^*$.

Note that we encode the information on whether a subformula $\beta \in \Gamma_{s,k}^* \cup \Delta_{s,k}^*$ is in $\Gamma_{s,k}^*$ or in $\Delta_{s,k}^*$ by putting the parent into the corresponding relation K_j if β is in $\Gamma_{s,k}^*$. The reason that we choose such a counter-intuitive encoding is that we need the information about the child at the parent in order to pick the right quantifier to access the child. The definition of the formulas ψ_{\wedge}^{s+1} and ψ_{\vee}^{s+1} below will clarify this.

The second part of the structure \mathcal{A} we are heading for is defined as in the proof of Proposition 4.5, except that now the leaves of the tree are formulas of the form $\bigvee_{i=1}^k \beta_i$ or $\bigwedge_{i=1}^k \beta_i$, where $\beta_1, \dots, \beta_k \in \Gamma_{1,k} \cup \Delta_{1,k}$, and we have to use Lemma 5.4 and Corollary 5.5 instead of Lemma 4.2 and Corollary 4.3.

We define formulas $\psi_{\wedge}^s(y, x_1, \dots, x_k)$ and $\psi_{\vee}^s(y, x_1, \dots, x_k)$ for $1 \leq s \leq t+1$, and formulas $\psi_{\wedge}^s(y, x_1, \dots, x_k)$ and $\psi_{\vee}^s(y, x_1, \dots, x_k)$ for $2 \leq s \leq t+1$ such that for every node $u \in T$ corresponding to a subformula β and for all $a_1, \dots, a_k \in \{1, \dots, m\}$ we have

- (i) If $\beta = \bigwedge_{i=1}^k \beta_i$, where $\beta_1, \dots, \beta_k \in \Gamma_{s,k} \cup \Delta_{s,k}$, then

$$\{X_{a_1}, \dots, X_{a_k}\} \text{ satisfies } \beta \iff \mathcal{A} \models \psi_{\wedge}^s(u, a_1, \dots, a_k).$$
- (ii) If $\beta = \bigvee_{i=1}^k \beta_i$, where $\beta_1, \dots, \beta_k \in \Gamma_{s,k} \cup \Delta_{s,k}$, then

$$\{X_{a_1}, \dots, X_{a_k}\} \text{ satisfies } \beta \iff \mathcal{A} \models \psi_{\vee}^s(u, a_1, \dots, a_k).$$
- (iii) If $\beta \in \Gamma_{s,k}^*$, then $(\{X_{a_1}, \dots, X_{a_k}\} \text{ satisfies } \beta \iff \mathcal{A} \models \psi_{\wedge}^s(u, a_1, \dots, a_k))$.
- (iv) If $\beta \in \Delta_{s,k}^*$, then $(\{X_{a_1}, \dots, X_{a_k}\} \text{ satisfies } \beta \iff \mathcal{A} \models \psi_{\vee}^s(u, a_1, \dots, a_k))$.

We let $\psi_{\wedge}^1(y, x_1, \dots, x_k)$ be the formula obtained from the formula $\psi_{\wedge,k}(x_1, \dots, x_k)$ of Lemma 5.4 by replacing each atomic subformula $Rx_1 \dots x_r$ by $R'yx_1 \dots x_r$ (compare this to the proof of Proposition 4.5). Similarly, we define $\psi_{\vee,k}^1(y, x_1, \dots, x_k)$ using the formula $\psi_{\vee}(x_1, \dots, x_k)$ of Corollary 5.5.

For $s \geq 1$, we let

$$\begin{aligned} \psi_{\wedge}^{s+1}(y, x_1, \dots, x_k) &:= \forall z (Eyz \rightarrow \psi_{\vee}^s(z, x_1, \dots, x_k)), \\ \psi_{\vee}^{s+1}(y, x_1, \dots, x_k) &:= \exists z (Eyz \wedge \psi_{\wedge}^s(z, x_1, \dots, x_k)), \\ \psi_{\wedge}^{s+1}(y, x_1, \dots, x_k) &:= \bigwedge_{i=1}^k \left((K_i y \rightarrow \forall z (E_i y z \rightarrow \psi_{\wedge}^{s+1}(z, x_1, \dots, x_k))) \right. \\ &\quad \left. \wedge (\neg K_i y \rightarrow \exists z (E_i y z \wedge \psi_{\vee}^{s+1}(z, x_1, \dots, x_k))) \right), \\ \psi_{\vee}^{s+1}(y, x_1, \dots, x_k) &:= \bigvee_{i=1}^k \left((K_i y \wedge \forall z (E_i y z \rightarrow \psi_{\wedge}^{s+1}(z, x_1, \dots, x_k))) \right. \\ &\quad \left. \vee (\neg K_i y \wedge \exists z (E_i y z \wedge \psi_{\vee}^{s+1}(z, x_1, \dots, x_k))) \right). \end{aligned}$$

It is easy to see now that these formulas satisfy (i)–(iv). Furthermore, ψ_{\wedge}^1 and ψ_{\vee}^1 are quantifier-free and, by a simultaneous induction on $s \geq 1$,

- ψ_{\wedge}^{s+1} can be transformed into a formula of the form $\forall y \chi$, where $\chi \in \Theta_{s-1,2}$;
- ψ_{\vee}^{s+1} can be transformed into a formula of the form $\exists z \chi$, where $\chi \in \Theta_{s-1,2}$;
- ψ_{\wedge}^{s+1} and ψ_{\vee}^{s+1} can easily be transformed into a formula in $\Theta_{s,2}$.

We let

$$\varphi := \exists x_1 \dots \exists x_k \exists y \left(\bigwedge_{1 \leq i < j \leq k} x_i \neq x_j \wedge \text{Root } y \wedge \psi_{\vee}^{t+1}(y, x_1, \dots, x_k) \right).$$

It is easy to see that φ is equivalent to a formula in $\Sigma_{t,2}^*$. □

Proposition 5.7. For all $t, u \geq 1$, $p\text{-MC}(\Sigma_{t,u}^*) \leq \text{WSAT}^*(\Omega_t)$.

Proof. The proof essentially duplicates the arguments of the proof of Proposition 4.10. The additional disjunctions and conjunctions between blocks of quantifiers in a $\Sigma_{t,u}^*$ -formula φ yield additional connectives in the propositional formula we look for. \square

By Lemma 5.3 and the preceding propositions we get:

Theorem 5.8. For $t, u \geq 1$,

$$\mathbf{W}^*[t] = [p\text{-MC}(\Sigma_{t,u}^*)]^{\text{fpt}} = [p\text{-MC}(\Sigma_{t,1}^*[2])]^{\text{fpt}}.$$

Corollary 5.9. $\mathbf{W}^*[1] = \mathbf{W}[1]$.

Proof. Since $\Sigma_{1,u}^* = \Sigma_{1,u} = \Sigma_1$, this is immediate by Theorem 4.13 and Theorem 5.8.

Corollary 5.10. $\mathbf{W}^*[2] = \mathbf{W}[2]$.

Proof. Again by Theorem 4.13 and Theorem 5.8, it suffices to show that

$$p\text{-MC}(\Sigma_{2,u}^*) \leq p\text{-MC}(\Sigma_{2,u}).$$

So let \mathcal{A} be a structure and φ a $\Sigma_{2,u}^*$ -sentence. We can assume that φ has the form

$$\exists x_1 \dots \exists x_\ell \bigvee_{i \in I} \bigwedge_{j \in J_i} \psi_{ij},$$

where I and the J_i are finite sets and the ψ_{ij} are formulas in $\Sigma_1 \cup \Pi_1$ with quantifier block of length $\leq u$. First we replace the disjunction $\bigvee_{i \in I}$ in φ by an existential quantifier. For this purpose, we add to the vocabulary τ of \mathcal{A} unary relation symbols R_i for $i \in I$ and consider an expansion $(\mathcal{A}, (R_i^A)_{i \in I})$ of \mathcal{A} , where $(R_i^A)_{i \in I}$ is a partition of A into nonempty disjoint sets. Then

$$\mathcal{A} \models \varphi \iff (\mathcal{A}, (R_i^A)_{i \in I}) \models \exists x_1 \dots \exists x_\ell \exists y \bigwedge_{j \in J_i} (\neg R_i y \vee \psi_{ij}).$$

Altogether, we can assume that φ has the form

$$\exists x_1 \dots \exists x_\ell \bigwedge_{j=1}^m \psi_j,$$

where for some quantifier-free χ_j

$$\psi_j = \exists \bar{y}_j \chi_j \quad \text{for } j = 1, \dots, s$$

and

$$\psi_j = \forall \bar{z} \chi_j \quad \text{for } j = s+1, \dots, m.$$

Here, $\bar{y}_1, \dots, \bar{y}_s, \bar{z}$ are sequences of length $\leq u$ and we can assume that any two of them have no variable in common. But then φ is equivalent to the $\Sigma_{2,u}$ -formula:

$$\exists x_1 \dots \exists x_\ell \exists \bar{y}_1 \dots \exists \bar{y}_s \forall \bar{z} \bigwedge_{j=1}^m \chi_j.$$

\square

Unfortunately, the argument of the preceding proof cannot be extended to an inductive proof of $W^*[t] = W[t]$ for all $t \geq 2$. To see this, observe that for an instance (\mathcal{A}, φ) of $p\text{-MC}(\Sigma_{3,u}^*)$, in the same way we would obtain an equivalent formula

$$\varphi' := \exists x_1 \dots \exists x_\ell \exists \bar{y}_1 \dots \exists \bar{y}_s \forall \bar{z} \bigwedge_{j=1}^m \chi_j,$$

where now the χ_j are Boolean combinations of $\Sigma_{2,u}^* \cup \Pi_{2,u}^*$ -formulas with all quantifier blocks of length at most u . But now the existential quantifiers in the χ_j cannot be transferred to the leading existential block in φ' , they are blocked by the universal quantifiers.

5.2. The A-hierarchy. Originally, the A-hierarchy was defined by means of halting problems: $A[\ell]$ (where $\ell \in \mathbb{N}$) has as complete problem the halting problem for alternating Turing machines with $\ell - 1$ alternations (and existential starting state), parameterized by the number of steps. In [11], it was shown that $A[\ell] = [\{p\text{-MC}(\Sigma_\ell[r]) \mid r \geq 1\}]^{\text{fpt}}$. In view of part 2 of the Normalisation Lemma this yields

$$A[\ell] = [p\text{-MC}(\Sigma_\ell)]^{\text{fpt}},$$

which, in this paper, we take as definition of the A-hierarchy. Since $\Sigma_{1,u} = \Sigma_1$ and $\Sigma_{\ell,u} \subseteq \Sigma_\ell$, we have

$$W[1] = A[1] \quad \text{and for } \ell \geq 2: W[\ell] \subseteq A[\ell].$$

In this section we derive a characterisation of the A-hierarchy in terms of weighted satisfiability problems for classes of propositional formulas.

We saw in the preceding sections that a single universal quantifier (or equivalently, a block of bounded length of universal quantifiers) in a first-order formula translates into a \bigwedge in the corresponding propositional formula, and similarly, an existential quantifier translates into a \bigvee . As the proof of Proposition 4.10 shows the leading (unbounded) block $\exists x_1 \dots \exists x_k$ yields, on the side of propositional logic, the weight or parameter k and the propositional variables $X_{i,a}$ (with $1 \leq i \leq k$ and with a ranging over the universe of the given structure). Since in $A[\ell]$ we have ℓ alternating (unbounded) blocks, we have to consider alternating weighted satisfiability problems for classes of propositional formulas. Such problems were already introduced by Abrahamson, Downey, and Fellows in [1] when they considered quantified boolean (propositional) logic.

Let Γ be a set of propositional formulas (as defined in Section 2) and $\ell \geq 1$. The ℓ -alternating weighted satisfiability problem $\text{AWSAT}_\ell(\Gamma)$ for formulas in Γ is the following problem:

<p>$\text{AWSAT}_\ell(\Gamma)$ <i>Input:</i> $\alpha \in \Gamma$ and a partition $I_1 \dot{\cup} \dots \dot{\cup} I_\ell$ of the propositional variables of α. <i>Parameter:</i> $k_1, \dots, k_\ell \in \mathbb{N}$. <i>Problem:</i> Decide if there is a size k_1 subset S_1 of I_1 such that for every size k_2 subset S_2 of I_2 there exists \dots such that the truth value assignment $S_1 \cup \dots \cup S_\ell$ satisfies α.</p>

Thus, $\text{AWSAT}_1(\Gamma) = \text{WSAT}(\Gamma)$. Generalising the definition

$$W[t] := [\{\text{WSAT}(\Omega_{t,d}) \mid d \geq t\}]^{\text{fpt}}$$

of the classes of the W-hierarchy on the alternating level, we define the parameterized complexity class $A[\ell, t]^{\text{fpt}}$ by

$$A[\ell, t] := [\{\text{AWSAT}_\ell(\Omega_{t,d}) \mid d \geq t\}]^{\text{fpt}}.$$

Thus, $W[t] = A[1, t]$ and as the main result of this section will show, $A[\ell] = A[\ell, 1]$, which yields the desired characterisation of the A-hierarchy in terms of propositional logic. Thus, the family of classes $A[\ell, t]$, which we may call the *A-matrix*, contains the classes of the W-hierarchy and the classes of the A-hierarchy.

We turn to a model-checking characterisation of this family: The propositional formulas in the defining problem of $A[\ell, t]$ contain ℓ “weighted alternations” and at most t (nested) big conjunctions or big disjunctions. As we remarked above, the ℓ weighted alternations translate into ℓ alternating blocks of quantifiers and the t (nested) big conjunctions or big disjunctions into t further quantifiers; the first of them can be merged with the last alternating block, so we expect that

$$A[\ell, t] = [p\text{-MC}(\Sigma^{\ell, t-1})]^{\text{fpt}},$$

where for $\ell \geq 1$ and $m \geq 0$ we denote by $\Sigma^{\ell, m}$ the class of first-order formulas of the form

$$\exists \bar{x}_1 \forall \bar{x}_2 \dots Q_\ell \bar{x}_\ell Q_{\ell+1} x_{\ell+1} \dots Q_{\ell+m} x_{\ell+m} \psi$$

where ψ is quantifier-free, all $Q_i \in \{\exists, \forall\}$, and $Q_i \neq Q_{i+1}$. Note that $\bar{x} \dots$ denotes a finite sequence of variables, thus the formula starts with ℓ unbounded blocks of quantifiers. Hence,

- $\Sigma^{\ell, 0} = \Sigma_\ell$.
- For $t \geq 1$, $\Sigma^{1, t-1} = \Sigma_{t, 1}$.

It should be clear how the class $\Pi^{\ell, m}$ of formulas is defined.

We call a $\Sigma^{\ell, m}$ -formula *strict* if each atomic subformula contains at most one variable from the first ℓ , the unbounded blocks of quantifiers. Again, part 1 of the First-Order Normalisation Lemma generalizes (with essentially the same proof) to $\Sigma^{\ell, m}$. We state the result and leave its verification to the reader:

Lemma 5.11. *For $\ell, m \geq 1$, $p\text{-MC}(\Sigma^{\ell, m}) \leq p\text{-MC}(\text{strict-}\Sigma^{\ell, m}[2])$.*

Now, we are able to prove the main result of this section.

Theorem 5.12. *For all $\ell, t \geq 1$*

$$A[\ell, t] = [p\text{-MC}(\Sigma^{\ell, t-1})]^{\text{fpt}} = [p\text{-MC}(\Sigma^{\ell, t-1}[2])]^{\text{fpt}}$$

Moreover, we have

- if ℓ is odd, then

$$A[\ell, t] = [\text{AWSAT}_\ell(\Gamma_{t, 2})]^{\text{fpt}} \quad \text{and for } t \geq 2, \quad A[\ell, t] = [\text{AWSAT}_\ell(\Gamma_{t, 1})]^{\text{fpt}};$$

- if ℓ is even, then

$$A[\ell, t] = [\text{AWSAT}_\ell(\Delta_{t, 2})]^{\text{fpt}} \quad \text{and for } t \geq 2, \quad A[\ell, t] = [\text{AWSAT}_\ell(\Delta_{t, 1})]^{\text{fpt}}.$$

Before proving this theorem, we state two consequences; the first one is the characterisation of the A-hierarchy by means of propositional logic:

Corollary 5.13. $A[\ell] = A[\ell, 1]$, i.e., $A[\ell] = [\{\text{AWSAT}_\ell(\Omega_{1, d}) \mid d \geq 1\}]$.

Corollary 5.14. For $\ell \geq 1$ and $t \geq 2$, $A[\ell, t] \subseteq A[\ell + 1, t - 1]$.

Proof. Since $\Sigma^{\ell, t-1} \subseteq \Sigma^{\ell+1, t-2}$, the claim follows from Theorem 5.12. □

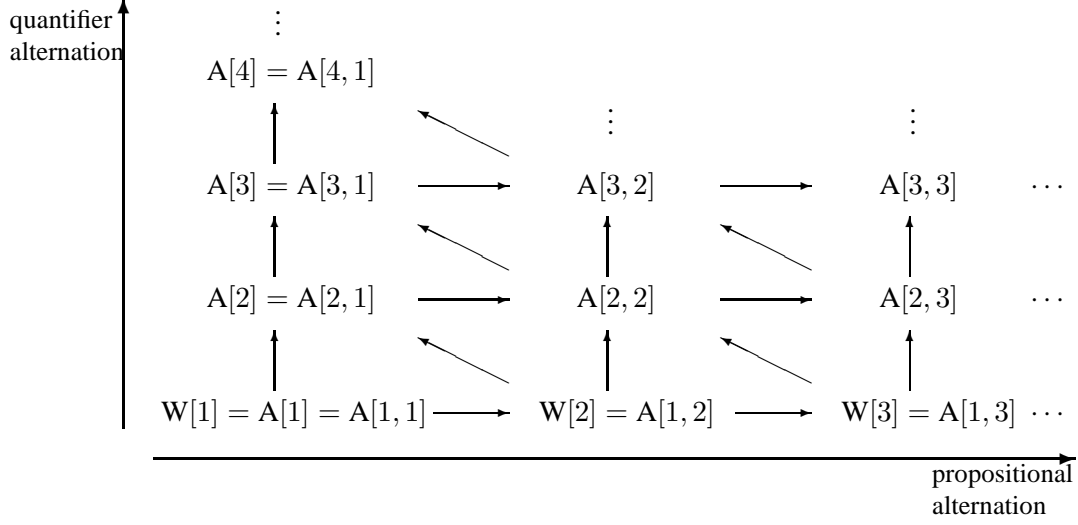


Figure 1: The A-matrix of parameterized complexity classes. Arrows indicate containment.

Figure 1 shows the matrix and the containment relations known to hold between the classes.

Since $W[t] = A[1, t]$ and $\Sigma^{1, t-1} = \Sigma_{t, 1}$, Theorem 5.12 (partly) generalises Theorem 4.13 and, in fact, its proof extends the argument given there.

Proof of Theorem 5.12: We first prove that $\text{AWSAT}_\ell(\Omega_{t, d}) \leq p\text{-MC}(\Sigma^{\ell, t-1})$. Let

$$((\alpha, I_1, \dots, I_\ell), (k_1, \dots, k_\ell))$$

be an instance of $\text{AWSAT}_\ell(\Omega_{t, d})$. By Lemma 3.1, we may actually assume that $\alpha \in \Delta_{t+1, d}$. Let $k := k_1 + \dots + k_\ell$ and $\{X_1, \dots, X_m\}$ the set of variables of α .

Let us first assume that ℓ is odd. We construct a structure \mathcal{A} and a formula ψ according to Lemma 4.7. We expand \mathcal{A} by unary relation V_1, \dots, V_ℓ such that $V_i^{\mathcal{A}} := \{j \mid X_j \in I_i\}$. For simplicity, we denote the resulting structure by \mathcal{A} again. We let

$$\begin{aligned}
\varphi := & \exists x_1 \dots \exists x_{k_1} \left(\bigwedge_{i=1}^{k_1} V_1 x_i \wedge \bigwedge_{\substack{i, j=1 \\ i \neq j}}^{k_1} x_i \neq x_j \wedge \right. \\
& \forall x_{k_1+1} \dots \forall x_{k_1+k_2} \left(\left(\bigwedge_{i=k_1+1}^{k_1+k_2} V_2 x_i \wedge \bigwedge_{\substack{i, j=k_1+1 \\ i \neq j}}^{k_1+k_2} x_i \neq x_j \right) \rightarrow \right. \\
& \dots \\
& \left. \exists x_{k_1+\dots+k_{\ell-1}+1} \dots \exists x_k \left(\bigwedge_{i=k_1+\dots+k_{\ell-1}+1}^k V_\ell x_i \wedge \bigwedge_{\substack{i, j=k_1+\dots+k_{\ell-1}+1 \\ i \neq j}}^k x_i \neq x_j \wedge \right. \right. \\
& \left. \left. \psi \right) \dots \right).
\end{aligned}$$

It is straightforward to verify that $\mathcal{A} \models \varphi$ if and only if, $((\alpha, I_1, \dots, I_\ell), (k_1, \dots, k_\ell))$ is a ‘yes’-instance of $\text{AWSAT}_\ell(\Omega_{t,d})$ and that φ is equivalent to a $\Sigma^{\ell,t-1}$ -formula.

If ℓ is even, we assume that $\alpha \in \Gamma_{t+1,d}$ and observe that Lemma 4.7 has a corresponding version for such formulas.

By Lemma 5.11, it remains to get a reduction from $p\text{-MC}(\Sigma^{\ell,t-1}[2])$ to $\text{AWSAT}_\ell(\Omega_{t,d})$ for some d (and to prove the additional claims of the theorem).

First, we treat the case $t = 1$ and for notational simplicity, assume $\ell = 3$. Let φ be a $\Sigma^{3,0}[2]$ -formula. By Lemma 3.4(2), we may assume that φ is a simple Σ_3 -sentence,

$$\varphi := \exists x_1 \dots \exists x_h \forall y_1 \dots \forall y_k \exists z_1 \dots \exists z_m (\lambda_1 \wedge \dots \wedge \lambda_s)$$

with literals λ_i and \mathcal{A} a structure in the corresponding vocabulary.

We first construct a propositional formula $\alpha' \in \Omega_{1,d}$ for some d . For the partition of its propositional variables into the three sets

$$I_1 := \{X_{i,a} \mid i = 1, \dots, h, a \in A\}, \quad I_2 := \{Y_{i,a} \mid i = 1, \dots, k, a \in A\},$$

and

$$I_3 := \{Z_{i,a} \mid i = 1, \dots, m, a \in A\},$$

and for the natural numbers h, k, m , we will see that

$$\mathcal{A} \models \varphi \iff ((\alpha', I_1, I_2, I_3), (h, k, m)) \in p\text{-AWSAT}_\ell(\Omega_{1,d}). \quad (5.3)$$

Clearly, the intended meaning of $X_{i,a}$ is “ x_i gets the value a ” and similarly for the other variables.

The formula α' has the form $(\bigwedge \dots \vee \bigvee \dots)$. The “big” conjunction takes care of existentially quantified variables: it contains as conjuncts $(\neg X_{i,a} \vee \neg X_{i,b})$ for $i = 1, \dots, h, a, b \in A, a \neq b$ and $(\neg Z_{i,a} \vee \neg Z_{i,b})$ for $i = 1, \dots, m, a, b \in A, a \neq b$. The “big” disjunction takes care of universally quantified variables; in fact, it *only* contains as disjuncts $(Y_{i,a} \wedge Y_{i,b})$ for $i = 1, \dots, k, a, b \in A, a \neq b$. So far, it should be clear that any satisfying assignment of α' of “weight h, k, m ” sets

- for every i exactly one variable $X_{i,a}$ to TRUE and similarly for the $Z_{i,a}$

or

- it sets $Y_{i,a}, Y_{i,b}$ to TRUE for some i and some $a, b \in A, a \neq b$.

Finally, we take care of the quantifier-free part of φ by adding to the big conjunction for every λ_i , say $\lambda_i(x_3, y_2)$ (recall that the arity of the vocabulary is ≤ 2), and every $(a, b) \in A$ with $\mathcal{A} \not\models \lambda_i(a, b)$ as conjunct the formula $(\neg X_{3,a} \vee \neg Y_{2,b})$. We leave the verification of (5.3) to the reader.

Now, we show how to get rid of the big disjunction in α' , thus proving the additional claim

$$p\text{-MC}(\Sigma^{3,0}[2]) \leq \text{AWSAT}_3(\Gamma_{1,2}).$$

Besides the propositional variables of α' , the formula α we aim at has additional propositional variables, namely the variables

$$C, Y_1, \dots, Y_k, Z_1, \dots, Z_m.$$

The partition of the variables of α consists of three sets, namely of I_1 and I_2 as above, i.e.,

$$I_1 := \{X_{i,a} \mid i = 1, \dots, h, a \in A\}, \quad I_2 := \{Y_{i,a} \mid i = 1, \dots, k, a \in A\},$$

and of J_3 that contains the variables of I_3 and the new variables, i.e.,

$$J_3 := \{Z_{i,a} \mid i = 1, \dots, m, a \in A\} \cup \{C, Y_1, \dots, Y_k, Z_1, \dots, Z_m\}.$$

The “parameters” are $h, k, m + 1$. In fact we will have

$$\begin{aligned} & ((\alpha', I_1, I_2, I_3), (h, k, m)) \in \text{AWSAT}_3(\Omega_{1,d}) \\ \iff & ((\alpha, I_1, I_2, J_3), (h, k, m + 1)) \in \text{AWSAT}_3(\Gamma_{1,2}). \end{aligned} \quad (5.4)$$

To understand the construction of α better, we briefly explain the meaning or role of the new propositional variables: C essentially signalizes that the big conjunction in α' is satisfied, Y_i that no variable $Y_{i,a}$ with $a \in A$ has been chosen; finally, in case the big disjunction in α' is satisfied, then Z_1, \dots, Z_m , but no $Z_{i,a}$, will be set to TRUE.

Let α be obtained from α' by

- eliminating the big disjunction;
- adding to the big conjunction the formulas (the indices always range over all possible values)
 - (1) $\neg C \vee \neg Y_i$,
 - (2) $\neg C \vee \neg Z_i$
 - (3) $\neg Z_i \vee \neg Z_{j,a}$
 - (4) $\neg Y_i \vee \neg Z_{j,a}$
 - (5) $\neg Y_i \vee \neg Y_{i,a}$
 - (6) $\neg Y_i \vee \neg Y_j$ for $i \neq j$.

Then, α is in $\Gamma_{1,2}$. We verify (5.4). Assume first that $(\alpha', V_1, V_2, V_3, h, k, m) \in \text{AWSAT}_3(\text{PROP})$. To verify the right hand side of (5.4), we choose $S_1 \subseteq I_1$ as it is done when verifying the left side. Now let S_2 be any size k subset of I_2 ; if S_2 does not satisfy $\bigvee \dots (Y_{i,a} \wedge Y_{i,b})$, then we select $S'_3 \subseteq I_3$ as when verifying for S_1, S_2 the left hand side. Then, we can set $S_3 := S'_3 \cup \{C\}$ and verify that $S_1 \cup S_2 \cup S_3$ satisfies α . If S_2 satisfies $\bigvee \dots (Y_{i,a} \wedge Y_{i,b})$, then there is some i_0 such that $Y_{i_0,a} \notin S_2$ for all $a \in A$. We set $S_3 := \{Y_{i_0}, Z_1, \dots, Z_m\}$ and again verify that $S := S_1 \cup S_2 \cup S_3$ satisfies α . Clearly, S satisfies all clauses (1)–(6). And, it also satisfies all old conjuncts, since they are not of the form $(\neg Y_{i,a} \vee \neg Y_{i,b})$.

Conversely, assume that the right hand side of (5.4) holds. For α' we choose S_1 as it is done for α when verifying the right hand side. Let S_2 be any size k subset of I_2 ; if S_2 satisfies $\bigvee \dots (Y_{i,a} \wedge Y_{i,b})$ we are done. Otherwise, we choose for S_1, S_2 a size $m + 1$ subset S_3 of J_3 such that $S_1 \cup S_2 \cup S_3$ satisfies α . By the formulas (5), S_3 does not contain any Y_i . By the clauses (2), S_3 at most contains m variables from $\{C, Z_1, \dots, Z_m\}$. Therefore, for some j there is at least one $a \in A$ such that $Z_{j,a} \in S_3$. But then, by the clauses (3), the set S_3 contains no Z_i . Thus, S_3 contains C and for every j exactly one $Z_{j,a}$ (recall that the big conjunction in α' and hence, the one in α , contains the conjuncts $(\neg Z_{i,a} \vee \neg Z_{i,b})$ for $i = 1, \dots, m$ and $a, b \in A$ with $a \neq b$). Therefore, setting $S'_3 := S_3 \cap I_3$, we have $S_1 \cup S_2 \cup S'_3$ satisfies α' .

Now, let us assume that $t \geq 2$ and, say, ℓ is odd. We aim at a reduction to $\text{AWSAT}_\ell(\Gamma_{t,1})$. The formula φ has the form

$$\exists \bar{x}_1 \forall \bar{x}_2 \dots \exists \bar{x}_\ell \forall x_{\ell+1} \dots Q_{\ell+(t-1)} x_{\ell+(t-1)} \psi,$$

i.e., the first “short” quantifier block (consisting of a single quantifier) is universal. Moreover, we can assume that φ is strict, that is, that every atomic subformula contains at most one variable of the unrestricted block. The unrestricted blocks are treated in the propositional formula as above and the short blocks and the quantifier-free part as in the proof of Proposition 4.10. In particular, to the big conjunction of the propositional formula $\alpha' = (\bigwedge \dots \vee \bigvee \dots)$ constructed for $t = 1$, we add conjuncts corresponding to the quantifier $\forall x_{\ell+1}$. Below this big conjunction there is a layer of big

disjunctions. (In case $t \geq 3$ this layer can also be used to eliminate the big disjunction of

$$\alpha' = \left(\bigwedge \dots \vee \bigvee_{i=1, \dots, k; a, b \in A, a \neq b} (Y_{i,a} \wedge Y_{i,b}) \right),$$

which is treated as a $\Delta_{2,1}$ -formula.) We argue as above to get rid of the big disjunction of α' .

Altogether, we obtain a reduction to $\text{AWSAT}_\ell(\Gamma_{t,1})$. Similarly, one argues in case ℓ is even: Then the first “short” quantifier block is existential, and therefore one obtains a reduction to $\text{AWSAT}_\ell(\Delta_{t,1})$. \square

Arguing as in the derivation of Corollary 4.8, one obtains

Remark 5.15. For $t \geq 2$ and $d \geq 1$,

- if ℓ is odd, then $\text{AWSAT}_\ell(\Delta_{t,d}) \in \text{A}[\ell, t - 1]$;
- if ℓ is even, then, $\text{AWSAT}_\ell(\Gamma_{t,d}) \in \text{A}[\ell, t - 1]$.

Remark 5.16. As for the W-hierarchy one can obtain improvements restricting the propositional formulas to monotone or antimonotone ones. We leave the details to the reader.

Remark 5.17. For the A-hierarchy there are two more or less natural ways to define a starred version $\text{A}^*[1], \text{A}^*[2], \dots$. From the point of view of first-order logic, we introduce the classes of formulas Σ_t^* by induction

$$\begin{aligned} \Sigma_0^* &:= \text{the set of quantifier-free formulas} \\ \Sigma_{t+1}^* &:= \text{formulas of the form } \exists y_1 \dots \exists y_u \psi, \text{ where } \psi \text{ is a} \\ &\quad \text{Boolean combination of formulas in } \Sigma_t^*, \end{aligned}$$

and set

$$\text{A}^*[t] := [p\text{-MC}(\Sigma_t^*)]^{\text{fpt}}.$$

But since every formula in Σ_t^* is logically equivalent to a formula in Σ_t , we immediately get $\text{A}^*[t] = \text{A}[t]$.

From the point of view of propositional logic we imitate the definition of W^* in the alternating context: For a set Γ of propositional formulas let

$\text{AWSAT}_\ell^*(\Gamma)$	
<i>Input:</i>	$\alpha \in \Gamma, k \in \mathbb{N}$ such that the depth of α is at most k , and a partition $I_1 \dot{\cup} \dots \dot{\cup} I_\ell$ of the propositional variables of α .
<i>Parameter:</i>	$k_1, \dots, k_\ell \in \mathbb{N}$ with $k = k_1 + \dots + k_\ell$.
<i>Problem:</i>	Decide if there is a size k_1 subset S_1 of I_1 such that for every size k_2 subset S_2 of I_2 there exists \dots such that the assignment $S_1 \cup \dots \cup S_\ell$ satisfies α .

And set

$$\text{A}^*[t] := [\{\text{AWSAT}_t^*(\Omega_{1,d}) \mid d \geq 1\}]^{\text{fpt}}.$$

Clearly, $\text{AWSAT}_t(\Omega_{1,d}) \leq \text{AWSAT}_t^*(\Omega_{1,d})$. On the other hand, essentially the proof of Proposition 5.12 shows that $\text{AWSAT}_t^*(\Omega_{1,d}^*) \leq p\text{-MC}(\Sigma_t^*)$, so that again we obtain $\text{A}^*[t] = \text{A}[t]$.

5.3. The AW-hierarchy. Downey and Fellows [6] introduced the AW-hierarchy and showed its collapse. Again this result can easily be derived (and slightly be improved) with the techniques developed in this paper.

To define this hierarchy, for a set Γ of propositional formulas, we introduce the *alternating weighted satisfiability problem* $\text{AWSAT}(\Gamma)$ (in contrast to $\text{AWSAT}_\ell(\Gamma)$ defined in the preceding section we have no restriction on the number of alternations):

$\text{AWSAT}(\Gamma)$ <i>Input:</i> $\alpha \in \Gamma$, $\ell \geq 1$, and a partition $I_1 \dot{\cup} \dots \dot{\cup} I_\ell$ of the propositional variables of α . <i>Parameter:</i> $k_1, \dots, k_\ell \in \mathbb{N}$. <i>Problem:</i> Decide if there is a size k_1 subset S_1 of I_1 such that for every size k_2 subset S_2 of I_2 there exists \dots such that the truth assignment $S_1 \cup \dots \cup S_\ell$ satisfies α .
--

Hence, given the input $(\alpha, \ell, I_1 \dot{\cup} \dots \dot{\cup} I_\ell)$ and the parameter (k_1, \dots, k_ℓ) we have the equivalence

$$\begin{aligned} & ((\alpha, \ell, I_1, \dots, I_\ell), (k_1, \dots, k_\ell)) \in \text{AWSAT}(\Gamma) \\ \iff & ((\alpha, I_1, \dots, I_\ell), (k_1, \dots, k_\ell)) \in \text{AWSAT}_\ell(\Gamma) \end{aligned} \quad (5.5)$$

(note that on the left side of the equivalence the number ℓ is part of the input and is not fixed in advance).

Definition 5.18. For $t \geq 1$, $\text{AW}[t] := [\{\text{AWSAT}(\Gamma_{t,d}) \mid d \geq 1\}]^{\text{fpt}}$.

In a very informal way the core of the proof of the following theorem can be described in the following form:

$$\begin{aligned} \text{AW}[t] &= [\{\bigcup_{\ell \geq 1} \text{AWSAT}_\ell(\Gamma_{t,d}) \mid d \geq 1\}]^{\text{fpt}} && \text{by (5.5)} \\ &= [p\text{-MC}(\bigcup_{\ell \geq 1} \Sigma^{\ell, t-1})]^{\text{fpt}} && \text{by Theorem 5.12.} \end{aligned}$$

Since $\bigcup_{\ell \geq 1} \Sigma^{\ell, t-1} = \bigcup_{\ell \geq 1} \Sigma^{\ell, 0} = \text{FO}$, we get $\text{AW}[1] = \text{AW}[t] = [p\text{-MC}(\text{FO})]^{\text{fpt}}$, which essentially is the statement of the following theorem.

Theorem 5.19. For $t \geq 1$,

$$\text{AW}[1] = \text{AW}[t] = [\text{AWSAT}(\Gamma_{1,2})]^{\text{fpt}} = [p\text{-MC}(\text{FO})]^{\text{fpt}} = [p\text{-MC}(\text{FO}[2])]^{\text{fpt}}.$$

Proof. Clearly, $\text{AW}[1] \subseteq \text{AW}[t]$. Consider an instance of $\text{AWSAT}(\Gamma_{t,d})$ consisting of the input

$$(\alpha, \ell, I_1, \dots, I_\ell)$$

and the parameter (k_1, \dots, k_ℓ) . In the proof of Theorem 5.12 we saw how to proceed in order to obtain a structure \mathcal{A} and a formula $\varphi \in \Sigma^{\ell, t-1}$ such that

$$((\alpha, \ell, I_1, \dots, I_\ell), (k_1, \dots, k_\ell)) \in \text{AWSAT}(\Gamma_{t,d}) \iff \mathcal{A} \models \varphi.$$

Clearly, this procedure is uniform in ℓ and an fpt-reduction from $\text{AWSAT}(\Gamma_{t,d})$ to $p\text{-MC}(\text{FO})$. By part (3) of the First-Order Normalisation Lemma, we know that $p\text{-MC}(\text{FO}) \leq p\text{-MC}(\text{FO}[2])$. Finally, let \mathcal{A} be a structure and $\varphi \in \text{FO}[2]$ a formula, say $\varphi \in \Sigma_\ell = \Sigma^{\ell, 0}$. We may assume that ℓ is odd. Then the proof of Theorem 5.12 shows how to obtain a formula $\alpha \in \Gamma_{1,2}$, a partition $I_1 \dot{\cup} \dots \dot{\cup} I_\ell$ of its variables, and k_1, \dots, k_ℓ such that

$$\mathcal{A} \models \varphi \iff ((\alpha, I_1, \dots, I_\ell), (k_1, \dots, k_\ell)) \in \text{AWSAT}_\ell(\Gamma_{1,2}),$$

i.e., such that

$$\mathcal{A} \models \varphi \iff ((\alpha, \ell, I_1, \dots, I_\ell), (k_1, \dots, k_\ell)) \in \text{AWSAT}(\Gamma_{1,2}).$$

Hence, we have an fpt-reduction from p -MC(FO[2]) to AWSAT($\Gamma_{1,2}$). \square

6. CONCLUSIONS

We hope to have demonstrated that the correspondence between propositional and first-order logic, or more precisely, weighted satisfiability and model-checking problems, is very fruitful. We see this correspondence at the core of structural parameterized complexity theory. Once it is established, many other results follow quite easily.

Several problems remain open, the most important being the question of whether the W -hierarchy and the W^* -hierarchy coincide. Even though our results clarify what is known, we have failed to make any definite progress on this problem.

Another nagging open question is whether the First-Order Normalisation Lemma can be extended to vocabularies with function symbols. A positive answer would greatly simplify the machine characterisation of the classes of the W -hierarchy given in [2].

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