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MODEL THEORY AND PROOF THEORY OF COALGEBRAIC PREDICATE LOGIC

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ABSTRACT. We propose a generalization of first-order logic originating in a neglected work by C.C. Chang: a natural and generic correspondence language for any types of structures which can be recast as Set-coalgebras. We discuss axiomatization and completeness results for several natural classes of such logics. Moreover, we show that an entirely general completeness result is not possible. We study the expressive power of our language, both in comparison with coalgebraic hybrid logics and with existing first-order proposals for special classes of Set-coalgebras (apart from relational structures, also neighbourhood frames and topological spaces). Basic model-theoretic constructions and results, in particular ultraproducts, obtain for the two classes that allow completeness—and in some cases beyond that. Finally, we discuss a basic sequent system, for which we establish a syntactic cut-elimination result.

Dedicated to Jiří Adámek on the occasion of his seventieth birthday

1. INTRODUCTION

Modal logics are traditionally a core formalism in computer science. Classically, their semantics is relational, i.e. a model typically comes with a set of states and one or several binary accessibility relations on the state set. However, non-relational semantics of various descriptions have come to play an increasing role, e.g. in concurrency, reasoning about knowledge and agency, description logics and ontologies. Models may involve such diverse features as concurrent games, as in coalition logic and alternating-time temporal logic [AHK02, Pau02]; probabilities [LS91, FH94, HM01]; integer weights as in the multigraph semantics of graded modal logic [DV02]; neighbourhoods [Che80]; and selection functions or preference orderings as in the different variants of conditional logic [Lew73, Che80]. *Coalgebraic modal logic* serves as a unifying framework for such non-relational modal logics [CKP+11].

Key words and phrases: coalgebraic predicate logic, coalgebraic model theory, coalgebraic proof theory.

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Relational modal logic can be seen as a subset of first-order logic, specifically as the bisimulation-invariant fragment as shown by van Benthem for arbitrary models and later shown for finite models by Rosen [vB76, Ros97]. An analogous first-order counterpart for coalgebraic modal logic has been introduced in previous work by two of the authors [SP10a]. The language described there does support a van Benthem/Rosen-style theorem. It is quite expressive but has a fairly complex syntax with three sorts, modelling states, sets of states, and composite states, respectively, and is equipped with a carefully tuned Henkin-style semantics. In the current work we develop *coalgebraic predicate logic (CPL)*, a first-order correspondence language for coalgebraic predicate logic that is slightly less expressive than the language proposed originally but has a simpler syntax and a straightforward semantics that does not require any design decisions. The naturality of CPL is further corroborated by the fact that CPL is expressively equivalent to hybrid logic (see the overview article by Areces and ten Cate [AtC07]) with satisfaction operators and universal quantification (equivalently with the downarrow binder $\downarrow$ and a global modality). Thus, CPL not only serves as a correspondence language for coalgebraic modal logic but also arises by adding a standard set of desirable expressive features widely used in specification and knowledge representation.

Our proposal originates in a largely forgotten paper by C.C. Chang [Cha73] who introduces a first-order logic of *Scott-Montague neighbourhood frames*, which in coalgebraic terms can be seen as coalgebras for the doubly contravariant powerset functor. Chang's original motivation was to simplify model theory for what Montague called *pragmatics* and to replace Montague's many-sorted setting by a single-sorted one. Chang's contributions were primarily of a model-theoretic nature. He provided adaptations of the notions of (elementary) submodel/extension, elementary chain of models and ultraproduct and established a Tarski-Vaught theorem as well as downward and upward Löwenheim-Skolem theorems. Our syntax is a notational variant of Chang's syntax; semantically, we generalize from neighbourhood frames to coalgebras for an underlying set functor, thus capturing the full range of non-relational modalities indicated above.

Our semantics naturally extends coalgebraic modal logic in that it is parametrized over an interpretation of the modal operators as predicate liftings [Pat03, Sch08]. It can thus be instantiated with modalities such as, for instance, the standard relational $\Diamond$; with neighbourhood-based modalities as in Chang's original setup; with probabilistic operators L_p 'with probability at least p'; or with a binary conditional $\Rightarrow$ 'if – then normally'. We incorporate a unary $\heartsuit$ into a first-order language by allowing formulas of the form

$$t\heartsuit\lceil z : \phi\rceil$$

where t is a term, ϕ is a formula of coalgebraic predicate logic, z is a (comprehension) variable. Such a formula stipulates that t satisfies $\heartsuit$, applied to the set of all z that satisfy ϕ. For example, in standard modal logic over relational semantics, the formula $x\Diamond\lceil z : z = y\rceil$ says that x has y as a (relational) successor. In the probabilistic setting, the formula $xL_p\lceil y : y \neq x\rceil$ states that the probability of moving from x to a different state is at least p.

As indicated above, CPL supports a van Benthem / Rosen type result stating essentially that coalgebraic modal logic is the bisimulation-invariant fragment of CPL both over the class of all structures and over the class of finite structures; this result is proved in a companion paper [SPL17], which also establishes a Gaifman-type theorem for CPL. In the current paper, we establish the following results on CPL:

- We give a Hilbert-style axiomatization that we prove strongly complete for two particular classes of coalgebraic structures, viz. structures that are either neighbourhood-like or *bounded*, where the latter type includes the relational and the graded case as well as positive Presburger modalities. As usual in model theory, strong completeness can be supplemented with a suitable variant of the Omitting Types Theorem.
- While boundedness is a rather strong condition on structures, we show that the condition is fairly essential for completeness in the sense that within a much broader type of *ω-bounded* structures, the bounded structures are the only ones that allow for strong completeness.
- As indicated above, we establish the equivalence of CPL and several natural variants of coalgebraic hybrid logic.
- We prove some basic model-theoretic results. Specifically, we show that, under the same (alternative) assumptions as for our completeness result, ultraproducts exist and a downward Löwenheim-Skolem theorem holds; in fact, it turns our that the latter is applicable more broadly, requiring as it does only ω-boundedness in place of boundedness in its corresponding variant.
- We give sequent systems complementing the above-mentioned Hilbert system, and establish completeness, under the same (alternative) assumptions as for the Hilbert system, and more interestingly, syntactic cut-elimination for the "neighbourhood-like" case.

The material is organized as follows. In §2 we introduce the syntax and semantics of CPL and give a number of intuitive examples. In § 3 we discuss the Hilbert-style axiomatization and associated completeness results. We proceed to clarify the relationship between CPL and several variants of coalgebraic modal and hybrid logic in § 4. In § 5 we take first steps in the model theory of CPL, and § 6 deals with proof theory. While our presentation of the basic definitions in coalgebraic logic is self-contained in principle, we do import some of its basic results. Additional information is found in work on coalgebraic finite models [Sch07], one-step rules [SP09, SP10c], and modularization of coalgebraic logics [SP11].

1.1. **Related Work.** As already discussed, the syntax of our logic follows Chang's first-order logic of neighbourhood frames [Cha73]. An alternative, two-sorted language for neighbourhood frames has been proposed by Hansen et al. [HKP09]. Over neighbourhood frames, the language studied in the present work is a fragment of the two-sorted one; we give details in § 2.

First-order formalisms have also been considered for topological spaces, which are particular instances of neighbourhood frames when defined in terms of local neighbourhood bases. In particular, Sgro [Sgr80] studies interior operator logic in topology with interior modalities for finite topological powers of the space. This language is the weakest one in the hierarchy of topological languages considered in an early overview by Ziegler [Zie85]. Makowsky and Marcja [MM77] prove a range of completeness theorems for topological logics, including a completeness result for the Chang language itself, i.e., a special version of our Theorem 3.19. See also ten Cate et al. [CGS09] for a more contemporary reference. Despite the fact that CPL combines quantifiers and modalities, it should not be confused with what is usually termed quantified or first-order modal logic; see Remark 2.1.

As mentioned above, our logic is less expressive but more naturally defined than the correspondence language used in the first van Benthen/Rosen type characterization result for coalgebraic modal logic [SP10a]. Axiomatizations and model-theoretic results as we develop here are not currently available for the more expressive language of [SP10a].

A different generic first-order logic largely concerned with the Kleisli category of a monad rather than with coalgebras for a functor is introduced and studied in [Jac10]. Of all the languages discussed above, this one seems least related to the present one; indeed, the study of connections with languages like that of the original, three-sorted variant [SP10a] is mentioned by Jacobs [Jac10] as a subject for future research.

This paper is based on results first announced in earlier conference papers [LPSS12, LPS13]. Compared to the conference versions, it features full proofs and additional examples. Some results previously only mentioned such as the Omitting Types Theorem (Theorem 3.30) are explicitly stated and proved here for the first time. We also corrected a number of errors and typos. Most notably, as reconstructing the proof of cut-elimination for the $\mathsf{G3c}$-style system proposed in [LPS13] proved problematic, we replaced it with a $\mathsf{G1c}$-style system in this version, with a different treatment of equality and provided all the proof details.

2. Syntax, Semantics and Examples

We proceed to give a formal definition of *coalgebraic predicate logic* (CPL). We fix a set Σ of predicate symbols and a modal similarity type Λ, i.e. a set of modal operators. Modal operators $\heartsuit \in \Lambda$ and predicate symbols $P \in \Sigma$ both come with fixed *arities* $\mathsf{ar}\heartsuit, \mathsf{ar}P \in \mathbb{N}$. The set $\mathrm{CPL}(\Lambda, \Sigma)$ of CPL *formulas* over Λ and Σ is given by the grammar

$$\mathrm{CPL}(\Lambda, \Sigma) \ni \phi, \psi \ ::= \ y_1 = y_2 \mid P(\vec{x}) \mid \bot \mid \phi \to \psi \mid \forall x.\phi \mid x\heartsuit\lceil y_1 : \phi_1\rceil \dots \lceil y_n : \phi_n\rceil$$

where $\heartsuit \in \Lambda$ is an n-ary modal operator and $P \in \Sigma$ a k-ary predicate symbol, x, y_i are variables from a fixed set iVar we keep implicit. We just write $\mathrm{CPL}(\Lambda)$ for $\mathrm{CPL}(\Lambda, \emptyset)$ and sometimes we omit Λ and Σ altogether. Booleans and the existential quantifier are defined in the standard way. We do not include function symbols, which can be added in a standard way [Cha73]. We adopt the usual convention that the scope of a quantifier extends as far to the right as possible. In the $\lceil y_i : \phi_i\rceil$ component, y_i is used as a comprehension variable, i.e., $\lceil y_i : \phi_i\rceil$ denotes a subset of the carrier of the model, to which modal operators can be applied in the usual way. In $x\heartsuit\lceil y_1 : \phi_1\rceil \dots \lceil y_n : \phi_n\rceil$, x is free and y_i is bound in ϕ_i, otherwise the notions of freeness and boundedness are standard. We write $\mathsf{FV}(\phi)$ for the set of free variables of a formula ϕ. A variable is *fresh* for a formula if it does not have free occurrences in it; to save space, we will also sometimes say that $x \in \mathsf{iVar}$ is fresh for $y \in \mathsf{iVar}$ whenever it is distinct from it. A *sentence*, as usual, is a formula without free variables.

As usual, some care is needed when defining substitution to avoid, on the one hand, capture of newly substituted variables by quantifiers and on the other hand, substituting for a bound variable. We take as our model the discussion in Enderton's monograph [End01, p. 112–113]. As we now have two ways in which a variable can become bound and the binder $\heartsuit$ involves also a variable/term in a non-binding way, it is desirable to spell out details. We thus define—prima facie not necessarily capture-avoiding—substitution $\alpha[t/x]$ with $t, x \in \mathsf{iVar}$ (had we allowed for function symbols, t could be any term) as replacing x with t in atomic formulas and commuting with implication (and of course other Boolean connectives, were they taken as primitives). For binders, the clauses are:

$$(\forall x.\phi)[t/y] = \begin{cases} \forall x.\phi & x = y \\ \forall x.\phi[t/y] & \text{otherwise,} \end{cases}$$

$$(x\heartsuit\lceil z_1 : \phi_1\rceil \dots \lceil z_n : \phi_n\rceil)[t/y] = u\heartsuit\lceil z_1 : \phi'_1\rceil \dots \lceil z_n : \phi'_n\rceil$$

$$\text{where } \phi'_i = \begin{cases} \phi_i & y = z_i \\ \phi'_i[t/y] & \text{otherwise,} \end{cases} \quad \text{and } u = \begin{cases} t & x = y \\ x & \text{otherwise.} \end{cases}$$

This of course cannot work without restrictions, so we follow Enderton in defining the notion of *substitutability* of t for x in a term. There are no restrictions on substitutability in atomic formulas, and for implications, it is defined as substitutability in the two argument formulas. Finally, t is substitutable for x in $z\heartsuit\lceil y_1 : \phi_1\rceil \dots \lceil y_n : \phi_n\rceil$ whenever for every i, t is either

- fresh for $\lceil y_i : \phi_i\rceil$ (this includes the case $t = y_i$) or
- different from y_i and substitutable for x in ϕ_i.

Note that in the first alternative, substituting t for x has no effect on $\lceil y_i : \phi_i\rceil$. In a language with more general terms, the second alternative would require that y_i is fresh for t rather than different from t. Substitutability of t for x in $\forall y_1.\phi_1$ is defined similarly (and standardly).

We depart from Enderton's conventions by restricting, from now on, the usage of the $\alpha[t/x]$ notation to the case where t is substitutable for x in α (as usual, when this is not the case the substitution can still be applied after suitably renaming bound variables in α). For example, the axiom scheme $\forall\vec{y}.(\forall x.\phi \to \phi[z/x])$ denoted as En2 in Table 1 has as its valid instances only those formulas where z is substitutable for x.

The semantics of CPL is parametrized over the choice of an endofunctor T on Set that determines the underlying system type: models are based on T-*coalgebras*, i.e. pairs $(C, \gamma : C \to TC)$ consisting of a carrier set C of worlds or *states* and a *transition function* γ. We think of the elements of TC as being *composite states*; e.g. if T is the identity functor then a composite state is just a state, and if T is powerset, then a composite state is a set of states. Thus, the transition function assigns to each state c a composite state $\gamma(c)$ that represents the successors of c and that we correspondingly refer to as the *composite successor* of c. E.g. in case T is powerset, a T-coalgebra assigns to each state a set of successor states, and hence is essentially a Kripke frame.

To interpret the modal operators, we extend T to a Λ-*structure*, i.e. we associate to every n-ary modal operator $\heartsuit \in \Lambda$ a set-indexed family of mappings

$$[\![\heartsuit]\!]_C : (\mathcal{Q}C)^n \to \mathcal{Q}TC$$

where $\mathcal{Q}$ denotes the contravariant powerset functor, subject to *naturality*, i.e.

$$(Tf)^{-1} \circ [\![\heartsuit]\!]_C = [\![\heartsuit]\!]_D \circ (f^{-1})^n$$

for every set-theoretic function $f : C \to D$. In categorical parlance, this means that $[\![\heartsuit]\!]$ is a natural transformation $\mathcal{Q}^n \to \mathcal{Q} \circ T^{op}$; we recall that the contravariant powerset functor $\mathcal{Q}$ maps a set X to the powerset of X and a map $X \to Y$ to the preimage map $\mathcal{Q}f : \mathcal{Q}Y \to \mathcal{Q}X$, i.e. $\mathcal{Q}f(A) = f^{-1}[A]$ for $A \subseteq Y$. Each such $[\![\heartsuit]\!]$ is called a *predicate lifting* [Pat03, Sch08]. Formally speaking, we should define a Λ-structure as a pair $(T, ([\![\heartsuit]\!])_{\heartsuit\in\Lambda})$, but to avoid cumbersome notation and terminology, we will speak about a Λ-structure *based* on T (or a Λ-structure *over* T) and suppress the second component of the pair whenever $([\![\heartsuit]\!])_{\heartsuit\in\Lambda}$ is clear from the context.

A triple $\mathfrak{M} = (C, \gamma, I)$ consisting of a coalgebra $\gamma : C \to TC$ and a predicate interpretation $I : \Sigma \to \bigcup_{n\in\omega} \mathcal{Q}(C^n)$ respecting arities of symbols will be called a *(coalgebraic) model*. In other words, a coalgebraic model consists simply of a Set-coalgebra and an ordinary first-order model whose universe coincides with the carrier of the coalgebra. Given a model $\mathfrak{M} = (C, \gamma, I)$ and a valuation $v : \mathsf{iVar} \to C$, we define satisfaction $\mathfrak{M}, v \models \phi$ in the standard way for first-order connectives and for $\heartsuit$ by the clause

$$\mathfrak{M}, v \models x\heartsuit\lceil y_1 : \phi_1\rceil \dots \lceil y_n : \phi_n\rceil \iff \gamma(v(x)) \in [\![\heartsuit]\!]_C([\![\phi_1]\!]^{y_1}_{\mathfrak{M}}, \dots, [\![\phi_n]\!]^{y_n}_{\mathfrak{M}})$$

where

$$[\![\phi]\!]^y_{\mathfrak{M}} = \{c \in C \mid \mathfrak{M}, v[c/y] \models \phi\} \tag{2.1}$$

and $v[c/y]$ is v modified by mapping y to c.

Remark 2.1. *Quantified* or *first-order modal logic* in the sense used widely in the literature (see, e.g., [Gar01]) combines quantification and modalities in a two-sorted and, effectively, two-dimensional semantics: One has an underlying set of worlds as well as an underlying set of individuals, with modalities interpreted as moving between worlds and quantification interpreted as ranging over individuals in the current world. We emphasize that although CPL also combines modalities and quantifiers, it is not a quantified modal logic in this sense: it is interpreted over a single set of individuals, and both the modalities and the quantifiers move within this set. In particular, the instance of CPL induced by the standard modalities equipped with their usual predicate liftings is standard first-order logic rather than a quantified modal logic, as we discuss below in some detail.

In a companion paper on the van Benthem-Rosen theorem for CPL [SPL17] and in conference papers the present work is based upon [LPSS12, LPS13], we have focused on Chang's original motivation for this language [Cha73]. Namely, Chang saw his setup as a modification of Montague's account of *pragmatics*, tailored to reasoning about social situations and relationships between an individual and sets of individuals. We have proposed a series of examples kept in the same spirit, utilizing Facebook, Twitter and social networks. In the present paper, we offer examples based on, so to say, networking of a more low-level character, especially *delay-* or *disruption-tolerant* networking (*DTN*). We do not claim to be very accurate with respect to specifications of concrete protocols; our examples are of purely inspirational and illustrative character. It is worth mentioning, though, that such routing and forwarding protocols can be backed by social insights [HCY08], so in a sense we are still following the spirit of our original examples.[1]

Neighbourhood Frames. Scott/Montague *neighbourhood semantics* is captured coalgebraically using $\Lambda = \{\Box\}$ and putting $TC = \mathcal{Q}\mathcal{Q}C$ (the doubly contravariant powerset functor), which extends to a Λ-structure by

$$[\![\Box]\!]_C(A) = \{\sigma \in TC \mid A \in \sigma\}.$$

A T-coalgebra then associates to each state a set of sets of states, i.e. a system of neighbourhoods; thus, T-coalgebras are just neighbourhood frames. In the presence of a binary relation $S(x, y)$ that we read as 'node/router y is in the forwarding table of x' and interpreting $\Box$ as 'is a recognized subcommunity', the formula

$$\exists y. x\Box\lceil z : S(z, y)\rceil$$

[1] In particular, Hui et al. [HCY08] gave us the idea of using *subcommunities* in this context.

reads as 'there exists a certain y such that amongst the subcommunities recognized by x, there is one formed exactly by those having y in its forwarding table'.

The instance of CPL that we obtain in this way is, up to quite minor syntactic differences, Chang's original language [Cha73]. As mentioned in § 1, it embeds as a fragment into Hansen et al.'s two-sorted correspondence language [HKP09]. We refrain from giving full syntactic details; roughly, the setup is as follows. The two-sorted language has sorts s for states and n for neighbourhoods, and features binary infix predicates N and E respectively modelling the neighbourhood relation between states and neighbourhoods, and the inverse elementhood relation $\ni$ between neighbourhoods and states. Then our $x\heartsuit\lceil y : \phi(y)\rceil$ can be translated as $\exists u.(x\mathsf{N}u \wedge \forall y.(u\mathsf{E}y \leftrightarrow \phi(y)))$.

Relational first-order logic. Instantiating CPL with the usual modalities of relational modal logic, specifically the logic K, we obtain a notational variant of ordinary FOL over relational structures, that is, of the usual correspondence language. The main idea has already been indicated in the introduction: we encode the successor relation in formulas of the form $x\Diamond\lceil z : y = z\rceil$, which states that y is a successor of x. Formally, we capture the standard modality and the propositional atoms of the relational modal logic K in the similarity type

$$\Lambda = \{\Diamond\} \cup \mathsf{At}$$

where At is a set of propositional atoms; as expected, $\Diamond$ is unary, and $a \in \mathsf{At}$ is nullary. We interpret these operators over the functor T given on objects by

$$TX = \mathcal{P}X \times \mathcal{P}\mathsf{At}$$

where $\mathcal{P}$ denotes the covariant powerset functor. That is, a coalgebra $\gamma : C \to TC$ assigns to each state $c \in C$ a set of successors as well as a set of propositional atoms valid in c. The interpretation is defined by means of predicate liftings

$$[\![\Diamond]\!]_X(A) = \{(Y,U) \in \mathcal{P}X \times \mathcal{P}\mathsf{At} \mid A \cap Y \neq \emptyset\}$$
$$[\![a]\!]_X = \{(Y,U) \in \mathcal{P}X \times \mathcal{P}\mathsf{At} \mid a \in U\}$$

where, corresponding to the arity of the modal operators, the predicate lifting for $\Diamond$ is unary and the predicate liftings for the $a \in \mathsf{At}$ are nullary. These predicate liftings capture precisely the standard semantics of both $\Diamond$ and the propositional atoms. In particular, the above-mentioned formula $x\Diamond\lceil z : y = z\rceil$ really does say that y is a successor of x. (Notice that in the nullary case, our syntax instantiates to formulas $x\,a$ saying that x satisfies the propositional atom a).

The standard first-order correspondence language of modal logic has unary predicates a for the atoms $a \in \mathsf{At}$ and a binary predicate R to represent the successor relation. We translate CPL(Λ) as defined above into the standard correspondence language by just extending the standard translation of modal logic to CPL, with the modification that the current state is represented by an explicit variable in CPL so that it is no longer necessary to index the standard translation with a variable name. That is, our translation ST is defined in the modal cases (which by our conventions include the case of propositional atoms) by

$$ST(x\Diamond\lceil y : \phi\rceil) = \exists y.\, R(x,y) \wedge ST(\phi)$$
$$ST(x\,a) = a(x) \qquad (a \in \mathsf{At})$$

and by commutation with all other constructs. In the converse direction, we translate $R(x,y)$ into $x\Diamond\lceil z : z = y\rceil$ and $a(x)$ into $x\,a$. In summary, *CPL over* $\Lambda = \{\Diamond\} \cup \mathsf{At}$ *with the above* *semantics is expressively equivalent to the standard first-order correspondence language of modal logic.*

Graded Modal Logic. We obtain a variant of graded modal logic [Fin72] if we consider the similarity type $\Lambda = \{\langle k \rangle \mid k \geq 0\}$ where $\langle k \rangle$ reads as 'more than k successors satisfy …'. We interpret the ensuing logic over *multigraphs* [DV02], which are coalgebras for the *multiset functor* $\mathcal{B}$ given on objects by

$$\mathcal{B}X = \{\mu : X \to \mathbb{N} \cup \{\infty\} \mid f \text{ a map}\}.$$

We see such a map $\mu : X \to \mathbb{N} \cup \{\infty\}$ as an integer-valued discrete measure on X, i.e. we write $\mu(A) = \sum_{x \in A} \mu(x)$ for $A \subseteq X$. Then, $\mathcal{B}$ acts on maps $f : X \to Y$ by taking image measures; i.e. $\mathcal{B}f(\mu)(B) = \mu(f^{-1}[B])$ for $B \subseteq Y$. We extend $\mathcal{B}$ to a Λ-structure by stipulating

$$[\![\langle k \rangle]\!]_X(A) = \{\mu \in \mathcal{B}X \mid \mu(A) > k\}$$

to express that more than k successors (counted with multiplicities) have property A. Note that over Kripke frame, graded operators can be coded into standard first order logic; the difference with standard first-order logic arises through the multigraph semantics, for which the requisite expressive means arise only through the graded operators.

Continuing our line of routing examples, we can, given a $\mathcal{B}$-coalgebra $\gamma : C \to \mathcal{B}C$, think of $\gamma(c)(c')$ as the number of packets forwarded from c to c' in the past hour. In the presence of a binary relation $S(x, y)$ interpreted as above, the formula

$$\neg \exists y.(x \langle k \rangle \lceil z : S(y, z) \rceil)$$

then expresses that there is no router y s.t. the total number of packets sent by x to nodes in y's forwarding table in the past hour exceeds k.

Presburger modal logic and arithmetic. A more general set of operators than graded modal logic is that of positive Presburger modal logic [DL06], which admits integer linear inequalities $\sum_{i=1}^{n} a_i \cdot \#(\phi_i) > k$ among formulas where $a_i \geq 0$ for all i. We see such a formula as an application of an n-ary modality $L_k(a_1, \ldots, a_n)$ to formulas $\phi_1, \ldots, \phi_n$, and interpret this modality over the multiset functor $\mathcal{B}$ as introduced above by the n-ary predicate lifting

$$[\![L_k(a_1, \ldots, a_n)]\!]_X(A_1, \ldots, A_n) = \{\mu \in \mathcal{B}X \mid \textstyle\sum_{i=1}^{n} a_i \cdot \mu(A_i) > k\}.$$

In addition to the binary predicate S, let us also introduce unary predicate $O(x)$ expressing that x is an overloaded node. The formula

$$\forall x.((x L_{10{,}000}(1, 3) \lceil y : S(x, y) \rceil \lceil y : O(y) \rceil) \to O(x))$$

means that, if the weighted number of packets sent by x to overloaded nodes combined with packets x sends to all nodes in its forwarding table exceeds 10,000, then x itself is overloaded.

Combination of Frame Classes. Frame classes can be combined: we can take $T = \mathcal{B}\times\mathcal{Q}\mathcal{Q}$ and combine operators for packet counting and subcommunity recognition. A formula

$$\neg x\Box\lceil y : y\langle 30\rangle\lceil u : u \neq z\rceil\rceil$$

expresses then that the collection of those nodes which have forwarded more than 30 packages to servers different than z in the past hour is not a subcommunity recognized by x.

Probabilistic Modal Logic. The *discrete distribution functor* $\mathcal{D}$ is defined on objects by

$$\mathcal{D}X = \{\mu : X \to [0,1] \mid \textstyle\sum_x \mu(x) = 1\},$$

and on morphisms by taking image measures exactly as for the multiset functor $\mathcal{B}$ discussed earlier. Coalgebras for $\mathcal{D}$ thus associate to every state a probability distribution over successor states; such structures are known as *Markov chains*, or *probabilistic transition systems*, or *type spaces*. Taking the similarity type $\Lambda = \{\langle p\rangle \mid p \in [0,1]\cap\mathbb{Q}\}$, with $\langle p\rangle$ read as 'with probability more than p' (thus departing from the choice of operators L_p ' with probability at least p' that we used in the introduction), we formally interpret $\langle p\rangle$ using the predicate lifting

$$[\![\langle p\rangle]\!]_X(A) = \{\mu \in \mathcal{D}X \mid \mu(A) > p\}$$

over $\mathcal{D}$. We thus obtain a form of probabilistic first-order logic for probabilistic transition systems that extends probabilistic modal logic [LS91, FH94, HM01]. Continuing our line of routing examples, if we interpret the transition probabilities as the likelihood of a server forwarding any given packet to another, then the formula

$$\forall x, y.(x\langle 1/2\rangle\lceil z : z = y\rceil \to y\langle 1/2\rangle\lceil z : z = x\rceil)$$

expresses a partial form of symmetric connectivity: whenever a server x prefers the connection to y in the sense that it will more likely than not route any given packet through y, then the same will hold in the other direction.

We obtain a version of this logic with finitely many modal operators in situations where all possible probabilities are contained in some finite set of rationals (such as when rolling a fair die). We then consider substructures of the form

$$\mathcal{D}_kX = \{\mu \in \mathcal{D}(X) \mid \mu(x) \in \{i/k \mid i = 0,\dots,k\}\},$$

restricting the modal operators to come from $\Lambda_k = \{\langle n/k\rangle \mid n = 0,\dots,k\}$.

Non-Monotonic Conditionals. An example of a binary modality is provided by (conditional) implication $>$. Such operators are interpreted over a variety of semantic structures; one of these involves *selection function frames*, which in our terminology can be defined as coalgebras for the *selection function functor* $\mathcal{S}$. The latter acts on objects by

$$\mathcal{S}X = \{f : \mathcal{Q}X \to \mathcal{P}X\}$$

and, correspondingly, on maps $f : X \to Y$ by $\mathcal{S}f = \mathcal{Q}f \to \mathcal{P}f : \mathcal{S}X \to \mathcal{S}Y$, i.e. $\mathcal{S}f(g)(A) = f[g(f^{-1}[A])]$ for $A \subseteq X$ (recall that $\mathcal{Q}$ denotes the contravariant powerset functor). We think of $f_x \in \mathcal{S}X$ as selecting the set $f_x(A)$ of worlds which x sees as 'most typical' given a condition $A \subseteq X$. Over this functor, we interpret the conditional $>$ by the predicate lifting

$$[\![>]\!]_X(A, B) = \{f \in \mathcal{S}X \mid f(A) \cap B \neq \emptyset\}.$$

The formula $\phi > \psi$ expresses that ψ is typically possible under condition ϕ. This presentation of conditional logic is dual to the standard presentation [Che80] in terms of a binary operator $\Rightarrow$ 'if – then normally', related to $>$ by $a > b \equiv \neg(a \Rightarrow \neg b)$. For our purposes, $>$ has the technical advantage of being bounded in the second argument in a sense that we will introduce in § 3.

We continue to interpret our examples in the context of routing: Given an $\mathcal{S}$-coalgebra (i.e. selection function frame) $\gamma : C \to \mathcal{S}C$, we may read $\gamma(c)(A)$ as the set of those servers through which server c will normally route an incoming packet if c is currently active in the subcommunity $A \subseteq C$. Then a formula of the form

$$\forall x, u.(\phi(u) \to x > \lceil y : \phi(y) \rceil \lceil z : z = u \rceil)$$

says that if x is currently active in a subcommunity delineated by the formula $\phi(y)$ and u belongs to that subcommunity, then u is normally a possible target for packets forwarded by x.

We have not mentioned propositional atoms other than in the example on relational first-order logic. We introduce an explicit notion of propositional atom as part of a modal signature:

Definition 2.2. A nullary modality $p \in \Lambda$ is a *propositional atom* if T decomposes as $T = T' \times 2$ and under this decomposition, $[\![p]\!]_X = T'X \times \{\top\}$.

Remark 2.3. Propositional atoms can easily be added to all our examples by just extending the functor with a component for their interpretation, as indicated in the above definition. Explicitly, if At is a set of propositional atoms and T' is a functor, then the atoms $p \in \mathsf{At}$ give rise to nullary modalities p, interpreted over $T = T' \times \mathcal{P}(\mathsf{At})$ by $[\![p]\!]_X = \{(t, U) \in T'X \times \mathcal{P}(\mathsf{At}) \mid p \in U\}$. In fact, this is an instance of what we have called *combination of frame classes* above: Systems featuring only propositional atoms with pointwise valuations, and no further transition structure, are captured as coalgebras for the constant functor $\mathcal{P}(\mathsf{At})$, and this system type can be freely combined with others. A coalgebraic framework for such combinations of system types and modalities is afforded by multisorted coalgebra, essentially following the principle of converting components of the coalgebraic type functor into sorts [SP11]. General results in this framework imply that completeness properties transfer smoothly to such logic combinations (essentially, to *fusions* in standard terminology). In particular, completeness will always extend without further ado under adding propositional atoms to a logic, so we will continue to largely elide them in the discussion of examples.

3. Completeness

In § 3.2 below, we propose an axiom system for CPL, sound wrt arbitrary structures (Theorem 3.17) and in § 3.3 we show its completeness wrt structures s.t. each operator on every coordinate is either "neighbourhood-like" or "Kripke-like" (Theorem 3.19). As discussed in § 3.5, even a mild relaxation of these conditions makes a generic completeness result impossible.

However, not only for the proof, but even for the statements of our completeness result, or of the axiomatization itself, we need some spadework.

3.1. **S1SC and Boundedness.** In order to state our axiomatization and completeness results, we need several notions from coalgebraic model theory. The first of them, central to the entire edifice, is that of *one-step satisfiability.*

Definition 3.1 (One-step logic)**.**

- Given a supply of primitive symbols D (which can be any set), define the set $\mathsf{Prop}(D)$ of *Boolean D-formulas* (or *propositions*) as
$$\boldsymbol{A}, \boldsymbol{B} ::= d \mid \boldsymbol{A} \to \boldsymbol{B} \mid \bot$$
where $d \in D$, and the set $\Lambda(D)$ of *modalized D-formulas* as
$$\Lambda(D) = \{\heartsuit d_1 \dots d_n \,|\, d_1, \dots, d_n \in D \text{ and } \heartsuit \in \Lambda \text{ is } n\text{-ary}\}.$$
Then the set $\mathsf{Rank1}(D)$ of *rank-1 D-formulas* is defined as
$$\mathsf{Rank1}(D) = \mathsf{Prop}(\Lambda(\mathsf{Prop}(D)));$$
in other words, a rank-1 formula is a Boolean combination of formulas consisting of a modality from Λ applied to Boolean combinations of atoms from D.
- Given a set C and a valuation $\tau : D \to \mathcal{P}(C)$, we extend τ to $\mathsf{Prop}(D)$ using the Boolean algebra structure of $\mathcal{P}(C)$, and then write $C, \tau \models \boldsymbol{A}$ if $\tau(\boldsymbol{A}) = C$, for $A \in \mathsf{Prop}(D)$.
- Given the same data, we define the extension $[\![\phi]\!]_{TC,\tau} \subseteq TC$ of $\phi \in \mathsf{Rank1}(D)$ by extending the assignment
$$[\![\heartsuit \boldsymbol{A}_1 \dots \boldsymbol{A}_n]\!]_{TX,\tau} = [\![\heartsuit]\!]_C(\tau(\boldsymbol{A}_1), \dots, \tau(\boldsymbol{A}_n))$$
using the Boolean algebra structure of $\mathcal{P}(TC)$.
- We then write $TC, \tau \models \phi$ if $[\![\phi]\!]_{TC,\tau} = TC$, and $t \models_{TC,\tau} \phi$ if $t \in [\![\phi]\!]_{TC,\tau}$.
- If $D \subseteq \mathcal{P}(C)$ and τ is just the inclusion, we will usually drop it from the notation; in particular, for subsets $Y_1, \dots, Y_n \subseteq C$ and $\heartsuit \in \Lambda$ n-ary, we write $t \models \heartsuit(Y_1, \dots, Y_n)$ to mean $t \in [\![\heartsuit]\!]_C(Y_1, \dots, Y_n)$.
- A set $\Xi \subseteq \mathsf{Rank1}$ is *one-step satisfiable* w.r.t. τ if $\bigcap_{\phi \in \Xi} [\![\phi]\!]_{TC,\tau} \neq \emptyset$.

Just like in the case of coalgebraic modal logic (see § 4 below), proof systems for CPL are best described in terms of rank-1 rules – or, more precisely, rule schemes –, which describe the geometry of the Λ-structure under consideration [Sch07].

Definition 3.2 (One-Step Rules)**.** Fix a collection sVar of schematic variables $\boldsymbol{p}, \boldsymbol{q}, \boldsymbol{r} \dots$

- A *one-step rule* is of the form $\boldsymbol{A}/\boldsymbol{P}$ where $\boldsymbol{A} \in \mathsf{Prop}(\mathsf{sVar})$ and $\boldsymbol{P} \in \mathsf{Rank1}(\mathsf{sVar})$ is a disjunctive clause over $\Lambda(\mathsf{sVar})$, i.e. a finite disjunction of formulas that are either in $\Lambda(\mathsf{sVar})$ or negations of formulas in $\Lambda(\mathsf{sVar})$. As usual, we impose moreover that every schematic variable is mentioned at most once in $\boldsymbol{P}$, and every schematic variable occurring in $\boldsymbol{A}$ occurs also in $\boldsymbol{P}$.
- A rule $\boldsymbol{A}/\boldsymbol{P}$ is *one-step sound* if
$$TC, \tau \models \boldsymbol{P} \text{ whenever } C, \tau \models \boldsymbol{A} \text{ for a valuation } \tau : \mathsf{sVar} \to \mathcal{P}(C).$$
- Given a set $\mathcal{R}$ of one-step rules and a valuation $\tau : \mathsf{sVar} \to \mathcal{P}(C)$, a set $\Xi \subseteq \mathsf{Rank1}(\mathsf{sVar})$ is *one-step consistent* (*with respect to* τ) [SP10c] if the set
$$\Xi \cup \{\boldsymbol{P}\sigma \mid \sigma : \mathsf{sVar} \to \mathsf{Prop}(\mathsf{sVar}) \text{ and } \boldsymbol{A}/\boldsymbol{P} \text{ is a rule in } \mathcal{R} \text{ s.t. } C, \tau \models \boldsymbol{A}\sigma\}$$
is propositionally consistent, where $\boldsymbol{A} \mapsto \boldsymbol{A}\sigma$ and $\boldsymbol{P} \mapsto \boldsymbol{P}\sigma$ denote the obvious inductive extensions of σ to $\mathsf{Prop}(\mathsf{sVar})$ and $\mathsf{Rank1}(\mathsf{sVar})$, respectively (in other words, we see σ as a substitution, and use postfix notation to denote application of substitutions).

Assumption 3.3. *For purposes of the technical development (excluding the examples), we fix from now on a set $\mathcal{R}$ of one-step sound one-step rules.*

We next introduce the two variants of *one-step* completeness that we need for our global completeness proof. By *one-step*, we mean that the completeness assumption is only made for a very simple logic that precludes nesting of modal operators, and hence can be interpreted over single elements of TC rather than full coalgebraic models.

Definition 3.4 (Strong One-Step Completeness [SP10c])**.** The rule set $\mathcal{R}$ is *strongly one-step complete* (*S1SC, neighbourhood-like*) for a Λ-structure if

$$\text{for every } C \in \mathsf{Set},\ \Xi \subseteq \mathsf{Rank1}(\mathsf{sVar}),\ \text{and } \tau : \mathsf{sVar} \to \mathcal{P}(C),$$
$$\Xi \text{ is one-step satisfiable wrt } \tau \text{ whenever } \Xi \text{ is one-step consistent wrt } \tau.$$

Similarly, $\mathcal{R}$ is *finitary S1SC* if the same condition holds for τ restricted to be of type $\mathsf{sVar} \to \mathcal{P}_{\mathit{fin}}(C)$.

By the usual argument, both forms of strong one-step completeness imply corresponding forms of compactness:

Definition 3.5 (One-step compactness)**.** Λ-structure is *one-step compact* if for every set X, every finitely satisfiable set $\Phi \subseteq \mathsf{Prop}(\Lambda(\mathcal{P}X))$ of one-step formulas is satisfiable. Similarly, a Λ-structure is *finitary one-step compact* if for every set X, every finitely satisfiable set $\Phi \subseteq \mathsf{Prop}(\Lambda(\mathcal{P}_{\mathit{fin}}X))$ of one-step formulas is satisfiable.

Lemma 3.6. *Every (finitary) S1SC Λ-structure is (finitary) one-step compact.* □

Remark 3.7. Every (finitary) S1SC rule set derives the *congruence rule*

$$\frac{\boldsymbol{p}_1 \leftrightarrow \boldsymbol{q}_1 \quad \dots \quad \boldsymbol{p}_{\mathsf{ar}\heartsuit} \leftrightarrow \boldsymbol{q}_{\mathsf{ar}\heartsuit}}{\heartsuit(\boldsymbol{p}_1, \dots, \boldsymbol{p}_{\mathsf{ar}\heartsuit}) \to \heartsuit(\boldsymbol{q}_1, \dots, \boldsymbol{q}_{\mathsf{ar}\heartsuit})}$$

for every modal operator $\heartsuit \in \Lambda$. The reason is that the congruence rule is clearly one-step sound, and even under the much weaker asssumption of *one-step completeness* (obtained by restricting Ξ to be finite in Definition 3.4), all one-step sound rules are derivable [SV].

Example 3.8.

- Modal neighbourhood semantics is axiomatised by

  $$\frac{\boldsymbol{p} \leftrightarrow \boldsymbol{q}}{\Box \boldsymbol{p} \to \Box \boldsymbol{q}}\ \mathsf{C}$$

  which expresses that $\Box$ is a congruential operator (where we write $\Box \boldsymbol{p} \to \Box \boldsymbol{q}$ in place of $\neg \Box \boldsymbol{p} \vee \Box \boldsymbol{q}$ for readability). This is the paradigmatic example of S1SC; see the discussion in Remark 3.10 below.
- The rule set for the normal modal logic K consists of the rules

  $$\frac{\boldsymbol{p} \to \boldsymbol{q}_1 \vee \dots \vee \boldsymbol{q}_n}{\Diamond \boldsymbol{p} \to \Diamond \boldsymbol{q}_1 \vee \dots \vee \Diamond \boldsymbol{q}_n}\ \mathsf{K_n}$$

  for all $n \geq 0$. As we are going to see in Lemma 3.9, $\mathcal{R}$ is finitary S1SC. On the other hand, for reasons detailed in Remark 3.10, $\mathcal{R}$ is not S1SC. An S1SC semantics for this particular rule set would be provided by *normal neighbourhood (i.e., filter) frames.*

- For graded modal logic, the proof of one-step completeness of the rule set

$$\frac{\boldsymbol{p} \to \boldsymbol{q}}{\Diamond_{n+1}\boldsymbol{p} \to \Diamond_n \boldsymbol{q}}\ \mathsf{RG1} \qquad \frac{\boldsymbol{r} \to \boldsymbol{p} \vee \boldsymbol{q}}{\Diamond_{n_1+n_2}\boldsymbol{r} \to \Diamond_{n_1}\boldsymbol{p} \vee \Diamond_{n_2}\boldsymbol{q}}\ \mathsf{A1}$$

$$\frac{\boldsymbol{p} \to \boldsymbol{r} \quad \boldsymbol{q} \to \boldsymbol{r} \quad \boldsymbol{p} \wedge \boldsymbol{q} \to \boldsymbol{s}}{\Diamond_{n_1}\boldsymbol{p} \vee \Diamond_{n_2}\boldsymbol{q} \to \Diamond_{n_1+n_2+1}\boldsymbol{r} \vee \Diamond_0 \boldsymbol{s}}\ \mathsf{A2} \qquad \frac{\neg \boldsymbol{p}}{\neg \Diamond_0 \boldsymbol{p}}\ \mathsf{RN}$$

  [SV, Lemma 3.10] upon inspection effectively establishes finitary S1SC.
- For positive Presburger modal logic, we similarly have that the proof of one-step completeness of a natural rule system for full Presburger modal logic [KP10, Lemma 4.5] straightforwardly adapts to a) positive Presburger modal logic, and b) finitary S1SC, provided that one generalizes the semantics to infinite multisets as we do here.
- Conditional logic provides a curious mixed case, due to its being neighbourhood-like in one coordinate and Kripke-like in another. We will introduce adequate apparatus and analyse it further in Example 3.16.
- In Remark 3.11, we discuss the issue of S1SC of coalition logic interpreted over effectivity functions.

Lemma 3.9. *The Λ-structure for relational modal logic (Section 2) is finitary S1SC.*

Proof. Consider any consistent $\Xi \subseteq \mathsf{Prop}(\Lambda(\mathcal{P}_{\mathit{fin}} X))$, where $\Lambda = \{\Diamond\}$; w.l.o.g., Ξ is maximally consistent. We need to find $A \in \mathcal{P}X$ s.t. $A \vDash_{\mathcal{P}X} \Xi$. Let us choose

$$A = \{x \mid \Diamond\{x\} \in \Xi\}.$$

One shows by induction on formulas that

$$A \vDash_{\mathcal{P}X} \phi \text{ iff } \phi \in \Xi.$$

The only non-trivial case is the modal one. For any $B = \{b_1, \dots, b_n\} \in \mathcal{P}_{\mathit{fin}} X$, we have that

$$\begin{aligned} A \vDash_{\mathcal{P}X} \Diamond B &\text{ iff } \exists i \leq n. A \vDash_{\mathcal{P}X} \Diamond\{b_i\} \\ &\text{ iff } \exists i \leq n. \Diamond\{b_i\} \in \Xi \\ &\text{ iff } \Diamond B = \Diamond\{b_1\} \cup \dots \cup \Diamond\{b_n\} \in \Xi. \end{aligned}$$ □

Remark 3.10. As noted in [SP10c, Remark 55], we can give a more abstract characterization of S1SC, recognizable also to readers familiar with more categorical presentations of coalgebraic modal logic (cf., e.g., [KR12]). Every signature Λ together with a given set of one-step axiom schemes (equivalently, one-step rules) can be encoded disregarding concrete syntax by its *functorial presentation* [KKP04] (cf. also [SP10c, Definition 28]) as an endofunctor L_Λ on the category BA of Boolean algebras. BA is dually adjoint to Set, with the adjunction given by the contravariant powerset functor[2] $\overline{\mathcal{Q}}$ and the functor $\mathcal{S}$ taking a Boolean algebra to the set of its ultrafilters:

$$L_\Lambda \circlearrowright \mathsf{BA} \underset{\mathcal{S}}{\overset{\overline{\mathcal{Q}}}{\rightleftarrows}} \mathsf{Set} \circlearrowleft T \tag{3.1}$$

The information contained in each Λ-structure can be then more abstractly encoded by $\delta : L_\Lambda \overline{\mathcal{Q}} \to \overline{\mathcal{Q}} T$ [KKP04] and the *canonical* structure for Λ is given by $M_\Lambda = \mathcal{S} L_\Lambda \overline{\mathcal{Q}}$. Coalgebras for the canonical structure can equivalently be described as generalized neighbourhood frames (where by *generalized* we mean that for every n-ary modality in Λ, we have n-ary

[2] We write here $\overline{\mathcal{Q}}$ to stress that we change the target category.

neighbourhoods, i.e. subsets of the n-th power of the state set), subject to satisfaction of the frame conditions embodied in the given one-step rules [SP10c, Remark 34]. For every Λ-structure, we can define a canonical *structure morphism* [SP10c, p. 1121] to M_Λ by composing the counit of the above adjunction with $\mathcal{S}\delta$, and S1SC effectively requires that this structure morphism is surjective. In other words, a Λ-structure is S1SC iff its functor surjects onto the canonical neighbourhood semantics; it is for this reason that we refer to the S1SC case as "neighbourhood-like". In fact, as we explain in the next remark, we do not currently have an in-the-wild example of an S1SC structure that is not actually isomorphic to the canonical neighbourhood semantics.

Remark 3.11. Coalition logic [Pau02] and, essentially equivalently, the next-step fragment of alternating-time temporal logic [AHK02], have modalities $[Q]$ indexed over *coalitions* Q, which are subsets of a fixed finite set N of *agents*; the operator $[Q]$ reads 'the coalition Q of players can enforce ... in the next step'. The semantics is formulated over structures called *game frames* or *concurrent game structures*, i.e., coalgebras for the functor

$$\mathcal{G}X = \{((S_i)_{i\in N}, f : \left(\prod_{i\in N} S_i\right) \to X) \mid \emptyset \neq S_i \subseteq \mathbb{N}\}$$

where S_i is thought of as the set of moves available to agent $i \in N$ and f is an *outcome function* that determines the next state of the game, depending on the moves chosen by the agents (we restrict to finitely many moves per agent as in alternating-time temporal logic). For notational convenience, given a coalition $Q = \{q_1, \dots, q_k\} \subseteq N$ and moves $s_{q_1} \in S_{q_1}$, $\dots$, $s_{q_k} \in S_{q_k}$, we write $s_Q = (s_q)_{q\in Q}$ and $S_Q = S_{q_1} \times \dots \times S_{q_k}$ (so that $s_Q \in S_Q$). Given $s_Q \in S_Q$ and $s_{N\setminus Q} \in S_{N\setminus Q}$, we write $(s_Q, s_{N\setminus Q})$ for the evident induced element of S_N.

An alternative semantics of the coalitional operators is provided by *effectivity functions*. These are functions E assigning to each coalition Q a set $E(Q)$ of properties that Q can enforce. Explicitly, a concurrent game $G = ((S_i)_{i\in P}, f) \in \mathcal{G}X$ induces an effectivity function E_G by

$$E_G(Q) = \{A \subseteq X \mid \exists s_Q \in S_Q. \forall s_{N\setminus Q} \in S_{N\setminus Q}. f(s_Q, s_{N\setminus Q}) \in A\}.$$

Effectivity functions congregate into a functor $\mathcal{E}$, a subfunctor of a product of neighbourhood functors. The modal operators $[Q]$ are interpreted over effectivity functions in the usual style of neighbourhood semantics, i.e. by

$$[\![[Q]]\!]_X(A) = \{E \in \mathcal{E}X \mid A \in E(Q)\}.$$

Composing this semantics with the above-defined projection from concurrent games to effectivity functions yields the interpretation of the coalitional modalities $[Q]$ over $\mathcal{G}$; this reproduces the standard semantics of coalition logic and alternating time temporal logic.

Now Theorem 3.2 in [Pau02] states that an effectivity function $E \in \mathcal{E}(X)$ is of the form E_G for some $G \in \mathcal{G}X$ iff it is *playable*, i.e. satisfies the following properties:

- For all Q, $\emptyset \notin E(Q) \ni X$
- E is *outcome-mononotic*, i.e. each $E(Q)$ is upwards closed under set inclusion.
- E is N-*maximal*, i.e. for all $A \subseteq X$, either $X \setminus A \in E(\emptyset)$ or $A \in E(N)$.
- E is *superadditive*, i.e. whenever $A_1 \in E(Q_1)$ and $A_2 \in E(Q_2)$ for disjoint coalitions Q_1, Q_2, then $A_1 \cap A_2 \in E(Q_1 \cup Q_2)$.

If this were the case, then coalition logic interpreted over either concurrent games or playable effectivity functions would be S1SC, and we claimed as much in the conference version [LPSS12]: The above conditions amount to playable effectivity functions being just neighbourhood systems that satisfy a set of one-step rules:

$$\neg[Q]\bot \qquad [Q]\top \qquad \frac{\boldsymbol{p} \to \boldsymbol{q}}{[Q]\boldsymbol{p} \to [Q]\boldsymbol{q}}$$

$$\frac{\boldsymbol{p} \vee \boldsymbol{q}}{[\emptyset]\boldsymbol{p} \vee [N]\boldsymbol{q}} \qquad \frac{\boldsymbol{p} \wedge \boldsymbol{q} \to \boldsymbol{r}}{[Q_1]\boldsymbol{p} \wedge [Q_2]\boldsymbol{q} \to [Q_1 \cup Q_2]\boldsymbol{r}} \text{ for } Q_1 \cap Q_2 = \emptyset,$$

and such structures are always S1SC (this is by the development in [SP10c], see in particular Remarks 34 and 55 in op. cit., and also Remark 3.10 above). However, it turns out that Theorem 3.2 in [Pau02] is not in fact entirely correct, and once fixed no longer implies that coalition logic is S1SC. To see this, note that for every effectivity function of the form E_G, $E_G(\emptyset)$ must have a least element, equivalently be closed under intersections: every element of $E_G(\emptyset)$ must contain the set

$$A = \{f(s_N) \mid s_N \in S_N\},$$

and this set is itself in $E_G(\emptyset)$. This condition is however not satisfied by all playable effectivity functions in the above sense: take X to be some infinite set, pick a non-principal ultrafilter U on X, and put $E(Q) = U$ for all coalitions Q. This defines a playable effectivity function but $E(\emptyset) = U$ has no least element. Adding the condition that $E(\emptyset)$ has a least element to the definition of playability does fix the theorem, but this condition is not expressible by a finitary one-step axiom and hence we do not obtain S1SC for coalition logic as a corollary.

As indicated above, we have alternative conditions that ensure completeness [SP10b]:

Definition 3.12. A modal operator $\heartsuit$ is *k-bounded* in the i-th argument for $k \in \mathbb{N}$ and with respect to a Λ-structure T if for every $C \in \mathsf{Set}$ and every $\vec{A} \subseteq C$,

$$[\![\heartsuit]\!]_C(A_1, \dots, A_n) = \bigcup_{B \subseteq A_i, \# B \leq k} [\![\heartsuit]\!]_C(A_1, \dots, A_{i-1}, B, A_{i+1}, \dots, A_n). \tag{3.2}$$

Boundedness of $\heartsuit$ in the i-th argument implies in particular that $\heartsuit$ is monotonic in the i-th argument. We can replace the assumption that the rule set $\mathcal{R}$ is S1SC with the weaker assumption that $\mathcal{R}$ is finitary S1SC, provided that modal operators are bounded on respective coordinates. The technical details of a suitably general setup are as follows:

Definition 3.13.

- A *boundedness signature* for Λ is a function $\flat\Lambda$ assigning to every $\heartsuit \in \Lambda$ a vector of elements of $\mathbb{N} \cup \{\infty\}$ of length $\mathsf{ar}\heartsuit$, i.e. an element of $(\mathbb{N} \cup \{\infty\})^{\mathsf{ar}\heartsuit}$.
- Being ∞-*bounded* is a condition trivially satisfied by all operators, i.e., every operator is "∞-bounded" in each coordinate.
- We say that $\flat\Lambda$ is *adequate* for a Λ-structure over T if every modal operator $\heartsuit \in \Lambda$ is $\flat\Lambda(\heartsuit)(i)$-bounded in i every $i \leq \mathsf{ar}\heartsuit$.
- We say that Λ is $\flat\Lambda$-*bounded* w.r.t T if $\flat\Lambda$ is adequate for the structure in question and the codomain of $\flat\Lambda$ does not contain ∞.
- We say that Λ is *bounded* w.r.t. T if it is $\flat\Lambda$-*bounded* w.r.t. T for *some* $\flat\Lambda$. That is, every modal operator $\heartsuit \in \Lambda$ for every $i \leq \mathsf{ar}\heartsuit$ is $k_{\heartsuit,i}$-bounded in i for some $k_{\heartsuit,i} < \infty$.

Example 3.14. Here are some examples of boundedness signatures adequate for structures under consideration:

- for the neighbourhood case, $\flat\Lambda(\Box) = (\infty)$,

- for the Kripke case, $\flat\Lambda(\Diamond) = (1)$:
  if $B \in [\![\Diamond]\!]_C(A)$, then $B \cap A \neq \emptyset$ by definition. So fix $c \in B \cap A$; then $B \in [\![\Diamond]\!]_C(\{c\})$,
- for graded modalities, $\flat\Lambda(\langle k\rangle) = (k+1)$:
  if $\mu \in [\![\langle k\rangle]\!]_C(A)$, then $\mu(A) > k$. Hence there exist (not necessarily distinct) elements $c_1, \dots, c_{k+1} \in C$ such that $\mu(\{c_1, \dots, c_{k+1}\}) > k$, i.e. $\mu \in [\![\langle k\rangle]\!]_C(\{c_1, \dots, c_{k+1}\})$,
- for positive Presburger logic,
  $$\flat\Lambda(L_k(a_1, \dots, a_n)) = ((k+1) \text{ div } a_1 + 1, \dots, (k+1) \text{ div } a_n + 1),$$
- for the discrete distribution functor $\mathcal{D}$, $\flat\Lambda(\langle p\rangle) = (\infty)$,
- for its finite variant $\mathcal{D}_k$, $\flat\Lambda(\langle n/k\rangle) = (n)$,
- for non-monotonic conditionals, $\flat\Lambda(>) = (\infty, 1)$.

Note that, e.g., the neighbourhood modality clearly fails to be bounded. Boundedness allows us to broaden the scope of our completeness results to setups where full S1SC would be too much to ask, i.e., to leave the neighbourhood-like setting. This is done by requiring S1SC on suitable coordinates only for valuations of schematic variables in *finite* sets. Modalities which are both finitary S1SC and bounded will be called *Kripke-like*. In order to make this precise so that we can cover mixed cases, such as those of non-monotonic conditionals, some care is needed.

Definition 3.15.

- The *colouring* function $\flat\!\flat : \mathbb{N} \cup \{\infty\} \to \{\mathit{fin}, \infty\}$ assigns *fin* to elements of $\mathbb{N}$ and $\flat\!\flat(\infty) = \infty$. It is extended pointwise to $(\mathbb{N} \cup \{\infty\})^{\mathsf{ar}\heartsuit}$ and $\flat\!\flat\Lambda$ is defined as the composition of $\flat\Lambda$ with this pointwise extension.
- Let $\mathfrak{c} : \mathsf{sVar} \to \{\mathit{fin}, \infty\}$ be a colouring of the set of schematic variables. Define the set of $\flat\!\flat\Lambda, \mathfrak{c}$-*coloured modalities* as
  $$\flat\!\flat\Lambda_{\mathfrak{c}} = \{\heartsuit \boldsymbol{p}_1 \dots \boldsymbol{p}_{\mathsf{ar}\Lambda} \mid \boldsymbol{p}_1 \dots \boldsymbol{p}_{\mathsf{ar}\Lambda} \in \mathsf{sVar} \text{ and } (\mathfrak{c}\boldsymbol{p}_1, \dots, \mathfrak{c}\boldsymbol{p}_{\mathsf{ar}\Lambda}) = \flat\!\flat\Lambda(\heartsuit)\}.$$
- A valuation $\tau : \mathsf{sVar} \to \mathcal{P}(C)$ respects $\mathfrak{c}$ iff $\tau(\boldsymbol{p}_i) \in \mathcal{P}_{\mathfrak{c}\boldsymbol{p}_i}$, where we recall that $\mathcal{P}_{\mathit{fin}}$ is finite powerset, and $\mathcal{P}_\infty$ is simply $\mathcal{P}$.
- A Gentzen-style one-step rule R is $\flat\!\flat\Lambda, \mathfrak{c}$-*compatible* if it is of the form
  $$\frac{\Gamma_1 \Rightarrow \Delta_1 \qquad \cdots \qquad \Gamma_k \Rightarrow \Delta_k}{\Gamma_R \Rightarrow \Delta_R}\,(R)$$
  where
  – $\Gamma_1, \dots, \Gamma_k, \Delta_1, \dots, \Delta_k$ are multisets of elements of sVar,
  – Γ_R and Δ_R are multisets of elements of $\flat\!\flat\Lambda_{\mathfrak{c}}$.
- For a Gentzen-style rule, write $\flat\!\flat\Lambda_{\mathfrak{c}}(R)$ for the set of $\flat\!\flat\Lambda, \mathfrak{c}$-compatible variants of R obtained by renaming of schematic variables. For a Hilbert-style rule, $\flat\!\flat\Lambda_{\mathfrak{c}}(R)$ is obtained via its Gentzen-style counterpart. Finally, $\flat\!\flat\Lambda_{\mathfrak{c}}(\mathcal{R}) = \{\flat\!\flat\Lambda_{\mathfrak{c}}(R) \mid R \in \mathcal{R}\}$.
- A set $\Xi \subseteq \mathsf{Rank1}(\mathsf{sVar})$ is $\mathfrak{c}$-*consistent wrt* τ if its union with the set
  $$\{\boldsymbol{P}\sigma \mid \sigma : \mathsf{sVar} \to \mathsf{Prop}(\mathsf{sVar}) \text{ and } \boldsymbol{A}/\boldsymbol{P} \in \flat\!\flat\Lambda_{\mathfrak{c}}(\mathcal{R}) \text{ s.t. } C, \tau \models \boldsymbol{A}\sigma\}$$
  is propositionally consistent.
- We say that a set of rules $\mathcal{R}$ is $\flat\Lambda$-*S1SC* if

  for every $C \in \mathsf{Set}$, any $\Xi \subseteq \mathsf{Rank1}(\mathsf{sVar})$, any colouring $\mathfrak{c}$ and any τ respecting $\mathfrak{c}$, Ξ is one-step satisfiable wrt τ whenever it is $\mathfrak{c}$-consistent wrt τ.

Table 1: Hilbert-style Calculus $\mathcal{HR}$

The axioms are modelled after those of Enderton [End01].

Everywhere below, $\forall\vec{y}.$ denotes a sequence of universal quantifiers of arbitrary length, possibly empty.

EN1: all propositional tautologies. These can be axiomatized, e.g., by

- $\forall\vec{y}.\,(\phi \to (\psi \to \phi))$
- $\forall\vec{y}.\,((\phi \to (\psi \to \chi)) \to ((\phi \to \psi) \to (\phi \to \chi)))$
- $\forall\vec{y}.\,(\bot \to \phi)$
- $\forall\vec{y}.\,(((\phi \to \bot) \to \bot) \to \phi)$

For EN2 and ALPHA below, recall that whenever we write a substitution we implicitly impose the assumption that the substituted term is actually substitutable.

EN2: $\forall\vec{y}.(\forall x.\phi \to \phi[z/x])$
EN3: $\forall\vec{y}.(\forall x.(\phi \to \psi) \to (\forall x.\phi \to \forall x.\psi))$
EN4: $\forall\vec{y}.(\phi \to \forall x.\phi)$ if x is fresh for ϕ
EN5: $\forall\vec{y}.(x = x)$
EN6.1: $\forall\vec{y}.(x = z \to P(\vec{u}, x, \vec{v}) \to P(\vec{u}, z, \vec{v}))$ for $P \in \Sigma \cup \{=\}$
EN6.2: $\forall\vec{y}.(x = z \to x\heartsuit\lceil y_1 : \phi_1\rceil \ldots \lceil y_n : \phi_n\rceil \to z\heartsuit\lceil y_1 : \phi_1\rceil \ldots \lceil y_n : \phi_n\rceil)$
ALPHA: $\forall\vec{y}((x\heartsuit \ldots \lceil z : \phi\rceil \ldots) \to (x\heartsuit \ldots \lceil u : \phi[u/z]\rceil \ldots))$
ONESTEP($\mathcal{R}$): $\forall\vec{y}.\forall z.(\forall x.\boldsymbol{A}\sigma \to [\sigma, x, z]\boldsymbol{P})$ where

- $\boldsymbol{A}/\boldsymbol{P}$ ranges over the one-step rules in $\mathcal{R}$, and
- σ is a substitution sending each $\boldsymbol{p}_i$ to a formula of $\mathcal{L}$, and $[\sigma, x, z]$ is the inductive extension of the map sending each $\heartsuit_i\vec{\boldsymbol{p}_i}$ to $z\heartsuit_i\lceil x : \sigma(\boldsymbol{p}_i^1)\rceil \cdots \lceil x : \sigma(\boldsymbol{p}_i^{a(i)})\rceil)$

BDPL$_{\flat\Lambda}$: An additional axiom scheme when $\flat\Lambda(\heartsuit)(i) \neq \infty$, with $\vec{z}$ fresh for $y_i, \vec{\phi}$

$$\forall\vec{y}.(x\heartsuit \ldots \lceil y_i : \phi_i\rceil \ldots \leftrightarrow \exists z_1 \ldots z_{\flat\Lambda(i)}.(\bigwedge_{j\leq\flat\Lambda(\heartsuit)(i)} \phi_i[z_j/y_i] \wedge x\heartsuit \ldots \lceil y_i : \bigvee_{j\leq\flat\Lambda(\heartsuit)(i)} y_i = z_j\rceil \ldots))$$

That is, the notion of $\flat\Lambda$-S1SC instantiates to finitary S1SC in bounded arguments, and to S1SC in unbounded ones.

Example 3.16. In the case of the modal signature $\{>\}$ of non-monotonic conditionals with $\flat\Lambda(>) = (\infty, 1)$, the associated one-step completeness condition may be called (*S1SC, finitary S1SC*) [SP10b]. A suitable axiomatization can be extracted from existing references [Che80, SP10c, PS10, SP10b]:

$$\frac{\boldsymbol{p} \to \boldsymbol{q}_1 \vee \ldots \vee \boldsymbol{q}_n}{\boldsymbol{r} > \boldsymbol{p} \to \boldsymbol{r} > \boldsymbol{q}_1 \vee \ldots \vee \boldsymbol{r} > \boldsymbol{q}_n}\ \mathsf{RCK} \qquad \frac{\boldsymbol{p} \leftrightarrow \boldsymbol{q}}{\boldsymbol{p} > \boldsymbol{r} \to \boldsymbol{q} > \boldsymbol{r}}\ \mathsf{RE}$$

A proof that this axiomatization is indeed $\flat\Lambda$-S1SC combines the neighbourhood argument in the first coordinate (cf. Remark 3.10) with the Kripke argument in the second (cf. Lemma 3.9).

3.2. **Hilbert-style Calculus.** We are finally ready to present our axioms for CPL in Table 1. Axioms EN1–EN6 are just those of Enderton, with EN6.2 an additional clause to cover the case of modal formulas. The α-renaming axiom ALPHA is needed because our syntax features separate comprehension variables. Our ONESTEP($\mathcal{R}$) axiom scheme generalizes what was originally just the congruence rule (Remark 3.7) in Chang's formalism, corresponding to the fact that the S1SC rule system for neighbourhood semantics consists of only the congruence rule. Axiom BdPL$_{\flat\Lambda}$ applies to operators that are bounded in suitable coordinates. It is important to notice that boundedness is not expressible as a *sentence* or *formula* in weak frameworks; in languages like $\mathrm{H}_\Lambda(@)$, it can only be expressed by a non-standard rule [SP10b].

Let $\Gamma, \Delta \subseteq \mathrm{CPL}(\Lambda, \Sigma)$, let $\mathcal{R}$ be a set of one-step rules and $\phi \in \mathrm{CPL}(\Lambda, \Sigma)$. Write $\Gamma \vdash^{\mathcal{HR}}_{\flat\Lambda} \phi$ if there are $\gamma_1, \ldots, \gamma_n \in \Gamma$ s.t. $\gamma_1 \to \ldots \to \gamma_n \to \phi$ can be deduced from En1–En6, Alpha, Onestep($\mathcal{R}$) and BdPL$_{\flat\Lambda}$ in Table 1 using **only Modus Ponens**. This clearly defines *a finitary deducibility relation* in the sense of Goldblatt [Gol93, Sec. 8.1] and being $\vdash^{\mathcal{HR}}_{\flat\Lambda}$*-consistent* is equivalent with being *finitely* $\vdash^{\mathcal{HR}}_{\flat\Lambda}$*-consistent* in his sense, that is, $\Gamma \vdash^{\mathcal{HR}}_{\flat\Lambda} \bot$ iff there is $\Gamma_0 \subseteq_{fin} \Gamma$ s.t. $\Gamma_0 \vdash^{\mathcal{HR}}_{\flat\Lambda} \bot$.

Theorem 3.17 (Soundness)**.** *Whenever a Λ-structure over T is adequate for $\flat\Lambda$, all the axioms in Table 1 hold in every coalgebraic Λ-model and the set of formulas valid in such a model is closed under $\vdash^{\mathcal{HR}}_{\flat\Lambda}$.*

Recall that by Convention 3.3, one-step soundness of $\mathcal{R}$ is not mentioned explicitly.

Definition 3.18. For any Λ, $\mathcal{R}$ and $\flat\Lambda$, we say that the inference system given by $\vdash^{\mathcal{HR}}_{\flat\Lambda}$ is *strongly complete* for a given Λ-structure based on T if for any set of sentences $\Gamma \in \mathrm{CPL}(\Lambda, \Sigma)$, $\Gamma \nvdash^{\mathcal{HR}}_{\flat\Lambda} \bot$ holds **if and only if** there is a coalgebraic Λ-model for Γ.

Theorem 3.19 (Strong Completeness)**.** *Whenever a set of rules $\mathcal{R}$ is* $\flat\Lambda$-S1SC *for a Λ-structure over T that is adequate for $\flat\Lambda$, then $\vdash^{\mathcal{HR}}_{\flat\Lambda}$ is strongly complete for this structure.*

Example 3.20. For concrete instances of this completeness result, combine Examples 3.8, 3.14 and 3.16.

3.3. **Proof of The Completeness Theorem.**

First, we introduce machinery proposed in [Gol93]. Consider any $Fr \subseteq \mathrm{CPL}(\Lambda, \Sigma)$ closed under propositional connectives. Fr can be, for example, the set of all formulas whose free variables are contained in a fixed subset of iVar, the set of all sentences and the entire $\mathrm{CPL}(\Lambda, \Sigma)$ itself being the two borderline cases. Any set $Inf \subseteq \mathcal{P}(Fr) \times Fr$ will be called, following Goldblatt, *a set of inferences*. For any $inf = (\Pi, \chi) \in Inf$ and any $\Gamma \subseteq Fr$, we say that

- Γ *respects inf* if $\Gamma \vdash^{\mathcal{HR}}_{\flat\Lambda} \chi$ whenever $\Gamma \vdash^{\mathcal{HR}}_{\flat\Lambda} \phi$ for all $\phi \in \Pi$,
- Γ *is closed under inf* if $\chi \in \Gamma$ whenever $\Pi \subseteq \Gamma$,
- Γ respects Inf iff it respects each member of Inf,
- Γ is closed under Inf iff it is closed under each member of Inf.

Theorem 3.21 (Goldblatt's Abstract Henkin Principle [Gol93])**.** *If Inf is a set of inferences in Fr of an infinite cardinality κ and Γ is a $\vdash^{\mathcal{HR}}_{\flat\Lambda}$-consistent subset of Fr satisfying in addition:*

$$\forall X \subseteq Fr. |X| < \kappa \text{ implies that } \Gamma \cup X \text{ respects } Inf \tag{3.3}$$

(i.e., every κ-finite extension of Γ respects Inf*), then Γ has a maximally $\vdash^{\mathcal{HR}}_{\flat\Lambda}$-consistent extension in Fr which is closed under Inf.*

Remark 3.22. We emphasize that speaking about *inferences* in Goldblatt's sense being infinite sets does *not* mean that *deductions* in the axiom system for for CPL use infinitary rules. As stated above, the only inference rule in our system is ordinary Modus Ponens.

Even the one-step rules defined above (which are not infinitary anyway) can be written as sentence schemes ONESTEP thanks to the use of quantifiers.

We further point out that an Enderton-style axiomatization does not involve the generalization rule: if x is a free variable in ϕ, it is not necessarily the case that $\phi \vdash^{\mathcal{HR}}_{\flat\Lambda} \forall x.\phi$ (this is not in contradiction to completeness: the rule is sound in the sense that validity of the premise implies validity of the conclusion, but its conclusion is not a logical consequence of its premise). This makes it enjoy a rather rare property for an axiomatization of FOL: a deduction theorem in exactly the same form as propositional logic, i.e., $\Gamma \cup \{\phi\} \vdash^{\mathcal{HR}}_{\flat\Lambda} \psi$ iff $\Gamma \vdash^{\mathcal{HR}}_{\flat\Lambda} \phi \to \psi$ (cf. [End01, p. 118]). This will also allow us to give our Henkin-style proofs *without introducing additional constants*—the role of Henkin constants for existentially quantified variables will be played by the variables themselves.[3] The only disadvantage of this approach would be that if we consider uncountable Λ or Σ, we would also need to allow uncountably many elements of iVar, something we highlight in the statement of several lemmas and claims below.

Let us recall the crucial ingredient in Henkin-style completeness proofs: the notion of quasi-Henkin model and its associated Truth Lemma. This is inspired by previously announced completeness proofs for coalgebraic hybrid logic [SP10b]; we discuss the relationship in detail in Remark 3.29 and § 4 below.

Definition 3.23. Let Γ be a maximal consistent set (MCS) of formulas. Define $C_\Gamma = \{|x| : x \text{ is a variable}\}$, where $|x| = \{z : x = z \in \Gamma\}$, and put $I_\Gamma(P) = \{(|x_1|, \ldots, |x_n|) : P(x_1, \ldots, x_n) \in \Gamma\}$. Set $\widehat{\phi}^{y_i} = \{|z| : \phi[z/y_i] \in \Gamma\}$, to be thought of as *the set of variables satisfying* ϕ *according to* Γ (when y_i is taken to be the *argument variable* or the *context hole*). Given a T-coalgebra structure $\gamma : C_\Gamma \to TC_\Gamma$, we say that $(C_\Gamma, \gamma, I_\Gamma)$ is a *quasi-Henkin coalgebraic model* if, for any variables x, y_1, ..., y_n and any formulas ψ, ϕ_1, ..., ϕ_n,

$$\exists x.\psi \in \Gamma \implies \text{ for some } y_i, y_i \in \widehat{\psi}^x. \tag{3.4}$$

(note that the converse implication holds for any MCS) and

$$x\heartsuit\lceil y_1 : \phi_1\rceil \cdots \lceil y_n : \phi_n\rceil \in \Gamma \iff \gamma(|x|) \in [\![\heartsuit]\!]_{C_\Gamma}(\widehat{\phi_1}^{y_1}, \ldots, \widehat{\phi_n}^{y_n}). \tag{3.5}$$

In a quasi-Henkin model, define the *canonical variable assignment* v_Γ by $v_\Gamma(x) = |x|$.

Lemma 3.24 (Truth Lemma). *Let Γ be a maximal consistent set of formulas and $\mathfrak{M}_\Gamma = (C_\Gamma, \gamma, I_\Gamma)$ a quasi-Henkin coalgebraic model. Then, for every formula ϕ,*

$$\mathfrak{M}_\Gamma, v_\Gamma \models \phi \iff \phi \in \Gamma. \tag{3.6}$$

Proof. By induction on ϕ. An auxiliary fact we need is that whenever ϕ satisfies the inductive claim (3.6), then

$$[\![\phi]\!]^y_{\mathfrak{M}_\Gamma} = \widehat{\phi}^y \tag{3.7}$$

(recall $[\![\phi]\!]^y$ is defined in (2.1)), which can be shown in the following way. Let $|z| \in C_\Gamma$. Then we have

$$\begin{aligned}\mathfrak{M}_\Gamma, v_\Gamma[|z|/y] \models \phi &\iff \mathfrak{M}_\Gamma, v_\Gamma \models \phi[z/y]\\ &\iff \phi[z/y] \in \Gamma \qquad \text{by (3.6)},\end{aligned}$$

[3]Recall that we have been working in a setup without functions symbols (including 0-ary ones) anyway; extending our original syntax with constants just for the sake of this particular proof does not even seem particularly hygienic.

as desired.

The base case of induction for atomic formulas follows now from the definitions of C_Γ and I_Γ, the Boolean cases from the fact that we are dealing with a MCS, and the case for quantifiers directly from Condition 3.4. For the modal case, where $\phi \equiv x\heartsuit\lceil y_1 : \phi_1\rceil \dots \lceil y_n : \phi_n\rceil$:

$$\begin{aligned}
\mathfrak{M}_\Gamma, v_\Gamma \models x\heartsuit\lceil y_1 : \phi_1\rceil \dots \lceil y_n : \phi_n\rceil &\iff \gamma(v_\Gamma(x)) \in [\![\heartsuit]\!]_{\mathfrak{M}_\Gamma}([\![\phi_1]\!]^{y_1}_{\mathfrak{M}_\Gamma}, \dots, [\![\phi_n]\!]^{y_n}_{\mathfrak{M}_\Gamma}) && \text{by def.}\\
&\iff \gamma(|x|) \in [\![\heartsuit]\!]_{\mathfrak{M}_\Gamma}(\widehat{\phi_1}^{y_1}, \dots, \widehat{\phi_n}^{y_n}) && \text{by (3.7)}\\
&\iff x\heartsuit\lceil y_1 : \phi_1\rceil \cdots \lceil y_n : \phi_n\rceil \in \Gamma && \text{by (3.5).}
\end{aligned}$$

□

Next, we need to find a suitable candidate for an MCS from which to build our quasi-Henkin model. Consider the following sets of inferences:

$$\begin{aligned}
\mathit{Inf}_{\text{NAME}a} &= \{\langle \{\phi[z/x] \mid z \in \mathsf{iVar}\}, \forall x.\phi\rangle \mid \phi \in \mathrm{CPL}(\Lambda, \Sigma), x \in \mathsf{iVar}\}\\
\mathit{Inf}_{\text{NAME}b} &= \{\langle \{(\phi_1 \leftrightarrow \psi_1)[z/x], \dots, (\phi_n \leftrightarrow \psi_n)[z/x] \mid z \in \mathsf{iVar}\},\\
&\qquad \forall x.\, (x\heartsuit\lceil x : \phi_1\rceil \dots \lceil x : \phi_n\rceil \leftrightarrow x\heartsuit\lceil x : \psi_1\rceil \dots \lceil x : \psi_n\rceil)\rangle \mid \vec{\phi}, \vec{\psi} \in \mathrm{CPL}(\Lambda, \Sigma), x \in \mathsf{iVar}\}\\
\mathit{Inf}_{\text{NAME}} &= \mathit{Inf}_{\text{NAME}a} \cup \mathit{Inf}_{\text{NAME}b}\\
\mathit{Inf}_{\flat\Lambda} &= \{\langle \{ \bigwedge_{j \le \flat\Lambda(\heartsuit)(i)} \phi_i[z_j/y_i] \to \neg x\heartsuit \dots \lceil y_i : \bigvee_{j \le \flat\Lambda(\heartsuit)(i)} y_i = z_j\rceil \cdots \mid \vec{z} \in \mathsf{iVar}\},\\
&\qquad \neg x\heartsuit \dots \lceil y_i : \phi_i\rceil \dots\rangle \mid \vec{\phi} \in \mathrm{CPL}(\Lambda, \Sigma), x \in \mathsf{iVar}, \heartsuit \in \Lambda, \flat\Lambda(\heartsuit)(i) \neq \infty\}\\
\mathit{Inf} &= \mathit{Inf}_{\text{NAME}} \cup \mathit{Inf}_{\flat\Lambda}
\end{aligned}$$

Let us begin with

Claim 3.25. Assume $|\mathsf{iVar}| = \kappa \geq |\Lambda \cup \Sigma \cup \omega|$. Then any $\vdash^{\mathcal{HR}}_{\flat\Lambda}$-consistent set of formulas Γ s.t. $|\{x \in \mathsf{iVar} \mid x \text{ fresh for } \Gamma\}| = \kappa$ (in particular, any consistent set of sentences) satisfies condition 3.3 of Theorem 3.21 for $\mathit{Inf}_{\text{NAME}}$.

Proof. We begin by observing that

(a) If $\Gamma' \vdash^{\mathcal{HR}}_{\flat\Lambda} \phi[z/x]$ and z is fresh for Γ', x, ϕ, then $\Gamma' \vdash^{\mathcal{HR}}_{\flat\Lambda} \forall x.\phi$

The proof of this fact is perfectly standard, but working with an Enderton-style axiomatization is particularly convenient for such reasoning: We have a finite $\Gamma'_0 \subseteq_{fin} \Gamma'$ s.t. $\Gamma'_0 \vdash^{\mathcal{HR}}_{\flat\Lambda} \phi[z/x]$. Then one uses the Deduction Theorem (cf. Remark 3.22) to obtain $\vdash^{\mathcal{HR}}_{\flat\Lambda} \bigwedge\Gamma'_0 \to \phi[z/x]$. However, even with an Enderton-style axiomatization it is still the case[4] that $\vdash^{\mathcal{HR}}_{\flat\Lambda} \chi$ implies $\vdash^{\mathcal{HR}}_{\flat\Lambda} \forall z.\chi$, hence $\vdash^{\mathcal{HR}}_{\flat\Lambda} \forall z.(\bigwedge\Gamma'_0 \to \phi[z/x])$. The rest is an easy exercise using En3, En4 and renaming of bound variables thanks to En2.

The condition (a) tells us that Γ itself does respect $\mathit{Inf}_{\text{NAME}a}$ by assumption. But if $|X| < \kappa$, then there are κ-many $z \in \mathsf{iVar}$ that are fresh for $\Gamma \cup X \cup \{\phi\}$. For any such z, (a) would hold also for $\Gamma' = \Gamma \cup X$. This gives condition 3.3 for $\mathit{Inf}_{\text{NAME}a}$.

For $\mathit{Inf}_{\text{NAME}b}$, let us observe that (a) allows to infer that

If $\Gamma' \vdash^{\mathcal{HR}}_{\flat\Lambda} (\phi_1 \leftrightarrow \psi_1)[z/x] \wedge \cdots \wedge (\phi_n \leftrightarrow \psi_n)[z/x]$ and z fresh for $\Gamma', \vec{\phi}, \vec{\psi}, x$, then $\Gamma \vdash^{\mathcal{HR}}_{\flat\Lambda} \forall x.\, ((\phi_1 \leftrightarrow \psi_1) \wedge \cdots \wedge (\phi_n \leftrightarrow \psi_n))$.

[4]In fact, a variant of the Generalization Theorem is available even for non-empty contexts as long as the quantified variable does not occur freely therein, cf. [End01, p. 117].

Now an application of the congruence rule (Remark 3.7) completes the proof of the claim. □

Claim 3.26. Assume $|\mathsf{iVar}| = \kappa \geq |\Lambda \cup \Sigma \cup \omega|$. Then any $\vdash^{\mathcal{HR}}_{\flat\Lambda}$-consistent set of formulas Γ s.t. $|\{x \in \mathsf{iVar} \mid x \text{ fresh for } \Gamma\}| = \kappa$ (in particular, a consistent set of *sentences*) satisfies condition 3.3 of Theorem 3.21 for $Inf_{\flat\Lambda}$.

Proof. We begin by observing that

(b) If $\Gamma' \vdash^{\mathcal{HR}}_{\flat\Lambda} \neg(\bigwedge_{j\leq\flat\Lambda(\heartsuit)(i)} \phi_i[z_j/y_i] \wedge x\heartsuit \ldots \lceil y_i : \bigvee_{j\leq\flat\Lambda(\heartsuit)(i)} y_i = z_j\rceil \ldots)$ for some $\vec{z}$ fresh for $\Gamma', x, y_i, \vec{\phi}$, then $\Gamma' \vdash^{\mathcal{HR}}_{\flat\Lambda} \neg x\heartsuit \ldots \lceil y_i : \phi_i\rceil \ldots$

This is shown by first following the proof of (a) and finding a finite $\Gamma'_0 \subseteq_{fin} \Gamma'$ s.t.

$$\vdash^{\mathcal{HR}}_{\flat\Lambda} \Gamma'_0 \to \neg\exists z_1, \ldots, z_{\flat\Lambda(\heartsuit)(i)}.(\bigwedge_{j\leq\flat\Lambda(\heartsuit)(i)} \phi_i[z_j/y_i] \wedge x\heartsuit \ldots \lceil y_i : \bigvee_{j\leq\flat\Lambda(\heartsuit)(i)} y_i = z_j\rceil \ldots).$$

Applying $\mathrm{BdPL}_{\flat\Lambda}$ proves (b).

The condition (b) tells us that Γ itself does respect $Inf_{\flat\Lambda}$ by assumption. But if $|X| < \kappa$, then there are κ-many $z \in \mathsf{iVar}$ which are fresh for $\Gamma \cup X \cup \{\phi_1, \ldots, \phi_{\mathsf{ar}\heartsuit}\}$ and distinct from x and $\vec{y}$. For any tuple of such z's, (b) would hold also for $\Gamma' = \Gamma \cup X$. This gives condition 3.3 for $Inf_{\flat\Lambda}$. □

Claim 3.27. Assume $|\mathsf{iVar}| = \kappa \geq |\Lambda \cup \Sigma \cup \omega|$. Then any $\vdash^{\mathcal{HR}}_{\flat\Lambda}$-consistent set of formulas Γ s.t. $|\{x \in \mathsf{iVar} \mid x \text{ fresh for } \Gamma\}| = \kappa$ (in particular, a consistent set of *sentences*) can be extended to a maximally $\vdash^{\mathcal{HR}}_{\flat\Lambda}$-consistent set of formulas Γ' s.t.

- whenever $\exists x.\phi \in \Gamma'$, then $\phi[z/x] \in \Gamma'$ for some $z \in \mathsf{iVar}$
- whenever $\exists x.\,(x\heartsuit\lceil x : \phi_1\rceil \ldots \lceil x : \phi_n\rceil \wedge \neg x\heartsuit\lceil x : \psi_1\rceil \ldots \lceil x : \psi_n\rceil) \in \Gamma'$, then there is $z \in \mathsf{iVar}$ and $i \leq n$ s.t. $\neg\,(\phi_i \leftrightarrow \psi_i)\,[z/x] \in \Gamma'$.
- whenever $x\heartsuit \ldots \lceil y_i : \phi_i\rceil \cdots \in \Gamma'$ and $\flat\Lambda(\heartsuit)(i) \neq \infty$, then there are $z_1, \ldots, z_{\flat\Lambda(\heartsuit)(i)}$ s.t. $x\heartsuit \ldots \lceil y_i : \bigvee_{j\leq\flat\Lambda(\heartsuit)(i)} y_i = z_j\rceil \cdots \in \Gamma'$ and moreover $\phi_i[z_j/y_i] \in \Gamma'$ for each $j \leq \flat\Lambda(\heartsuit)(i)$.

Proof. This immediately follows from the preceding Claim, Theorem 3.21 and the fact Γ' is a MCS. □

Proof of Theorem 3.19. Recall Definition 3.23. We will build our quasi-Henkin model using Γ'. Satisfaction of condition 3.4 follows then directly from the first item in Claim 3.27, i.e., from being closed under Inf_{NAMEa}. Hence, we just need to define a transition structure γ on $C_{\Gamma'}$ and for this purpose, we need to find for each $|x|$ a suitable $t \in TC_{\Gamma'}$ s.t. when $\gamma(|x|)$ is defined as t, the condition 3.5 is satisfied.

Assume then $\mathsf{Rank1}$ has enough schematic variables to name all elements of $\mathrm{CPL}(\Lambda, \Sigma)$; let $\boldsymbol{p}_\phi$ be the schematic variable corresponding to ϕ under some fixed assignment. For each x, we can define an evaluation $\tau_x(\boldsymbol{p}_\psi) = \widehat{\psi}^x$. Note that for each pair of distinct x and y we have that $\tau_x(\boldsymbol{p}_{x=y})$ is a singleton, thanks to the definition of $C_{\Gamma'}$.

Thus, let us define for each $x \in \mathsf{iVar}$ the set

$$\Psi_x := \{\epsilon\heartsuit\boldsymbol{p}_{\psi_1^\circ} \ldots \boldsymbol{p}_{\psi_n^\circ} \mid \psi_1, \ldots, \psi_n \in \mathrm{CPL}(\Lambda, \Sigma) \text{ and } \epsilon(x\heartsuit\lceil x{:}\psi_1^\circ\rceil \ldots \lceil x{:}\psi_n^\circ\rceil) \in \Gamma'\},$$

where ϵ is either nothing or negation and for each $i \leq \mathsf{ar}\heartsuit$, ψ_i° is either:

- ψ_i itself, if $\flat\Lambda(\heartsuit)(i) = \infty$ or
- $\bigvee_{j\leq\flat\Lambda(\heartsuit)(i)} x = z_j$ otherwise, where $z_1, \ldots, z_{\flat\Lambda(\heartsuit)(i)}$ are s.t.

– $x\heartsuit \dots \lceil x : \bigvee_{j \leq \flat\Lambda(\heartsuit)(i)} x = z_j \rceil \dots \in \Gamma'$ and moreover
– $\psi_i[z_j/x] \in \Gamma'$ for each $j \leq \flat\Lambda(\heartsuit)(i)$.

Furthermore, let us define a colouring $\mathfrak{c}_x$ of schematic variables which assigns fin to every $\boldsymbol{p}_{\bigvee_{j \leq m} x = z_m}$, where $z_1, \dots, z_m$ is any finite sequence of variables and ∞ to every other $\boldsymbol{p}_\psi$. It is clear that τ_x respects $\mathfrak{c}_x$. We have:

Claim 3.28. Ψ_x is $\mathfrak{c}_x$-consistent wrt τ_x, i.e.,

$$\Psi_x \cup \{\boldsymbol{P}\sigma \mid \sigma : \mathsf{sVar} \to \mathsf{Prop}(\mathsf{sVar}) \text{ and } \boldsymbol{A}/\boldsymbol{P} \in \flat\!\flat\Lambda_{\mathfrak{c}}(\mathcal{R}) \text{ s.t. } C_{\Gamma'}, \tau_x \models \boldsymbol{A}\sigma\}.$$

is propositionally consistent.

Proof. Assume it is not. By compactness of the classical propositional calculus, a contradiction can be then derived already from a certain finite subset of Ψ_x and finitely many $\boldsymbol{P}\sigma$ s.t. $\boldsymbol{A}/\boldsymbol{P} \in \flat\!\flat\Lambda_{\mathfrak{c}}(\mathcal{R})$ and $C_{\Gamma'}, \tau_x \models \boldsymbol{A}\sigma$. Given the definition of τ_x and Ψ_x, this directly contradicts the fact that Γ' is supposed to be a MCS closed under all instances of axioms OneStep($\mathcal{R}$) and BdPL$_{\flat\Lambda}$ in Table 1. □

By Definition 3.15 of $\flat\Lambda$-S1SC, Claim 3.28 implies that Ψ_x is one-step satisfiable wrt τ_x, i.e.,

$$\bigcap_{\phi \in \Psi_x} [\![\phi]\!]_{TC_{\Gamma'}, \tau_x} = \{\epsilon[\![\heartsuit]\!]_{C'_\Gamma} \widehat{\psi_1^\circ}^x \dots \widehat{\psi_n^\circ}^x \mid \psi_1, \dots, \psi_n \in \mathrm{CPL}(\Lambda, \Sigma) \text{ and } \epsilon(x\heartsuit\lceil x : \psi_1^\circ \rceil \dots \lceil x : \psi_n^\circ \rceil) \in \Gamma'\} \neq \emptyset.$$

Now use the Axiom of Choice to define $\gamma(|x|)$ to be a representative of this non-empty intersection for every $|x|$ (strictly speaking, for an arbitrarily chosen representative of this equivalence relation), automatically yielding the condition 3.5 of Definition 3.23. □

Remark 3.29. The similarities and differences between CPL and languages like $\mathrm{H}_\Lambda(@)$ and its extensions to be discussed in § 4 are best appreciated by comparing the proof of Theorem 3.19 with earlier hybrid ones [SP10b]. In the predicate case:

- not only one-step rules, but also non-standard naming and pasting rules of [SP10b] can be expressed as ordinary first-order axioms.
- As we are going to discuss now, the Henkin-style completeness proof directly leads to the Omitting Types theorem. It is not clear how to obtain such a result for a language like $\mathrm{H}_\Lambda(@)$ studied in [SP10b]; the presence of a binding and/or quantification mechanism seems essential in the proof. Recall again that the presence of such a mechanism also allowed us to reuse (equivalence classes of) variables as building block of models instead of Henkin-style constants.

3.4. **Omitting Types Theorem.**

The Omitting Types Theorem is a standard result of model theory. Goldblatt [Gol93, § 8.2] shows how to establish it using the Abstract Henkin Principle. Here is a more detailed description how to obtain it in our setting. In this section, we assume that the entire $\mathrm{CPL}(\Lambda, \Sigma)$ is countable and we keep these countable Λ and Σ fixed and implicit.

Fix a finite subset of iVar $\{x_1, \dots, x_k\}$ and denote the set of all formulas whose free variables are contained in $\{x_1, \dots, x_k\}$ as $\mathrm{CPL}(k)$. Thus, the set of sentences can be written as $\mathrm{CPL}(0)$. Recall that a *k-type* (sometimes called a *complete type*) is a maximal consistent subset of CPL(k); sometimes, one also uses the term *partial type* for consistent yet not maximal subsets of CPL(k). For any given $\Gamma \subseteq$ CPL(0) and any (partial or total) k-type Δ, say that Δ is *principal over* Γ if there is $\phi \in$ CPL(k) consistent with Γ s.t. $\forall \psi \in \Delta. \Gamma \vdash^{\mathcal{HR}}_{\flat\Lambda} \phi \to \psi$. Note here that for complete types, we can assume that $\phi \in \Delta$. Say that a model $\mathfrak{M} = (C, \gamma, I)$ *realizes* a (partial or total) k-type Δ if $\bigcap_{\psi \in \Delta} [\![\psi]\!]_C^{x_1,\dots,x_k} \neq \emptyset$, where as before

$$[\![\phi]\!]_C^{x_1,\dots,x_k} = \{(c_1, \dots, c_k) \in C \mid \mathfrak{M}, v[c_1/x_1]\dots[c_k/x_k] \models \phi\};$$

a k-type is *omitted* by $\mathfrak{M}$ if it is not realized by it.

Note that for complete types, one consequence of being non-principal is that Δ is neither entailed by Γ nor inconsistent with it (maximal consistent sets are closed under finite conjuctions).

Theorem 3.30 (Omitting Types)**.** *Whenever a set of rules $\mathcal{R}$ is* $\flat\Lambda$-S1SC *for a Λ-structure over T that is adequate for $\flat\Lambda$, Γ is a consistent set of sentences and Δ is a* (*complete or partial*) *k-type non-principal over Γ, Γ has a model omitting Δ.*

Proof. We only need to refine somewhat the proof of the completeness theorem by using a richer set of inferences than Inf. Consider

$$Inf_\Delta = Inf \cup \{\langle\{\sigma[z_1 \dots z_k/x_1 \dots x_k] \mid \sigma \in \Delta\}, \bot\rangle \mid z_1, \dots, z_k \text{ distinct els. of } \mathsf{iVar}\}$$

(we have not formally defined simultaneous substitution, but it should be clear how to extend conventions introduced in § 2). We claim that the condition 3.3 of Theorem 3.21 is satisfied with $\kappa = \omega$. For assume it is not. Then there exists a finite set $\Delta \subseteq$ CPL and a finite tuple of distinct variables $z_1, \dots, z_k$ s.t. (*) $\Gamma \cup \Delta$ is consistent but

$$\Gamma \vdash^{\mathcal{HR}}_{\flat\Lambda} \bigwedge \Delta \to \sigma[z_1 \dots z_k/x_1 \dots x_k] \text{ for every } \sigma \in \Delta.$$

Let $\vec{z'}$ be a sequence containing all the variables in Δ different from $z_1 \dots z_k$ and $\delta = \exists \vec{z'}. \bigwedge \Delta$. Then we have

$$(**)\ \Gamma \vdash^{\mathcal{HR}}_{\flat\Lambda} \delta \to \sigma[z_1 \dots z_k/x_1 \dots x_k] \text{ for every } \sigma \in \Delta$$

and consequently, setting δ' to be $\delta[x_1 \dots x_k/z_1 \dots z_k]$

$$(***)\ \Gamma \vdash^{\mathcal{HR}}_{\flat\Lambda} \delta' \to \sigma \text{ for every } \sigma \in \Delta$$

(in deriving (**) and (***) we obviously use the fact that Γ is a set of *sentences*).

At the same time, Γ being a set of sentences yields that δ' is consistent with Γ by virtue of (*). This entails a contradiction with non-principality of Δ over Γ. The rest proceeds as in the completeness proof. □

Remark 3.31. Goldblatt [Gol93, § 8.2] points out this can be extended to simultaneously omitting a countable set of non-principal types.

Here is a typical application adapted from the monograph of Chang and Keisler [CK90, Ch 2.2, p. 83]: ω-logic, ω-rule and ω-completeness.

Given any T, Λ and $\mathcal{R}$ within the scope of our completeness result, extend Λ with a countable family of propositional atoms (cf. Definition 2.2) $\{\mathsf{is}\,\underline{n} \mid n \in \mathbb{N}\} \cup \{\mathsf{is\,nat}\}$. By similar considerations as in Remark 2.3, the completeness result is not affected by such extensions.

Remark 3.32. Of course, we could alternatively extended Σ with corresponding predicate symbols, which would be even easier from the point of view of directly applying Theorems 3.21 and 3.30, but would also have a less coalgebraic flavour.

Consider now the following set of sentences in the extended language:

$$\begin{aligned}\Gamma_N = &\{\forall x.x \,\mathsf{is}\, \underline{n} \to (x \,\mathsf{is}\,\mathsf{nat} \wedge \neg x \,\mathsf{is}\, \underline{m}) \mid n, m \in \mathbb{N}, n \neq m\} \cup \\ &\{\forall x, y.x \,\mathsf{is}\, \underline{n} \wedge y \,\mathsf{is}\, \underline{n} \to x = y \mid n \in \mathbb{N}\} \cup \\ &\{\exists x.x \,\mathsf{is}\, \underline{n} \mid n \in \mathbb{N}\}.\end{aligned}$$

An ω-*model* is any model of Γ_N where, moreover, the denotation of $\mathsf{is}\,\mathsf{nat}$ is the set-theoretical sum of all "$\mathsf{is}\,\underline{n}$". A theory Γ is ω-*complete* if $\Gamma \cup \Gamma_N$ is closed under the ω-*rule*:

$$\frac{\forall x.x \,\mathsf{is}\, \underline{n} \to \phi(x) \quad n \in \mathbb{N}}{\forall x.x \,\mathsf{is}\,\mathsf{nat} \to \phi(x)}\ \omega$$

One can use the Omitting Types theorem to show the following:

Corollary 3.33. *Assume that a set of rules $\mathcal{R}$ is $\flat\Lambda$-S1SC for a Λ-structure over T is adequate for $\flat\Lambda$, and Γ is a set of sentences s.t. $\Gamma \cup \Gamma_N$ is consistent. If Γ is ω-complete, then Γ has an ω-model.*

Sketch. Consider

$$\Delta(x) = \{\neg(x \,\mathsf{is}\, \underline{n}) \mid n \in \mathbb{N}\} \cup \{x \,\mathsf{is}\,\mathsf{nat}\}.$$

Pick any $\phi(x) \in \mathrm{CPL}(1)$ s.t. $x \,\mathsf{is}\,\mathsf{nat} \wedge \phi(x)$ is consistent with $\Gamma \cup \Gamma_N$. Then

$$\begin{aligned} & \Gamma \cup \Gamma_N \nvdash^{\mathcal{HR}}_{\flat\Lambda} \forall x.(x \,\mathsf{is}\,\mathsf{nat} \to \neg\phi(x)) \\ \Longrightarrow\quad & \Gamma \cup \Gamma_N \nvdash^{\mathcal{HR}}_{\flat\Lambda} \forall x.(x \,\mathsf{is}\, \underline{n} \to \neg\phi(x)) \text{ for some } n \in \mathbb{N} \quad (\text{by } \omega-\text{completeness}), \\ \Longleftrightarrow\quad & \Gamma \cup \Gamma_N \nvdash^{\mathcal{HR}}_{\flat\Lambda} \phi(x) \to \neg(x \,\mathsf{is}\, \underline{n}) \text{ for some } n \in \mathbb{N}. \end{aligned}$$

On the other hand, if

$$\Gamma \cup \Gamma_N \vdash^{\mathcal{HR}}_{\flat\Lambda} x \,\mathsf{is}\,\mathsf{nat} \to \neg\phi(x)$$

and at the same time

$$\Gamma \cup \Gamma_N \vdash^{\mathcal{HR}}_{\flat\Lambda} \phi(x) \to x \,\mathsf{is}\,\mathsf{nat},$$

then $\phi(x)$ is inconsistent with $\Gamma \cup \Gamma_N$. Thus, $\Delta(x)$ is non-principal over $\Gamma \cup \Gamma_N$ and hence, $\Gamma \cup \Gamma_N$ has a model omitting it. Such a model is an ω-model. □

As discussed by Chang and Keisler [CK90, Ch 2.2], it follows that we can extend our deductive apparatus with Γ_N as additional axioms and ω-rule as an additional rule of proof and consistency in this extended system is equivalent to the existence of an ω-model.

Such examples are worth contrasting with the incompleteness result (Theorem 3.37) we are going to present next.

3.5. **ω-boundedness and Failure of Completeness.** In this subsection, we show that there is a substantial gap between S1SC and finitary S1SC as conditions allowing for strong completeness, by proving that within a larger class of *ω-bounded* structures, the bounded structures are the only ones that satisfy compactness. Here, ω-boundedness of an operator means informally that its satisfaction can always be established by looking only at a finite subset of the successors, without however requiring a fixed bound on their number. In examples for this property, we concentrate on cases additionally satisfying finitary one-step compactness (Definition 3.5), a condition essentially necessary for overall compactness and that will moreover become important in our forays into model theory (§ 5). In the whole subsection, to keep things simple we work with unary $\heartsuit \in \Lambda$.

Definition 3.34 (ω-Bounded operators)**.** A modal operator $\heartsuit$ is *ω-bounded* if for each set X and each $A \subseteq X$,

$$[\![\heartsuit]\!]_X(A) = \bigcup_{B \subseteq_{\mathit{fin}} A} [\![\heartsuit]\!]_X(B).$$

Example 3.35 (Nonstandard subdistributions)**.** We generally write $\mathcal{S}$ for the discrete subdistribution functor, i.e. $\mathcal{S}(X)$ consists of real-valued discrete measures μ on X such that $\mu(X) \leq 1$, and for maps f, $\mu(f)$ takes image measures. As a variant of this functor, we consider the the discrete subdistributions functor $\mathcal{S}^{rc}$ where measures take values in real-closed fields. Explicitly: we intend to model Markov chains with non-standard probabilities; these consist of a set X of states, and at each state x an R_x-valued transition distribution μ_x, where R_x is a real-closed field (i.e. a model of the first-order theory of the reals). These structures are coalgebras for the functor T which maps a set X to the set of pairs (R, μ) where R is a real-closed field and μ is an R-valued discrete subdistribution on X (again meaning that $\mu(X) \leq 1$). This functor is in fact class-valued, which however does not affect the applicability of our coalgebraic analysis (which never requires iterated application of the coalgebraic type functor, e.g. it does not use the terminal sequence). We take the modal signature Λ to consist of the operators $\langle p \rangle$ ('with probability more than p') for $p \in [0,1] \cap \mathbb{Q}$.

We show that the $\langle p \rangle$ are ω-bounded and that the arising logic $\mathcal{L}$ is finitary one-step compact. To see the former, let $(R, \mu) \in TX$ and let $A \subseteq X$ such that $\mu \models \langle p \rangle A$, i.e. $\sum_{x \in A} \mu(x) > p$. Then there exists $B \subseteq_{\mathit{fin}} A$ such that $\sum_{x \in B} \mu(x) > p$, i.e. $\mu \models \langle p \rangle B$. Since $\langle p \rangle$ is clearly monotone, this implies that $[\![\langle p \rangle]\!]_X(A) = \bigcup_{B \subseteq_{\mathit{fin}} A} [\![\langle p \rangle]\!]_X(B)$, as required.

To show that $\mathcal{L}$ is finitary one-step compact, let $\Phi \subseteq \mathsf{Prop}(\Lambda(\mathcal{P}_{\mathit{fin}}(X)))$ be finitely satisfiable. Extend the standard language of real arithmetic with a constant symbol c_x for each element of X, obtaining a language L. Then satisfaction of a formula in $\mathsf{Prop}(\Lambda(\mathcal{P}_{\mathit{fin}}(X)))$ by $\mu \in TX$ translates into a first-order formula over L with c_x representing $\mu(x)$; specifically, the translation t commutes with the Boolean connectives and translates formulas $\langle p \rangle A$ with $A \in \mathcal{P}_{\mathit{fin}}(X)$ into $\sum_{x \in A} c_x > p$. Applying t to Φ and introducing additional formulas $c_x \geq 0$ for all $x \in X$ and $\sum_{x \in A} c_x \leq 1$ for all $A \in \mathcal{P}_{\mathit{fin}}(X)$ thus produces a finitely satisfiable, and hence satisfiable, set of first-order formulas over L. A model of this set consists of a real-closed field R and interpretations $\hat{c}_x \in R$ of the constants c_x such that putting $\mu(x) = \hat{c}_x$ defines a discrete subdistribution (note that $\sum_{x \in A} \hat{c}_x \leq 1$ for all $A \in \mathcal{P}_{\mathit{fin}}(X)$ implies $\sum_{x \in X} \hat{c}_x \leq 1$), which then yields a model (R, μ) of Φ.

Example 3.36 (Zero-dimensional subdistributions)**.** Fix a zero-dimensional closed (hence compact) subset $Z \subseteq [0,1]$, e.g. a discrete set or the Cantor space, and let $\mathcal{S}^Z$ be the associated *zero-dimensional discrete subdistributions functor*, i.e. the subfunctor of the subdistribution functor $\mathcal{S}$ where probabilities of finite sets of states are restricted to take values in Z:

$$\mathcal{S}^Z(X) = \{\mu \in \mathcal{S}(X) \mid \forall A \in \mathcal{P}_{\mathit{fin}}(X).\, \mu(A) \in Z\}.$$

Moreover, we restrict the probabilities p in operators $\langle p \rangle$ to be such that $(p, 1] \cap Z$ is clopen in Z; since Z is zero-dimensional, there exist enough such p to separate all values in Z. As before, all these operators are ω-bounded. It remains to show that the logic is finitary one-step compact. So let $\Phi \subseteq \mathsf{Prop}(\Lambda(\mathcal{P}_{\mathit{fin}}(X)))$ be finitely satisfiable. Note that the space Z^X, equipped with the product topology, is compact. We equip $\mathcal{S}^Z(X)$ with the subspace topology in Z^X. Observe that the condition $\forall A \in \mathcal{P}_{\mathit{fin}}(X).\, \mu(A) \in Z$ already implies $\mu(X) \leq 1$; since for $A \in \mathcal{P}_{\mathit{fin}}(X)$, the summation map $Z^A \to Z$ is continuous (this would fail for infinite A), it follows that $\mathcal{S}^Z(X)$ is closed in Z^X, hence compact.

By the restriction placed on the indices p in modal operators $\langle p \rangle$, and again using continuity of finite summation, we have that for every formula $\langle p \rangle A$ with $A \in \mathcal{P}_{\mathit{fin}}(X)$, the extension

$$[\![\langle p \rangle A]\!] = \{\mu \in \mathcal{S}^Z(X) \mid \mu(A) > p\}$$

is clopen in $\mathcal{S}^Z(X)$. As clopen sets are closed under Boolean combinations, we thus have that the extension of every formula in $\mathsf{Prop}(\Lambda(\mathcal{P}_{\mathit{fin}}(X)))$ is clopen in $\mathcal{S}^Z(X)$. Let $\mathfrak{A}$ denote the family of clopens induced in this way by formulas in Φ. Finite satisfiability of Φ implies that $\mathfrak{A}$ has the finite intersection property, and hence has non-empty intersection by compactness of $\mathcal{S}^Z(X)$. It follows that Φ is satisfiable.

Structures which are ω-bounded without being k-bounded fail strong completeness. To state this observation in full generality, we require the notion of *propositional atom* as defined previously (Definition 2.2).

Theorem 3.37. *Whenever a Λ-structure makes some $\heartsuit \in \Lambda$ ω-bounded without being k-bounded for any $k \in \omega$, strong completeness fails whenever either Σ contains a predicate symbol of positive arity or Λ contains a propositional atom.*

Proof. Assume $P \in \Sigma$ is a predicate symbol of positive arity, w.l.o.g. unary, and $\heartsuit \in \Lambda$ is as in the statement of theorem. Consider

$$\begin{aligned}\Delta(x) =& \{\neg x \heartsuit \lceil y : P(y) \rceil\} \cup \\ & \{\forall y_1, ..., y_k.(P(y_1) \wedge \cdots \wedge P(y_k) \to \neg x \heartsuit \lceil y : y = y_1 \vee ... \vee y = y_k \rceil) \mid k \in \omega\}.\end{aligned}$$

Clearly, every finite subset of $\Delta(x)$ is satisfiable in a model based on a coalgebra witnessing the failure of k-boundedness for a suitably large k. However, a coalgebraic model satisfying the whole $\Delta(x)$ would witness the failure of ω-boundedness. This means that $\Delta(x)$ is a counterexample to compactness, and hence no finitary deduction system can be strongly complete. The proof for the case where Λ contains a propositional atom is entirely analogous. ∎

Example 3.38. The probabilistic instances of CPL given by interpreting the probabilistic modalities $\langle p \rangle$ over nonstandard or zero-dimensional subdistributions, respectively, and are ω-bounded but fail to be k-bounded for any k. Hence they fail to be compact by Theorem 3.37 (once equipped with propositional atoms) although they satisfy finitary one-step compactness (Examples 3.35 and 3.36).

4. Correspondence with Coalgebraic Modal Logic

We next compare the expressivity of CPL with that of various coalgebraic modal and hybrid logics.

4.1. **Coalgebraic Standard Translation for CML.** The formulas $\mathrm{CML}_\Lambda\Sigma$ of pure (coalgebraic) modal logic in the modal signature Λ over Σ (now all elements of Σ are assumed to be of arity 1) are given by the grammar:

$$\mathrm{CML}_\Lambda\Sigma \quad \phi, \psi ::= P \mid \bot \mid \phi \to \psi \mid \heartsuit(\phi_1, \dots, \phi_n),$$

where $P \in \Sigma$.

Satisfaction is defined with respect to $\mathfrak{M} = (C, \gamma, I)$ and a specific point $c \in C$ in a standard way, see e.g. [SP10a, SP10b].

Definition and Proposition 4.1. *Define the* coalgebraic standard translation *as*

$$\begin{aligned} ST_x(P) &= P(x), \\ ST_x(\heartsuit(\phi_1, \dots, \phi_n)) &= x\heartsuit\lceil x : ST_x(\phi_1)\rceil \dots \lceil x : ST_x(\phi_n)\rceil, \\ ST_x(\bot) &= \bot, \\ ST_x(\phi \to \psi) &= ST_x(\phi) \to ST_x(\psi). \end{aligned}$$

Then for any $\phi \in \mathrm{CML}_\Lambda\Sigma$ *and any* $\mathfrak{M} = (C, \gamma, I), v, c$, *we have* $\mathfrak{M}, c \vDash \phi$ *iff* $\mathfrak{M}, v[x \mapsto c] \vDash ST_x(\phi)$.

For example, $ST_x(\heartsuit\heartsuit P) = x\heartsuit\lceil x : x\heartsuit\lceil x : P(x)\rceil\rceil$. This definition is more straightforward than the standard translation into FOL of modal logic over ordinary Kripke frames. Moreover, ST_x uses only one variable from iVar, namely x itself. In fact, we can immediately observe that

Proposition 4.2. *Whenever* Σ *consists entirely of* unary *predicate symbols, the subset of* $\phi \in \mathrm{CPL}(\Sigma)$ *obtained as the image of* ST_x *for a fixed* $x \in \mathsf{iVar}$ *consists precisely of equality-free and quantifier-free formulas in the variable* x.

4.2. **Hybrid Languages.** In this section, we establish the equivalence of CPL with the hybrid languages $\mathrm{H}_\Lambda(\downarrow, \mathsf{A})$ and $\mathrm{H}_\Lambda(\forall, @)$. Both correspondences also hold for ordinary predicate logic over relational structures (FOL) and extend to CPL. We take this as yet another indication that CPL is natural and well-designed both as a generalization of FOL and "the" predicate logic cousin of existing coalgebraic formalisms.

This is our main, but not the only motivation. We progress towards this result step-by-step, extending the modal language gradually with new hybrid constructs. In this way, we reveal that a similar correspondence exists between natural fragments of CPL and weaker hybrid languages, most importantly between quantifier-free CPL and $\mathrm{H}_\Lambda(\downarrow, @)$.

Again, obviously the correspondence between fragments of CPL and extensions of CML is tighter than in the case of FOL and ML only due to the modal flavour of CPL. However, results such as Corollary 4.5 are useful spadework: any model-theoretic tool to be developed—say, a variant of E-F games—would be adequate for an extended coalgebraic modal formalism (e.g., $\mathrm{H}_\Lambda(\downarrow, @)$) **iff** it is adequate for the corresponding fragment of CPL (e.g., the variable-free fragment), so we are free to work with whichever formalism we find more convenient at a given moment. The straightforward correspondence also provides a good starting point for an extension of research programme sketched in [Cat05]—see Remark 4.9 at the end of this section.

Given a supply of *world variables* wVar that we are going to keep fixed and implicit—in fact, as stated below, *near* identical to iVar—we define the following *coalgebraic hybrid languages*

$$\begin{array}{ll}
\mathrm{H}_\Lambda(\downarrow,@) & \phi,\psi ::= \ z \mid P \mid \bot \mid \phi\to\psi \mid \heartsuit(\phi_1,\dots,\phi_n) \mid @_z\phi \ \mid \downarrow z.\phi \\
\mathrm{H}_\Lambda(\downarrow,\mathsf{A}) & \phi,\psi ::= \ z \mid P \mid \bot \mid \phi\to\psi \mid \heartsuit(\phi_1,\dots,\phi_n) \mid \mathsf{A}\phi \ \mid \downarrow z.\phi \\
\mathrm{H}_\Lambda(\forall,@) & \phi,\psi ::= \ z \mid P \mid \bot \mid \phi\to\psi \mid \heartsuit(\phi_1,\dots,\phi_n) \mid @_z\phi \ \mid \forall z.\phi
\end{array}$$

where $z \in \mathsf{wVar}$. We refer the reader to, e.g, [SP10b, BC06, Cat05] for the semantics. The extension of the standard translation to these formalism is unproblematic in some cases, just like in the case of ordinary hybrid logic over Kripke frames:

$$ST_x(z) = x = z, \quad ST_x(\mathsf{A}\phi) = \forall x.ST_x(\phi), \quad ST_x(\forall z.\phi) = \forall z.ST_x(\phi).$$

One is tempted to put forward also

$$ST_x(@_z\phi) = ST_x(\phi)[z/x], \quad ST_x(\downarrow z.\phi) = ST_x(\phi)[x/z].$$

However, with other clauses remaining the same, this would violate our convention that $[z/x]$ is used only when z is *substitutable* for x; we would need to interpret it as *capture-avoiding* substitution. Sadly, this in turn would entail forsaking the luxury of using just one designated variable for comprehension. Guillame Malod (see [CF05]) observed that if we restrict the supply of variables, a translation along the above lines—indeed first proposed in the literature, which also goes to show that the present discussion is less trivial than it might seem—would fail even when embedding the hybrid logic over Kripke frames in the two-variable fragment of FOL. Malod's counterexample used nesting of modalities of level two, but as our translation uses just one designated variable, ST would go wrong already on formulas of depth one. Just consider $ST_x(\downarrow z.\Diamond z)$: were we careless about capture of bound variables, we would obtain $x\Diamond\lceil x : x = x\rceil$, which is a formula with a completely different meaning. There are two ways out. First is to redefine

$$STmod_x(@_z\phi) = \forall x.(x = z \to ST_x(\phi)), \tag{4.1}$$

$$STmod_x(\downarrow z.\phi) = \forall z.(x = z \to ST_x(\phi)). \tag{4.2}$$

The second is to keep ST for hybrid formulas as defined above and change the modal clause instead:

$$ST_x(\heartsuit(\phi_1,\dots,\phi_n)) = x\heartsuit\lceil y : ST_y(\phi_1)\rceil \dots \lceil y : ST_y(\phi_n)\rceil, \tag{4.3}$$

where y is the first (in some fixed enumeration) variable *not used* in $ST_x(\phi_1),\dots,ST_x(\phi_n)$; by *not used* here we mean both free and bound usage. Furthermore, to ensure that the translation works correctly, we have to assume that *neither x nor y* appears in wVar. While the requirement to use more bound variables can be cumbersome—particularly for infinite sets of formulas—we prefer this option, as it makes it easier to characterize weaker hybrid languages as suitable syntactic fragments of CPL.

We can now state a generalization of both Proposition 4.1 and corresponding results from the hybrid logic literature—see, e.g., [BC06] for references:

Table 2: Coalgebraic Hybrid Translation from quantifier-free CPL to $\mathrm{H}_\Lambda(\downarrow, @)$

$$HT(P(x)) = @_x P \qquad HT(x = y) = @_x y$$
$$HT(\bot) = \bot \qquad HT(\phi \to \psi) = HT(\phi) \to HT(\psi)$$
$$HT(x \heartsuit \lceil y_1 : \phi_1 \rceil \ldots \lceil y_n : \phi_n \rceil) = @_x \heartsuit(\downarrow y_1.HT(\phi_1), \ldots, \downarrow y_n.HT(\phi_n))$$

Proposition 4.3. *For any hybrid formula ϕ and any $\mathfrak{M} = (C, \gamma, I), v, c$, we have $\mathfrak{M}, v, c \vDash \phi$ iff $\mathfrak{M}, v[x \mapsto c] \vDash ST_x(\phi)$.*

As is well-known in the hybrid logic community—see again [BC06] for references—there is also a translation in the reverse direction for sufficiently expressive hybrid languages. This also generalizes to our setting, see Table 2.

Proposition 4.4. *For any $\phi \in$ CPL and any $\mathfrak{M} = (C, \gamma, I), v, c$, we have*

$$\mathfrak{M}, v, c \vDash HT(\phi) \text{ iff } \mathfrak{M}, v[x \mapsto c] \vDash \phi.$$

Combining Propositions 4.4 and 4.3, we get:

Corollary 4.5. *Whenever Σ consists purely of unary predicates (and no function symbols), $\mathrm{H}_\Lambda(\downarrow, @)$ is expressively equivalent to the quantifier-free fragment of* CPL*, assuming* iVar *contains* wVar *plus a disjoint infinite supply of additional individual variables (used for comprehension).*

Remark 4.6 (Quantifier-free CPL as the bounded fragment of FOL)**.** In the case of ordinary FOL, the fragment equivalent to $\mathrm{H}_\Lambda(\downarrow, @)$ is characterized as the *bounded fragment*, see, e.g., [AtC07]. In fact, our formula $x \heartsuit \lceil y : \phi \rceil$, despite being quantifier-free on the surface, can be described as a form of bounded quantification. This can be formalized as a result stating that over coalgebras for the covariant powerset functor (Kripke frames), quantifier-free CPL is equivalent to the bounded-fragment of ordinary FOL, where the role of $\heartsuit$ in CPL is played by the binary relation symbol R in FOL; details are left to the reader.

Remark 4.7 (Chang's original syntax)**.** As already mentioned, our syntax is slightly different to the original one proposed by Chang [Cha73]. In that paper, there were no explicit comprehension variables and even in the enriched syntax which allowed constants and function terms, the term on the left-hand side of $\heartsuit$ had to be a variable. This variable was reused then on the right side of $\heartsuit$ as the comprehension variable. In other words, Chang's $x \heartsuit \phi(x)$ was equivalent to ours $x \heartsuit \lceil x : \phi(x) \rceil$. In presence of quantifiers, which can be used to simulate the effect of capture-avoiding substitution as in $STmod$ (this trick in fact stems back to Alfred Tarski), the two languages are obviously equivalent. But when considering fragments, as we do here, the equivalence breaks down; without quantifiers, Chang's syntax does not allow (4.2) and simple renaming of the comprehension variable on the right-hand side of $\heartsuit$ as in (4.3) is not possible either.

There are two usual routes in hybrid logic to achieve full first-order expressivity. One is to add universal quantifiers over wVar in presence of the satisfaction operator @. The other is to add the global modality A in presence of the downarrow binder $\downarrow$. The hybrid translation is extended then as follows:

$$HT_{\forall @}(\forall x.\phi) = \forall x.HT(\phi)$$
$$HT_{\mathsf{A}\downarrow}(\forall x.\phi) = \downarrow y.\mathsf{A} \downarrow x.\mathsf{A}(y \to \phi)$$

In $HT_{\mathsf{A}\downarrow}$ we need the proviso that y is not occurring in the whole formula.

Theorem 4.8. $\mathrm{H}_\Lambda(\downarrow, \mathsf{A})$*,* $\mathrm{H}_\Lambda(\forall, @)$ *and* CPL *are expressively equivalent.*

As we can use $STmod_x$ now and keep reusing x as the comprehension variable, it is enough to assume that $\mathsf{iVar} = \mathsf{wVar} \cup \{x\}$. Since $@_z\phi$ is definable in presence of A (as $\mathsf{A}(z \to \phi)$), $\downarrow$ is definable by the universal quantifier over wVar (as $\forall z.(z \to \phi)$) and A is definable by combination of $\forall$ and $@$ (as $\forall y.@_y\phi$, where y is not used in ϕ), we get in fact seven equivalent languages: CPL, Chang's original language, $\mathrm{H}_\Lambda(\downarrow, \mathsf{A})$, $\mathrm{H}_\Lambda(\forall, @)$, $\mathrm{H}_\Lambda(\downarrow, \mathsf{A})$ with $@$, $\mathrm{H}_\Lambda(\forall, @)$ with $\downarrow$ and the jumbo hybrid language with all connectives introduced above.

Remark 4.9. The equivalences stated here extend to the case of hybrid languages and CPL enriched with quantification over predicates (i.e., second-order languages). It would be interesting to follow more thoroughly the program of *coalgebraic abstract model theory* both above and below CPL. See Ten Cate's PhD Thesis [Cat05] for spadework in abstract model theory below first-order logic.

4.3. **Semantic Correspondence: The Van Benthem-Rosen Theorem.** Our Proposition 4.2 provides a *syntactic* characterization of the modal fragment of our language. In a companion paper [SPL17], we develop a *semantic*, Van Benthem-Rosen style characterization. To compare these two characterizations, let us briefly recall the details.

In the context of standard Kripke models, expressiveness of modal logic is characterized by van Benthem's theorem: modal logic is the bisimulation invariant fragment of first-order logic in the corresponding signature. The finitary analogue of this theorem [Ros97] states that every formula that is bisimulation invariant *over finite models* is equivalent *over finite models* to a modal formula. In the coalgebraic context, replace bisimilarity with behavioural equivalence [Sta11]. Moreover, we need to assume that the language has 'enough' expressive power; e.g., we cannot expect that bisimulation invariant formulas are equivalent to CML formulas over the empty similarity type. This is made precise as follows:

Definition 4.10. A Λ-structure is *separating* if, for every set X, every element $t \in TX$ is uniquely determined by the set $\{(\heartsuit, A) \mid \heartsuit \in \Lambda \ n\text{-ary}, A \in \mathcal{P}(X)^n, t \in [\![\heartsuit]\!]_X(A)\}$.

Separation is in general a less restrictive condition than those we needed for completeness proofs. In particular, separation automatically obtains for Kripke semantics. It was first used to establish the Hennessy-Milner property for coalgebraic logics [Pat04, Sch08].

Theorem 4.11 ([SPL17])**.** *Suppose that the structure is separating and $\phi(x)$ is a CPL formula with one free variable. Then ϕ is invariant under behavioural equivalence* (*over finite models*) *iff it is equivalent to an infinitary CML formula with finite modal rank* (*over finite models*)*.*

If we deal with finite similarity types only, the conclusion can be strengthened:

Theorem 4.12 ([SPL17])**.** *Suppose that the structure is separating, Λ is finite and $\phi(x)$ is a CPL formula with one free variable. Then ϕ is invariant under behavioural equivalence* (*over finite models*) *iff ϕ is equivalent to a **finite** CML formula* (*over finite models*)*.*

In fact, we can combine Theorem 4.12 with the syntactic characterization of Proposition 4.2 to obtain

Corollary 4.13. *Whenever Σ consists entirely of* unary *predicate symbols and the structure is separating, the behaviourally-invariant* (*over finite structures*) *formulas of* CPL *in one-free variable are up to equivalence* (*over finite structures*) *precisely the equality-free and quantifier-free formulas in the single-variable fragment of* CPL.

5. First Steps in Coalgebraic Model Theory

We proceed to outline the beginning of coalgebraic model theory, taking a look at ultraproducts and the downwards Löwenheim-Skolem theorem. In the course of the technical development, we will import a result on one-step cutfree complete rule sets established in earlier work [Sch07, PS10].

Recall that if $\mathfrak{U}$ is an ultrafilter on an index set I and (X_i) is an I-indexed family of nonempty sets, then the *ultraproduct* $\prod_{\mathfrak{U}} X_i$ is defined as

$$\prod_{\mathfrak{U}} X_i = \big(\prod_{i\in I} X_i\big)/\sim$$

where $\sim$ is the equivalence relation on $\prod_{i\in I} X_i$ defined by

$$(x_i) \sim (y_i) \iff \{i \in I \mid x_i = y_i\} \in \mathfrak{U}.$$

One may regard $\mathfrak{U}$ as a $\{0,1\}$-valued measure on I; under this reading, the above definition says that (x_i) and (y_i) are identified under $\sim$ if they are almost everywhere equal. We write elements of $\prod_{\mathfrak{U}} X_i$ and $\prod_{i\in I} X_i$ just as x, omitting notation for equivalence classes and accessing the i-th component as x_i.

Observe that if $X = \prod_{\mathfrak{U}} X_i$ is an ultraproduct of sets and (A_i) is a family of subsets $A_i \subseteq X_i$, then

$$A = \prod_{\mathfrak{U}} A_i := \{x \mid \{i \mid x_i \in A_i\} \in \mathfrak{U}\} \tag{5.1}$$

is a well-defined subset of X (this is in fact just the way unary predicates are standardly extended from the components to the ultraproduct). Subsets of ultraproducts that are of this form are called *admissible*.

Lemma 5.1. *All finite subsets of ultraproducts are admissible.*

Ultraproducts of coalgebras will not be determined uniquely; instead, we give a property-oriented definition and later show existence.

Definition 5.2 (Quasi-Ultraproducts of Coalgebras)**.** Let $(C_i) = (X_i, \xi_i)_{i\in I}$ be a family of T-coalgebras, and let $\mathfrak{U}$ be an ultrafilter on I. A coalgebra ξ on the set-ultraproduct $X = \prod_{\mathfrak{U}} X_i$ is called a *quasi-ultraproduct* of the C_i if for every family (A_i) of subsets $A_i \subseteq X_i$, every $x \in \prod_{\mathfrak{U}} X_i$, and every $\heartsuit \in \Lambda$,

$$\xi(x) \in [\![\heartsuit]\!]_X \prod_{\mathfrak{U}} A_i \iff \{i \in I \mid \xi_i(x_i) \in [\![\heartsuit]\!]_{C_i}(A_i)\} \in \mathfrak{U}. \tag{5.2}$$

The notion of quasi-ultraproduct extends naturally to coalgebraic models using the standard definition to extend the interpretation of predicates (as indicated above, Equation (5.1) recalls the case of unary predicates).

The definition of quasi-ultraproducts is designed in such a way that Łoś's theorem, which in the view of ultrafilters as $\{0,1\}$-valued measures states that the ultraproduct satisfies exactly those formulas that hold in almost all its components, extends to coalgebras:

Theorem 5.3 (Coalgebraic Łoś's Theorem)**.** *If* $\mathfrak{M} = (C, \gamma, V)$ *is a quasi-ultraproduct of* $\mathfrak{M}_i = (C_i, \gamma_i, V_i)$ *for the ultrafilter* $\mathfrak{U}$*, then for every tuple* $(a^1, \dots, a^n)$ *of states in* C*, where* $a^k = (a_i^k)_{i \in I}$*, and for every CPL formula* $\phi(x_1, \dots, x_n)$*,* $C \models \phi(a^1, \dots, a^n) \iff \{i \mid C_i \models \phi(a_i^1, \dots, a_i^k)\} \in \mathfrak{U}$*.*

Proof. Induction over formulas. The cases for Boolean operators and quantifiers are as in the classical case, and the case for modal operators is exactly by the quasi-ultraproduct property. □

From this theorem, we obtain the usual applications, in particular compactness (the latter with literally the same proof as in the classical case). The question is, of course, when quasi-ultraproducts exist. A core observation is

Lemma 5.4. *In the notation of Definition 5.2, the demands placed on* $\xi(x)$ *by Condition* (5.2) *constitute a finitely satisfiable set of one-step formulas.*

In the proof of this lemma, and on several further occasions, we will need the fact that the set of all one-step sound one-step rules is *one-step cutfree complete* [PS10]. Instead of repeating the definition of this term, we state the relevant property directly.

Lemma 5.5. [Sch07, Theorem 18 with proof] *Let* Φ *be a finite subset of* $\Lambda(\mathcal{P}(C)) \cup \neg\Lambda(\mathcal{P}(C))$*, where* $\neg\Lambda(\mathcal{P}(C)) = \{\neg\heartsuit A \mid \heartsuit A \in \Lambda(\mathcal{P}(C))\}$*. If* Φ *is one-step unsatisfiable, then there exists a sound one-step rule* $\boldsymbol{A}/\boldsymbol{P}$ *and a valuation* $\tau : \mathsf{sVar} \to \mathcal{P}(C)$ *such that* $C \models \boldsymbol{A}\tau$ *and* $\boldsymbol{P}\tau = \neg \bigwedge \Phi$*.*

We then proceed as follows with the open proof of Lemma 5.4:

Proof (Lemma 5.4). Fix finitely many instances of (5.2) (keeping the same notation) for families of sets $(A_i^j)_{i \in I}$ and sets $A^j = \prod_{\mathfrak{U}} A_i^j$, $j = 1, \dots, k$. We regard these sets as extensions of unary predicates P^j over the X_i and over X, respectively. If the corresponding instances of (5.2) do not have a solution $\xi(x)$ in TX, then by Lemma 5.5 we have a sound one-step rule $\boldsymbol{A}/\boldsymbol{P}$ and a valuation $\tau : \mathsf{sVar} \to \mathcal{P}(X)$ such that $X \models \boldsymbol{A}\tau$ but the instances of (5.2) for $A^1, \dots, A^k$ demand $\xi(x) \models \neg\boldsymbol{P}\tau$. Let $\boldsymbol{p}_1, \dots, \boldsymbol{p}_k$ be the schematic variables appearing in $\boldsymbol{A}/\boldsymbol{P}$; w.l.o.g. $\tau(\boldsymbol{p}_j) = A^j$ for $j = 1, \dots, k$. Then X satisfies the first-order sentence $\forall z.(\boldsymbol{A}\sigma)$ where $\sigma(\boldsymbol{p}_j) = P^j(y)$. By Łoś's theorem (in fact already by its classical version), there exists $B \in \mathfrak{U}$ such that $X_i \models \forall z.(\boldsymbol{A}\sigma)$ and hence $X_i \models \boldsymbol{A}\tau_i$ for all $i \in B$, where $\tau_i(\boldsymbol{p}_j) = A_i^j$. By one-step soundness of $\boldsymbol{A}/\boldsymbol{P}$ this implies $TX_i \models \boldsymbol{P}\tau_i$ for all $i \in B$. But our formulation above that the instances of (5.2) for $A^1, \dots, A^k$ demand $\xi(x) \models \neg\boldsymbol{P}\tau$ means more explicitly (and using the fact that $\mathfrak{U}$ is an ultrafilter) that $\{i \in I \mid \xi_i(x_i) \models \neg\boldsymbol{P}\tau_i\} \in \mathfrak{U}$, so that we have a contradiction. □

From Lemma 5.4, our first existence criterion for quasi-ultraproducts is immediate:

Theorem 5.6. *If a* Λ*-structure is one-step compact* (*Definition 3.5*)*, then it has quasi-ultraproducts.*

Example 5.7. The above criterion applies in particular to all neighbourhood-like logics. It thus subsumes Chang's original ultraproduct construction [Cha73]

Like for our completeness results, an alternative is to require bounded operators:

Theorem 5.8. *If a* Λ*-structure is finitary one-step compact* (*Definition 3.5*) *and all its operators are bounded, then it has quasi-ultraproducts.*

The proof needs the following lemma.

Lemma 5.9. *Let* $(C_i) = (X_i, \xi_i)_{i \in I}$ *be a family of* T*-coalgebras, and let* $\mathfrak{U}$ *be an ultrafilter on* I*. Let* X *be the ultraproduct* $\prod_{\mathfrak{U}} X_i$*, and let* $x \in X$*. Then the set*

$$\Psi = \{\epsilon\heartsuit\{y^1, \dots, y^k\} \mid \{i \mid \xi_i(x_i) \models \epsilon\heartsuit\{y_i^1, \dots, y_i^k\}\} \in \mathfrak{U}\}$$

of one-step formulas (where $\heartsuit$ *ranges over* Λ*, the* y^i *range over* X*, and* ϵ *stands for either negation or nothing) is finitely satisfiable.*

Proof. Analogous to Lemma 5.4, using atoms of the form $z = c$ in place of unary predicates. In more detail: if a finite subset Φ of Ψ is one-step unsatisfiable, then by Lemma 5.5 there exist a one-step sound rule $\boldsymbol{A}/\boldsymbol{P}$ and a valuation τ such that $\boldsymbol{P}\tau = \neg \bigwedge \Phi$ and $X \models \boldsymbol{A}\tau$. Now $X \models \boldsymbol{A}\tau$ is semantically equivalent to $X \models \forall z.\boldsymbol{A}_0$ where $\boldsymbol{A}_0$ is propositional formula over atoms of the form $z = c$, where c ranges over constants denoting elements of the involved finite subsets of X in an extended first-order structure based on X. Then $\{i \mid X_i \models \forall z.\, \boldsymbol{A}_0\} \in \mathfrak{U}$ by (the classical version of) Łoś's theorem, where we interpret constants in X_i by taking the i-th component of the interpretation in X (this is just the way the interpretation of constants in the factors relates to that in the ultraproduct, classically). Hence $\{i \mid X_i \models \boldsymbol{A}\sigma_i\} \in \mathfrak{U}$, where σ_i replaces $\{y^1, \dots, y^k\}$ with $\{y_i^1, \dots, y_i^k\}$, and hence $\{i \mid TX_i \models \boldsymbol{W}\} \in \mathfrak{U}$, contradiction as in Lemma 5.4. □

Proof (Theorem 5.8). By Lemma 5.9 and finitary one-step compactness, there exists $\xi(x)$ satisfying the set Ψ from Lemma 5.9. To show 5.2 for $A \subseteq X$, we regard A as the extension of a unary predicate P. Then $\xi(x) \models \heartsuit A$ is equivalent to

$$x \models \exists y^1, \dots, y^k.\, (P(y^1) \wedge \dots \wedge P(y^k) \wedge x\heartsuit\lceil z : z = y^1, \dots, z = y^k \rceil).$$

Thus it suffices to prove the Łoś equivalence for formulas $x\heartsuit\lceil z : z = y^1, \dots, z = y^k\rceil$. This, however, is exactly what satisfaction of Ψ by $\xi(x)$ guarantees. □

For operators that are ω-bounded but not k-bounded for any k, the ultraproduct construction cannot be available, in consequence of Theorem 3.37. However, the downward Löwenheim-Skolem theorem does survive under the weaker assumption of ω-boundedness:

Theorem 5.10 (Downward Löwenheim-Skolem Theorem)**.** *Over* ω*-bounded finitary one-step compact* Λ*-structures,* $\mathrm{CPL}(\Lambda, \Sigma)$ *satisfies the downward Löwenheim-Skolem theorem; that is, every model of infinite cardinality* κ *has, for every infinite cardinal* $\lambda \leq \kappa$*, an elementary substructure of cardinality* λ*.*

Here, we use the term *elementary substructure* in the usual way to designate first-order substructures whose elements satisfy the same formulas as they do in the original model; we explicitly do *not* require that the coalgebra structure on the substructure forms a subcoalgebra.

The proof needs the following simple lemma.

Lemma 5.11. *Let* Y *be an infinite subset of* X*,* $\tau : \mathsf{sVar} \to \mathcal{P}_{\mathit{fin}}(Y)$ *and* $\boldsymbol{A} \in \mathsf{Prop}(\mathsf{sVar})$*. Then* $Y \models \boldsymbol{A}\tau$ *iff* $X \models \boldsymbol{A}\tau$*.*

Proof. Only finitely many $\boldsymbol{p} \in \mathsf{sVar}$ are relevant, so we can, for the rest of the proof, assume that sVar is finite. Define the τ-valuation of $x \in X$ as the valuation $\kappa : \mathsf{sVar} \to 2$ given by $\kappa(\boldsymbol{p}) = \top$ iff $x \in \tau(\boldsymbol{p})$. Then the claim of the lemma is equivalent to saying that every τ-valuation occurring in X occurs also in Y. Now if $x \in X \setminus Y$, then the τ-valuation of x is everywhere false; this valuation occurs also in Y, as sVar and the $\tau(\boldsymbol{p})$ are finite. □

Proof (Theorem 5.10). Let $\mathfrak{M} = (C, \gamma, I)$ be a coalgebraic model of cardinality κ. Pick Skolem functions for all formulas $\exists x.\, \phi$ as usual, and for every formula $x\heartsuit\lceil y : \phi\rceil$ a finitely non-deterministic Skolem function $f_{x\heartsuit\lceil y:\phi\rceil} : C^{\mathsf{FV}(x\heartsuit\lceil y:\phi\rceil)} \to \mathcal{P}_{\mathit{fin}}(C)$ with the property that for every valuation $\eta \in C^{\mathsf{FV}(x\heartsuit\lceil y:\phi\rceil)}$, $f_{x\heartsuit\lceil y:\phi\rceil}(\eta) \subseteq_{\mathit{fin}} [\![\phi]\!]_{C,\eta}$ and

$$C, \eta \models x\heartsuit\lceil y : \phi\rceil \iff \gamma(\eta(x)) \models \heartsuit f_{x\heartsuit\lceil y:\phi\rceil}(\eta).$$

(Such a function $f_{x\heartsuit\lceil y:\phi\rceil}$ exists because $\heartsuit$ is ω-bounded.) Pick a countably infinite subset $Y_0 \subseteq C$ and let Y be the closure of Y_0 under the Skolem functions, in the case of the non-deterministic Skolem functions $f_{x\heartsuit\lceil y:\phi\rceil}$ in the sense that $f_{x\heartsuit\lceil y:\phi\rceil}[Y] \subseteq Y$. Then Y is countable: it consists of the possible values of countably many finitely non-deterministic finite Skolem terms.

It remains to define a coalgebra structure ζ on $c \in Y$ in such a way that

$$\zeta(c) \models \heartsuit A \iff \gamma(c) \models \heartsuit A \tag{5.3}$$

for all $A \subseteq_{\mathit{fin}} Y$; that is, we have to prove that the set

$$\Psi = \{\epsilon\heartsuit A \mid \gamma(c) \models \epsilon\heartsuit A\}$$

of one-step formulas over $\mathcal{P}_{\mathit{fin}}(Y)$ is satisfiable over Y (where $\heartsuit$ ranges over Λ, A ranges over $\mathcal{P}_{\mathit{fin}}(Y)$, and ϵ ranges over $\{\cdot, \neg\}$). By finitary one-step compactness, it suffices to prove that Ψ is finitely satisfiable. Assume the contrary; then by Lemma 5.5 there exists a sound one-step rule $\boldsymbol{A}/\boldsymbol{P}$ valuation $\tau : \mathsf{sVar} \to \mathcal{P}(Y)$ such that $Y \models \boldsymbol{A}\tau$ and $\boldsymbol{P}\tau$ propositionally contradicts some finite subset Ψ_0 of Ψ. By Lemma 5.11, $C \models \boldsymbol{A}\tau$, and hence $C \models \boldsymbol{P}\tau$; therefore, Ψ_0 is unsatisfiable over C, in contradiction to the fact that $\gamma(c)$ satisfies Ψ by construction.

Since Ψ is satisfiable, we have a coalgebra structure ζ satisfying (5.3). It follows by induction over the formula structure that for every coalgebraic first-order formula ϕ and every valuation v in Y,

$$\mathfrak{N}, v \models \phi \quad \text{iff} \quad \mathfrak{M}, v \models \phi :$$

where $\mathfrak{N} = (Y, \zeta, J)$ and J is the induced substructure obtained by restricting I to Y. The Boolean cases are trivial. The case for existential quantification is as in the classical case. The case $x\heartsuit\lceil y : \phi\rceil$ is as follows: $\mathfrak{N}, v \models x\heartsuit\lceil y : \phi\rceil$ iff $\zeta(v(x)) \models \heartsuit[\![\phi]\!]_{\mathfrak{N},v} = \heartsuit([\![\phi]\!]_{\mathfrak{M},v} \cap Y)$ (where the equality holds by induction) iff (by ω-boundedness) $\zeta(v(x)) \models \heartsuit A$ for some $A \subseteq_{\mathit{fin}} [\![\phi]\!]_{\mathfrak{M},v} \cap Y$, equivalently $\gamma(v(x)) \models \heartsuit A$ by (5.3). The latter implies $\mathfrak{M}, v \models x\heartsuit\lceil y : \phi\rceil$ by monotonicity; conversely, $\mathfrak{M}, v \models x\heartsuit\lceil y : \phi\rceil$ implies $\gamma(v(x)) \models \heartsuit f_{x\heartsuit\lceil y:\phi\rceil}(v)$ by construction, and $f_{x\heartsuit\lceil y:\phi\rceil}(v) \subseteq_{\mathit{fin}} [\![\phi]\!]_{\mathfrak{M},v} \cap Y$. ☐

Example 5.12. The above version of the downward Löwenheim-Skolem theorem applies to our main bounded examples (relational, graded, and positive Presburger modalities) as well as to probabilistic modalities over non-standard or zerodimensional subdistributions, respectively, which are ω-bounded but not k-bounded for any k (Examples 3.35 and 3.36).

Finally, we note that the downward Löwenheim-Skolem theorem holds also for the one-step compact case; this is in mild generalization of a corresponding result for the neighbourhood case proved already by Chang [Cha73].

Theorem 5.13. *Over one-step compact Λ-structures,* CPL(Λ, Σ) *satisfies the downward Löwenheim-Skolem theorem* (*in the same formulation as in Theorem 5.10*).

Proof. Let Φ be a set of coalgebraic first-order formulas in $\mathrm{CPL}(\Lambda, \Sigma)$, and let $\mathfrak{M} = (C, \gamma, I)$ be such that $\mathfrak{M} \models \Phi$. Pick Skolem functions for all formulas $\exists x.\, \phi$ as usual, and for every one-step sound one-step rule $R = \boldsymbol{A}/\boldsymbol{P}$ fix a Skolem function that given an element $x \in C$ satisfying some instance of $\neg \boldsymbol{P}$ picks an element of C that satisfies the corresponding instance of $\neg \boldsymbol{A}$. More precisely: let $\sigma : \mathsf{iVar} \to \mathrm{CPL}(\Lambda, \Sigma)$ be a substitution, let x, y be variables with x fresh, let $\boldsymbol{P}^{x,y}\sigma$ be the formula obtained by replacing in $\boldsymbol{P}$ each modal operator application $\heartsuit \boldsymbol{p}$ with $x\heartsuit\lceil y : \sigma(\boldsymbol{p})\rceil$, and let v be a valuation such that $\mathfrak{M}, v \models \neg \boldsymbol{P}^{x,y}\sigma$. Then there exists a y-variant v' of v such that $\mathfrak{M}, v' \models \neg \boldsymbol{A}\sigma$, and the Skolem function $f_{R,\sigma}$ assigns such a $v'(y)$ to $v|_{\mathsf{FV}(\boldsymbol{P}^{x,y}\sigma)}$. As $\mathsf{FV}(\boldsymbol{P}^{x,y}\sigma)$ is finite, $f_{R,\sigma}$ is a finitary function, so that closing a given subset $Y_0 \subseteq C$ of cardinality $|Y_0| = \lambda$ under the Skolem functions yields a set $Y \subseteq C$ of the same cardinality $|Y| = \lambda$.

The coalgebra structure ζ that we are to define on Y has to satisfy the *coherence* condition

$$\zeta(c) \models \heartsuit(\llbracket\rho\rrbracket_v^y \cap Y) \text{ iff } \gamma(c) \models \heartsuit\llbracket\rho\rrbracket_v^y$$

for all $c \in Y$, all formulas ρ, and all valuations v in Y, where the second condition is by definition equivalent to $c \in \llbracket x\heartsuit\lceil y : \rho\rceil\rrbracket_v^x$. Once this is established, we can show as usual that $\mathfrak{N} = (Y, \zeta, J)$, with J interpreting Σ by restricting I to Y, is an elementary substructure of (C, γ, I), and we are done.

Now assume that $\zeta(c)$ as required fails to exist, which means that the set Φ of constraints of the form $\epsilon\heartsuit(\llbracket\rho\rrbracket_v^y \cap Y)$ (where ϵ stands for either nothing or negation) on $\zeta(c)$ determined by the coherence condition is one-step unsatisfiable. By one-step compactness, already some finite subset Φ_0 of Φ is unsatisfiable. By Lemma 5.5 and the format of Φ, there exist a sound one-step rule $R = \boldsymbol{A}/\boldsymbol{P}$ and a substitution $\sigma : \mathsf{sVar} \to \mathrm{CPL}(\Lambda, \Sigma)$ such that $Y \models \boldsymbol{A}\tau$ and $\boldsymbol{P}\tau = \neg \bigwedge \Phi_0$ where $\tau(\boldsymbol{p}) = \llbracket\sigma(\boldsymbol{p})\rrbracket_v^y \cap Y$. However, by construction of Φ we have $\gamma(c) \in \llbracket\neg \boldsymbol{P}\rrbracket\hat{\tau}$, where $\hat{\tau}$ is the $\mathcal{P}(C)$-valuation sending $\boldsymbol{p} \in \mathsf{sVar}$ to $\llbracket\sigma(\boldsymbol{p})\rrbracket_v^y$, and hence $\mathfrak{M}, v \models \neg \boldsymbol{P}^{x,y}\sigma$ where x is fresh and we assume w.l.o.g. that $v(x) = c$. Then $f_{R,\sigma}(v|_{\mathsf{FV}(\boldsymbol{P}^{x,y}\sigma)}) \in \llbracket\neg \boldsymbol{A}\rrbracket\tau$, in contradiction to $Y \models \boldsymbol{A}\tau$. ☐

Example 5.14. Besides the plain neighbourhood case, Theorem 5.13 covers all instances of CPL defined by imposing rank-1 frame conditions on neighbourhood frames, e.g. CPL over monotone neighbourhood frames and various deontic logics.

6. Proof theory

6.1. **Sequent system for CPL.** In § 3, we have seen a complete Hilbert calculus for coalgebraic predicate logic. The present goal is a cut-free, complete sequent calculus. Our basis is the system $\mathsf{G1c}$ of [TS96] that we extend with modal rules describing the (fixed) Λ-structure. Our treatment of equality, on the other hand, is inspired by Kanger [Kan57], Degtyarev and Voronkov [DV01] and Seligman [Sel01]. In fact, the syntactic cut-elimination proof presented here is based on Seligman's ideas.

We take *sequents* to be pairs (Γ, Δ), written $\Gamma \Rightarrow \Delta$ where $\Gamma, \Delta \subseteq \mathcal{L}$ are finite multisets. The sequent calculus for coalgebraic predicate logic contains four types of rules: the standard logical and structural rules for first-order logic, rules for equality and rules for the modal operators. The logical rules are standard as in Table 3. The formula introduced in the conclusion of a logical rule is called the *principal* formula of the rule. This applies, in particular, to the structural rules in Table 3: the formula ϕ in the conclusion is the principal one. Note that, somewhat counterintuitively, in the equality rules the formula $x = y$ in the conclusion is the *context*, i.e., the only *non-principal* formula and all the remaining ones are *principal*!

To account for the modal operators, we incorporate the one-step rules $\mathcal{R}$ into the sequent system. In principle, we just generate a sequent rule within CPL from every one-step rule in $\mathcal{R}$. Only for presentational purposes, we factor this process through an alternative modal rule format where propositional operators are fully dissolved into sequents:

Definition 6.1. A rule

$$\frac{\Gamma_1 \Rightarrow \Delta_1 \cdots \Gamma_k \Rightarrow \Delta_k}{\heartsuit_1 \vec{\boldsymbol{p}}_1, \ldots, \heartsuit_n \vec{\boldsymbol{p}}_n \Rightarrow \heartsuit_{n+1} \vec{\boldsymbol{p}}_{n+1}, \ldots, \heartsuit_{n+m} \vec{\boldsymbol{p}}_{n+m}}$$

represents a one-step rule $\boldsymbol{A}/\boldsymbol{P}$ *in sequent format* if $\boldsymbol{A}$ is propositionally equivalent to $\bigwedge_{i=1}^{k}((\bigwedge \Gamma_i) \to (\bigvee \Delta_i)$, and $\boldsymbol{P}$ is propositionally equivalent to $(\bigwedge_{j=1}^{n} \heartsuit_j \vec{\boldsymbol{p}}_j) \to (\bigvee_{j=n+1}^{m} \heartsuit_j \vec{\boldsymbol{p}}_j)$. We transfer the existing syntactic restrictions on one-step rules according to Definition 3.2 to this format by requiring that every schematic variable occurring in the premise occurs also in the conclusion, and every schematic variable occurs at most once in the conclusion.

It is clear that every one-step rule can be represented in sequent format (just transform the premise into conjunctive normal form and then trivially translate disjunctive clauses into sequents in both premise and conclusion). Subsequently, we generate a sequent rule $\mathcal{S}(R)$ in CPL syntax, adding weakening contexts Σ, Θ to both the conclusion and all the premises:

$$\frac{\Sigma, \Gamma_1 \sigma^y_{\boldsymbol{x}} \Rightarrow \Delta_1 \sigma^y_{\boldsymbol{x}}, \Theta \quad \cdots \quad \Sigma, \Gamma_k \sigma^y_{\boldsymbol{x}} \Rightarrow \Delta_k \sigma^y_{\boldsymbol{x}}, \Theta}{\Sigma, z\heartsuit_1 \lceil \boldsymbol{x_1} \boldsymbol{:} \boldsymbol{\phi_1} \rceil, \ldots, z\heartsuit_n \lceil \boldsymbol{x_n} \boldsymbol{:} \boldsymbol{\phi_n} \rceil \Rightarrow z\heartsuit_{n+1} \lceil \boldsymbol{x_{n+1}} \boldsymbol{:} \boldsymbol{\phi_{n+1}} \rceil, \ldots, z\heartsuit_{n+m} \lceil \boldsymbol{x_{n+m}} \boldsymbol{:} \boldsymbol{\phi_{n+m}} \rceil \boldsymbol{,} \boldsymbol{\Theta}}$$

with syntactic details as summarized in Table 3. The formulas $z\heartsuit_i \lceil \boldsymbol{x} \boldsymbol{:} \boldsymbol{\phi_i} \rceil$ are the principal formulas of $\mathcal{S}(R)$.

Example 6.2. Recall from Example 3.8 that the rule set for the normal modal logic K consists of rules that are represented in sequent format as

$$\frac{\boldsymbol{p} \Rightarrow \boldsymbol{q}_1, \ldots, \boldsymbol{q}_n}{\Diamond \boldsymbol{p} \Rightarrow \Diamond \boldsymbol{q}_1, \ldots, \Diamond \boldsymbol{q}_n} \ \mathsf{K_n}$$

for all $n \geqslant 0$. We obtain the following first-order version

$$\frac{\Sigma, \phi_0[y/x_0] \Rightarrow \phi_1[y/x_1], \ldots, \phi_n[y/x_n], \Theta}{\Sigma, z\Diamond \lceil x_0 : \phi_0 \rceil \Rightarrow z\Diamond \lceil x_1 : \phi_1 \rceil, \ldots, z\Diamond \lceil x_n : \phi_n \rceil, \Theta} \ \mathcal{S}(\mathsf{K_n})\dagger_y$$

(where y is fresh in the conclusion) by the previous definition. Recall also that modal neighbourhood semantics is axiomatised by a one-step rule that is represented in sequent format as

$$\frac{\boldsymbol{p} \Rightarrow \boldsymbol{q} \quad \boldsymbol{q} \Rightarrow \boldsymbol{p}}{\Box \boldsymbol{p} \Rightarrow \Box \boldsymbol{q}} \ \mathsf{C},$$

which expresses that $\Box$ is a congruential operator. The first order version of C then reads

$$\frac{\Sigma, \phi_0[y/x_0] \Rightarrow \phi_1[y/x_1], \Theta \quad \Sigma, \phi_1[y/x_1] \Rightarrow \phi_0[y/x_0], \Theta}{\Sigma, z\Box \lceil x_0 : \phi_0 \rceil \Rightarrow z\Box \lceil x_1 : \phi_1 \rceil, \Theta} \ \mathcal{S}(\mathsf{C})\dagger_y$$

(where y is fresh in the conclusion) which provides a complete and, as we are going to see below, cut-free axiomatisation of Chang's original logic.

Table 3: Sequent System of Coalgebraic Predicate Logic

In all the rules below, $\dagger_y$ means that y is **fresh in the conclusion**.

Axioms

$$\frac{}{\phi \Rightarrow \phi}\ \mathsf{Ax} \qquad \frac{}{\bot \Rightarrow}\ \mathsf{L}\bot \qquad \frac{}{\Rightarrow x = x}\ \mathsf{R=}$$

Logical Rules

$$\frac{\phi, \Gamma \Rightarrow \Delta, \psi}{\Gamma \Rightarrow \Delta, \phi \to \psi}\ \mathsf{R}\to \qquad \frac{\Gamma \Rightarrow \Delta, \phi \quad \psi, \Gamma \Rightarrow \Delta}{\phi \to \psi, \Gamma \Rightarrow \Delta}\ \mathsf{L}\to$$

$$\frac{\Gamma \Rightarrow \Delta, \phi[y/x]}{\Gamma \Rightarrow \Delta, \forall x.\phi}\ \mathsf{R}\forall\dagger_y \qquad \frac{\phi[z/x], \Gamma \Rightarrow \Delta}{\forall x.\phi, \Gamma \Rightarrow \Delta}\ \mathsf{L}\forall$$

Equality Rules

$$\frac{x = y, \Gamma[x/z] \Rightarrow \Delta[x/z]}{x = y, \Gamma[y/z] \Rightarrow \Delta[y/z]}\ \mathsf{L=}_1 \qquad \frac{x = y, \Gamma[y/z] \Rightarrow \Delta[y/z]}{x = y, \Gamma[x/z] \Rightarrow \Delta[x/z]}\ \mathsf{L=}_2$$

Modal Rules $\mathcal{S}(\mathcal{R})$: for every one-step rule $R \in \mathcal{R}$,

$$\frac{\Sigma, \Gamma_1\sigma^y_{\boldsymbol{x}} \Rightarrow \Delta_1\sigma^y_{\boldsymbol{x}}, \Theta \quad \cdots \quad \Sigma, \Gamma_k\sigma^y_{\boldsymbol{x}} \Rightarrow \Delta_k\sigma^y_{\boldsymbol{x}}, \Theta}{\begin{array}{c}\Sigma, z\heartsuit_1\lceil \boldsymbol{x_1} : \boldsymbol{\phi_1}\rceil, \dots, z\heartsuit_n\lceil \boldsymbol{x_n} : \boldsymbol{\phi_n}\rceil \Rightarrow \\ z\heartsuit_{n+1}\lceil \boldsymbol{x_{n+1}} : \boldsymbol{\phi_{n+1}}\rceil, \dots, z\heartsuit_{n+m}\lceil \boldsymbol{x_{n+m}} : \boldsymbol{\phi_{n+m}}\rceil, \Theta\end{array}}\ \mathcal{S}(R)\dagger_y$$

where

- R is represented in sequent format as $\dfrac{\Gamma_1 \Rightarrow \Delta_1 \quad \cdots \quad \Gamma_k \Rightarrow \Delta_k}{\heartsuit_1\vec{\boldsymbol{p}}_1, \dots, \heartsuit_n\vec{\boldsymbol{p}}_n \Rightarrow \heartsuit_{n+1}\vec{\boldsymbol{p}}_{n+1}, \dots, \heartsuit_{n+m}\vec{\boldsymbol{p}}_{n+m}}$
- $\lceil \boldsymbol{x_i} : \boldsymbol{\phi_i}\rceil = \lceil x^1_i : \phi^1_i\rceil \dots \lceil x^{\mathsf{ar}\heartsuit}_i : \phi^{\mathsf{ar}\heartsuit}_i\rceil$ is a finite sequence of comprehension formulas according to $\mathsf{ar}\heartsuit_i$, and
- the substitution $\sigma^y_{\boldsymbol{x}}$ sends $\boldsymbol{p}^j_i$ to the formula $\phi^j_i[y/x^j_i]$ of $\mathcal{L}$.

Structural Rules

$$\frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \phi}\ \mathsf{RW} \qquad \frac{\Gamma \Rightarrow \Delta}{\phi, \Gamma \Rightarrow \Delta}\ \mathsf{LW}$$

$$\frac{\Gamma \Rightarrow \Delta, \phi, \phi}{\Gamma \Rightarrow \Delta, \phi}\ \mathsf{RC} \qquad \frac{\phi, \phi, \Gamma \Rightarrow \Delta}{\phi, \Gamma \Rightarrow \Delta}\ \mathsf{LC}$$

Cut Rule (optional)

$$\frac{\Gamma \Rightarrow \Delta, \phi \quad \phi, \Sigma \Rightarrow \Theta}{\Gamma, \Sigma \Rightarrow \Delta, \Theta}\ \mathsf{Cut}$$

We write $\mathcal{SR} \vdash \Gamma \Rightarrow \Delta$ if $\Gamma \Rightarrow \Delta$ can be derived using the logical rules, equality rules, and axiom rules of Table 3, together with the rules $\mathcal{S}(R)$ from Table 3 for every rule $R \in \mathcal{R}$.We write $\mathcal{SR}\mathsf{Cut} \vdash \Gamma \Rightarrow \Delta$ if the *cut rule* Cut of Table 3 is used additionally. If $\mathfrak{M} = (C, \gamma, I)$ is a first-order model over a Λ-structure, we write $\mathfrak{M}, v \models \Gamma \Rightarrow \Delta$ if $\mathfrak{M}, v \models \bigwedge \Gamma \to \bigvee \Delta$ and, as usual $\mathfrak{M} \models \Gamma \Rightarrow \Delta$ if $\mathfrak{M}, v \models \Gamma \Rightarrow \Delta$ for all variable assignments v and finally $\models \Gamma \Rightarrow \Delta$ if $\mathfrak{M} \models \Gamma \Rightarrow \Delta$ for all first-order models $\mathfrak{M}$ over the corresponding structure, which we elide in the notation.

Proposition 6.3. *For any one-step rule $R \in \mathcal{R}$ and any model $\mathfrak{M} = (C, \gamma, I)$, $\mathcal{S}(R)$ preserves the validity on $\mathfrak{M}$.*

Proof. Let $R \in \mathcal{R}$ be represented in sequent format as

$$\frac{\Gamma_1 \Rightarrow \Delta_1 \quad \cdots \quad \Gamma_k \Rightarrow \Delta_k}{\heartsuit_1 \vec{\boldsymbol{p}}_1, \ldots, \heartsuit_n \vec{\boldsymbol{p}}_n \Rightarrow \heartsuit_{n+1} \vec{\boldsymbol{p}}_{n+1}, \ldots, \heartsuit_{n+m} \vec{\boldsymbol{p}}_{n+m}} .$$

By Assumption 3.3, R is one-step sound. Let $\mathfrak{M} = (C, \gamma, I)$ be a model. To show that $\mathcal{S}(R)$ preserves validity, assume that all of $\Sigma, \Gamma_i \sigma^y_{\boldsymbol{x}} \Rightarrow \Delta_i \sigma^y_{\boldsymbol{x}}, \Theta$ $(1 \leqslant i \leqslant k)$ are valid in $\mathfrak{M}$. Fix any variable assignment v on C. To show that the conclusion of $\mathcal{S}(R)$ is true at M, v, assume that $M, v \models \bigwedge \Sigma$ and $M, v \not\models \bigvee \Theta$. Our goal is to show that

$$M, v \models \bigwedge_{1 \leqslant i \leqslant n} z \heartsuit_i \lceil \boldsymbol{x_i} : \boldsymbol{\phi_i} \rceil \to \bigvee_{1 \leqslant j \leqslant m} z \heartsuit_{n+j} \lceil \boldsymbol{x_{n+j}} : \boldsymbol{\phi_{n+j}} \rceil,$$

i.e.,

$$\begin{array}{c} \text{if } \gamma(v(z)) \in \bigcap_{1 \leqslant i \leqslant n} [\![\heartsuit_i]\!]_{C,v} ([\![\phi^1_i]\!]^{x^1_i}_{C,v}, \ldots, [\![\phi^{\mathsf{ar}\heartsuit_i}_i]\!]^{x^{\mathsf{ar}\heartsuit_i}_i}_{C,v}) \\ \text{then } \gamma(v(z)) \in \bigcup_{1 \leqslant j \leqslant m} [\![\heartsuit_{n+j}]\!]_{C,v} ([\![\phi^1_{n+j}]\!]^{x^1_{n+j}}_{C,v}, \ldots, [\![\phi^{\mathsf{ar}\heartsuit_{n+j}}_{n+j}]\!]^{x^{\mathsf{ar}\heartsuit_{n+j}}_{n+j}}_{C,v}) . \end{array}$$

Let us define a valuation $\tau : \mathsf{sVar} \to \mathcal{P}(C)$ by $\tau(\boldsymbol{p^j_i}) = [\![\phi^j_i [y/x^j_i]]\!]^y_{C,v}$. To show that $C, \tau \models \bigwedge \Gamma_i \to \bigvee \Delta_i$ for all $1 \leqslant i \leqslant k$, let us fix any $c \in C$. Since y is fresh in the conclusion of $\mathcal{S}(R)$, it follows from $M, v \models \bigwedge \Sigma$ and $M, v \not\models \bigvee \Theta$ that $M, v[c/y] \models \bigwedge \Sigma$ and $M, v[c/y] \not\models \bigvee \Theta$. Then from our assumption of the validity of all premises of $\mathcal{S}(R)$ on a pair (M, v), we obtain $M, v[c/y] \models \Gamma_i \sigma^y_{\boldsymbol{x}} \Rightarrow \Delta_i \sigma^y_{\boldsymbol{x}}$, which implies $c \in \tau(\bigwedge \Gamma_i \to \bigvee \Delta_i)$, as desired. Since R is one-step sound, we have that $TC, \tau \models \bigwedge_{1 \leqslant i \leqslant n} \heartsuit_i \vec{\boldsymbol{p}}_i \to \bigvee_{1 \leqslant j \leqslant m} \heartsuit_{n+j} \vec{\boldsymbol{p}}_{n+j}$. Because $\tau(\boldsymbol{p^j_i}) = [\![\phi^j_i [y/x^j_i]]\!]^y_{C,v} = [\![\phi^j_i]\!]^{x^j_i}_{C,v}$ by freshness of y, we can conclude our desired implication above. □

We show soundness and completeness of the sequent system $\mathcal{SR}$ by translating into, and from, the Hilbert system $\mathcal{HR}$ which is known to be (semantically) complete when $\mathcal{R}$ is strongly one-step complete. Note that $\mathcal{HR}$ does not include the the $\mathrm{BDPL}_{\flat\Lambda}$-axioms. Before showing that both systems $\mathcal{HR}$ and $\mathcal{SR}\mathsf{Cut}$ have the same deductive power, we note one consequence of the congruence rule (Remark 3.7) provided that the rules absorb congruence. We introduce the concept of absorption in a slightly more general form which will be used later.

Definition 6.4. We say that a finite set $\mathbb{S}$ of sequents *covers* a finite set $\mathbb{S}'$ of sequents if each element $\Gamma \Rightarrow \Delta$ of $\mathbb{S}'$ contains some element $\Pi \Rightarrow \Sigma$ of $\mathbb{S}$ in the sense that $\Pi \subseteq \Gamma$ and $\Sigma \subseteq \Delta$. We write $\mathbb{S} \rhd \mathbb{S}'$ if $\mathbb{S}$ covers $\mathbb{S}'$ where we identify sequents with singleton sets. A set $\mathcal{R}$ of rules *absorbs* a rule $\Sigma_1 \Rightarrow \Theta_1, \cdots, \Sigma_m \Rightarrow \Theta_m / \Sigma \Rightarrow \Theta$ if there exists a rule $R = \Gamma_1 \Rightarrow \Delta_1, \cdots, \Gamma_n \Rightarrow \Delta_n / \Gamma_R \Rightarrow \Delta_R \in \mathcal{R}$ such that

$$\{\Sigma_1 \Rightarrow \Theta_1, \ldots, \Sigma_m \Rightarrow \Theta_m\} \rhd \{\Gamma_1 \Rightarrow \Delta_1, \ldots, \Gamma_n \Rightarrow \Delta_n\}$$

and $\Gamma_R \Rightarrow \Delta_R \rhd \Sigma \Rightarrow \Theta$. A rule set *absorbs congruence* if it absorbs the rule

$$\frac{p_1 \Rightarrow q_1 \quad \cdots \quad p_n \Rightarrow q_n \quad q_1 \Rightarrow p_1 \quad \cdots \quad q_n \Rightarrow p_n}{\heartsuit(p_1, \ldots, p_n) \Rightarrow \heartsuit(q_1, \ldots, q_n)} \ \mathsf{Cong}\heartsuit$$

and it *absorbs monotonicity of* $\heartsuit$ *in the* i*-th argument* if the rule

$$\frac{p_i \Rightarrow q_i}{\heartsuit(p_1, \ldots, p_n) \Rightarrow \heartsuit(p_1, \ldots, p_{i-1}, q_i, p_{i+1}, \ldots p_n)} \ \mathsf{Mon}_i$$

is absorbed.

Lemma 6.5. *When $\mathcal{R}$ absorbs congruence, the following congruence rule*

$$\frac{\{\Sigma, \phi_0^j[y/x_0^j] \Rightarrow \phi_1^j[y/x_1^j], \Theta \quad \Sigma, \phi_1^j[y/x_1^j] \Rightarrow \phi_0^j[y/x_0^j], \Theta \mid 1 \leqslant j \leqslant n\}}{\Sigma, z\heartsuit\lceil \boldsymbol{x_0} : \boldsymbol{\phi_0}\rceil \Rightarrow z\heartsuit\lceil \boldsymbol{x_1} : \boldsymbol{\phi_1}\rceil, \Theta}\ \mathsf{Cong}\heartsuit$$

(*where y is fresh in the conclusion and n is the arity of $\heartsuit$*) *is admissible in $\mathcal{SR}$ and $\mathcal{SR}\mathsf{Cut}$.*

Proof. Since $\mathcal{R}$ absorbs congruence, we can find a one-step rule $R = \Gamma_1 \Rightarrow \Delta_1, \cdots, \Gamma_m \Rightarrow \Delta_m / \Gamma_R \Rightarrow \Delta_R \in \mathcal{R}$ such that $\{p_j \Rightarrow q_j, q_j \Rightarrow p_j \mid 1 \leqslant j \leqslant n\} \rhd \{\Gamma_1 \Rightarrow \Delta_1, \ldots, \Gamma_m \Rightarrow \Delta_m\}$ and $\Gamma_R \Rightarrow \Delta_R \rhd \heartsuit(p_1, \ldots, p_n) \Rightarrow \heartsuit(q_1, \ldots, q_n)$. Fix such a one-step rule R. Assume that $\Sigma, \phi_0^j[y/x_0^j] \Rightarrow \phi_1^j[y/x_1^j], \Theta$ and $\Sigma, \phi_1^j[y/x_1^j] \Rightarrow \phi_0^j[y/x_0^j], \Theta$ are derivable in $\mathcal{SR}$ for all $1 \leqslant j \leqslant n$. Let us define the substitution $\sigma_{\boldsymbol{x}}^y$ which sends each $\boldsymbol{p}_i^j$ to a formula $\phi_i^j[y/x_i^j]$, where $i = 0$ or 1. Since $\{p_j \Rightarrow q_j, q_j \Rightarrow p_j \mid 1 \leqslant j \leqslant n\} \rhd \{\Gamma_1 \Rightarrow \Delta_1, \ldots, \Gamma_m \Rightarrow \Delta_m\}$, we can obtain the derivability of $\Sigma, \Gamma_k\sigma_{\boldsymbol{x}}^y \Rightarrow \Delta_k\sigma_{\boldsymbol{x}}^y, \Theta$ ($1 \leqslant k \leqslant m$) in $\mathcal{SR}$ with the help of weakening rules. Since $R \in \mathcal{R}$, the covering $\Gamma_R \Rightarrow \Delta_R \rhd \heartsuit(p_1, \ldots, p_n) \Rightarrow \heartsuit(q_1, \ldots, q_n)$ and the weakening rules allow us to obtain the derivability of $\Sigma, z\heartsuit\lceil \boldsymbol{x_0} : \boldsymbol{\phi_0}\rceil \Rightarrow z\heartsuit\lceil \boldsymbol{x_1} : \boldsymbol{\phi_1}\rceil, \Theta$ in $\mathcal{SR}$, as required. □

By our equality rules, the following lemma is immediate.

Lemma 6.6. *The replacement axiom $x = y, \phi[x/z] \Rightarrow \phi[y/z]$ is derivable in $\mathcal{SR}$.*

One direction of the translation between the two proof systems can now be given as follows:

Theorem 6.7. *Suppose that $\mathcal{R}$ absorbs congruence and let $\mathcal{HR} \vdash \phi$. Then $\mathcal{SR}\mathsf{Cut} \vdash \Rightarrow \phi$.*

Proof. First, we demonstrate admissibility of modus ponens in $\mathcal{SR}\mathsf{Cut}$ by

$$\dfrac{\Rightarrow \phi \qquad \dfrac{\Rightarrow \phi \to \psi \quad \phi \to \psi, \phi \Rightarrow \psi}{\phi \Rightarrow \psi}\ \mathsf{Cut}}{\Rightarrow \psi}\ \mathsf{Cut}$$

where the derivability of $\phi \to \psi, \phi \Rightarrow \psi$ is easily established by $\mathsf{L}\to$. Note that this is the only place in this proof where we need Cut. Hence, it suffices to show that all the axioms of $\mathcal{HR}$ (recall Table 3) are derivable in cut-free $\mathcal{SR}$. All of equality axioms En5, En6.1 and En6.2 are derivable by the equality axiom $\mathsf{R}=$ and the equality rules $\mathsf{L}=_{\mathsf{i}}$. Moreover, since this is easy to show for logical but non-modal axioms, we focus on Cong, Alpha and Onestep($\mathcal{R}$). Firstly, the derivability of Cong follows from Lemma 6.5. Secondly, for Alpha we have the following derivation:

$$\frac{\{\phi_j[y/x_j] \Rightarrow \phi_j[y/x_j] \mid j \neq i\} \quad \phi_i[y/x_i] \Rightarrow \phi_i[u/x_i][y/u] \quad \phi_i[u/x_i][y/u] \Rightarrow \phi_i[y/x_i]}{z\heartsuit\lceil x_0 : \phi_0\rceil \cdots \lceil x_i : \phi_i\rceil \cdots \lceil x_n : \phi_n\rceil \Rightarrow z\heartsuit\lceil x_0 : \phi_0\rceil \cdots \lceil u : \phi_i[u/x_i]\rceil \cdots \lceil x_n : \phi_n\rceil}\ \mathsf{Cong}\heartsuit$$

where we note that $\mathsf{Cong}\heartsuit$ is admissible by Lemma 6.5. All the premises are axioms since u is assumed to be fresh in ϕ_i. Finally, let us move to the provability of Onestep($\mathcal{R}$). Suppose that $R = \Gamma_1 \Rightarrow \Delta_1, \ldots, \Gamma_k \Rightarrow \Delta_k / \Gamma_R \Rightarrow \Delta_R$ is a one-step rule as in Definition 3.2. With the help of contraction rules, we note that the following are derivable rules in $\mathcal{SR}$: for any finite multiset Θ,

$$\frac{\Theta, \Gamma \Rightarrow \Delta}{\bigwedge \Theta, \Gamma \Rightarrow \Delta}\ \mathsf{L}\wedge \quad \frac{\Gamma \Rightarrow \Delta, \Theta}{\Gamma \Rightarrow \Delta, \bigvee \Theta}\ \mathsf{R}\vee.$$

We obtain the following derivation where $N = \{1, ..., n\}$, $M = \{n+1, ..., n+m\}$ and π_i is an abbreviation of $(\bigwedge \Gamma_i \to \bigvee \Delta_i)\sigma$:

$$\dfrac{\dfrac{\dfrac{\{\pi_1[y/x] \wedge \cdots \wedge \pi_n[y/x], (\Gamma_i\sigma)[y/x] \Rightarrow (\Delta_i\sigma)[y/x] \mid 1 \leqslant i \leqslant k\}}{\{\forall x.(\pi_1 \wedge \cdots \wedge \pi_n), (\Gamma_i\sigma)[y/x] \Rightarrow (\Delta_i\sigma)[y/x] \mid 1 \leqslant i \leqslant k\}}\ \mathsf{L}\forall}{\forall x.(\pi_1 \wedge \cdots \wedge \pi_n), \{x\heartsuit_i\lceil \boldsymbol{x} : \boldsymbol{\phi_i}\rceil \mid i \in N\} \Rightarrow \{x\heartsuit_i\lceil \boldsymbol{x} : \boldsymbol{\phi_i}\rceil \mid i \in M\}}\ \mathcal{S}(\mathsf{R})}{\forall x.(\pi_1 \wedge \cdots \wedge \pi_n), \bigwedge\{x\heartsuit_i\lceil \boldsymbol{x} : \boldsymbol{\phi_i}\rceil \mid i \in N\} \Rightarrow \bigvee\{x\heartsuit_i\lceil \boldsymbol{x} : \boldsymbol{\phi_i}\rceil \mid i \in M\}}\ \mathsf{L}\wedge, \mathsf{R}\wedge$$

which shows derivability of the axiom OneStep($\mathcal{R}$) as the top sequent is readily seen to be derivable in $\mathcal{SR}$. □

For the converse direction, absorption of congruence is not required.

Theorem 6.8. *Suppose that* $\mathcal{SR}\mathsf{Cut} \vdash \Gamma \Rightarrow \Delta$. *Then* $\mathcal{HR} \vdash \bigwedge \Gamma \to \bigvee \Delta$.

Proof. It suffices to show that all the translations of the axioms and rules of $\mathcal{SR}$ are derivable in $\mathcal{HR}$. We can easily handle the cases of the axioms and rules for logical connectives of first-order logic. The provability of the translation of $\mathsf{L} =_{\mathsf{l}}$ follows from the provability of $x = y \to (\phi[x/w] \to \phi[y/w])$. As for $\heartsuit \in \Lambda$, the provability of the translation of $\mathcal{S}(R)$ follows from OneStep($\mathcal{R}$) and Alpha. □

As a corollary, we obtain (for the time being, in a calculus with cut) both soundness and completeness of the sequent calculus.

Corollary 6.9. *Suppose that* $\mathcal{R}$ *is strongly one-step complete. Then* $\mathcal{SR}\mathsf{Cut} \vdash \Gamma \Rightarrow \Delta$ *iff* $\models \Gamma \Rightarrow \Delta$.

Proof. By Theorems 6.7 and 6.8 in conjunction with soundness and completeness of $\mathcal{HR}$ (Theorem 3.19). The absorption of congruence was shown in [PS10, Proposition 5.12]. □

A paradigmatic example of a set of rules satisfying the assumptions of Corollary 6.9 is C and its CPL translation $\mathcal{S}(\mathsf{C})$ from Example 6.2 above.

As we have seen in § 3, the assumption of *strongly* one-step complete rule sets limits available examples to "essentially neighbourhood-like" ones. This is why we also gave a complete Hilbert-style axiomatisation also for *bounded* operators (recall Definition 3.12).

Note that k-boundedness of i-th argument of Definition 3.12 implies in particular that $\heartsuit$ is monotonic in the i-th argument. Examples of bounded modalities include the standard $\Diamond$ of relational modal logic interpreted over Kripke frames, graded modalities over multigraphs and we refer to [SP10b] for more examples. In the Hilbert-calculus, boundedness was reflected syntactically by the axiom

$$\begin{aligned}\text{BdPL}_{k,i}\ \forall \vec{y}.(x\heartsuit\lceil y_1 : \phi_1\rceil \ldots \lceil y_n : \phi_n\rceil \leftrightarrow \exists z_1 \ldots z_k.(x\heartsuit\lceil y_1 : \phi_1\rceil \ldots \lceil y_{i-1} : \phi_{i-1}\rceil \\ \lceil y_i : y_i = z_1 \vee \cdots \vee y_i = z_k\rceil\lceil y_{i+1} : \phi_{i+1}\rceil \ldots \lceil y_n : \phi_n\rceil \wedge \bigwedge_{j \leqslant k} \phi_i[y_i/z_j]))\end{aligned}$$

where each z_i is fresh for all the y_is and ϕ_is. The derivability predicate induced by extending the Hilbert calculus $\mathcal{HR}$ by the boundedness axiom above gives completeness under weaker conditions.

Definition 6.10. We write $\mathcal{BHR} \vdash \phi$ if ϕ is derivable in $\mathcal{HR}$ where additionally $\text{BdPL}_{k,i}$ is used for every operator that is k-bounded in the i-th argument.

Strictly speaking, the derivability predicate $\mathcal{BHR}$ should include information about precisely which operators are assumed to be k-bounded in the i-th argument, but this will always be clear from the context. In the presence of boundedness, completeness of the Hilbert-calculus has been established under weaker conditions (see Theorem 3.19).
We can reflect boundedness in the sequent calculus by adding a paste rule, similar in spirit to the paste rule of hybrid logic [BdRV01, § 7] which was generalised to a coalgebraic setting in [SP10b]. In a sequent setting, this rule takes the form

$$\frac{\begin{array}{c}\Gamma, x\heartsuit\lceil x_1 : \phi_1\rceil \cdots \lceil x_{i-1} : \phi_{i-1}\rceil\lceil y : \bigvee_{1\leqslant j\leqslant k} y = z_j\rceil\lceil x_{i+1} : \phi_{i+1}\rceil \cdots \lceil x_n : \phi_n\rceil, \\ \phi[z_1/y], ..., \phi[z_k/y] \Rightarrow \Delta \qquad z_1, \dots, z_k \text{ fresh}\end{array}}{\Gamma, x\heartsuit\lceil x_1 : \phi_1\rceil \cdots \lceil x_{i-1} : \phi_{i-1}\rceil\lceil y : \phi\rceil\lceil x_{i+1} : \phi_{i+1}\rceil \cdots \lceil x_n : \phi_n\rceil \Rightarrow \Delta}\ \mathsf{Paste}_i^k,$$

where z_1, ..., z_k are pairwise distinct fresh variables. Additional use of the above paste-rule in the system $\mathcal{SR}$ is denoted by $\mathcal{BSR}$, that is, we write $\mathcal{BSR} \vdash \Gamma \Rightarrow \Delta$ if $\Gamma \Rightarrow \Delta$ is derivable in $\mathcal{SR}$ where Paste_i^k may additionally be applied for every modality that is k-bounded in the i-th argument.

When $\mathcal{R}$ absorbs congruence and monotonicity of all operators that are k-bounded in the i-th argument, we note that Lemmas 6.5 and 6.6 hold also for $\mathcal{BSR}$.

Theorem 6.11. *Suppose that $\mathcal{R}$ absorbs congruence and monotonicity in the i-th argument of every operator that is k-bounded in the i-th argument. Then $\mathcal{BHR} \vdash \phi$ implies that $\mathcal{BSR}\mathsf{Cut} \vdash \Rightarrow \phi$.*

Proof. First of all, if $\mathcal{R}$ absorbs monotonicity in the i-th argument of $\heartsuit \in \Lambda$, the rule

$$\frac{\Sigma, \phi_i[y/x_i] \Rightarrow \psi[y/x], \Theta}{\Sigma, z\heartsuit\lceil \boldsymbol{x} : \boldsymbol{\phi}\rceil \Rightarrow z\heartsuit\lceil x_1 : \phi_1\rceil \dots \lceil x_{i-1} : \phi_{i-1}\rceil\lceil x_i : \psi\rceil\lceil x_{i+1} : \phi_{i+1}\rceil \dots \lceil x_n : \phi_n\rceil, \Theta}\ \mathsf{Mon}_i$$

(where y is fresh in the conclusion) is admissible in $\mathcal{BSR}$ (and $\mathcal{BSR}\mathsf{Cut}$). Almost all the arguments are the same as the proof of Theorem 6.7, except that we need to show the provability of BdPL by Paste (note that the only place we need the cut rule is the derivability of Modus Ponens). More precisely, we can show the left-to-right implication of BdPL by means of Paste_i^k and Mon_i gives the reverse direction. For example, when $\heartsuit$ is unary and 1-bounded, the derivability of the right-to-left direction of BdPL is demonstrated as follows.

$$\cfrac{\cfrac{\cfrac{v = w, \phi[w/y] \Rightarrow \phi[v/y]}{x\heartsuit\lceil y : y = w\rceil, \phi[w/y] \Rightarrow x\heartsuit\lceil y : \phi\rceil}\ \mathsf{Mon}}{x\heartsuit\lceil y : y = w\rceil \wedge \phi[w/y]) \Rightarrow x\heartsuit\lceil y : \phi\rceil}\ \mathsf{L}\wedge}{\exists z.(x\heartsuit\lceil y : y = z\rceil \wedge \phi[z/y]) \Rightarrow x\heartsuit\lceil y : \phi\rceil}\ \mathsf{L}\exists,$$

where the top sequent is the replacement axiom, which is derivable by Lemma 6.6. □

The reverse direction of Theorem 6.11 is established analogously to Theorem 6.8 and again absorption properties are not needed.

Theorem 6.12. $\mathcal{BSR}\mathsf{Cut} \vdash \Gamma \Rightarrow \Delta$ *only if* $\mathcal{BHR} \vdash \bigwedge \Gamma \to \bigvee \Delta$.

Proof. The only difference from the proof of Theorem 6.7 is to need to care about the translation of Paste. However, we can easily establish this by the axiom BDLP. □

As in the non-bounded case we obtain semantic soundness and completeness, but under weaker coherence conditions.

Corollary 6.13. *Suppose that $\mathcal{R}$ is strongly finitary one-step complete. Then $\mathcal{BSR}\mathsf{Cut} \vdash \Gamma \Rightarrow \Delta$ iff $\models \Gamma \Rightarrow \Delta$.*

Proof. By Theorems 6.11 and 6.12 in conjunction with soundness and completeness of $\mathcal{BHR}$ (Theorem 3.19). Note that absorption of congruence and monotonicity follows from (strong, finitary) one-step completeness as in [PS10, Proposition 5.12]. □

A canonical example of a rule set satisfying the assumptions of the above corollary can be obtained by taking K of Example 6.2 and extending it with Paste_i^k for $i = k = n = 1$.

6.2. **Admissibility of Cut.** When we try to prove the admissibility of Cut in first-order logic (or $\mathcal{SR}$), we encounter difficulties with the rules of contraction. That is, the following derivation:

$$\mathcal{D} = \dfrac{\dfrac{\overset{\mathcal{D}'}{\Gamma \Rightarrow \Delta, \phi, \phi}}{\Gamma \Rightarrow \Delta, \phi}\,\mathsf{RC} \qquad \overset{\mathcal{D}''}{\phi, \Sigma \Rightarrow \Theta}}{\Gamma, \Sigma \Rightarrow \Delta, \Theta}\,\mathsf{Cut},$$

may be transformed into:

$$\dfrac{\dfrac{\dfrac{\overset{\mathcal{D}'}{\Gamma \Rightarrow \Delta, \phi, \phi} \quad \overset{\mathcal{D}''}{\phi, \Sigma \Rightarrow \Theta}}{\Gamma, \Sigma \Rightarrow \Delta, \Theta, \phi}\,\mathsf{Cut} \qquad \overset{\mathcal{D}''}{\phi, \Sigma \Rightarrow \Theta}}{\Gamma, \Sigma, \Sigma \Rightarrow \Delta, \Theta, \Theta}\,\mathsf{Cut}}{\Gamma, \Sigma \Rightarrow \Delta, \Theta}\,\mathsf{LC}, \mathsf{RC},$$

but this derivation does not provide us with a reduction in terms of the number of sequents above the application of Cut in $\mathcal{D}$. This is why Gentzen introduced the following generalized form of Cut:

$$\dfrac{\Gamma \Rightarrow \Delta, \phi^m \quad \phi^n, \Sigma \Rightarrow \Theta}{\Gamma, \Sigma \Rightarrow \Delta, \Theta}\,\mathsf{Mcut}$$

where n, $m \geqslant 1$ and ϕ^k stands for k copies of ϕ and "*Mcut*" is a shorthand of "multi-cut" (sometimes also called "mix"). Since Cut is a special case of the new rule of Mcut, it suffices for us to prove the admissibility of Mcut in a given sequent system to obtain the admissibility of Cut in the system.

Moreover, we note that *a priori* we cannot expect that an application of Mcut between two instances of modal rules can be moved up without changing the conclusion: the set $\mathcal{R}$ of one-step rules can possibly consist of a single rule, and an application of Mcut between this rule and itself may not be derivable. We therefore need to impose an additional requirement to deal with this case.

Definition 6.14. Let $\mathbb{S}$ be a finite set of sequents. The set of all sequents that can be derived from premises in $\mathbb{S}$ using (only) *one application* of Mcut is denoted by $\mathsf{MCut}(\mathbb{S})$. A rule set $\mathcal{R}$ *absorbs multicut*, if for all pairs (R_1, R_2) of rules in $\mathcal{R}$:

$$\dfrac{\Gamma_{11} \Rightarrow \Delta_{11} \quad \cdots \quad \Gamma_{1r_1} \Rightarrow \Delta_{1r_1}}{\Gamma_{R_1} \Rightarrow \Delta_{R_1}, (\heartsuit\vec{\boldsymbol{p}})^m}\,R_1 \quad \dfrac{\Gamma_{21} \Rightarrow \Delta_{21} \quad \cdots \quad \Gamma_{2r_2} \Rightarrow \Delta_{2r_2}}{(\heartsuit\vec{\boldsymbol{p}})^n, \Gamma_{R_2} \Rightarrow \Delta_{R_2}}\,R_2$$

there is a rule $R = \Gamma_1 \Rightarrow \Delta_1, \cdots, \Gamma_k \Rightarrow \Delta_k / \Gamma_R \Rightarrow \Delta_R \in \mathcal{R}$ such that:

$\mathsf{MCut}(\Gamma_{11} \Rightarrow \Delta_{11}, \ldots, \Gamma_{1r_1} \Rightarrow \Delta_{1r_1}, \Gamma_{21} \Rightarrow \Delta_{21}, \ldots, \Gamma_{2r_2} \Rightarrow \Delta_{2r_2}) \rhd \{\Gamma_1 \Rightarrow \Delta_1, \ldots, \Gamma_k \Rightarrow \Delta_k\}$

and $\Gamma_R \Rightarrow \Delta_R \rhd \Gamma_{R_1}, \Gamma_{R_2} \Rightarrow \Delta_{R_1}, \Delta_{R_2}$.

Lemma 6.15. *If $\mathcal{SR} \vdash \Gamma \Rightarrow \Delta$ and y is fresh in Γ and Δ, then $\mathcal{SR} \vdash \Gamma[y/x] \Rightarrow \Delta[y/x]$ with the same height of derivation.*

Lemma 6.16 (Hauptsatz)**.** *Let $\mathcal{D}$ be a derivation in the system $\mathcal{SR}$ extended with* Mcut *in the following form:*

$$\dfrac{\begin{matrix}\mathcal{D}_L \\ \Gamma \Rightarrow \Delta, \phi^m\end{matrix} \quad \begin{matrix}\mathcal{D}_R \\ \phi^n, \Sigma \Rightarrow \Theta\end{matrix}}{\Gamma, \Sigma \Rightarrow \Delta, \Theta}\ \mathsf{Mcut},$$

where $\mathcal{D}_L$ and $\mathcal{D}_R$ contain no application of Mcut*, the last rule of $\mathcal{D}$ is the only application of* Mcut *in $\mathcal{D}$. Then $\Gamma, \Sigma \Rightarrow \Delta, \Theta$ is derivable in $\mathcal{SR}$.*

Proof. First of all, we introduce some terminology used only in this proof. Let $\mathcal{D}$ be the derivation in question. We say that ϕ is a *cut formula* of $\mathcal{D}$, and we define the complexity $\mathtt{c}(\mathcal{D})$ as the complexity of the cut formula ϕ, i.e., the length or the number of connectives including the logical and modal connectives. Moreover, we define $\mathtt{w}(\mathcal{D})$ as the total number of sequents in $\mathcal{D}_\mathtt{L}$ and $\mathcal{D}_\mathtt{R}$. Our proof of the statement of the claim is shown by the double induction on $(\mathtt{c}(\mathcal{D}), \mathtt{w}(\mathcal{D}))$ (note that $\mathtt{c}(\mathcal{D}) \geqslant 0$ and $\mathtt{w}(\mathcal{D}) \geqslant 2$). Let us denote the last applied rule (or axiom, possibly) of a derivation $\mathcal{E}$ by $\mathtt{rule}(\mathcal{E})$. We divide our argument into the following (exhaustive) cases:

(1) One of $\mathtt{rule}(\mathcal{D}_\mathtt{L})$ and $\mathtt{rule}(\mathcal{D}_\mathtt{R})$ is an axiom.
(2) One of $\mathtt{rule}(\mathcal{D}_\mathtt{L})$ and $\mathtt{rule}(\mathcal{D}_\mathtt{R})$ is a structural rule.
(3) One of $\mathtt{rule}(\mathcal{D}_\mathtt{L})$ and $\mathtt{rule}(\mathcal{D}_\mathtt{R})$ is a logical rule or a modal rule and the cut formula is not principal in the rule.
(4) Both $\mathtt{rule}(\mathcal{D}_\mathtt{L})$ and $\mathtt{rule}(\mathcal{D}_\mathtt{R})$ are logical rules for the same logical connective and the cut formula is principal in each of the rules.
(5) Both $\mathtt{rule}(\mathcal{D}_\mathtt{L})$ and $\mathtt{rule}(\mathcal{D}_\mathtt{R})$ are modal rules and the cut formula is principal in each of the rules.
(6) One of $\mathtt{rule}(\mathcal{D}_\mathtt{L})$ and $\mathtt{rule}(\mathcal{D}_\mathtt{R})$ is an equality rule.

Let us check each case one by one.

(1) One of $\mathtt{rule}(\mathcal{D}_\mathtt{L})$ and $\mathtt{rule}(\mathcal{D}_\mathtt{R})$ is an axiom: We have four cases since it is impossible that $\mathtt{rule}(\mathcal{D}_\mathtt{L})$ is $\mathsf{L}\bot$ or $\mathtt{rule}(\mathcal{D}_\mathtt{L})$ is $\mathsf{R}=$. Firstly, when $\mathtt{rule}(\mathcal{D}_\mathtt{L})$ is Ax, let the derivation be

$$\dfrac{\dfrac{}{\phi \Rightarrow \phi}\ \mathsf{Ax} \quad \begin{matrix}\mathcal{D}_\mathtt{R} \\ \phi^n, \Sigma \Rightarrow \Theta\end{matrix}}{\phi, \Sigma \Rightarrow \Theta}\ \mathsf{Mcut}.$$

When $n = 1$, we already obtain the derivability of $\phi, \Sigma \Rightarrow \Theta$ in $\mathcal{SR}$. When $n \geqslant 2$, $\phi, \Sigma \Rightarrow \Theta$ is obtained from $\phi^n, \Sigma \Rightarrow \Theta$ by finitely many applications of LC.

Secondly, when $\mathtt{rule}(\mathcal{D}_\mathtt{R})$ is Ax, the argument is similar to the previous case where $\mathtt{rule}(\mathcal{D}_\mathtt{L})$ is Ax.

Thirdly, when $\texttt{rule}(\mathcal{D}_{\mathtt{L}})$ is $\mathsf{R}=$, we need to look at what the last rule $\texttt{rule}(\mathcal{D}_{\mathtt{R}})$ is, where $\mathcal{D}$ is of the following form:

$$\dfrac{\dfrac{}{\Rightarrow x = x}\,\mathsf{R}= \qquad \begin{array}{c}\mathcal{D}_{\mathtt{R}}\\ (x=x)^n, \Sigma \Rightarrow \Theta\end{array}}{\Sigma \Rightarrow \Theta}\,\mathsf{Mcut}.$$

If $\texttt{rule}(\mathcal{D}_{\mathtt{R}})$ is an axiom, then it should be Ax and we have already checked this case in our second case of this item. Otherwise, $\texttt{rule}(\mathcal{D}_{\mathtt{R}})$ is a structural rule, a logical rule, a modal rule or an equality rule. These cases will be discussed below (especially (2), (3) and (6)), so we leave them out for now. It is, however, noted that the cut formula $x = x$ is not principal in the case (3).

Fourthly, when $\texttt{rule}(\mathcal{D}_{\mathtt{R}})$ is $\mathsf{L}\bot$, then we need to look at the last rule $\texttt{rule}(\mathcal{D}_{\mathtt{L}})$, where $\mathcal{D}$ is

$$\dfrac{\begin{array}{c}\mathcal{D}_{\mathtt{L}}\\ \Gamma \Rightarrow \Delta, \bot^m\end{array} \qquad \dfrac{}{\bot \Rightarrow}\,\mathsf{L}\bot}{\Gamma \Rightarrow \Delta}\,\mathsf{Mcut}.$$

If $\texttt{rule}(\mathcal{D}_{\mathtt{L}})$ is an axiom, it should be Ax and we have already checked such case in the first case of this item. Otherwise, $\texttt{rule}(\mathcal{D}_{\mathtt{L}})$ must be a structural rule, a logical rule, an modal rule or an equality rule. Again these cases will be discussed below (especially (2), (3) and (6), where it is noted that the cut formula $\bot$ is not principal in the case (3)), so we leave them out for now.

(2) One of $\texttt{rule}(\mathcal{D}_{\mathtt{L}})$ and $\texttt{rule}(\mathcal{D}_{\mathtt{R}})$ is a structural rule: all arguments for this case are standard, so we deal only with the case where $\texttt{rule}(\mathcal{D}_{\mathtt{L}})$ is RC, i.e., $\mathcal{D}$ is of the following form:

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{D}'_{\mathtt{L}}\\ \Gamma \Rightarrow \Theta, \phi^{m+1}\end{array}}{\Gamma \Rightarrow \Delta, \phi^m}\,\mathsf{RC} \qquad \begin{array}{c}\mathcal{D}_{\mathtt{R}}\\ \phi^n, \Sigma \Rightarrow \Theta\end{array}}{\Gamma, \Sigma \Rightarrow \Delta, \Theta}\,\mathsf{Mcut},$$

since multicut plays an essential role. This derivation is transformed into:

$$\dfrac{\begin{array}{c}\mathcal{D}'_{\mathtt{L}}\\ \Gamma \Rightarrow \Delta, \phi^{m+1}\end{array} \qquad \begin{array}{c}\mathcal{D}_{\mathtt{R}}\\ \phi^n, \Sigma \Rightarrow \Theta\end{array}}{\Gamma, \Sigma \Rightarrow \Delta, \Theta}\,\mathsf{Mcut}$$

where the application of Mcut is eliminable since the complexity of the derivation is the same as $\mathtt{c}(\mathcal{D})$ and the weight of the derivation is smaller than $\mathtt{w}(\mathcal{D})$.

(3) One of $\texttt{rule}(\mathcal{D}_{\mathtt{L}})$ and $\texttt{rule}(\mathcal{D}_{\mathtt{R}})$ is a logical rule or a modal rule and the cut formula is not principal in the rule: Our argument for logical rules are standard, so we focus on the case where one of the rules is a modal rule $\mathcal{S}(R)$. Let $\texttt{rule}(\mathcal{D}_{\mathtt{L}})$ is $\mathcal{S}(R)$. Then our derivation $\mathcal{D}$ is of the following form:

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{D}_{\mathtt{L}_1}\\ \Sigma', (\Gamma_1\sigma)[y/x] \Rightarrow (\Delta_1\sigma)[y/x], \Theta', \phi^m\end{array} \cdots \begin{array}{c}\mathcal{D}_{\mathtt{L}_k}\\ \Sigma', (\Gamma_k\sigma)[y/x] \Rightarrow (\Delta_k\sigma)[y/x], \Theta', \phi^m\end{array}}{\begin{array}{c}\Sigma', z\heartsuit_1\lceil \boldsymbol{x}\,\boldsymbol{:}\,\boldsymbol{\phi_1}\rceil, \ldots, z\heartsuit_n\lceil \boldsymbol{x}\,\boldsymbol{:}\,\boldsymbol{\phi_n}\rceil \Rightarrow\\ z\heartsuit_{n+1}\lceil \boldsymbol{x}\,\boldsymbol{:}\,\boldsymbol{\phi_{n+1}}\rceil, \ldots, z\heartsuit_{n+m}\lceil \boldsymbol{x}\,\boldsymbol{:}\,\boldsymbol{\phi_{n+m}}\rceil, \Theta', \phi^m\end{array}}\,\mathcal{S}(R)\dagger_y \qquad \begin{array}{c}\mathcal{D}_{\mathtt{R}}\\ \phi^n, \Sigma \Rightarrow \Theta\end{array}}{\begin{array}{c}\Sigma, \Sigma', z\heartsuit_1\lceil \boldsymbol{x}\,\boldsymbol{:}\,\boldsymbol{\phi_1}\rceil, \ldots, z\heartsuit_n\lceil \boldsymbol{x}\,\boldsymbol{:}\,\boldsymbol{\phi_n}\rceil \Rightarrow\\ z\heartsuit_{n+1}\lceil \boldsymbol{x}\,\boldsymbol{:}\,\boldsymbol{\phi_{n+1}}\rceil, \ldots, z\heartsuit_{n+m}\lceil \boldsymbol{x}\,\boldsymbol{:}\,\boldsymbol{\phi_{n+m}}\rceil, \Theta', \Theta\end{array}}\,\mathsf{Mcut}$$

where $\dagger_y$ in the application of $\mathcal{S}(R)$ means that y is fresh in the conclusion. For each $\mathcal{D}_{\mathtt{L}_i}$, we apply height-preserving substitution $[z/y]$ for a fresh variable z in the conclusion of $\mathcal{D}$ and we obtain the following derivation:

$$\dfrac{\overset{\mathcal{D}_{\mathtt{L}_i}[z/y]}{\Sigma', (\Gamma_i\sigma)[z/x] \Rightarrow (\Delta_i\sigma)[z/x], \Theta', \phi^m} \quad \overset{\mathcal{D}_{\mathtt{R}}}{\phi^n, \Sigma \Rightarrow \Theta}}{\Sigma, \Sigma', (\Gamma_i\sigma)[z/x] \Rightarrow (\Delta_i\sigma)[z/x], \Theta', \Theta}\ \mathsf{Mcut}.$$

We can eliminate the last application of Mcut since the complexity of the derivation is the same as $\mathtt{c}(\mathcal{D})$ and the weight of the derivation is smaller than $\mathtt{w}(\mathcal{D})$. Finally we apply the same rule $\mathcal{S}(R)$ to obtain the desired conclusion. When $\mathtt{rule}(\mathcal{D}_{\mathtt{R}})$ be $\mathcal{S}(R)$, the argument is similar to the case just discussed.

(4) Both $\mathtt{rule}(\mathcal{D}_{\mathtt{L}})$ and $\mathtt{rule}(\mathcal{D}_{\mathtt{R}})$ are logical rules for the same logical connective and the cut formula is principal in each of the rules: We have two cases, i.e., two cases where the cut formula is of the form $\phi \to \psi$ or of the form $\forall x.\phi$. Here we only deal with the case where the cut formula is of the form $\forall x.\phi$. Then the derivation $\mathcal{D}$ is of the following form:

$$\dfrac{\dfrac{\overset{\mathcal{D}'_{\mathtt{L}}}{\Gamma \Rightarrow \Delta, (\forall x.\phi)^{m-1}, \phi[y/x]}}{\Gamma \Rightarrow \Delta, (\forall x.\phi)^m}\ \mathsf{R}\forall\dagger_y \quad \dfrac{\overset{\mathcal{D}'_{\mathtt{R}}}{\phi[z/x], (\forall x.\phi)^{n-1}, \Sigma \Rightarrow \Theta}}{(\forall x.\phi)^n, \Sigma \Rightarrow \Theta}\ \mathsf{L}\forall}{\Gamma, \Sigma \Rightarrow \Delta, \Theta}\ \mathsf{Mcut}.$$

With the help of our height-preserving substitution, we can consider a multicut between $\mathcal{D}'_{\mathtt{L}}$ and $\mathcal{D}_{\mathtt{R}}$:

$$\dfrac{\overset{\mathcal{D}'_{\mathtt{L}}[z/y]}{\Gamma \Rightarrow \Delta, (\forall x.\phi)^{m-1}, \phi[z/x]} \quad \overset{\mathcal{D}_{\mathtt{R}}}{(\forall x.\phi)^n, \Sigma \Rightarrow \Theta}}{\Gamma, \Sigma \Rightarrow \Delta, \Theta, \phi[z/x]}\ \mathsf{Mcut},$$

and then by induction hypothesis (the complexity of this derivation is the same as $\mathcal{D}$ but the weight is smaller than the original $\mathcal{D}$) we now know that $\Gamma, \Sigma \Rightarrow \Delta, \Theta, \phi[z/x]$ is derivable in $\mathcal{SR}$ without multicuts by a derivation $\mathcal{E}_1$. Let us also consider a multicut between $\mathcal{D}_{\mathtt{L}}$ and $\mathcal{D}'_{\mathtt{R}}$:

$$\dfrac{\overset{\mathcal{D}_{\mathtt{L}}}{\Gamma \Rightarrow \Delta, (\forall x.\phi)^m} \quad \overset{\mathcal{D}'_{\mathtt{R}}}{\phi[z/x], (\forall x.\phi)^{n-1}, \Sigma \Rightarrow \Theta}}{\phi[z/x], \Gamma, \Sigma \Rightarrow \Delta, \Theta}\ \mathsf{Mcut},$$

and then by induction hypothesis (the complexity of this derivation is the same as $\mathcal{D}$ but the weight is smaller than the original $\mathcal{D}$) we now know that $\phi[z/x], \Gamma, \Sigma \Rightarrow \Delta, \Theta$ is derivable in $\mathcal{SR}$ without multicuts by a derivation $\mathcal{E}_2$. Now let us take a cut between $\mathcal{E}_1$ and $\mathcal{E}_2$:

$$\dfrac{\overset{\mathcal{E}_1}{\Gamma, \Sigma \Rightarrow \Delta, \Theta, \phi[z/x]} \quad \overset{\mathcal{E}_2}{\phi[z/x], \Gamma, \Sigma \Rightarrow \Delta, \Theta}}{\Gamma, \Gamma, \Sigma, \Sigma \Rightarrow \Delta, \Delta, \Theta, \Theta}\ \mathsf{Mcut}$$

and the conclusion of this derivation is derivable in $\mathcal{SR}$ without multicuts by induction hypothesis because the complexity of this derivation (i.e., the length of $\phi[z/x]$) is strictly smaller than $\mathtt{c}(\mathcal{D})$. Finally, finitely many applications of contraction rules enables us to obtain the derivability of $\Gamma, \Sigma \Rightarrow \Delta, \Theta$ in $\mathcal{SR}$, as desired.

(5) Both $\mathtt{rule}(\mathcal{D}_{\mathtt{L}})$ and $\mathtt{rule}(\mathcal{D}_{\mathtt{R}})$ are modal rules and the cut formula is principal in each of the rules: Let $\mathtt{rule}(\mathcal{D}_{\mathtt{L}}) = \mathcal{S}(R_1)$ and $\mathtt{rule}(\mathcal{D}_{\mathtt{R}}) = \mathcal{S}(R_2)$ where we can assume:

$$R_1 = \dfrac{\Gamma_{11} \Rightarrow \Delta_{11} \quad \cdots \quad \Gamma_{1k} \Rightarrow \Delta_{1k}}{\heartsuit_1\vec{\boldsymbol{p}}_1, \ldots, \heartsuit_a\vec{\boldsymbol{p}}_a \Rightarrow \heartsuit_{a+1}\vec{\boldsymbol{q}}_1, \ldots, \heartsuit_{a+b}\vec{\boldsymbol{q}}_b, (\heartsuit\vec{\boldsymbol{p}})^n},$$

$$R_2 = \frac{\Gamma_{21} \Rightarrow \Delta_{21} \quad \cdots \quad \Gamma_{2l} \Rightarrow \Delta_{2l}}{(\heartsuit\vec{\boldsymbol{p}})^m, \spadesuit_1\vec{\boldsymbol{p}}'_1, \ldots, \spadesuit_c\vec{\boldsymbol{p}}'_c \Rightarrow \spadesuit_{c+1}\vec{\boldsymbol{q}}'_1, \ldots, \spadesuit_{c+d}\vec{\boldsymbol{q}}'_d},$$

because the cut formula is principal in both rules. In what follows, we assume that all of $\vec{\boldsymbol{p}}_i$, $\vec{\boldsymbol{q}}_j$, $\vec{\boldsymbol{p}}'_i$, $\vec{\boldsymbol{q}}'_j$ are distinct. So $\mathcal{D}_{\mathtt{L}}$ is of the following form:

$$\dfrac{\begin{matrix}\mathcal{D}'_{\mathtt{L}_1} & & \mathcal{D}'_{\mathtt{L}_k}\\ \Sigma_1, \Gamma_{11}\sigma^{y_1}_{\boldsymbol{x}} \Rightarrow \Delta_{11}\sigma^{y_1}_{\boldsymbol{x}}, \Theta_1 & \cdots & \Sigma_1, \Gamma_{1k}\sigma^{y_1}_{\boldsymbol{x}} \Rightarrow \Delta_{1k}\sigma^{y_1}_{\boldsymbol{x}}, \Theta_1\end{matrix}}{\begin{matrix}\Sigma_1, z\heartsuit_1\lceil \boldsymbol{x_1} : \boldsymbol{\phi_1}\rceil, \ldots, z\heartsuit_a\lceil \boldsymbol{x_a} : \boldsymbol{\phi_a}\rceil \Rightarrow \\ z\heartsuit_{a+1}\lceil \boldsymbol{x_{a+1}} : \boldsymbol{\phi_{a+1}}\rceil, \ldots, z\heartsuit_{a+b}\lceil \boldsymbol{x_{a+b}} : \boldsymbol{\phi_{a+b}}\rceil, (z\heartsuit\lceil \boldsymbol{x} : \boldsymbol{\phi}\rceil)^m, \Theta_1\end{matrix}}\ \mathcal{S}(R_1)$$

and $\mathcal{D}_{\mathtt{R}}$ is of the following form:

$$\dfrac{\begin{matrix}\mathcal{D}'_{\mathtt{R}_1} & & \mathcal{D}'_{\mathtt{R}_l}\\ \Sigma_2, \Gamma_{21}\tau^{y_2}_{\boldsymbol{x}} \Rightarrow \Delta_{21}\tau^{y_2}_{\boldsymbol{x}}, \Theta_2 & \cdots & \Sigma_2, \Gamma_{2l}\tau^{y_2}_{\boldsymbol{x}} \Rightarrow \Delta_{2l}\tau^{y_2}_{\boldsymbol{x}}, \Theta_2\end{matrix}}{\begin{matrix}\Sigma_2, (z\heartsuit\lceil \boldsymbol{x} : \boldsymbol{\phi}\rceil)^n, z\spadesuit_1\lceil \boldsymbol{x_1} : \boldsymbol{\psi_1}\rceil, \ldots, z\spadesuit_c\lceil \boldsymbol{x_c} : \boldsymbol{\psi_c}\rceil \Rightarrow \\ z\spadesuit_{c+1}\lceil \boldsymbol{x_{c+1}} : \boldsymbol{\psi_{c+1}}\rceil, \ldots, z\spadesuit_{c+d}\lceil \boldsymbol{x_{c+d}} : \boldsymbol{\psi_{c+d}}\rceil, \Theta_2\end{matrix}}\ \mathcal{S}(R_2).$$

We also note that the conclusion of $\mathcal{D}$ is:

$$\Sigma_1, \Sigma_2, \{z\heartsuit_i\lceil \boldsymbol{x_i} : \boldsymbol{\phi_i}\rceil\}_{1 \leqslant i \leqslant a}, \{z\spadesuit_j\lceil \boldsymbol{x_j} : \boldsymbol{\psi_j}\rceil\}_{1 \leqslant j \leqslant c} \Rightarrow \\ \{z\heartsuit_{a+i}\lceil \boldsymbol{x_{a+i}} : \boldsymbol{\phi_{a+i}}\rceil\}_{1 \leqslant i \leqslant b}, \{z\spadesuit_{c+j}\lceil \boldsymbol{x_{c+j}} : \boldsymbol{\psi_{c+j}}\rceil\}_{1 \leqslant j \leqslant d}, \Theta_1, \Theta_2.$$

Let y be a fresh variable not occurring in this conclusion. By height-preserving substitution, we can obtain derivations $\mathcal{D}'_{\mathtt{L}_i}[z/y_1]$ and $\mathcal{D}'_{\mathtt{R}_j}[z/y_2]$ ($1 \leqslant i \leqslant k$ and $1 \leqslant j \leqslant l$). Since $\mathcal{R}$ absorbs multicut, we can find a rule $R = \Gamma_1 \Rightarrow \Delta_1, \cdots, \Gamma_e \Rightarrow \Delta_e / \Gamma_R \Rightarrow \Delta_R \in \mathcal{R}$ such that

($*_1$) $\mathsf{MCut}(\{\Gamma_{1i} \Rightarrow \Delta_{1i}\}_{1 \leqslant i \leqslant k}, \{\Gamma_{2j} \Rightarrow \Delta_{2j}\}_{1 \leqslant j \leqslant l}) \rhd \{\Gamma_1 \Rightarrow \Delta_1, \ldots, \Gamma_e \Rightarrow \Delta_e\}$ and

($*_2$) $\Gamma_R \Rightarrow \Delta_R \rhd \{\heartsuit_i\vec{\boldsymbol{p}}_i\}_{1 \leqslant i \leqslant a}, \{\spadesuit_j\vec{\boldsymbol{p}}'_j\}_{1 \leqslant j \leqslant c} \Rightarrow \{\heartsuit_{a+i}\vec{\boldsymbol{q}}_i\}_{1 \leqslant i \leqslant b}, \{\spadesuit_{c+j}\vec{\boldsymbol{q}}'_j\}_{1 \leqslant j \leqslant d}$.

By the clause ($*_1$) and our derivations $\mathcal{D}'_{\mathtt{L}_i}[z/y_1]$, $\mathcal{D}'_{\mathtt{R}_j}[z/y_2]$, we now use the induction hypothesis (the complexity is the same but the weight becomes smaller than that of $\mathcal{D}$) and weakening rules to obtain the derivability in $\mathcal{SR}$ (without multicuts) of

$$\Gamma_i\sigma \Rightarrow \Delta_i\sigma \quad (1 \leqslant i \leqslant e)$$

where σ is a substitution which is the union of $\sigma^{y_1}_{\boldsymbol{x}}$ and $\tau^{y_2}_{\boldsymbol{x}}$. It follows from the rule $\mathcal{S}(R)$, the clause ($*_2$) and weakening rules that the conclusion of $\mathcal{D}$ is derivable in $\mathcal{SR}$ without multicuts, as desired.

(6) One of `rule`($\mathcal{D}_{\mathtt{L}}$) and `rule`($\mathcal{D}_{\mathtt{R}}$) is an equality rule: There are three cases that we need to consider. In the first case, `rule`($\mathcal{D}_{\mathtt{R}}$) is $\mathsf{L} =_{\mathsf{i}}$ where at least one occurrence of the cut formulas is not principal in $\mathsf{L} =_{\mathsf{i}}$ and so the cut formula is of the form $x = y$. In the second case, `rule`($\mathcal{D}_{\mathtt{L}}$) is $\mathsf{L} =_{\mathsf{i}}$ but all occurrences of the cut formula are principal. In the third case, `rule`($\mathcal{D}_{\mathtt{R}}$) is $\mathsf{L} =_{\mathsf{i}}$ and all occurrences of the cut formula are principal. Since our argument for the third case is almost similar to the one for the second case, we focus on the first and the second cases in what follows.

Firstly, consider the case when `rule`($\mathcal{D}_{\mathtt{R}}$) is $\mathsf{L} =_{\mathsf{i}}$ and at least one occurrence of the cut formulas is not principal in $\mathsf{L} =_{\mathsf{i}}$. Without loss of generality, we assume that $i = 1$.

Then our derivation $\mathcal{D}$ is of the following form:

$$\dfrac{\dfrac{\mathcal{D}_{\mathtt{L}}}{\Gamma \Rightarrow \Delta, (x=y)^m} \quad \dfrac{\dfrac{\mathcal{D}'_{\mathtt{R}}}{x=y, \phi'_1[x/w], \ldots, \phi'_{n-1}[x/w], \Sigma'[x/w] \Rightarrow \Theta'[x/w]}}{x=y, \phi'_1[y/w], \ldots, \phi'_{n-1}[y/w], \Sigma'[y/w] \Rightarrow \Theta'[y/w]}\ \mathsf{L}=_1}{\Gamma, \Sigma \Rightarrow \Delta, \Theta}\ \mathsf{Mcut}$$

where $\Sigma'[y/w] = \Sigma$, $\Theta'[y/w] = \Theta$, $\phi'_i[y/w]$ is $x = y$ and so $x = y, \phi'_1[y/w], \ldots, \phi'_{n-1}[y/w]$ is the same as $(x = y)^n$. In this case we need to check what is the last rule $\mathtt{rule}(\mathcal{D}_{\mathtt{L}})$. If $\mathtt{rule}(\mathcal{D}_{\mathtt{L}})$ is Ax (it cannot be $\mathsf{L}\bot$), or a structural rule, or a logical or modal rule, we can use the same argument in the items (1), (2), (3). If $\mathtt{rule}(\mathcal{D}_{\mathtt{L}})$ is an equality rule $\mathsf{L}=_\mathsf{i}$, then our argument is the same as in the second case below. The remaining case is $\mathtt{rule}(\mathcal{D}_{\mathtt{L}})$ is an axiom $\mathsf{R}=$. Then our derivation above $\mathcal{D}$ has the following form:

$$\dfrac{\dfrac{}{\Rightarrow x=x}\ \mathsf{R}= \quad \dfrac{\dfrac{\mathcal{D}'_{\mathtt{R}}}{x=x, \phi'_1[x/w], \ldots, \phi'_{n-1}[x/w], \Sigma'[x/w] \Rightarrow \Theta'[x/w]}}{x=x, \phi'_1[x/w], \ldots, \phi'_{n-1}[x/w], \Sigma'[x/w] \Rightarrow \Theta'[x/w]}\ \mathsf{L}=_1}{\Sigma \Rightarrow \Theta}\ \mathsf{Mcut}.$$

Then this derivation is transformed into:

$$\dfrac{\dfrac{}{\Rightarrow x=x}\ \mathsf{R}= \quad \dfrac{\mathcal{D}'_{\mathtt{R}}}{x=x, \phi'_1[x/w], \ldots, \phi'_{n-1}[x/w], \Sigma'[x/w] \Rightarrow \Theta'[x/w]}}{\Sigma \Rightarrow \Theta}\ \mathsf{Mcut}$$

and this last application of multicut is eliminable since the complexity is the same as that of $\mathcal{D}$ but the weight becomes smaller.

Secondly, let $\mathtt{rule}(\mathcal{D}_{\mathtt{L}})$ be $\mathsf{L}=_\mathsf{i}$ and assume that all occurrences of the cut formula are principal. In this case the derivation $\mathcal{D}$ is of the following form:

$$\dfrac{\dfrac{\dfrac{\mathcal{D}'_{\mathtt{L}}}{x=y, \Gamma''[x/w] \Rightarrow \Delta''[x/w], \phi'_1[x/w], \ldots, \phi'_m[x/w]}}{x=y, \Gamma''[y/w] \Rightarrow \Delta''[y/w], \phi'_1[y/w], \ldots, \phi'_m[y/w]}\ \mathsf{L}=_1 \quad \dfrac{\mathcal{D}_{\mathtt{R}}}{\phi^n, \Sigma \Rightarrow \Theta}}{x=y, \Gamma', \Sigma \Rightarrow \Delta, \Theta}\ \mathsf{Mcut}$$

where $\Gamma''[y/w] = \Gamma'$, $\Delta'[y/w] = \Delta$ and $\phi'_i[y/w] = \phi$ ($1 \leqslant i \leqslant m$). Before transforming this derivation into a multicut-free derivation, we remark that $\phi'_i[x/w][y/x] = \phi'_i[y/w][y/x] = \phi[y/x]$, $\Gamma''[x/w][y/x] = \Gamma''[y/w][y/x] = \Gamma'[y/x]$, $\Delta'[x/w][y/x] = \Delta'[y/w][y/x] = \Delta[y/x]$. With the help of this remark, the derivation $\mathcal{D}$ is transformed into:

$$\dfrac{\dfrac{\dfrac{\dfrac{\mathcal{D}'_{\mathtt{L}}[y/x]}{y=y, \Gamma'[y/x] \Rightarrow \Delta[y/x], (\phi[y/x])^m} \quad \dfrac{\mathcal{D}_{\mathtt{R}}[y/x]}{(\phi[y/x])^n, \Sigma[y/x] \Rightarrow \Theta[y/x]}}{y=y, \Gamma'[y/x], \Sigma[y/x] \Rightarrow \Delta[y/x], \Theta[y/x]}\ \mathsf{Mcut}}{x=y, y=y, \Gamma'[y/x], \Sigma[y/x] \Rightarrow \Delta[y/x], \Theta[y/x]}\ \mathsf{RW}}{\dfrac{x=y, x=y, \Gamma', \Sigma \Rightarrow \Delta, \Theta}{x=y, \Gamma', \Sigma \Rightarrow \Delta, \Theta}\ \mathsf{LC}}\ \mathsf{L}=_2$$

where we note that the first application of multicut is eliminable since the complexity is the same as that of $\mathcal{D}$ but the weight is smaller than that of $\mathcal{D}$ by height-preserving substitution $[y/x]$. □

Theorem 6.17 (Cut Elimination)**.** *Suppose that $\mathcal{R}$ absorbs multicut. Then the rule* Mcut *is admissible in $\mathcal{SR}$. Therefore,* Cut *is also admissible in $\mathcal{SR}$.*

Proof. Suppose that a sequent is derivable in the system $\mathcal{SR}$ extended with MCut. Let $\mathcal{E}$ be such a derivation. Then we focus on one of the topmost applications of MCut to show that such application of MCut is eliminable, i.e., we show that the derivation whose last applied rule is such multicut can be replaced with a multicut-free derivation of $\mathcal{SR}$. This is done using Lemma 6.16. Once we eliminate one of the topmost applications of MCut, we repeat the same argument for the remaining topmost applications with the help of Lemma 6.16 to get rid of all applications of MCut in the original derivation $\mathcal{E}$. □

In what follows, we introduce the notion of *absorption of contraction and cut* and show that jointly they provide a sufficient condition of absorption of multicut.

Definition 6.18. Let $\mathbb{S}$ be a finite set of sequents. The set of sequents that can be derived from premises $\mathbb{S}$ using (only) the *contraction rules* is denoted by $\mathsf{Con}(\mathbb{S})$. Similarly, the set of all sequents that can be derived from premises in $\mathbb{S}$ using (only) *one application* of the *cut rule* is denoted by $\mathsf{Cut}(\mathbb{S})$. A rule set $\mathcal{R}$ *absorbs contraction* if, for all rules $R = \Gamma_1 \Rightarrow \Delta_1, \cdots, \Gamma_k \Rightarrow \Delta_k / \Gamma_R \Rightarrow \Delta_R \in \mathcal{R}$ and all $\Gamma' \Rightarrow \Delta' \in \mathsf{Con}(\Gamma_R \Rightarrow \Delta_R)$ there exists a rule $S = \Sigma_1 \Rightarrow \Theta_1, \cdots, \Sigma_l \Rightarrow \Theta_l / \Gamma_S \Rightarrow \Delta_S \in \mathcal{R}$ such that

$$\mathsf{Con}(\{\Gamma_1 \Rightarrow \Delta_1, \dots, \Gamma_k \Rightarrow \Delta_k\}) \rhd \{\Sigma_1 \Rightarrow \Theta_1, \dots, \Sigma_l \Rightarrow \Theta_l\}$$

and $\Gamma_S \Rightarrow \Delta_S \rhd \Gamma' \Rightarrow \Delta'$. A rule set $\mathcal{R}$ *absorbs cut*, if for all pairs (R_1, R_2) of rules in $\mathcal{R}$:

$$\dfrac{\Gamma_{11} \Rightarrow \Delta_{11} \quad \cdots \quad \Gamma_{1r_1} \Rightarrow \Delta_{1r_1}}{\Gamma_{R_1} \Rightarrow \Delta_{R_1}, \heartsuit\vec{\boldsymbol{p}}}\ R_1 \quad \dfrac{\Gamma_{21} \Rightarrow \Delta_{21} \quad \cdots \quad \Gamma_{2r_2} \Rightarrow \Delta_{2r_2}}{\heartsuit\vec{\boldsymbol{p}}, \Gamma_{R_2} \Rightarrow \Delta_{R_2}}\ R_2$$

there is a rule $R = \Gamma_1 \Rightarrow \Delta_1, \cdots, \Gamma_k \Rightarrow \Delta_k / \Gamma_R \Rightarrow \Delta_R \in \mathcal{R}$ such that:

$$\mathsf{Cut}(\Gamma_{11} \Rightarrow \Delta_{11}, \dots, \Gamma_{1r_1} \Rightarrow \Delta_{1r_1}, \Gamma_{21} \Rightarrow \Delta_{21}, \dots, \Gamma_{2r_2} \Rightarrow \Delta_{2r_2}) \rhd \{\Gamma_1 \Rightarrow \Delta_1, \dots, \Gamma_k \Rightarrow \Delta_k\}$$

and $\Gamma_R \Rightarrow \Delta_R \rhd \Gamma_{R_1}, \Gamma_{R_2} \Rightarrow \Delta_{R_1}, \Delta_{R_2}$.

Informally, absorption of cut and contraction of a rule set allows us to replace an application of cut or contraction to the conclusions of rules in $\mathcal{R}$ by a possibly different rule with possibly weaker premises and stronger conclusion. While these definitions are purely syntactic, a semantic characterisation has been given in [PS10] in terms of *one-step cut-free completeness*. For many Λ-structures including those for the modal logic K and the logic of (monotone) neighbourhood frames, one-step cut-free complete rule sets are known. In particular, these rule sets satisfy absorption of cut, contraction and congruence [PS10, § 5].

Lemma 6.19. *If the rule set $\mathcal{R}$ absorbs contraction and cut then $\mathcal{R}$ also absorbs multicut.*

By Theorem 6.17 and Lemma 6.19, we obtain the following.

Corollary 6.20. *Suppose that $\mathcal{R}$ absorbs contraction and cut. Then* Cut *is also admissible in $\mathcal{SR}$.*

As an immediate corollary, we obtain completeness of the cut-free calculus assuming that $\mathcal{R}$ is *strongly* one-step complete:

Corollary 6.21. *Suppose that $\mathcal{R}$ is strongly one-step complete. Then $\models \Gamma \Rightarrow \Delta$ iff $\mathcal{SR} \vdash \Gamma \Rightarrow \Delta$.*

Proof. This follows from Theorem 6.17 with the help of Proposition 5.11 and 5.12 of [PS10], the latter asserting precisely the absorption of cut and congruence. □

The situation is more complex in presence of bounded operators where completeness of the Hilbert calculus is only guaranteed in presence of BdPL, and completeness of the associated sequent calculus relies on Paste_i^k. The difficulty in a proof of cut-elimination is a cut-end derivation where a cut is performed on $x\heartsuit\lceil y_1 : \phi_1\rceil \dots \lceil y_n : \phi_n\rceil$ which is introduced by Paste_i^k and a (one-step) rule where the same formula is principal. We leave this as an open problem:

Problem 6.22. Is there a way to modify the rules of $\mathcal{BSR}$ so that completeness with respect to $\mathcal{BHR}$ holds and cut is admissible?

7. Conclusions and Further Work

We have introduced *coalgebraic predicate logic*, a natural first-order formalism that incorporates coalgebraic modalities and thus serves as an expressive language for coalgebras. As instances, it subsumes both standard relational first-order logic and Chang's first-order logic of neighbourhood systems [Cha73]; other instances include a first-order logic of nonmonotone conditionals as well as first-order logics of integer-weighted relations that include weighted or (positive) Presburger modalities. We have provided the foundations for proof theory and model theory of CPL.

In terms of future research, a promising avenue appears to be coalgebraic finite model theory; in fact, the first result in this direction is the existing finite version of the coalgebraic van Benthem-Rosen theorem [SP10a, LPSS12, SPL17]. It is worth observing that van Benthem-Rosen is a rare instance of a model-theoretic characterization of a fragment of first-order predicate logic that remains valid over finite models. The only other major result of this type we are aware of is the characterization of existential-positive formulas as exactly those preserved under homomorphisms [Ros08]. The result is relevant to constraint satisfaction problems and to database theory, as existential-positive formulas correspond to unions of conjunctive queries. Interestingly, the proof of Rossman's result relies on Gaifman graphs, which also play a central role in the proof of the coalgebraic Rosen theorem.

Embedding modal operators into a first-order syntax opens up the possibility of applying modalities to predicates of arity greater than 1; operators of this type are found, e.g., in Halpern's Type-1 probabilistic first-order logic [Hal90]. We leave the ramifications of this option to future investigation.

Possible directions in coalgebraic model theory over unrestricted models include generalizations of standard results of classical model theory like Beth definability or interpolation and the Keisler-Shelah characterization theorem.

It remains to be seen which results of *modal model theory* building upon the interplay between modal and predicate languages can be generalized. Specific potential examples include Sahlqvist-type results for suitably well-behaved structures and analogues of results by Fine (does elementary generation imply canonicity, at least wherever the coalgebraic Jónsson-Tarski theorem [KKP05] obtains?)[5] or Hodkinson [Hod06] (is there an algorithm generating a CML axiomatization for CPL-definable classes of coalgebras?).

Finally, a natural direction of investigation will be to study models based on coalgebras for endofunctors on categories other than Set and corresponding variants of CPL with non-Boolean propositional bases.

[5] Recently, first results in this direction have been announced by Kentarô Yamamoto, UC Berkeley.

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