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FIRST ORDER THEORIES OF SOME LATTICES OF OPEN SETS

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Abstract. We show that the first order theory of the lattice of open sets in some natural topological spaces is m-equivalent to second order arithmetic. We also show that for many natural computable metric spaces and computable domains the first order theory of the lattice of effectively open sets is undecidable. Moreover, for several important spaces (e.g., $\mathbb{R}^n$, $n \geq 1$, and the domain $P\omega$) this theory is m-equivalent to first order arithmetic.

1. Introduction

From the very beginning of Model Theory, the study of (un)decidability of first order theories became a central and popular topic. As a result, for virtually all structures $\mathbb{A} = (A; \sigma)$ of a given signature σ (in the Russian literature structures are also known as algebraic systems) occurring naturally in mathematics their first order theories $FO(\mathbb{A})$ has been shown to be decidable or undecidable (among vast literature on the subject we mention [TMR53, ELTT65, Er80], as examples).

More recently, several researchers in Computability Theory have been working on the problem of characterizing the algorithmic complexity of undecidable first order theories (the complexity is usually measured by the m-degree [Ro67, So87] of the theory that in this context coincides with the 1-1-degree, i.e. with the type of computable isomorphism of the theory). For many natural structures $\mathbb{A}$ with undecidable theory the theory $FO(\mathbb{A})$ turns out to be m-equivalent either to first order arithmetic $FO(\mathbb{N})$ or to second order arithmetic $FO(\mathbb{N}_2)$ where $\mathbb{N} := (\omega; +, \times)$ and $\mathbb{N}_2 := (\omega \cup P(\omega); \omega, P(\omega), \in, +, \times)$, see e.g. [NS80, NSS96, Ni98]. As is well known (see e.g. [Ro67]), $FO(\mathbb{N})$ is m-equivalent to the ω'th iteration $\emptyset^{(\omega)}$ of the Turing jump starting from the empty set.

Decidability issues for topological spaces seem to have been studied less systematically than for structures arising in algebra, logic and discrete mathematics, probably because first order language is not well suited for topology. Nevertheless, there was some important work for structures related to a topological space X, the most natural of which is the lattice

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$\mathbf{\Sigma}^0_1(X)$ of open sets in X. To our knowledge, A. Grzegorczyk [Gr51] was the first to consider decidability issues in topology. One of the results in [Gr51] (Corollary 2) interprets first order arithmetic in $\mathbf{\Sigma}^0_1(\mathbb{R}^n)$ for each $n \geq 2$ which implies that $FO(\mathbb{N})$ is m-reducible to $FO(\mathbf{\Sigma}^0_1(\mathbb{R}^n))$ and hence the latter theory is undecidable. The question of whether $FO(\mathbf{\Sigma}^0_1(\mathbb{R}))$ is decidable was left open. M. Rabin [Ra69] answered the question affirmatively (as well as the analogous question for the Cantor and Baire spaces) as a corollary of his result on decidability of the monadic second order theory of the binary tree.

A systematic model-theoretic study of structures arising in a topological setting was undertaken in [HJRT77] where it is shown, in particular, that $FO(\mathbb{N}_2) \leq_m FO(\mathbf{\Sigma}^0_1(X))$ for many Hausdorff spaces X, and the above mentioned Grzegorczyk's estimate was improved to $FO(\mathbb{N}_2) \equiv_m FO(\mathbf{\Sigma}^0_1(\mathbb{R}^n))$ for each $n \geq 2$. Note that in fact the papers [Gr51, HJRT77] work with the lattices of closed sets rather than with the lattices of opens but for our purposes this is clearly equivalent.

Another facet of the relationship between Topology and Computability Theory is the study of effectivity in a topological setting as developed in Computable Analysis [Wei00] and Effective Descriptive Set Theory [Mo09, Se06]. An important object of study here is the lattices $\Sigma^0_1(X)$ of the so called effectively open sets in topological spaces X satisfying some effectivity conditions (see the next section for more details). The lattice $\Sigma^0_1(X)$ is certainly the most important sublattice of the lattice $\mathbf{\Sigma}^0_1(X)$ of open sets, hence it is natural and instructive to study also definability and (un)decidability issues for the lattices of effectively open sets.

This study is interesting and non-trivial even for the discrete space ω of natural numbers since in this case the lattice $\Sigma^0_1(\omega)$ coincides with the lattice $\mathcal{E}$ of computably enumerable (c.e.) subsets of ω which is an important and popular object of study in Computability Theory [Ro67, So87]. A principal fact about this lattice is the undecidability of $FO(\mathcal{E})$ [He83, He84]. Moreover, $FO(\mathcal{E})$ is known [HN98] to be m-equivalent to first-order arithmetic $FO(\mathbb{N})$. Note that the first order theory of $\mathbf{\Sigma}^0_1(\omega) = P(\omega)$ is decidable (because $P(\omega)$ is a Boolean algebra).

It seems that not much is known about (un)decidability of first order theories of the lattices of effectively open sets except for what is known about the lattice $\Sigma^0_1(\omega)$ and its relativizations. To our knowledge, only the cases of the Cantor space $\mathcal{C}$ and the Baire space $\mathcal{N}$ have been studied to some extent, in the context of the theory of Π^0_1-classes. In [Ni00] (see the discussion of Main Theorem in the Introduction and Section 3) it is shown that $FO(\Pi^0_1(\mathcal{C}))$ is m-equivalent to $FO(\mathbb{N})$. Since the lattice $\Pi^0_1(\mathcal{C})$ formed by the complements of effectively open sets is anti-isomorphic to $\Sigma^0_1(\mathcal{C})$, this settles the question for $\mathcal{C}$. To our knowledge, similar questions for $\mathcal{N}$ (even the decidability of $FO(\Sigma^0_1(\mathcal{N}))$) are open.

In this paper, we make further steps in the study of (un)decidability issues for the theories of $\Sigma^0_1(X)$ and $\mathbf{\Sigma}^0_1(X)$. After recalling some necessary preliminaries we reprove in Section 3 the estimate from [HJRT77] $FO(\mathbb{N}_2) \equiv_m FO(\mathbf{\Sigma}^0_1(\mathbb{R}^n))$, $n \geq 2$, using the original approach of A. Grzegorczyk which is different from the approach in [HJRT77]. We also establish the same estimate for some natural domains. In Section 4 we first show that for many natural effective spaces X (including computable metric spaces without isolated points and many natural computable domains) the theory $FO(\Sigma^0_1(X))$ is undecidable. Then we show that $FO(\Sigma^0_1(\mathbb{R}^n))$, $n \geq 1$, is m-equivalent to first order arithmetic. The same estimate also holds for some natural domains. We conclude in Section 4.3 with a discussion of remaining open questions.

The methods of this paper apply mainly to second countable locally compact spaces. A precise estimate of the complexity of $FO(\mathbf{\Sigma}^0_1(X))$ and $FO(\Sigma^0_1(X))$ turns out to be subtle and depends strongly on the topology of X. For many natural spaces X we still have a big gap between the known lower and upper bounds for $FO(\mathbf{\Sigma}^0_1(X))$ and $FO(\Sigma^0_1(X))$. In particular, for the Baire space we currently only know the estimate $\emptyset' \leq_m FO(\Sigma^0_1(X)) \leq_m O^{(\omega)}$ where O is the Kleene ordinal notation system which is a Π^1_1-complete set.

Our upper bounds for the m-degree of $FO(\mathbf{\Sigma}^0_1(X))$ and $FO(\Sigma^0_1(X))$ are obtained by a straightforward application of the Tarski-Kuratowski algorithm, while the lower bounds use a suitable interpretation of one of the structures $\mathcal{E}, \mathbb{N}, \mathbb{N}_2$ in the lattice under consideration.

This paper is an extended version of the conference paper [KS16] that contains, in particular, new results on the lattices of all open sets and an essentially modified proof of Theorem 4.4 for $n > 1$.

2. Preliminaries

Here we briefly recall some notions and notation relevant to this paper. We freely use the standard set-theoretic notation like $|X|$ for the cardinality of X, $X \times Y$ for the cartesian product of sets and topological spaces, $P(X)$ for the set of all subsets of X, $\overline{A}$ for the complement $X \setminus A$ of a subset A of a space X.

We assume the reader to be familiar with basic notions of topology (see e.g. [En89]). We often abbreviate "topological space" by "space". By $Cl(S)$ (resp. $Int(S)$) we denote the closure (resp. the interior) of a set $S \subseteq X$ in a space X. A space X is *Polish* if it is separable and metrizable with a metric d such that (X, d) is a complete metric space. We denote the set of open subsets of a space X by $\mathbf{\Sigma}^0_1(X)$. This is the first among the finite levels $\{\mathbf{\Sigma}^0_n(X)\}$ of the Borel hierarchy [Ke95, dBr13] which is formed by applying the operations of complementation and countable union to the open sets.

Let ω be the space of non-negative integers with the discrete topology. The space $\omega \times \omega = \omega^2$ is homeomorphic to ω, the homeomorphism being realized by the Cantor pairing function $\langle x, y\rangle$.

Let ω^ω be the set of all infinite sequences of natural numbers (i.e., of all functions $\xi : \omega \to \omega$). Let $\omega^{<\omega}$ be the set of finite sequences of elements of ω, including the empty sequence. For $\sigma \in \omega^{<\omega}$ and $\xi \in \mathcal{N}$, we write $\sigma \sqsubseteq \xi$ to denote that σ is an initial segment of the sequence ξ. By $\sigma\xi = \sigma \cdot \xi$ we denote the concatenation of σ and ξ, and by $\sigma \cdot \mathcal{N}$ the set of all extensions of σ in $\mathcal{N}$. For $x \in \omega^\omega$, we can write $x = x(0)x(1)\cdots$ where $x(i) \in \omega$ for each $i < \omega$. For $x \in \mathcal{N}$ and $n < \omega$, let $x[n] = x(0)\cdots x(n-1)$ denote the initial segment of x of length n. Notations in the style of regular expressions like 0^ω, $0^{<\omega}1$ or 0^m1^n have the obvious standard meaning. Define the topology on ω^ω by taking arbitrary unions of sets of the form $\sigma \cdot \omega^\omega$, where $\sigma \in \omega^{<\omega}$, as the open sets. The space $\mathcal{N} = \omega^\omega$ with this topology known as the *Baire space*, is of primary importance for Descriptive Set Theory and Computable Analysis.

For any finite alphabet A (usually we assume without loss of generality that $A = k = \{0, \ldots, k-1\}$ where $0 < k < \omega$), let A^ω be the set of ω-words over A. This set may be topologized similarly to the Baire space. The resulting spaces, which for $k \geq 2$ are all (computably) homeomorphic among themselves, are known as *Cantor spaces* (usually the term Cantor space is applied to the space $\mathcal{C} = 2^\omega$ of infinite binary sequences). Note that the Cantor space is compact while the Baire space is not.

Next we recall some definitions related to domain theory (for more details see e.g. [AJ94, Er93, GH+03]).

Let X be a T_0-space. For $x, y \in X$, let $x \leq y$ denote that $x \in U$ implies $y \in U$, for all open sets U. The relation $\leq$ is a partial order known as the *specialization order*. Let $F(X)$ be the set of *finitary elements* of X (known also as *compact elements*), i.e. elements $p \in X$ such that the upper cone $\uparrow p = \{x \mid p \leq x\}$ is open. Such open cones are called *f-sets*. The space X is called a *φ-space* if every open set is a union of f-sets. Note that every non-discrete φ-space is not Hausdorff. A φ-space is *complete* if any non-empty directed set has a supremum w.r.t. the specialization order.

A φ-space X is an *f-space* if any compatible elements $c, d \in F(X)$ have a least upper bound w.r.t. $\leq$ (compatibility means that c, d have an upper bound in $F(X)$). An f-space X is an *f_0-space* if $F(X)$ has a least element.

Let $\omega^{\leq\omega} = \omega^{<\omega} \cup \omega^{\omega}$ be the set of all finite and infinite strings of natural numbers with the topology generated by the sets $\{x \mid u \sqsubseteq x\}$ where $u \in \omega^{<\omega}$. For every $1 \leq k < \omega$, the space $k^{\leq\omega}$ of all finite and infinite words over the alphabet $\{0, \ldots, k-1\}$ is defined in the same way. Let $P\omega$ be the powerset of ω with the topology generated by the sets $\uparrow F := \{A \mid F \subseteq A \subseteq \omega\}$ where F is a finite subset of ω.

Let $\omega_\perp = \omega \cup \{\perp\}$ be the the space with the topology generated by $\{n\}$, $n \in \omega$. Let $\omega_\perp^\omega$ be the space of partial functions on ω with the topology generated by the sets $\{g \mid f \subseteq g\}$ where f is a function with finite graph and $\subseteq$ is the subgraph relation (as usual, we identify a partial function g on ω with the total function $\tilde{g} : \omega \to \omega_\perp = \omega \cup \{\perp\}$ where $g(x)$ is undefined iff $\tilde{g}(x) = \perp$). For each k, $2 \leq k < \omega$, let $k_\perp^\omega$ be the space of partial functions $g : \omega \rightharpoonup \{0, \ldots, k-1\}$ defined similarly to $\omega_\perp^\omega$.

As is well known, $\omega^{\leq\omega}, k^{\leq\omega}, P\omega, \omega_\perp, \omega_\perp^\omega, k_\perp^\omega$ are complete f_0-spaces where the sets of f-elements are respectively $\omega^{<\omega}$, $k^{<\omega}$,the finite subsets of ω, $\omega_\perp$, the finite partial functions on ω, the finite partial functions from ω to $\{0, \ldots, k-1\}$.

As is well known (see e.g. [Er72]), for any (complete) f_0-spaces X, Y the space Y^X of continuous functions from X to Y with the topology of pointwise convergence is again a (complete) f_0-space. Therefore, any space of continuous partial functionals over ω of a finite type is a complete f_0-space. In particular, this applies to the spaces $\mathbb{F}_n$ defined by induction as $\mathbb{F}_0 := \omega_\perp$, $\mathbb{F}_{n+1} := \omega_\perp^{\mathbb{F}_n}$.

Next we explain what we mean by effectively open sets. For any countably based topological space X and any numbering β of a base of X, define a function $\pi : \omega \to P(X)$ by $\pi(n) = \bigcup \beta[W_n]$ where $\{W_n\}$ is the standard numbering of the c.e. sets [Ro67, So87]) and $\beta[W_n] = \{\beta(a) \mid a \in W_n\}$. The sets in $\pi[\omega]$ are called *effectively open sets in X*. Thus, the set of effectively open sets $\Sigma_1^0(X)$ is always equipped with the induced numbering π, hence it makes sense to speak about computable sequences of effectively open sets.

For many reasonable spaces X with effectivity conditions, one can define in a natural way the (finite levels of the) effective Borel hierarchy $\{\Sigma_n^0(X)\}$ and the effective Luzin hierarchy $\{\Sigma_n^1(X)\}$ (see e.g. [Mo09, Se06, Se15] for details) which are reasonable effective versions of the classical Borel and Luzin hierarchies. (Note that the definition of effective hierarchies in a given effective space depends on the chosen numbering of a base of the space.) In particular, $\{\Sigma_n^0(\omega)\}$ and $\{\Sigma_n^1(\omega)\}$ (taken with a natural numbering of a base in ω) coincide with the arithmetical and analytical hierarchies of subsets of ω which are central objects of study in Computability Theory [Ro67].

We define some particular classes of effective spaces relevant to this paper. A *computable metric space* [Wei00] is a triple (X, d, ν), where (X, d) is a metric space and $\nu : \omega \to X$ is a numbering of a dense subset $rng(\nu)$ of X such that the set

$$\{(i,j,k,l) \mid \varkappa_k < d(\nu(i),\nu(j)) < \varkappa_l\}$$

is c.e. Here $\varkappa$ is the conventional numbering of the set $\mathbb{Q}$ of rationals. Any computable metric space (X,d,ν) gives rise to a numbering β of the standard base $\beta_{\langle m,n\rangle} = B(\nu_m,\varkappa_n)$ where $\langle m,n\rangle$ is the Cantor pairing and $B(\nu_m,\varkappa_n)$ is the basic open ball with center ν_m and radius $\varkappa_n$ (if $\varkappa_n \leq 0$ the "ball" is empty).

By a *strongly computable metric space* (SCMS) we mean a computable metric space such that there exists an infinite computable sequence $\{B_n\}$ of pairwise disjoint basic open balls. The metric spaces ω, $\mathbb{Q}$, $\mathcal{C}$, $\mathcal{N}$, $\mathbb{R}^n$ (where $\mathbb{R}$ is the space of real numbers) equipped with the standard metrics and with natural numberings of dense subsets are SCMS. Any computable metric space without isolated points is an SCMS. Working with the Euclidean spaces $\mathbb{R}^n$, we denote by d the Euclidean metric, by 0 the zero-vector $(0,\ldots,0) \in \mathbb{R}^n$, non-empty open (resp. closed) rational balls by $B(a,r)$ (resp. $C(a,r)$) where $a \in \mathbb{Q}^n, r \in \mathbb{Q}^+$. Sometimes it is convenient to use also the empty ball $B(a,0)$.

By a *computable φ-space* we mean a pair (X,δ) consisting of a φ-space X and a numbering $\delta : \omega \to F(X)$ of all the finitary elements such that the specialization order is c.e. on the finitary elements (i.e., the relation $\delta_x \leq \delta_y$ is c.e.). Setting $\beta(n) := \uparrow\delta_n$ we obtain a numbering of a topological base of X. Thus, we have a notion of an effective open set in every computable φ-space.

By a *strongly computable φ-space* (SCΦS) we mean a computable φ-space X such that the specialization order is computable on the finitary elements, and there is a computable sequence $\{c_n\}$ of pairwise incomparable finitary elements. An SCΦS X is a *strongly computable f_0-space (SCF$_0$S)* if it is an f_0-space, the relation of compatibility is computable on $F(X)$, and the supremum of compatible finitary elements is computable. Although the restrictions imposed on SCΦSs and SCF$_0$Ss are rather strong, many popular domains are SCF$_0$Ss. In particular, this applies to all concrete examples of φ-spaces mentioned above in this section. (For the space Y^X of continuous functions, a close inspection of the corresponding proofs [Er72] shows that if X, Y are (complete) SCF$_0$Ss then so is Y^X. Therefore, any space of continuous partial functionals over ω of a finite type is a complete SCF$_0$S.) Note that the "strong" variations above are rather ad hoc and do not pretend to be fundamental notions in the field.

We conclude this section by briefly recalling of some notions from logic. We consider only structures of finite relational signatures (when a functional symbol is used, as e.g. in the structure $\mathbb{N}$, we identify the corresponding function with its graph). For a σ-structure $\mathbb{A} = (A;\sigma)$, a relation $R \subseteq A^k$ is *definable in* $\mathbb{A}$ if there is a first order σ-formula $\phi(x_1,\ldots,x_k,p_1,\ldots,p_l)$ and (possibly) some values $p_1,\ldots,p_l \in A$ of parameters such that

$$R = \{(x_1,\ldots,x_k) \in A^k \mid \mathbb{A} \models \phi(x_1,\ldots,x_k,p_1,\ldots,p_l)\}.$$

If the list of parameters is empty then we speak about *definability without parameters*. Thus, $R \subseteq A^k$ is *definable in* $\mathbb{A}$ *without parameters* if there is a first order σ-formula $\phi(x_1,\ldots,x_k)$ with

$$R = \{(x_1,\ldots,x_k) \in A^k \mid \mathbb{A} \models \phi(x_1,\ldots,x_k)\}.$$

A function on A is definable (with or without parameters) if its graph is definable. An element of A is definable if the corresponding singleton set $\{a\}$ is definable. A structure is definable if its universe and all signature predicates are definable.

E.g., if $\mathbb{A} = (A; \cup, \cap, 0, 1)$ is a bounded distributive lattice then any of $\cup, \cap, 0, 1$ is definable without parameters in $(A; \leq)$ where $\leq$ is the induced partial order on A, and $\leq$ is definable in $(A, \cup)$. Moreover, in this case we can even speak about arbitrary Boolean terms of elements of A (meaning their values in a Boolean algebra extending $\mathbb{A}$). Thus, dealing with our lattices $\mathbf{\Sigma}^0_1(X)$ and $\Sigma^0_1(X)$ we can mean any of the signatures $\{\subseteq\}$, $\{\cup, \cap, \emptyset, X\}$, or even $\{\cup, \cap, \bar{}, \emptyset, X\}$. For simplicity, we omit the signature symbols in the notation of these structures. We use in our formulas some standard abbreviations, in particular the bounded quantifiers $(\forall x)_{\phi(x)}\psi := \forall x(\phi(x) \to \psi)$ and $(\exists x)_{\phi(x)}\psi := \exists x(\phi(x) \wedge \psi)$ or the "quantifier" $\exists! x\psi$ meaning "there exists a unique x satisfying ψ".

The first order theory $FO(\mathbb{A})$ of the structure $\mathbb{A}$ is the set of σ-sentences true in $\mathbb{A}$. Along with first order logic, in logical theories some other logics are considered, in particular the (monadic) second order logic where one can use, along with the usual variables, also variables ranging over the (unary) relations on A. Accordingly, one can consider the monadic second order theory $MSO(\mathbb{A})$, or the full second order theory $SO(\mathbb{A})$, of $\mathbb{A}$ (in the latter case one needs variables for relations of any arity). Since the Cantor coding is definable in $\mathbb{N}$ without parameters, $SO(\mathbb{N}) \equiv_m MSO(\mathbb{N})$. The theory $MSO(\mathbb{A})$ may be considered as the first order theory $FO(\mathbb{A}^{mso})$ of the extended structure

$$\mathbb{A}^{mso} := (A \cup P(A); A, P(A), \in, \sigma)$$

obtained from $\mathbb{A}$ by adjoining the powerset of A to the universe, the unary predicates for A and $P(A)$, and the membership relation to the list of relations. In particular, we have $\mathbb{N}_2 = \mathbb{N}^{mso}$ and $SO(\mathbb{N}) \equiv_m FO(\mathbb{N}_2)$.

An important tool to compare algorithmic complexity of theories is the notion of interpretability of one theory or structure in another. In fact, there are many versions of this notion (see e.g. [TMR53, ELTT65, Er80] of which we briefly recall a couple of those used in the sequel.

A τ-structure $\mathbb{B}$ is *interpretable in a σ-structure $\mathbb{A}$ without parameters* if some isomorphic copy of $\mathbb{B}$ is definable in $\mathbb{A}$ without parameters. A weaker version of this is the notion of c-interpretability (where c comes from "congruence"). We say that $\mathbb{B}$ is c-interpretable in $\mathbb{A}$ without parameters if there exist a τ-structure $\mathbb{C}$ and a congruence $\sim$ on $\mathbb{C}$ such that both $\mathbb{C}$ and $\sim$ are definable in $\mathbb{A}$ without parameters and the quotient-structure $\mathbb{C}/{\sim}$ (whose elements are the equivalence classes $c/{\sim}, c \in C$) is isomorphic to $\mathbb{B}$. Interpretability with parameters is introduced in the same manner.

As is well known (see e.g. [TMR53, ELTT65]), if $\mathbb{B}$ is c-interpretable in $\mathbb{A}$ without parameters then $FO(\mathbb{B}) \leq_m FO(\mathbb{A})$. The same is true for definability with parameters provided that the set of "defining" parameters may be chosen definable. The latter notion means that there is a non-empty definable (without parameters) set $P \subseteq A^l$ of parameters such that for any value of parameters in P the corresponding structure is isomorphic to $\mathbb{B}$. For future reference, we formulate some of the mentioned facts as a lemma.

Lemma 2.1. *Let $\mathbb{B}$ be c-interpretable in $\mathbb{A}$ without parameters (or with a non-empty set of parameters which is itself definable without parameters). Then $FO(\mathbb{B}) \leq_m FO(\mathbb{A})$.*

When no non-empty set of eligible parameters is definable, the relation $FO(\mathbb{B}) \leq_m FO(\mathbb{A})$ is not true in general but there is some version of undecidability which is preserved also by such interpretations. A theory (not necessarily complete) of signature σ is *hereditarily undecidable* if any of its subtheories of signature σ is undecidable. It is well known (see e.g. [Er80]) that if $FO(\mathbb{B})$ is hereditarily undecidable and $\mathbb{B}$ is c-interpretable in $\mathbb{A}$ with parameters then $FO(\mathbb{A})$ is hereditarily undecidable.

Additional information about interpretations may be found on page 215 of [Ho93].

3. The lattices of open sets

Here we give precise estimates of the algorithmic complexity of $FO(\mathbf{\Sigma}^0_1(X))$ for some spaces X.

3.1. **An upper bound.** First we establish a natural upper bound that applies to many countably based locally compact spaces. We need a technical notion related to local compactness. By *analytically locally compact space* (AnLCS) we mean a triple (X, β, κ) consisting of a topological space X, a numbering β of a base in X containing the empty set (the presence of the empty set is not principal and maybe removed by using slight modification of the notion of AnLCS), and a numbering κ of some compact sets in X such that any set β_n is a union of some sets in $\{\kappa_i \mid i < \omega\}$, and the relation $\kappa_i \subseteq \bigcup \beta[D_n]$ (where $\{D_n\}$ is the canonical numbering of finite subsets of ω [Ro67]) is analytical, i.e. it is in $\bigcup_n \Sigma^1_n(\omega)$.

Note that although AnLCSs are not automatically locally compact, many locally compact spaces may be considered as AnLCSs. In particular, the computable φ-spaces, the finite dimensional Euclidean spaces, and the Cantor space are AnLCSs (for instance, for a computable φ-space (X, δ) we can set $\kappa_n := \beta_n := \uparrow\delta_n$ which is compact; the relation $\kappa_i \subseteq \bigcup \beta[D_n]$ in this case is c.e.).

Proposition 3.1. *If (X, β, κ) is an AnLCS then $FO(\mathbf{\Sigma}^0_1(X)) \leq_m FO(\mathbb{N}_2)$.*

Proof. Define a surjection $\tau : \mathcal{N} \to \mathbf{\Sigma}^0_1(X)$ by $\tau(p) = \bigcup_n \beta_{p(n)}$. The surjection τ has several nice properties, in particular it is an admissible representation of the hyperspace of open sets in X (see e.g. [Se13] for additional details).

It suffices to show that the relation $\tau(p) \subseteq \tau(q)$ is an analytical subset of $\mathcal{N} \times \mathcal{N}$ because then the elementary diagram of the represented structure $(\mathbf{\Sigma}^0_1(X); \subseteq, \tau)$, and hence also $FO(\mathbf{\Sigma}^0_1(X))$, are m-reducible to $FO(\mathbb{N}_2)$.

Obviously, $\tau(p) \subseteq \tau(q)$ is equivalent to $\forall n(\kappa_n \subseteq \tau(p) \to \kappa_n \subseteq \tau(q))$, hence it suffices to show that the relation $\kappa_n \subseteq \tau(p)$ is analytical. We have $\kappa_n \subseteq \tau(p)$ iff $\kappa_n \subseteq \bigcup_i \beta_{p(i)}$ iff $\exists m(\kappa_n \subseteq \beta_{p(0)} \cup \cdots \cup \beta_{p(m)})$, by compactness of κ_n. The set

$$\{(m, n, p) \mid \kappa_n \subseteq \beta_{p(0)} \cup \cdots \cup \beta_{p(m)}\})$$

is, by the definition of AnLCS, an analytical subset of $\omega \times \omega \times \mathcal{N}$, hence the relation $\kappa_n \subseteq \tau(p)$ is analytical. $\square$

3.2. **Lattices of opens in Euclidean spaces.** The next result improves the estimate from [Gr51] mentioned in the Introduction, providing a different proof of this result compared with [HJRT77].

Theorem 3.2. *For any $n \geq 2$, $FO(\mathbf{\Sigma}^0_1(\mathbb{R}^n)) \equiv_m FO(\mathbb{N}_2)$.*

Proof. Since the upper bound holds by Proposition 3.1, we only have to prove the lower bound $FO(\mathbb{N}_2) \leq_m FO(\mathbf{\Sigma}^0_1(\mathbb{R}^n))$. For this, we extend Grzegorczyk's interpretation [Gr51] of $\mathbb{N}$ in $\mathbf{\Sigma}^0_1(\mathbb{R}^n)$ to an interpretation of $\mathbb{N}_2$ in $\mathbf{\Sigma}^0_1(\mathbb{R}^n)$. We start with a brief sketch of the Grzegorczyk interpretation (with slightly different notation).

Since $\mathbf{\Sigma}^0_1(\mathbb{R}^n)$ is a distributive lattice, we can use in the definitions not only the symbol of inclusion but also the symbols of Boolean operations and the constants $\emptyset, \mathbb{R}^n$. Note that for any $x \in \mathbb{R}^n$ the set $\mathbb{R}^n \setminus \{x\}$ is open, and the class of such co-singleton sets is definable in $\mathbf{\Sigma}^0_1(\mathbb{R}^n)$ as the class of sets maximal w.r.t. inclusion among the sets strictly below $\mathbb{R}^n$. We also use some other observations from [HJRT77].

The key observation of Grzegorczyk was the definability in $\mathbf{\Sigma}^0_1(\mathbb{R}^n)$ (without parameters) of the set Cof of cofinite subsets of $\mathbb{R}^n$, as well as of the following relations on Cof:

- $U \approx V$ iff $|\overline{U}| = |\overline{V}|$ where $\overline{U} := \mathbb{R}^n \setminus U$,
- $P_+(U, V, W)$ iff $|\overline{U}| + |\overline{V}| = |\overline{W}|$,
- $P_\times(U, V, W)$ iff $|\overline{U}| \times |\overline{V}| = |\overline{W}|$.

Grzegorczyk's interpretation is now given by the natural isomorphism $k \mapsto U_k/{\approx}$ from $\mathbb{N}$ onto the quotient of $(Cof; P_+, P_\times)$ modulo the definable congruence $\approx$ where $U_k := \mathbb{R}^n \setminus \{(i, \dots, i) \mid i < k\}$.

The definability of $Cof, \approx, P_+, P_\times$ makes heavy use of the notions of connected open sets and open connected components which are definable, respectively, by the formulas

$$Con(U) := U \neq \emptyset \wedge \neg \exists V, W(U \subseteq V \cup W \wedge V \cap W = \emptyset \wedge U \cap V \neq \emptyset \wedge U \cap W \neq \emptyset)$$

and

$$Cmp(U, V) := U \subseteq V \wedge Con(U) \wedge \forall U'(U' \subseteq V \wedge Con(U') \wedge U \cap U' \neq \emptyset \to U' \subseteq U).$$

Let now $\xi(U, V)$ be a formula saying in $\mathbf{\Sigma}^0_1(\mathbb{R}^n)$ that U is coinfinite, V is cofinite, $U \cap F = \emptyset$ (where F is the finite set of all points in the complement of V), $U \cup F$ is open, and V is the smallest element of $(Cof; \subseteq)$ with these properties. In other words, $\mathbf{\Sigma}^0_1(\mathbb{R}^n) \models \xi(U, V)$ iff U is obtained from the open set $U \cup F$ by removing the points from F; Note that, by the minimality of V, for any coinfinite open set U there is at most one cofinite open set V with $\mathbf{\Sigma}^0_1(\mathbb{R}^n) \models \xi(U, V)$.

Let $\mathcal{P}$ be the definable subset of $\mathbf{\Sigma}^0_1(\mathbb{R}^n)$ formed by the coinfinite sets W satisfying $(\forall U)_{Cmp(U,W)} \exists V \xi(U, V)$. Then $\mathcal{P}$ is disjoint with Cof and we can associate with any $W \in \mathcal{P}$ the set

$$A_W := \{k \in \omega \mid (\exists U)_{Cmp(U,W)} \exists V(\xi(U, V) \wedge |\overline{V}| = k)\}.$$

Note that for any $A \subseteq \omega$ there is $W = W(A) \in \mathcal{P}$ with $A = A_W$. Indeed, we can take the set $W := \bigcup\{S_a \mid a \in A\}$ where S_a is obtained from the open set $(a, a+1) \times \mathbb{R}^{n-1}$ by removing a points (note that S_a are the connected components of W). Note also that the relations

$$\epsilon(V, W) \Leftrightarrow V \in Cof \wedge W \in \mathcal{P} \wedge |\overline{V}| \in A_W$$

and

$$W \equiv W' \Leftrightarrow W \in \mathcal{P} \wedge W' \in \mathcal{P} \wedge A_W = A_{W'}$$

are definable in $\mathbf{\Sigma}^0_1(\mathbb{R}^n)$ without parameters.

These remarks mean that the maps $k \mapsto U_k/{\approx}$, $A \mapsto W(A)/{\equiv}$ give an isomorphism from $\mathbb{N}_2$ onto the quotient of $(Cof \cup \mathcal{P}; Cof, \mathcal{P}, \epsilon, P_+, P_\times)$ modulo the definable congruence on $Cof \cup \mathcal{P}$ induced by $\approx$ and $\equiv$. Thus, $\mathbb{N}_2$ is c-interpretable in $\mathbf{\Sigma}^0_1(\mathbb{R}^n)$ without parameters. By Lemma 2.1, $FO(\mathbb{N}_2) \leq_m FO(\mathbf{\Sigma}^0_1(\mathbb{R}^n))$. $\square$

3.3. **Lattices of opens in domains.** Here we give a similar estimate for some natural domains.

Theorem 3.3. *For any* $X \in \{P\omega, k_\perp^\omega \mid 2 \leq k \leq \omega\}$, $FO(\mathbf{\Sigma}_1^0(X)) \equiv_m FO(\mathbb{N}_2)$.

Proof. First consider the space $P\omega$ (which is homeomorphic to $1_\perp^\omega$). Since the upper bound holds by Proposition 3.1, we only have to prove the lower bound $FO(\mathbb{N}_2) \leq_m FO(\mathbf{\Sigma}_1^0(P\omega))$. By an observation of R. Robinson mentioned in the beginning of the proof of Lemma 7.2 in [HJRT77], $\times$ is definable in $(\omega; +)$ by a monadic second order formula, so if suffices to prove $FO(\mathbb{N}_2') \leq_m FO(\mathbf{\Sigma}_1^0(P\omega))$ where $\mathbb{N}_2' := (\omega \cup P(\omega); \omega, P(\omega), \in, +)$.

We show that the class $\mathcal{F}$ of f-sets (i.e., sets of the form $\uparrow F$ where $F \subseteq \omega$ is a finite set), is definable in $\mathbf{\Sigma}_1^0(P\omega)$ without parameters. Indeed, a defining formula is

$$ir(V) := V \neq \emptyset \wedge \forall U, U'(V \subseteq U \cup U' \to V \subseteq U \vee V \subseteq U')$$

which says in $\mathbf{\Sigma}_1^0(P\omega)$ that V is a non-zero join-irreducible element. Obviously, any set $V = \uparrow F$ satisfies this formula. Conversely, let an element $V \in \mathbf{\Sigma}_1^0(P\omega)$ satisfy the formula. Since $V \neq \emptyset$, for some sequence $\{F_n\}$ of finite sets we have $V = \bigcup_n \uparrow F_n$. Since the partial order $(\{F_n \mid n < \omega\}; \subseteq)$ is well founded, it has a minimal element F. Then of course $\uparrow F \subseteq V$, so it suffices to show that $V \subseteq \uparrow F$. Let $S := \bigcup\{\uparrow F_n \mid F \not\subseteq F_n\}$. Since $S \in \mathbf{\Sigma}_1^0(P\omega)$, $V \subseteq \uparrow F \cup S$ and V is join-irreducible, it suffices to show that $V \not\subseteq S$. Suppose the contrary, then $F \in S$, so $F_n \subseteq F$ for some n with $F \not\subseteq F_n$, contradicting the minimality of F.

Let $\mathcal{G}$ be the set of finite subsets of ω. Since $(\mathcal{G}; \subseteq)$ is isomorphic to $(\mathcal{F}; \supseteq)$ via $G \mapsto \uparrow G$, we can interpret $\mathbb{N}_2'$ in $\mathbf{\Sigma}_1^0(P\omega)$ similarly to the previous proof. Define the relations $\approx, P_+$ on $\mathcal{F}$ as follows:

- $\uparrow F \approx \uparrow G$ iff $|F| = |G|$,
- $P_+(\uparrow F, \uparrow G, \uparrow H)$ iff $|F| + |G| = |H|$.

We show that $\approx, P_+, P_\times$ are definable in $\mathbf{\Sigma}_1^0(P\omega)$ without parameters.

First we show that the set $\mathcal{V}_n := \{\uparrow F : |F| = n\}$ is definable in $\mathbf{\Sigma}_1^0(P\omega)$ for each $n < \omega$. A sequence $\{\phi_n(V)\}$ of defining formulas is given by induction on n as follows: $\phi_0(V) := \forall U(U \subseteq V)$ (saying that V is the largest element of a lattice), and

$$\phi_{n+1}(V) := \neg\phi_0(V) \wedge \cdots \wedge \neg\phi_n(V) \wedge (\forall U)_{ir(V)}(V \subset U \to \phi_0(V) \vee \cdots \vee \phi_n(V))$$

(saying in $\mathbf{\Sigma}_1^0(P\omega)$ that V is a maximal join-irreducible element among those not in the set defined by the formula $\phi_0(V) \vee \cdots \vee \phi_n(V)$). By induction on n, ϕ_n defines $\mathcal{V}_n$ for each n.

Note that any $U \in \mathbf{\Sigma}_1^0(P\omega)$ is uniquely representable as the union of its maximal (w.r.t. inclusion) f-subsets. Clearly, the relation $Maxf(C, U)$ meaning that "C is a maximal f-subset of U" is definable in $\mathbf{\Sigma}_1^0(P\omega)$.

Let $A \approx_U B$ be the definable (in $\mathbf{\Sigma}_1^0(P\omega)$) relation "$A = \uparrow F, B = \uparrow G \in \mathcal{F}$, $F \cap G = \emptyset$, $|F \cap H| = |G \cap H| = 1$ for each maximal f-subset $\uparrow H$ of U, $H \cap H' = \emptyset$, for all distinct maximal f-subsets $\uparrow H, \uparrow H'$ of U, and $F \cup G \subseteq \bigcup\{H \mid Maxf(\uparrow H, U)\}$". For instance, $\uparrow\emptyset \approx_\emptyset \uparrow\emptyset$. Clearly, $\uparrow F \approx_U \uparrow G$ implies $|F| = |G|$, and for all $F, G \in \mathcal{F}$ we have: $|F| = |G|$ iff $|F \setminus G| = |G \setminus F|$ iff $\exists U(\uparrow(F \setminus G) \approx_U \uparrow(G \setminus F))$. This yields the definability of $\approx$.

Since $|F| + |G| = |H|$ iff there are disjoint $F', G' \in \mathcal{G}$ such that $|F'| = |F|$, $|D'| = |G|$ and $|F' \cup G'| = |H|$, the relation P_+ is definable.

Since $k \mapsto \uparrow\{0, \ldots, k-1\}/{\approx}$ is an isomorphism from $(\omega; +)$ onto the quotient of $(\mathcal{F}; P_+)$ modulo $\approx$, we get a c-interpretation of $(\omega; +)$ in $\mathbf{\Sigma}_1^0(P\omega)$ without parameters. As in the proof of the previous theorem, this interpretation can be easily extended to a c-interpretation of $\mathbb{N}_2$ in $\mathbf{\Sigma}_1^0(P\omega)$ without parameters.

Namely, let $\mathcal{P} := \mathbf{\Sigma}^0_1(P\omega)\setminus\mathcal{F}$, so any $W \in \mathcal{P}$ is either empty or has at least two maximal f-subsets. Then $\mathcal{P}$ is definable and we can associate with any $W \in \mathcal{P}$ the set

$$A_W := \{k-1 \in \omega \mid \exists K \in \mathcal{G}(Maxf(\uparrow K, W) \wedge k = |K|)\}.$$

Note that for any $A \subseteq \omega$ there is $W = W(A) \in \mathcal{P}$ with $A = A_W$. If A is empty, we have to take $W = \emptyset$. Otherwise, fix $a_0 \in A$ and choose a countable partition $Q, R_0, R_1, \dots$ of ω into infinite sets with $Q = \{q(0) < q(1) < \cdots\}$ and $R_i = \{r_i(0) < r_i(1) < \cdots\}$; now it suffices to set

$$W(A) := \uparrow\{q(0), \dots, q(a_0)\} \cup \bigcup\{\uparrow\{r_a(0), \dots, r_a(a)\} \mid a \in A\}.$$

Furthermore, the relations

$$\epsilon(\uparrow F, W) \Leftrightarrow F \in \mathcal{G} \wedge W \in \mathcal{P} \wedge |F| \in A_W$$

and

$$W \equiv W' \Leftrightarrow W \in \mathcal{P} \wedge W' \in \mathcal{P} \wedge A_W = A_{W'}$$

are definable in $\mathbf{\Sigma}^0_1(P\omega)$ without parameters. Therefore, the maps $k \mapsto \uparrow\{0, \dots, k-1\}/{\approx}$, $A \mapsto W(A)/{\equiv}$ give an isomorphism from $\mathbb{N}_2'$ onto the quotient of $(\mathcal{F}\cup\mathcal{P}; \mathcal{F}, \mathcal{P}, \epsilon, P_+)$ modulo the definable congruence on $\mathcal{F}\cup\mathcal{P}$ induced by $\approx$ and $\equiv$. Thus, $\mathbb{N}_2'$ is c-interpretable in $\mathbf{\Sigma}^0_1(P\omega)$ without parameters. By Lemma 2.1, $FO(\mathbb{N}_2') \leq_m FO(\mathbf{\Sigma}^0_1(P\omega))$, completing the proof for $P\omega$.

Now consider the space $k^\omega_\perp$. Since again $FO(\mathbf{\Sigma}^0_1(k^\omega_\perp)) \leq_m FO(\mathbb{N}_2)$ by Proposition 3.1, it suffices to show $FO(\mathbb{N}_2) \leq_m FO(\mathbf{\Sigma}^0_1(k^\omega_\perp))$. By the just established estimate for $P\omega$, it suffices to show $FO(\mathbf{\Sigma}^0_1(P\omega)) \leq_m FO(\mathbf{\Sigma}^0_1(k^\omega_\perp))$, i.e., to interpret $\mathbf{\Sigma}^0_1(P\omega)$ in $\mathbf{\Sigma}^0_1(k^\omega_\perp)$ with a non-empty definable set of parameters.

For any total function $f : \omega \to k$, let A_f be the set of partial subfunctions of f. Then A_f is a closed subset of $k^\omega_\perp$ which, taken with the subspace topology, is homeomorphic to $1^\omega_\perp$, and hence also to $P\omega$. Therefore, $\mathbf{\Sigma}^0_1(P\omega)$ is isomorphic to $\mathbf{\Sigma}^0_1(A_f)$ for each $f : \omega \to k$, so it suffices to show that $\mathcal{A} := \{A_f \mid f : \omega \to k\}$ is definable in $\mathbf{\Sigma}^0_1(k^\omega_\perp)$ without parameters. This follows from the following assertion that is easy to check: $A \in \mathcal{A}$ iff A is a maximal (w.r.t. inclusion) closed subset of $k^\omega_\perp$ satisfying

$$\forall U, V \in \mathbf{\Sigma}^0_1(k^\omega_\perp)(U \cap A \neq \emptyset \wedge V \cap A \neq \emptyset \to U \cap V \cap A \neq \emptyset). \qquad \square$$

As mentioned in the Introduction, M. Rabin has shown that $FO(\mathbf{\Sigma}^0_1(X))$ is decidable for $X \in \{\mathcal{C}, \mathcal{N}, \mathbb{R}\}$. The same holds for $X \in \{[0,1], (0,1]\}$, with a slight modification of the proof in [Ra69]. We conclude this section with a natural variation of this for domains.

Proposition 3.4. *For any $1 \leq k \leq \omega$, $FO(\mathbf{\Sigma}^0_1(k^{\leq\omega}))$ is decidable.*

Proof. Relate to any $A \subseteq k^{<\omega}$ the open set $U_A := \bigcup_{a\in A} a \cdot k^{\leq\omega}$. Then $A \mapsto U_A$ is a surjection from $P(k^{<\omega})$ onto $\mathbf{\Sigma}^0_1(k^{\leq\omega})$. Let $\preceq$ be the preorder on $P(k^{<\omega})$ defined by: $A \preceq B$ iff $U_A \subseteq U_B$. Then the lattice $\mathbf{\Sigma}^0_1(k^{\leq\omega})$ is isomorphic to the quotient order of $(P(k^{<\omega}); \preceq)$ modulo the induced congruence $\approx$, hence $\mathbf{\Sigma}^0_1(k^{\leq\omega})$ is c-interpretable in $(P(k^{<\omega}); \preceq)$ without parameters and therefore $FO(\mathbf{\Sigma}^0_1(k^{\leq\omega})) \leq_m FO(P(k^{<\omega}); \preceq)$.

Since $A \preceq B$ iff $\forall a \in A(a \cdot k^{\leq\omega} \subseteq U_B)$ iff $\forall a \in A(a \in U_B)$ iff $\forall a \in A \exists b \in B(b \sqsubseteq a)$, the structure $(P(k^{<\omega}); \preceq)$ is interpretable in $(k^{<\omega}; \sqsubseteq)^{mso}$ without parameters and therefore $FO(P(k^{<\omega}); \preceq) \leq_m MSO(k^{<\omega}; \sqsubseteq)$. Since the last theory is decidable by [Ra69], $FO(\mathbf{\Sigma}^0_1(k^{\leq\omega}))$ is decidable. $\square$

4. The lattices of effectively open sets

In this section we examine algorithmic complexity of the first order theories of lattices of effectively open sets.

4.1. **Preliminary results.** We start by showing that for many natural effective spaces their theories of the effectively open sets are hereditarily undecidable.

Theorem 4.1. *Let X be an SCMS or an SCΦS. Then $FO(\Sigma^0_1(X))$ is hereditarily undecidable.*

Proof. Since the theory $FO(\mathcal{E})$ is hereditarily undecidable [He83, He84], it suffices to interpret $\mathcal{E}$ in $\Sigma^0_1(X)$ with parameters; in our case two parameters V, W will suffice.

Consider the formulas $\phi(U, V, W) := V \subseteq U \wedge U \subseteq W$ and

$$\phi_{\subseteq}(U, U', V, W) := \phi(U, V, W) \wedge \phi(U', V, W) \wedge U \subseteq U'.$$

For any values $V, W \in \Sigma^0_1(X)$ for parameters with $V \subseteq W$, the formulas define the sublattice $\mathcal{D}$ of $\Sigma^0_1(X)$ formed by the sets lying between V and W. Therefore, it suffices to find effectively open sets V, W such that the lattice $\mathcal{D}$ is isomorphic to $\mathcal{E}$.

Let first X be an SCMS. Let $\{B_n\}$ be the sequence of basic open balls from the definition of SCMS and let B'_n be obtained from the ball B_n by removing its center c_n. Then $\{B'_n\}$ is a computable sequence of effectively open sets, hence $V := \bigcup_n B'_n$ and $W := \bigcup_n B_n$ are effectively open and $W \setminus V = \{c_0, c_1, \ldots\}$. From the definition of SCMS it is easy to see that the function $D \mapsto \{n \mid c_n \in D\}$ is a desired isomorphism between $\mathcal{D}$ and $\mathcal{E}$.

Now let X be an SCΦS. Let $\{c_n\}$ be the sequence of finitary elements from the definition of SCΦS. This time we take as parameters the values $W := \bigcup_n \uparrow c_n$ and $V := \bigcup_n (\uparrow c_n \setminus \{c_n\})$. From the definition of SCΦS it is easy to see that $\{\uparrow c_n \setminus \{c_n\}\}$ is a computable sequence of effectively open sets, hence again V and W are effectively open and $W \setminus V = \{c_0, c_1, \ldots\}$. Moreover, the function $D \mapsto \{n \mid c_n \in D\}$ is again an isomorphism between $\mathcal{D}$ and $\mathcal{E}$. $\square$

Next we aim to obtain precise estimates of the algorithmic complexity of $FO(\Sigma^0_1(X))$ for some spaces X. First we establish a natural upper bound that applies to many countably based locally compact spaces. *Arithmetically locally compact spaces* (ArLCS) are defined precisely as AnLCS in the previous section but this time the relation $\kappa_i \subseteq \bigcup \beta[D_n]$ is required to be arithmetical, i.e. to be in $\bigcup_n \Sigma^0_n(\omega)$. Again, many locally compact spaces may be considered as ArLCSs. In particular, the computable φ-spaces, the finite dimensional Euclidean spaces, and the Cantor space are ArLCSs.

The following proposition is an easy variation of Proposition 3.1.

Proposition 4.2. *If (X, β, κ) is an ArLCS then $FO(\Sigma^0_1(X)) \leq_m FO(\mathbb{N})$.*

Proof. Recall that $\pi_i := \bigcup \beta[W_i]$ is the natural numbering of effectively open sets. It suffices to show that the relation $\pi_i \subseteq \pi_j$ is arithmetical because then the elementary diagram of the numbered structure $(\Sigma^0_1(X); \subseteq, \pi)$, and hence also $FO(\Sigma^0_1(X))$, is m-reducible to $FO(\mathbb{N})$.

Obviously, $\pi_i \subseteq \pi_j$ is equivalent to $\forall n(\kappa_n \subseteq \pi_i \to \kappa_n \subseteq \pi_j)$, hence it suffices to show that the relation $\kappa_n \subseteq \pi_i$ is arithmetical. We have: $\kappa_n \subseteq \pi_i$ iff $\kappa_n \subseteq \bigcup \beta[W_i]$ iff $\exists m(D_m \subseteq W_i \wedge \kappa_n \subseteq \bigcup \beta[D_m])$ (the last equivalence holds by compactness of κ_n). The last relation is arithmetical by the definition of ArLCS. $\square$

In the next subsection we give precise estimates of the m-degrees of $FO(\Sigma^0_1(\mathbb{R}^n))$ for which we need the following lemma. Recall that the definition of $\Sigma^0_1(\mathbb{R}^n))$ depends on the natural numbering of rational open balls in $\mathbb{R}^n$).

Lemma 4.3. *Any connected component of an effectively open set in $\mathbb{R}^n$ is effectively open.*

Proof. Let U be a connected component of $V \in \Sigma^0_1(\mathbb{R}^n)$ and let a be a rational point in U. Then

$$U = \bigcup \{B(b,r) \mid b \in \mathbb{Q}^n \wedge r \in \mathbb{Q}^+ \wedge \exists a_1 \cdots a_{m-1} \in \mathbb{Q}^n \exists r_1 \cdots r_{m-1} \in \mathbb{Q}^+ \\ (\bigwedge_{i=1}^{m} C(a_i,r_i) \subseteq V \wedge \bigwedge_{i=1}^{m-1} (B(a_i,r_i) \cap B(a_{i+1},r_{i+1}) \neq \emptyset) \wedge a \in B(a_1,r_1))\}$$

where $b = a_m$ and $r = r_m$. Since we can computably enumerate the basic closed balls $C(b,r) \subseteq V$ [KK07] with $b \in \mathbb{Q}^n, r \in \mathbb{Q}^+$, U is effectively open. □

4.2. **Effectively open sets in Euclidean spaces.** Now we prove the main result of this paper:

Theorem 4.4. *For any $n \geq 1$, $FO(\Sigma^0_1(\mathbb{R}^n)) \equiv_m FO(\mathbb{N})$.*

Since the upper bound holds by Proposition 4.2, we only have to prove the lower bound $FO(\mathbb{N}) \leq_m FO(\Sigma^0_1(\mathbb{R}^n))$. Since $\Sigma^0_1(\mathbb{R}^n)$ is a distributive lattice, we can use in the definitions not only the symbol of inclusion but also the symbols of Boolean operations and the constants $\emptyset, \mathbb{R}^n$.

Recall that a point $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ is computable iff any real $x_i, i = 1, \ldots, n$, is computable, i.e. $x_i = \lim_n p_n = \lim_n q_n$ for some computable sequences $\{p_n\}, \{q_n\}$ of rationals with $p_0 < p_1 < \cdots x_i \cdots < q_1 < q_0$. Thus, x is computable iff $\mathbb{R}^n \setminus \{x\}$ is effectively open, hence we can use (to simplify notation) in our defining formulas the computable points (as the complements of effectively open sets maximal w.r.t. inclusion among the effectively open sets strictly below $\mathbb{R}^n$).

More precisely, we can use a variable x to range over the computable points, the atomic formulas $x \in U$ (where U range as usual through $\Sigma^0_1(\mathbb{R}^n)$), and we can quantify over x. Our proof for $n \geq 2$ is closely related to that in [Gr51] while the proof for $n = 1$ is based on quite different ideas, so we consider the two cases separately. To keep our notation as close as possible to that of [Gr51], we let our formulas use, along with the "usual" variables U, V, W, G ranging over $\Sigma^0_1(\mathbb{R}^n)$, the variables a, x, y ranging over the computable points of $\mathbb{R}^n$, and variables A, B, C, D ranging over $\Pi^0_1(\mathbb{R}^n)$.

Proof for $n \geq 2$. In this case we find an interpretation of $\mathbb{N}$ in $\Sigma^0_1(\mathbb{R}^n)$ without parameters, which is sufficient by Lemma 2.1. Let $\widetilde{Cof}$ be the unary relation on $\Sigma^0_1(\mathbb{R}^n)$ which is true precisely on the complements in $\mathbb{R}^n$ of finite sets of computable points (equivalently, we will use the relation $\widetilde{Fin}$ on $\Pi^0_1(\mathbb{R}^n)$ which is true precisely on the finite sets of computable points). Furthermore, let $CON(U)$ and $CMP(U,V)$ denote formulas of signature $\{\subseteq\}$ such that for all $U, V \in \Sigma^0_1(\mathbb{R}^n)$ we have: $\Sigma^0_1(\mathbb{R}^n) \models CON(U)$ iff U is connected, and $\Sigma^0_1(\mathbb{R}^n) \models CMP(U,V)$ iff U is a connected component of V. With these $\widetilde{Fin}, CON, CMP$ at hand (we define them later in the proof), it is easy to interpret $\mathbb{N}$ in $\Sigma^0_1(\mathbb{R}^n)$ similarly to [Gr51] or to the proof of Theorem 3.2.

Namely, first we show that Axiom A6 in [Gr51] holds also in the effective setting, i.e. for any finite disjoint sets A, B of computable points the following formula is true in $\Sigma^0_1(\mathbb{R}^n)$:

$$(A \cup B \subseteq U \wedge CON(U)) \to \exists V \exists W (A \subseteq V \wedge B \subseteq W \\ \wedge V \cap W = \emptyset \wedge V \cup W \subseteq U \wedge CON(V) \wedge CON(W)).$$

Indeed by the proof of the topological version of Axiom 6, given A, B we can find open sets V, W with the specified properties; moreover, these sets are obtained as finite unions of rational open balls, so V, W are effectively open and the effective version of Axiom 6 holds.

Let now $A \approx_G B$ be the ternary relation meaning that A, B are finite disjoint sets of computable points and G is an effectively open set such that

$$A \cup B \subseteq G \wedge \forall H (CMP(H, G) \to (|H \cap A| = 1 \wedge |H \cap B| = 1)).$$

Then $A \approx_G B$ implies that A, B are of the same cardinality, $|A| = |B|$. Note that for any finite sets A, B of computable points we have: $|A| = |B|$ iff $\Sigma^0_1(\mathbb{R}^n) \models Eq(A, B)$ where $Eq(A, B)$ is the formula

$$\widetilde{Fin}(A) \wedge \widetilde{Fin}(B) \wedge \exists C, D, G(\widetilde{Fin}(C) \wedge \widetilde{Fin}(D) \wedge C \approx_G D \wedge C = A \setminus B \wedge D = B \setminus A).$$

As in [Gr51], it is now straightforward to interpret the structure $\mathbb{N}$ in the structure $(\Sigma^0_1(\mathbb{R}^n); \widetilde{Cof})$ without parameters by interpreting natural numbers as the cardinalities of finite sets A, B of computable points (i.e., by taking the quotient-set of all such A under the equivalence relation Eq) and interpreting $+, \times$ as follows: $|A| + |B| = |C|$ iff

$$\exists A', B'(Eq(A', A) \wedge Eq(B', B) \wedge A' \cap B' = \emptyset \wedge Eq(A' \cup B', C),$$

and $|A| \times |B| = |C|$ iff

$$\exists U \exists F (Eq(B, F) \wedge F, C \subseteq U \wedge \forall V (CMP(V, U) \to (|V \cap F| = 1 \wedge |V \cap C| = |A|))).$$

It remains to define $\widetilde{Fin}$, CON, CMP from the beginning of the proof. We will use the formulas $Con(U)$ and $Cmp(U, V)$ from the proof of Theorem 3.2 which now have a different meaning because the set variables range now over the effectively open sets rather than the open sets. If $\Sigma^0_1(\mathbb{R}^n) \models Con(U)$ we say that U is *effectively connected*. Note that if two effectively open sets U, V are effectively connected and $U \cap V \neq \emptyset$ then $U \cup V$ is also effectively connected. Note also that connected effectively open sets are effectively connected. If $\Sigma^0_1(\mathbb{R}^n) \models Cmp(U, V)$ we say that U is an *effective connected component* of V.

Let $\Phi(V)$ be the formula $\forall x \in V \exists U \subseteq V (x \in U \wedge Cmp(U, V))$ saying that any computable point of V belongs to some effective component of V. Since the computable points are dense in $\mathbb{R}^n$, $\Sigma^0_1(\mathbb{R}^n) \models \Phi(V)$ implies that V is the union of its effective components.

Define the unary relation $\widetilde{Iso}$ on $\Pi^0_1(\mathbb{R}^n)$ (an effective analogue of the relation Iso from [Gr51]) as follows: $\widetilde{Iso}(A)$ iff

$$\exists V (A \subseteq V \wedge \Phi(V) \wedge \forall U (Cmp(U, V) \to \exists! x (x \in A \cap U))) \\ \text{and } \forall U (U \cap A \neq \emptyset \to \exists x (x \in U \cap A)).$$

Then the relation $\widetilde{Iso}$ is definable, $\widetilde{Iso}(A)$ implies that any point in A is computable and isolated (indeed, take a satisfying V, then any $a \in A$ is in a unique effective component U of V; choose the unique computable $x \in U \cap A$; since the computable points are dense in $\mathbb{R}^n$, $A \cap U = \{x\}$, hence $x = a$ is isolated), any infinite Π^0_1-set satisfying $\widetilde{Iso}$ is not bounded, and any finite set of computable points satisfies $\widetilde{Iso}$.

The definition of $\widetilde{Fin}$ looks as follows: $\widetilde{Fin}(A)$ iff

$$\widetilde{Iso}(A) \wedge \forall U(\forall a \in A \exists V(a \in V \subseteq \overline{U}) \to \exists W(A \subseteq W \subseteq \overline{U}))$$

(the second conjunct says that if the closure $Cl(U)$ of an effectively open set U is disjoint with A then A can be separated from U by an effectively open set W). From left to right this is easy: if $V_a \in \Sigma^0_1(\mathbb{R}^n)$ satisfies $a \in V_a \subseteq \overline{U}$ for each $a \in A$ and A is finite then $W := \bigcup\{V_a \mid a \in A\}$ is effectively open and separates A from U.

For the other direction, it suffices to show that if $\widetilde{Iso}(A)$ holds and A is infinite then there is an effectively open set U such that $\forall a \in A(a \notin Cl(U))$ and $A \subseteq W \in \Sigma^0_1(\mathbb{R}^n)$ implies $W \cap U \neq \emptyset$. Since $\widetilde{Iso}(A)$, $\overline{A}$ is effectively open, hence one can effectively enumerate the rational closed balls $C(b, q)$ contained in $\overline{A}$. Hence, there is a computable sequence $\{B(b_k, q_k)\}$ of all rational open balls such that $C(b_k, q_k) \subseteq \overline{A}$ and $q_k \leq 1$; note that $\bigcup_k B(b_k, q_k) = \overline{A} = \bigcup_k C(b_k, q_k)$. For any $k < \omega$, let $F(k) := 0$ if $d(b_k, 0) \leq q_k$, and let $F(k)$ be the integer part of $d(b_k, 0) - q_k$ otherwise; the function F is computable. As is well known [Ro67], there is a strictly increasing limitwise-monotone-computable function $t : \omega \to \{1, 2, \ldots\}$ that dominates all computable functions on ω (recall that t is limitwise-monotone-computable if for some computable function $g : \omega \times \omega \to \omega$ we have $g(k, s) \leq g(k, s+1)$ and $t(k) = \lim_s g(k, s)$, and that t dominates a function $f : \omega \to \omega$ if there is k_0 with $\forall k \geq k_0(f(k) < t(k))$).

We claim that the set $U := \bigcup_k B(b_k, r_k)$, where $r_k := q_k - \frac{1}{t(F(k))}$, has the desired properties (note that in the possible case $r_k \leq 0$ the "ball" $B(b_k, r_k)$ is empty). Obviously, $U \subseteq \overline{A}$. Since F is computable and t is limitwise-monotone-computable, U is effectively open. Let us check that $a \notin Cl(U)$ for each $a \in A$. Suppose the contrary: there is a sequence $\{c_i\}$ in U that converges to a, so in particular for some i_0 we have $\forall i \geq i_0(d(c_i, a) \leq 1)$. Choose a sequence of naturals $\{k_i\}$ with $c_i \in B(b_{k_i}, r_{k_i})$. Since $a \notin B(b_{k_i}, q_{k_i})$, we have $d(c_i, a) \geq \frac{1}{t(F(k_i))}$ for all i. Note that $\forall i(F(k_i) \leq m)$ for some m, because for all $i \geq i_0$ we have

$$F(k_i) \leq d(b_{k_i}, 0) \leq d(b_{k_i}, c_i) + d(c_i, 0) \leq q_{k_i} + d(c_i, 0) \leq 1 + d(c_i, a) + d(a, 0) \leq 1 + 1 + d(a, 0).$$

For any i we have $\frac{1}{t(F(k_i))} \geq \frac{1}{t(m)}$ and therefore $d(a, c_i) \geq \frac{1}{t(m)} > 0$, contradicting the convergence of $\{c_i\}$ to a.

Finally, let $A \subseteq W \in \Sigma^0_1(\mathbb{R}^n)$; we have to show that $W \cap U \neq \emptyset$. Since $W \setminus A$ is effectively open and A is infinite (hence unbounded), there is a computable subsequence $\{B(b_{k_i}, q_{k_i})\}$ of $\{B(b_k, q_k)\}$ such that $B(b_{k_i}, q_{k_i}) \subseteq W$ for each i and the sequence $\{F(k_i)\}$ is strictly increasing. Let $h : \omega \to \mathbb{Q}^+$ be a computable function such that $q_{k_i} = h(F(k_i))$ for each $i < \omega$. Since t dominates all computable functions, for some i_0 we have $\forall m \geq i_0(\frac{1}{h(m)} < t(m))$. Since $\{F(k_i)\}$ is strictly increasing, $F(k_i) \geq i_0$ for some i, hence $\frac{1}{h(F(k_i))} < t(F(k_i))$ and therefore $q_{k_i} > \frac{1}{t(F(k_i))}$. Thus, the non-empty ball $B(b_{k_i}, r_{k_i})$ is contained in $W \cap U$, so the latter set is non-empty. This completes the proof of definability of $\widetilde{Fin}$.

Let $Bound_\Sigma(U) := \forall A \subseteq U(\widetilde{Iso}(A) \to \widetilde{Fin}(A))$. Then for any $U \in \Sigma^0_1(\mathbb{R}^n)$ we have: $\Sigma^0_1(\mathbb{R}^n) \models Bound_\Sigma(U)$ iff U is bounded. Indeed, from left to right this follows from the properties of $\widetilde{Iso}$. Conversely, let U be non-bounded. Then there is a computable sequence $\{a_k\}$ of rational points in U such that $d(a_{k+1}, 0) \geq d(a_k, 0) + 1$ for each $k < \omega$. Then $A := \{a_k \mid k < \omega\}$ is an infinite Π^0_1-subset of U and $\widetilde{Iso}(A)$ holds, as desired.

Let $Bound_\Pi(A) := \exists U \supseteq A(Bound_\Sigma(U))$. Then clearly for any $A \in \Pi^0_1(\mathbb{R}^n)$ we have: $\Sigma^0_1(\mathbb{R}^n) \models Bound_\Pi(A)$ iff A is bounded.

Let

$$BCON(A) := Bound_\Pi(A) \wedge \neg\exists U, V(A \subseteq U \cup V \wedge U \cap V = \emptyset \wedge A \cap U \neq \emptyset \wedge A \cap V \neq \emptyset)).$$

Then for any $A \in \Pi^0_1(\mathbb{R}^n)$ we have: $\Sigma^0_1(\mathbb{R}^n) \models BCON(A)$ iff A is bounded and connected. Indeed, from right to left this is obvious. Conversely, it suffices to show that if A bounded and non-connected then

$$\Sigma^0_1(\mathbb{R}^n) \models \exists U, V(A \subseteq U \cup V \wedge U \cap V = \emptyset \wedge A \cap U \neq \emptyset \wedge A \cap V \neq \emptyset)).$$

Since A in non-connected, for some open sets U', V' we have

$$A \subseteq U' \cup V', U' \cap V' = \emptyset, A \cap U' \neq \emptyset, A \cap V' \neq \emptyset.$$

Representing U' and V' as the unions of some families $\mathcal{F}'$ and $\mathcal{G}'$ of rational open balls, we obtain an open cover $\mathcal{F}' \cup \mathcal{G}'$ of A. Since A is compact, there are finite subfamilies $\mathcal{F}$ and $\mathcal{G}$ of resp. $\mathcal{F}'$ and $\mathcal{G}'$ such that $\mathcal{F} \cup \mathcal{G}$ covers A. Then the effectively open sets $U := \bigcup \mathcal{F}$ and $V := \bigcup \mathcal{G}$ have the desired properties.

Let

$$CON(U) := \forall x, y \in U \exists B(x, y \in B \subseteq U \wedge BCON(B)).$$

Then for any $U \in \Sigma^0_1(\mathbb{R}^n)$ we have: $\Sigma^0_1(\mathbb{R}^n) \models CON(U)$ iff U is connected. Indeed, if U is connected and x, y are computable points in U, let B be a polygonal path with rational inner points that connects x and y inside U; then $x, y \in B \subseteq U$ and $BCON(B)$, as desired. Conversely, let U be non-connected. Choose computable points x, y from distinct connected components of U. Then clearly there is no connected set B with $x, y \in B \subseteq U$.

With the formula CON at hand, it is straightforward to write down the formula CMP, completing the proof of the theorem for the case $n > 1$.

In fact, the arguments above (except those concerning Axiom 6) work for any $n \geq 1$, so we have the following corollary which is interesting in its own right.

Corollary 4.5. *There are formulas $CON(U)$ and $CMP(U, V)$ of signature $\{\subseteq\}$ such that for all $n \geq 1$ and $U, V \in \Sigma^0_1(\mathbb{R}^n)$ we have: $\Sigma^0_1(\mathbb{R}^n) \models CON(U)$ iff U is connected, and $\Sigma^0_1(\mathbb{R}^n) \models CMP(U, V)$ iff U is a connected component of V.*

Proof of Theorem 4.4 for $n = 1$. We use the formulas CON and CMP from Corollary 4.5. Let $\xi(U, V)$ be the formula

$$U \neq \emptyset \wedge V \neq \emptyset \wedge U \cap V = \emptyset \wedge \forall U'(U' \cap V = \emptyset \to U' \subseteq U) \wedge \forall V'(V' \cap U = \emptyset \to V' \subseteq V)$$

saying in $\Sigma^0_1(\mathbb{R})$ that U, V are disjoint non-empty effectively open sets such that $U = Int(\mathbb{R} \setminus V)$ (where Int is the interior operator) and $V = Int(\mathbb{R} \setminus U)$.

Note that between any U-components $U_1 < U_2$ (where $U_1 < U_2$ means $\forall x \in U_0 \forall y \in U_1(x < y)))$ there is a V-component. Suppose the contrary, then the interval $W := (inf(U_1), sup(U_2))$ is disjoint with V, hence $W \subseteq U$. This is a contradiction because $sup(U_1) \in W \setminus U$. Symmetrically, between any two V-components there is a U-component.

Let $\{q_0, q_1, \cdots\}$ be a computable enumeration of the set $U \cap \mathbb{Q}$ without repetitions. Define the following equivalence relation $\sim$ on ω: $m \sim n$ iff q_m, q_n are in the same U-component. We claim that this relation is c.e. Indeed, since $m \sim n$ is equivalent to the disjunction of $q_m \leq q_n \wedge [q_m, q_n] \subseteq U$ and $q_n \leq q_m \wedge [q_n, q_m] \subseteq U$, so it suffices to check that the relation $q_m \leq q_n \wedge [q_m, q_n] \subseteq U$ is c.e. We have $U = \bigcup_i B_i$ for a computable sequence

$\{B_i\}$ of basic open balls (i.e., intervals with rational endpoints). Since closed intervals are compact, the relation $q_m \leq q_n \wedge [q_m, q_n] \subseteq U$ is equivalent to

$$\exists l \exists i_0, \ldots, i_l (q_m \in B_{i_0} \wedge q_n \in B_{i_l} \wedge B_{i_0} \cap B_{i_1} \neq \emptyset \wedge \cdots \wedge B_{i_{l-1}} \cap B_{i_l} \neq \emptyset).$$

Alternatively, the last assertion follows from the results in [KK07]. Therefore, $\sim$ is c.e. It is also co-c.e. because, by the previous paragraph, $m \not\sim n$ is equivalent to the disjunction of the predicates $q_m < q_n \wedge \exists r \in V \cap \mathbb{Q}(q_m < r < q_n)$ and $q_n < q_m \wedge \exists r \in V \cap \mathbb{Q}(q_n < r < q_m)$.

Let now

$$Cmp^*(U,V) := U \subseteq V \wedge \forall U'(CMP(U',V) \to (U' \subseteq U \vee\ U \cap U' = \emptyset)).$$

Then $\Sigma^0_1(\mathbb{R}) \models Cmp^*(U,V)$ iff U is an effectively open union of some connected components of V. Let

$$ICmp(V) := \exists U(Cmp^*(U,V) \wedge \neg\exists W(W = V \setminus U \wedge Cmp^*(W,V))).$$

Then $\Sigma^0_1(\mathbb{R}) \models ICmp(V)$ implies that V has infinitely many connected components. Suppose the contrary, then $V = V_0 \cup \cdots \cup V_n$ for some $n \geq 0$ and pairwise disjoint components $V_0, \ldots, V_n \in \Sigma^0_1(\mathbb{R})$. Then any U with $\Sigma^0_1(\mathbb{R}) \models Cmp^*(U,V)$ is a union of some of $V_0, \ldots, V_n$, hence $V \setminus U$ is the finite union of the remaining V_i, a contradiction.

Finally, let $\theta(U,V) := \xi(U,V) \wedge ICmp(U)$. Then $\Sigma^0_1(\mathbb{R}) \models \theta(U,V)$ iff both U, V have infinitely many connected components which are computable intervals, $U = Int(\mathbb{R} \setminus V)$ and $V = Int(\mathbb{R} \setminus U)$. For any such U, V, the relation $\sim$ is computable with infinitely many equivalence classes, hence the lattice $\mathcal{E}$ is isomorphic to the lattice $\mathcal{F}$ of c.e. sets closed under $\sim$. The lattice $\mathcal{F}$ is in turn isomorphic to the lattice $(\mathcal{G};\subseteq)$ where $\mathcal{G} := \{G \mid \Sigma^0_1(\mathbb{R}) \models Cmp^*(G,U)\}$ consists of effectively open subsets of U closed under components, for each U as above. This shows that $\mathcal{E}$ is c-definable in $\Sigma^0_1(\mathbb{R})$ with a non-empty definable set of parameters. By Lemma 2.1, $FO(\mathcal{E}) \leq_m FO(\Sigma^0_1(\mathbb{R}))$. □

4.3. **Effectively open sets in domains.** For some natural domains we have the following interpretability result:

Theorem 4.6. *For any $X \in \{P\omega, k^{\leq\omega}, k^{\omega}_{\perp} \mid 2 \leq k \leq \omega\}$, $\mathcal{E}$ is interpretable without parameters in $\Sigma^0_1(X)$.*

Proof. We give the proof only for the most popular space $P\omega$ (which is homeomorphic to $1^{\omega}_{\perp}$) but similar arguments work for the other spaces as well. First we check (just as in the proof of Theorem 3.3) that the set of finitary elements $\uparrow F$, for all finite $F \subseteq \omega$, is definable in $\Sigma^0_1(P\omega)$.

Next we check (again as in the proof of Theorem 3.3) that the set $\mathcal{V}_n := \{\uparrow F : |F| = n\}$ is definable in $\Sigma^0_1(P\omega)$ for each $n < \omega$ via the formula $\phi_n(V)$.

Now let $U_n := \{S \subseteq \omega : n \leq |S|\}$, so $U_0 = P\omega$ and $U_n = \bigcup \mathcal{V}_n$. Then the singleton set $\{U_{n+1}\}$ is defined by the formula

$$\psi_{n+1}(U) := \forall V(\phi_{n+1}(V) \to V \subseteq U) \wedge \neg\exists V(\phi_n(V) \wedge V \subseteq u).$$

By the proof of Theorem 4.1, the lattice $\mathcal{E}$ is isomorphic to the sublattice $\{S \in \Sigma^0_1(P\omega) \mid U_1 \subseteq S \subseteq U_2\}$ of $\Sigma^0_1(P\omega)$, and is thus definable without parameters. □

Corollary 4.7. *For any $X \in \{P\omega, k^{\leq\omega}, k^{\omega}_{\perp} \mid 2 \leq k \leq \omega\}$, $FO(\Sigma^0_1(X)) \equiv_m FO(\mathbb{N})$.*

Proof. The upper bound holds by Proposition 4.2, the lower bound by the previous theorem (alternatively, by the proof of Theorem 3.3). □

5. Conclusion

The results of this paper show that a satisfactory understanding of definability in the lattices of open and of effectively open sets is connected with intricate relationships between topological and algorithmic properties of the corresponding effective spaces. For this reason we believe that this research direction is interesting and deserves further developments. Many questions remain open. In particular, we are still far from understanding the border between decidability and undecidability of the theories of lattices of all open sets: in particular, we currently have no general sufficient condition giving undecidability of this theory (similar to Theorem 4.1 for the effectively open sets).

The methods in Section 4 are rather different when we distinguish between metric spaces and domains. It would be useful to develop unified methods applicable e.g. to many quasi-metric spaces [dBr13].

The methods of this paper work mainly for second countable locally compact spaces. It would be nice to investigate our questions for second countable non-locally compact spaces like the Baire space or for the space $\mathbb{R}^\omega$. The situation with non- second countable spaces is even less clear.

We guess that Theorem 4.6 and Corollary 4.7 hold also for the spaces of continuous partial functionals of finite types over ω but the given proofs should be modified considerably.

A natural generalization of the topic of this paper is the study of first order theories of lattices of other levels of the standard hierarchies, in particular of higher levels of Borel hierarchy $\mathbf{\Sigma}^0_n$ and of the effective Borel hierarchy Σ^0_n. As shown in [Ra69], the first order theories of the lattices of $\mathbf{\Sigma}^0_2$-sets in $\mathcal{C}$ and in $\mathbb{R}$ are decidable.

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