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MIXED POWERDOMAINS FOR PROBABILITY AND NONDETERMINISM

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Abstract. We consider mixed powerdomains combining ordinary nondeterminism and probabilistic nondeterminism. We characterise them as free algebras for suitable (in)equational theories; we establish functional representation theorems; and we show equivalencies between state transformers and appropriately healthy predicate transformers. The extended nonnegative reals serve as 'truth-values'. As usual with powerdomains, everything comes in three flavours: lower, upper, and order-convex. The powerdomains are suitable convex sets of subprobability valuations, corresponding to resolving nondeterministic choice before probabilistic choice. Algebraically this corresponds to the probabilistic choice operator distributing over the nondeterministic choice operator. (An alternative approach to combining the two forms of nondeterminism would be to resolve probabilistic choice first, arriving at a domain-theoretic version of random sets. However, as we also show, the algebraic approach then runs into difficulties.)

Rather than working directly with valuations, we take a domain-theoretic functional-analytic approach, employing domain-theoretic abstract convex sets called Kegelspitzen; these are equivalent to the abstract probabilistic algebras of Graham and Jones, but are more convenient to work with. So we define power Kegelspitzen, and consider free algebras, functional representations, and predicate transformers. To do so we make use of previous work on domain-theoretic cones (d-cones), with the bridge between the two of them being provided by a free d-cone construction on Kegelspitzen.

1. Introduction

In this paper we investigate mixed powerdomains combining ordinary and probabilistic nondeterminism. These can be defined generally as free algebras over dcpos (directed complete posets). The algebraic laws we consider in this regard are for the binary choice operators $\cup$ and $+_r$ of ordinary and probabilistic nondeterminism as well as a constant for nontermination (where $x +_r y$ expresses a choice of x with probability r versus one of y with

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probability $1 - r$) together with an axiom to the effect that probabilistic choice distributes over ordinary nondeterministic choice, viz.:

$$x +_r (y \cup z) = (x +_r y) \cup (x +_r z)$$

(see below for axioms for ordinary and probabilistic nondeterminism, or, for example, [16]). We characterise the free algebras as suitable convex sets of subprobability valuations in the case of domains (continuous dcpos). We do this for all three domain-theoretic notions of ordinary nondeterminism, viz. lower (or Hoare), upper (or Smyth), and convex (or Plotkin), though in the last case we need an additional assumption, that the domains are coherent.

We further give suitable notions of predicates and predicate transformers, obtaining a dual correspondence between predicate transformers and (mixed) nondeterministic functions (i.e., Kleisli category morphisms). The relevant notions of predicate use $\overline{\mathbb{R}}_+$, the domain of the non-negative reals extended with a point at infinity, or its convex powerdomain, as 'truth-values.' Our results on predicate transformers are obtained via functional characterisations of the mixed powerdomains, and are again obtained for all three notions of ordinary nondeterminism. As before these results obtain generally for domains except in the convex case where, additionally, coherence is again required.

In previous joint work with Regina Tix [61, 28] based on Tix's Ph.D. thesis [60], we carried out a similar programme for ordinary and so-called 'extended' probabilistic computation where valuations take values in $\overline{\mathbb{R}}_+$ rather than $[0, 1]$. The method we used was a kind of domain-theoretic functional analysis where, instead of working directly with domains of valuations, one works with an abstract domain-theoretic notion of cone, called a d-cone. We investigated powercones, which embody notions of non-determinism at the cone-theoretic level, and then applied our results to cones of valuations. The powercones were shown to be the free cones with a semilattice operation over which the cone operations distribute, with a further requirement that the semilattice be a join-semilattice in the lower case, and a meet-semilattice in the upper case; it immediately follows, although not remarked in [61], that the valuation powercones provide corresponding free constructions on domains. Predicate transformers and functional representations were also first investigated at the cone-theoretic level, with the results being again applied to cones of valuations.

We proceed analogously here, but replacing cones with a suitable domain-theoretic notion of abstract convex space, termed a Kegelspitze[1]. By embedding Kegelspitzen in d-cones we are able to make use of our previous results, thereby avoiding a good deal of work. This approach works particularly well when characterising the mixed powerdomains, but less well when considering functional representations. We do obtain strong enough abstract results for their intended application. However the general results require assumptions (taken from previous work) on the cones in which the Kegelspitzen are embedded, rather than natural assumptions directly concerning the Kegelspitzen themselves. One wonders to

[1]The German word Kegelspitze means 'tip of a cone', suggesting a convex set obtained by cutting off the top of a cone.

what extent one can succeed with more natural assumptions, and, indeed, to what extent one can proceed without making use of d-cones.

Several other authors have previously considered the combination of ordinary and probabilistic nondeterminism, making use of sets of distributions. In a domain-theoretic context, the pioneers were the Oxford Programming Research Group [44, 40, 39, 41]. That work was restricted to the case of countable discrete domains, as was that of Ying [69]. Later, Mislove [42] defined mixed powerdomains for all three notions of ordinary nondeterminism over continuous domains required to be coherent in the convex case. Mislove also considered nondeterministic powerdomains over 'abstract probabilistic algebras'. With the addition of a bottom element, these are the same as the identically named algebras of Graham and Jones [14, 18] and equivalent to our Kegelspitzen; however Kegelspitzen seem more in the spirit of domain theory, as we discuss below.

Goubault-Larrecq [9, 10, 11, 12] worked at a topological level, considering all three notions of ordinary nondeterminism combined with various classes of valuations: all valuations, subprobability valuations, and probability valuations, but without explicit consideration of the algebraic structures involved. He established functional representation results under quite weak assumptions on the underlying spaces. When specialised to domains and subprobability valuations, his results correspond to our Corollaries 4.4, 4.7, and 4.10. He worked directly with the valuation spaces rather than, as we do, making use of abstract structures such as cones and barycentric algebras. In [2, 3] Beaulieu worked algebraically; his results include free constructions of algebras satisfying the above laws over sets and partial orders, but not domains.

There has been some discussion of other ways to combine nondeterminism with probability. Categorical distributive laws provide a standard means of showing the composition of two monads form a third (see [38]). However there is no such law enabling one to compose the monad of ordinary nondeterminism with that of probabilistic nondeterminism — see the Appendix of [64]. For this reason, Varacca and Winskel [62, 63, 64] reject, or weaken, one of the axioms of (extended) probabilistic nondeterminism, viz. the distributivity law $(r + s)x = rx + sx$, and consider certain 'indexed valuations' in place of the more usual ones. As shown by Varacca in [63, Chapter 4] this approach applies to domains, where indexed valuations come in three flavours: Hoare, Smyth, and Plotkin. Categorical distributivity laws are obtained in several cases, and a freeness result (with the above equational distributive law) is given in the case of the combination of Hoare indexed valuations and the Hoare powerdomain.

In [8, 11], and see too [13], Goubault-Larrecq also combined the two types of monads in a different order, considering the probabilistic powerdomains over the nondeterministic powerdomains; however no algebraic aspects were discussed. This approach is in the spirit of what is known under the name of *random sets* in probability theory. The approach can be thought of as resolving probabilistic choice before nondeterministic choice; in contrast, approaches employing sets of distributions can rather be thought of as resolving nondeterministic choice first.

Algebraically, the above distributive law corresponds to resolving nondeterministic choice first. The other distributive law

$$x \cup (y +_r z) = (x \cup y) +_r (x \cup z)$$

corresponds to first resolving probabilistic choice. As pointed out in [43] this law leads to some odd consequences when combined with the other laws, particularly the idempotence of nondeterministic choice. In Appendix A we show that the equational theory consisting of the laws for nondeterministic and probabilistic choice together with this distributive law is equivalent to that of join-distributive bisemilattices [52], i.e., of algebras with two semilattice operations called join and meet, with join distributing over meet. It therefore has no quantitative content.

An algebraic treatment of this combination of the two forms of nondeterminism would therefore have to weaken some law. One natural possibility is to drop the idempotence of nondeterministic choice (this was done in a process calculus-oriented context in [68, 4]). There is a natural 'finite random sets' functor supporting models of this weaker theory, namely $\mathcal{D}_\omega \circ \mathcal{P}^+_\omega$ where $\mathcal{D}_\omega$ is the finite probability distributions monad, and $\mathcal{P}^+_\omega$ is the finite non-empty sets monad. The barycentric structure is evident, and the nondeterministic choice operations $\cup_X \colon \mathcal{D}_\omega\mathcal{P}^+_\omega(X)^2 \to \mathcal{D}_\omega\mathcal{P}^+_\omega(X)$ are given by:

$$(\sum_{i=1,\ldots,m} \alpha_i x_i) \cup_X (\sum_{i=1,\ldots,n} \beta_j y_j) = \sum_{\substack{i=1,\ldots,m \\ j=1,\ldots,n}} \alpha_i \beta_j (x_i \cup y_j)$$

Unsurprisingly, these finite random set algebras do not provide the free algebras for the weaker theory. More surprisingly, perhaps, they do not provide the free algebras for *any* equational theory over the relevant signature; this too is shown in Appendix A. As another point along these lines, we recall a result of Varacca [63, Proposition 3.1.3] that there is no distributive law of $\mathcal{P}^+_\omega$ over $\mathcal{D}_\omega$. Overall, it seems hard to see how there can be any satisfactory algebraic treatment of the combination of probability and nondeterminism in which probabilistic choice is resolved first.

In Section 2 we develop the theory of Kegelspitzen, beginning with a notion of (ordered) barycentric algebra. This notion is based on the equational theory of the barycentric operations $rx + (1-r)y$ (for real numbers r between 0 and 1) on convex sets in vector spaces. There is an extensive relevant literature, which we survey. We need this notion augmented with a compatible partial order and, in addition, with a distinguished element 0. Finally, specialising to directed complete partial orders and Scott-continuous operations, we introduce the central notion of Kegelspitzen, axiomatising subprobabilistic powerdomains. In order to relate these structures to our previous work on cones, we prove embedding theorems at the various levels, and then establish the preservation of crucial properties. In the case of Kegelspitzen, the embedding theorem is Theorem 2.35 and the property-preserving results are those of Propositions 2.42, 2.43 and 2.44, and Corollary 2.45; the properties preserved include continuity and coherence.

In Section 3 we define, and give universal algebraic characterisations of, the various mixed powerdomains, first doing the same for suitable notions of power Kegelspitze. The universal characterisations of the free power Kegelspitzen are given in Theorems 3.4, 3.9, and 3.14 (one for each notion of nondeterminism). The universal characterisations of the mixed powerdomains then follow, and are given in Corollaries 3.15, 3.16, and 3.17; the first two hold for any domain, and the third holds for any coherent domain.

In Section 4, we consider functional representations. The three functional representations of power Kegelspitzen are given by Theorems 4.2, 4.6, and 4.9; they largely follow straightforwardly from the corresponding results for cones in [28]. The corresponding three functional representations of the mixed powerdomains over domains are given by Corollaries 4.4, 4.7, and 4.10 and are derived from the corresponding results for Kegelspitzen.

In Section 5 we consider predicate transformers for domains, showing the equivalence of 'state transformers', i.e., Kleisli maps, and 'healthy' predicate transformers, viz. maps on predicates obeying suitable conditions. The conditions and equivalences follow from the functional representation theorems for domains, and are given by Corollaries 5.1, 5.2, and 5.4. There are related general results for Kegelspitzen; these follow from the functional representation theorems for Kegelspitzen, and are discussed briefly.

The results given in Sections 4 and 5 all make use of $\overline{\mathbb{R}}_+$, the d-cone of the non-negative reals augmented by a point at infinity; this is unsurprising given their derivation from the corresponding results for cones. However, in the context of Kegelspitzen, it is natural to further seek results replacing that cone by $\mathbb{I}$, the unit interval Kegelspitze. This is done for the mixed powerdomains in Section 6, where both functional and predicate transformer results are obtained in all three cases. The functional representation results are Corollaries 6.4, 6.6, and 6.9; the predicate transformer results are Corollaries 6.5, 6.7, and 6.10. With additional assumptions there are related general results for Kegelspitzen, which we discuss briefly.

Terminology. Throughout the paper, we assume familiarity with standard terminology and notation of domain theory as covered in, say, [7]. In particular, for partially ordered set we shortly say poset; the abbreviation *dcpo* stands for 'directed complete partially ordered set'; 'bounded directed complete' means that every upper-bounded directed set has a least upper bound; the way below relation in a dcpo C is written as $\ll_C$, or simply $\ll$; and a *domain* is a continuous dcpo, that is, a dcpo in which, for every element a, the elements way-below a form a directed set with supremum a. For any subset X of a poset C we write $\uparrow_C X$, or simply $\uparrow X$, for the set of elements above some element of X (and $\downarrow X$ is understood analogously); similarly, we write $\twoheaduparrow_C X$ for the set of elements way-above some element of X.

A *sub-dcpo* is a subset C of a dcpo D that is closed for suprema of directed sets, that is, the supremum of any directed subset of C belongs to C. Sub-dcpos should not be mixed up with Scott-closed sets which are sub-dcpos and, in addition, lower sets. The intersection of an arbitrary family of sub-dcpos is a sub-dcpo; thus, for any subset P of a dcpo there is a least sub-dcpo P^d containing P. The elements of P^d can also be obtained by closing under directed suprema repeatedly. We will say that P is *dense* in C, if $P^d = C$. Again, this notion of density is different from dense for the Scott topology.

We write $\mathbb{R}_+$ for the set of nonnegative real numbers with their usual order, and $\overline{\mathbb{R}}_+ =_{def} \mathbb{R}_+ \cup \{+\infty\}$ for the nonnegative real numbers augmented by a top element $+\infty$. Finally, $\mathbb{I} =_{def} [0,1]$ is the closed and $]0,1[$ the open unit interval.

Contents

1. Introduction 1
2. Kegelspitzen 7
2.1. Ordered cones and ordered barycentric algebras 7
2.2. Ordered pointed barycentric algebras 13
2.3. d-Cones and Kegelspitzen 18
2.4. Preservation results 26
2.5. Duality and the subprobabilistic powerdomain 28
3. Power Kegelspitzen 32
3.1. Lower power Kegelspitzen 33
3.2. Upper power Kegelspitzen 37
3.3. Convex power Kegelspitzen 44
3.4. Powerdomains combining probabilistic choice and nondeterminism 49
4. Functional representations 51
4.1. The lower power Kegelspitze 53
4.2. The upper power Kegelspitze 55
4.3. The convex power Kegelspitze 58
5. Predicate transformers 61
5.1. The lower case 62
5.2. The upper case 63
5.3. The convex case 64
6. The unit interval 67
6.1. The lower case 71
6.2. The upper case 72
6.3. The convex case 74
Acknowledgements 77
References 77
Appendix A. The other distributive law 80
Appendix B. A counterexample 83

2. Kegelspitzen

In this section we introduce the notion of a Kegelspitze, which provides the foundation for our later developments. We begin with its algebraic structure. This is that of an abstract convex set, which we call a barycentric algebra. We then enrich the structure, first with a compatible partial order, and then with a directed complete partial order. In order to make use of our previous work [28] on d-cones, we prove embedding theorems of barycentric algebras in abstract cones at each stage. We conclude by recalling the properties of the subprobabilistic powerdomain and showing how it fits within our framework.

2.1. Ordered cones and ordered barycentric algebras.

For our work there are two basic notions abstracted from substructures in real vector spaces, an abstract notion of a cone and an abstract notion of a convex set.

In a real vector space V, a subset C is understood to be a cone, if $x + y \in C$ and $r \cdot x \in C$ for all $x, y \in C$ and every nonnegative real number r. Generalising, we obtain an abstract notion of a cone:

Definition 2.1. An *(abstract) cone* is a set C together with a commutative associative addition $(x, y) \mapsto x + y \colon C \times C \to C$ that admits a neutral element 0, and a scalar multiplication $x \mapsto r \cdot x \colon C \to C$ by real numbers $r > 0$ satisfying the usual laws for scalar multiplication in vector spaces, that is, the following equational laws hold for all $x, y, z \in C$ and all real numbers $r > 0, s > 0$:

$$
\begin{array}{rclcrcl}
x + (y + z) & = & (x + y) + z & \qquad & (rs) \cdot x & = & r \cdot (s \cdot x) \\
x + y & = & y + x & & (r + s) \cdot x & = & r \cdot x + s \cdot x \\
x + 0 & = & x & & r \cdot (x + y) & = & r \cdot x + r \cdot y \\
 & & & & 1 \cdot x & = & x
\end{array}
$$

Preserving all the above laws we may extend (and we will tacitly always do so) the scalar multiplication on a cone to real numbers $r \geq 0$ by defining

$$0 \cdot x =_{def} 0 \text{ for all } x \in C$$

A map $f \colon C \to D$ between cones is said to be:

homogeneous	if $f(r \cdot x) = r \cdot f(x)$	for all $r \in \mathbb{R}_+$ and all $x \in C$,
additive	if $f(x + y) = f(x) + f(y)$	for all $x, y \in C$,
linear	if f is homogeneous and additive.	

In a cone all the equational laws for addition and scalar multiplication that hold in vector spaces also hold, except that we restrict scalar multiplication to nonnegative real numbers and elements x need not have negatives $-x$. Thus, we may calculate in cones just as we do in vector spaces, except that we have to avoid negatives. As usual, we generally write scalar multiplication $r{\cdot}x$ as rx.

The notion of a barycentric algebra captures the equational properties of convex sets. A subset A of a real vector space is *convex* if

$$ra + (1-r)b \in A \text{ for all } a, b \in A, r \in [0,1] \tag{Conv}$$

We may use the same property for defining convexity of subsets in an (abstract) cone. On every convex set A we may define for every real number $r \in [0,1]$ a binary operation $+_r$, the convex combination $(a,b) \mapsto a +_r b =_{def} r \cdot a + (1-r) \cdot b$. Straightforward calculations show that these operations satisfy the following equational laws:

$$a +_1 b = a \tag{B1}$$

$$a +_r a = a \tag{B2}$$

$$a +_r b = b +_{1-r} a \tag{SC}$$

$$(a +_p b) +_r c = a +_{pr} (b +_{\frac{r-pr}{1-pr}} c) \quad \text{provided } r < 1, p < 1 \tag{SA}$$

SC stands for *skew commutativity* and SA for *skew associativity.*

Definition 2.2. An *abstract convex set* or *barycentric algebra* is a set A endowed with a binary operation $a +_r b$ for every real number r in the unit interval $[0,1]$ such that the above equational laws (B1), (B2), (SC), (SA) hold. A map $f\colon A \to B$ between barycentric algebras is *affine* if $f(a +_r b) = f(a) +_r f(b)$ for all $a, b \in A$ and all $r \in [0,1]$.

Barycentric algebras A are *entropic* (or *commutative*) in the sense that all the operations $+_r$ are affine maps from $A \times A \to A$, that is, for all r and s in the unit interval we have the entropic identity

$$(a +_r b) +_s (c +_r d) = (a +_s c) +_r (b +_s d). \tag{E}$$

If $c = d$ this reduces to the distributivity law

$$(a +_r b) +_s c = (a +_s c) +_r (b +_s c). \tag{D}$$

The entropic identity (E) can be verified by direct calculation. However, calculations in barycentric algebras are quite tedious as the skew associativity law (SA) is awkward to apply. A simple proof is indicated below after Lemma 2.3.

Since barycentric algebras (resp., cones) are equationally defined classes of algebras, there are free barycentric algebras (resp., free cones), and every barycentric algebra (resp., cone) is the image of a free one under an affine (resp., linear) map.

These free objects have a simple description. For any set I we write $\mathbb{R}^{(I)}$ for the direct sum of I copies of $\mathbb{R}$, that is the vector space of all I-tuples $x = (x_i)_{i\in I}$ of real numbers x_i such that $x_i \neq 0$ for finitely many indices i. For $i \in I$ we write $\delta(i)$ for $(\delta_{ij})_{j\in I}$, the canonical basis vector, where δ_{ij} is the Kronecker symbol. Thus, δ maps I to a basis of $\mathbb{R}^{(I)}$. Analogously to the the fact that $\mathbb{R}^{(I)}$ is the free vector space over the set I, we have:

Lemma 2.3.

(1) *The positive cone $\mathbb{R}^{(I)}_+$ of all $x \in \mathbb{R}^{(I)}$ with nonnegative entries is the free cone over I with unit δ.*

(2) *The simplex P_I of all $x \in \mathbb{R}_+^{(I)}$ such that $\sum_i x_i = 1$ is the free barycentric algebra over I with unit δ.* □

The first claim means that, for every map f from I to a cone C, there is a unique linear map $\overline{f}\colon \mathbb{R}_+^{(I)} \to C$ such that $f = \overline{f} \circ \delta$, namely $\overline{f}(x) = \sum_i x_i f(i)$. Similarly, the second claim tells us that, for every map f from I to a barycentric algebra A, there is a unique affine map $\overline{f}\colon P_I \to A$ such that $f = \overline{f} \circ \delta$.

In an equationally definable class of algebras, an equational law holds if and only if it holds in the free algebras. Thus, for the entropic identity (E) to hold in all barycentric algebras, it suffices to verify this property for the free barycentric algebras; and an easy calculation shows that (E) holds in any convex subset of a real vector space. The same calculation can be done after embedding a barycentric algebra in a cone as a convex subset:

Standard Construction 2.4. Let A be a barycentric algebra. On

$$\mathbb{R}_{>0} \times A = \{(r, a) \mid 0 < r \in \mathbb{R},\ a \in A\}$$

define addition and multiplication by scalars $t > 0$ by:

$$(r, a) + (s, b) =_{def} (r + s, a +_{\frac{r}{r+s}} b), \qquad t(r, a) =_{def} (tr, a)$$

Adjoin a new element 0:

$$C_A =_{def} \{0\} \cup \{(r, a) \mid r > 0, a \in A\} = \{0\} \cup (\mathbb{R}_{>0} \times A)$$

so that 0 is a neutral element for addition and $t \cdot 0 = 0$. Simple calculations show that C_A thereby becomes a cone, and the map $e = (a \mapsto (1, a))\colon A \to C_A$ is an embedding of a barycentric algebra in a cone (i.e., it is affine and injective).

We identify elements $a \in A$ with the corresponding elements $e(a) = (1, a) \in C_A$, thus identifying A with the convex subset $\{1\} \times A$ of C_A. In this way A becomes a *base* of the cone C_A in the sense that A is convex and that every element $x = (r, a) \neq 0$ in C_A can be written in the form $x = ra$, where r and a are uniquely determined by x.

We want to add a partial order to our algebraic structure:

Definition 2.5. An *ordered (abstract) cone* is a cone equipped with a partial order $\leq$ such that addition and scalar multiplication are monotone, that is,

$$a \leq a' \implies a + b \leq a' + b\ , \quad ra \leq ra'\ .$$

A map $f\colon C \to D$ between ordered cones is said to be:

subadditive	if $f(a + b) \leq f(a) + f(b)$ for all $a, b \in C$,
superadditive	if $f(a + b) \geq f(a) + f(b)$ for all $a, b \in C$,
sublinear	if f is homogeneous and subadditive,
superlinear	if f is homogeneous and superadditive.

The linear maps are those that are sublinear and superlinear.

Definition 2.6. An *ordered barycentric algebra* is a barycentric algebra with a partial order $\leq$ such that the barycentric operations $+_r$ are monotone, that is,

$$a \leq a' \implies a +_r b \leq a' +_r b$$

for $0 \leq r \leq 1$. A map $f \colon A \to B$ between ordered barycentric algebras is said to be:
convex if $f(a +_r b) \leq f(a) +_r f(b)$ for $0 \leq r \leq 1$ and $a, b \in C$,
concave if $f(a +_r b) \geq f(a) +_r f(b)$ for $0 \leq r \leq 1$ and $a, b \in C$.

The affine maps are those that are both convex and concave.

Every barycentric algebra can be understood to be ordered by the discrete order $a \leq b$ iff $a = b$ and, if A and B both are discretely ordered, the convex maps between them are the affine ones.

Convex subsets of ordered vector spaces and ordered cones are ordered barycentric algebras with respect to the induced order.

In complete analogy to Standard Construction 2.4, we can construct embeddings of ordered barycentric algebras in ordered cones, by which we mean monotone affine maps which are also order embeddings (a monotone map between partial orders is an order embedding if it reflects the partial order).

Standard Construction 2.7. For an ordered barycentric algebra A we use the embedding of A in the abstract cone C_A as in Standard Construction 2.4 and we extend the order on A by defining an order $\leq$ on C_A by $0 \leq 0$ and:

$$(r, a) \leq (s, b) \iff r = s \text{ and } a \leq b \text{ in } A$$

With this order, C_A becomes an ordered (abstract) cone and the affine map $e = a \mapsto (1, a)$ from A to C_A is an affine order embedding.

The following surprising lemma will be used in later sections:

Lemma 2.8. *Let a, b, c be elements of an ordered barycentric algebra A. If $a +_p c \leq b +_p c$ holds for some p with $0 < p < 1$, then this holds for all such p.*

Proof. We may view A as a convex subset of an ordered cone C according to 2.7. Suppose now that a, b, c are elements of A and that the inequality $a +_p c \leq b +_p c$ holds for some $0 < p < 1$.

We first claim that $a +_q c \leq b +_q c$ for $q = \frac{2p}{1+p}$: By the above hypothesis we have the inequality $pa + (1-p)c \leq pb + (1-p)c$. We use this inequality twice for establishing this first claim: $qa + (1-q)c = \frac{1}{1+p}(2pa + (1-p)c) = \frac{1}{1+p}(pa + pa + (1-p)c) \leq \frac{1}{1+p}(pa + pb + (1-p)c) \leq \frac{1}{1+p}(pb + pb + (1-p)c) = \frac{1}{1+p}(2pb + (1-p)c) = qb + (1-q)c$.

Secondly, we define recursively $p_0 = p$ and $p_{n+1} = \frac{2p_n}{1+p_n}$. Our first claim allows to conclude $a +_{p_n} b \leq b +_{p_n} c$ for all n. As the p_n form an increasing sequence converging to 1, for every $q < 1$, there is an n such that $q < p_n$. Thus the following third claim finishes the proof of our lemma.

Claim: If $a +_p c \leq b +_p c$ holds for some p, then it also holds for all $q \leq p$. Indeed, $a +_q c = qa + (1-q)c = \frac{q}{p}(pa + (1-p)c) + \frac{p-q}{p}c \leq \frac{q}{p}(pb + (1-p)c) + \frac{p-q}{p}c = qb + (1-q)c = b +_q c$. □

Remarks 2.9. (**Historical Notes and References**) (1) The axiomatization of convex sets arising in vector spaces over the reals has a long history. The first axiomatization seems to be due to M. H. Stone [56]. Independently, H. Kneser [32] gave a similar axiomatization motivated by von Neumann's and Morgenstern's work on game theory [46]. Stone's and Kneser's results are not restricted to the reals; they axiomatise convex sets embeddable in vector spaces over linearly ordered skew fields. Such an axiomatization cannot be equational. For a barycentric algebra to be embeddable in a vector space one has to add a cancellation axiom:

$$a +_r c = b +_r c \implies a = b \quad (\text{for } 0 < r < 1) \tag{C1}$$

Similarly, one can show that an ordered barycentric algebra is embeddable in an ordered vector space if, and only if, it satisfies the order cancellation axiom:

$$a +_r c \leq b +_r c \implies a \leq b \quad (\text{for } 0 < r < 1) \tag{OC1}$$

(2) Several authors independently developed the equational theory of convex sets and the corresponding notion of an abstract convex set (while ignoring each other). Our extension to ordered structures seems to be new. We now try to give as complete as possible an account of these developments. The reader should be aware that we always stay within the equational theory of convex sets in real vector spaces, and we do not go into generalities, such as abstract convexities in the sense of, for example, van de Vel [65].

(3) W. Neumann [45] seems to be the first to have looked at the equational theory of convex sets. He remarked that barycentric algebras may be very different from convex sets in vector spaces. Indeed $\vee$-semilattices become examples of barycentric algebras if we define $a +_r b =_{def} a \vee b$ for $0 < r < 1$. Neumann [45] noticed that the semilattices form the only proper nontrivial equationally definable subclass of the class of all barycentric algebras. Every barycentric algebra has a greatest homomorphic image which is a semilattice. This semilattice is significant; indeed, for a convex subset of a real vector space this greatest homomorphic semilattice image is the (semi-)lattice of its faces. W. Neumann also characterised the free barycentric algebras, a characterisation that we reproduced in Lemma 2.3.

The equational axioms (B1), (B2), (SC), (SA) that we use in our definition of barycentric algebras are due to Świrszcz [58] and have been reproduced by Romanowska and Smith [53, Section 5.8]; the same axioms have also been used by Graham [14] and by Jones and Plotkin [19, 18] when they introduced the notion of an *abstract probabilistic powerdomain*. Romanowska and Smith introduced the term *barycentric algebra* for an abstract convex set; their monograph, cited above, is an exhaustive source on barycentric algebras and related structures.

(4) The notion of an abstract cone has emerged under the name of *quasilinear space* in interval mathematics in works of O. Mayer [37]; one may also consult papers by W.

Schirotzek, in particular [54], where ordered quasilinear spaces appear, the closed intervals in the reals with the Egli-Milner order forming a prime example. The embedding of a barycentric algebra as a convex subset in the abstract cone C_A is due to J. Flood [6]. The surprising lemma 2.8 is due to W. Neumann [45] in the unordered case. The possibility of embedding a barycentric algebra in a cone makes calculations much easier as already remarked by J. Flood [6]. The proof of Lemma 2.8 illustrates this advantage when compared with Neumann's original proof in the unordered case.

Let us note in passing that every $\vee$-semilattice with a zero becomes an abstract cone by defining $a+b = a\vee b$ and $ra = a$ for $r > 0$, but $ra = 0$ for $r = 0$. The class of $\vee$-semilattices with 0 is the only proper nontrivial equationally definable subclass of the class of all abstract cones.

(5) Another early axiomatisation of abstract convex sets is due to Gudder [15]. He uses as operations convex combinations of two and three elements and axiomatises these operations. Without introducing the notion of an abstract cone he gives the construction of the embedding of an abstract convex set into a cone viewed as a convex set.

(6) Equivalent to the equational one, there is another approach to abstract convex sets initiated by T. Świrszcz [58] who characterises them as the Eilenberg-Moore algebras of the monad P of probability distributions with finite support over the category of sets. In functional analysis this approach has been rediscovered by G. Rodé [51] and developed further by H. König [35] without any background in category theory. The approach has been pursued further in a series of papers by Pumplün, Röhrl, Kemper and others (see e.g. [48, 49, 50, 30, 67]). We summarise it as follows:

For all natural numbers $m > 0$, the set P_m of all probability measures $\mathbf{q} = (q_1, \ldots, q_m)$ on an m-element set is a compact convex set and, for every finite set $\mathbf{q}_1, \ldots, \mathbf{q}_n \in P_m$, the convex combination $\sum_{i=1}^n p_i\mathbf{q}_i$ is again an element of P_m for every $\mathbf{p} = (p_1, \ldots, p_n) \in P_n$. The sum $\sum_{i=1}^n p_i\mathbf{q}_i$ is the *barycenter* of masses $p_1, \ldots, p_n$ placed at points $\mathbf{q}_1, \ldots, \mathbf{q}_n$. The extreme points of P_n are the Dirac measures δ_i, $i = 1, \ldots, n$, given by the Kronecker symbol δ_{ki}.

A *convex space* is a nonempty set X together with a family of mappings $\mathbf{p}^X : X^n \to X$ for $\mathbf{p} \in P_n$, $n = 1, 2, \ldots$, satisfying for every $x = (x_1, \ldots, x_m) \in X^m$ the identities

$$\delta_i^X(x) = x_i, \qquad \mathbf{p}^X(\mathbf{q}_1^X(x), \ldots, \mathbf{q}_n^X(x)) = \big(\sum_{i=1}^n p_i\mathbf{q}_i\big)^X(x)$$

for all $\mathbf{p} \in P_n$ and $\mathbf{q}_1, \ldots, \mathbf{q}_n \in P_m$. Using the formal notation $\sum_{i=1}^n p_ix_i = \mathbf{p}^X(x)$ the two equations take the form

$$\sum_{i=1}^n \delta_{ik}x_i = x_k \tag{A1}$$

where δ_{ik} is the Kronecker symbol, and

$$\sum_{i=1}^n p_i\big(\sum_{k=1}^m q_{ik}x_k\big) = \sum_{k=1}^m \big(\sum_{i=1}^n p_iq_{ik}\big)x_k \tag{A2}$$

for $\mathbf{p} = (p_1, \ldots, p_n) \in P_n$, and $\mathbf{q}_i = (q_{i1}, \ldots, q_{im}) \in P_m$, $i = 1, \ldots, n$.

It should be added that the main interest of the authors using the latter approach was not in convex but in *superconvex* spaces. For those one replaces the sets P_n of probability measures on finite sets by the set P_I of discrete probability measures on an infinite countable set I, using the same defining identities as above replacing finite by infinite sums.

2.2. **Ordered pointed barycentric algebras.** Convex sets containing 0 in vector spaces and in (abstract) cones are pointed barycentric algebras in the following sense:

Definition 2.10. A *pointed barycentric algebra* is a barycentric algebra A with a distinguished element usually written 0. A map $f\colon A \to B$ between pointed barycentric algebras is *0-affine* or *linear* if it is affine and preserves the distinguished element: $f(0) = 0$.

If A is a convex set containing 0 in a vector space or in an abstract cone, we have $ra \in A$ for all $a \in A$ and all $r \in [0, 1]$, that is, we have a multiplication by scalars $r \in [0, 1]$. Similarly, we can define a multiplication by scalars $r \in [0, 1]$ for an arbitrary pointed barycentric algebra A. For an element $a \in A$ and $r \in [0, 1]$ define

$$r \cdot a =_{def} a +_r 0$$

Scalar multiplication has the usual properties:

$$0 \cdot a = 0 = r \cdot 0,\ 1 \cdot a = a,\ (rs) \cdot a = r \cdot (s \cdot a),\ r \cdot (a +_s b) = r \cdot a +_s r \cdot b$$

as is straightforward to check (the last property follows from the distributive law (D)). Every linear map $f\colon A \to B$ of pointed barycentric algebras is *homogeneous* in the sense that $f(r \cdot a) = r \cdot f(a)$ for all $a \in A$ and $0 \le r \le 1$. Indeed, we have: $f(r \cdot a) = f(a +_r 0) = f(a) +_r f(0) = f(a) +_r 0 = r \cdot f(a)$.

Every cone is a pointed barycentric algebra. Thus, for a map f between cones we have two notions of homogeneity, one where r ranges over all nonnegative real numbers, and another where r only ranges over the unit interval. But the two notions are equivalent for cones; indeed, if $f(ra) = rf(a)$ for all r in the unit interval, then $f(a) = f(r^{-1}ra) = r^{-1}f(ra)$ for all $r \ge 1$, whence $rf(a) = f(ra)$ for all $r \ge 1$, too. This shows that our terminology of linearity for maps between cones and pointed barycentric algebras, respectively, is consistent.

As for barycentric algebras (see 2.3), there is a simple description for the free pointed barycentric algebra over a set I:

Lemma 2.11. *The convex set*

$$S_I = \{x = (x_i)_{i \in I} \in \mathbb{R}_+^{(I)} \mid \sum_i x_i \le 1\} \subseteq \mathbb{R}^{(I)}$$

of all finitely supported subprobability distributions on I with $(0)_{i \in I}$ as distinguished element is the free pointed barycentric algebra over I with unit δ.

Proof. Let A be a pointed barycentric algebra with distinguished element 0 and $f\colon I \to A$ an arbitrary function. We add a new element i_0 to I and extend f to this new element by $f(i_0) = 0$. There is a unique affine map g from the free barycentric algebra $P_{I\cup\{i_0\}}$ to A such that $g \circ \delta = f$. We now compose g with the affine bijection from S_I to $P_{I\cup\{i_0\}}$ that maps the subprobability distribution $x = (x_i)_{i\in I}$ on I to the probability distribution y on $I \cup \{i_0\}$ with $y_i = x_i$ for $i \in I$, and $y_{i_0} = 1 - \sum_{i\in I} x_i$. In this way we obtain an affine map $\overline{f}\colon S_I \to A$ such that $f = \overline{f} \circ \delta$ and $f(0) = 0$. Clearly, $\overline{f}$ is unique with these properties. □

We now add a partial order:

Definition 2.12. An *ordered pointed barycentric algebra* is an ordered barycentric algebra with a distinguished element 0. A map $f\colon A \to B$ of ordered pointed barycentric algebras is *sublinear* (resp., *superlinear*) if it is homogeneous and convex (resp., concave).

The linear maps are those that are both sublinear and superlinear.

For general reasons, there is a free ordered cone $\mathsf{Cone}(A)$ over any ordered pointed barycentric algebra A. We need a concrete construction of this free object. The idea is to stretch the multiplication by scalars from those in the unit interval to all nonnegative reals:

Standard Construction 2.13. Let A be an ordered pointed barycentric algebra. Since every pointed barycentric algebra is the image of a free one under a linear map, by Lemma 2.11 there is a linear surjection $f\colon S_I \to A$, where S_I is the pointed barycentric algebra of finitely supported subprobability measures on some set I.

For every $x \in \mathbb{R}_+^{(I)}$ there is a greatest real number $0 < r \leq 1$ such that $rx \in S_I$, and for every real number s with $0 < s \leq r$ we also have $sx \in S_I$. We define a relation $\precsim$ on $\mathbb{R}_+^{(I)}$:

$$x \precsim x' \text{ if } f(rx) \leq f(rx') \text{ for some } r,\ 0 < r \leq 1, \text{ such that } rx \in S_I \text{ and } rx' \in S_I.$$

If this holds for some $0 < r \leq 1$, then it also holds for all s with $0 < s \leq r$, since f is homogeneous.

Lemma 2.14. *The relation $\precsim$ is a preorder on the cone $\mathbb{R}_+^{(I)}$ compatible with the cone operations, that is, $x \precsim x'$ implies $x + y \precsim x' + y$ and $rx \precsim rx'$ for every y in the cone and every nonnegative real number r .*

Proof. Clearly, the relation $\precsim$ is reflexive and transitive. For compatibility, consider elements $x \precsim x'$ in S_I. There is an r with $0 < r \leq 1$ such that $rx, rx' \in S_I$ and $f(rx) \leq f(rx')$. Given y, choose s such that $0 < s \leq r$ and $s(x+y), s(x'+y) \in S_I$. Using that f is linear on S_I we obtain $f(\frac{s}{2}(x+y)) = f(s(x+_{\frac{1}{2}} y)) = f(sx+_{\frac{1}{2}} sy) = f(sx)+_{\frac{1}{2}} f(sy) \leq f(sx')+_{\frac{1}{2}} f(sy) = f(\frac{s}{2}(x'+y))$, that is $x + y \precsim x' + y$. The property that $tx \precsim tx'$ for $t \in \mathbb{R}_+$ is straightforward. □

Corollary 2.15. *The relation $x \sim x'$ if $x \precsim x'$ and $x' \precsim x$ is a congruence relation on the cone $\mathbb{R}_+^{(I)}$.* □

We write $\mathsf{Cone}(A)$ for $\mathbb{R}_+^{(I)}/\!\sim$, the quotient cone ordered by $\rho(x) \leq \rho(y)$ if $x \precsim y$, where $\rho\colon \mathbb{R}_+^{(I)} \to \mathbb{R}_+^{(I)}/\!\sim$ is the (linear monotone) quotient map. Restricted to S_I, the quotient map factors through f. Indeed, if x, y are elements of S_I such that $f(x) \leq f(y)$, then $x \precsim y$ by the definition of $\precsim$. Thus, there is a unique monotone linear map $u\colon A \to \mathsf{Cone}(A)$ such that $\rho|_{S_I} = u \circ f$.

We note two important properties of this map:

Lemma 2.16.
(1) *If $u(a) \leq u(b)$, for $a, b \in A$, then $ra \leq rb$ for some $0 < r \leq 1$.*
(2) *Every $b \in \mathsf{Cone}(A)$ has the form $ru(a)$ for some $a \in A$ and $r \geq 1$.*

Proof.
(1) Suppose that $u(f(x)) \leq u(f(y))$, for $x, y \in S_I$. Then $\rho(x) \leq \rho(y)$. So $x \precsim y$ and we have $f(rx) \leq f(ry)$ for some $0 < r \leq 1$, which is to say that $rf(x) \leq rf(y)$ for some $0 < r \leq 1$.
(2) Choose $x \in S_I$ such that $\rho(x) = b \in \mathsf{Cone}(A)$. There is a $y \in S_I$ and an $r \geq 1$ such that $x = ry$. Set $a =_{def} f(y)$. Then: $ru(a) = ru(f(y)) = r\rho(y) = \rho(ry) = \rho(x) = b$. □

The two properties of the map $u\colon A \to \mathsf{Cone}(A)$ picked out in this lemma yield a characterisation of when a monotone linear map from a pointed ordered barycentric algebra to an ordered cone is universal as expressed by the freeness property in the following proposition:

Proposition 2.17. *Let $u : A \to C$ be a monotone linear map from a pointed ordered barycentric algebra to an ordered cone C. Then C is the free ordered cone over A with unit u (in the sense that every monotone homogeneous map h from A into an ordered cone D has a unique monotone homogeneous extension $\widetilde{h}\colon C \to D$ along u) if, and only if, the following two properties hold of u:*
(1) *If $u(a) \leq u(b)$, for $a, b \in A$, then $ra \leq rb$ for some $0 < r \leq 1$.*
(2) *Every $x \in C$ has the form $ru(a)$ for some $a \in A$ and $r \geq 1$.*

For such universal maps u, the extension $\widetilde{h}$ is sublinear, superlinear, linear, respectively if, and only if, h is. Moreover, $\widetilde{h} \leq \widetilde{g}$ if and only if $h \leq g$.

Proof. Suppose the two properties hold. For uniqueness of the extension use the second assumption and note that, for such an $\widetilde{h}$, we have:

$$\widetilde{h}(ru(a)) = r\widetilde{h}(u(a)) = rh(a)$$

We then use this property to define $\widetilde{h}$; to show $\widetilde{h}$ well-defined and monotone we chose $a, a' \in A$ and $r, r' \geq 1$ and prove that if $ru(a) \leq r'u(a')$ then $rh(a) \leq r'h(a')$. For if $ru(a) \leq r'u(a')$ then $u(rs^{-1}a) \leq u(r's^{-1}a')$, where $s =_{def} \max(r, r')$; so, by the first assumption, $trs^{-1}a \leq tr's^{-1}a'$ for some $0 < t \leq 1$. So, in turn, we have $trs^{-1}h(a) = h(trs^{-1}a) \leq h(tr's^{-1}a') = tr's^{-1}h(a)$, and so $rh(a) \leq r'h(a)$ as required. It is straightforward that $\widetilde{h}$ is homogeneous.

That the two properties are necessary follows from Lemma 2.16, which provides an example of a map $u\colon A \to \mathsf{Cone}(A)$ possessing them. Suppose indeed that $u'\colon A \to C$ has the universal property. Then there are mutually inverse monotone homogeneous maps $\widetilde{u}\colon C \to \mathsf{Cone}(A)$ and $\widetilde{u'}\colon \mathsf{Cone}(A) \to C$ such that $u = \widetilde{u} \circ u'$ and $u' = \widetilde{u'} \circ u$. For property 1, take $a \leq b$ in A such that $u'(a) \leq u'(b)$. Then $u(a) = \widetilde{u}(u'(a)) \leq \widetilde{u}(u'(b)) = u(b)$, hence $ra \leq rb$ for some $0 < r \leq 1$. For property 2, take $x \in C$. Then $\widetilde{u}(x) = ru(a)$ for some $a \in A$ and $r \geq 1$. Applying $\widetilde{u'}$ yields $x = \widetilde{u'}(ru(a)) = r\widetilde{u'}(u(a)) = ru'(a)$.

Given such a map $u : A \to C$, suppose h is sublinear. Then so is $\widetilde{h}$. To see this choose $x, x' \in C$. By the first property $x = ru(a)$ and $x' = r'u(a')$ for some $a, a' \in A$ and $r, r' \geq 1$. Then set $s =_{def} r + r'$ and calculate: $\widetilde{h}(x + y) = \widetilde{h}(ru(a) + r'u(a')) = \widetilde{h}(su(a +_{r/s} a')) = sh(a +_{r/s} a') \leq s(h(a) +_{r/s} h(a'))$(as h is sublinear) $= rh(a) + r'h(a') = \widetilde{h}(x) + \widetilde{h}(y)$. The converse is evident as u is linear. The assertion for superlinearity is proved similarly, and then the assertion for linearity follows.

Finally we show that $\widetilde{h} \leq \widetilde{g}$ if $h \leq g$ (the converse is obvious). Assuming $h \leq g$, for any $a \in A$ and $r \geq 1$ we need only calculate: $\widetilde{h}(ru(a)) = rh(a) \leq rg(a) = \widetilde{g}(ru(a))$. □

Together with Lemma 2.16 we now have:

Theorem 2.18. *For any ordered pointed barycentric algebra A, $\mathsf{Cone}(A)$ is the free ordered cone over A with unit u in the following strong sense: Every monotone homogeneous map h from A into an ordered cone D has a unique monotone homogeneous extension $\widetilde{h}\colon \mathsf{Cone}(A) \to D$ along u. The extension $\widetilde{h}$ is sublinear, superlinear, linear, respectively if, and only if, h is. Moreover, $\widetilde{h} \leq \widetilde{g}$ if and only if $h \leq g$.* □

It would be nice, if the monotone linear map u from A into the universal cone would be an order embedding. But this is not always the case. We give an example of an ordered pointed barycentric algebra that cannot be embedded in any ordered cone at all. Indeed, in an ordered cone, if for some $0 < r < 1$ we have $rx \leq ry$, then $x \leq y$ by multiplying with the scalar r^{-1}. This need not be true in an ordered pointed barycentric algebra:

Example 2.19. We consider the unit interval and replace the element 1 by two elements $1_1, 1_2$. On each of the sets $[0, 1[\cup 1_i,\ i = 1, 2$, we take convex combinations as usual in the unit interval, and the set $\{1_1, 1_2\}$ is considered as a join-semilattice with $x +_r y = 1_2$ whenever $x \neq y$ and $0 < r < 1$. In this way we obtain a pointed barycentric algebra with 0 as distinguished element. Clearly, $rx = r = ry$ whenever $0 < r < 1$ and $x, y \in \{1_1, 1_2\}$.

We therefore pay attention to the order cancellation law:

$$rx \leq ry \implies x \leq y \quad (\text{for } 0 < r < 1) \tag{OC2}$$

and its specialised form

$$rx = ry \implies x = y \quad (\text{for } 0 < r < 1) \tag{C2}$$

These laws are particular instances of the cancellation laws (OC1) and (C1). For example (C2) can be rewritten as $x +_r 0 = y +_r 0 \implies x = y$.

In view of the properties of scalar multiplication, the map $x \mapsto rx$ of A into itself is linear and monotone. The axiom (OC2) is equivalent to the statement that this map is also an order embedding whenever $0 < r \leq 1$, which implies that the image rA is an ordered pointed barycentric algebra isomorphic to A.

Property (OC2) is less restrictive than it seems. Indeed, using Lemma 2.8 for the case $c = 0$ we obtain:

Remark 2.20. If in an ordered pointed barycentric algebra $ra \leq rb$ for some $0 < r < 1$ then this holds for all such r.

We now answer the question which ordered pointed barycentric algebras can be embedded into ordered cones, where by embedding we mean a linear order embedding:

Proposition 2.21. *For an ordered pointed barycentric algebra A satisfying* (OC2)*, the universal map $u\colon A \to \mathsf{Cone}(A)$ is an embedding. An ordered pointed barycentric algebra can be embedded into an ordered cone if, and only if, it satisfies* (OC2)*.*

Proof. Consider $u : A \to \mathsf{Cone}(A)$, and, given $a, a' \in A$, suppose that $u(a) \leq u(a')$. Then by Lemma 2.16.1, $ra \leq ra'$ for some $0 < r \leq 1$, and so $a \leq a'$, if A satisfies (OC2). Thus u is an order embedding. Thus, if A satisfies (OC2), it can be embedded in an ordered cone. Conversely, if an ordered pointed barycentric algebra can be embedded in an ordered cone, it satisfies (OC2), since (OC2) is satisfied in every ordered cone. □

We can also identify which embeddings are universal:

Corollary 2.22. *An embedding $u\colon A \to C$ of an ordered pointed barycentric algebra A in an ordered cone C is universal if, and only if, every $x \in C$ has the form $r \cdot u(a)$, for some $a \in A$ and $r \geq 1$.* □

Under the assumption (OC2), we may identify an ordered pointed barycentric A with its image in $\mathsf{Cone}(A)$ under the embedding u by the previous proposition and we will do so without mentioning in the sequel. With this identification for every $c \in \mathsf{Cone}(A)$, there is an $0 < r \leq 1$ such that $rc \in A$; we will frequently use this fact.

It would be desirable to embed an ordered pointed barycentric algebra in an ordered cone as a lower set. If an ordered pointed barycentric algebra A is embedded in an ordered cone C as a lower set, then one has for all $a, b \in A$:

$$a \leq rb \implies \exists a' \in A.\ a = ra' \quad (\text{for } 0 < r < 1) \tag{OC3}$$

Indeed, if $a \leq rb$ for $a, b \in A$ and $0 < r < 1$, then $a' =_{def} \frac{1}{r}a \leq b$ in C. Then $a' \in A$, as A is a lower set in C, and $a = ra'$. Property (OC3) is not satisfied for all ordered pointed barycentric algebras. As an example, let A be the convex hull of the points $(0,0), (1,0), (0,1), (2,2)$ in $\mathbb{R}^2_+$. Then A is a pointed barycentric algebra embedded in the ordered cone $\mathbb{R}^2_+$, but A is not a lower set. As it does not satisfy Property (OC3), A cannot be embedded in any ordered cone as a lower set. We also have a converse:

Lemma 2.23. *An ordered pointed barycentric algebra A satisfying order cancellation* (OC2) *is embedded in the ordered cone* $\mathsf{Cone}(A)$ *as a lower set if, and only if, it satisfies Property* (OC3).

Proof. By Proposition 2.21 we can embed A into the ordered cone $\mathsf{Cone}(A)$. To show that A is embedded as a lower set, suppose that x is an element of $\mathsf{Cone}(A)$ such that $x \leq b$ for some $b \in A$. Then $rx \in A$ for some r with $0 < r < 1$ and $rx \leq rb$. By Property (OC3) there is an $x' \in A$ such that $rx' = rx$ which implies $x = x' \in A$ by multiplying by $\frac{1}{r}$. □

In the presence of the order cancellation property (OC2) one has $a' \leq b$ for the element a' whose existence is postulated in (OC3).

Remark 2.24. (**Historical Notes and References**) As far as we know, pointed barycentric algebras have not attracted much attention. They are identical to the *finitely positively convex spaces* in the sense of Wickenhäuser, Pumplün, Röhrl, and Kemper [67, 48, 49, 50, 30]. To define them, one uses the same setting and the identities used for convex spaces (see historical remark 2.9), but replaces the convex sets P_n of probability measures on n-element sets by the pointed convex sets

$$S_n = \{(q_1, \ldots, q_n) \in [0,1]^n \mid \sum_{i=1}^{n} q_i \leq 1\}$$

of subprobability measures on n-element sets. As before, the main interest of these authors was directed towards the positively *superconvex* spaces and their applications in functional analysis, where the S_n are replaced by the set S of subprobability measures on an infinite countable set. Our standard construction 2.13 for constructing the free ordered cone over an ordered pointed barycentric algebra is simpler than Pumplün's construction of a free cone over an (unordered) positively superconvex space (see [48, Definition 4.17 ff.]).

In the same way as join-semilattices can be considered to be barycentric algebras, join-semilattices with a distinguished element can be considered to be pointed barycentric algebras.

Recently, convex and positively convex spaces were taken up by A. Sokolova and H. Woracek [55]. These authors are particularly interested in finitely generated barycentric and pointed barycentric algebras, that is, homomorphic images of polyhedra and pointed polyhedra in finite dimensional vector spaces, and they prove that finitely generated barycentric and pointed barycentric algebras, respectively, are finitely presented.

2.3. **d-Cones and Kegelspitzen.** We now endow partially ordered sets with their Scott topology. In particular, the sets $\mathbb{R}_+$, $\overline{\mathbb{R}}_+$, and the unit interval $[0,1]$ are endowed with their usual order and the corresponding Scott topology. Maps are restricted to Scott-continuous ones.

Thus, for an ordered cone C it is natural to ask for addition $(a,b) \mapsto a+b \colon C \times C \to C$ and scalar multiplication $(r,a) \mapsto ra \colon \mathbb{R}_+ \times C \to C$ to be Scott-continuous (in both arguments). Note that the continuity of scalar multiplication in the first argument implies that 0 is the least element of C; indeed, Scott-continuity with respect to scalars implies that $r \mapsto ra\colon \mathbb{R}_+ \to C$ is monotone, whence $0 \leq 1$ implies $0 = 0 \cdot a \leq 1 \cdot a = a$.

Definition 2.25. An ordered cone in which addition $(a, b) \mapsto a + b\colon C \times C \to C$ and scalar multiplication $(r, a) \mapsto ra\colon \mathbb{R}_+ \times C \to C$ are Scott-continuous (in both arguments) will be called an *s-cone*. If in addition the order is directed complete (resp., bounded directed complete), we say that C is a *d-cone* (resp., a *b-cone*).

We are heading towards a similar connection between the algebraic and the order structure on ordered pointed barycentric algebras. For this we have to restrict the scalars r to the unit interval:

Definition 2.26. A *Kegelspitze* is a pointed barycentric algebra K equipped with a directed complete partial order such that, for every r in the unit interval, convex combination $(a, b) \mapsto a +_r b\colon C \times C \to C$ and scalar multiplication $(r, a) \mapsto ra\colon [0, 1] \times C \to C$ are Scott-continuous in both arguments. (An alternative name would be *pointed barycentric d-algebra*.)

We only need to require scalar multiplication to be continuous in its first argument in the definition, since $a \mapsto ra = a +_r 0$ is required to be continuous anyway. The minimalistic definition of a Kegelspitze above may look artificial. In the Historical Notes 2.39 below we discuss an equivalent definition that looks more natural.

Since Scott-continuous maps are monotone, every Kegelspitze is an ordered pointed barycentric algebra. As for d-cones, 0 will be the least element. It is noteworthy that property (OC2) is always satisfied:

Lemma 2.27. *Every Kegelspitze satisfies the order cancellation property* (OC2)*: If $ra \leq rb$ for some $0 < r < 1$, then $a \leq b$.*

Proof. Indeed, if $ra \leq rb$ for some $0 < r < 1$, the order theoretical version of Neumann's lemma 2.8 implies that $ra \leq rb$ for all $r < 1$ which implies $a \leq b$ by the Scott continuity of the map $r \mapsto ra\colon [0, 1] \to K$. □

We would like to embed every Kegelspitze K into a d-cone, where embeddings of Kegelspitzen in d-cones are Scott-continuous linear maps which are order embeddings. We proceed in two steps. In a first step we use the embedding of K (considered as a pointed barycentric algebra) in the ordered cone $\mathsf{Cone}(K)$ according to Standard Construction 2.13. By Proposition 2.21 this Standard Construction yields indeed a linear order embedding u of K in $\mathsf{Cone}(K)$, since K satisfies (OC2) by Lemma 2.27. By the following lemma, this embedding is Scott-continuous:

Lemma 2.28. *Let u be a homogeneous order embedding of a Kegelspitze K in an s-cone C in such a way that for every element $y \in C$ there is an r, $0 < r < 1$, such that $ry \in u(K)$. Then u is Scott-continuous.*

Proof. Let $(x_i)_i$ be a directed family in K and x its supremum in K. Since u is monotone, clearly $u(x_i) \leq u(x)$. In order to show that $u(x)$ is the supremum of the $u(x_i)$ in C, consider any upper bound $y \in C$ of the $u(x_i)$. Choose an r, $0 < r < 1$, such that $ry \in u(K)$ and $y' \in K$ with $u(y') = ry$. Then $ry = u(y')$ is an upper bound of the directed family $ru(x_i) = u(rx_i)$ (using homogeneity of u). Since u is an order embedding, y' is an upper bound of the rx_i in K. By the Scott continuity of scalar multiplication, rx is the least upper bound of the rx_i in K, whence $rx \leq y'$. Using homogeneity and monotonicity of u, we deduce $ru(x) = u(rx) \leq u(y') = ry$ which implies $u(x) \leq y$. □

We now want to show that scalar multiplication and addition on $\mathsf{Cone}(K)$ are Scott-continuous. This is no problem for scalar multiplication:

We first recall that $a \mapsto ra \colon \mathsf{Cone}(K) \to \mathsf{Cone}(K)$ is Scott-continuous for every $r > 0$, since this map is monotone and has a monotone inverse, multiplication by r^{-1}.

We now verify that $r \mapsto ra \colon \mathbb{R}_+ \to \mathsf{Cone}(K)$ is Scott-continuous. Suppose indeed that r_i is an increasing family in $\mathbb{R}_+$ with $r = \sup_i r_i$. Choose an $s, 0 < s < 1$, such that $sr \leq 1$ and $sa \in K$. We then use the continuity of $r \mapsto ra \colon [0,1] \to K$ to obtain $\sup_i(sr_i)(sa) = (\sup_i sr_i)(sa) = (sr)(sa)$. Now in $\mathsf{Cone}(K)$ we have $s^2 \sup_i r_i a = \sup_i(sr_i)(sa) = srsa = s^2(ra)$ which implies $\sup_i r_i a = ra$.

We now turn to addition. To prove that $a \mapsto a + b \colon \mathsf{Cone}(K) \to \mathsf{Cone}(K)$ is Scott-continuous for every fixed $b \in \mathsf{Cone}(K)$, we have to show: If a_i is a directed system in $\mathsf{Cone}(K)$ which has a sup $a = \sup_i a_i$ then the family $a_i + b$ has a sup and $a + b = \sup_i(a_i + b)$. For this we choose an s, $0 < s < 1$, such that $sa \in K$ and $sb \in K$. Because $sa_i \leq sa$, we would like to conclude that $sa_i \in K$, since then the Scott continuity of convex combination in K implies that $\sup_i(\frac{1}{2}(sa_i) + \frac{1}{2}(sb)) = \frac{1}{2}\sup_i(sa_i) + \frac{1}{2}(sb) = \frac{1}{2}(sa) + \frac{1}{2}(sb)$, whence in $\mathsf{Cone}(K)$ we have $\sup_i(a_i + b) = 2s^{-1}\sup_i(\frac{1}{2}(sa_i) + \frac{1}{2}(sb)) = 2s^{-1}(\frac{1}{2}(sa) + \frac{1}{2}(sb) = a + b$ as desired.

Thus, we would like to use that K is a lower set in $\mathsf{Cone}(K)$. By Lemma 2.23, this is equivalent to the requirement that K satisfies Property (OC3). As we often use this property we make a definition:

Definition 2.29. A Kegelspitze is said to be *full* if it satisfies Property (OC3).

We now can state:

Proposition 2.30. *For a full Kegelspitze K, the free cone $\mathsf{Cone}(K)$ over K according to Standard Construction 2.13 is a b-cone and K is Scott-continuously embedded in $\mathsf{Cone}(K)$ as a Scott-closed convex set.*

The b-cone $\mathsf{Cone}(K)$ is the free b-cone over K w.r.t. Scott-continuous homogeneous maps as, for every such map f from K into a b-cone D, the unique homogeneous extension $\widetilde{f} \colon \mathsf{Cone}(K) \to D$ is Scott-continuous. Moreover, $\widetilde{f}$ is sublinear, superlinear, or linear, if, and only if, f is. Further, $f \leq g$ if and only if $\widetilde{f} \leq \widetilde{g}$.

Proof. In the presence of (OC3), Lemma 2.23 shows that we have a linear order embedding of K in $\mathsf{Cone}(K)$ as a lower set and convexity is evident. The embedding is Scott-continuous by Lemma 2.28 and so K is embedded as a Scott-closed set.

The arguments preceding the statement of the proposition show that addition and scalar multiplication are Scott-continuous on $\mathsf{Cone}(K)$. We even have a b-cone: Indeed, if $(x_i)_i$ is a directed family in $\mathsf{Cone}(K)$ bounded above by some x, choose an r, $0 < r < 1$, such that $rx \in K$. Using that K is a lower set, $(rx_i)_i$ is a directed family in K and so has a sup $y = \sup_i rx_i$ in K. We conclude that the $(x_i)_i$ have a sup namely $r^{-1}y = \sup_i x_i$.

Now let $f\colon K \to D$ be a homogeneous function into a b-cone D. By Theorem 2.18 it has a unique homogeneous extension $\widetilde{f}\colon \mathsf{Cone}(K) \to D$. If f is Scott-continuous, then $\widetilde{f}$ is Scott-continuous, too: Indeed, let $(x_i)_i$ be a bounded directed family in $\mathsf{Cone}(K)$ with $x = \sup_i x_i$. Choose any $r > 0$ such that $rx \in K$. Since K is a lower set in $\mathsf{Cone}(K)$ we also have $rx_i \in K$ and $rx = \sup_i rx_i$ in K. By the continuity of f on K we have $f(rx) = \sup_i f(rx_i)$, whence $\widetilde{f}(x) = r^{-1}\widetilde{f}(rx) = r^{-1}f(rx) = r^{-1}\sup_i f(rx_i) = r^{-1}\sup_i \widetilde{f}(rx_i) = \sup_i \widetilde{f}(x_i)$.

The remaining claims follow directly from Theorem 2.18. □

In a second step we use a completion procedure following Zhang and Fan [70], Keimel and Lawson [26, 27] and Jung, Moshier, and Vickers [21], in order to embed the b-cone $\mathsf{Cone}(K)$ in a d-cone.

Standard Construction 2.31. A *universal* (or *free*) *dcpo-completion*[2] of a poset P consists of a dcpo $\overline{P}$ and a Scott-continuous map $\xi\colon P \to \overline{P}$ enjoying the universal property that every Scott-continuous map f from P to a dcpo Q has a unique Scott-continuous extension $\overline{f}\colon \overline{P} \to Q$ satisfying $f = \overline{f} \circ \xi$. A universal dcpo-completion, if it exists, is evidently unique up to a canonical isomorphism.

Let us extract relevant information about universal dcpo-completions of a poset P from the literature:

(1) *Every poset P has a universal dcpo-completion.* One may, for example [70, Theorem 1], take the least sub-dcpo $\overline{P}$ of the dcpo of all nonempty Scott-closed subsets of P (ordered by inclusion) containing the principal ideals ${\downarrow}x$ with $x \in P$, and $\xi =_{def} (x \mapsto {\downarrow}x)\colon P \to \overline{P}$ as canonical map.

(2) *Let $\xi\colon P \to D$ be a topological embedding (for the respective Scott topologies) of a poset P into a dcpo D. Then the least sub-dcpo $\overline{P}$ of D containing the image $\xi(P)$ together with the corestriction $\xi\colon P \to \overline{P}$ is a universal dcpo-completion and the Scott topology of $\overline{P}$ is the subspace topology induced by the Scott topology on D.* (See [26, Theorem 7.4]).

(3) *A function $\xi\colon P \to D$ of a poset P into a dcpo D is a universal dcpo-completion if, and only if,*
(i) ξ is a topological embedding (for the Scott topologies) and

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[2] In the literature [70, 26, 27] the term *dcpo-completion* is used instead of *universal dcpo-completion*. For our purposes we prefer the the latter terminology, since there are Scott-continuous order embeddings of posets into dcpos which are dense for directed suprema but not universal, for example the embedding of $\mathbb{R}^2_+$ into the dcpo obtained by adding a top element.

(ii) the image $\xi(P)$ is dense in D

(This follows from the previous item, since any two universal dcpo-completions are isomorphic.)

(4) *Let $P_1, \ldots, P_n, Q$ be posets and let $\overline{P_1}, \ldots, \overline{P_n}, \overline{Q}$ be universal dcpo-completions thereof. Then every Scott-continuous function $f\colon P_1 \times \cdots \times P_n \to Q$ has a unique Scott-continuous extension $\overline{f}\colon \overline{P_1} \times \cdots \times \overline{P_n} \to \overline{Q}$. Further, if g is another such function, then $\overline{f} \leq \overline{g}$ if, and only if, $f \leq g$.* (Here the products are understood to have the product order. Thus, the claim follows from [27, Proposition 5.6], since functions defined on finite products are Scott-continuous if, and only if, they are separately Scott-continuous in each of their arguments.)

(5) *The universal dcpo-completion of a finite direct product of posets is the direct product of the universal dcpo-completions of its factors. More precisely, if $\xi_i\colon P_i \to \overline{P_i}$ $(i = 1, \ldots, n)$ are universal dcpo-completions, then so is*

$$\xi =_{def} \xi_1 \times \cdots \times \xi_n \colon P_1 \times \cdots \times P_n \to \overline{P_1} \times \cdots \times \overline{P_n}$$

(This follows directly from the previous item.)

In the characterisation 2.31(3) of universal dcpo-completions above, the first condition — being a topological embedding — is the critical one. As we now show, it holds automatically in many situations.

To begin with, we remark that, for a Scott-closed subset C of a poset P, the canonical embedding of C into P is topological, that is, the intrinsic Scott topology on C is the subspace topology induced by the Scott topology on P.

But even on a lower subset P of a dcpo Q, the intrinsic Scott topology of P may be strictly finer than the subspace topology induced by the Scott topology of Q. A simple example for this phenomenon is given by $P = \mathbb{R}_+ \times \mathbb{R}_+$ with the coordinatewise order and $Q = P^\top$, the dcpo obtained by attaching to P a top element. Here $[0, 1] \times \mathbb{R}_+$ is Scott-closed in P, but the Scott closure of this subset in Q is all of Q.

Following [36] we say that a dcpo P is *meet continuous* if for any $x \in P$ and any directed set $D \subseteq P$ with $x \leq \bigvee^\uparrow D$, x is in the Scott closure of $\downarrow x \cap \downarrow D$. All domains are meet continuous as are all dcpos with a Scott-continuous meet operation.

Lemma 2.32. *The canonical embedding of a lower subset P of a meet continuous dcpo Q is a topological embedding (for the respective Scott topologies).*

Proof. One checks that every directed sup in P is also a directed sup in Q. Consequently the inclusion of P in Q is Scott-continuous. So, for every subset V of Q that is Scott-open in Q, $V \cap P$ is open for the Scott topology on P.

In the other direction we have to show: Let U be a subset of P which is open for the Scott topology on P. Then there is a Scott-open subset V of Q such that $U = V \cap P$. Since $\uparrow U \cap P = U$, it suffices to show that $\uparrow U$ is Scott-open in Q, where $\uparrow U$ is the upper set in Q generated by U. For this take any directed subset D in Q with supremum d in $\uparrow U$. We have to show that there is a $c \in D$ with $c \in \uparrow U$.

To this end, as d in $\uparrow U$, there is an $x \in U$ with $x \leq d$. By meet continuity, x is in the Scott closure w.r.t. Q of $\downarrow x \cap \downarrow D$. Following the remark above on the Scott topologies of closed subsets of partial orders, we see that the intrinsic Scott topology on $\downarrow x$ agrees with both the subspace topology induced by the Scott topology on Q, and the subspace topology induced by the Scott topology on P. As the P-closure of the set $\downarrow x \cap \downarrow D$ intersects with the P-open set U, the set $\downarrow x \cap \downarrow D$ itself intersects U. For an element x' in the intersection, one has $x' \in U$ and $x' \leq c$ for some $c \in D$, whence $c \in \uparrow U$, and we see that this c has the required properties. □

The previous lemma yields a sufficient condition for the universality of dcpo-completions:

Corollary 2.33. *The canonical embedding of a lower subset P of a meet continuous dcpo Q is a universal dcpo-completion if, and only if, P is dense in Q.* □

We now consider the universal dcpo-completion of an s-cone.

Proposition 2.34. *Let C be an s-cone and $\overline{C}$ a universal dcpo-completion. Then addition and scalar multiplication on C extend uniquely to Scott-continuous operations on the dcpo-completion $\overline{C}$ which thus becomes a d-cone. The unique Scott-continuous extension $\overline{f}\colon \overline{C} \to D$ of a Scott-continuous function f from C to a d-cone D is homogeneous, sublinear, superlinear, or linear, respectively, if f is. Moreover, $f \leq g$ if and only if $\overline{f} \leq \overline{g}$.*

Proof. By property 2.31(4) of universal dcpo-completions, addition and scalar multiplication on C extend uniquely to Scott-continuous operations on $\overline{C}$. As a consequence of [27, Proposition 8.1], the extended operations obey the same equational laws as in C, that is, $\overline{C}$ is a d-cone.

Now let D be any d-cone and $f\colon C \to D$ a Scott-continuous map. By the universal property, f has a unique Scott-continuous extension $\overline{f}\colon \overline{C} \to D$. If f is subadditive, let us show that $\overline{f}$ is subadditive, too. For this we consider the two maps $g : (a, b) \mapsto f(a + b)$ and $h : (a, b) \mapsto f(a) + f(b)$ from $C \times C$ to D. Clearly, $(a, b) \mapsto \overline{f}(a + b)$ and $(a, b) \mapsto \overline{f}(a) + \overline{f}(b)$ are Scott-continuous extensions of g and h to $\overline{C} \times \overline{C}$. Since $\overline{C} \times \overline{C}$ is the dcpo-completion of $C \times C$ by property 2.31(4) of dcpo-completions, these are the unique Scott-continuous extensions $\overline{g}$ and $\overline{h}$, respectively. The subadditivity of f is equivalent to the statement that $g \leq h$ which, by the last part of property 2.31(4) of dcpo-completions, is equivalent to $\overline{g} \leq \overline{h}$ which again is equivalent to the subadditivity of $\overline{f}$. The argument for superadditivity is similar and for homogeneity it is even simpler. □

We now consider a full Kegelspitze K and apply the completion procedure above to $\mathsf{Cone}(K)$ which, by Proposition 2.30, is the universal b-cone over K; we write $\mathsf{d\text{-}Cone}(K)$ for its universal dcpo-completion $\overline{\mathsf{Cone}(K)}$. We know that K is embedded in $\mathsf{Cone}(K)$ as a Scott-closed convex set. Since $\mathsf{Cone}(K)$ is bounded directed complete, it is a lower set in its dcpo-completion. Thus K is a lower set in $\overline{\mathsf{Cone}(K)}$, too. Further, since K is embedded in $\mathsf{Cone}(K)$ and since universal dcpo-completions preserve existing directed sups, K is a sub-dcpo of $\overline{\mathsf{Cone}(K)}$. So K is Scott-closed in the universal dcpo-completion $\overline{\mathsf{Cone}(K)} = \mathsf{d\text{-}Cone}(K)$.

Using Propositions 2.30 and 2.34 we then obtain:

Theorem 2.35. *Let K be a full Kegelspitze. The universal dcpo-completion $\mathsf{d\text{-}Cone}(K)$ of the b-cone $\mathsf{Cone}(K)$ according to the Standard Construction 2.13 is a d-cone and K is embedded in this d-cone as a Scott-closed convex set.*

The embedding of K into $\mathsf{d\text{-}Cone}(K)$ is universal in the sense that every Scott-continuous homogeneous map f from K into a d-cone D has a unique Scott-continuous homogeneous extension $\overline{f}\colon \mathsf{d\text{-}Cone}(K) \to D$. Moreover, the extension $\overline{f}$ is sublinear, superlinear, or linear if, and only if, f is. Moreover, $f \le g$ if, and only if, $\overline{f} \le \overline{g}$. □

Below, we wish to identify some naturally occurring embeddings as universal. To that end, we begin with a proposition characterising universal embeddings. The *standard factorisation* of a Scott-continuous linear order embedding $K \xrightarrow{e} C$ of a Kegelspitze in a d-cone is

$$K \xrightarrow{u} B \xrightarrow{\xi} C$$

where B is the sub-cone of C with carrier $\{re(a) \mid r > 1, a \in K\}$ and the induced order, u is the co-restriction of e, and ξ is the inclusion.

Lemma 2.36. *Let $K \xrightarrow{u} B \xrightarrow{\xi} C$ be the standard factorisation of a Scott-continuous linear order embedding $K \xrightarrow{e} C$ of a full Kegelspitze K in a d-cone. Then B is the free b-cone over K, with unit u, with respect to Scott continuous homogeneous maps, and ξ is a Scott-continuous linear map.*

Proof. Clearly $K \xrightarrow{u} B$ is an embedding of an ordered pointed barycentric algebra in an ordered cone. Using Corollary 2.22 and Proposition 2.30, we then see that B is the free b-cone over K, with unit u, with respect to both monotone homogeneous maps and Scott-continuous homogeneous maps.

The map ξ is a monotone homogeneous map from B into C extending e along u (i.e., $\xi \circ u = e$). As e is Scott-continuous and B is the free b-cone over K, with unit u, with respect to both monotone and Scott-continuous homogeneous maps, ξ is in fact Scott-continuous. It is evidently linear. □

Proposition 2.37. *Let $K \xrightarrow{u} B \xrightarrow{\xi} C$ be the standard factorisation of an embedding $K \xrightarrow{e} C$ of a full Kegelspitze K in a d-cone. Then e is universal for Scott-continuous homogeneous maps if, and only if, C is the universal dcpo-completion over B, with unit ξ.*

Proof. In one direction, suppose that C is the universal dcpo-completion over B, with unit ξ. By Lemma 2.36 we also have that B is the free b-cone over K, with unit u. It then follows from Proposition 2.34 that C is the free d-cone over B, with unit ξ. So C is the free d-cone over K, with unit $\xi \circ u = e$, as required.

In the other direction assume that e is universal, and consider the standard construction of the free d-cone over K:

$$K \xrightarrow{u_s} \mathsf{Cone}(K) \xrightarrow{\xi_s} \mathsf{d\text{-}Cone}(K)$$

As $\mathsf{Cone}(K)$ is the free b-cone over K with unit u_s and, by Lemma 2.36, B is the free b-cone over K with unit u, there is a homogeneous dcpo-isomorphism $\alpha\colon \mathsf{Cone}(K) \cong B$ such that $\alpha \circ u_s = u$. Next, as $\mathsf{d\text{-}Cone}(K)$ is the free d-cone over $\mathsf{Cone}(K)$ with unit ξ_s, and as α and ξ are Scott-continuous homogeneous maps (the latter by Lemma 2.36), there is a Scott-continuous homogeneous map $\beta\colon \mathsf{d\text{-}Cone}(K) \to C$ such that $\beta \circ \xi_s = \xi \circ \alpha$. We then have $\beta \circ (\xi_s \circ u_s) = e$. But, as $\mathsf{d\text{-}Cone}(K)$ is the free d-cone over K with unit $\xi_s \circ u_s$ and, by assumption, C is the free d-cone over K with unit e, it follows that β is a dcpo-isomorphism.

Putting all this together, we see that we have dcpo-isomorphisms $\alpha : \mathsf{Cone}(K) \cong B$ and $\beta : \mathsf{d\text{-}Cone}(K) \cong C$ such that $\beta \circ \xi_s = \alpha \circ \xi$. Then, as $\mathsf{Cone}(K) \xrightarrow{\xi_s} \mathsf{d\text{-}Cone}(K)$ is a dcpo-completion, it follows that $B \xrightarrow{\xi} C$ is a dcpo-completion, as required. □

As a first application of the proposition, we next give a sufficient condition for an embedding of a Kegelspitze in a d-cone to be universal in the meet continuous case. The criterion applies in particular to continuous d-cones as indicated just before the statement of Lemma 2.32.

Proposition 2.38. *Let $e\colon K \to C$ be a Scott-continuous linear order embedding of a full Kegelspitze K in a meet continuous d-cone. Suppose that:*

(1) $e(K)$ is a lower subset of C, and

(2) $B =_{def} \{re(a) \mid a \in K, r > 0\}$ is dense in C.

Then e is universal.

Proof. Endowing B with the ordered cone structure induced by C, we obtain $K \xrightarrow{u} B \xrightarrow{\xi} C$, the standard factorisation of e. We may suppose, w.l.o.g., that u, and so e, is an inclusion. As $e(K)$ is a lower subset of C, so is B. Then, by Proposition 2.37, Corollary 2.33 and the assumption that B is dense in C, we obtain the desired result. □

As another application of Proposition 2.37, we check that, given two universal embeddings $K_i \xrightarrow{e_i} C_i$ $(i = 1, 2)$ of full Kegelspitzen in d-cones, their product $K_1 \times K_2 \xrightarrow{e_1 \times e_2} C_1 \times C_2$ is itself a universal embedding (we use the evident definitions of the products of Kegelspitzen and of d-cones). Let $K_i \xrightarrow{u_i} B_i \xrightarrow{\xi_i} C_i$ be the standard factorisation of e_i. Then one checks that the standard factorisation of $e_1 \times e_2$ is $K_1 \times K_2 \xrightarrow{u_1 \times u_2} B_1 \times B_2 \xrightarrow{\xi_1 \times \xi_2} C_1 \times C_2$ (we use the evident definition of the product of ordered cones). By Proposition 2.37, the $B_i \xrightarrow{\xi_i} C_i$ are universal dcpo-completions, and so, by 2.31(4), $B_1 \times B_2 \xrightarrow{\xi_1 \times \xi_2} C_1 \times C_2$ is also a universal dcpo-completion. Using Proposition 2.37 again, we see that $e_1 \times e_2$ is universal.

Remark 2.39. (**Historical Notes and References**) The *abstract probabilistic algebras* of Graham and Jones [14, 18] are barycentric algebras on a dcpo P with a bottom element 0 such that the map $+ : [0,1] \times P^2 = (r,(x,y)) \mapsto x +_r y$ is continuous, taking the Hausdorff topology on $[0,1]$, the Scott topology on P^2 , and the product topology on $[0,1] \times P^2$.

Let us add that Jones [18, Section 4.2] proves that an abstract probabilistic algebra can equivalently be defined to be a dcpo P together with Scott-continuous maps $S_n \times P^n \to P$, $n \in \mathbb{N}$, informally written $\sum_{i=1}^n q_i x_i$, satisfying the equations (A1) and (A2) in remark 2.9, where S_n is the domain of subprobability measures on an n-element set as in remark 2.24 with the Scott topology. Jones explicitly adds the requirement that the operations $\sum_{i=1}^n q_i x_i$ are commutative in the sense that they are invariant under any permutation of the indices, a property that other authors (see the remarks 2.9 and 2.24) silently hide in the suggestive notation of a sum.

The notion of an abstract probabilistic algebra P of C. Jones is equivalent to our notion of a Kegelspitze. Indeed, one can show that a barycentric algebra over a dcpo is an algebra in this sense if, and only if, it is a Kegelspitze in our sense, so the two notions are equivalent. However, with our definition one can directly use domain-theoretic methods, for example for completing Kegelspitzen to d-cones.

2.4. **Preservation results.** We now turn to the question as to which additional properties of a Kegelspitze K are inherited by the universal d-cone $\mathsf{d\text{-}Cone}(K)$ over K. First we consider continuity. We define a Kegelspitze K to be *continuous* if it is continuous as a dcpo.

We will use the following standard lemma:

Lemma 2.40. *Let P be a Scott-closed subset of a continuous poset Q. Then P is a continuous poset and the way-below relation $\ll_P$ on P is the restriction of the way-below relation $\ll_Q$ on Q.*

Proof. Let x, y be elements of P. Clearly $x \ll_Q y$ implies $x \ll_P y$. Thus, if Q is continuous, the same holds for P. Conversely, let $x \ll_P y$. Since Q is continuous, the elements $z \ll_Q y$ form a directed set with supremum y. Since this directed set is in P, there is some $z \ll_Q y$ such that $x \le z$, whence $x \ll_Q y$. □

In any d-cone, $x \mapsto rx$ is an order isomorphism for $r > 0$, whence $x \ll y \iff rx \ll ry$ for every $r > 0$. This statement is not true for Kegelspitzen, in general.

For elements x, y in a Kegelspitze K and $0 < r < 1$ we always have that $rx \ll ry$ implies $x \ll y$. Indeed if $y \le \sup_i x_i$, then $ry \le r \sup_i x_i = \sup_i rx_i$, whence $rx \le rx_i$ for some i which implies $x \le x_i$ by Lemma 2.27. But $x \ll y$ does not imply $rx \ll ry$ in general, as the counterexample in Appendix B shows. Fortunately, this difficulty disappears when we require property (OC3):

Lemma 2.41. *In a continuous full Kegelspitze K we have $x \ll y$ if and only if $rx \ll ry$ for every r with $0 < r < 1$.*

Proof. Let K be a continuous Kegelspitze and $0 < r < 1$. By Property (OC2), the map $x \mapsto rx$ is an order isomorphism from K onto rK and rK is a sub-dcpo of K. Moreover, if x and y are elements of K such that $x \ll y$ in K, then $rx \ll ry$ in rK. Property (OC3) implies that rK is a lower set, hence, a Scott-closed subset of K. By Lemma 2.40, the way-below relation of the dcpo rK is the restriction of the way-below relation on K. Thus $rx \ll ry$ in K. □

Proposition 2.42. *For a full Kegelspitze K, the cone* $\mathsf{d\text{-}Cone}(K)$ *is continuous if and only if K is a continuous Kegelspitze. If this is the case, then every element $x \in \mathsf{d\text{-}Cone}(K)$ is the supremum of a directed family of elements a_i in* $\mathsf{Cone}(K)$ *with $a_i \ll x$.*

Proof. If the d-cone $\mathsf{d\text{-}Cone}(K)$ is continuous, then the Scott-closed subset K is continuous by Lemma 2.40. Suppose conversely that K is a continuous Kegelspitze satisfying (OC3). Then K may be considered to be a Scott-closed convex subset of the ordered cone $\mathsf{Cone}(K)$.

For x, y in $\mathsf{Cone}(K)$, we have $x \ll_{\mathsf{Cone}(K)} y$ iff $rx, ry \in K$ and $rx \ll_K ry$ for some $r > 0$. Indeed, suppose that $x \ll_{\mathsf{Cone}(K)} y$. Let $r > 0$ be such that $rx, ry \in K$ and let $u_i \in K$ be a directed family with $ry \leq \sup_i u_i$. Then $y \leq r^{-1} \sup_i u_i = \sup_i r^{-1} u_i$ whence $x \leq r^{-1} u_i$ for some i which implies that $rx \leq u_i$. Conversely, suppose that $rx \ll_K ry$ and consider a directed family $v_i \in \mathsf{Cone}(K)$ such that $y \leq \sup_i v_i$. There is an $s > 0$ such that $s \sup_i v_i \in K$. Moreover, we may choose $s < r$ and $s < 1$. By Lemma 2.41, we know that $sx = \frac{s}{r} rx \ll_K \frac{s}{r} ry = sy$. Since $sy \leq s \sup_i v_i = \sup_i sv_i$ we conclude that that $sx \leq sv_i$ for some i, whence $x \leq v_i$ by Lemma 2.27. We conclude that $\mathsf{Cone}(K)$ is a continuous b-cone.

For the continuous b-cone $\mathsf{Cone}(K)$, the universal dcpo-completion $\mathsf{d\text{-}Cone}(K)$ agrees with the round ideal completion which is a continuous d-cone by [27, Corollary 8.3]. The elements of a round ideal constitute a directed family of elements way below the element defined by the round ideal. □

Recall that we say that the way-below relation on an ordered cone is *additive* if

$$a \ll b \text{ and } a' \ll b' \implies a + a' \ll b + b'$$

Similarly, in a Kegelspitze we say that convex combinations preserve the way-below-relation if

$$a \ll b \text{ and } a' \ll b' \implies a +_r a' \ll b +_r b'$$

Proposition 2.43. *Let K be a continuous full Kegelspitze. Then* $\mathsf{d\text{-}Cone}(K)$ *has an additive way-below relation if and only if convex combinations preserve the way-below-relation in K.*

Proof. Suppose first that convex combinations preserve the way-below relation on K. Choose $a \ll a'$ and $b \ll b'$ in $\mathsf{d\text{-}Cone}(K)$. Interpolate elements $a \ll a'' \ll a'$ and $b \ll b'' \ll b'$. Then a'' and b'' belong to $\mathsf{Cone}(K)$, since, by Proposition 2.42, a' and b' are suprema of directed families in $\mathsf{Cone}(K)$, and since $\mathsf{Cone}(K)$ is a lower set in $\mathsf{d\text{-}Cone}(K)$. Thus we can find an r with $0 < r < 1$ such that $ra'' \in K$ and $rb'' \in K$. Since $ra \ll ra''$ and $rb \ll rb''$ in K, we use the property that convex combinations preserve the way-below relation to conclude that $\frac{1}{2}ra + \frac{1}{2}rb \ll \frac{1}{2}ra'' + \frac{1}{2}rb''$. Multiplying by $2r$ yields $a + b \ll a'' + b'' \leq a' + b'$. Thus the way-below relation is additive on $\mathsf{d\text{-}Cone}(K)$. The converse is straightforward. □

Another noteworthy property is coherence. Recall that a dcpo is called *coherent* if the intersection of any two Scott-compact saturated subsets is Scott-compact.

Proposition 2.44. *Let K be a continuous full Kegelspitze. Then the d-cone* $\mathsf{d\text{-}Cone}(K)$ *is coherent if, and only if, K is coherent.*

Proof. Clearly, if $\mathsf{d\text{-}Cone}(K)$ is coherent, then the Scott-closed subset K is coherent, too. For the converse recall that a continuous poset is said to have property M with respect to a basis B if, for any $x_1, x_2, y_1, y_2 \in B$ with $y_1 \ll x_1$ and $y_2 \ll x_2$ there is a finite set $F \subset B$ such that $\uparrow x_1 \cap \uparrow x_2 \subseteq \uparrow F \subseteq \uparrow y_1 \cap \uparrow y_2$. By [7, Proposition III-5.12] the following are equivalent for a continuous dcpo P:

(1) P is coherent.
(2) P satisfies M for every basis B.
(3) P satisfies M for some basis B.

Now let K be a continuous full Kegelspitze which is coherent. Thus, K satisfies property M with $B = K$. We conclude that $\mathsf{Cone}(K)$ satisfies property M with $B = \mathsf{Cone}(K)$. Suppose indeed that x_1, x_2, y_1, y_2 are elements of $\mathsf{Cone}(K)$ such that $x_1 \ll y_1$ and $x_2 \ll y_2$. We can find an r with $0 < r < 1$ in such a way that $ry_1 \in K$ and $ry_2 \in K$. As $rx_1 \ll ry_1$ and $rx_1 \ll ry_1$ hold in K, by the coherence of K we can find a finite subset F of K such that $\uparrow rx_1 \cap \uparrow rx_2 \subseteq \uparrow F \subseteq \uparrow ry_1 \cap \uparrow ry_2$. We then have $\uparrow x_1 \cap \uparrow x_2 \subseteq \uparrow r^{-1}F \subseteq \uparrow y_1 \cap \uparrow y_2$.

Since $\mathsf{Cone}(K)$ is a basis of $\mathsf{d\text{-}Cone}(K)$, we conclude that $\mathsf{d\text{-}Cone}(K)$ is coherent. □

By [7, Corollary II-5.13] we conclude:

Corollary 2.45. *Let K be a continuous full Kegelspitze. Then* $\mathsf{d\text{-}Cone}(K)$ *is Lawson compact if, and only if, K is Lawson compact.* □

2.5. **Duality and the subprobabilistic powerdomain.** We next give a brief introduction to duality for d-cones and Kegelspitzen, together with our main examples, function spaces and probabilistic powerdomains. For any dcpos P and Q we write Q^P for the dcpo of all Scott-continuous maps $f : P \to Q$ with the pointwise order, and note that it is a continuous lattice whenever P is a domain and Q is a continuous lattice (see, e.g., [7]).

Example 2.46. (d-Cone function spaces) Let P be a dcpo. For every d-cone $(C, +, 0, \cdot)$, the dcpo C^P is a d-cone when equipped with the pointwise sum and scalar multiplication:

$$(f + g)(x) =_{def} f(x) + g(x), \qquad (r \cdot f)(x) =_{def} r \cdot f(x)$$

We write $\mathcal{L}P$ for the d-cone $\overline{\mathbb{R}}_+^P$ of all Scott-continuous functions $f : P \to \overline{\mathbb{R}}_+$. We will use the following properties of the d-cone $\mathcal{L}P$ later on:

(a) If P is a domain then $\mathcal{L}P$ is a continuous lattice, hence a continuous d-cone.
(b) For any domain P, the way-below relation of $\mathcal{L}P$ is additive, if, and only if, P is coherent (Tix [61, Propositions 2.28 and 2.29]).

Example 2.47. (Kegelspitze function spaces) For every Kegelspitze $(K, +_r, 0)$ the dcpo K^P becomes a Kegelspitze with the pointwise barycentric operations:

$$(f +_r g)(x) =_{def} f(x) +_r g(x)$$

and with the constant function with value 0 as distinguished element.

Taking K to be the unit interval $\mathbb{I} = [0, 1]$, we obtain the Kegelspitze $\mathcal{L}_{\leq 1}(P) =_{def} \mathbb{I}^P$. It is a Scott-closed convex subset of $\mathcal{L}P$ forming a sub-Kegelspitze of $\mathcal{L}P$ (considered as a Kegelspitze), and therefore full. Further, $\mathcal{L}P$ is the universal d-cone over the Kegelspitze $\mathcal{L}_{\leq 1}P$, that is $\mathcal{L}P \cong \mathsf{d\text{-}Cone}(\mathcal{L}_{\leq 1}P)$. The inclusion $\mathcal{L}_{\leq 1}(P) \subseteq \mathcal{L}(P)$ is the universal embedding, as follows from Proposition 2.38, noting that $\mathcal{L}P$ has a continuous meet, and every $f \in \mathcal{L}P$ is the sup of the sequence of bounded functions $p_n \wedge f$, where p_n is the projection sending $\overline{\mathbb{R}}_+$ to $[0, n]$. Finally, $\mathcal{L}_{\leq 1}P$ is a domain if P is, and its way-below relation is inherited from that of $\mathcal{L}P$ and is preserved by the barycentric operations if, and only if, P is coherent.

For any two d-cones C and D, the Scott-continuous linear functions $f : C \to D$ form a sub-d-cone $\mathcal{L}_{\text{lin}}(C, D)$ of the function space D^C. Similarly, for any Kegelspitze K and d-cone D, the set $\mathcal{L}_{\text{lin}}(K, D)$ of Scott-continuous linear functions is a sub-d-cone of D^K. Note that if K is C, regarded as a Kegelspitze, then $\mathcal{L}_{\text{lin}}(C, D)$ and $\mathcal{L}_{\text{lin}}(K, D)$ are identical.

Duality will play an important rôle: every d-cone C and Kegelspitze K has a dual d-cone, viz. $C^* =_{def} \mathcal{L}_{\text{lin}}(C, \overline{\mathbb{R}}_+)$ and $K^* =_{def} \mathcal{L}_{\text{lin}}(K, \overline{\mathbb{R}}_+)$, respectively.

By Theorem 2.35, if K is full then every Scott-continuous linear functional $f : K \to \overline{\mathbb{R}}_+$ has a unique Scott-continuous linear extension $\widetilde{f} : C \to \overline{\mathbb{R}}_+$, where $C = \mathsf{d\text{-}Cone}(K)$, and so $f \mapsto \widetilde{f}$ is a natural d-cone isomorphism between K^* and C^*. We will use this isomorphism freely.

Duality leads to the *weak Scott topology* on a d-cone C, which, as the name implies, is coarser than the Scott topology. It is the coarsest topology on C for which the Scott-continuous linear functionals on C remain continuous. The sets

$$U_f =_{def} \{x \in C \mid f(x) > 1\}, \ f \in C^*$$

form a subbasis of this topology.

We need the notion of a reflexive cone. For any d-cone C we have a canonical map ev_C from C into the bidual C^{**}. It assigns the evaluation map $f \mapsto f(x)$ to every $x \in C$ and is Scott-continuous and linear. We say that C is *reflexive* if ev_C is an isomorphism of d-cones. If C is a reflexive d-cone, then its dual C^* is also reflexive, and its dual is (isomorphic to) C.

We will need, as we did in [28], the notion of a *convenient* d-cone. This is a continuous reflexive d-cone C whose weak Scott topology agrees with its Scott topology, and whose dual C^* is continuous and has an additive way-below relation. The valuation powerdomain construction, which we consider next, exemplifies these strong requirements.

Example 2.48. (Probabilistic powerdomains) Probability measures on dcpos are modelled by valuations, which assign a probability to Scott-open subsets in a continuous way. This permits the development of a satisfactory theory of integration of Scott-continuous functions.

For large classes of dcpos, Scott-continuous valuations correspond to regular Borel probability measures and so it does not make much difference whether one works with Scott-continuous valuations or probability measures. For the sake of applications in semantics it is useful not to restrict to probabilities, where the whole space has probability 1, but to admit subprobabilities, where the whole space has probability of at most 1. For theoretical purposes, on the other hand, it is more convenient to extend the notion of probability to that of a measure, where the measure of the whole space can be any nonnegative real number, or even $+\infty$.

A *valuation* on a dcpo P is a map μ defined on the complete lattice $\mathcal{O}P$ of Scott-open subsets of P taking nonnegative real values, including $+\infty$, which (replacing finite additivity) is strict and modular:

$$\begin{array}{rcll} \mu(\emptyset) & = & 0 & \\ \mu(U \cup V) + \mu(U \cap V) & = & \mu(U) + \mu(V) & \text{for all } U, V \in \mathcal{O}P \end{array}$$

The *point* (or *Dirac*) valuations δ_x ($x \in P$) are given by:

$$\delta_x(U) = \begin{cases} 1 & (x \in U) \\ 0 & (x \notin U) \end{cases}$$

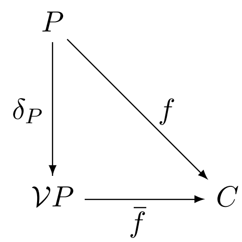
The *simple* valuations are the finite linear combinations of point valuations. The set $\mathcal{V}P$ of all Scott-continuous valuations $\mu\colon \mathcal{O}P \to \overline{\mathbb{R}}_+$ forms a sub-d-cone of the function space $\overline{\mathbb{R}}_+^{\mathcal{O}P}$, with the pointwise partial order and with pointwise addition and scalar multiplication:

$$(\mu + \nu)(U) =_{def} \mu(U) + \nu(U), \quad (r \cdot \mu)(U) =_{def} r \cdot \mu(U)$$

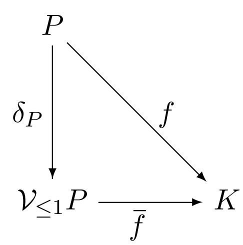
We call it the *valuation powerdomain* over P (it is also known as the *extended probabilistic powerdomain*). The Scott-continuous valuations μ such that $\mu(P) \leq 1$ are called *subprobability valuations*; they form a Scott-closed convex subset $\mathcal{V}_{\leq 1}P$ of the d-cone $\mathcal{V}P$, and hence a full sub-Kegelspitze. We call it the *subprobabilistic powerdomain* over P. We will use the following results later on:

(a) Let P be a domain. The valuation powerdomain $\mathcal{V}P$ is a continuous d-cone with an additive way-below relation and (hence) the subprobabilistic powerdomain $\mathcal{V}_{\leq 1}P$ is a continuous Kegelspitze in which the barycentric operations preserve its way-below relation, which is inherited from $\mathcal{V}P$ (Jones [18, Theorem 5.2 and Corollary 5.4], Kirch [31], Tix [59] and see [7, Theorem IV-9.16]).
(b) Let P be a coherent domain. Then the powerdomains $\mathcal{V}P$ and $\mathcal{V}_{\leq 1}P$ are also coherent (Jung and Tix [22], Jung [20]). In each case the relevant simple valuations form a basis.
(c) For any dcpo P there is a canonical Scott-continuous map $\delta_P\colon P \to \mathcal{V}_{\leq 1}P \subseteq \mathcal{V}P$ which assigns to any $x \in P$ the Dirac measure $\delta_P(x)$ which has value 1 for all Scott-open neighbourhoods U of x and value 0 otherwise. Then, if P is a domain, $\mathcal{V}P$ is the free d-cone over P (Kirch [31], and see [7, Theorem IV-9.24]), and $\mathcal{V}_{\leq 1}P$ is the free Kegelspitze over P (Jones [18, Theorem 5.9]). That is, for every d-cone C and every

Scott-continuous function $f\colon P \to C$, there is a unique Scott-continuous linear map $\overline{f}\colon \mathcal{V}P \to C$ such that the following diagram commutes:



and for every Kegelspitze K and every Scott-continuous map $f\colon P \to K$ there is a unique Scott-continuous linear map $\overline{f}\colon \mathcal{V}_{\leq 1}P \to K$ such that the following diagram commutes:



(d) Let P be a domain. Then the valuation powerdomain $\mathcal{V}P$ is the universal d-cone over the Kegelspitze $\mathcal{V}_{\leq 1}P$, that is: $\mathcal{V}P \cong \mathsf{d\text{-}Cone}(\mathcal{V}_{\leq 1}P)$ (the inclusion $\mathcal{V}_{\leq 1}P \subseteq \mathcal{V}P$ is the universal arrow). This is because $\mathcal{V}P$ and $\mathcal{V}_{\leq 1}P$ are, respectively, the free d-cone and the free Kegelspitze over P.

(e) For any dcpo P, the valuation powerdomain $\mathcal{V}P$ is the dual of $\mathcal{L}P$ up to isomorphism, and, if P is a domain, the d-cone $\mathcal{L}P$ is the dual of $\mathcal{V}P$, which implies that both $\mathcal{V}P$ and $\mathcal{L}P$ are reflexive (Kirch [31, Satz 8.1 and Lemma 8.2], and Tix [59, Theorem 4.16]).

The proof is based on the appropriate notion of an integral of a Scott-continuous function $f\colon P \to \overline{\mathbb{R}}_+$ with respect to a Scott-continuous valuation μ. Among the various ways to define this integral, the most elegant is via the Choquet integral:

$$\int f\,d\mu =_{def} \int_0^{+\infty} \mu\big(f^{-1}(]r, +\infty])\big)\,dr$$

The Choquet integral should be read as the generalised Riemann integral of the nonnegative monotone-decreasing function $r \in [0, +\infty[\mapsto \mu\big(f^{-1}\big(]r, +\infty]\big)\big)$, which may take infinite values. The integral was originally defined as a Lebesgue integral by Jones [18]; Tix later proved the Choquet definition equivalent [59].

The isomorphism between $\mathcal{V}P$ and $(\mathcal{L}P)^*$, the dual of $\mathcal{L}P$, is the map $\mu \mapsto \lambda f.\ \int f\,d\mu$, establishing a kind of Riesz representation theorem. And if P is a domain, the isomorphism between $\mathcal{L}P$ and the dual $(\mathcal{V}P)^*$ is the map $f \mapsto \lambda\mu.\ \int f\,d\mu$.

(f) For a domain P, the Scott topology and the weak Scott topology agree on both $\mathcal{L}P$ and $\mathcal{V}P$ (Tix [59, Satz 4.10]).

Summarising the properties reported in Examples 2.46 and 2.48, we can say that the valuation powerdomain $\mathcal{V}P$ over a continuous coherent domain P is a convenient d-cone.

3. Power Kegelspitzen

We now construct three kinds of power Kegelspitzen, proceeding analogously to the constructions of the three types of powercone in [61]. Under various assumptions on a given Kegelspitze K, we will construct its *lower, upper*, and *convex power Kegelspitzen*, $\mathcal{H}K$, $\mathcal{S}K$, and $\mathcal{P}K$. Power Kegelspitzen and powercones have a choice operation, so we begin by defining the three kinds of Kegelspitzen and cones enriched with a choice operation that thereby arise.

A *Kegelspitze semilattice* is a Kegelspitze equipped with a Scott-continuous semilattice operation $\cup$ over which convex combinations distribute, that is, for all $x, y, z \in K$ and $r \in [0, 1]$ we have:

$$x +_r (y \cup z) = (x +_r y) \cup (x +_r z)$$

It is a *Kegelspitze join-semilattice* if $\cup$ is the binary supremum operation (equivalently, if $x \leq x \cup y$ always holds). It is a *Kegelspitze meet-semilattice* if $\cup$ is the binary infimum operation (equivalently, if $x \cup y \leq x$ always holds). A morphism of Kegelspitze semilattices is a morphism of Kegelspitzen which also preserves the semilattice operation. Using the distributivity axiom it is straightforward to show that the semilattice operation is homogeneous and that the barycentric operations are $\subseteq$-monotone (a function between two semilattices is said to be *$\subseteq$-monotone* if it preserves the partial order $\subseteq$ naturally associated to semilattices). Further, the following *convexity* identity holds

$$x \cup (x +_r y) \cup y = x \cup y \tag{CI}$$

This can be proved beginning with the equation $x \cup y = (x \cup y) +_r (x \cup y)$, then expanding out the right-hand side using the distributivity of $+_r$ over $\cup$, and then using the inclusion $\subseteq$ associated with the semilattice operation.

There is an analogous notion of *d-cone semilattice*. This is a d-cone C equipped with a Scott-continuous semilattice operation $\cup$ over which the cone operations distribute, i.e., for all $x, y, z \in C$ and $r \in \mathbb{R}_+$ we have:

$$x + (y \cup z) = (x + y) \cup (x + z) \qquad r \cdot (x \cup y) = r \cdot x \, \cup \, r \cdot y$$

Such a cone is a *d-cone join-semilattice* (*d-cone meet-semilattice*) if $\cup$ is the binary supremum operation (respectively, the binary infimum operation). Every d-cone semilattice (d-cone join-semilattice, d-cone meet-semilattice) can be regarded as a Kegelspitze semilattice (respectively, Kegelspitze join-semilattice, Kegelspitze meet-semilattice). A morphism of d-cone semilattices is a morphism of d-cones which also preserves the semilattice operation. Much as before, distributivity implies that $\cup$ is homogeneous, and that the d-cone operations are $\subseteq$-monotone.

The lower, upper, and convex power Kegelspitzen will be, respectively, Kegelspitze join-semilattices, meet-semilattices, and semilattices. Possibly under further assumptions, they will be the free such Kegelspitze semilattices on K. The analogous result in the lower case for powercones was proved in [61]. Freeness results were also proved in the other two cases, but of a different character, having weaker assumptions and weaker conclusions.

In order to verify the properties of the various power Kegelspitzen, we will embed the Kegelspitzen in d-cones, and then use the embeddings to transfer results from [61] about powercones to the power Kegelspitzen.

Another way to proceed is to view the power Kegelspitzen as retracts of the powerdomains of the domains underlying the Kegelspitzen. One can then transfer results about the powerdomains (such as the preservation of continuity) to the Kegelspitzen. Similar uses of retracts already occur in [61, 9, 10, 11, 12]. We have not explored this option, but it is certainly possible to strengthen some of our results in this way. However, the results we present are sufficient for their intended application in Section 3.4 to mixed powerdomains combining probabilistic choice and nondeterminism.

3.1. **Lower power Kegelspitzen.** We first investigate the *convex lower* (or *Hoare*) *power Kegelspitze*. We need some closure properties of convex sets:

Lemma 3.1. *Let K be a Kegelspitze. Then:*
(1) *Any directed union of convex subsets of K is convex.*
(2) *The Scott closure of a convex subset of K is convex.*
(3) *If X and Y are convex subsets of K then so is*
$$X +_r Y =_{def} \{x +_r y \mid x \in X, y \in Y\}$$

Proof. The first statement is immediate. For the second, note that the Scott closure of a set is obtained by repeating the operations of downwards closure and taking directed unions transfinitely many times. As each of these operations can be seen to preserve convexity, and as taking directed unions does too, it follows that the Scott closure of a convex set is convex. Finally the third statement follows using the entropic law (E). □

The lower power Kegelspitze $\mathcal{H}K$, of a given full Kegelspitze $(K, +_r, 0)$, that is, a Kegelspitze satisfying Property (OC3), consists of the collection of non-empty Scott-closed convex subsets of K, ordered by subset, with zero $\{0\}$ and with convex combination operators $+_{rH}$ given by:
$$X {+_{rH}} Y =_{def} \overline{X {+_r} Y}$$
for $r \in [0, 1]$, (where, for any $X \subseteq K$, $\overline{X}$ is the closure of X in the Scott topology). That these operators are well-defined follows from Lemma 3.1.

As we said above, in order to verify the properties of $\mathcal{H}K$ we will make use of the embedding of K into a d-cone C and the properties of the lower powercone (or Hoare powercone) of C. So let us begin by reviewing the definition and properties of the lower powercone $\mathcal{H}C$ of a d-cone $(C, +, 0, \cdot)$ [61, Section 4.1]. As a partial order, it is the collection of all nonempty Scott-closed convex subsets of C ordered by inclusion $\subseteq$. It has arbitrary suprema, given by:

$$\bigvee_{i \in I} X_i = \overline{\mathrm{conv} \bigcup_{i \in I} X_i}$$

with directed suprema given by:

$$\bigvee_{i \in I}^{\uparrow} X_i = \overline{\bigcup_{i \in I}^{\uparrow} X_i}$$

Addition and scalar multiplication are lifted from C to $\mathcal{H}C$ as follows:

$$X +_H Y =_{def} \overline{X + Y} \qquad r \cdot_H X =_{def} r \cdot X$$

where $X + Y = \{x + y \mid x \in X, y \in Y\}$, and $r \cdot X = \{r \cdot x \mid x \in X\}$. Convex combinations are then given by $r \cdot_H X +_H (1 - r) \cdot_H Y = \overline{X +_r Y}$. Further, the following is proved in [61, Section 4.1]:

Theorem 3.2. *Let $(C, +, 0, \cdot)$ be a d-cone. Then $(\mathcal{H}C, +_H, 0, \cdot_H)$ is also a d-cone, and, equipped with binary suprema, it forms a d-cone join-semilattice.*

If C is continuous, then so is $\mathcal{H}C$. The non-empty finitely generated convex Scott-closed sets $\overline{\mathrm{conv} F}$, where F is a finite, non-empty subset of C, form a basis for $\mathcal{H}C$; further, for any $X, Y \in \mathcal{H}C$, $X \ll_{\mathcal{H}C} Y$ if, and only if, $X \subseteq \overline{\mathrm{conv} F}$ and $F \subseteq \mathop{\downarrow\!\!\downarrow} Y$, for some such F. If, in addition, the way-below relation of C is additive, so is that of $\mathcal{H}C$. □

We can now show:

Theorem 3.3. *Let $(K, +_r, 0)$ be a full Kegelspitze. Then $(\mathcal{H}K, +_{rH}, \{0\})$ is a full Kegelspitze. It has arbitrary suprema, given by:*

$$\bigvee_{i \in I} X_i = \overline{\mathrm{conv} \bigcup_{i \in I} X_i}$$

with directed suprema given by:

$$\bigvee_{i \in I}^{\uparrow} X_i = \overline{\bigcup_{i \in I}^{\uparrow} X_i}$$

and, equipped with binary suprema, it forms a Kegelspitze join-semilattice.

If, further, K is a continuous Kegelspitze, then so is $\mathcal{H}K$. The non-empty finitely generated convex Scott-closed sets $\overline{\mathrm{conv} F}$, where F is a finite, non-empty subset of K, form a basis for $\mathcal{H}K$; further, for any $X, Y \in \mathcal{H}K$, $X \ll_{\mathcal{H}K} Y$ if, and only if, $X \subseteq \overline{\mathrm{conv} F}$ and $F \subseteq \mathop{\downarrow\!\!\downarrow} Y$, for some such F. If, in addition, the way-below relation of K is closed under convex combinations, so is that of $\mathcal{H}K$.

Proof. Using Theorem 2.35 we can regard K as a Scott-closed convex subset of the d-cone $C =_{def} \textsf{d-Cone}(K)$, with its partial order and algebraic structure inherited from that of C.

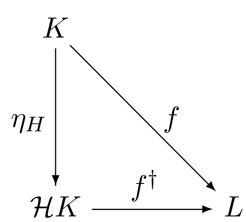
It is then immediate that $\mathcal{H}K$ embeds as a sub-partial order of $\mathcal{H}C$. Further, as K is Scott-closed and convex, one easily shows, using the above formulas for the suprema and convex combinations of $\mathcal{H}C$, that $\mathcal{H}K$ is a Scott-closed convex subset of $\mathcal{H}C$, bounded above by K. Therefore $\mathcal{H}K$ has all sups and a Kegelspitze structure, and they are inherited from $\mathcal{H}C$. Explicitly, arbitrary sups and and directed sups are given by the claimed formulas, as the corresponding formulas hold for $\mathcal{H}C$ (we may, equivalently, take Scott closure of subsets of K with respect to C or K). The inherited convex combinations and zero agree with those of $\mathcal{H}C$. Convex combinations distribute over the semilattice operation (here binary sups), as $+$ and $r \cdot -$ do. So, equipped with binary suprema, $\mathcal{H}K$ is a Kegelspitze join-semilattice. It automatically satisfies Property (OC3) as it is embedded in the d-cone $\mathcal{H}C$ as a Scott-closed subset.

Next, suppose that K is a continuous Kegelspitze. Then, by Proposition 2.42, C is continuous, and so, by Theorem 3.2, $\mathcal{H}C$ is too. As $\mathcal{H}K$ is a Scott-closed subset of $\mathcal{H}C$, it follows that $\mathcal{H}K$ is continuous with way-below relation the restriction of that of $\mathcal{H}C$ to $\mathcal{H}K$. That the non-empty finitely-generated Scott-closed sets form a basis of $\mathcal{H}K$, and the characterisation of the way-below relation of $\mathcal{H}K$ then follow from the corresponding parts of Theorem 3.2. Further, as $r \cdot -$ preserves the way-below relation of $\mathcal{H}C$ ($r \cdot -$ is an order-isomorphism of cones), it also preserves the restriction to $\mathcal{H}K$, and so $\mathcal{H}K$ is continuous as a Kegelspitze.

Finally, suppose additionally that K's way-below relation is closed under convex combinations. Then, by Proposition 2.43, C has an additive way-below relation. We then see from [61] that the d-cone operations of $\mathcal{H}C$ preserve its way-below relation, and so that convex combinations preserve the way-below relation of $\mathcal{H}K$ (it is the restriction of that of $\mathcal{H}C$ as $\mathcal{H}K$ is a closed subset of $\mathcal{H}C$). □

We now show that $\mathcal{H}K$ is the free Kegelspitze join-semilattice over any full Kegelspitze K, with unit the evident Kegelspitze morphism $\eta_H : K \to \mathcal{H}K$, where $\eta_H(x) = {\downarrow}x$.

Theorem 3.4. *Let K be a full Kegelspitze. Then the map η_H is universal. That is, for every Kegelspitze join-semilattice L and Kegelspitze morphism $f : K \to L$ there is a unique Kegelspitze semilattice morphism $f^\dagger : \mathcal{H}K \to L$ such that the following diagram commutes:*



The morphism is given by:

$$f^\dagger(X) = \bigvee\nolimits_L f(X)$$

Proof. To show uniqueness, choose an $X \in \mathcal{H}K$. It can be written as a non-empty sup, viz. $\bigvee_{x \in X} \eta_H(x)$. Then, noting that continuous binary join morphisms preserve all non-empty sups (for such sups are directed sups of finite non-empty sups, and finite non-empty sups are iterated binary ones) we calculate:

$$\begin{array}{rcl} f^\dagger(X) & = & f^\dagger(\bigvee_{x\in X} \eta_H(x)) \\ & = & \bigvee_{x\in X} f^\dagger(\eta_H(x)) \\ & = & \bigvee_{x\in X} f(x) \end{array}$$

To show existence we therefore set

$$f^\dagger(X) = \bigvee_{x\in X} f(x)$$

and verify that it makes the diagram commute and is both a Kegelspitze and a semilattice map. The first of these requirements holds as we calculate:

$$f^\dagger(\eta_H(x)) = f^\dagger(\{y \mid y \leq x\}) = \bigvee_{y\leq x} f(y) = f(x)$$

For the second we need to show that $f^\dagger$ is strict and continuous and preserves convex combinations. Strictness is a consequence of the diagram commuting, as both η_H and f are strict. For continuity we first show that

$$\bigvee f(\overline{A}) = \bigvee f(A)$$

for any $A \subseteq K$. Choose $A \subseteq K$. We evidently have $\bigvee f(A) \leq \bigvee f(\overline{A})$, and it remains to prove the converse inequality $\bigvee f(\overline{A}) \leq \bigvee f(A)$. By the continuity of f, we have $f(\overline{A}) \subseteq \overline{f(A)}$ whence $\bigvee f(\overline{A}) \leq \bigvee \overline{f(A)}$. So we only have to show that $x \leq \bigvee f(A)$ for any $x \in \overline{f(A)}$. This follows from the fact that the Scott closure of a set is obtained by transfinitely many repetitions of the operations of downwards closure and taking directed sups.

We can now calculate:

$$\begin{array}{rcl} f^\dagger(\bigvee^{\uparrow}_{i\in I} X_i) & = & f^\dagger(\overline{\bigcup^{\uparrow}_{i\in I} X_i}) \\ & = & \bigvee f(\overline{\bigcup^{\uparrow}_{i\in I} X_i}) \\ & = & \bigvee f(\bigcup^{\uparrow}_{i\in I} X_i) \\ & = & \bigvee^{\uparrow}_{i\in I} \bigvee_{x\in X_i} f(x) \\ & = & \bigvee^{\uparrow}_{i\in I} f^\dagger(X_i) \end{array}$$

For convex combinations we calculate:

$$\begin{array}{rcl} f^\dagger(X {+_r}_H Y) & = & f^\dagger(\overline{X +_r Y}) \\ & = & f^\dagger(X +_r Y) \\ & = & \bigvee_{z\in X+_r Y} f(z) \\ & = & \bigvee_{x\in X,\, y\in Y} f(x +_r y) \\ & = & \bigvee_{x\in X,\, y\in Y} f(x) +_r f(y) \\ & = & (\bigvee_{x\in X} f(x)) +_r (\bigvee_{y\in Y} f(y)) \\ & = & f^\dagger(X) +_r f^\dagger(Y) \end{array}$$

The second equation holds as f is Scott-continuous. The sixth equation holds as convex combinations in L distribute over arbitrary non-empty sups, as we may see by analysing such sups as before, i.e., as directed sups of iterated binary ones.

Finally, we need to show that binary sups are also preserved, and so calculate:

$$\begin{array}{rcl} f^\dagger(X \vee Y) & = & f^\dagger(\overline{\mathrm{conv}(X \cup Y)}) \\ & = & f^\dagger(\mathrm{conv}(X \cup Y)) \\ & = & \bigvee_{x \in X,\, y \in Y} \bigvee_{0 \le r \le 1} f(x +_r y) \\ & = & \bigvee_{x \in X,\, y \in Y} \bigvee_{0 < r < 1} (f(x) \vee f(x +_r y) \vee f(y)) \\ & = & \bigvee_{x \in X,\, y \in Y} \bigvee_{0 < r < 1} (f(x) \vee (f(x) +_r f(y)) \vee f(y)) \\ & = & \bigvee_{x \in X,\, y \in Y} f(x) \vee f(y) \\ & = & \bigvee_{x \in X} f(x) \vee \bigvee_{y \in Y} f(y) \\ & = & f^\dagger(X) \vee f^\dagger(Y) \end{array}$$

where the sixth equation follows using the convexity identity (CI). □

3.2. **Upper power Kegelspitzen.** We next investigate the *convex upper* (or *Smyth*) *power Kegelspitze* $\mathcal{S}K$, of a given continuous full Kegelspitze $(K, +_r, 0)$. The restriction to the continuous case is necessary as in the case of powercones. It consists of the collection of non-empty Scott-compact convex saturated (= upper) subsets of K, ordered by reverse inclusion $\supseteq$, with zero K and with convex combination operators $+_{rS}$ given by:

$$X +_{rS} Y =_{def} \uparrow(X +_r Y)$$

for $r \in [0, 1]$. To see that these operators are well-defined, first note that $X +_r Y$ is Scott-compact, as it is the image under $+_r$ of $X \times Y$, which is compact in the product topology on $K \times K$, which latter is the same as the Scott topology, as K is continuous. Then note that the upper closure of a Scott-compact (convex) set is Scott-compact (respectively convex).

The upper power Kegelspitze has binary infima, which make it a Kegelspitze meet-semilattice. They are given by:

$$X \wedge Y = \uparrow\mathrm{conv}(X \cup Y)$$

In order to verify the properties of $\mathcal{S}K$, we follow our general methodology, using the embedding of K into a d-cone C and the properties of the upper powercone (or Smyth powercone) of C. Let us begin by recalling the definition and properties of the upper powercone $\mathcal{S}C$ of a continuous d-cone $(C, +, 0, \cdot)$ [61, Section 4.2]. It consists of all nonempty Scott-compact convex saturated subsets ordered by reverse inclusion $\supseteq$. It has directed suprema given by intersection:

$$\bigvee_{i \in I}^{\uparrow} X_i = \bigcap_{i \in I}^{\downarrow} X_i$$

and binary infima given by:

$$X \wedge Y = \uparrow\mathrm{conv}(X \cup Y)$$

Addition and scalar multiplication are lifted from C to $\mathcal{S}C$ as follows:

$$X +_S Y =_{def} \uparrow(X + Y) \qquad r \cdot_S X =_{def} \uparrow(r \cdot X)$$

Note that $r \cdot_S X = \uparrow\{0\} = C$ if $r = 0$ and $r \cdot_S X = r \cdot X$ if $r > 0$. Convex combinations are given by $r \cdot_S X +_S (1 - r) \cdot_S Y = \uparrow(r \cdot X + (1 - r) \cdot Y)$. Further, the following is proved in [61, Section 4.2]:

Theorem 3.5. *Let $(C, +, 0, \cdot)$ be a continuous d-cone. Then $(\mathcal{S}C, +_S, C, \cdot_S)$ is a continuous d-cone and, equipped with binary infima, it forms a d-cone meet-semilattice.*

The non-empty finitely generated convex saturated Scott-compact sets $\uparrow\mathrm{conv}\, F$, where F is a finite, non-empty subset of C, form a basis for $\mathcal{S}C$; further, for any $X, Y \in \mathcal{S}C$, $X \ll_{\mathcal{S}C} Y$ if, and only if, $X \supseteq \uparrow\mathrm{conv}\, F$ and $\Uparrow F \supseteq Y$, for some such F. If the way-below relation of C is additive, so is that of $\mathcal{S}C$. □

For the proof of the next theorem we recall that a *(monotone) retraction pair* between two partial orders P and Q is a pair of monotone maps

$$P \xrightarrow{e} Q \xrightarrow{r} P$$

such that $r \circ e = \mathrm{id}_P$; it is a *(monotone) closure pair* if, additionally, $e \circ r \geq \mathrm{id}_Q$. We can now show:

Theorem 3.6. *Let $(K, +_r, 0)$ be a continuous full Kegelspitze. Then $(\mathcal{S}K, +_{rS}, K)$ is a continuous Kegelspitze meet-semilattice. Directed suprema are given by intersection:*

$$\bigvee_{i \in I}^{\uparrow} X_i = \bigcap_{i \in I}^{\downarrow} X_i$$

and binary infima are given by:

$$X \wedge Y = \uparrow\mathrm{conv}(X \cup Y)$$

The non-empty finitely generated convex saturated Scott-compact sets $\uparrow\mathrm{conv}\, F$, where F is a finite, non-empty subset of K, form a basis for $\mathcal{S}K$; further, for any $X, Y \in \mathcal{S}K$, $X \ll_{\mathcal{S}K} Y$ if, and only if, $X \supseteq \uparrow\mathrm{conv}\, F$ and $\Uparrow F \supseteq Y$, for some such F. The way-below relation $\ll_{\mathcal{S}K}$ is preserved by $r \cdot_{\mathcal{S}K} -$, and if the way-below relation of K is closed under convex combinations, so is that of $\mathcal{S}K$. If K is coherent then $\mathcal{S}K$ is a bounded-complete domain, hence coherent too.

Proof. Using Theorem 2.35, we can regard K as a Scott-closed convex subset of the d-cone $C =_{def} \mathsf{d\text{-}Cone}(K)$, with its partial order and algebraic structure inherited from that of C. By Proposition 2.42, C is continuous, and so, by Lemma 2.40, the way-below relation of K is also inherited from that of C.

To relate $\mathcal{S}K$ to $\mathcal{S}C$ we first define a Scott-closed convex subset L of $\mathcal{S}C$ and then show that $\mathcal{S}K$ is a closure of L (with partial ordering inherited from $\mathcal{S}C$). This enables us to transport structure from $\mathcal{S}C$ to $\mathcal{S}K$ via L. We take L to be the collection of elements of $\mathcal{S}C$ intersecting K. It is evidently a lower set (for $\leq\, = \,\supseteq$). If X_i is a directed subset of L

then $X_i \cap K$ is a $\subseteq$-filtered collection of non-empty saturated Scott-compact subsets of K and so has a non-empty intersection by a consequence of the Hofmann-Mislove Theorem [7, Corollary II-1.22]. This shows that $\bigvee_{\mathcal{S}C} X_i$ is in L. The convexity of L follows from that of K.

We order L by reverse inclusion $\supseteq$, i.e., as a sub-partial order of $\mathcal{S}C$. As L is a Scott-closed subset of $\mathcal{S}C$, it is a continuous sub-dcpo of $\mathcal{S}C$, with way-below relation inherited from that of $\mathcal{S}C$, and with basis $B \cap L$, for any basis B of $\mathcal{S}C$. Further, as L is a convex Scott-closed subset of $\mathcal{S}C$, it inherits a Kegelspitze structure from $\mathcal{S}C$, with zero C, and with convex combinations given by $X {+_r}_L Y = r \cdot_S X +_S (1-r) \cdot Y = \uparrow(r \cdot X + (1-r) \cdot Y)$. Binary infima are also inherited by L, and as these distribute over $+_C$ and $\cdot_C$, L forms a Kegelspitze meet-semilattice.

We now define a monotone closure pair:

$$\mathcal{S}K \xrightarrow{e} L \xrightarrow{c} \mathcal{S}K$$

by setting $e(X) =_{def} \uparrow_C X$ and $c(Y) =_{def} Y \cap K$.

Since (e, c) is a monotone closure pair, the existence of directed suprema in L implies the existence of directed suprema in $\mathcal{S}K$ and c preserves these directed suprema, that is, c is Scott-continuous. The following shows that the supremum of a directed collection X_i in $\mathcal{S}K$ is calculated as expected:

$$\bigvee_i^{\uparrow} X_i = c(\bigvee_i^{\uparrow} e(X_i)) = (\bigcap_i^{\downarrow} \uparrow_C X_i) \cap K = \bigcap_i^{\downarrow} (\uparrow_C X_i) \cap K = \bigcap_i^{\downarrow} X_i$$

For the continuity of e, we have to show for any directed collection X_i in $\mathcal{S}K$ that:

$$\uparrow_C(\bigcap_i X_i) \supseteq (\bigcap_i \uparrow_C X_i)$$

(the other direction holds as e is monotone). Suppose, for the sake of contradiction, that there is a $y \in C$, with y in every $\uparrow_C X_i$, but not in $\uparrow_C(\bigcap_i X_i)$. We then have that $\bigcap_i X_i \subseteq \{x \in C \mid x \not\leq y\} \cap K$. As the latter set is Scott-open in K, and as K is well-filtered (this follows from the Hofmann-Mislove theorem, see [7, Theorem II-1.21]), there is an i such that $X_i \subseteq \{x \in C \mid x \not\leq y\} \cap K$, which contradicts our assumption. Thus (c, e) is a Scott-continuous closure pair and we can conclude from the continuity of L that $\mathcal{S}K$ is a continuous dcpo.

Turning to the characterisation of the way-below relation on $\mathcal{S}K$, choose X, Y in $\mathcal{S}K$. By general properties of retractions we know that $X \ll_{\mathcal{S}K} Y$ holds iff there is a $U \in L$ such that $X \leq c(U)$ and $U \ll_L \uparrow_C Y$ (equivalently $e(X) \leq U$ and $U \ll_L \uparrow_C Y$, as (e, c) is an adjoint pair). As the way-below relation on L is the restriction of that on $\mathcal{S}C$, Theorem 3.5 tells us that $U \ll_L \uparrow_C Y$ holds iff there is a non-empty finite subset F of C such that $U \supseteq \uparrow_C \mathrm{conv}\, F$ and $\Uparrow_C F \supseteq \uparrow_C Y$. Putting these together, we have that $X \ll_{\mathcal{S}K} Y$ holds iff there is a non-empty finite subset F of C such that $\uparrow_C X \supseteq \uparrow_C \mathrm{conv}\, F$ and $\Uparrow_C F \supseteq \uparrow_C Y$ (equivalently, $\uparrow_C X \supseteq \mathrm{conv}\, F$ and $\Uparrow_C F \supseteq Y$).

As $Y \subseteq K$ and $\uparrow_C F \supseteq Y$, and K is a Scott-closed subset of C, $F \cap K$ is non-empty. So we have that $X \ll_{\mathcal{S}K} Y$ holds if, and only if, there is a non-empty finite subset F' of K such that $\uparrow_C X \supseteq \mathrm{conv}\, F'$ and $\uparrow_C F' \supseteq Y$ (in one direction, given F, set $F' = F \cap K$; in the other direction, given F', take $F = F'$). For such an F' we have $\uparrow_C X \supseteq \mathrm{conv}\, F'$ iff $X \supseteq \mathrm{conv}\, F'$ (as K is a convex subset of C) and $\uparrow_C F' \supseteq Y$ iff $\uparrow_K F' \supseteq Y$, and we have established the desired characterisation of $\ll_{\mathcal{S}K}$.

Using this characterisation, and the fact that K is continuous, it follows immediately that $\ll_{\mathcal{S}K}$ is preserved by $r \cdot_{\mathcal{S}K} -$. It also follows immediately that the non-empty finitely generated convex saturated sets form a basis of $\mathcal{S}K$.

Next, a calculation now shows that e preserves the convex combination operation:

$$\begin{array}{rcl} e(X) +_{rS} e(Y) & = & \uparrow_C (r \cdot (\uparrow_C X) + (1-r) \cdot (\uparrow_C Y)) \\ & = & \uparrow_C (\uparrow_C r \cdot X + \uparrow_C (1-r) \cdot Y) \\ & = & \uparrow_C (r \cdot X + (1-r) \cdot Y) \\ & = & \uparrow_C \uparrow_K (r \cdot X + (1-r) \cdot Y) \\ & = & e(X +_{rK} Y) \end{array}$$

It follows that the convex combination operation on $\mathcal{S}K$ can be defined in terms of that on L as we have: $X +_{rS} Y = ce(X +_{rS} Y) = c(e(X) +_{rL} e(Y))$. So, as $e, c, +_{rL}$ are all Scott-continuous, so is $+_{r\mathcal{S}K}$.

As (c, e) is a closure pair and L has binary infima, so does $\mathcal{S}K$ and e preserves them. For any $X, Y \in \mathcal{S}K$ we can then calculate:

$$\begin{array}{rcl} X \wedge Y & = & c(e(X) \wedge e(Y)) \\ & = & (\uparrow_C \mathrm{conv}(\uparrow_C X \cup \uparrow_C Y)) \cap K \\ & = & (\uparrow_C \mathrm{conv} \uparrow_C (X \cup Y)) \cap K \\ & = & (\uparrow_C \mathrm{conv}(X \cup Y)) \cap K \\ & = & \uparrow_K \mathrm{conv}(X \cup Y) \end{array}$$

showing that binary meets are given as required. We have also seen that $\wedge_{\mathcal{S}K}$ can be defined as a composition of $e, c, \wedge_L$, and so is Scott-continuous.

As e is an order-mono (i.e., it reflects the partial order) and as it preserves convex combinations and binary meets, any inequations between these operations holding in L also hold in $\mathcal{S}K$. So (also using the fact that both convex combinations and binary meets are monotone) $\mathcal{S}K$ is an ordered barycentric algebra, binary meets form a meet-semilattice, and convex combinations distribute over binary meets. Therefore, as convex combinations and binary meets are both Scott-continuous, and as $r \mapsto ra$ is Scott-continuous, we see that $\mathcal{S}K$ is a Kegelspitze meet semilattice with zero K. We also know that $\mathcal{S}K$ is continuous and so it is a continuous Kegelspitze.

Next, we show that convex combinations in $\mathcal{S}K$ preserve its way-below relation, assuming the same is true of K. Suppose that $X \ll_{\mathcal{S}K} Y$ and $X' \ll_{\mathcal{S}K} Y'$. Using the characterisation of $\ll_{\mathcal{S}K}$ we see that there are finite, non-empty $F, F' \subseteq K$ such that $X \supseteq \uparrow\mathrm{conv}\, F$, $\uparrow F \supseteq Y$, $X' \supseteq \uparrow\mathrm{conv}\, F'$, and $\uparrow F' \supseteq Y'$, and then that we need only show that

$X{+_r}_S X' \supseteq \uparrow\mathrm{conv}(F{+_r}F')$ and $\Uparrow(F{+_r}F') \supseteq Y{+_r}_S Y'$, i.e., that $\uparrow(X{+_r}X') \supseteq \uparrow\mathrm{conv}(F{+_r}F')$ and $\Uparrow(F{+_r}F') \supseteq \uparrow(Y{+_r}Y')$. The first of these requirements holds as we have:

$$\uparrow(X{+_r}X') \supseteq \uparrow(\uparrow\mathrm{conv}\, F{+_r}\uparrow\mathrm{conv}\, F') \supseteq \uparrow(\mathrm{conv}\, F{+_r}\,\mathrm{conv}\, F') \supseteq \uparrow\mathrm{conv}(F{+_r}F')$$

(with the last inclusion holding because of the entropic law). The second holds as convex combinations in K preserve $\ll_K$.

Finally if K is also (Scott-)coherent, i.e., if the intersection of two Scott-compact saturated sets is again such, then $\mathcal{S}K$ is a bounded complete dcpo. So, as it is also continuous, it is coherent (see [7, Proposition III-5.12]). □

We note that the proof also establishes that the map

$$u = X \;\mapsto\; \uparrow X \colon \mathcal{S}K \to \mathcal{S}(\mathsf{d\text{-}Cone}(K))$$

is a d-cone meet semilattice embedding (that is, it is a d-cone semilattice morphism that reflects the partial order).

Theorem 3.6 does not assert that $\mathcal{S}K$ satisfies Property (OC3), but only that $r \cdot_{\mathcal{S}K} -$ preserves $\ll_{\mathcal{S}K}$. Indeed, $\mathcal{S}K$ need not satisfy Property (OC3):

Fact 3.7. The upper power Kegelspitze of the subprobabilistic powerdomain $\mathcal{V}_{\leq 1}\{0,1\}$ of the two-element discrete partial order does not satisfy Property (OC3).

Proof. For notational convenience we replace $\mathcal{V}_{\leq 1}\{0,1\}$ by the isomorphic Kegelspitze obtained from $\mathbb{S} = \{(r,s) \in [0,1]^2 \mid r+s \leq 1\}$ by ordering it coordinatewise and equipping it with the evident convex combination operators.

Set $Y \in \mathcal{S}\mathbb{S}$ to be $\{(1,0)\}$, and take any $0 < r < 1$ and set $X' \in \mathcal{S}\mathbb{S}$ to be the saturated convex closure of $\{(0,1),(r,0)\}$, which is:

$$\{(\overline{r}(1-s), s) \mid s \in [0,1], r \leq \overline{r} \leq 1\}$$

Clearly $X' \supseteq \uparrow(r \cdot Y)$, i.e., $X' \leq r \cdot_S Y$. Suppose, for the sake of contradiction, that $X' = r \cdot_S X = \uparrow(r \cdot X)$ for some $X \in \mathcal{S}\mathbb{S}$. Then, as $(0,1) \in X' = \uparrow r \cdot X$ we have $(0,1) \geq r \cdot x$ for some $x \in X$. As $r \cdot x \in r \cdot_S X = X'$, $r \cdot x$ has the form $(\overline{r}(1-s), s)$ for some $s \in [0,1]$ and $r \leq \overline{r} \leq 1$. Then as $(0,1) \geq r \cdot x = (\overline{r}(1-s), s)$, and so $0 \geq \overline{r}(1-s)$, we see that $s = 1$ and so that $r \cdot x = (0,1)$. But this cannot be the case as $r < 1$. □

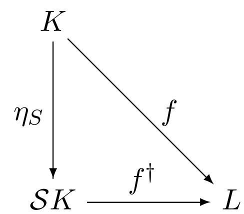
The failure of Property (OC3) is a priori a problem, as it obstructs the iteration of the upper Kegelspitze construction $\mathcal{S}$. However it is not a problem for this paper as iterating the upper mixed powerdomain does not involve iterating $\mathcal{S}$.

We next show that $\mathcal{S}K$ is the free Kegelspitze meet-semilattice over any Kegelspitze K satisfying suitable assumptions. The unit $\eta_S : K \to \mathcal{S}K$ is the evident Kegelspitze morphism $\eta_S(x) =_{def} \uparrow_K x$.

Lemma 3.8. *Let K be a continuous full Kegelspitze in which convex combinations preserve the way-below relation. Suppose that F, G are non-empty subsets of K such that $\Uparrow G \supseteq F$. Then $\uparrow\mathrm{conv}\, G \ll_{\mathcal{S}K} \uparrow\mathrm{conv}\, F$.*

Proof. As convex combinations preserve $\ll_K$ and $\uparrow G \supseteq F$, we have $\uparrow \mathrm{conv}\, G \supseteq \uparrow\mathrm{conv}\, F$. Then, using the compactness of $\uparrow\mathrm{conv}\, F$, we see that there is a non-empty finite subset H of $\mathrm{conv}\, G$ such that $\uparrow H \supseteq \uparrow\mathrm{conv}\, F$. The conclusion follows by the characterisation of $\ll_{\mathcal{S}K}$ given in Theorem 3.6. □

Theorem 3.9. *Let K be a continuous full Kegelspitze in which convex combinations preserve the way-below relation. Then the map η_S is universal. That is, for every Kegelspitze meet-semilattice L and Kegelspitze morphism $f : K \to L$ there is a unique Kegelspitze semilattice morphism $f^\dagger : \mathcal{S}K \to L$ such that the following diagram commutes:*



The morphism is given by:

$$f^\dagger(X) = \bigvee^{\uparrow}\{\bigwedge f(F) \mid F \subseteq_{\mathrm{fin}} K, F \neq \emptyset, \uparrow\mathrm{conv}\, F \ll_{\mathcal{S}K} X\}$$

Proof. Using the basis of $\ll_{\mathcal{S}K}$ given in Theorem 3.6, for any $X \in \mathcal{S}K$ we have

$$X = \bigvee^{\uparrow}\{\bigwedge \eta_S(F) \mid F \subseteq_{\mathrm{fin}} K, F \neq \emptyset, \uparrow\mathrm{conv}\, F \ll_{\mathcal{S}K} X\}$$

where we make use of the easily proved fact that for any finite non-empty subset F of K, we have: $\uparrow\mathrm{conv}\, F = \bigwedge_{b \in F} \eta_S(b)$.

It then follows for any Kegelspitze semilattice morphism $f^\dagger$ which makes the diagram commute that

$$f^\dagger(X) = \bigvee^{\uparrow}\{\bigwedge f(F) \mid F \subseteq_{\mathrm{fin}} K, F \neq \emptyset, \uparrow\mathrm{conv}\, F \ll_{\mathcal{S}K} X\}$$

establishing uniqueness.

For existence we define $f^\dagger$ by means of this formula and verify that it makes the diagram commute and is both a Kegelspitze and a semilattice map. For continuity, it is evident that $f^\dagger$ is monotone, and so it suffices to show that for any directed set X_i, $i \in I$, in $\mathcal{S}K$ and any finite, non-empty $F \subseteq K$ with $\uparrow\mathrm{conv}\, F \ll_{\mathcal{S}K} \bigvee_i X_i$ we have:

$$\bigwedge f(F) \leq \bigvee_i \{\bigwedge f(G) \mid G \subseteq_{\mathrm{fin}} K, G \neq \emptyset, \uparrow\mathrm{conv}\, G \ll_{\mathcal{S}K} X_i\}$$

This holds as if $\uparrow\mathrm{conv}\, F \ll_{\mathcal{S}K} \bigvee_i X_i$ then $\uparrow\mathrm{conv}\, F \ll_{\mathcal{S}K} X_i$, for some i.

Next, it is helpful to prove that

$$f^\dagger(\bigwedge \eta_S(F)) = \bigwedge f(F) \qquad (*)$$

for any finite non-empty set F, that is, that:

$$\bigvee^{\uparrow}\{\bigwedge f(G) \mid G \subseteq_{\mathrm{fin}} K, G \neq \emptyset, \uparrow\mathrm{conv}\, G \ll_{\mathcal{S}K} \uparrow\mathrm{conv}\, F\} = \bigwedge f(F)$$

To show the left-hand side is $\leq$ the right-hand side, suppose we have a non-empty finite subset G of K such that $\uparrow\mathrm{conv}\, G \ll_{\mathcal{S}K} \uparrow\mathrm{conv}\, F$. Then, by the characterisation of $\ll_{\mathcal{S}K}$ given in Theorem 3.6, there is a finite non-empty $H \subseteq K$ such that $\uparrow\mathrm{conv}\, G \supseteq \uparrow\mathrm{conv}\, H$ and $\uparrow H \supseteq \uparrow\mathrm{conv}\, F$ So, for any $a \in F$, there is a $b \in H$ such that $b \ll a$, and so a $c \in \mathrm{conv}\, G$ such that $c \leq a$. Let G_1 be the set of such c's. We then have:

$$\bigwedge f(G) = \bigwedge f(G) \wedge \bigwedge f(G_1) \leq \bigwedge f(G_1) \leq \bigwedge f(F)$$

where the equality follows from the convexity identity (CI). This shows the left-hand side is $\leq$ the right-hand side.

Conversely, given $F = \{a_1, \ldots, a_n\}$, with $n > 0$, choose $b_1 \ll a_1, \ldots, b_n \ll a_n$ and take $G = \{b_1, \ldots, b_n\}$. By Lemma 3.8 we have $\uparrow\mathrm{conv}\, G \ll_{\mathcal{S}K} \uparrow\mathrm{conv}\, F$. So the left-hand side is $\geq \bigwedge f(G)$, and so $\geq \bigwedge f(F)$, as G consists of an arbitrary choice of an elements way-below each element of F.

Taking F to be a singleton in $(*)$, we see that, as required, the diagram commutes; this, in turn, implies that $f^\dagger$ is strict, as η_S and f are. As regards preservation of the semilattice operation $\wedge$, as every element is a directed supremum of non-empty finite infima of elements of the form $\eta_S(b)$and as $\wedge$ is Scott-continuous, we need only verify it for such non-empty finite infima, and that follows immediately from $(*)$.

We finally show that $f^\dagger$ preserves convex combinations. Since $f^\dagger$ is Scott-continuous and every element of $\mathcal{S}K$ is a directed supremum of meets of the form $\bigwedge_{b\in F} \eta_S(b)$ ($F \subseteq K$ non-empty and finite), it suffices to show that $f^\dagger$ preserves convex combinations of such finite meets. To that end, given $F, G \subseteq K$ non-empty and finite, we calculate:

$$\begin{array}{rcl}
f^\dagger(\bigwedge_{b\in F} \eta_S(b) +_{r\mathcal{S}K} \bigwedge_{c\in G} \eta_S(c)) & = & f^\dagger(\bigwedge_{b\in F, c\in G}(\eta_S(b) +_{r\mathcal{S}K} \eta_S(c))) \\
& = & f^\dagger(\bigwedge_{b\in F, c\in G} \eta_S(b +_r c)) \\
& = & \bigwedge_{b\in F, c\in G} f(b +_r c) \\
& = & \bigwedge_{b\in F, c\in G}(f(b) +_r f(c)) \\
& = & \bigwedge_{b\in F} f(b) +_r \bigwedge_{c\in G} f(c) \\
& = & f^\dagger(\bigwedge_{b\in F} \eta_S(b)) +_r f^\dagger(\bigwedge_{c\in G} \eta_S(c))
\end{array}$$

where the third and sixth equalities follow from $(*)$, and the first and fifth follow from distributivity. □

This result contrasts with the corresponding universality result for upper powercones in [61]. There the assumptions are weaker, but so are the conclusions: there is no assumption of preservation of the way-below relation, but the universality relates only to continuous d-cone semilattices, not to all of them. Further the proof methods for the two theorems are different. It would be interesting to know if the assumption made in Theorem 3.9 that convex combinations preserve the way-below relation is needed.

3.3. **Convex power Kegelspitzen.** We next investigate the *convex* (or *Plotkin*) *power Kegelspitze* $\mathcal{P}K$, of a given continuous and coherent (so Lawson compact) full Kegelspitze. Note that we have to suppose not only continuity but also coherence in order to prove the desired results. First we need some definitions from [61]. Nonempty Lawson-compact order-convex subsets of a Lawson-compact domain are called *lenses*. Both Scott-closed sets and saturated Scott-compact sets are lenses, as they are both Lawson-compact, and every lens X can be written as the intersection of a non-empty Scott-closed convex set and a non-empty Scott-compact saturated convex one, as we have: $X = \overline{X} \cap {\uparrow}X$. We also have $\overline{X} = {\downarrow}X$ for any lens X. If a lens X of a continuous Lawson-compact Kegelspitze is also convex, then so are ${\downarrow}X$ and ${\uparrow}X$. The Egli-Milner ordering is defined on order-convex subsets of a partial order $\leq$ by:

$$X \leq_{\mathrm{EM}} Y \ \equiv_{\mathrm{def}} \ \forall x \in X.\, \exists y \in Y.\, x \leq y \ \wedge \ \forall y \in Y.\, \exists x \in X.\, x \leq y$$

which can equivalently be written as

$$X \leq_{\mathrm{EM}} Y \ \equiv \ {\downarrow}X \subseteq {\downarrow}Y \ \wedge \ {\uparrow}Y \subseteq {\uparrow}X$$

We define $\mathcal{P}K$ to be the collection of convex lenses of K ordered by the Egli-Milner ordering, with zero $\{0\}$ and with convex combination operators $+_{rP}$ given by:

$$X{+_{rP}}Y =_{def} ({\downarrow}X{+_{rH}}{\downarrow}Y) \cap ({\uparrow}X{+_{rS}}{\uparrow}Y)$$

for $r \in [0,1]$. It follows from the above remarks on lenses that this operator is well-defined. Using the explicit definitions of convex combinations for the lower and upper power Kegelspitzen, one sees that $X{+_{rP}}Y = \overline{X +_r Y} \cap {\uparrow}(X +_r Y)$. The convex power Kegelspitze is a Kegelspitze semilattice when equipped with the semilattice operator $\cup_P$ defined by:

$$X \cup_P Y =_{def} ({\downarrow}X \vee_{\mathcal{H}K} {\downarrow}Y) \cap ({\uparrow}X \wedge_{\mathcal{S}K} {\uparrow}Y)$$

Using the explicit definitions of the semilattice operations for the lower and upper power Kegelspitzen one sees that $X \cup_P Y = \overline{\mathrm{conv}(X \cup Y)} \cap {\uparrow}\mathrm{conv}(X \cup Y)$; note too that ${\downarrow}(X \cup Y) = {\downarrow}X \vee_{\mathcal{H}K} {\downarrow}Y$ and ${\uparrow}(X \cup_P Y) = {\uparrow}X \vee_{\mathcal{S}K} {\uparrow}Y$.

In order to verify the properties of $\mathcal{P}K$ we proceed as before, via embeddings into cones. Let us begin by recalling the definition and properties of the convex powercone $\mathcal{P}C$ of a continuous Lawson-compact d-cone $(C, +, 0, \cdot)$ [61, Section 4.3]. This is the collection of all convex lenses of C partially ordered by the Egli-Milner ordering. It has directed suprema given by:

$$\bigvee_{i \in I}^{\uparrow} X_i = (\bigvee_{i \in I}^{\uparrow} {\downarrow}X_i) \cap (\bigvee_{i \in I}^{\downarrow} {\uparrow}X_i)$$

where, on the right, we take directed suprema in $\mathcal{H}C$ and $\mathcal{S}C$, respectively. More explicitly, we have:

$$\bigvee_{i \in I}^{\uparrow} X_i = (\overline{\bigcup_{i \in I}^{\uparrow} {\downarrow}X_i}) \cap (\bigcap_{i \in I}^{\downarrow} {\uparrow}X_i)$$

Addition and scalar multiplication are lifted from C to $\mathcal{P}C$ as follows:

$$X +_P Y =_{def} ({\downarrow}X +_H {\downarrow}Y) \cap ({\uparrow}X +_S {\uparrow}Y) \qquad r \cdot_P X =_{def} (r \cdot_H {\downarrow}X) \cap (r \cdot_S {\uparrow}X)$$

Using the explicit definitions of addition and scalar multiplication in the lower and upper powercones, these definitions simplify to:

$$X +_P Y = \overline{X + Y} \cap \uparrow(X + Y) \qquad r \cdot_P X = r \cdot X$$

Convex combinations are given by:

$$r \cdot_P X \ +_P \ (1 - r) \cdot_P Y = \overline{r \cdot X + (1 - r) \cdot Y} \cap \uparrow(r \cdot X + (1 - r) \cdot Y)$$

There is also a Scott-continuous semilattice operation. It is defined by:

$$X \cup_P Y =_{def} (\downarrow X \vee_{\mathcal{H}C} \downarrow Y) \cap (\uparrow X \vee_{\mathcal{S}C} \uparrow Y)$$

which simplifies to $X \cup_P Y = \overline{\mathrm{conv}(X \cup Y)} \cap \uparrow\mathrm{conv}(X \cup Y)$. Further, the following is proved in [61, Section 4.3]:

Theorem 3.10. *Let $(C, +, 0, \cdot)$ be a continuous coherent d-cone. Then $(\mathcal{P}C, +_P, \{0\}, \cdot_P)$ is also a continuous coherent d-cone, and, equipped with the semilattice operation $\cup_P$, it forms a d-cone semilattice.*

The finitely generated convex lenses $k_C(F) =_{def} \overline{\mathrm{conv} F} \cap \uparrow\mathrm{conv}\, F$, where F is a finite, non-empty subset of C, form a basis for $\mathcal{P}C$, and, for any $X, Y \in \mathcal{P}C$, we have $X \ll_{\mathcal{P}C} Y$ if, and only if, $X \leq_{\mathrm{EM}} k_C(F)$ and $F \subseteq \mathrel{\rlap{\downarrow}{\downarrow}} Y$ and $\mathrel{\rlap{\uparrow}{\uparrow}} F \supseteq Y$ (i.e., $F \ll_{\mathrm{EM}} Y$) for some such F. If the way-below relation of C is additive, so is that of $\mathcal{P}C$. □

We can now show:

Theorem 3.11. *Let $(K, +_r, 0)$ be a continuous coherent full Kegelspitze. Then $(\mathcal{P}K, +_{rP}, \{0\})$ is also a continuous coherent full Kegelspitze and, equipped with the Scott-continuous semilattice operation $\cup_P$, it forms a Kegelspitze semilattice. Directed suprema are given by:*

$$\bigvee^{\uparrow}_{i \in I} X_i = (\overline{\bigcup^{\uparrow}_{i \in I} \downarrow X_i}) \cap (\bigcap^{\downarrow}_{i \in I} \uparrow X_i)$$

The finitely generated convex lenses $k_K(F) =_{def} \overline{\mathrm{conv} F} \cap \uparrow\mathrm{conv}\, F$, where F is a finite, nonempty subset of K, form a basis for $\mathcal{P}K$, and, for any $X, Y \in \mathcal{P}K$, we have $X \ll_{\mathcal{P}K} Y$ if, and only if, $X \leq_{\mathrm{EM}} k_K(F)$ and $F \subseteq \mathrel{\rlap{\downarrow}{\downarrow}} Y$ and $\mathrel{\rlap{\uparrow}{\uparrow}} F \supseteq Y$ (i.e., $F \ll_{\mathrm{EM}} Y$) for some such F. If, in addition, the way-below relation of K is closed under convex combinations, so is that of $\mathcal{P}K$.

Proof. Applying Theorem 2.35 we can regard K as a Scott-closed convex subset of the d-cone $C =_{def} \textsf{d-Cone}(K)$, with its partial order and algebraic structure inherited from that of C. Applying Propositions 2.42 and 2.44, we see that C is continuous and coherent. It follows that K is a sub-dcpo of C, that its way-below relation is inherited from that of C, that a subset of K is Scott-compact in the topology of K if, and only if, it is Scott-compact in C, and that a subset of K is a lens of K if, and only if, it is a lens of C.

We therefore see that $\mathcal{P}K$ is a subset of $\mathcal{P}C$. It also evidently inherits its partial order from that of $\mathcal{P}C$. We next show that $\mathcal{P}K$ is a Scott-closed convex subset of K. To see it is a lower set, suppose $X \leq_{\mathrm{EM}} Y \in \mathcal{P}K$. Then $X \subseteq {\downarrow_C} X \subseteq {\downarrow_C} Y \subseteq K$ (the last as $Y \subseteq K$ and K is a lower set), and so $X \in \mathcal{P}K$. For closure under directed suprema, suppose X_i is a directed subset of $\mathcal{P}K$. Then $\bigvee^{\uparrow}_{i\in I} X_i = Y \cap Z$, where

$$Y =_{def} \overline{(\bigcup^{\uparrow}_{i\in I} {\downarrow_C} X_i)} \qquad \text{and} \qquad Z =_{def} \bigcap^{\downarrow}_{i\in I} {\uparrow_C} X_i$$

with the closure being taken in C. As $X_i \subseteq K$, Y is in fact a Scott-closed subset of K. Therefore the directed supremum is a subset of K and so a lens of K, as required, and we have shown that $\mathcal{P}K$ is a Scott-closed subset of $\mathcal{P}C$. It follows in particular that $\mathcal{P}K$ is a sub-dcpo of $\mathcal{P}C$. Noting that we can write Y equivalently taking the lower closure and the topological closure in K, and that we can write $Z \cap K$ as $\bigcap_{i\in I} {\uparrow_K} X_i$ we further see that directed suprema in $\mathcal{P}K$ are given as claimed.

For convex closure, recall that convex combinations are given by

$$\overline{r\cdot X + (1-r)\cdot Y} \cap {\uparrow_C}(r\cdot X + (1-r)\cdot Y)$$

where the closure is taken in C. Taking $X, Y \in \mathcal{P}K$ we see that $r\cdot X + (1-r)\cdot Y$ is a subset of K, as K is a convex subset of C. So the closure can equivalently be taken in K and we find that the convex combination is a subset of K and so, as required, a lens in K. Intersecting ${\uparrow_C}(r\cdot X + (1-r)\cdot Y)$ with K, we note that we can write the convex combination equivalently as $\overline{r\cdot X + (1-r)\cdot Y} \cap {\uparrow_K}(r\cdot X + (1-r)\cdot Y)$ with the closure taken in K. Thus the convex combination operators of $\mathcal{P}K$ are the same as those inherited from $\mathcal{P}C$.

As $\mathcal{P}K$ is a Scott-closed convex subset of $\mathcal{P}C$, it inherits a continuous coherent Kegelspitze structure satisfying Property (OC3) from $\mathcal{P}C$, with way-below relation the restriction of that of $\mathcal{P}C$, and with basis $B \cap \mathcal{P}K$, where B is any basis of $\mathcal{P}C$. As the zero of $\mathcal{P}K$ is evidently that of $\mathcal{P}C$, we see from the above that the partial order and algebraic structure defined on $\mathcal{P}K$ is that inherited from $\mathcal{P}C$, and so $\mathcal{P}K$ is indeed a continuous coherent Kegelspitze satisfying Property (OC3).

One checks that the operation $\cup$ defined on $\mathcal{P}K$ is the restriction of $\cup_P$ to $\mathcal{P}K$. It is therefore, as claimed a Scott-continuous semilattice operation. Further as $+_P$ and $r\cdot_P -$ both distribute over $\cup_P$, we see that, equipped with $\cup$, $\mathcal{P}K$ is, as claimed, a Kegelspitze semilattice.

The finitely generated convex lenses $k_C(F) = \overline{\mathrm{conv} F} \cap {\uparrow_C} \mathrm{conv}\, F$ that are in $\mathcal{P}K$ form a basis of $\mathcal{P}K$. As then $F \subseteq k_C(F) \subseteq \mathcal{P}K$, the closure can be equivalently be taken in K, and we see that $k_C(F) = k_K(F)$. So $\mathcal{P}K$ has a basis as claimed. The characterisation of $\ll_{\mathcal{P}K}$ can then be read off from the characterisation of $\ll_{\mathcal{P}C}$, as $\ll_{\mathcal{P}K}$ is the restriction of $\ll_{\mathcal{P}C}$ to $\mathcal{P}K$.

If $\ll_K$ is closed under convex combinations, we can assume by Proposition 2.43 that $\ll_C$ is closed under sums. Then $\ll_{\mathcal{P}C}$ is also closed under sums, and so, too, under convex combinations. As $\mathcal{P}K$ inherits convex combinations and its way-below relation from $\mathcal{P}C$, we see that $\ll_{\mathcal{P}K}$ is closed under convex combinations, concluding the proof. □

We next show that $\mathcal{P}K$ is the free Kegelspitze semilattice over any Kegelspitze K satisfying suitable assumptions. The unit $\eta_P : K \to \mathcal{P}K$ is the evident Kegelspitze morphism $\eta_P(x) =_{def} \{x\}$. We first need two lemmas.

Lemma 3.12. *Let K be a continuous coherent full Kegelspitze. Suppose that H, G, F are non-empty finite subsets of K such that $H \ll_{\mathrm{EM}} G$ and $k_K(G) \ll_{\mathcal{P}K} k_K(F)$. Then there are finite sets $H_1 \subseteq \operatorname{conv} H$ and $F_1 \subseteq \operatorname{conv} F$ such that $H \cup H_1 \leq_{\mathrm{EM}} F \cup F_1$.*

Proof. By the characterisation of $\ll_{\mathcal{P}K}$ given in Theorem 3.11, there is a non-empty finite $I \subseteq_{\mathrm{fin}} K$ such that $k_K(G) \leq_{\mathrm{EM}} k_K(I)$ and $I \ll_{\mathrm{EM}} k_K(F)$.

We first show:

$$\forall c \in H.\, \exists a \in \operatorname{conv} F.\, c \leq a \tag{$*$}$$

Choosing $c \in H$, as $H \ll_{\mathrm{EM}} G$ we find a $b \in G$ with $c \ll b$. Then, as $k_K(G) \leq_{\mathrm{EM}} k_K(I)$ we find an i' in the closed set $\overline{\operatorname{conv} I}$ such that $b \leq i'$. So, as $c \ll i' \in \overline{\operatorname{conv} I}$, there is an $i \in \operatorname{conv} I$ such that $c \ll i$, and so $c \leq i$. As convex combinations are monotone, it is then enough to show that every $i'' \in I$ is below an element of $\operatorname{conv} F$. This follows as, since $I \ll_{\mathrm{EM}} k_K(F)$, every such i'' is way-below an element of the closed set $\overline{\operatorname{conv} F}$.

We next show:

$$\forall i \in I.\, \exists c \in \operatorname{conv} H.\, c \leq i \tag{$**$}$$

Choosing $i \in I$, as $k_K(G) \leq_{\mathrm{EM}} k_K(I)$ there is a $b \in \operatorname{conv} G$ with $b \leq i$. As $H \ll_{\mathrm{EM}} G$ and convex combinations are monotone, we then find the required $c \in \operatorname{conv} H$.

We now build a finite set H_1 of elements of $\operatorname{conv} H$ by picking one below each element of I, as guaranteed by $(**)$. We then have:

$$\forall c \in H \cup H_1.\, \exists a \in \operatorname{conv} F.\, c \leq a \tag{$***$}$$

Indeed, for $c \in H$, the conclusion is given by $(*)$, and, for $c \in H_1$, we use that $c \leq i$ for some $i \in I$ and that every element of I is below an element of $\operatorname{conv} F$, as in the argument proving $(*)$. We next build a finite set F_1 of elements of $\operatorname{conv} F$ by picking one above each element of $H \cup H_1$, as guaranteed by $(***)$.

We claim that $H \cup H_1 \leq_{\mathrm{EM}} F \cup F_1$, as required. This follows as, on the one hand, by $(***)$, every element of $H \cup H_1$ is below an element of F_1, and, on the other hand, as $I \ll_{\mathrm{EM}} k_K(F)$, every element of $\operatorname{conv} F$ (and so of $F \cup F_1$) is above an element of I and so, by the choice of H_1, above an element of H_1. □

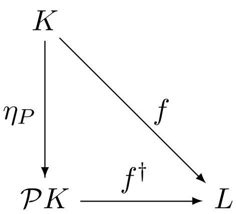
Lemma 3.13. *Let K be a continuous coherent full Kegelspitze in which convex combinations preserve the way-below relation, Suppose that F, G are finite non-empty subsets of K such that $G \ll_{\mathrm{EM}} F$. Then $k_K(G) \ll_{\mathcal{P}K} k_K(F)$.*

Proof. By the characterisation of $\ll_{\mathcal{P}K}$ given in Theorem 3.11 it suffices to find a finite set H such that $k_K(G) \leq_{\mathrm{EM}} k_K(H)$, $H \subseteq \mathop{\downarrow\!\!\downarrow} k_K(F)$ and $\mathop{\uparrow\!\!\uparrow} H \supseteq k_K(F)$.

As convex combinations preserve $\ll_K$ and $G \ll_{\mathrm{EM}} F$, we have $\operatorname{conv} G \ll_{\mathrm{EM}} \operatorname{conv} F$. Then, using the compactness of $\operatorname{conv} F$, we see that there is a non-empty finite subset G' of $\operatorname{conv} G$ such that $\mathop{\uparrow\!\!\uparrow} G' \supseteq \operatorname{conv} F$. Let $H = G \cup G'$. Since $G \subseteq H \subseteq \operatorname{conv} G$, we have

$k_K(G) = k_K(H)$. We also have $H \subseteq \downarrow \mathrm{conv}\, F \subseteq \downarrow k_K(F)$, whence $H \subseteq \downarrow k_K(F)$. Finally, $\uparrow H \supseteq \uparrow G' \supseteq \mathrm{conv}\, F$, and so we have $\uparrow H \supseteq \uparrow \mathrm{conv}\, F \supseteq k_K(F)$. □

Theorem 3.14. *Let K be a continuous coherent full Kegelspitze and in which convex combinations preserve the way-below relation. Then the map η_P is universal. That is, for every Kegelspitze semilattice L and Kegelspitze morphism $f : K \to L$ there is a unique Kegelspitze semilattice morphism $f^\dagger : \mathcal{P}K \to L$ such that the following diagram commutes:*



The morphism is given by:

$$f^\dagger(X) = \bigvee^{\uparrow}\{\bigcup\nolimits_L f(F) \mid F \subseteq_{\mathrm{fin}} K, F \neq \emptyset, F \ll_{\mathrm{EM}} X\}$$

Proof. For any non-empty finite set $F \subseteq K$ we have

$$k_K(F) = \bigcup\nolimits_P \eta_P(F)$$

as $\downarrow \bigcup_P \eta_P(F) = \bigvee_{\mathcal{H}K}\{\downarrow \eta_P(b) \mid b \in F\} = \bigvee_{\mathcal{H}K}\{\downarrow b \mid b \in F\} = \overline{\mathrm{conv} F} = \downarrow k_K(F)$, and (proved similarly) $\uparrow \bigcup_P \eta_P(F) = \uparrow k_K(F)$.

Using this and the basis given in Theorem 3.11, for any $X \in \mathcal{P}K$ we then have:

$$X = \bigvee^{\uparrow}\{\bigcup\nolimits_P \eta_P(F) \mid F \subseteq_{\mathrm{fin}} K, F \neq \emptyset, k_K(F) \ll_{\mathcal{P}K} X\}$$

It follows that

$$f^\dagger(X) = \bigvee^{\uparrow}\{\bigcup\nolimits_L f(F) \mid F \subseteq_{\mathrm{fin}} K, F \neq \emptyset, k_K(F) \ll_{\mathcal{P}K} X\}$$

establishing uniqueness.

For existence we define $f^\dagger$ by means of this formula and then verify that it makes the diagram commute and is both a Kegelspitze and a semilattice map. It is clearly continuous. Next, as in the proof of Theorem 3.9 , it helpful to prove that, for any non-empty $F \subseteq_{\mathrm{fin}} K$, we have:

$$f^\dagger(\bigcup\nolimits_P \eta_P(F)) = \bigcup\nolimits_L f(F) \qquad (*)$$

that is, that:

$$\bigvee^{\uparrow}\{\bigcup\nolimits_L f(G) \mid G \subseteq_{\mathrm{fin}} K, G \neq \emptyset, k_K(G) \ll_{\mathcal{P}K} k_K(F)\} = \bigcup\nolimits_L f(F)$$

To show that the left-hand side is $\leq \bigcup_L f(F)$, suppose given $G = \{b_1, \ldots, b_n\} \subseteq K$, with $n > 0$, such that $k_K(G) \ll_{\mathcal{P}K} k(F)$. Choose $c_1 \ll b_1, \ldots, c_n \ll b_n$ and set $H = \{c_1, \ldots, c_n\}$. By Lemma 3.12, there are finite sets $H_1 \subseteq \mathrm{conv}\, H$ and $F_1 \subseteq \mathrm{conv}\, F$ such that $H \cup H_1 \leq_{\mathrm{EM}} F \cup F_1$. We then have:

$$\bigcup_L f(H) = \bigcup_L f(H) \cup_L \bigcup_L f(H_1) \leq \bigcup_L f(F) \cup_L \bigcup_L f(F_1) = \bigcup_L f(F)$$

where the two equalities follow using the fact that L satisfies the convexity identity (CI) several times. So $\bigcup_L f(G) \leq \bigcup_L (F)$ as H consists of an arbitrary choice of elements way-below each element of G.

Conversely, supposing $F = \{a_1, \ldots, a_n\}$, with $n > 0$, choose $b_1 \ll a_1, \ldots, b_n \ll a_n$ and take $G = \{b_1, \ldots, b_n\}$. By Lemma 3.13 we have $k_K(G) \ll_{\mathcal{P}K} k_K(F)$. So the left-hand side is $\geq \bigcup_L f(G)$, and so $\geq \bigcup_L f(F)$, as G consists of an arbitrary choice of elements way-below each element of F.

Given $(*)$, the rest of the proof follows exactly as did that of Theorem 3.9. □

Similarly to the case of upper semilattices, this universality result contrasts with the corresponding universality result for convex powercones in [61]. As before, it would be interesting to know if the preservation assumption made here is needed.

3.4. **Powerdomains combining probabilistic choice and nondeterminism.** Powerdomains combining probabilistic choice and nondeterminism exist on arbitrary dcpos for general reasons. That is, there is always a free Kegelspitze semilattice over any dcpo, and the same is true for Kegelspitze join- and meet-semilattices. This is because each of these kinds of structure can be axiomatised by inequations over a signature of finitary operations, possibly (Scott-)continuously parameterised by an auxiliary dcpo, and free algebras over dcpos satisfying such inequations always exist (this can be shown using the General Adjoint Functor Theorem, and see [17]). These various free semilattices over a dcpo are automatically continuous if the dcpo is, as follows from [57] (but not from the less general results on free algebras in [1], which do not apply when there is parameterisation).

Free Kegelspitze semilattices are given by the inequational theory with: a binary operation symbol $+_r$, for each $r \in [0, 1]$; a unary operation symbol $\cdot_r$, continuously parameterised by r, ranging over the dcpo $[0, 1]$; a binary operation symbol $\cup$; and a constant 0. The equations consist of: equations for a Kegelspitze, by which we mean the barycentric algebra equations for $+_r$, as given in Section 2 and the equation $\cdot_r(x) = x +_r 0$; equations asserting that $\cup$ is associative, commutative, and idempotent; and the equation

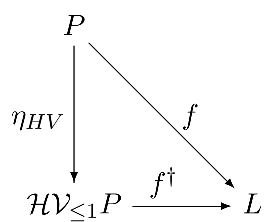
$$x +_r (y \cup z) = (x +_r y) \cup (x +_r z)$$

saying that $+_r$ distributes over $\cup$ in its second argument, for any $r \in [0, 1]$ (and so also in its first one). For Kegelspitze join-semilattices one adds the inequation $x \leq x \cup y$; for meet-semilattices one instead adds the inequation $x \cup y \leq x$.

While we do not know any general characterisation of these various free constructions, by making use of our previous results we can characterise them for domains (assumed also coherent in the convex case). From the discussion in Section 2.5 we know that the subprobabilistic power domain $\mathcal{V}_{\leq 1}P$ over a dcpo is a full Kegelspitze; that, in case P is a domain, it is a continuous Kegelspitze with convex combinations preserving the way-below relation; and that, in case P is also coherent, then so is $\mathcal{V}_{\leq 1}P$. We further know that, if P is a domain, then the subprobabilistic powerdomain $\mathcal{V}_{\leq 1}P$ is the free Kegelspitze over P, with unit $x \mapsto \delta_x$, where δ_x is the Dirac distribution, with mass 1 at x (given a Scott-continuous $f : P \to K$, we write $\overline{f} : \mathcal{V}_{\leq 1}P \to K$ for its extension to a Kegelspitze map).

Therefore we can form the three power Kegelspitzen $\mathcal{HV}_{\leq 1}P$, $\mathcal{SV}_{\leq 1}P$, and $\mathcal{PV}_{\leq 1}P$, assuming that P is a domain (and a coherent one, in the convex case); it is immediate from the above remarks on the subprobabilistic powerdomain and Theorems 3.4, 3.9, and 3.14 that these yield, respectively, the free Kegelspitze join-semilattice, the free Kegelspitze meet-semilattice, and the free Kegelspitze semilattice over a given domain. We record these results as corollaries.

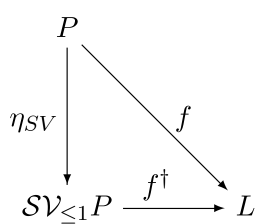
Corollary 3.15. *Let P be a domain. Then the map $\eta_{HV} =_{def} {\downarrow}\delta_x : P \to \mathcal{HV}_{\leq 1}P$ is universal. That is, for every Kegelspitze join-semilattice L and Scott-continuous map $f : P \to L$ there is a unique Kegelspitze semilattice morphism $f^\dagger : \mathcal{HV}_{\leq 1}P \to L$ such that the following diagram commutes:*



The morphism is given by:

$$f^\dagger(X) = \bigvee \overline{f}(X)$$ □

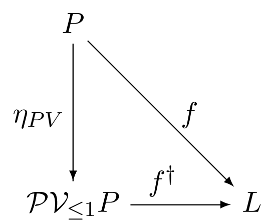
Corollary 3.16. *Let P be a domain. Then the map $\eta_{SV} =_{def} {\uparrow}\delta_x : P \to \mathcal{SV}_{\leq 1}P$ is universal. That is, for every Kegelspitze meet-semilattice L and Scott-continuous map $f : P \to L$ there is a unique Kegelspitze semilattice morphism $f^\dagger : \mathcal{SV}_{\leq 1}P \to L$ such that the following diagram commutes:*



The morphism is given by:

$$f^\dagger(X) = \bigvee^{\uparrow} \{\bigwedge \overline{f}(F) \mid F \subseteq_{\text{fin}} \mathcal{V}_{\leq 1}P, F \neq \emptyset, {\uparrow}F \supseteq X\}$$ □

Corollary 3.17. *Let P be a coherent domain. Then the map $\eta_{PV} =_{def} \{\delta_x\} : P \to \mathcal{PV}_{\leq 1}P$ is universal. That is, for every Kegelspitze semilattice L and Scott-continuous map $f : P \to L$ there is a unique Kegelspitze semilattice morphism $f^\dagger : \mathcal{PV}_{\leq 1}P \to L$ such that the following* *diagram commutes:*



The morphism is given by:

$$f^\dagger(X) = \bigvee^{\uparrow}\{\bigcup_L \overline{f}(F) \mid F \subseteq_{\mathrm{fin}} \mathcal{V}_{\leq 1}P, F \neq \emptyset, \uparrow F \supseteq X\} \qquad \square$$

4. Functional representations

In [28, Sections 4 and 6], the various powercones over a d-cone were represented by functionals. We will use those results to obtain similar functional representations of the corresponding power Kegelspitzen, and then deduce corresponding functional representations for mixed powerdomains.

Some context may help. For a functional representation of a monad T one chooses a test space O, say, and represents an object $T(X)$ by a suitable collection of functionals, with domain a space of 'test functions' from X to O and range O. One general such method is to work in a symmetric monoidal closed category, when one has available the 'continuation' or 'double-dualisation' monad $[[X, O], O]$ (writing $[X, Y]$ for the function space). Assuming that the monad T is strong, there is then a 1-1 correspondence between T-algebras $\alpha : T(O) \to O$ and morphisms $T \to [[-, O], O]$ of strong monads [29, 33, 34]. If there are sufficiently many test functions, this morphism will be a monomorphism, and, perhaps with further restrictions on the functionals, it may corestrict to an isomorphism; one may also have to restrict to certain objects X.

In our case, we would work with the category of Kegelspitzen and continuous linear maps, when $[K, L]$ would be the Kegelspitze formed from such maps with the pointwise order and algebraic structure, and the extended reals $\overline{\mathbb{R}}_+$ provide a natural test space. As we will see below, there is a natural choice of functionals for all three of our power-Kegelspitzen, but in no case are such functionals generally linear: for example in the Hoare case they are rather sublinear. When, later, we apply our results to obtain functional representations of mixed powerdomain monads, we are working in the cartesian-closed category of dcpos, the above general framework does apply and our functional representations are, in fact, submonads of the relevant continuation monads (but, for non-essential reasons, with some minor differences in the convex case).

Throughout this section, we generally work with full Kegelspitzen, that is, those satisfying Property (OC3). We consider such Kegelspitzen K to be embedded in their universal d-cones $C = \textsf{d-Cone}(K)$ as Scott-closed convex sets (Theorem 2.35) and we recall that the universal d-cones are continuous whenever the Kegelspitzen are (Proposition 2.42). From Section 2.5

we recall that the subprobabilistic powerdomain $\mathcal{V}_{\leq 1}P$ of a dcpo P is a full Kegelspitze, that $\mathsf{d\text{-}Cone}(\mathcal{V}_{\leq 1}P) \cong \mathcal{V}P$, the valuation powerdomain of P, and that, in case P is a domain, $\mathcal{V}_{\leq 1}P$ is continuous.

We will make use of *norms* on d-cones C, taking them to be Scott-continuous sublinear functionals $\|\text{-}\|\colon C \to \overline{\mathbb{R}}_+$ such that $\|x\| > 0$ for every $x \neq 0$; *normed d-cones* are then d-cones equipped with a norm. A map $f\colon C \to D$ from one normed d-cone C to another D is *nonexpansive* if $\|f(x)\| \leq \|x\|$ for all $x \in C$; it is a morphism of normed d-cones if it is nonexpansive and a morphism of d-cones (i.e., Scott-continuous and linear).

Various function space d-cones will be involved in our development. As well as those considered in Section 2.5 we note that, for any d-cones C and D, the subsets $\mathcal{L}_{\mathrm{sub}}(C, D)$, and $\mathcal{L}_{\mathrm{sup}}(C, D)$ of D^C of, respectively, the sublinear, and superlinear functions form sub-d-cones of D^C. Regarding d-cone semilattices, if D is a d-cone semilattice (respectively, join-semilattice, meet-semilattce) then, with the pointwise structure, so is D^P, for any dcpo P. Further, for any d-cone C, if D is a join-semilattice (meet-semilattice) then, as is easily checked, $\mathcal{L}_{\mathrm{sub}}(C, D)$ (respectively, $\mathcal{L}_{\mathrm{sup}}(C, D)$) is a sub-d-cone join-semilattice (respectively, sub-d-cone meet-semilattice) of D^C.

Supposing additionally the cones C and D to be normed, the collection $\mathcal{L}^{\leq 1}_{\mathrm{sub}}(C, D)$ of all Scott-continuous sublinear nonexpansive functions, with D a d-cone join-semilattice, forms a Kegelspitze join-semilattice; indeed it is a sub-Kegelspitze join-semilattice of $\mathcal{L}_{\mathrm{sub}}(C, D)$, regarding the latter as a Kegelspitze join-semilattice.

A trivial example of a normed cone is $\overline{\mathbb{R}}_+$ with norm the identity function:

$$\|x\| = x$$

A less trivial example is provided by the dual cone K^* of a Kegelspitze K equipped with the sup norm, defined by:

$$\|f\|^*_K = \sup_{x \in K} f(x)$$

where the index K indicates the dependency of this norm on the d-cone K^* on the Kegelspitze K. Notice that $\|f\|^*_K = +\infty$ if there is an $x \in K$ such that $f(x) = +\infty$.

If K satisfies Property (OC3) then we can define a norm on C^*, where $C =_{def} \mathsf{d\text{-}Cone}(K)$ by:

$$\|f\|^*_K =_{def} \|f \restriction K\|^*_K = \sup_{x \in K} f(x)$$

where the index now indicates the dependency of this norm on C^* on K. With this norm, the d-cone isomorphism between K^* and C^* given in Section 2.5, Example 2.47 becomes an isomorphism of normed d-cones.

Recall that, for any element $x \in C = \mathsf{d\text{-}Cone}(K)$, the evaluation map $\mathrm{ev}_C(x)\colon C^* \to \overline{\mathbb{R}}_+$ sends f to $f(x)$. We note that $\mathrm{ev}_C(x) \leq \|\text{-}\|^*_K$ if $x \in K$, with the converse holding if K is continuous. For, if $x \in K$ then we have $\mathrm{ev}_C(x) \leq \|\text{-}\|^*_K$, since $f(x) \leq \sup_{x \in K} f(x) = \|f\|^*_K$, for $f \in C^*$. And if $x \notin K$, then using the Strict Separation Theorem [61, Theorem 3.8], we obtain an $f \in C^*$ such that $f(y) \leq 1$ for $y \in K$ but $f(x) > 1$, whence $\|f\|^*_K = \sup_{y \in K} f(y) \leq 1 < f(x) = \mathrm{ev}_C(x)(f)$, and so $\mathrm{ev}_C(x) \not\leq \|\text{-}\|^*_K$. Thus, if the d-cone $C = \mathsf{d\text{-}Cone}(K)$ is continuous and reflexive, the Scott-continuous linear functionals $\varphi \leq \|\text{-}\|_K^*$ on C^* are given by evaluations at points $x \in K$.

4.1. **The lower power Kegelspitze.** We regard $\overline{\mathbb{R}}_+$ as a d-cone join-semilattice with the semilattice operation $r \vee s = \max(r, s)$ (as such it is isomorphic to $\mathcal{H}\overline{\mathbb{R}}_+$, now regarding $\overline{\mathbb{R}}_+$ as a d-cone). We take an arbitrary d-cone C with its lower powercone $\mathcal{H}C$ and its dual C^*. Consider the map $\Lambda_C \colon \mathcal{H}C \to \overline{\mathbb{R}}_+^{C^*}$ where:

$$\Lambda_C(X)(f) =_{def} \sup_{x \in X} f(x)$$

Fixing $f \in C^*$, we obtain the map $\Lambda_C(-)(f) \colon \mathcal{H}C \to \overline{\mathbb{R}}_+$, which is the unique d-cone join-semilattice morphism extending f along the canonical embedding $\eta \colon C \to \mathcal{H}C$ by [28, Proposition 3.2].

Fixing $X \in \mathcal{H}C$, we obtain the functional

$$\Lambda_C(X) \colon C^* \to \overline{\mathbb{R}}_+$$

where:

$$\Lambda_C(X)(f) = \sup_{x \in X} f(x)$$

As the pointwise supremum of the Scott-continuous linear functionals $\mathrm{ev}_C(x)$ $(x \in X)$, $\Lambda_C(X)$ is Scott-continuous and sublinear. In this way we obtain a d-cone join-semilattice morphism

$$\Lambda_C \colon \mathcal{H}C \longrightarrow \mathcal{L}_{\mathrm{sub}}(C, \overline{\mathbb{R}}_+)$$

which represents the lower convex powercone by the Scott-continuous sublinear functionals on the dual cone C^*.

Theorem 4.1 ([28, Proposition 6.1 and Theorem 6.2])**.** *Let C be a d-cone. Then we have a d-cone join-semilattice morphism $\Lambda_C \colon \mathcal{H}C \to \mathcal{L}_{\mathrm{sub}}(C, \overline{\mathbb{R}}_+)$, where:*

$$\Lambda_C(X) = \sup_{x \in X} f(x)$$

If, in addition, C is continuous then Λ_C is an order embedding; if, further, C is reflexive with a continuous dual then it is an isomorphism. □

To apply the above considerations to the universal d-cone $C = \mathsf{d\text{-}Cone}(K)$ over K we now consider a full Kegelspitze K. The power Kegelspitze $\mathcal{H}K$ is a Scott-closed convex join-subsemilattice of the powercone $\mathcal{H}C$. The functionals $\Lambda_C(X)$ representing Scott-closed convex subsets X of K are the sublinear functionals $\Lambda_C(X)$ dominated by the norm $\|\text{-}\|_K^* = \Lambda_C(K)$, and all of them if K is continuous. For certainly if $X \subseteq K$ then $\Lambda_C(X) \leq \Lambda_C(K) = \|\text{-}\|_K^*$, and, assuming the converse, for any $x \in X$ we have $\mathrm{ev}_C(x) \leq \Lambda_C(X) \leq \|\text{-}\|_K^*$, and so $x \in K$ by the above discussion (assuming K continuous).

Recalling that $\mathcal{L}_{\mathrm{sub}}^{\leq 1}(K^*, \overline{\mathbb{R}}_+)$ is the collection of nonexpansive functionals in $\mathcal{L}_{\mathrm{sub}}(K^*, \overline{\mathbb{R}}_+)$, we therefore have a Kegelspitze join-semilattice morphism

$$\Lambda_K \colon \mathcal{H}K \to \mathcal{L}_{\mathrm{sub}}^{\leq 1}(K^*, \overline{\mathbb{R}}_+)$$

viz. the composition

$$\mathcal{H}K \xrightarrow{\Lambda_C \restriction \mathcal{H}K} \mathcal{L}^{\leq 1}_{\mathrm{sub}}(C^*, \overline{\mathbb{R}}_+) \cong \mathcal{L}^{\leq 1}_{\mathrm{sub}}(K^*, \overline{\mathbb{R}}_+)$$

of the restriction of Λ_C to $\mathcal{H}K$ with the isomorphism $\mathcal{L}^{\leq 1}_{\mathrm{sub}}(C^*, \overline{\mathbb{R}}_+) \cong \mathcal{L}^{\leq 1}_{\mathrm{sub}}(K^*, \overline{\mathbb{R}}_+)$ arising from the normed d-cone isomorphism between K^* and C^*. Theorem 4.1 then yields the desired functional representation theorem, adapting its hypotheses to Kegelspitzen:

Theorem 4.2. *Let K be a full Kegelspitze. Then we have a Kegelspitze join-semilattice morphism $\Lambda_K \colon \mathcal{H}K \to \mathcal{L}^{\leq 1}_{\mathrm{sub}}(K^*, \overline{\mathbb{R}}_+)$. It is given by:*

$$\Lambda_K(X)(f) =_{def} \sup_{x \in X} f(x)$$

If K is continuous then Λ_K is an order embedding. If, further, the dual cone K^* is continuous and the universal d-cone $\mathsf{d\text{-}Cone}(K)$ is reflexive, then Λ_K is an isomorphism. □

With the aid of this theorem we can obtain a corresponding result for the lower mixed powerdomain. For any dcpo P, making use of Section 2.5, we see that the predicate extension and restriction maps

$$\mathrm{EXT}_P =_{def} f \mapsto \overline{f} \colon \mathcal{L}P \to (\mathcal{V}_{\leq 1}P)^* \quad \text{and} \quad \mathrm{RES}_P =_{def} f \mapsto f \circ \delta \colon (\mathcal{V}_{\leq 1}P)^* \to \mathcal{L}P$$

are d-cone morphisms, and mutually inverse isomorphisms if P is a domain.

Next, for any dcpo P, we equip the d-cone $\mathcal{L}P$ with the sup norm, i.e., the one defined by: $\|f\|_\infty = \sup_{x \in P} f(x)$.

Lemma 4.3. *Let P be a dcpo. Then the extension map $\mathrm{EXT}_P \colon \mathcal{L}P \to (\mathcal{V}_{\leq 1}P)^*$ preserves the norm. The restriction map $\mathrm{RES}_P \colon (\mathcal{V}_{\leq 1}P)^* \to \mathcal{L}P$ is nonexpansive and preserves the norm if P is a domain. So EXT_P and RES_P are normed d-cone morphisms, and mutually inverse isomorphisms if P is a domain.*

Proof. We wish first to show that $\|f\|_\infty = \|\overline{f}\|^*_{(\mathcal{V}_{\leq 1}P)}$ for a given $f \in \mathcal{L}P$ (where $\overline{f}(\mu) = \int f\, d\mu$). In one direction, we have $\|f\|_\infty \leq \|\overline{f}\|^*_{(\mathcal{V}_{\leq 1}P)}$ as, for any $x \in P$, $f(x) = \int f\, d\delta_x = \overline{f}(\delta_x)$. In the other direction it suffices to show that $\overline{f}(\mu) \leq \|f\|_\infty$ for all $\mu \in \mathcal{V}_{\leq 1}P$. This holds as we have: $\overline{f}(\mu) = \int f\, d\mu \leq \int (x \mapsto \|f\|_\infty)\, d\mu = \mu(P)\|f\|_\infty \leq \|f\|_\infty$. Next, the restriction map is evidently nonexpansive. If P is continuous it preserves the norm as then it is right inverse to the extension map, and that preserves the norm. □

We will make use of the mapping:

$$\Phi_P \colon \overline{\mathbb{R}}_+^{(\mathcal{V}_{\leq 1}P)^*} \to \overline{\mathbb{R}}_+^{\mathcal{L}P}$$

where P is a dcpo and $\Phi_P(F) = F \circ \mathrm{EXT}_P$. It is a d-cone morphism, and preserves pointwise joins and meets. If P is a domain it is an isomorphism, with inverse $\Phi^r_P =_{def} F \mapsto F \circ \mathrm{RES}_P$.

Corollary 4.4. *Let P be a dcpo. Then we have a Kegelspitze join-semilattice morphism:*

$$\Lambda_P \colon \mathcal{H}\mathcal{V}_{\leq 1}P \longrightarrow \mathcal{L}^{\leq 1}_{\mathrm{sub}}(\mathcal{L}P, \overline{\mathbb{R}}_+)$$

It is given by:

$$\Lambda_P(X)(f) =_{def} \sup_{\mu \in X} \int f \, d\mu$$

If P is a domain then Λ_P is an isomorphism.

Proof. We first check that both Φ_P and Φ_P^r preserve sublinearity and nonexpansiveness, the latter by Lemma 4.3. So Φ_P cuts down to a morphism

$$\mathcal{L}_{\mathrm{sub}}^{\leq 1}((\mathcal{V}_{\leq 1}P)^*, \overline{\mathbb{R}}_+) \to \mathcal{L}_{\mathrm{sub}}^{\leq 1}(\mathcal{L}P, \overline{\mathbb{R}}_+)$$

of Kegelspitze join-semilattices that is an isomorphism if P is a domain.

Next, as discussed in Section 2.5, $\mathcal{V}_{\leq 1}P$ is a full Kegelspitze, and, if P is a domain, then $\mathcal{V}_{\leq 1}P$ is continuous and $\mathsf{d\text{-}Cone}(\mathcal{V}_{\leq 1}P) \cong \mathcal{V}P$; further, if P is continuous then $(\mathcal{V}P)^*$ (which is isomorphic to $(\mathcal{V}_{\leq 1}P)^*$) is continuous (being isomorphic to $\mathcal{L}P$), and $\mathcal{V}P$ is reflexive. So if P is a domain then $\mathcal{V}_{\leq 1}P$ satisfies all the other various hypotheses of Theorem 4.2.

An easy calculation then displays Λ_P as the following composition of Kegelspitze join-semilattice morphisms that are isomorphisms if P is a domain:

$$\mathcal{H}\mathcal{V}_{\leq 1}P \xrightarrow{\Lambda_K} \mathcal{L}_{\mathrm{sub}}^{\leq 1}((\mathcal{V}_{\leq 1}P)^*, \overline{\mathbb{R}}_+) \xrightarrow{\Phi_P} \mathcal{L}_{\mathrm{sub}}^{\leq 1}(\mathcal{L}P, \overline{\mathbb{R}}_+)$$ □

We remark that nonexpansiveness has a simple formulation for monotone homogeneous functionals $F\colon \mathcal{L}P \to \overline{\mathbb{R}}_+$, viz. that

$$F(\mathbf{1}_P) \leq 1$$

where $\mathbf{1}_P$ is the constant function on P with value 1. The condition is evidently is a special case of nonexpansiveness, as $\|\mathbf{1}_P\|_\infty = 1$. Conversely, for any $g \in \mathcal{L}P$, noting that $g \leq \|g\|_\infty \mathbf{1}_P$, we have: $\|F(g)\| \leq \|F(\|g\|_\infty \mathbf{1}_P)\| = \|g\|_\infty \|F(\mathbf{1}_P)\| \leq \|g\|_\infty$.

4.2. **The upper power Kegelspitze.** We regard $\overline{\mathbb{R}}_+$ as a d-cone meet-semilattice with the semilattice operation $r \wedge s = \min(r, s)$ (as such it is isomorphic to $\mathcal{S}\overline{\mathbb{R}}_+$, now regarding $\overline{\mathbb{R}}_+$ as a d-cone). Take a continuous d-cone C with its upper powercone $\mathcal{S}C$ and its dual C^*. Consider the map $\Lambda_C\colon \mathcal{S}C \to \overline{\mathbb{R}}_+^{C^*}$ where:

$$\Lambda_C(X)(f) =_{def} \inf_{x \in X} f(x)$$

Fixing $f \in C^*$, we obtain the map $\Lambda_C(-)(f)\colon \mathcal{S}C \to \overline{\mathbb{R}}_+$, which is the unique Scott-continuous linear meet-semilattice homomorphism extending f along the canonical embedding $\eta\colon C \to \mathcal{S}C$ which maps x to $\uparrow x$ (this follows from [28, Proposition 3.5], using the above isomorphism).

Fixing $X \in \mathcal{S}C$, we obtain the functional

$$\Lambda_C(X)\colon C^* \to \overline{\mathbb{R}}_+$$

where:

$$\Lambda_C(X)(f) = \inf_{x \in X} f(x)$$

As the pointwise infimum of linear functionals, $\Lambda_C(X)$ is superlinear. It is also Scott-continuous. Indeed, for a Scott-compact set X, the image $f(X)$ is Scott-compact in $\overline{\mathbb{R}}_+$, hence has a smallest element $\min f(X) = \inf_{x\in X} f(x) = \Lambda_C(X)(f)$; thus $\uparrow\Lambda_C(X)(f) = \mathcal{S}(f)(X)$; since $f \mapsto \mathcal{S}(f)$ is Scott-continuous, $f \mapsto \mathcal{S}(f)(X)\colon C^* \to \mathcal{S}(\overline{\mathbb{R}}_+)$ is Scott-continuous, too; composing with the isomorphism $\mathcal{S}(\overline{\mathbb{R}}_+) \cong \overline{\mathbb{R}}_+$ yields the Scott continuity of $\Lambda_C(X)$.

In this way we obtain a d-cone meet-semilattice morphism

$$\Lambda_C\colon\mathcal{S}C \longrightarrow \mathcal{L}_{\sup}(C^*, \overline{\mathbb{R}}_+)$$

representing the upper powercone by the Scott-continuous superlinear functionals on the dual cone C^*. We need the quite strong hypothesis of a convenient d-cone (see Section 2.5) to obtain the analogue of Theorem 4.1:

Theorem 4.5 ([28, Proposition 6.4 and Theorem 6.5])**.** *Suppose that C is a continuous d-cone. Then we have a d-cone meet-semilattice morphism $\Lambda_C\colon\mathcal{S}C \to \mathcal{L}_{\sup}(C^*, \overline{\mathbb{R}}_+)$, which is an order embedding, where:*

$$\Lambda_C(X)(f) = \inf_{x\in X} f(x)$$

Further, if C is convenient then Λ_C is an isomorphism. □

We now consider a continuous full Kegelspitze K. The universal d-cone $C = \mathsf{d\text{-}Cone}(K)$ is also then continuous and we apply the above considerations to it. The upper power Kegelspitze $\mathcal{S}K$ consists of all nonempty Scott-compact saturated convex subsets X of K. As discussed in Section 3.2 the map $u\colon \mathcal{S}K \to \mathcal{S}C$, where $u(X) = \uparrow X$ is a d-cone meet-semilattice morphism which is an order-embedding. The functions $\Lambda_C(u(X))\colon C^* \to \overline{\mathbb{R}}_+$ $(X \in \mathcal{S}K)$ are Scott-continuous and superlinear. We want to characterise the Scott-continuous and superlinear functionals F on C^* that represent the elements of $\mathcal{S}K$ in this way. It turns out that, unlike the case of the lower power Kegelspitze, being nonexpansive is not sufficient. We notice that, for any $X \in \mathcal{S}K$, the representing functional $F\colon C^* \to \overline{\mathbb{R}}_+$ has a remarkable property: it is *strongly nonexpansive*, by which we mean that

$$F(f+g) \le F(f) + \|g\|_K^*$$

holds for all $f, g \in C^*$ (setting $f = 0$, we see that strong nonexpansiveness implies non-expansiveness). Indeed, we have: $F(f+g) = \inf_{x\in X}(f+g)(x) = \inf_{x\in X}(f(x)+g(x)) \le \inf_{x\in X}(f(x) + \sup_{x\in K} g(x)) = \inf_{x\in X}(f(x) + \|g\|_K^*) = F(f) + \|g\|_K^*$.

For any normed d-cone D we write $\mathcal{L}^{\mathrm{sne}}_{\sup}(D, \overline{\mathbb{R}}_+)$ for the collection of all Scott-continuous superlinear functionals $F\colon D \to \overline{\mathbb{R}}_+$ that are *strongly nonexpansive* in the sense that:

$$F(x+y) \;\le\; F(x) + \|y\|$$

holds for all $x, y \in D$. The collection forms a Kegelspitze meet-semilattice, indeed it is a sub-Kegelspitze meet-semilattice of $\mathcal{L}_{\sup}(D, \overline{\mathbb{R}}_+)$. One easily checks that $\mathcal{L}^{\mathrm{sne}}_{\sup}(C^*, \overline{\mathbb{R}}_+)$ and $\mathcal{L}^{\mathrm{sne}}_{\sup}(K^*, \overline{\mathbb{R}}_+)$ are isomorphic as Kegelspitze meet-semilattices via the normed d-cone isomorphism between K^* and C^*.

Putting all this together, we define

$$\Lambda_K \colon \mathcal{S}K \longrightarrow \mathcal{L}^{\mathrm{sne}}_{\sup}(K^*, \overline{\mathbb{R}}_+)$$

to be the Kegelspitze meet-semilattice morphism given by the composition:

$$\mathcal{S}K \xrightarrow{\Lambda_C \circ u} \mathcal{L}^{\mathrm{sne}}_{\sup}(C^*, \overline{\mathbb{R}}_+) \cong \mathcal{L}^{\mathrm{sne}}_{\sup}(K^*, \overline{\mathbb{R}}_+)$$

Theorem 4.5 then yields the desired functional representation theorem, adapting its hypotheses to Kegelspitzen:

Theorem 4.6. *Let K be a continuous full Kegelspitze. Then we have a Kegelspitze meet-semilattice morphism $\Lambda_K \colon \mathcal{S}K \to \mathcal{L}^{\mathrm{sne}}_{\sup}(K^*, \overline{\mathbb{R}}_+)$. It is given by:*

$$\Lambda_K(X)(f) =_{def} \inf_{x \in X} f(x)$$

If, further, $\mathsf{d\text{-}Cone}(K)$ *is convenient, then Λ_K is an isomorphism.*

Proof. We note first that, for any $X \in \mathcal{S}K$ and $f \in K^*$, we have:

$$\Lambda_K(X)(f) = \Lambda_C(\uparrow X)(\widetilde{f}) = \inf_{x \in \uparrow X} \widetilde{f}(x) = \inf_{x \in X} \widetilde{f}(x) = \inf_{x \in X} f(x)$$

Next, as $\mathsf{d\text{-}Cone}(K)$ is continuous since K is, Theorem 4.5 tells us that Λ_C is an order-embedding; so, as u is also one, so too is Λ_K.

For the isomorphism, assuming that $\mathsf{d\text{-}Cone}(K)$ is convenient, we need then only show that $\Lambda_C \circ u \colon \mathcal{S}K \to \mathcal{L}^{\mathrm{sne}}_{\sup}(C^*, \overline{\mathbb{R}}_+)$ is onto. So take any strongly nonexpansive Scott-continuous superlinear $F \colon C^* \to \overline{\mathbb{R}}_+$. By Theorem 4.5 we know that there is a $Y \in \mathcal{S}C$ such that $\Lambda_C(Y) = F$, that is such that $F(f) = \inf_{y \in Y} f(y)$ for all $f \in C^*$. Let $X = Y \cap K$. Clearly, X is a Scott-compact convex set saturated in K. We want to show that X is non-empty and $\Lambda_C(u(X)) = \Lambda_C(Y)$, that is, $\inf_{y \in Y} f(y) = \inf_{x \in X} f(x)$ for all $f \in C^*$.

Since the d-cone $K^* \cong C^*$ is assumed to be continuous, strong nonexpansiveness allows us to apply the Main Lemma [28, Lemma 5.1(1)] to C^*. We learn that $\Lambda_C(Y)(f) = \inf \varphi(f)$, where φ ranges over the Scott-continuous linear functionals on C^* such that $\Lambda_C(Y) \leq \varphi \leq \|\text{-}\|^*_K$. Using the hypotheses of continuity and reflexivity for C, and the discussion at the beginning of this section, this can be rewritten in the form $\Lambda_C(Y)(f) = \inf_{x \in Q} f(x)$, where Q is the set of those elements $x \in K$ that satisfy $\Lambda_C(Y)(f) \leq f(x)$ for all $f \in C^*$. Note that Q is non-empty, as, taking f to be constantly 0, we have $\inf_{x \in Q} f(x) = \Lambda_C(Y)(f) = 0$ (recalling that Y is non-empty).

As Q is non-empty and $\Lambda_C(Y)(f) = \inf_{x \in Q} f(x)$, it only remains, therefore, to show that $X = Q$. Clearly, $X \subseteq Q$. For the reverse containment, suppose that $x \in Q \subseteq K$. We cannot have $x \notin Y$ as otherwise, by the Strict Separation Theorem [61, Theorem 3.8], there is an $r > 1$ such that $f(x) \leq 1$ and $f(y) > r$ for every $y \in Y$, whence $\Lambda_C(Y)(f) = \inf_{y \in Y} f(y) \geq r \not\leq f(x)$. So $x \in K \cap Y = X$ as required. □

We can now specialise to domains:

Corollary 4.7. *Let P be a domain. Then we have a Kegelspitze meet-semilattice isomorphism*

$$\Lambda_P\colon \mathcal{S}\mathcal{V}_{\leq 1}P \;\cong\; \mathcal{L}^{\mathrm{sne}}_{\sup}(\mathcal{L}P, \overline{\mathbb{R}}_+)$$

It is given by:

$$\Lambda_P(X)(f) =_{def} \inf_{\mu\in X}\int f\,d\mu$$

Proof. Using Lemma 4.3, we can check that both Φ_P and its inverse preserve strong nonexpansiveness, and so that Φ_P cuts down to an isomorphism

$$\mathcal{L}^{\mathrm{sne}}_{\sup}((\mathcal{V}_{\leq 1}P)^*, \overline{\mathbb{R}}_+) \cong \mathcal{L}^{\mathrm{sne}}_{\sup}(\mathcal{L}P, \overline{\mathbb{R}}_+)$$

of Kegelspitze meet-semilattices. Next, from the discussion of the valuation powerdomain in Section 2.5, we know that $\mathsf{d\text{-}Cone}(\mathcal{V}_{\leq 1}P) \cong \mathcal{V}P$ is convenient, and so we can apply Theorem 4.6. One then displays Λ_P as the following composition of Kegelspitze meet-semilattice isomorphisms:

$$\mathcal{S}\mathcal{V}_{\leq 1}P \xrightarrow{\Lambda_{(\mathcal{V}_{\leq 1}P)}} \mathcal{L}^{\mathrm{sne}}_{\sup}((\mathcal{V}_{\leq 1}P)^*, \overline{\mathbb{R}}_+) \xrightarrow{\Phi_P} \mathcal{L}^{\mathrm{sne}}_{\sup}(\mathcal{L}P, \overline{\mathbb{R}}_+)$$ □

Strong nonexpansiveness has a simple formulation for Scott-continuous homogeneous functionals $F\colon \mathcal{L}P \to \overline{\mathbb{R}}_+$, viz. that

$$F(f + \mathbf{1}_P) \leq F(f) + 1$$

holds for all $f \in \mathcal{L}P$.[3] Clearly a strongly nonexpansive functional satisfies this condition, since $\|\mathbf{1}\|_\infty = 1$. Suppose conversely that the second condition is satisfied and take any $f, g \in \mathcal{L}P$. For $g = 0$ there is nothing to prove. So let $g \neq 0$, and suppose that g is bounded, i.e., that $\|g\|_\infty < \infty$. Then, using homogeneity and then monotonicity and then the simplified condition, we have:

$$F(f+g) = \|g\|F(\frac{1}{\|g\|}f + \frac{1}{\|g\|}g) \leq \|g\|F(\frac{1}{\|g\|}f + \mathbf{1}_P) \leq \|g\|(F(\frac{1}{\|g\|}f) + 1) = F(f) + \|g\|$$

As every non-zero $g \in \mathcal{L}P$ is the directed sup of bounded non-zero such g's, using the continuity of F we then see that F is strongly nonexpansive.

4.3. **The convex power Kegelspitze.** Here our representations employ functionals with values not in $\overline{\mathbb{R}}_+$, but rather in $\mathcal{P}\overline{\mathbb{R}}_+$, the convex powercone of the extended nonnegative reals; this consists of the closed intervals $a = [\underline{a}, \overline{a}]$, with $\underline{a} \leq \overline{a}$ in $\overline{\mathbb{R}}_+$, ordered by the Egli-Milner order, where: $[\underline{a}, \overline{a}] \leq_{\mathrm{EM}} [\underline{b}, \overline{b}]$ if $\underline{a} \leq \underline{b}$ and $\overline{a} \leq \overline{b}$, with addition given by $[\underline{a}, \overline{a}] + [\underline{b}, \overline{b}] = [\underline{a} + \underline{b}, \overline{a} + \overline{b}]$, and scalar multiplication given by $r[\underline{a}, \overline{a}] = [r\underline{a}, r\overline{a}]$. The semilattice operation on $\mathcal{P}\overline{\mathbb{R}}_+$ is $[\underline{a}, \overline{a}] \cup [\underline{b}, \overline{b}] = [\min(\underline{a}, \underline{b}), \max(\overline{a}, \overline{b})]$; the semilattice order is containment $\subseteq$. We define a norm on $\mathcal{P}\overline{\mathbb{R}}_+$ by setting $\|a\| =_{def} \overline{a}$.

[3] Goubault-Larrecq calls such functionals *subnormalised previsions* in [9, 10, 11, 12].

We also use notation adapted from [28, Section 4] setting up, for any dcpo P, a bijection between functions $F\colon P \to \mathcal{P}\overline{\mathbb{R}}_+$ and pairs of functions $G, H\colon P \to \overline{\mathbb{R}}_+$ with $G \leq H$. In one direction, given such an F, we write $\underline{F}(x)$ and $\overline{F}(x)$ for the lower and upper ends of the image $F(x)$ of any $x \in P$, obtaining such a pair of functions $\underline{F}$ and $\overline{F}$; conversely, given such a pair of functions G and H, we set $[G, H](x)$ equal to the interval $[G(x), H(x)]$. The function F is Scott-continuous if and only if both $\underline{F}$ and $\overline{F}$ are. In case we are considering functionals $F\colon D \to \mathcal{P}\overline{\mathbb{R}}_+$ where D is a d-cone, then F is said to be $\subseteq$-*sublinear*, if F is homogeneous and $F(x+y) \subseteq F(x) + F(y)$ for all $x, y \in D$ (which is equivalent to $\underline{F}$ being superlinear and $\overline{F}$ sublinear), and it is said to be *medial*[4] if we have

$$\underline{F}(x+y) \leq \underline{F}(x) + \overline{F}(y) \leq \overline{F}(x+y)$$

for all $x, y \in D$.

Let us recall the (diagonal) functional representation of the convex powercone $\mathcal{P}C$ of a coherent continuous d-cone C from [28, Sections 4 and 6]. It combines the representations of the lower and upper powercones. Take a continuous coherent d-cone C with its its dual C^* and its convex powercone $\mathcal{P}C$. Consider the map $\Lambda_C\colon \mathcal{P}C \to \mathcal{P}\overline{\mathbb{R}}_+^{C^*}$ where:

$$\Lambda_C(X)(f) =_{def} [\inf_{x\in X} f(x), \sup_{x\in X} f(x)]$$

Fixing $f \in C^*$, we obtain the map $\Lambda_C(-)(f)\colon \mathcal{P}C \to \mathcal{P}\overline{\mathbb{R}}_+$, which is is the unique d-cone semilattice morphism $\mathcal{P}f\colon \mathcal{P}C \to \mathcal{P}\overline{\mathbb{R}}_+$ extending $\eta_{\overline{\mathbb{R}}_+} \circ f$ along the canonical embedding η_C, where, for any continuous coherent cone D, the canonical embedding $\eta_D\colon D \to \mathcal{S}D$ maps x to $\{x\}$ (this follows from [28, Proposition 3.8]).

Fixing $X \in \mathcal{P}C$ we obtain the Scott-continuous $\subseteq$-sublinear medial functional

$$\Lambda_C(X)\colon C^* \to \mathcal{P}\overline{\mathbb{R}}_+$$

where

$$\underline{\Lambda_C(X)}(f) = \inf_{x\in X} f(x), \quad \overline{\Lambda_C(X)}(f) = \sup_{x\in X} f(x)$$

The collection $\mathcal{L}_{\subseteq,\mathrm{med}}(D, \mathcal{P}\overline{\mathbb{R}}_+)$ of all Scott-continuous $\subseteq$-sublinear medial functionals $F : D^* \to \mathcal{P}\overline{\mathbb{R}}_+$ forms a d-cone semilattice, for any d-cone D; indeed, it is a sub-d-cone semilattice of $\mathcal{P}\overline{\mathbb{R}}_+^{D}$.

In this way we obtain a d-cone semilattice morphism

$$\Lambda_C\colon \mathcal{P}C \longrightarrow \mathcal{L}_{\subseteq,\mathrm{med}}(C^*, \mathcal{P}\overline{\mathbb{R}}_+)$$

that represents the convex powercone by the Scott-continuous $\subseteq$-sublinear medial functionals $F\colon C^* \to \mathcal{P}\overline{\mathbb{R}}_+$.

[4]This property is called 'canonicity' in [28, Section 4.3], and 'Walley's condition' in [11, Definition 7.1]. Indeed, Walley includes this property among those of his 'coherent previsions' in his book on reasoning with imprecise probabilities [66, Section 2.6].

Theorem 4.8 ([28, Proposition 6.4, Theorem 6.8])**.** *Suppose that C is a continuous coherent d-cone. Then we have a d-cone semilattice morphism $\Lambda_C\colon \mathcal{P}C \longrightarrow \mathcal{L}_{\subseteq,\mathrm{med}}(C^*, \mathcal{P}\overline{\mathbb{R}}_+)$, which is an order embedding, where:*

$$\Lambda_C(X)(f) = [\inf_{x\in X} f(x), \sup_{x\in X} f(x)]$$

Further, if C is convenient then Λ_C is an isomorphism. □

We now consider a continuous coherent full Kegelspitze K. The continuous universal d-cone $C = \mathsf{d\text{-}Cone}(K)$ is then also continuous and coherent (this last by Proposition 2.44). As in the proof of Theorem 3.11, the power Kegelspitze $\mathcal{P}K$ can be considered to be the collection of all convex lenses in C that are contained in K, which latter is itself a convex lens of C. In this way $\mathcal{P}K$ can be seen as a Scott-closed convex $\cup$-subsemilattice of $\mathcal{P}C$.

We then note that a convex lens X is in $\mathcal{P}K$ iff $\overline{\Lambda_C(X)}$ is dominated by the norm functional $\|\text{-}\|^*_K$ (for X is in $\mathcal{P}K$ iff $\downarrow X \subseteq K$ iff $\overline{\Lambda}_C(\downarrow X) \leq \|\text{-}\|^*_K$, by the discussion in Section 4.1, and we have $\Lambda_C(X) = \Lambda_C(\downarrow X)$). Now, for any normed d-cone D, the collection $\mathcal{L}^{\leq 1}_{\subseteq,\mathrm{med}}(D, \mathcal{P}\overline{\mathbb{R}}_+)$ of all Scott-continuous and $\subseteq$-sublinear functionals $F\colon D \to \mathcal{P}\overline{\mathbb{R}}_+$ with F nonexpansive and medial forms a Kegelspitze semilattice, indeed it is a sub-Kegelspitze semilattice of $\mathcal{L}_{\subseteq,\mathrm{med}}(D, \mathcal{P}\overline{\mathbb{R}}_+)$ (regarding the latter as a Kegelspitze semilattice).

We therefore have a Kegelspitze semilattice morphism

$$\Lambda_K\colon \mathcal{P}K \longrightarrow \mathcal{L}^{\leq 1}_{\subseteq,\mathrm{med}}(K^*, \mathcal{P}\overline{\mathbb{R}}_+)$$

viz. the composition:

$$\mathcal{P}K \xrightarrow{\Lambda_C\restriction K} \mathcal{L}^{\leq 1}_{\subseteq,\mathrm{med}}(C^*, \mathcal{P}\overline{\mathbb{R}}_+) \cong \mathcal{L}^{\leq 1}_{\subseteq,\mathrm{med}}(K^*, \mathcal{P}\overline{\mathbb{R}}_+)$$

From Theorem 4.8 we then immediately obtain the following functional representation theorem:

Theorem 4.9. *Let K be a continuous coherent full Kegelspitze. Then we have a Kegelspitze semilattice morphism $\Lambda_K\colon \mathcal{P}K \longrightarrow \mathcal{L}^{\leq 1}_{\subseteq,\mathrm{med}}(K^*, \mathcal{P}\overline{\mathbb{R}}_+)$ which is an order embedding. It is given by:*

$$\Lambda_K(X)(f) =_{def} [\inf_{x\in X} f(x)\,, \sup_{x\in X} f(x)\,]$$

Further, if $\mathsf{d\text{-}Cone}(K)$ is convenient then Λ_K is an isomorphism. □

We specialise this result to domains. For any domain P

$$(\Phi_c)_P\colon \mathcal{P}\overline{\mathbb{R}}_+^{(\mathcal{V}_{\leq 1}P)^*} \to \mathcal{P}\overline{\mathbb{R}}_+^{\mathcal{L}P}$$

where $(\Phi_c)_P(F) = F \mapsto F \circ \mathrm{EXT}_P$, is a d-cone semilattice isomorphism with inverse $F \mapsto F \circ \mathrm{RES}_P$. We then obtain:

Corollary 4.10. *Let P be a coherent domain. Then we have a Kegelspitze semilattice isomorphism*

$$\Lambda_P\colon \mathcal{P}\mathcal{V}_{\leq 1}P \;\cong\; \mathcal{L}^{\leq 1}_{\subseteq,\mathrm{med}}(\mathcal{L}P, \mathcal{P}\overline{\mathbb{R}}_+)$$

It is given by:

$$\Lambda_P(X)(f) =_{def} [\inf_{\mu \in X} \int f \, d\mu \, , \sup_{\mu \in X} \int f \, d\mu \,]$$

Proof. Checking that $(\Phi_c)_P$ and its inverse preserve $\subseteq$-sublinearity and mediality, we see that Φ_{cP} cuts down to an isomorphism

$$\mathcal{L}^{\leq 1}_{\subseteq,\mathrm{med}}((\mathcal{V}_{\leq 1}P)^*, \mathcal{P}\overline{\mathbb{R}}_+) \cong \mathcal{L}^{\leq 1}_{\subseteq,\mathrm{med}}(\mathcal{L}P, \mathcal{P}\overline{\mathbb{R}}_+)$$

of Kegelspitze semilattices. As P is a coherent domain, then, following Section 2.5, we see that $\mathcal{V}_{\leq 1}P$ is continuous and coherent, and that $\mathsf{d\text{-}Cone}(\mathcal{V}_{\leq 1}P)$ is convenient. So we may apply Theorem 4.9. One then displays Λ_P as the following composition of Kegelspitze semilattice isomorphisms:

$$\mathcal{P}\mathcal{V}_{\leq 1}P \xrightarrow{\Lambda_{(\mathcal{V}_{\leq 1}P)}} \mathcal{L}^{\leq 1}_{\subseteq,\mathrm{med}}((\mathcal{V}_{\leq 1}P)^*, \mathcal{P}\overline{\mathbb{R}}_+) \xrightarrow{(\Phi_c)_P} \mathcal{L}^{\leq 1}_{\subseteq,\mathrm{med}}(\mathcal{L}P, \mathcal{P}\overline{\mathbb{R}}_+) \qquad \square$$

5. Predicate transformers

We are ready now to achieve a goal that we can summarise under the slogan *'the equivalence of state transformer and predicate transformer semantics'*. Let us begin by describing the general framework for the lower and upper cases. In Section 3, in both these cases, we modelled mixed probabilistic and nondeterministic phenomena by a monad S over a full subcategory C of the category of dcpos and Scott-continuous maps. The Kleisli category of S has as morphisms the Scott-continuous maps

$$s \colon P \to S(Q)$$

We name these *state transformers*.

In Section 4, we considered functional representations of S, which led to a monad T with isomorphisms $S(P) \cong T(P)$. This monad is a submonad of the continuation monad $\overline{\mathbb{R}}_+^{\overline{\mathbb{R}}_+^{\,P}}$, and so its Kleisli category is faithfully embedded in the Kleisli category of the continuation monad whose morphisms are the Scott-continuous maps

$$t \colon P \to \overline{\mathbb{R}}_+^{\overline{\mathbb{R}}_+^{\,Q}}$$

These morphisms provide our general notion of state transformer. The collection $(\overline{\mathbb{R}}_+^{\overline{\mathbb{R}}_+^{\,Q}})^P$ of such state transformers can be regarded as either a d-cone join-semilattice or a d-cone meet-semilattice with respect to the pointwise structure obtained from $\overline{\mathbb{R}}_+$, depending on whether $\overline{\mathbb{R}}_+$ is viewed as a d-cone join-semilattice or a d-cone meet-semilattice.

In this setting, it makes sense to think of $\overline{\mathbb{R}}_+$ as a space of *truthvalues* and then to call Scott-continuous maps $f \colon P \to \overline{\mathbb{R}}_+$ on a dcpo P *predicates*, so that the function space $\overline{\mathbb{R}}_+^P = \mathcal{L}P$ becomes the dcpo of predicates on P. A *predicate transformer* is then a Scott-continuous map

$$p \colon \mathcal{L}Q \to \mathcal{L}P$$

and, as before, the collection $(\mathcal{L}P)^{\mathcal{L}Q}$ of such predicate transformers can be regarded as either a d-cone join-semilattice or a d-cone meet-semilattice depending on how $\overline{\mathbb{R}}_+$ is viewed.

There is an evident natural bijection $\mathrm{PT}\colon (\overline{\mathbb{R}}_+^{\overline{\mathbb{R}}_+^Q})^P \cong (\mathcal{L}P)^{\mathcal{L}Q}$, where:

$$\mathrm{PT}(t)(g)(x) =_{def} t(x)(g) \quad (g \in \mathcal{L}Q, x \in P)$$

This bijection is both a d-cone join-semilattice isomorphism and a d-cone meet-semilattice isomorphism, depending on which of the above semilattice structures are taken on the state and predicate transformers. It is then our aim to characterise the 'healthy' predicate transformers, that is, those p that correspond to the state transformers $t\colon P \to T(Q) \subseteq \overline{\mathbb{R}}_+^{\overline{\mathbb{R}}_+^Q}$ arising from the two monads for mixed nondeterminism.

For the convex case there is a natural modification of this general framework where the role of $\overline{\mathbb{R}}_+$ is taken by over by $\mathcal{P}\overline{\mathbb{R}}_+$, the convex powercone over $\overline{\mathbb{R}}_+$. As signalled in Section 4, the uniformity at hand is that, up to isomorphism, we are making use of the three powercones $\mathcal{H}\overline{\mathbb{R}}_+$, $\mathcal{S}\overline{\mathbb{R}}_+$, and $\mathcal{P}\overline{\mathbb{R}}_+$. All three are based on $\overline{\mathbb{R}}_+$, which is $\mathcal{V}\mathbf{1}$, the free valuation powerdomain on the one-point dcpo.

In all cases considered here the 'healthy' predicate transformers do not preserve the natural algebraic operations on the function spaces, that is, they are not homomorphisms. In particular, in the lower and upper cases they are respectively sublinear and superlinear. This phenomenon is explained from a general point of view in [24, 25].

As indicated above, we restrict ourselves to predicate transformers for the power Kegelspitzen over domains. There are, nevertheless, related results for Kegelspitzen more generally. For example, in the lower and upper cases, one takes predicates on a Kegelspitze K to be elements of K^*, the sub-Kegelspitze of $\overline{\mathbb{R}}_+^K$ of all Scott-continuous linear functionals. Predicate transformers are suitable Scott-continuous maps

$$p\colon L^* \to K^*$$

and state transformers are linear Scott-continuous maps

$$t\colon K \to \overline{\mathbb{R}}_+^{L^*}$$

In all three cases one obtains Kegelspitze isomorphisms between Kegelspitzen of state transformers and Kegelspitzen of suitably healthy predicate transformers. This differs from the domain case, where one rather obtains Kegelspitze *semilattice* isomorphisms; the difference arises as there seems to be no general reason why, for example, a dual Kegelspitze K^* should be a semilattice, whereas, if $K = \mathcal{V}P$ then K^* is $\mathcal{L}P$ which is a semilattice.

5.1. **The lower case.** Consider two dcpos P and Q. For any state transformer $t\colon P \to \overline{\mathbb{R}}_+^{\mathcal{L}Q}$, the corresponding predicate transformer $\mathrm{PT}(t)\colon \mathcal{L}Q \to \mathcal{L}P$ is given by $\mathrm{PT}(t)(g)(x) = t(x)(g)$ for $g \in \mathcal{L}Q$ and $x \in P$. One can check directly that $\|\mathrm{PT}(t)(g)\|_\infty \leq \|g\|_\infty$ for every $g \in \mathcal{L}Q$ if and only if $t(x) \leq \|\text{-}\|_\infty$ for every $x \in P$, and that $\mathrm{PT}(t)$ is sublinear if and only if $t(x)$ is sublinear for every $x \in P$. Thus the state transformers $t\colon P \to \mathcal{L}^{\leq 1}_{\mathrm{sub}}(\mathcal{L}Q, \overline{\mathbb{R}}_+)$ correspond bijectively via PT to the nonexpansive sublinear predicate transformers $p\colon \mathcal{L}Q \to \mathcal{L}P$. So PT cuts down to a Kegelspitze join-semilattice isomorphism

$$\mathcal{L}^{\leq 1}_{\text{sub}}(\mathcal{L}Q, \overline{\mathbb{R}}_+)^P \cong \mathcal{L}^{\leq 1}_{\text{sub}}(\mathcal{L}Q, \mathcal{L}P)$$

taking the pointwise Kegelspitze join-semilattice structure on $\mathcal{L}^{\leq 1}_{\text{sub}}(\mathcal{L}Q, \overline{\mathbb{R}}_+)^P$. Finally, to make the link between state transformers and the healthy predicate transformers, we set $\text{PT}_{P,Q}(s) =_{def} \text{PT}(\Lambda_Q \circ s)$, $(s\colon P \to \mathcal{HV}_{\leq 1}Q)$, where Λ_Q is as in Section 4.1, and calculate that

$$\text{PT}_{P,Q}(s)(g)(x) = \text{PT}(\Lambda_Q \circ s)(g)(x) = \Lambda_Q(s(x))(g) = \sup_{\mu \in s(x)} \int g \, d\mu$$

Combining the above discussion with Corollary 4.4 we then obtain:

Corollary 5.1. *Let P and Q be dcpos. To every state transformer $s\colon P \to \mathcal{HV}_{\leq 1}Q$ we can assign a predicate transformer $\text{PT}_{P,Q}(s)\colon \mathcal{L}Q \to \mathcal{L}P$ by:*

$$\text{PT}_{P,Q}(s)(g)(x) =_{def} \sup_{\mu \in s(x)} \int g \, d\mu \quad (g \in \mathcal{L}Q, x \in P)$$

The predicate transformer $\text{PT}_{P,Q}(s)$ is sublinear and nonexpansive. The assignment $\text{PT}_{P,Q}$ is a Kegelspitze join-semilattice morphism

$$(\mathcal{HV}_{\leq 1}Q)^P \longrightarrow \mathcal{L}^{\leq 1}_{\text{sub}}(\mathcal{L}Q, \mathcal{L}P)$$

If Q is a domain then it is an isomorphism. □

Similar to the simplification of nonexpansiveness for functionals discussed in Section 4.1 (and an immediate consequence of it), the condition of nonexpansiveness has a simple formulation for homogeneous predicate transformers $p\colon \mathcal{L}Q \to \mathcal{L}P$, viz. $p(\mathbf{1}_Q) \leq \mathbf{1}_P$.

5.2. **The upper case.** Consider two dcpos, P and Q. In agreement with the terminology introduced in Section 4.2 we will say that a predicate transformer $p\colon \mathcal{L}Q \to \mathcal{L}P$ is *strongly nonexpansive* if we have

$$p(f+g) \leq p(f) + \|g\|_\infty \cdot \mathbf{1}_P$$

for all $f, g \in \mathcal{L}Q$, where $\mathbf{1}_P$ is the constant function on P with value 1. Strongly nonexpansive predicate transformers are nonexpansive. Indeed, for $f = 0$ the inequality yields $p(g)(x) \leq \|g\|_\infty$ for all $x \in P$, whence $\|p(g)\|_\infty \leq \|g\|_\infty$. In the case of homogeneous predicate transformers this can be simplified to the equivalent condition :

$$p(f + \mathbf{1}_Q) \leq p(f) + \mathbf{1}_P \quad (\text{for all } f \in \mathcal{L}Q)$$

as follows immediately from the corresponding simplification for functionals in Section 4.2.

For any state transformer $t\colon P \to \overline{\mathbb{R}}_+^{\mathcal{L}Q}$, we have $\text{PT}(t)(g)(x) = t(x)(g)$, $(g \in \mathcal{L}Q, x \in P)$. This firstly implies that $t(x)$ is superlinear for every x if, and only if, $\text{PT}(t)$ is superlinear. It secondly implies that $t(x)$ is strongly nonexpansive for every x if, and only if, $\text{PT}(t)$ is strongly nonexpansive. For we have $t(x)(f+g) \leq t(x)(f) + \|g\|_\infty$ for every $x \in P$ if, and only if, $\mathrm{PT}(t)(f+g)(x) \leq \mathrm{PT}(t)(f)(x) + \|g\|_\infty$ for every $x \in P$, that is, if, and only if, $\mathrm{PT}(t)(f+g) \leq \mathrm{PT}(t)(f) + \|g\|_\infty \cdot \mathbf{1}_P$.

We write $\mathcal{L}^{\mathrm{sne}}_{\mathrm{sup}}(\mathcal{L}Q, \mathcal{L}P)$ for the set of strongly nonexpansive superlinear predicate transformers. and note that it forms a sub-Kegelspitze meet-semilattice of $(\mathcal{L}P)^{\mathcal{L}Q}$ (taking the pointwise Kegelspitze meet-semilattice structure on $(\mathcal{L}P)^{\mathcal{L}Q}$). So PT cuts down to a Kegelspitze meet-semilattice isomorphism

$$\mathcal{L}^{\mathrm{sne}}_{\mathrm{sup}}(\mathcal{L}Q, \overline{\mathbb{R}}_+)^P \cong \mathcal{L}^{\mathrm{sne}}_{\mathrm{sup}}(\mathcal{L}Q, \mathcal{L}P)$$

Finally, to make the link between state transformers and the healthy predicate transformers, we set $\mathrm{PT}_{P,Q}(s) =_{def} \mathrm{PT}(\Lambda_Q \circ s)$ $(s\colon P \to \mathcal{SV}_{\leq 1}Q)$, where Λ_Q is as in Section 4.2 (and assuming now that Q is a domain), and calculate that

$$PT_{P,Q}(s)(g)(x) = \inf_{\mu \in s(x)} \int g \, d\mu$$

Combining the above discussion with Corollary 4.7 we then obtain:.

Corollary 5.2. *Let P be a dcpo and let Q be a domain. To every state transformer $s\colon P \to \mathcal{SV}_{\leq 1}Q$ we can assign a predicate transformer $\mathrm{PT}_{P,Q}(s)\colon \mathcal{L}Q \to \mathcal{L}P$ by:*

$$\mathrm{PT}_{P,Q}(s)(g)(x) =_{def} \inf_{\mu \in s(x)} \int g \, d\mu \quad (g \in \mathcal{L}Q, x \in P)$$

The predicate transformer $\mathrm{PT}_{P,Q}(s)$ is superlinear and strongly nonexpansive. The assignment $\mathrm{PT}_{P,Q}$ is a Kegelspitze meet-semilattice isomorphism

$$(\mathcal{SV}_{\leq 1}Q)^P \cong \mathcal{L}^{\mathrm{sne}}_{\mathrm{sup}}(\mathcal{L}Q, \mathcal{L}P) \qquad \square$$

5.3. **The convex case.** For this case we have to modify our framework. First we need a function space construction. For d-cone semilattices C and D, the collection $\mathcal{L}_{\mathrm{mon}}(C, D)$ of Scott-continuous $\subseteq$-monotone maps from C to D is a sub-d-cone semilattice of the d-cone semilattice of all Scott-continuous maps from C to D equipped with the pointwise d-cone semilattice structure.

In the modified framework, the rôle of $\overline{\mathbb{R}}_+$, considered as join- and meet-semilattices in the lower and upper cases, is taken over by $\mathcal{P}\overline{\mathbb{R}}_+$, the convex powercone over $\overline{\mathbb{R}}_+$. Predicates on a dcpo P are no longer functionals with values in $\overline{\mathbb{R}}_+$ but are now rather Scott-continuous functionals of the form $f\colon P \to \mathcal{P}\overline{\mathbb{R}}_+$; they form a d-cone semilattice with the pointwise structure. We define a norm on predicates $f : P \to \mathcal{P}\overline{\mathbb{R}}_+$ by: $\|f\| =_{def} \|\overline{f}\|_\infty (= \bigvee_{x \in P} \overline{f}(x))$. Employing the notation of Section 4.3 we have a bijection $f \to (\underline{f}, \overline{f})$ between predicates and pairs of linear functionals $g, h \in \mathcal{L}P$ with $g \leq h$. Note that $(\underline{f}, \overline{f}) \leq (\underline{f'}, \overline{f'})$ if, and only if, $\underline{f} \leq \underline{f'}$ and $\overline{f} \leq \overline{f'}$, that $(\underline{f}, \overline{f}) \cup (\underline{f'}, \overline{f'}) = (\underline{f} \wedge \underline{f'}, \overline{f} \vee \overline{f'})$, and that $(\underline{f}, \overline{f}) \subseteq (\underline{f'}, \overline{f'})$ if, and only if, $\underline{f} \geq \underline{f'}$ and $\overline{f} \leq \overline{f'}$.

We take general state transformers to be maps:

$$t\colon P \to \mathcal{P}\overline{\mathbb{R}}_+^{\overline{\mathbb{R}}_+^Q}$$

One might rather have expected, $t\colon P \to \mathcal{P}\overline{\mathbb{R}}_+^{\mathcal{P}\overline{\mathbb{R}}_+{}^Q}$, uniformly replacing $\overline{\mathbb{R}}_+$ with $\mathcal{P}\overline{\mathbb{R}}_+$; we chose our definition to be closer to the functional representation.

Predicate transformers are taken to be Scott-continuous maps

$$p\colon (\mathcal{P}\overline{\mathbb{R}}_+)^Q \to (\mathcal{P}\overline{\mathbb{R}}_+)^P$$

which are, in addition, required to preserve the partial order $\subseteq$, i.e., to be $\subseteq$-monotone. This requirement is a technical condition to achieve an isomorphism between general state transformers and predicate transformers (see below). The predicate transformers form a d-cone semilattice $\mathcal{L}_{\mathrm{mon}}((\mathcal{P}\overline{\mathbb{R}}_+)^Q, (\mathcal{P}\overline{\mathbb{R}}_+)^P)$. Note that a predicate transformer p is nonexpansive if, and only if, $\|\overline{p}(f)\|_\infty \leq \|\overline{f}\|_\infty$, for any predicate f.

We link these predicate transformers to general state transformers via 'predicate transformers of diagonal form' which we take to be Scott-continuous functions:

$$q\colon \mathcal{L}Q \longrightarrow (\mathcal{P}\overline{\mathbb{R}}_+)^P$$

State transformers $t\colon P \to (\mathcal{P}\overline{\mathbb{R}}_+)^{\mathcal{L}Q}$ are connected to predicate transformers of diagonal form by the map

$$\mathrm{T}\colon (\mathcal{P}\overline{\mathbb{R}}_+^{\mathcal{L}Q})^P \longrightarrow ((\mathcal{P}\overline{\mathbb{R}}_+)^P)^{\mathcal{L}Q}$$

where $\mathrm{T}(t)(g)(x) =_{def} t(x)(g)$. This map is evidently an isomorphism of d-cone semilattices, with respect to the pointwise structures.

To connect predicate transformers of diagonal form to predicate transformers we first extend some definitions from predicates to functions $F\colon D \to (\mathcal{P}\overline{\mathbb{R}}_+)^P$, with D a d-cone. Let F be such a function. We define $\underline{F}, \overline{F}\colon D \to \mathcal{L}P$ by setting $\underline{F}(x) = \underline{F(x)}$ and $\overline{F}(x) = \overline{F(x)}$, for $x \in D$. Then define a map P between the two kinds of predicate transformers:

$$\mathrm{P}\colon ((\mathcal{P}\overline{\mathbb{R}}_+)^P)^{\mathcal{L}Q} \longrightarrow \mathcal{L}_{\mathrm{mon}}((\mathcal{P}\overline{\mathbb{R}}_+)^P, (\mathcal{P}\overline{\mathbb{R}}_+)^Q)$$

by $P(q)(f) =_{def} [\underline{q(\underline{f})}, \overline{q(\overline{f})}])$.

Lemma 5.3. P *is an isomorphism of d-cone semilattices.*

Proof. It is routine to verify that P is a morphism of d-cone semilattices. To see that P is an order embedding suppose that $\mathrm{P}(q) \leq \mathrm{P}(q')$ and choose $g \in L^*$ to show that $q(g) \leq q'(g)$. Then we have $[\underline{q(\underline{f})}, \overline{q(\overline{f})}]) \leq [\underline{q'(\underline{f})}, \overline{q'(\overline{f})}])$ where $f = [g, g]$. So $\underline{q(g)} \leq \underline{q'(g)}$ and $\overline{q(g)} \leq \overline{q'(g)}$, and so $q(g) \leq q'(g)$, as required.

To see that P is onto, choose a predicate transformer p to find a q with $p = \mathrm{P}(q)$. We claim that $\underline{p([\underline{f}, \overline{f}])} = \underline{p([\underline{f}, \underline{f}])}$ and $\overline{p([\underline{f}, \overline{f}])} = \overline{p([\overline{f}, \overline{f}])}$. For the first of these claims, as $[\underline{f}, \underline{f}] \leq [\underline{f}, \overline{f}]$ we have $\underline{p([\underline{f}, \underline{f}])} \leq \underline{p([\underline{f}, \overline{f}])}$, since p preserves the order $\leq$, and as $[\underline{f}, \underline{f}] \subseteq [\underline{f}, \overline{f}]$ we have $\underline{p([\underline{f}, \underline{f}])} \geq \underline{p([\underline{f}, \overline{f}])}$, since p preserves the order $\subseteq$. The proof of the second of these claims is similar: as $[\underline{f}, \overline{f}] \leq [\overline{f}, \overline{f}]$ we have $\overline{p([\underline{f}, \overline{f}])} \leq \overline{p([\overline{f}, \overline{f}])}$, since p preserves the order $\leq$, and as $[\underline{f}, \overline{f}] \supseteq [\overline{f}, \overline{f}]$ we have $\overline{p([\underline{f}, \overline{f}])} \geq \overline{p([\overline{f}, \overline{f}])}$, since p preserves the order $\subseteq$.

Defining $q = g \mapsto p([g, g])$, we then see that $p = \mathrm{P}(q)$, as required. $\square$

So we have a Kegelspitze semilattice isomorphism between general state transformers and predicate transformers:

$$\mathrm{P}\circ\mathrm{T}\colon (\mathcal{P}\overline{\mathbb{R}}_+^{\mathcal{L}Q})^P \cong \mathcal{L}_{\mathrm{mon}}((\mathcal{P}\overline{\mathbb{R}}_+)^P, (\mathcal{P}\overline{\mathbb{R}}_+)^Q)$$

and we seek the relevant healthiness conditions on the predicate transformers.

Define a function $F\colon D\to(\mathcal{P}\overline{\mathbb{R}}_+)^P$, with D a d-cone and P a dcpo, to be *medial* if:

$$\underline{F}(x+y)\leq\underline{F}(x)+\overline{F}(y)\leq\overline{F}(x+y)$$

for all $x,y\in D$, and define a function $F : D\to C$, where D is a d-cone and C is a d-cone semilattice, to be $\subseteq$*-sublinear* if it is homogeneous and $F(x+y)\subseteq F(x)+F(y)$, for all $x,y\in D$ (this generalises the definition of $\subseteq$-sublinearity in Section 4.3, and in the case where C is $(\mathcal{P}\overline{\mathbb{R}}_+)^P$, it is equivalent to $\overline{F}$ being sublinear and $\underline{F}$ being superlinear).

Now fix a state transformer s and set $q=\mathrm{T}(s)$ and $p=\mathrm{P}(q)$. We have $\overline{t(x)}$ sublinear for every $x\in P$ iff $\overline{q}$ is sublinear iff $\overline{p}$ is sublinear and, similarly, $\underline{t(x)}$ is superlinear for every $x\in P$ iff $\underline{p}$ is superlinear. So $t(x)$ is $\subseteq$-sublinear for every $x\in P$ iff q is $\subseteq$-sublinear iff p is $\subseteq$-sublinear. Next, $t(x)$ is medial for all $x\in P$ iff q is medial iff p is medial. Finally, $t(x)$ is nonexpansive for all $x\in P$ iff $\overline{t(x)}\leq\|\text{-}\|_\infty$ for all $x\in P$ iff $\|\overline{q}(g)\|_\infty\leq\|g\|_\infty$ for all $g\in\mathcal{L}Q$, iff $\|\overline{p}(f)\|_\infty\leq\|\overline{f}\|_\infty$ for all predicates f, that is, iff p is nonexpansive.

We write $\mathcal{L}^{\leq 1}_{\mathrm{mon},\subseteq,\mathrm{med}}((\mathcal{P}\overline{\mathbb{R}}_+)^Q,(\mathcal{P}\overline{\mathbb{R}}_+)^P)$ for the set of $\subseteq$-monotone, $\subseteq$-sublinear, medial, nonexpansive predicate transformers. As is straightforwardly checked, it forms a sub-Kegelspitze semilattice of $\mathcal{L}_{\mathrm{mon}}((\mathcal{P}\overline{\mathbb{R}}_+)^Q,(\mathcal{P}\overline{\mathbb{R}}_+)^P)$, and, from the above considerations, we see that $\mathrm{P}\circ\mathrm{T}$ cuts down to a Kegelspitze semilattice isomorphism:

$$\mathcal{L}^{\leq 1}_{\subseteq,\mathrm{med}}(\mathcal{L}Q,\mathcal{P}\overline{\mathbb{R}}_+)^P\cong\mathcal{L}^{\leq 1}_{\mathrm{mon},\subseteq,\mathrm{med}}((\mathcal{P}\overline{\mathbb{R}}_+)^Q,(\mathcal{P}\overline{\mathbb{R}}_+)^P)$$

Finally, to make the link between state transformers and predicate transformers we set $\mathrm{PT}_{P,Q}(s) =_{def} \mathrm{P}(\mathrm{T}(\Lambda_Q\circ s))$, $(s\colon P\to\mathcal{P}\mathcal{V}_{\leq 1}Q\ )$, where Λ_Q is as in Section 4.3 (and assuming now that Q is a coherent domain), and calculate that

$$\mathrm{PT}_{P,Q}(s)(g)(x)=[\inf_{\mu\in s(x)}\int\underline{g}\,d\mu,\ \sup_{\mu\in s(x)}\int\overline{g}\,d\mu\ ]$$

Combining the above discussion with Corollary 4.10, we then obtain:

Corollary 5.4. *Let P be a dcpo and let Q be a coherent domain. To every state transformer $s\colon P\to\mathcal{P}\mathcal{V}_{\leq 1}Q$ we can assign a predicate transformer $\mathrm{PT}_{P,Q}(s)\colon\mathcal{P}\overline{\mathbb{R}}_+^Q\to\mathcal{P}\overline{\mathbb{R}}_+^P$ by:*

$$\mathrm{PT}_{P,Q}(s)(g)(x)=_{def}[\inf_{\mu\in s(x)}\int\underline{g}\,d\mu,\ \sup_{\mu\in s(x)}\int\overline{g}\,d\mu\ ]\quad(g\in\mathcal{P}\overline{\mathbb{R}}_+^Q, x\in P)$$

The predicate transformer $\mathrm{PT}_{P,Q}(s)$ is nonexpansive, $\subseteq$-monotone, $\subseteq$-sublinear, and medial. The assignment $\mathrm{PT}_{P,Q}$ is a Kegelspitze semilattice isomorphism

$$(\mathcal{P}\mathcal{V}_{\leq 1}Q)^P\cong\mathcal{L}^{\leq 1}_{\mathrm{mon},\subseteq,\mathrm{med}}(\mathcal{P}\overline{\mathbb{R}}_+^Q,\mathcal{P}\overline{\mathbb{R}}_+^P)\qquad\square$$

6. The unit interval

In this section we consider replacing the extended positive reals $\overline{\mathbb{R}}_+$ by the unit interval $\mathbb{I}$. In the lower and upper cases, functional representations will involve maps to $\mathbb{I}$; $\mathbb{I}$ will play the rôle of truth values for predicates; and predicate transformers will be functions from $\mathcal{L}_{\leq 1}Q$ to $\mathcal{L}_{\leq 1}P$. In the convex case, functional representations will involve maps to $\mathcal{P}\mathbb{I}$; $\mathcal{P}\mathbb{I}$ will play the rôle of truth values; and predicate transformers will be functions from $\mathcal{P}\mathbb{I}^Q$ to $\mathcal{P}\mathbb{I}^P$. As we shall see, the results obtained are the same as those with $\overline{\mathbb{R}}_+$, except that nonexpansiveness requirements are dropped.

First, we slightly weaken the notion of a norm introduced in Section 4 deleting the requirement that nonzero elements have nonzero norm. A *seminorm* on a Kegelspitze K is defined to be a Scott-continuous sublinear map from K to $\overline{\mathbb{R}}_+$ and a *seminorm* on a cone C is a Scott-continuous sublinear map from C to $\overline{\mathbb{R}}_+$. A seminormed Kegelspitze is a Kegelspitze equipped with a seminorm $\|-\|$, and similarly for seminormed cones. A function $f : K \to L$ between seminormed Kegelspitzen is *nonexpansive* if, for all $a \in K$ we have:

$$\|f(a)\| \leq \|a\|$$

and similarly for seminormed cones. A *seminormed d-cone semilattice* is a d-cone semilattice equipped with a seminorm such that the operation $\cup$ is nonexpansive, by which we mean that $\|a \cup b\| \leq \max(\|a\|, \|b\|)$.

We next need some function space constructions. For any Kegelspitze K and Kegelspitze L (Kegelspitze semilattice L) we write $\mathcal{L}_{\mathrm{hom}}(K, L)$ for the Scott-continuous homogeneous functions from K to L. Equipped with the pointwise structure, $\mathcal{L}_{\mathrm{hom}}(K, L)$ forms a sub-Kegelspitze (respectively, sub-Kegelspitze semilattice) of L^K; further, for any d-cone C (d-cone semilattice C), $\mathcal{L}_{\mathrm{hom}}(K, C)$ (regarding C as a Kegelspitze) forms a sub-d-cone (respectively sub-d-cone semilattice) of C^K when equipped with the pointwise structure.

For seminormed d-cones C and D, we write $\mathcal{L}^{\leq 1}_{\mathrm{hom}}(C, D)$ for the collection of all Scott-continuous, homogeneous, nonexpansive functions from C to D. Equipped with the pointwise structure it forms a sub-Kegelspitze of $\mathcal{L}_{\mathrm{hom}}(C, D)$; further, if D is a seminormed d-cone semilattice, $\mathcal{L}^{\leq 1}_{\mathrm{hom}}(C, D)$ forms a sub-Kegelspitze semilattice of $\mathcal{L}_{\mathrm{hom}}(C, D)$.

We have a basic function space isomorphism as an immediate consequence of the universal embedding in a d-cone of a full Kegelspitze given by Theorem 2.35. Let $e : K \to C$ be a universal Kegelspitze embedding (in the sense of Section 2) of a Kegelspitze K in a d-cone C. Then Theorem 2.35 tells us that, for any d-cone D, function extension $f \mapsto \overline{f}$ yields a dcpo isomorphism

$$\mathcal{L}_{\mathrm{hom}}(K, D) \cong \mathcal{L}_{\mathrm{hom}}(C, D)$$

with inverse given by restriction $g \mapsto g \circ e$ along the universal arrow. Moreover, as restriction preserves the pointwise structure, the isomorphism is an isomorphism of d-cones; further, if D is additionally equipped with a semilattice structure, then the isomorphism is an isomorphism of d-cone semilattices.

To connect nonexpansiveness and Kegelspitzen we make use of particular seminorms. For any Scott-closed convex subset X of a cone C, we define the *(lower) Minkoswki functional* $\nu_X : C \to \overline{\mathbb{R}}_+$ by:

$$\nu_X(a) =_{def} \inf\{r \in \mathbb{R}_+ \mid a \in r \cdot X\}$$

Minkowski functionals were previously considered in [47] and in [23].

Proposition 6.1. *Let X be a Scott-closed convex subset of a d-cone C. Then:*

(1) *ν_X is Scott-continuous and sublinear.*
(2) *If $0 < \nu_X(a) < \infty$ then, for some $x \in X$, we have $a = \nu_X(a) \cdot x$.*
(3) *$X = \{a \in C \mid \nu_X(a) \leq 1\}$.*
(4) *If C is a d-cone semilattice and X also a subsemilattice, then we have:*

$$\nu_X(a \cup b) \leq \max(\nu_X(a), \nu_X(b))$$

Proof.

(1) (a) For monotonicity, suppose $a \leq b \in C$. Then if $b \in r \cdot X$, we have $a \in r \cdot X$, since $r \cdot X$ is a lower set and so $\nu_X(a) \leq \nu_X(b)$.
  (b) Having established monotonicity, for continuity it remains to show that $\nu_X(\bigvee_i a_i) \leq \bigvee_i \nu_X(a_i)$, for any directed set a_i $(i \in I)$ of elements of C. Suppose that $\bigvee_i \nu_X(a_i) < r$ for some $r \in \mathbb{R}_+$. Then, for every i, $\nu_X(a_i) < r$ and so $a_i \in r \cdot X$. Since $r \cdot X$ is Scott-closed, we also have $\bigvee_i a_i \in r \cdot X$ and consequently $\nu_X(\bigvee_i a_i) \leq r$. As r is an arbitrary element of $\mathbb{R}_+$ with $\bigvee_i \nu_X(a_i) < r$ this shows that $\nu_X(\bigvee_i a_i) \leq \bigvee_i \nu_X(a_i)$, as required.
  (c) For homogeneity, choose $a \in C$ and $r \in]0, 1[$ to show that $\nu_X(r \cdot a) = r \cdot \nu_X(a)$. This follows from the observation that, for any positive $s \in \mathbb{R}_+$, $a \in s \cdot X$ iff $r \cdot a \in rs \cdot X$.
  (d) For subadditivity, choose a, b in C and $r > \nu_X(a)$, $s > \nu_X(b)$. Then $a \in r \cdot X$ and $b \in s \cdot X$ whence $a + b \in r \cdot X + s \cdot X = (r + s) \cdot X$, since X is convex. So we have $r + s > \|a + b\|$ and, since this holds for all $r > \nu_X(a)$ and $s > \nu_X(b)$, we conclude that $\nu_X(a + b) \leq \nu_X(a) + \nu_X(b)$.

(2) As $0 < \nu_X(a) < \infty$ there are sequences $r_n \in \mathbb{R}_+$ and $x_n \in X$, with r_n decreasing and positive, such that $a = r_n \cdot x_n$ and $\nu_X(a) = \inf r_n$. So $x_n = r_n^{-1} \cdot a$ is an increasing sequence, and taking sups we see that $\sup x_n = (\sup r_n^{-1}) \cdot a = \nu_X(a)^{-1} \cdot a$. We have $x =_{def} \sup x_n \in X$, as X is Scott-closed, and so $\nu_X(a) \cdot x = a$.

(3) Evidently $X \subseteq \{a \in C \mid \nu_X(a) \leq 1\}$. Conversely, suppose that we have $a \in C$ with $\nu_X(a) \leq 1$. If $\nu_X(a) < 1$ then clearly $a \in 1 \cdot X = X$. Otherwise we have $\nu_X(a) = 1$. In this case, by the second part we have $a = \nu_X(a) \cdot x$ for some $x \in X$. Then, as $\nu_X(a) = 1$, we see that $a \in X$, as required.

(4) As for subadditivity, choose any real number $r > \max(\nu_X(a), \nu_X(b))$. Then a and b are both in $r \cdot X$. Since X is supposed to be a subsemilattice and $x \mapsto r \cdot x$ is a semilattice homomorphism, $r \cdot X$ is subsemilattice, too, so that $a \cup b \in r \cdot X$, that is $\nu_X(a \cup b) \leq r$. Since this holds for all $r > \max(\nu_X(a), \nu_X(b))$, we conclude that $\nu_X(a \cup b) \leq \max(\nu_X(a), \nu_X(b))$. ∎

So all Minkowski functionals are seminorms. For every full Kegelspitze K and every universal Kegelspitze embedding $K \overset{e}{\to} C$, we write $\|\text{-}\|_K$ for the seminorm $\nu_{e(K)}\colon C \to \overline{\mathbb{R}}_+$, and on K we use the same notation for the seminorm $\|a\|_K =_{def} \|e(a)\|_K$. When we do not mention below which seminorm we use we mean the relevant one of these. We have the following pleasant facts:

Fact 6.2. For full Kegelspitzen K and L, let $K \overset{e}{\to} C$ and $L \overset{e'}{\to} D$ be universal Kegelspitze embeddings. Suppose that $f\colon K \to L$ and and $g\colon C \to D$ are homogeneous maps such that the following diagram commutes:

$$\begin{array}{ccc} C & \xrightarrow{\;g\;} & D \\ {\scriptstyle e}\uparrow & & \uparrow{\scriptstyle e'} \\ K & \xrightarrow[\;f\;]{} & L \end{array}$$

Then both f and g are nonexpansive.

Proof. It suffices to show that g is nonexpansive. So, for $a \in C$ we have to show that $\|g(a)\|_L \leq \|a\|_K$. This is certainly true if $\|a\|_K = +\infty$. Otherwise take any real number $r > \|a\|_K$. Then $a \in r \cdot e(K)$. We deduce that $g(a) \in g(r \cdot e(K)) = r \cdot g(e(K)) = r \cdot e'(f(K)) \subseteq r \cdot e'(L)$ which implies that $\|g(a)\|_L \leq r$. Since this holds for all $r > \|a\|_K$, we have the desired inequality. □

The next proposition is at the root of our results for the unit interval. It enables nonexpansiveness requirements to be dropped when using the unit interval. First define $K \overset{e}{\to} C$ to be a *universal Kegelspitze semilattice embedding* if K is a full Kegelspitze semilattice, D is a d-cone semilattice, e preserves the semilattice operation, and $K \overset{e}{\to} C$ is a universal Kegelspitze embedding. Note that then the norm $\|\text{-}\|_K$ on C satisfies property (4) of Proposition 6.1.

Proposition 6.3. *For full Kegelspitzen K and L, let $K \overset{e}{\to} C$ and $L \overset{e'}{\to} D$ be universal Kegelspitze embeddings. Then there is a Kegelspitze isomorphism:*

$$\mathcal{L}_{\mathrm{hom}}(K, L) \cong \mathcal{L}^{\leq 1}_{\mathrm{hom}}(C, D)$$

The isomorphism sends $f \in \mathcal{L}_{\mathrm{hom}}(K, L)$ to $\overline{e' \circ f}$; its inverse sends $g \in \mathcal{L}^{\leq 1}_{\mathrm{hom}}(C, D)$ to the restriction of $g \circ e$ along e'; and f and g are related by the isomorphism if, and only if, $g \circ e = e' \circ f$.

In case $L \overset{e'}{\to} D$ is additionally a universal Kegelspitze semilattice embedding, the isomorphism is a Kegelspitze semilattice isomorphism.

Proof. Composing with the Kegelspitze embedding e' and then function extension, viewed as a Kegelspitze isomorphism, we obtain a Kegelspitze embedding:

$$\mathcal{L}_{\mathrm{hom}}(K, L) \xrightarrow{e' \circ -} \mathcal{L}_{\mathrm{hom}}(K, D) \overset{\bar{-}}{\to} \mathcal{L}_{\mathrm{hom}}(C, D)$$

We see from Fact 6.2 that every function in the range of the embedding is nonexpansive. Conversely let $g : C \to D$ be a nonexpansive Scott-continuous homogeneous function. Then, in particular, for every $a \in K$ we have $\|g(e(a))\| \le \|e(a)\| \le 1$ and so there is a (necessarily unique) $b \in L$ such that $g(e(a)) = e'(b)$. So we have a function $f : K \to L$ such that $e'(f(a)) = g(e(a))$ for all $a \in K$; this function is Scott-continuous and homogeneous as g is and e and e' are Kegelspitze embeddings. As g extends $f \circ e'$ along e, the Kegelspitze embedding $f \mapsto \overline{f \circ e'}$ of $\mathcal{L}_{\mathrm{hom}}(K, L)$ in $\mathcal{L}_{\mathrm{hom}}(C, D)$ sends f to g, and so cuts down to a bijection, and so a Kegelspitze isomorphism, between $\mathcal{L}_{\mathrm{hom}}(K, L)$ and $\mathcal{L}^{\le 1}_{\mathrm{hom}}(C, D)$, with inverse as claimed.

That f and g are related by the isomorphism if, and only if, $g \circ e = e' \circ f$ is clear.

With the extra semilattice assumptions, D is a d-cone semilattice and so $\mathcal{L}^{\le 1}_{\mathrm{hom}}(C, D)$ is a Kegelspitze semilattice; further, the isomorphism preserves the semilattice structure as e' does. □

We will typically apply this result by first restricting to a subclass (e.g., to sublinear functions in the lower case) and then specialising to dcpos or domains.

We could also obtain general results for Kegelspitzen K by adding to the assumptions considered above the assumption that the evident embedding of K^* in $\mathsf{d\text{-}Cone}(K)^*$ is universal, where now by K^* we mean the Kegelspitze of Scott-continuous linear functions from K to $\mathbb{I}$. By Proposition 2.38, an equivalent assumption, assuming $\mathsf{d\text{-}Cone}(K)^*$ continuous, is that every Scott continuous linear function from K to $\mathbb{R}_+$ is a directed sup of bounded such functions.

One obtains general functional representation and predicate transformer results, except for predicate transformer results in the convex case. (The obstacle in that case is that the equational proof below that mediality transfers in the domain case is not available at a general level, since there seems to be no general reason why dual Kegelspitzen or dual d-cones should be semilattices.)

As remarked in the introduction, one might prefer a development not involving d-cones at all. Another improvement, perhaps easier to achieve, would be a development where the assumptions on Kegelspitzen involved only $\mathbb{I}$.

There is a pleasant induction principle for d-cones given by Kegelspitze universal embeddings. Say that a property of a d-cone is *upper homogeneous* if it is closed under all actions $r\cdot-$ with $r \ge 1$. Then, for any full Kegelspitze K and any universal Kegelspitze embedding $K \xrightarrow{e} C$, if a property of C is closed under directed sups, and is upper homogeneous, then it holds for all of C if it holds for all of $e(K)$. This can be proved by reference to the construction of universal embeddings in Section 2.

There is an n-ary version of this induction principle. Given $n > 0$ universal embeddings $K_i \xrightarrow{e_i} C_i$ of full Kegelspitzen K_i $(i = 1, \dots, n)$, if a relation on $C_1, \dots, C_n$ is closed under directed sups and is upper homogeneous (in an evident sense), then the relation holds for all of $C_1 \times \dots \times C_n$ if it holds for all of $e_1(K_1) \times \dots \times e_n(K_n)$. This follows from the unary principle since, as shown in Section 2.3, universal Kegelspitze embeddings are closed under finite non-empty products.

6.1. **The lower case.** For any full Kegelspitzen K and L we take $\mathcal{L}_{\mathrm{sub}}(K, L)$ to be the collection of Scott-continuous sublinear functions from K to L equipped with the pointwise structure and so forming a sub-Kegelspitze of L^K, and a sub-Kegelspitze join-semilattice if L is a Kegelspitze join-semilattice.

Let $e\colon K \to C$ and $e'\colon L \to D$ be universal Kegelspitze embeddings, where, additionally, L is a Kegelspitze join-semilattice and D is a d-cone join-semilattice (when e is automatically a Kegelspitze semilattice morphism and so $e'\colon L \to D$ is a universal Kegelspitze semilattice embedding). Then the Kegelspitze semilattice isomorphism of Proposition 6.3 restricts to a Kegelspitze semilattice isomorphism

$$\mathcal{L}_{\mathrm{sub}}(K, L) \cong \mathcal{L}^{\leq 1}_{\mathrm{sub}}(C, D)$$

as an $f \in \mathcal{L}_{\mathrm{hom}}(K, L)$ is sublinear iff $e' \circ f$ is iff (using Theorem 2.35) $\overline{e' \circ f}$ is.

We saw in Section 2.5 that the inclusion $\mathcal{L}_{\leq 1}P \subseteq \mathcal{L}P$ is a universal embedding for any dcpo P, and it is easy to check that the norm $\|\text{-}\|_\infty$ is the Minkowski seminorm, thereby ensuring consistency with the previous two sections. Further, $\mathcal{L}_{\leq 1}P$ is a Kegelspitze join-semilattice and $\mathcal{L}P$ is a d-cone join-semilattice. The analogous remarks apply to the inclusion $\mathbb{I} \subseteq \overline{\mathbb{R}}_+$.

So, for any dcpo P we obtain the Kegelspitze semilattice isomorphism:

$$\mathcal{L}_{\mathrm{sub}}(\mathcal{L}_{\leq 1}P, \mathbb{I}) \cong \mathcal{L}^{\leq 1}_{\mathrm{sub}}(\mathcal{L}P, \overline{\mathbb{R}}_+)$$

and for any dcpos P and Q we obtain the Kegelspitze semilattice isomorphism:

$$\mathcal{L}_{\mathrm{sub}}(\mathcal{L}_{\leq 1}Q, \mathcal{L}_{\leq 1}P) \cong \mathcal{L}^{\leq 1}_{\mathrm{sub}}(\mathcal{L}Q, \mathcal{L}P)$$

As immediate consequences of Corollaries 4.4 and 5.1 we then obtain:

Corollary 6.4. *Let P be a dcpo. Then we have a Kegelspitze join-semilattice morphism:*

$$\Lambda_P \colon \mathcal{H}\mathcal{V}_{\leq 1}P \longrightarrow \mathcal{L}_{\mathrm{sub}}(\mathcal{L}_{\leq 1}P, \mathbb{I})$$

It is given by:

$$\Lambda_P(X)(f) =_{def} \sup_{\nu \in X} \int f \, d\nu$$

If P is a domain then Λ_P is an isomorphism. □

Corollary 6.5. *Let P and Q be dcpos. To every state transformer $s\colon P \to \mathcal{HV}_{\leq 1}Q$ we can assign a predicate transformer $\mathrm{PT}_{P,Q}(s)\colon \mathcal{L}_{\leq 1}Q \to \mathcal{L}_{\leq 1}P$ by:*

$$\mathrm{PT}_{P,Q}(s)(g)(x) =_{def} \sup_{\nu \in s(x)} \int g\, d\nu \quad (g \in \mathcal{L}_{\leq 1}Q, x \in P)$$

The predicate transformer $\mathrm{PT}_{P,Q}(s)$ is sublinear. The assignment $\mathrm{PT}_{P,Q}$ is a Kegelspitze join-semilattice morphism

$$(\mathcal{HV}_{\leq 1}Q)^P \longrightarrow \mathcal{L}_{\mathrm{sub}}(\mathcal{L}_{\leq 1}Q, \mathcal{L}_{\leq 1}P)$$

If Q is a domain then it is an isomorphism. □

6.2. **The upper case.** Suppose we are given full Kegelspitzen K and L and universal Kegelspitze embeddings $e\colon K \to C$ and $e'\colon L \to D$, where L has a top element (which we write as 1). Then we say that a function $f\colon K \to L$ is *strongly nonexpansive* if:

$$f(a +_r b) \;\leq\; f(a) +_r \|b\|_K \cdot 1$$

holds for all $a, b \in K$ and $r \in [0,1]$, and that a function $g\colon C \to D$ is *strongly nonexpansive* if:

$$g(x + y) \;\leq\; g(x) + \|y\|_K \cdot e'(1)$$

holds for all $x, y \in C$; note that this last definition is consistent with the corresponding definitions in previous sections. Also, a homogeneous function $g : C \to D$ is strongly nonexpansive iff it is when considered as a function between Kegelspitzen.

We write $\mathcal{L}^{\mathrm{sne}}_{\mathrm{sup}}(C, D)$ for the collection of all Scott-continuous superlinear strongly nonexpansive functions $g\colon C \to D$; this collection forms a Kegelspitze with the pointwise structure, and a Kegelspitze meet-semilattice if D is a d-cone meet-semilattice. We further write $\mathcal{L}^{\mathrm{sne}}_{\mathrm{sup}}(K, L)$ for the collection of all Scott-continuous superlinear strongly nonexpansive functionals $f\colon K \to L$; this collection forms a Kegelspitze with the pointwise structure, and a Kegelspitze meet-semilattice if L is one.

Now suppose, additionally, that L is a Kegelspitze meet-semilattice and D is a d-cone meet-semilattice (when e' is automatically a Kegelspitze semilattice morphism and so $e'\colon L \to D$ is a universal Kegelspitze semilattice embedding). Then the Kegelspitze semilattice isomorphism of Proposition 6.3 restricts to a Kegelspitze semilattice isomorphism:

$$\mathcal{L}^{\mathrm{sne}}_{\mathrm{sup}}(K, L) \cong \mathcal{L}^{\mathrm{sne}}_{\mathrm{sup}}(C, D)$$

which sends f to $g =_{def} \overline{f \circ e'}$. Regarding superadditivity, f is superadditive iff $f \circ e'$ is iff (by Theorem 2.35) $\overline{f \circ e'}$ is. Also g is strongly nonexpansive iff $g(x +_r y) \;\leq\; g(x) +_r \|y\|_K \cdot e'(1)$ for all $x, y \in C$ and $r \in [0,1]$, iff, by the binary induction principle for universal embeddings, $g(e(a) +_r e(b)) \;\leq\; g(e(a)) +_r \|e(b)\|_K \cdot e'(1)$ for $a, b \in K$, $r \in [0,1]$, iff $e'(f(a +_r b)) \;\leq\; e'(f(a) +_r \|b\|_K \cdot 1)$, for $a, b \in K$, $r \in [0,1]$ (as $g \circ e = e' \circ f$), iff f is strongly nonexpansive.

The Kegelspitzen $\mathbb{I}$, $\overline{\mathbb{R}}_+$, $\mathcal{L}_{\leq 1}P$ and $\mathcal{L}P$ (P a dcpo) are all Kegelspitze meet-semilattices and so the inclusions $\mathbb{I} \subseteq \overline{\mathbb{R}}_+$ and $\mathcal{L}_{\leq 1}P \subseteq \mathcal{L}P$ are universal Kegelspitze semilattice embeddings. So, in particular, for any dcpo P we obtain a Kegelspitze semilattice isomorphism:

$$\mathcal{L}^{\mathrm{sne}}_{\mathrm{sup}}(\mathcal{L}_{\leq 1}P, \mathbb{I}) \cong \mathcal{L}^{\mathrm{sne}}_{\mathrm{sup}}(\mathcal{L}P, \overline{\mathbb{R}}_+)$$

and for any dcpos P and Q we obtain a Kegelspitze semilattice isomorphism:

$$\mathcal{L}^{\mathrm{sne}}_{\mathrm{sup}}(\mathcal{L}_{\leq 1}Q, \mathcal{L}_{\leq 1}P) \cong \mathcal{L}^{\mathrm{sne}}_{\mathrm{sup}}(\mathcal{L}Q, \mathcal{L}P)$$

As immediate consequences of Corolleries 4.7 and 5.2 we then obtain:

Corollary 6.6. *Let P be a domain. Then we have a Kegelspitze meet-semilattice isomorphism*

$$\Lambda_P \colon \mathcal{S}\mathcal{V}_{\leq 1}P \;\cong\; \mathcal{L}^{\mathrm{sne}}_{\mathrm{sup}}(\mathcal{L}_{\leq 1}P, \mathbb{I})$$

It is given by:

$$\Lambda_P(X)(f) =_{def} \inf_{\nu \in X} \int f \, d\nu \qquad \square$$

Corollary 6.7. *Let P be a dcpo and let Q be a domain. To every state transformer $s \colon P \to \mathcal{S}\mathcal{V}_{\leq 1}Q$ we can assign a predicate transformer $\mathrm{PT}_{P,Q}(s) \colon \mathcal{L}_{\leq 1}Q \to \mathcal{L}_{\leq 1}P$ by:*

$$\mathrm{PT}_{P,Q}(s)(g)(x) =_{def} \inf_{\nu \in s(x)} \int g \, d\nu \quad (g \in \mathcal{L}_{\leq 1}Q, x \in P)$$

The predicate transformer $\mathrm{PT}_{P,Q}(s)$ is superlinear and strongly nonexpansive. The assignment $\mathrm{PT}_{P,Q}$ is a Kegelspitze meet-semilattice isomorphism

$$(\mathcal{S}\mathcal{V}_{\leq 1}Q)^P \cong \mathcal{L}^{\mathrm{sne}}_{\mathrm{sup}}(\mathcal{L}_{\leq 1}Q, \mathcal{L}_{\leq 1}P) \qquad \square$$

Finally we show that, as in previous cases, the strong nonexpansiveness condition can be simplified for homogeneous functions.

Fact 6.8. Let $K \xrightarrow{e} C$ and $L \xrightarrow{e'} D$ be universal embeddings where both K and L have top elements and where $\|\text{-}\|_K$ is a norm on C. Then:

(1) A homogeneous function $f : K \to L$ is strongly nonexpansive iff $f(x +_r 1) \leq f(x) +_r 1$, for all $x \in K$ and $r \in [0, 1]$

(2) A homogeneous function $g : C \to D$ is strongly nonexpansive iff $g(x + 1) \leq g(x) + 1$, for all $x \in C$

Proof. We assume, without loss of generality, that the embeddings are inclusions.

(1) We have to show that $f(a +_r b) \;\leq\; f(a) +_r \|b\|_K \cdot 1$ for all $a, b \in K$ and $r \in [0, 1]$. If $\|b\|_K = 0$ then, as $\|\text{-}\|_K$ is a norm, $b = 0$. Otherwise, setting $s =_{def} \|b\|_K$, we see by Proposition 6.1 that $b = s \cdot c$ for some $c \in K$ and so that $b \leq s \cdot 1$. Then, taking $t =_{def} 1 - (1 - r)s$, we note that $r \leq t$ and calculate:

$$f(a +_r b) \leq f(a +_r s \cdot 1) = f(r/t \cdot a +_t 1) \leq f(r/t \cdot a) +_t 1 = r/t \cdot f(a) +_t 1 = f(a) +_r \|b\|_K \cdot 1$$

(2) We have to show that $g(x+y) \leq \ g(x) + \|y\|_K \cdot 1$ for all $x, y \in C$. This is trivial if $\|y\|_K = \infty$. If it is 0 then, as $\|\text{-}\|_K$ is a norm, $y = 0$. Otherwise, setting $s =_{def} \|y\|_K$ and noting that $s^{-1} \cdot y \in K$, we calculate:

$$\begin{aligned} g(x+y) = r \cdot g(s^{-1} \cdot x \,+\, s^{-1} \cdot y) \ &\leq r \cdot g(s^{-1} \cdot x \,+\, 1) \\ &\leq r \cdot (g(s^{-1} \cdot x) \,+\, 1) = g(x) + \|y\|_K \cdot 1 \end{aligned}$$ □

6.3. **The convex case.** We begin with some definitions. Given a full Kegelspitze K and a full Kegelspitze semilattice L, say that a function $f : K \to L$ is $\subseteq$-*sublinear* if it is homogeneous and, for all $a, b \in K$ and $r \in [0, 1]$ we have:

$$f(a +_r b) \subseteq f(a) +_r f(b)$$

Note that a function $g : C \to D$ from a d-cone to a d-cone semilattice is $\subseteq$-sublinear, as defined in Section 5.3, iff it is when considered as a function from a Kegelspitze to a Kegelspitze semilattice.

Next, given Kegelspitzen K, L, and M and two functions $\mathrm{d}_L, \mathrm{u}_L : L \to M$, we say that a function $f : K \to L$ is *medial (w.r.t.* $\mathrm{d}_L, \mathrm{u}_L$*)* if, for all $a, b \in K$ and $r \in [0, 1]$ we have:

$$f_{\mathrm{d}}(a +_r b) \leq f_{\mathrm{d}}(a) +_r f_{\mathrm{u}}(b) \leq f_{\mathrm{u}}(a +_r b)$$

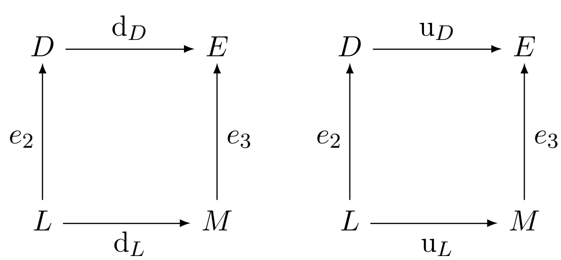
where $f_{\mathrm{d}} =_{def} \mathrm{d}_L \circ f$ and $f_{\mathrm{u}} =_{def} \mathrm{u}_L \circ f$. Similarly, given d-cones C, D, and E and two functions $\mathrm{d}_D, \mathrm{u}_D : D \to E$, we say that a function $g : D \to E$ is *medial (w.r.t.* $\mathrm{d}_D, \mathrm{u}_D$*)* if, for all $x, y \in C$, we have:

$$g_{\mathrm{d}}(x + y) \leq g_{\mathrm{d}}(x) + g_{\mathrm{u}}(y) \leq g_{\mathrm{u}}(x + y)$$

where $g_{\mathrm{d}} =_{def} \mathrm{d}_D \circ g$ and $g_{\mathrm{u}} =_{def} \mathrm{u}_D \circ g$; this is equivalent to it being medial when considered as a function between Kegelspitzen, provided that it is homogeneous, as are d_D, and u_D.

Now we suppose given:

- full Kegelspitzen K and M and a full Kegelspitze semilattice L,
- d-cones C and E and a d-cone semilattice D,
- universal Kegelspitze embeddings $K \xrightarrow{e_1} C$ and $M \xrightarrow{e_3} E$ and a universal Kegelspitze semilattice embedding $L \xrightarrow{e_2} D$, and
- Scott continuous homogeneous functions $\mathrm{d}_L, \mathrm{u}_L : L \to M$ and $\mathrm{d}_D, \mathrm{u}_D : D \to E$ such that both the following two diagrams commute:



Then, if $f : K \to L$ and $g : C \to D$ are Scott-continuous homogeneous functions such that $e_2 \circ f = g \circ e_1$ then f is $\subseteq$-sublinear iff g is, and f is medial iff g is. The proofs are straightforward using the binary induction principle for universal Kegelspitze embeddings.

We now suppose further that

- M is both a Kegelspitze meet- and join-semilattice, and d_L, u_L are both Kegelspitze semilattice morphisms, with M taken, accordingly, as a Kegelspitze meet- or join-semilattice, and
- E is both a d-cone meet-semilattice and join-semilattice, and d_D, u_D are both d-cone semilattice morphisms, with M taken, accordingly, as a d-cone meet- or join-semilattice.

We write $\mathcal{L}^{\leq 1}_{\subseteq,\mathrm{med}}(C, D)$ for the set of Scott-continuous, $\subseteq$-sublinear, medial, nonexpansive functions from D to E; equipped with the pointwise structure it forms a sub-Kegelspitze semilattice of $\mathcal{L}^{\leq 1}_{\mathrm{hom}}(C, D)$. We further write $\mathcal{L}_{\subseteq,\mathrm{med}}(K, L)$ for the set of Scott-continuous, $\subseteq$-sublinear, medial functions from K to L; equipped with the pointwise structure it forms a sub-Kegelspitze semilattice of $\mathcal{L}_{\mathrm{hom}}(K, L)$.

As we have seen that $\subseteq$-sublinearity and mediality transfer along the Kegelspitze semilattice isomorphism of Proposition 6.3, we now see that that, under our several suppositions, this isomorphism restricts to a Kegelspitze semilattice isomorphism:

$$\mathcal{L}_{\subseteq,\mathrm{med}}(K, L) \cong \mathcal{L}^{\leq 1}_{\subseteq,\mathrm{med}}(C, D)$$

Let us now consider the particular case where $K \xrightarrow{e_1} C$ is the inclusion $\mathcal{L}_{\leq 1}P \subseteq \mathcal{L}P$, for some dcpo P, $L \xrightarrow{e_2} D$ is the inclusion $\mathcal{P}\mathbb{I} \subseteq \mathcal{P}\overline{\mathbb{R}}_+$, $M \xrightarrow{e_3} E$ is the inclusion $\mathbb{I} \subseteq \overline{\mathbb{R}}_+$, $\mathrm{d}_{\mathcal{P}\overline{\mathbb{R}}_+}(x) =_{def} \underline{x}$, $\mathrm{u}_{\mathcal{P}\overline{\mathbb{R}}_+}(x) =_{def} \overline{x}$, and $\mathrm{d}_{\mathcal{P}\mathbb{I}}$ and $\mathrm{u}_{\mathcal{P}\mathbb{I}}$ are defined similarly.

Then we already know that e_1 is a universal Kegelspitze embedding and it is evident that e_3 is too; that e_2 is follows from Proposition 2.38, and so it is evidently a universal Kegelspitze semilattice embedding. It is then clear that all the above assumptions hold, and so we have a Kegelspitze semilattice isomorphism:

$$\mathcal{L}_{\subseteq,\mathrm{med}}(\mathcal{L}_{\leq 1}P, \mathcal{P}\mathbb{I}) \cong \mathcal{L}^{\leq 1}_{\subseteq,\mathrm{med}}(\mathcal{L}P, \mathcal{P}\overline{\mathbb{R}}_+)$$

where, on the left nonexpansiveness is defined relative to the Minkowski seminorms on $\mathcal{L}P$ and $\overline{\mathbb{R}}_+$. However these seminorms are the same as those considered before: we already know this for $\mathcal{L}P$, and it is easy to show that for $\overline{\mathbb{R}}_+$ (i.e., to show that $\|x\|_{\mathcal{P}\mathbb{I}} = \overline{x}$). As an immediate corollary of Corollary 4.10 we then obtain:

Corollary 6.9. *Let P be a coherent domain. Then we have a Kegelspitze semilattice isomorphism*

$$\Lambda_P \colon \mathcal{P}\mathcal{V}_{\leq 1}P \;\cong\; \mathcal{L}_{\subseteq,\mathrm{med}}(\mathcal{L}_{\leq 1}P, \mathcal{P}\mathbb{I})$$

It is given by:

$$\Lambda_P(X)(f) =_{def} [\inf_{\nu \in X} \int f\, d\nu \,, \sup_{\nu \in X} \int f\, d\nu\,] \qquad \square$$

To obtain a corresponding result for predicate transformers we change the above framework slightly: instead of supposing that K is a full Kegelspitze, C is a d-cone, and

$K \xrightarrow{e_1} C$ is a universal Kegelspitze embedding, we suppose that K is a full Kegelspitze semilattice, C is a d-cone semilattice, and $K \xrightarrow{e_1} C$ is a universal Kegelspitze semilattice embedding.

Then, for Scott-continuous homogeneous functions $f : K \to L$, $g : C \to D$ where $e_2 \circ f = g \circ e_1$, we additionally have that f is $\subseteq$-monotone iff g is. This is proved by the universal embedding induction principle, as usual, but noting that $\subseteq$-monotonicity can be expressed equationally: for example g is $\subseteq$-monotone if, and only if, for all $x, y \in C$, $g(x) \cup g(x \cup y) = g(x \cup y)$.

We now write $\mathcal{L}^{\leq 1}_{\mathrm{mon},\subseteq,\mathrm{med}}(C, D)$ for the set of Scott-continuous, $\subseteq$-monotone, $\subseteq$-sublinear, medial, nonexpansive functions from D to E; equipped with the pointwise structure it forms a sub-Kegelspitze semilattice of $\mathcal{L}^{\leq 1}_{\mathrm{hom}}(C, D)$. We further write $\mathcal{L}_{\mathrm{mon},\subseteq,\mathrm{med}}(K, L)$ for the set of Scott-continuous, $\subseteq$-monotone, $\subseteq$-sublinear, medial functions from K to L; equipped with the pointwise structure it forms a sub-Kegelspitze semilattice of $\mathcal{L}_{\mathrm{hom}}(K, L)$.

As we have seen that $\subseteq$-monotonicity, $\subseteq$-sublinearity, and mediality transfer along the Kegelspitze semilattice isomorphism of Proposition 6.3, we now see that that, under our several suppositions, this isomorphism restricts to a Kegelspitze semilattice isomorphism:

$$\mathcal{L}^{\leq 1}_{\mathrm{mon},\subseteq,\mathrm{med}}(C, D) \cong \mathcal{L}_{\mathrm{mon},\subseteq,\mathrm{med}}(K, L)$$

To apply this result, we note that, for any dcpo P, the inclusion $\mathcal{P}\mathbb{I}^P \subseteq \mathcal{P}\overline{\mathbb{R}}_+^P$ is a universal Kegelspitze semilattice embedding (for a proof again use Proposition 2.38, now following the same lines as the proof that the inclusion $\mathcal{L}_{\leq 1}P \subseteq \mathcal{L}P$ is universal) and that the Minkowski seminorm on $\mathcal{P}\overline{\mathbb{R}}_+^P$ is the same as the norm defined before, i.e., that $\|f\|_{\mathcal{P}\overline{\mathbb{R}}_+^P} = \sup_{x\in P} \overline{f(x)}$.

Now consider the particular case where $K \xrightarrow{e_1} C$ is the inclusion $\mathcal{P}\mathbb{I}^Q \subseteq \mathcal{P}\overline{\mathbb{R}}_+^Q$, for some dcpo Q, $L \xrightarrow{e_2} D$ is the inclusion $\mathcal{P}\mathbb{I}^P \subseteq \mathcal{P}\overline{\mathbb{R}}_+^P$, for some dcpo P, $M \xrightarrow{e_3} E$ is the inclusion $\mathbb{I}^P \subseteq \overline{\mathbb{R}}_+^P$, $\mathrm{d}_{\mathcal{P}\overline{\mathbb{R}}_+^P}(f) =_{def} \underline{f}$, $\mathrm{u}_{\mathcal{P}\overline{\mathbb{R}}_+^P}(f) =_{def} \overline{f}$, and $\mathrm{d}_{\mathcal{P}\mathbb{I}^P}$ and $\mathrm{u}_{\mathcal{P}\mathbb{I}^P}$ are defined similarly. Then all the above assumptions hold, and so we have a Kegelspitze semilattice isomorphism:

$$\mathcal{L}^{\leq 1}_{\mathrm{mon},\subseteq,\mathrm{med}}(\mathcal{P}\overline{\mathbb{R}}_+^Q, \mathcal{P}\overline{\mathbb{R}}_+^P) \cong \mathcal{L}_{\mathrm{mon},\subseteq,\mathrm{med}}(\mathcal{P}\mathbb{I}^Q, \mathcal{P}\mathbb{I}^P)$$

and then as an immediate corollary of Corollary 5.4 we obtain:

Corollary 6.10. *Let P be a dcpo and let Q be a coherent domain. To every state transformer $s\colon P \to \mathcal{P}\mathcal{V}_{\leq 1}Q$ we can assign a predicate transformer $\mathrm{PT}_{P,Q}(s)\colon \mathcal{P}\mathbb{I}^Q \to \mathcal{P}\mathbb{I}^P$ by:*

$$\mathrm{PT}_{P,Q}(s)(g)(x) =_{def} [\inf_{\nu\in s(x)} \int \underline{g}\, d\nu, \sup_{\nu\in s(x)} \int \overline{g}\, d\nu\,] \quad (g \in \mathcal{P}\mathbb{I}^Q, x \in P)$$

The predicate transformer $\mathrm{PT}_{P,Q}(s)$ is $\subseteq$-monotone, $\subseteq$-sublinear, and medial. The assignment $\mathrm{PT}_{P,Q}$ is a Kegelspitze semilattice isomorphism

$$(\mathcal{P}\mathcal{V}_{\leq 1}Q)^P \cong \mathcal{L}_{\mathrm{mon},\subseteq,\mathrm{med}}(\mathcal{P}\mathbb{I}^Q, \mathcal{P}\mathbb{I}^P) \qquad \square$$

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Appendix A. The other distributive law

We consider two equational theories: B, for barycentric algebras, and S, for semilattices. The first has binary operation symbols $+_r$ ($r \in [0,1]$) and axioms the equations for barycentric algebras (B1), (B2), (SC), and (SA) given in Section 2.1; the second has a single binary operation symbol $\cup$ and associativity, commutativity, and idempotency axioms. For any $r \in [0,1]$ we write r' for $1-r$.

We write $t_1 = u_1, \ldots, t_1 = u_1 \vdash_T t = u$ for a given equational theory T to mean that $t = u$ follows by equational reasoning from the theory T and the equations $t_1 = u_1, \ldots, t_n = u_n$. We need a proof-theoretic version of another lemma of Neumann: [45, Lemma 3] (with corrected bounds). We first show a lemma that can also be derived from general results due to Sokolova and Woracek [55, Theorem 4.4 and Example 4.6]:

Lemma A.1. *Let $\sim$ be a congruence on the barycentric algebra $[0,1]$. Then if two elements of the open interval $]0,1[$ are congruent, so are any other two.*

Proof. Let $\sim$ be such a congruence and suppose that we have $r \sim s$ with $0 < r < s < 1$. Set $\alpha = r/s < 1$. Then $\alpha r \sim \alpha s = r \sim s$ hence $\alpha r \sim s$. Repeating the argument yields $\alpha^n r \sim s$, for any $n \geq 0$, and so we get arbitrarily close to 0 with elements congruent to s.

In the other direction, we can define a 'symmetric' congruence relation $\sim'$, setting $p \sim' q$ to hold if, and only if, $p' \sim q'$, as the map $p \mapsto p'$ is an (involutive) automorphism of $[0,1]$. We then have $s' \sim' r'$ and $0 < s' < r' < 1$. So, by the above argument, we can get arbitrarily close to 0 with elements in the symmetric congruence relation with r', and so arbitrarily close to 1 with elements congruent to r.

As congruence classes are convex, we then see that any two elements of the open interval $]0,1[$ are congruent. □

Lemma A.2. *Let T be an equational theory extending* B. *Then for any terms t and u, and any $0 < r < s < 1$ and $0 < p < q < 1$ we have:*

$$t +_r u = t +_s u \vdash_T \; t +_p u = t +_q u$$

Proof. One fixes r and s with $0 < r < s < 1$ and then applies Lemma A.1 to the congruence $\sim$ where $p \sim q$ iff $t +_r u = t +_s u \vdash_T \; t +_p u = t +_q u$. □

The equational theory BSD$'$ has axioms those of B and S together with equations

$$x \cup (y +_r z) = (x \cup y) +_r (x \cup z) \qquad (r \in]0,1[) \tag{D$'$}$$

stating that $\cup$ distributes over each of the $+_r$. Recall that a *join-distributive bi-semilattice* [52] is an algebra with two semilattice operations $\cap$ and $\cup$, with $\cup$ distributing over $\cap$.

Theorem A.3. *The equational theory* BSD$'$ *is equivalent to that of join-distributive bi-semilattices.*

Proof. In the following we just write $t = u$ rather than $\vdash_{\mathrm{BSD}'} t = u$. Substituting $(y +_r z)$ for x in D′ (and using S and re-using D′) we get:

$$(y +_r z) = ((y +_r z) \cup y) +_r ((y +_r z) \cup z) = (y +_r (y \cup z)) +_r ((y \cup z) +_r z) \qquad \text{(A.1)}$$

for all $r \in]0, 1[$. Now, substituting $y \cup z$ for z (and using S and then B) we get:

$$y +_r (y \cup z) = (y +_r (y \cup z)) +_r ((y \cup z) +_r (y \cup z)) = y +_{r^2} (y \cup z)$$

for all $r \in]0, 1[$. So by Lemma A.2 we obtain:

$$y +_r (y \cup z) = y +_s (y \cup z) \qquad \text{(A.2)}$$

for all $r, s \in]0, 1[$. But now, for all $r \in]0, 1[$, we have:

$$\begin{aligned}
(y +_r z) &= (y +_r (y \cup z)) +_r ((y \cup z) +_r z) && \text{(by Equation (A.1))}\\
&= (y +_{r'} (y \cup z)) +_r ((y \cup z) +_{r'} z) && \text{(by Equation (A.2), and using B)}\\
&= (y +_r (y \cup z)) +_{r'} ((y \cup z) +_r z) && \text{(by the entropic identity)}\\
&= (y +_{r'} (y \cup z)) +_{r'} ((y \cup z) +_{r'} z) && \text{(by Equation (A.2), and using B)}\\
&= (y +_{r'} z) && \text{(by Equation (A.1), substituting } r' \text{ for } r)
\end{aligned}$$

and so we can apply Lemma A.2 again, and obtain:

$$y +_r z = y +_s z$$

for all $r, s \in]0, 1[$.

But then we have an equivalence of our theory with that of join-distributive bi-semilattices, where join and meet are $\cup$ and $\cap$. To translate from the theory of join-distributive bi-semilattices into our theory one translates $\cup$ as $x_0 \cup x_1$ and $\cap$ as (e.g.) $x_0 +_{1/2} x_1$. In the other direction, one translates $\cup$ as $x_0 \cup x_1$, $+_r$ as $x_0 \cap x_1$, for all $r \in]0, 1[$, and $+_0$ and $+_1$ as x_0 and x_1, respectively. □

We next consider a weaker theory: we drop the idempotence of $\cup$. Let C be the theory of commutative semigroups, that is of algebras with an associative and commutative multiplication operation (having dropped idempotence, the change to a multiplicative notation is natural). The equational theory CCSA of *convex commutative semigroup algebras* has as axioms those of B and C together with the following distributive laws, stating that multiplication distributes over the $+_r$:

$$x(y +_r z) = xy +_r xz \qquad (r \in]0, 1[)$$

The real interval $[0, 1]$ provides an example convex commutative semigroup algebra with the usual barycentric operations and multiplication.

Let $\mathcal{D}_\omega$ be the finite probability distributions monad. We regard $\mathcal{D}_\omega X$ as consisting of convex combinations $\sum_{i=1,m} p_i x_i$ of elements of X; with the evident barycentric operations it is the free barycentric algebra over X. Then the free convex commutative semigroup algebra over a given commutative semigroup M is provided by the barycentric algebra $\mathcal{D}_\omega M$, with the following multiplication:

$$(\sum_{i=1,\dots,m} p_i x_i)(\sum_{i=1,\dots,n} q_j y_j) = \sum_{\substack{i=1,\dots,m\\ j=1,\dots,n}} (p_i q_j) x_i y_j$$

and with unit the inclusion (this construction is a variant of the standard group algebra construction). In particular, writing $\mathcal{M}^+_\omega$ for the finite non-empty multisets monad, we see that $\mathcal{D}_\omega\mathcal{M}^+_\omega X$ is the free convex commutative semigroup algebra over X, as $\mathcal{M}^+_\omega$ is the free commutative semigroup monad.

We regard $\mathcal{D}_\omega\mathcal{M}^+_\omega X$ as consisting of all polynomials with no constants, with variables in X, and with coefficients in $[0,1]$ adding up to 1. In other words, it consists of all convex combinations of non-trivial polynomials with variables in X. The barycentric operations are the evident convex combinations of such polynomials, and multiplication is the usual polynomial multiplication. The unit $\eta\colon x \to \mathcal{D}_\omega\mathcal{M}^+_\omega X$ is the inclusion, and the extension $\overline{f}\colon \mathcal{D}_\omega\mathcal{M}^+_\omega X \to A$ to a CCSA-homomorphism of a map f from X to a convex commutative semigroup algebra A assigns to any polynomial $p(x_1,\dots,x_n)$ in $\mathcal{D}_\omega\mathcal{M}^+_\omega X$ its value $p(f(x_1),\dots,f(x_n)) \in A$ as obtained using the CCSA operations of A.

A convex commutative semigroup algebra A is *complete* if, for all CCSA-terms t and u we have:

$$A \models t = u \quad \Rightarrow \quad \vdash_{\mathrm{CCSA}} t = u$$

This holds if, and only if, distinct polynomials in $\mathcal{D}_\omega\mathcal{M}^+_\omega X$ can be separated by elements of A, that is, if for any such $p(x_1,\dots,x_n) \neq q(x_1,\dots,x_n)$, with variables in $x_1,\dots,x_n$, there are $a_1,\dots a_n \in A$ such that $p(a_1,\dots,a_n) \neq q(a_1,\dots,a_n)$. For example, $[0,1]$ is complete in this sense.

We now focus on the convex semigroup algebra $\mathcal{D}_\omega\mathcal{P}^+_\omega X$. We need two lemmas.

Lemma A.4. *Let X be a set with at least two elements. Then $\mathcal{D}_\omega\mathcal{P}^+_\omega X$ is complete.*

Proof. Let $p(x_1,\dots,x_n)$, $q(x_1,\dots,x_n)$ be distinct polynomials in $\mathcal{D}_\omega\mathcal{M}^+_\omega X$, and choose $r_1,\dots,r_n$ in $[0,1]$ separating them. Choose two distinct elements y, z in X. We can define a semigroup homomorphism $h\colon \mathcal{P}^+_\omega X \to [0,1]$ by:

$$h(u) = \begin{cases} 1 & (u = \{y\}) \\ 0 & (\text{otherwise}) \end{cases}$$

Then h has an extension to a CCSA-homomorphism $\overline{h}\colon \mathcal{D}_\omega\mathcal{P}^+_\omega X \to [0,1]$, and, taking $a_i \in \mathcal{D}_\omega\mathcal{P}^+_\omega X$ to be $\{y\} +_{r_i} \{z\}$, for $i = 1,n$, and noting that $\overline{h}(a_i) = h(\{y\}) +_{r_i} h(\{z\}) = r_i$, we see that:

$$\overline{h}(p(a_1,\dots,a_n)) = p(\overline{h}(a_1),\dots,\overline{h}(a_n)) = p(r_1,\dots,r_n) \neq q(r_1,\dots,r_n) = \overline{h}(q(a_1,\dots,a_n))$$

and so that $a_1,\dots,a_n$ are elements of $\mathcal{D}_\omega\mathcal{P}^+_\omega X$ separating p and q. ∎

Lemma A.5. *Let X be a nonempty set and let T be a subtheory of* CCSA*. Then $\mathcal{D}_\omega\mathcal{P}^+_\omega X$ is not the free T-algebra over X.*

Proof. First note that, for any $p \in \mathcal{D}_\omega\mathcal{M}^+_\omega X$, we have $p \neq pp$. Then, if $\mathcal{D}_\omega\mathcal{P}^+_\omega X$ were the free T-algebra over X, there would be a T-algebra homomorphism $h\colon \mathcal{D}_\omega\mathcal{P}^+_\omega X \to \mathcal{D}_\omega\mathcal{M}^+_\omega X$, as $\mathcal{D}_\omega\mathcal{M}^+_\omega X$ is a CCSA-algebra and so a T-algebra. Choosing any $y \in \mathcal{P}^+_\omega X$, we would then find $h(y) = h(yy) = h(y)h(y)$, a contradiction. □

So, in particular, for non-empty X, $\mathcal{D}_\omega\mathcal{P}^+_\omega X$ is, as may be expected, not the free CCSA-algebra. We can now prove:

Theorem A.6. *Let X be a set with at least two elements. Then $\mathcal{D}_\omega\mathcal{P}^+_\omega X$ is not the free T-algebra over X for any equational theory T with the same signature as that of* CCSA*.*

Proof. Suppose, for the sake of contradiction that $\mathcal{D}_\omega\mathcal{P}^+_\omega X$ is the free T-algebra over X for an equational theory T with the same signature as CCSA. Then, in particular, it is a T-algebra, and so all T-equations hold in it. But, by Lemma A.4 all equations holding in it are in CCSA. So $\mathcal{D}_\omega\mathcal{P}^+_\omega X$ is the free algebra for a subtheory of CCSA. However, by Lemma A.5, that cannot be the case. □

So we have shown that there is no algebraic (i.e., equational) account of the natural random set algebras $\mathcal{D}_\omega\mathcal{P}^+_\omega X$. It may even be that $\mathcal{D}_\omega \circ \mathcal{P}^+_\omega$ admits no monadic structure.

Appendix B. A counterexample

Referring to Lemma 2.41 and the preceding discussion, we give an example of a continuous Kegelspitze E whose scalar multiplication does not preserve the way-below relation $\ll_E$. That is, there are elements $a \ll_E b$ in E such that $ra \not\ll_E rb$ for some $r < 1$.

We consider the half-open unit interval $]0, 1]$ with its upper (= Scott) topology; the open subsets are the half-open intervals $]r, 1]$, $0 \leq r \leq 1$. Let $\mathcal{L}$ be the continuous d-cone of continuous functions $f\colon]0, 1] \to \overline{\mathbb{R}}_+$ (in classical analysis one would have said that the $f \in \mathcal{L}$ are the monotone increasing lower semicontinuous functions). Such a function f has a greatest value, namely $f(1)$. We write $\mathcal{L}_{\leq 1}$ and $\mathcal{L}_{\leq 2}$ for the Scott-closed, hence continuous, Kegelspitzen of functions $f \in \mathcal{L}$ such that $f(1) \leq 1$ and $f(1) \leq 2$, respectively. We write k_1 for the constant function 1 on $]0, 1]$.

Now let E be the collection of those $f \in \mathcal{L}_{\leq 2}$ which can be represented as a convex combination $f = q \cdot 2\mathrm{k}_1 + (1 - q) \cdot h$ $(0 \leq q \leq 1)$ of the constant function $2\mathrm{k}_1$ with some $h \in \mathcal{L}_{\leq 1}$. This provides our counterexample.

We first claim that E forms a sub-Kegelspitze of $\mathcal{L}_{\leq 2}$. It is clearly convex and contains the least element of $\mathcal{L}_{\leq 2}$. To show E is a sub-dcpo of $\mathcal{L}_{\leq 2}$ with the inherited ordering, we set

$$E' =_{def} \{f \in \mathcal{L}_{\leq 2} \mid 2f(1)\mathrm{k}_1 \leq f + 2\mathrm{k}_1\}$$

and, noting that E' forms a sub-dcpo, show that E and E' coincide.

It is clear that $E \subseteq E'$. Conversely, suppose $f \in E'$ so that $2f(1)\mathrm{k}_1 \leq f + 2\mathrm{k}_1$. If $f(1) = 2$, then $f = 2\mathrm{k}_1$ and so $f \in E$. If $f(1) \leq 1$, then trivially $f \in \mathcal{L}_{\leq 1} \subseteq E$. Thus we can suppose that $1 < f(1) < 2$. We can rewrite the condition for membership in E' as $2(f(1)-1)\mathrm{k}_1 \leq f$ and we let $\widehat{f} = f - 2(f(1)-1)\mathrm{k}_1$. Since $\widehat{f}(1) = f(1) - 2(f(1)-1) = 2 - f(1) \, (\neq 0)$, we have $\frac{\widehat{f}(1)}{2-f(1)} = 1$, so that $\frac{\widehat{f}}{2-f(1)} \in \mathcal{L}_{\leq 1}$. Letting $q = f(1) - 1$ we have $1 - q = 2 - f(1)$ and $0 < q < 1$, since $1 < f(1) < 2$, and f becomes a convex combination of the constant function $2\mathrm{k}_1$ and the function $\frac{\widehat{f}}{2-f(1)} \in \mathcal{L}_{\leq 1}$, namely $f = 2(f(1)-1)\mathrm{k}_1 + \widehat{f} = q \cdot 2\mathrm{k}_1 + (1-q) \cdot \frac{\widehat{f}}{1-q}$, so that $f \in E$.

We remark that E is not a full Kegelspitze. For example, writing χ_s for the characteristic function of the interval $]s, 1]$ $(0 < s < 1)$, we have $\chi_s \leq 1/2 \cdot (2\mathrm{k}_1)$. However, we cannot have $\chi_s = 1/2 \cdot f$ for any $f \in E$, as we would then have $f(1) = 2$ and so, by the defining property of E', $f = 2\mathrm{k}_1$. But, as $\chi_s = 1/2 \cdot f$, we have $f(r) = 0$ for any $0 < r < s$.

We next claim that E is continuous. For this, it is enough to show that each element f is the lub of a directed set of elements $\ll_E$ below it. In case $f(1) \leq 1$, any element in $\mathcal{L}_{\leq 2}$ below it is also in E, and we use the continuity of $\mathcal{L}_{\leq 2}$ to get a directed set of elements $\ll_{\mathcal{L}_{\leq 2}}$ below it, and so $\ll_E$ below it, as E is a sub-dcpo of $\mathcal{L}_{\leq 2}$.

Otherwise $f(1) > 1$, and we can again set $\widehat{f} = f - 2(f(1)-1)\mathrm{k}_1$. Since $\widehat{f}(1) = 2 - f(1) < 1$, we have $\widehat{f} \in \mathcal{L}_{\leq 1}$. Let

$$\rho_{p,g} = 2p\mathrm{k}_1 + g$$

For $0 < p < f(1) - 1$ and $g \ll_{\mathcal{L}_{\leq 1}} \widehat{f}$, we obtain a family of functions which clearly is directed and has $2(f(1)-1)\mathrm{k}_1 + \widehat{f} = f$ as its least upper bound. Each of these functions belongs to E, since it can be written as a convex combination $p \cdot 2\mathrm{k}_1 + (1-p) \cdot \frac{g}{1-p}$ and since $\frac{g}{1-p} \in \mathcal{L}_{\leq 1}$ (the latter as $g(1) \leq \widehat{f}(1) = 2 - f(1) < 1 - p$).

It remains to show that $\rho_{p,g} \ll_E f$. For this, let $(f_i)_i$ be a directed family in E such that $f \leq \bigvee^{\uparrow}_i f_i$. Then $f(1) \leq \bigvee^{\uparrow}_i f_i(1)$ so that there is an i_0 such that $p < f_{i_0}(1) - 1$, since $p < f(1) - 1$. We now restrict our attention to the indices i such that $f_i \geq f_{i_0}$. Since these f_i belong to $E = E'$, they satisfy $2(f_i(1)-1)\mathrm{k}_1 \leq f_i$. It follows that $2p\mathrm{k}_1 \leq 2(f_i(1)-1)\mathrm{k}_1 \leq f_i$, that is $0 \leq f_i - 2p\mathrm{k}_1 \in \mathcal{L}$. As $f - 2p\mathrm{k}_1 \leq (\bigvee^{\uparrow}_i f_i) - 2p\mathrm{k}_1 = \bigvee^{\uparrow}_i (f_i - 2p\mathrm{k}_1)$, we then have $f - 2p\mathrm{k}_1 \leq_{\mathcal{L}} \bigvee^{\uparrow}_i (f_i - 2p\mathrm{k}_1)$. Since $g \ll_{\mathcal{L}_{\leq 1}} \widehat{f} = f - 2(f(1)-1)\mathrm{k}_1 \leq_{\mathcal{L}} f - 2p\mathrm{k}_1$, we also have $g \ll_{\mathcal{L}} f - 2p\mathrm{k}_1$, and so $g \leq f_{i_1} - 2p\mathrm{k}_1$ for some $i_1 \geq i_0$. Thus $\rho_{p,g} \leq 2p\mathrm{k}_1 + g \leq f_{i_1}$.

Now that we have shown E to be a continuous Kegelspitze, it only remains to see that, as claimed, scalar multiplication does not preserve $\ll_E$. One the one hand, from the above discussion we have $\mathrm{k}_1 \ll_E 2\mathrm{k}_1$ (for, setting $f = 2\mathrm{k}_1$ and $p = 1/2$, we see that $\widehat{f} = \perp$ and $p < f(1) - 1$, and then that $\mathrm{k}_1 = \rho_{1/2,\perp}$). However, on the other hand, we have $1/2 \cdot \mathrm{k}_1 \not\ll_E \mathrm{k}_1$ (for $\mathrm{k}_1 = \bigvee^{\uparrow}_{n>0} \chi_{2^{-n}}$, but we have $1/2 \cdot \mathrm{k}_1 \leq \chi_{2^{-n}}$ for no $n > 0$).