FEEDBACK COMPUTABILITY ON CANTOR SPACE

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ABSTRACT. We introduce the notion of feedback computable functions from 2^ω to 2^ω, extending feedback Turing computation in analogy with the standard notion of computability for functions from 2^ω to 2^ω. We then show that the feedback computable functions are precisely the effectively Borel functions. With this as motivation we define the notion of a feedback computable function on a structure, independent of any coding of the structure as a real. We show that this notion is absolute, and as an example characterize those functions that are computable from a Gandy ordinal with some finite subset distinguished.

1. FEEDBACK MACHINES AND BOREL MAPS

One of the most important observations of (effective) descriptive set theory is that every continuous map between Polish spaces is computable with respect to some oracle, and every map which is computable with respect to some oracle is continuous. (See [Mos09, Ex. 3D.21].)

This fact allows one to transport results from computability theory to the theory of continuous functions, and vice versa. But it also is important because it provides a machine model for continuity. Specifically, it provides a way of thinking about a continuous map on a Polish space as a construction of the output from the input, instead of simply via the abstract definition (requiring the inverse image of open sets to be open).

We will show that this correspondence extends to feedback Turing computation [AFL15] and Borel functions. That is, we will show that feedback Turing computability provides a machine model for Borel functions on 2^ω, and conversely that every feedback computable function is itself Borel. These results should not be surprising, as it is already known (as reviewed below) that feedback computable reals are exactly the hyperarithmetic (i.e., Δ_1^1) reals, and it is an old result of descriptive set theory that the Borel sets are exactly the Δ_1^1-definable sets. So what we are doing here could be viewed as merely a type shift — from

Δ_1^1 reals, which are Δ_1^1 properties of natural numbers, to Δ_1^1 properties on reals (for sets, or on pairs of reals if thinking about the graph of a function).

1.1. **Notation.** It will be useful to let $\tau\colon \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ be a computable bijection. We will also need a computable bijection $\iota\colon \mathbb{N} \to 2^{<\omega}$. For $\sigma \in 2^{<\omega}$, define $[\sigma]:=\{x \in 2^\omega : \sigma \prec x\}$, i.e., the collection of elements of 2^ω extending σ.

For a set A, define $\mathcal{P}_{<\omega}(A)$ to be the finite powerset of A, i.e., the set of all finite subsets of A.

By a **countable ordinal** we will mean a well-founded linear order $(A, \lhd)$ such that $A \subseteq \omega$. In particular, this representation of countable ordinals will make it possible for oracle machines, as well as feedback machines, to access them as oracles.

Given an admissible ordinal α, let α^+ be the **next admissible after** α, i.e., the least admissible ordinal greater than α. An ordinal α is defined to be a **Gandy ordinal** [AS76] if it is admissible and for all $\gamma < \alpha^+$ there is an α-computable well-ordering of order type γ.

For more details on admissible sets and ordinals, and on (effective) descriptive set theory, see [Sac90], [Kec95], and [Mos09].

1.2. **Feedback computability.** We now review the notion of feedback computability studied in [AFL15]. The intuitive idea is that we want to make sense of the notion of a machine which can ask halting queries of *machines of the same type.*

The notation $\{e\}_F^X(n)$ denotes the eth Turing machine with oracle X and halting function F (which can also be interpreted as an oracle) on input n. When $\{e\}_F^X(n)$ queries X it is said to be making an **oracle query**, and when it queries F it is said to be making a **halting query**.

Definition 1.1. For any $X\colon \omega \to \{0,1\}$ define the set $H_X \subseteq \omega \times \omega$ to be the smallest collection for which there is a function $h_X\colon H_X \to \{\uparrow,\downarrow\}$ satisfying the following:

($\downarrow$) If $\{e\}_{h_X}^X(n)$ makes no halting queries outside of H_X and converges after a finite number of steps then $(e,n) \in H_X$ and $h_X(e,n) = \downarrow$, and conversely.

($\uparrow$) If $\{e\}_{h_X}^X(n)$ makes no halting queries outside of H_X and does not converge (i.e., runs forever) then $(e,n) \in H_X$ and $h_X(e,n) = \uparrow$, and conversely.

Furthermore, this h_X is unique.

Definition 1.2. A **feedback Turing machine** (or **feedback machine** for short) is a machine of the form $\{e\}_{h_X}^X$ for some $e \in \omega$. The notation $\langle e \rangle^X(n)$ is shorthand for $\{e\}_{h_X}^X(n)$.

The set H_X is the collection of **non-freezing** computations and the notation $\langle e \rangle^X(n) \Downarrow$ means $(e,n) \in H_X$. If $(e,n) \notin H_X$ then $\langle e \rangle^X(n)$ is **freezing**, written $\langle e \rangle^X(n) \Uparrow$.

One of the most important results about feedback computability is that feedback reducibility is equivalent to Δ_1^1- or hyperarithmetic reducibility. A set X is hyperarithmetically reducible to Y if X is in the least admissible set containing Y.

Theorem 1.3 ([AFL15, Theorem 16]). *For any $X, Y\colon \omega \to \{0,1\}$ the following are equivalent.*

- *X is $\Delta_1^1(Y)$.*
- *There is a feedback machine e such that $\langle e \rangle^Y(n) = X(n)$ for all $n \in \mathbb{N}$.*

Having recalled the notion of feedback computability, we now introduce the notion of a feedback computable function from 2^ω to 2^ω.

Definition 1.4. For $X \subseteq \mathbb{N}$, a map $f \colon 2^\omega \to 2^\omega$ is **feedback computable with respect to** X if there is a feedback machine with code e such that for all $Y \in 2^\omega$, the function $\langle e \rangle^{X,Y}$ is total and for all $n \in \omega$, $\langle e \rangle^{X,Y}(n) = f(Y)(n)$. In this case e is said to **code** (with respect to X) a feedback computable function from 2^ω to 2^ω.

In other words, a function is feedback computable if there is a feedback machine which, when given a description of a point in the domain, outputs a description of the image of the point in the range.

1.3. **Borel codes.** A *Borel code* of a Borel subset of 2^ω captures the way in which the Borel set was built up from basic open sets using the operations of countable union and complementation. There are many different types of Borel codes, all of which are, for practical purposes, equivalent. However, for our purposes it will be convenient to give a concrete coding system where each Borel code is a an element of $\mathbb{N}^{\mathbb{N}}$.

We begin with a couple of basic operations on functions from $\mathbb{N}$ to $\mathbb{N}$. Suppose $f \colon \mathbb{N} \to \mathbb{N}$. Let $f_* \colon \mathbb{N} \to \mathbb{N}$ be such that $f_*(n) := f(n+1)$ for all $n \in \mathbb{N}$. Also, for $m \in \mathbb{N}$, let $f_m \colon \mathbb{N} \to \mathbb{N}$ be such that $f_m(n) := f(\tau(m,n)+1)$ for all $n \in \mathbb{N}$.

Definition 1.5. Let $\mathrm{BC} \subseteq \mathbb{N}^{\mathbb{N}}$ be the collection of **Borel codes** (for 2^ω), defined by induction as follows.
- $\mathrm{BC}_0 := \{f : f(0) \geq 2\}$.
- If $\beta \in \mathrm{ORD}$ is greater than 0 then BC_β is the smallest set containing
  - $B_{\beta^*} \cup \{f : f(0) = 0 \wedge f_* \in \mathrm{BC}_{\beta^*}\}$ for $\beta^* < \beta$, and
  - all functions $f \colon \mathbb{N} \to \mathbb{N}$ such that $f(0) = 1$ and for all $m \in \mathbb{N}$ there is a $\beta_n < \beta$ such that $f_m \in B_{\beta_n}$.

Finally, set $\mathrm{BC} := \bigcup_{\alpha < \omega_1} \mathrm{BC}_\alpha$. If $f \in \mathrm{BC}$ then define the **rank** of f to be the least ordinal α such that $f \in \mathrm{BC}_\alpha$.

A Borel code (for 2^ω) has an associated Borel set, called its *realization*.

Definition 1.6. Suppose $\zeta \in \mathbb{N}^{\mathbb{N}}$ is a Borel code (for 2^ω). Define the **realization** of ζ, written $\boldsymbol{R}(\zeta)$, to be the Borel subset of 2^ω defined by induction on the rank of the ζ as follows.
- If $\zeta(0) \geq 2$, then $\boldsymbol{R}(\zeta) := [\sigma]$ where $\sigma = \iota(\zeta(0) - 2)$. Note that in this case, $\mathrm{rank}(\zeta) = 0$.
- If $\zeta(0) = 0$, then let $\boldsymbol{R}(\zeta) := 2^\omega \setminus \boldsymbol{R}(\zeta_*)$. Note that in this case, if $\mathrm{rank}(\zeta) = \beta + 1$ for some $\beta \in \mathrm{ORD}$, then $\mathrm{rank}(\zeta_*) = \beta$.
- If $\zeta(0) = 1$ then $\boldsymbol{R}(\zeta) := \bigcup_{m \in \mathbb{N}} \boldsymbol{R}(\zeta_m)$. Note that in this case, $\mathrm{rank}(\zeta_m) < \mathrm{rank}(\zeta)$ for all $m \in \mathbb{N}$.

By the *union* or *intersection* of a collection of Borel codes, we will mean the code for union or intersection of their realizations.

One of the first steps in showing that the Borel functions and the feedback computable functions coincide is to show that feedback computability interacts well with Borel codes.

For any ordinal $\alpha \in \omega_1$, define an **encoding** of α to be a linear ordering on a subset of ω of order type α.

Lemma 1.7. *There is a feedback machine* **bor** *such that for any $\alpha \in \omega_1$, any encoding $\widehat{\alpha}$ of α, any $X \in 2^\omega$, and any $n \in \mathbb{N}$ such that $\langle n \rangle^X$ is total, we have*

- $\langle \mathbf{bor} \rangle^{X,\widehat{\alpha}}(n) = 1$ *if* $\langle n \rangle^X \in \mathrm{BC}_\alpha$, *and*
- $\langle \mathbf{bor} \rangle^{X,\widehat{\alpha}}(n) = 0$ *if* $\langle n \rangle^X \notin \mathrm{BC}_\alpha$.

Proof. Let $\langle \mathbf{bor} \rangle^{X,\widehat{\alpha}}(n)$ do the following.

Step 1:

If $\langle n \rangle^X(0) \geq 2$ then return 1.

Step 2:

If $\langle n \rangle^X(0) = 0$ then search for a $\beta < \alpha$ such that $\langle \mathbf{bor} \rangle^{X,\widehat{\beta}}(n_*) = 1$ where $\widehat{\beta}$ is the encoding of β induced by $\widehat{\alpha}$, and where $\langle n_* \rangle^X(m) := \langle n \rangle^X(m+1)$ for all $m \in \mathbb{N}$. If there exists such a β then return 1, and otherwise return 0.

Step 3:

If $\langle n \rangle^X(0) = 1$ then for each $m \in \mathbb{N}$ search for a $\beta_m < \alpha$ such that $\langle \mathbf{bor} \rangle^{X,\widehat{\beta_m}}(n_m) = 1$ where $\widehat{\beta_m}$ is the encoding of β_m induced by $\widehat{\alpha}$, and where $\langle n_m \rangle^X(k) := \langle n \rangle^X(\tau(m,k)+1)$ for all $m \in \mathbb{N}$. If there exists such a β_m for each $m \in \mathbb{N}$ then return 1, and otherwise return 0.

It is an easy induction to show that **bor** is the desired code. $\qquad \square$

In other words, there is a feedback machine which, uniformly in α and an oracle X, can check whether or not a total function feedback computable in X is a Borel code of rank at most α.

Lemma 1.8. *There is a feedback machine* **in** *such that for any Borel code C, any $X \in 2^\omega$ and any $n \in \mathbb{N}$ such that $\langle n \rangle^X$ is total, we have*

- $\langle \mathbf{in} \rangle^{C,X}(n) = 1$ *if* $\langle n \rangle^X \in \mathbf{R}(C)$, *and*
- $\langle \mathbf{in} \rangle^{C,X}(n) = 0$ *if* $\langle n \rangle^X \notin \mathbf{R}(C)$.

Proof. Call C the first oracle and X the second oracle. We will need to unravel the code C. However, there is no mechanism for changing an oracle. This ends up not being a problem, because the changes we would like to make are simple — which we formalize using the Recursion Theorem.

Toward this end, suppose we have computer code e, which we will think of as instructions for a feedback machine $\langle e \rangle$. We will define new code e^*, as follows. The behavior of the computation $\langle e^* \rangle^{C,X}$ depends on the value of $C(0)$.

Case 1: $C(0) \geq 2$.

Let $\sigma := \iota(C(0) - 2)$. If $\langle n \rangle^X(m) = \sigma(m)$ for all $m \in \mathrm{dom}(\sigma)$, then return 1. Otherwise return 0.

Case 2: $C(0) = 0$.

Let $\langle e \rangle^{C_*,X}$ be the machine that runs just like $\langle e \rangle$ with oracle C, X, except that whenever e makes a query of k to the first oracle, a query of $k+1$ is made instead. Return $1 - \langle e \rangle^{C_*,X}$.

Case 3: $C(0) = 1$.

Let $\langle e \rangle^{C_m,X}$ be the machine that runs just like $\langle e \rangle$ with oracle C, X, except that whenever e makes a query of k to the first oracle, a query of $\tau(m, k) + 1$ is made instead. Let $\langle e^* \rangle^{C,X}$ search for an m such that $\langle e \rangle^{C_m,X} = 1$, and if it finds such an m it returns 1, else 0.

The function from e to e^* is computable. Let **in** be a fixed point. It is an easy induction on the rank the Borel code C to show that **in** has the desired properties. $\qquad\square$

In other words, there is a feedback machine which can determine whether or not a feedback computable function (relative to an oracle) is in the realization of a Borel code, uniformly in the Borel code and the oracle. Now we show, unsurprisingly, that if given a feedback computable collection of Borel codes, they can be combined to form a new Borel code.

Lemma 1.9. *There are feedback machines* **neg**, **cup** *such that*
- *if* $\langle n \rangle^X \in \mathrm{BC}$, *then* $\langle \mathbf{neg} \rangle^X(n, \cdot) \in \mathrm{BC}$ *with* $\boldsymbol{R}\big(\langle \mathbf{neg} \rangle^X(n, \cdot)\big) = 2^\omega \setminus \boldsymbol{R}\big(\langle n \rangle^X\big)$, *and*
- *if* $\langle n \rangle^X(m, \cdot) \in \mathrm{BC}$ *for all* $m \in \mathbb{N}$, *then* $\langle \mathbf{cup} \rangle^X(n, \cdot) \in \mathrm{BC}$ *with* $\boldsymbol{R}\big(\langle \mathbf{cup} \rangle^X(n, \cdot)\big) = \bigcup_{m \in \omega} \boldsymbol{R}\big(\langle n \rangle^X(m, \cdot)\big)$.

Proof. Let $\langle \mathbf{neg} \rangle^X(n, 0) = 0$ and $\langle \mathbf{neg} \rangle^X(n, m + 1) = \langle n \rangle^X(m)$ for all $m \in \mathbb{N}$. Let $\langle \mathbf{cup} \rangle^X(n, 0) = 1$ and $\langle \mathbf{cup} \rangle^X(n, m + 1) = \langle n \rangle^X(\tau^{-1}(m))$. $\qquad\square$

We now define the notion of $\Delta^1_1(X)$ function. We do this in terms of Borel codes, since the Borel subsets of 2^ω are exactly all the Δ^1_1-sets. For more on this notion of Δ^1_1-function, see [Mos09, Section 3D and Theorem 3E.5].

Definition 1.10. A map $f : 2^\omega \to 2^\omega$ is $\Delta^1_1(X)$ for $X \subseteq \mathbb{N}$ if there is a $\Delta^1_1(X)$ sequence of functions $(\gamma_\sigma)_{\sigma \in 2^{<\omega}}$ such that $\boldsymbol{R}(\gamma_\sigma) = f^{-1}([\sigma])$.

The following result is standard (see, e.g., [Mos09, Chapter 2]).

Lemma 1.11. *A map* $f : 2^\omega \to 2^\omega$ *is Borel if and only if* f *is* $\Delta^1_1(X)$ *for some* $X \subseteq \mathbb{N}$.

1.4. **A characterization of feedback computable functions.** We now show that we can isolate Borel codes for those oracles that cause a feedback computation to halt with a tree of subcomputations of height at most α.

Proposition 1.12. *There is a computable collection of codes for feedback machines,* $e_\downarrow$, $e_{\downarrow j}$, $e_\uparrow$, $e_{\uparrow j}$ *for* $j \in \omega$, *such that*
- *for all* $\alpha \in \omega_1$ *and encodings* $\widehat{\alpha}$,
- *for all* $X \subseteq \omega$,
- *for all* $\sigma \in 2^{<\omega}$,
- *for all* $f \in \omega$, *and*
- *for all* $x \in \{\downarrow, \uparrow\} \cup \{\downarrow, \uparrow\} \times \omega$,

$\langle e_x \rangle^{X, \widehat{\alpha}}(f, n, \sigma, \cdot)$ *is a Borel code, which we will refer to as* ζ_x.

Further, whenever $Y \in 2^\omega$ *with* $\sigma \prec Y$ *and* $j \in \omega$ *the following properties hold.*
- $Y \in \boldsymbol{R}(\zeta_\downarrow)$ *if and only if* $\langle f \rangle^{X,Y}(n)$ *halts with a tree of subcomputations of height* $\leq \alpha$.
- $Y \in \boldsymbol{R}(\zeta_{\downarrow j})$ *if and only if* $\langle f \rangle^{X,Y}(n)$ *halts with a tree of subcomputations of height* $\leq \alpha$ *and outputs* j.
- $Y \in \boldsymbol{R}(\zeta_\uparrow)$ *if and only if* $\langle f \rangle^{X,Y}(n)$ *does not halt and the tree of subcomputations is of height* $\leq \alpha$.

- $Y \in \mathbf{R}(\zeta_{\uparrow j})$ *if and only if* $\langle f \rangle^{X,Y}(n)$ *does not halt after j-many steps, and up to the jth step the tree of subcomputations has height $\leq \alpha$.*

  Finally, we always have $Y \notin \mathbf{R}(\zeta_x)$ if $\sigma \not\prec Y$.

Proof. We begin with some notation. Enumerate all triples (η, ν, k) such that η and ν are elements of $2^{<\omega}$ and $\{f\}^{X,\eta}_\nu(n)$ does not make any invalid oracle calls in the first k-many steps, i.e., does not make any oracle queries outside of η or ν. Call this collection B.

For each $\boldsymbol{\eta} = (\eta, \nu, k) \in B$, let $\langle (r^{\boldsymbol{\eta}}_i, m^{\boldsymbol{\eta}}_i) \rangle_{i \leq \ell^{\boldsymbol{\eta}}}$ be the sequence such that $\langle \tau(r^{\boldsymbol{\eta}}_i, m^{\boldsymbol{\eta}}_i) \rangle_{i \leq \ell^{\boldsymbol{\eta}}}$ are the queries made by $\{f\}^{X,\eta}_\nu(n)$ to ν. Note that the length of the sequence is $\ell^{\boldsymbol{\eta}}$. Recall these are called halting queries. In particular if ν is the correct response to every halting query made by $\langle f \rangle^{X,Y}(n)$, i.e., ν agrees with $h_{X,Y}$, then $\{f\}^{X,Y}_\nu(n) \cong \langle f \rangle^{X,Y}(n)$. With this notation we will use the convention that $\nu(\tau(a, b)) = 0$ means ν "believes" $\langle a \rangle^{X,Y}(b)$ halts and $\nu(\tau(a, b)) = 1$ means ν "believes" $\langle a \rangle^{X,Y}(b)$ does not halt.

Our goal will be to define Borel codes $C^{\widehat{\alpha}}_{\boldsymbol{\eta}}$ for all $\boldsymbol{\eta} \in B$ in such a way that $C^{\widehat{\alpha}}_{\boldsymbol{\eta}}$ is the Borel code of all Y extending η such that the behavior of halting calls of $\langle f \rangle^{X,Y}(n)$ on the first k-many steps agrees with that of ν, and for which the tree of computations has height at most α.

Let $B_\downarrow$ be the collection of triples $(\eta, \nu, k) \in B$ such that $\{f\}^{X,\eta}_\nu(n)$ halts in at most k steps. For $j \in \omega$, let $B_{\downarrow j}$ be the collection of triples in $B_\downarrow$ with output j. Let $B_{\uparrow j}$ be the collection of triples in $B \setminus B_\downarrow$ whose third coordinate is j.

For each $\boldsymbol{\eta} = (\eta, \nu, k) \in B$ we define a Borel code $C^{\widehat{\alpha}}_{\boldsymbol{\eta}}$ as follows.

(i) If $\langle f \rangle^{X,\eta}(n)$ makes no halting queries, then $C^{\widehat{\alpha}}_{\boldsymbol{\eta}}$ is a code for $[\eta]$.

(ii) If $\{f\}^{X,\eta}_\nu$ makes any halting queries and $\alpha = 0$, then $C^{\widehat{\alpha}}_{\boldsymbol{\eta}}$ is a Borel code for $\emptyset$.

(iii) Suppose $\nu(\tau(r^{\boldsymbol{\eta}}_i, m^{\boldsymbol{\eta}}_i)) = 0$, i.e., $\{f\}^{X,\eta}_\nu$ "thinks" $\langle r^{\boldsymbol{\eta}}_i \rangle^{X,Y}(m^{\boldsymbol{\eta}}_i)$ halts. Then let $D^{\widehat{\alpha}}_{\boldsymbol{\eta},i}$ be the Borel code which is the union of the Borel codes of $\langle e_\downarrow \rangle^{X,\widehat{\beta}}(r^{\boldsymbol{\eta}}_i, m^{\boldsymbol{\eta}}_i, \eta, \cdot)$ for $\beta < \alpha$, where $\widehat{\beta}$ is the encoding of β induced by $\widehat{\alpha}$. In other words $D^{\widehat{\alpha}}_{\boldsymbol{\eta},i}$ is a Borel code for the collection of those Y such that $\langle r^{\boldsymbol{\eta}}_i \rangle^{X,Y}(m^{\boldsymbol{\eta}}_i) = 0$ and has a tree of subcomputations of rank less than α. Or said another way, $D^{\widehat{\alpha}}_{\boldsymbol{\eta},i}$ is a Borel code for the collection of those Y such that $\langle f \rangle^{X,Y}(n)$ agrees with ν on the ith halting query and has a tree of subcomputations of rank less than α.

Suppose $\nu(\tau(r^{\boldsymbol{\eta}}_i, m^{\boldsymbol{\eta}}_i)) = 1$, i.e., $\{f\}^{X,\eta}_\nu$ "thinks" $\langle r^{\boldsymbol{\eta}}_i \rangle^{X,Y}(m^{\boldsymbol{\eta}}_i)$ does not halt. Then for $j \in \omega$ and $\beta < \alpha$, let $E^{\widehat{\alpha}}_{\boldsymbol{\eta},i,j,\widehat{\beta}}$ be the Borel code of the union of $\langle e_{\uparrow j} \rangle^{X,\widehat{\beta}}(r^{\boldsymbol{\eta}}_i, m^{\boldsymbol{\eta}}_i, \eta, \cdot)$. In other words $E^{\widehat{\alpha}}_{\boldsymbol{\eta},i,j,\beta}$ is a Borel code for those Y such that $\langle r^{\boldsymbol{\eta}}_i \rangle^{X,Y}(m^{\boldsymbol{\eta}}_i)$ hasn't halted by step j and which has a tree of subcomputations of height at most β. Now for $\gamma < \alpha$ let $D^{\widehat{\alpha},\widehat{\gamma}}_{\boldsymbol{\eta},i}$ be the Borel code of the intersection of $E^{\alpha}_{\boldsymbol{\eta},i,j,\beta}$ over $j \in \omega$ and $\beta < \gamma$, where $\widehat{\gamma}$ is the encoding of γ induced by $\widehat{\alpha}$. So $D^{\widehat{\alpha},\widehat{\gamma}}_{\boldsymbol{\eta},i}$ is a Borel code for those Y such that $\langle r^{\boldsymbol{\eta}}_i \rangle^{X,Y}(m^{\boldsymbol{\eta}}_i)$ doesn't halt and has a tree of subcomputations of height at most γ. Now let $D^{\widehat{\alpha}}_{\boldsymbol{\eta},i}$ be the union of $D^{\widehat{\alpha},\widehat{\gamma}}_{\boldsymbol{\eta},i}$ over $\gamma < \alpha$. So $D^{\widehat{\alpha}}_{\boldsymbol{\eta},i}$ is a Borel code for those Y such that $\langle r^{\boldsymbol{\eta}}_i \rangle^{X,Y}(m^{\boldsymbol{\eta}}_i)$ doesn't halt and has a tree of subcomputations of height at less than α.

Let $C^{\widehat{\alpha}}_{\boldsymbol{\eta}}$ be the intersection of $D^{\widehat{\alpha}}_{\boldsymbol{\eta},i}$ for $i \leq \ell^{\boldsymbol{\eta}}$. Note that as $\ell^{\boldsymbol{\eta}}$ is finite the rank of the tree of subcomputations of anything in $C^{\widehat{\alpha}}_{\boldsymbol{\eta}}$ bounded by the supremum of the

tree of subcomputations of $\langle r_i^{\boldsymbol{\eta}} \rangle^{X,Y}(m_i^{\boldsymbol{\eta}})$ plus 1. Therefore the rank of the tree of subcomputations of $\langle f \rangle^{X,Y}(n)$ is at most α for all $Y \in C_{\boldsymbol{\eta}}^{\widehat{\alpha}}$.

Finally for $j \in \omega$ we define the codes as follows.

(I) $\langle e_\downarrow \rangle^{X,\widehat{\alpha}}(f, n, \eta, \cdot)$ is the union of $C_{\boldsymbol{\eta}}^{\widehat{\alpha}}$ such that $\boldsymbol{\eta} \in B_\downarrow$.

(II) $\langle e_{\downarrow j} \rangle^{X,\widehat{\alpha}}(f, n, \eta, \cdot)$ is the union of $C_{\boldsymbol{\eta}}^{\widehat{\alpha}}$ such that $\boldsymbol{\eta} \in B_{\downarrow j}$.

(III) $\langle e_{\uparrow j} \rangle^{X,\widehat{\alpha}}(f, n, \eta, \cdot)$ is the union of $C_{\boldsymbol{\eta}}^{\widehat{\alpha}}$ such that $\boldsymbol{\eta} \in B_{\uparrow j}$.

(IV) $\langle e_\uparrow \rangle^{X,\widehat{\alpha}}(f, n, \eta, \cdot)$ is the intersection of the codes given by $\langle e_{\uparrow j} \rangle^{X,\widehat{\alpha}}(f, n, \eta, \cdot)$ for $j \in \omega$.

Claim 1.13. $C_{\boldsymbol{\eta}}^{\widehat{\alpha}}$ is a Borel code of the set of all Y extending η such that the behavior of $\langle f \rangle^{X,Y}(n)$ on the first k-many steps agrees with that of ν, and for which the tree of computations has height at most $\widehat{\alpha}$.

Proof. Our proof that these codes satisfy our theorem proceeds by induction on α. Notice by conditions (i) and (ii) that if $\alpha = 0$ then $C_{\boldsymbol{\eta}}^{\widehat{\alpha}}$ satisfies the claim.

But then by induction on α and condition (iii), $D_{\boldsymbol{\eta},i}^{\widehat{\alpha}}$ is a Borel code for those Y extending η for which the ith halting call agree with ν and the tree of subcomputations has height $< \alpha$. This then implies that $C_{\boldsymbol{\eta}}^{\widehat{\alpha}}$ satisfies the claim. $\square$

It is then straightforward to check from Claim 1.13 that these definitions satisfy the proposition. $\square$

Before moving on to the main application of Proposition 1.12, it is worth taking a moment to highlight the importance of the ordinal α. Specifically, if we did not have a uniform bound on the height of the tree of subcomputations we were considering, we might accidentally make a halting query which would cause our computation to freeze — causing the entire construction to break. However, we show in Lemma 1.14 that this is not an issue, as there will always be a single bound on all trees of subcomputations, which is itself feedback computable from X.

Lemma 1.14. *Suppose e is a code (with respect to X) for a feedback computable function from 2^ω to 2^ω. Then there is a countable ordinal α and an encoding $\widehat{\alpha}$ that is feedback computable in X such that for every $Y \in 2^\omega$ and $n \in \mathbb{N}$ the tree of subcomputations of $\langle e \rangle^{X,Y}(n)$ has height bounded by α.*

Proof. Let $P(T)$ be the statement "$(\exists Y \in 2^\omega)(\exists n \in \mathbb{N})$ such that T is locally the tree of subcomputations for $\langle e \rangle^{X,Y}(n)$". Then $P(\cdot)$ is a $\Sigma_1^1(X)$ predicate. However, for all $Y \in 2^\omega$ and $n \in \mathbb{N}$, the feedback computation $\langle e \rangle^{X,Y}(n)$ does not freeze and hence its tree of subcomputations is well-founded. In particular this implies that $P(T)$ holds only if T is a well-founded relation.

Hence, by a relativized version of [Sac90, II.5.9], there is some ordinal α hyperarithmetic in X such that α bounds the height of all T satisfying P. Further, by [AFL15, Theorem 16], some encoding $\widehat{\alpha}$ is feedback computable in X because α is hyperarithmetic in X. In particular this implies that α bounds the height of the tree of subcomputations of $\langle e \rangle^{X,Y}(n)$ for all $Y \in 2^\omega$ and $n \in \mathbb{N}$. $\square$

Proposition 1.15. *Suppose $f : 2^\omega \to 2^\omega$ is a feedback computable map (with respect to X). Then f is $\Delta_1^1(X)$.*

Proof. Let e be a code (with respect to X) for the map f, and let α and $\widehat{\alpha}$ be as in Lemma 1.14. By Proposition 1.12, there is a uniformly computable (in X and $\widehat{\alpha}$) collection

of Borel codes $\zeta_{\downarrow j}$ such that $\boldsymbol{R}(\zeta_{\downarrow j}) = \{Y : \langle e \rangle^{X,Y}(n) = j$ and $\langle e \rangle^{X,Y}(n)$ has a tree of subcomputations of height $< \alpha\}$. But then $\boldsymbol{R}(\zeta_{\downarrow j}) = \{Y : \langle e \rangle^{X,Y}(n) = j\}$, as the trees of subcomputations for feedback machines of the form $\langle e \rangle^{X,Y}(n)$ have height $< \alpha$. Hence there is a collection of Borel codes $(\gamma_\sigma)_{\sigma \in 2^{<\omega}}$, uniformly feedback computable in X, such that $\boldsymbol{R}(\gamma_\sigma) = f^{-1}([\sigma])$ for each $\sigma \in 2^{<\omega}$. But then by [AFL15, Theorem 16], the functions $\gamma_\sigma \colon \mathbb{N} \to \mathbb{N}$ are $\Delta_1^1(X)$ uniformly in σ. Therefore f is $\Delta_1^1(X)$. $\qquad\square$

Proposition 1.16. *Suppose $f \colon 2^\omega \to 2^\omega$ is $\Delta_1^1(X)$. Then f is feedback computable (with respect to X).*

Proof. There is a sequence $\langle \gamma_\sigma \rangle_{\sigma \in 2^{<\omega}}$ which is in $\Delta_1^1(X)$ such that for $\sigma \in 2^{<\omega}$ the real γ_σ is a Borel code for $f^{-1}([\sigma])$. By Theorem 1.3, we therefore have that $\langle \gamma_\sigma \rangle_{\sigma \in 2^{<\omega}}$ is feedback computable from X.

Then by Lemma 1.8 there is a feedback machine e such that $\langle e \rangle^{X,Y}(n) = 1$ if there is a $\sigma \in 2^{n+1}$ such that $\sigma(n) = 1$ and $Y \in \boldsymbol{R}(\gamma_\sigma)$, and $\langle e \rangle^{X,Y}(n) = 0$ otherwise. In other words, $\langle e \rangle^{X,Y}(n)$ is the value of $f(Y)(n)$. Hence e is a code (relative to X) for f. $\qquad\square$

At this point, we have accomplished our main purpose in this section, to provide a machine model for Borel functions from 2^ω to 2^ω, by showing that the Borel functions are exactly the feedback computable functions. To round this out, we include some other characterizations of feedback computable functions.

If f is feedback computable (mention of the parameter $X \subseteq \omega$ will be suppressed), then $f(Y)$ is in any admissible set containing Y. So f is uniformly Σ_1 definable over all admissible sets, as $f(Y) = Z$ iff within any admissible set containing Y there is a tree witnessing the computation of $f(Y)$ [AFL15, Proposition 4]. In fact, f can trivially be extended to a function on the entire universe V, by letting $f(Y)$ be the empty set whenever Y is not a real. Conversely, suppose $f \colon V \to V$ is uniformly Σ_1 definable over all admissible sets, and f takes reals to reals. Then, as a function on reals, f is Σ_1^1; namely, $f(Y) = Z$ iff there is a real coding an ω-standard admissible set modeling $f(Y) = Z$. (In a little more detail, it costs nothing to say the model is ω-standard, as you can insist that the members of the model's version of ω are given by the evens in their natural order. Furthermore, the ordinal standard part of an admissible set is itself admissible, so an ω-standard admissible set containing Y, even if non-standard, will also contain $f(Y)$.) It is folklore that every Σ_1^1 function is Δ_1^1, as $f(Y) \neq Z$ iff there is a W such that $f(Y) = W$ and $W \neq Z$.

For another characterization of the functions in question, van de Wiele [vdW82] showed that a function is uniformly Σ_1 over all admissible sets iff it is E-recursive. This was later extended by Slaman [Sla86] (see also [Lub88] for a different proof[1]) to include hereditarily countable parameters. Slaman's result is that for a hereditarily countable parameter p, a function f is uniformly $\Sigma_1(p)$ over all admissible sets iff f is ES_p recursive, where ES_p recursion is E-recursion augmented by selection from p, a schema first identified in [Hoo82] and further studied in [Sla85]. In our case, the parameter is a real X; by Gandy Selection, selection from a real follows from the regular E-recursion schema [Sla85], so that f is uniformly $\Sigma_1(X)$ over all admissible sets iff f is E-recursive in X.

Summarizing the above, we have the following theorem.

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[1]There is a minor mistake in the latter which can easily be corrected. Slaman's proof uses selection from the parameter p. It is mistakenly claimed in [Lub88] that selection from p is not necessary. In fact, the construction in [Lub88] is perfectly good; it's just that the use of selection from p in the construction was overlooked.

Theorem 1.17. *For any function $f: 2^\omega \to 2^\omega$ and any $X \subseteq \omega$ the following are equivalent.*
(a) *f is $\Delta^1_1(X)$.*
(b) *f is feedback computable with respect to X.*
(c) *f can be extended to a function on V which is uniformly $\Sigma_1(X)$ definable over all admissible sets.*
(d) *f can be extended to a function E-recursive in X.*

2. Feedback computability relative to a structure

In this section, we extend the notion of an oracle from a set of natural numbers to a structure (up to isomorphism). If the structure is countable, it can be coded as a set of natural numbers, however this cannot be done if the structure is uncountable. As such, we want our definition of computation from a structure to ultimately be independent of any coding of our structure. This can be seen as a feedback analogue of Medvedev reducibility on isomorphism classes of structures. (For a survey of Muchnik and Medvedev degrees, see [Hin12].) We will make use of the fact (Theorem 1.17) that Borel functions can be thought of as those that are feedback computable from an oracle, and the fact that the isomorphism classes of countable structures are Borel (see [Kec95, Theorem 16.6]). In fact, using Theorem 1.17, one can rephrase the results in this section in terms of notions from effective descriptive set theory, instead of feedback computability, if so desired.

Definition 2.1. A countable language L_0 is **feedback computable** if the sets of relation, function, and constant symbols in L_0, and their arities, are uniformly feedback computable.

A particular feedback computable enumeration of such data gives rise to a natural encoding of each countable L_0-structure with underlying set $\mathbb{N}$, which we will sometimes call its L_0**-encoding**, and often use implicitly. (For more details on such encodings, see, e.g., [Mon18].)

A structure is hereditarily countable when its underlying set and the sets of its relations, functions, and constants are hereditarily countable. Note that a structure is isomorphic to a hereditarily countable one if and only if it is a countable structure in a countable language. However, there are hereditarily countable structures that are uncountable in some admissible sets that contain them as an element. Hence we may think of a structure being hereditarily countable in an admissible set as a measure of its computability with respect to the admissible set.

We now give two definitions of different kinds of structures that can be computed from a structure independent of any coding.

2.1. **Feedback computing expansions.** We now introduce the notion of feedback computing an expansion of a structure.

Definition 2.2. For $j \in \{0, 1\}$, let L_j be a feedback computable language, and let $\mathcal{M}^*_j$ be an L_j-structure with underlying set $\mathbb{N}$ and natural L_j encoding i_j. We say that $e \in \mathbb{N}$ **feedback computes $\mathcal{M}^*_1$ from $\mathcal{M}^*_0$ using the natural encodings** if $\langle e \rangle^{i_0(\mathcal{M}^*_0)}$ is a total function with $\langle e \rangle^{i_0(\mathcal{M}^*_0)} = i_1(\mathcal{M}^*_1)$.

We will later use the following lemma.

Lemma 2.3. *For $j \in \{0,1\}$, let L_j be a feedback computable language, and let $\mathcal{M}_j^*$ be an L_j-structure with underlying set $\mathbb{N}$. The statement "e feedback computes $\mathcal{M}_1^*$ from $\mathcal{M}_0^*$ using the natural encodings" is Σ_1 over any admissible set containing both $\mathcal{M}_j^*$'s, as is the statement "there exists an $\mathcal{M}_1^*$ such that e feedback computes $\mathcal{M}_1^*$ from $\mathcal{M}_0^*$ using the natural encodings". Further, for every $\mathcal{M}_0^*$ and e, there is at most one $\mathcal{M}_1^*$ for which e feedback computes $\mathcal{M}_1^*$ from $\mathcal{M}_0^*$ using the natural encodings.*

Proof. Each convergent feedback computation $\langle e \rangle^{i_0(\mathcal{M}_0^*)}(n)$ has a witness to its convergence and value in any such admissible set. So if $\langle e \rangle^{i_0(\mathcal{M}_0^*)}$ is total, by admissibility a set of witnesses to the convergence of each $\langle e \rangle^{i_0(\mathcal{M}_0^*)}(n)$ can be formed, witnessing the totality of $\langle e \rangle^{i_0(\mathcal{M}_0^*)}$. It is then arithmetic in the output of $\langle e \rangle^{i_0(\mathcal{M}_0^*)}$ whether that function is the natural encoding of a structure, and what that structure would be, in particular whether it equals $\mathcal{M}_1^*$. So the Σ_1 sentence claimed to existence is the existence of a witness or a construction that $\langle e \rangle^{i_0(\mathcal{M}_0^*)} = i_1(\mathcal{M}_1^*)$. $\qquad\square$

Definition 2.4. Let $L_0 \subseteq L_1$ be feedback computable languages, $\mathcal{M}_0$ be a countable L_0-structure, and $\mathcal{M}_1$ a countable L_1-structure which is an expansion of $\mathcal{M}_0$, i.e., $\mathcal{M}_1|_{L_0} = \mathcal{M}_0$. Then e **feedback computes the expansion** $\mathcal{M}_1$ **of oracle** $\mathcal{M}_0$, written $\langle e \rangle^{\mathcal{M}_0} \asymp^+ \mathcal{M}_1$, if for every L_0-structure $\mathcal{M}_0^*$ with underlying set $\mathbb{N}$ and isomorphism $f \colon \mathcal{M}_0^* \to \mathcal{M}_0$, for all L_1-structures $\mathcal{M}_1^*$ such that e feedback computes $\mathcal{M}_1^*$ from $\mathcal{M}_0^*$ using the natural encodings, we have $\mathcal{M}_1^*|_{L_0} = \mathcal{M}_0^*$ and f is an isomorphism from $\mathcal{M}_1^* \to \mathcal{M}_1$.

As an example, consider the case where $L_0 = (\mathrm{id}, \times)$ and $L_1 = (\mathrm{id}, \times, (\cdot)^{-1})$ with id a constant, $\times$ a binary function, and $(\cdot)^{-1}$ a unary function. Suppose $\mathcal{M}_0$ is a group in the language L_0 and $\mathcal{M}_1$ is the same group but in the language L_1 (i.e., with the inverse function). Then, as we can feedback compute the inverse function when we are passed a group, there is a feedback machine which computes the expansion $\mathcal{M}_1$ of $\mathcal{M}_0$.

The following generalization of the relativized Lévy–Shoenfield Absoluteness Theorem [Jec97, Theorem 36'] will be used in Propositions 2.6 and 2.11. A version of it is stated as Exercise 15.14 of [Jec97] (in only the first two editions); we include its proof for completeness.

Theorem 2.5 ([Jec97, Ex. 15.14]). *Let A be a transitive admissible set containing the countable ordinals and in which p is hereditarily countable. If φ is a $\Sigma_1(p)$ sentence true in V, then φ is true in A.*

Proof. Let $\varphi(p)$ be $\exists x\, \psi(x)$, where ψ is Δ_0 and in prenex form. Replace the existential quantifiers in ψ by terms for Skolem functions. By Löwenheim–Skolem, and using the hereditary countability of p, there is a witness x to $\psi(x)$ in a countable transitive model. In particular, the height of this model is a countable ordinal, say γ. Work relative to some coding Z of p, which by hypothesis exists in A. Build the tree of finite structures (allowing the Skolem functions to be partial), which we think of as finite substructures of models of $\psi(x)$, with a ranking function (of the sets of the model) into γ. There are two additional constraints on this tree. A node on level n must contain (at least) the first n-many elements (according to Z) of the transitive closure of p, along with their elementary diagram, and no node may contain an element of the transitive closure of p not given by Z. (This way, in the model induced by any infinite path through the tree, the interpretation of the symbol "p" will be p itself.) Furthermore, for a node τ to be a child of a node σ, all of the Skolem functions must have a value at τ on any input from σ. (This way, the model induced by any infinite path through the tree will be a model of $\varphi(p)$.) Conversely, any countable model

of $\varphi(p)$ of height at most γ, along with a counting of that model, induces an infinite path through this tree. (To make choices among the successor of a node, you will in general need a choice function on this tree, which can be assumed to exist, since the tree can be constructed in L of some real parameters.)

Note that, for any γ, this tree is in A. Also, if it is well-founded, then it has a ranking function in any admissible set in which it is countable; and if it is not well-founded, then it has a branch definable over any such admissible set. It bears mention that there is such an admissible set in A, namely $L_\alpha[Z]$, where α is the least Z-admissible beyond γ. So if $\varphi(p)$ is true in V, then there is a countable γ such that the tree built on γ is not well-founded, and hence has a branch definable over $L_\alpha[Z]$, and therefore in A, from which a witness to $\varphi(p)$ can be built. $\square$

Observe that by the upwards persistence of $\Sigma_1(p)$ formulas, if φ is a $\Sigma_1(p)$ sentence true in A, then it is true in all supersets of A. Hence the absoluteness result Theorem 2.5 also holds for all supersets of a transitive admissible set containing the countable ordinals in which p is hereditarily countable.

Often the parameter in the statement of Theorem 2.5 is taken to be a real $Z \subseteq \mathbb{N}$, in which case the statement can be simplified to absoluteness between V and $L_{\omega_1^{L[Z]}}[Z]$. The formulation above is more general in that no coding $Z \subseteq \mathbb{N}$ of the transitive closure of p need be assumed specified, there being no canonical choice of such a Z. The formulation above is superficially weaker in that, once Z is chosen, A must be at least $L_{\omega_1}[Z]$, whereas in truth the potentially smaller set $L_{\omega_1^{L[Z]}}[Z]$ would suffice. However, this stronger version follows from the formulation given, since the theorem as stated gives absoluteness between V and $L[Z]$, and then the theorem could be interpreted in $L[Z]$.

Proposition 2.6. *Let $L_0 \subseteq L_1$ be countable languages, let $\mathcal{M}_0$ be an L_0-structure, and let $\mathcal{M}_1$ be an L_1-structure whose reduct to L_0 is $\mathcal{M}_0$. Let A be a transitive admissible set containing the countable ordinals and in which $\mathcal{M}_0$ and $\mathcal{M}_1$ are hereditarily countable. Then the statement $\langle e \rangle^{\mathcal{M}_0} \asymp^+ \mathcal{M}_1$ is absolute between V and all supersets of A.*

Proof. Consider the definition of $\asymp^+$, Definition 2.4. It is of the form "for all $\mathcal{M}_0^*$ and f and $\mathcal{M}_1^*$, where the $\mathcal{M}_j^*$'s are structures on $\mathbb{N}$, if e feedback computes $\mathcal{M}_1^*$ from $\mathcal{M}_0^*$ then $\mathcal{M}_1^*$ is an expansion of $\mathcal{M}_0^*$ and the isomorphism f extends to $\mathcal{M}_1^*$". Whether e feedback computes $\mathcal{M}_1^*$ from $\mathcal{M}_0^*$ is, by Lemma 2.3, Δ_1 expressible as "all admissible sets containing the $\mathcal{M}_j^*$'s satisfy a certain sentence" and "there is an admissible set containing the $\mathcal{M}_j^*$'s satisfying a certain sentence". What follows is arithmetic in the parameters. Hence the entire definition is Π_1. By Theorem 2.5, applied to the negation of this relation, it is absolute between V and A, and hence also with respect to all supersets of A. $\square$

It follows from Proposition 2.6 that the following definition is well-defined and doesn't depend on the specific forcing extension. Note that this can be seen as a feedback computability analogue of relations being uniformly relatively intrinsically (u.r.i.) computable (see [Mon18]).

Definition 2.7. Let $L_0 \subseteq L_1$ be languages, $\mathcal{M}_0$ be a (not necessarily countable) L_0-structure, and $\mathcal{M}_1$ an L_1-structure which is an expansion of $\mathcal{M}_0$, i.e., $\mathcal{M}_1|_{L_0} = \mathcal{M}_0$. Then e **feedback computes the expansion** $\mathcal{M}_1$ **of oracle** $\mathcal{M}_0$, written $\langle e \rangle^{\mathcal{M}_0} \asymp^+ \mathcal{M}_1$, if there is some forcing extension $V[G]$ of the universe in which $\mathcal{M}_0$ is countable and $V[G] \models \langle e \rangle^{\mathcal{M}_0} \asymp^+ \mathcal{M}_1$.

We now define what it means for a subset of a structure to be feedback computable.

Definition 2.8. Suppose L_0 is a language and $\mathcal{M}$ is a (not necessarily countable) L_0-structure. Suppose $U \subseteq \mathcal{M}$ is fixed by all automorphisms of $\mathcal{M}$ and $\mathcal{M}_U$ is the expansion of $\mathcal{M}$ which adds U as a new unary predicate. Then U is **feedback computable** from $\mathcal{M}$ if there is an $e \in \mathbb{N}$ such that $\langle e \rangle^{\mathcal{M}} \asymp^+ \mathcal{M}_U$.

Note that by Proposition 2.6 the specific forcing extension is irrelevant. The reason why we require U to be closed under automorphisms of $\mathcal{M}$ is so that the set U is uniquely defined by $\mathcal{M}_U$. Our main use of this notion is when $\mathcal{M}$ is of the form $(\gamma, \in_\gamma, A)$ for some ordinal γ and finite subset A, where $\in_\gamma$ denotes the relation $\in$ restricted to γ.

2.2. **Feedback computing a structure.** Having defined what it means for an expansion of a structure to be feedback computable, we now define what it means for a structure to be feedback computable from another structure.

Definition 2.9. Let L_j be a language and $\mathcal{M}_j$ a countable L_j-structure for $j \in \{0, 1\}$. Then e **feedback computes** $\mathcal{M}_1$ **from** $\mathcal{M}_0$, written $\langle e \rangle^{\mathcal{M}_0} \asymp \mathcal{M}_1$, if for every L_0-structure $\mathcal{M}_0^* \cong \mathcal{M}_0$ with underlying set $\mathbb{N}$ there is an L_1-structure $\mathcal{M}_1^* \cong \mathcal{M}_1$ with underlying set $\mathbb{N}$ such that e feedback computes $\mathcal{M}_1^*$ from $\mathcal{M}_0^*$ using the natural encodings.

As an example, let G be a group and H a normal subgroup. Consider the structure given by the group G along with a distinguished relation for H. We can feedback compute the group G/H by simply choosing a representative from each coset of H along with the group multiplication table for these representatives induced by multiplying the corresponding cosets.

Note that for a structure $\mathcal{M}$ with underlying set $\mathbb{N}$ and natural encoding i, the notation $\langle e \rangle^{i(\mathcal{M})}$ denotes the feedback computable (partial) function from $\mathbb{N}$ to $\mathbb{N}$ that takes as an oracle the natural encoding of $\mathcal{M}$. In contrast, $\langle e \rangle^{\mathcal{M}}$ will not be used on its own, and $\langle e \rangle^{\mathcal{M}} \asymp \mathcal{N}$ (or $\langle e \rangle^{\mathcal{M}} \asymp^+ \mathcal{N}$) can be thought of as saying that no matter what copy of $\mathcal{M}$ is passed as an oracle to $\langle e \rangle$, the output is always a copy of $\mathcal{N}$ (or in the case of $\asymp^+$ a copy of $\mathcal{N}$ which is also an expansion of $\mathcal{M}$).

It is worth noting that if $\langle e \rangle^{\mathcal{M}} \asymp^+ \mathcal{N}$ then we also have $\langle e \rangle^{\mathcal{M}} \asymp \mathcal{N}$. However the converse need not hold as if $\langle e \rangle^{\mathcal{M}} \asymp \mathcal{N}$ the output may be a structure whose restriction to $L_{\mathcal{M}}$ (the language of $\mathcal{M}$) is only isomorphic to $\mathcal{M}$ and not equal to it.

Lemma 2.10. *Let A be an admissible set in which models $\mathcal{M}$ and $\mathcal{N}$ (for the same language) are hereditarily countable. Then whether $\mathcal{M}$ and $\mathcal{N}$ are isomorphic in V is Π_1 definable over A, and if they are isomorphic then an isomorphism is definable over A.*

Proof. Observe that the set of isomorphisms is Σ_1^1 definable (in a real parameter coding $\mathcal{M}$ and $\mathcal{N}$ by any fixed standard way of coding a model by a real). Hence by the Kleene Basis Theorem, this set has a member computable in the hyperjump of the parameters (if non-empty) [Sac90, III.1.3]. □

At its core, the proof is essentially building, in A, the tree of finite partial isomorphisms between $\mathcal{M}$ and $\mathcal{N}$ (that is, isomorphisms between finite subsets of $\mathcal{M}$ and $\mathcal{N}$). They are isomorphic (in V) iff the tree is ill-founded iff there is no ranking function from the tree to the ordinals in A, a Π_1 statement over A; furthermore, an isomorphism can be built from any path through the tree, one of which is definable over A (when the tree is ill-founded).

Proposition 2.11. *Let L_0 and L_1 be languages, and let $\mathcal{M}_0$ be an L_0-structure and $\mathcal{M}_1$ an L_1-structure. Let A be a transitive admissible set containing the countable ordinals and in which $\mathcal{M}_0$ and $\mathcal{M}_1$ are hereditarily countable. Then the statement $\langle e \rangle^{\mathcal{M}_0} \asymp \mathcal{M}_1$ is absolute between V and all supersets of A.*

Proof. Consider the definition of $\asymp$, Definition 2.9. It is of the form "for all $\mathcal{M}_0^*$ isomorphic to $\mathcal{M}_0$ there is an $\mathcal{M}_1^*$ such that e feedback computes $\mathcal{M}_1^*$ from $\mathcal{M}_0^*$ and $\mathcal{M}_1^*$ and $\mathcal{M}_1$ are isomorphic". As above, whether e feedback computes $\mathcal{M}_1^*$ from $\mathcal{M}_0^*$ is, by Lemma 2.3, Δ_1 expressible as "all admissible sets containing the $\mathcal{M}_j^*$'s satisfy a certain sentence" and "some admissible set containing the $\mathcal{M}_j^*$'s satisfies a certain sentence". Also, by Lemma 2.10, so is the assertion "$\mathcal{M}_1^*$ and $\mathcal{M}_1$ are isomorphic", as a definable assertion over any admissible set in which the parameters are hereditarily countable. It bears observation that there is such an admissible set in A; namely, if $R \in A$ is a real coding (in some simple, standard way) both $\mathcal{M}_1$ and $\mathcal{M}_1^*$, then $L_{\omega_1^R}[R] \in A$, as A contains all countable ordinals and $L_{\omega_1^R}[R] \subseteq A$. This provides a Π_1 definition of $\asymp$. By Theorem 2.5, applied to the negation of this relation, it is absolute between V and A, and hence also with respect to all supersets of A. $\square$

The absoluteness of the relation of feedback reducibility between structures (Definition 2.9) allows us to make sense of feedback reducibility between structures even when those structures happen to be uncountable by considering the question of reducibility in a forcing extension where the structures are countable.

It follows from Proposition 2.11 that the following definition is well-defined and doesn't depend on the specific forcing extension.

Definition 2.12. Let L_0, L_1 be languages, $\mathcal{M}_0$ be a (not necessarily countable) L_0-structure, and $\mathcal{M}_1$ a (not necessarily countable) L_1-structure. Then e **feedback computes $\mathcal{M}_1$ from $\mathcal{M}_0$**, written $\langle e \rangle^{\mathcal{M}_0} \asymp \mathcal{M}_1$, if there is some forcing extension $V[G]$ of the universe in which $\mathcal{M}_0$ is countable and $V[G] \models \langle e \rangle^{\mathcal{M}_0} \asymp \mathcal{M}_1$.

The relative computability of uncountable structures was studied using generic extensions and Muchnik degrees in [KMS16]. Our consideration of the feedback reducibility of uncountable structures can be seen as a feedback analogue of these notions, except using the analogue of Medvedev degrees instead of Muchnik degrees, because of the uniformity of our reductions.

Furthermore, both notions of feedback reducibility that take a structure as an oracle (Definitions 2.4 and 2.9) allow us to perform computation in a way that ignores the particular instantiations of the structures. This is important, as there are times when there is more computable information that can be obtained by the encoding of the structure than can be obtained intrinsically from the structure.

2.3. **Example: Functions from ω to ω.** There is one example of computing one structure from another which is particularly important. Note that if $(W, \lhd, c)$ is a well-ordering of order type ω and c is a constant in W, then, uniformly in an encoding of an oracle $(W, \lhd, c)$, we can return the element of $\mathbb{N}$ which c represents. Therefore if $\langle e \rangle^{\mathcal{M}} \asymp (W, \lhd, c)$ there is little harm in identifying the output with the number c represents. In particular we can define $\langle e \rangle^{\mathcal{M}}(n) = m$ if $\langle e \rangle^{\mathcal{M} \times (\omega, \in, n)} \asymp (\omega, \in, m)$. Hence we can think of $\langle e \rangle^{\mathcal{M}}(n) = m$ as saying that whenever e is handed a copy of $\mathcal{M}$ as an oracle, along with the natural number n, it outputs the natural number m.

As an example of this, suppose G is a torsion group. There is a feedback computable function $\langle e \rangle$ which takes n, computes the nth prime p and returns the smallest $m > 0$ such that there is a subgroup of size p^m if such an m exists and returns 0 otherwise. While the input depends on the specific group, it does not depend on the encoding of the group.

The following is then immediate from Proposition 2.11.

Corollary 2.13. *Let $\mathcal{M}$ be an L-structure. The statement $\langle e \rangle^{\mathcal{M}}(n) = m$ is absolute among all transitive admissible sets containing the countable ordinals and in which $\mathcal{M}$ is hereditarily countable.*

2.4. **Example: α-infinite time Turing machines.** For an admissible ordinal α, the (α, α)-infinite time Turing machines (ITTMs) provide a different model which captures the α-computable sets. We now show how to represent this model via oracle feedback computation, which highlights the value of feedback computation as a machine model for Borel maps.

The following definition is a straightforward generalization of the (∞, ω)-Turing machines of [HL00] and the (α, α)-Turing machines in [KS09].

Definition 2.14. Let α, β be ordinals. A (run of a) α-**time**, β-**space Turing machine** (referred to as an (α, β)-ITTM) consists of the following data.

- A function $T \colon \alpha \times \beta \to \{0, 1\}$, called the **tape**. For $\gamma \in \alpha$, the function $T(\gamma, \cdot) \colon \beta \to \{0, 1\}$ is called the **values** of the tape at time γ. The values at time 0 are called the **initial values**.
- A function $H \colon \alpha \to \beta$, called the **head location**.
- A function $S \colon \alpha \to E$, called the **state space**, where E is a finite linearly ordered set containing a special starting state s and halting state h. For $\gamma \in \alpha$, the value $S(\gamma)$ is called the **state** of the machine at time γ.
- A function $C \colon E \times \{0, 1\} \to E \times \{0, 1\} \times \{\text{LEFT}, \text{RIGHT}, \text{STAY}\}$, called the **lookup table**. It can be thought of as taking the state of the machine and the symbol written under the tape and returning the new state, the new symbol, and whether to move the head left or right, or to have it stay where it is.

  This data is required to satisfy the following conditions.

- $C(h, z) = (h, z, \text{STAY})$ for all $z \in \{0, 1\}$.
- $H(0) = 0$ and $S(0) = s$.
- If $\gamma \in \alpha$ is a limit ordinal then $H(\gamma) = \liminf_{\zeta \in \gamma} H(\zeta)$ and $S(\gamma) = \liminf_{\zeta \in \gamma} S(\zeta)$.
- If $\gamma + 1 \in \alpha$ and $(e, z, M) = C\big(S(\gamma), T(\gamma, H(\gamma))\big)$, then the following hold.
  - $S(\gamma + 1) = e$.
  - $T(\gamma + 1, H(\gamma)) = z$.
  - $T(\gamma + 1, \eta) = T(\gamma, \eta)$ for $\eta \neq H(\gamma)$.
  - If $M = \text{STAY}$ then $H(\gamma + 1) = H(\gamma)$.
  - If $M = \text{RIGHT}$ then $H(\gamma + 1) = H(\gamma) + 1$.
  - If $M = \text{LEFT}$ and $H(\gamma) = p + 1$ then $H(\gamma + 1) = p$.
  - If $M = \text{LEFT}$ and $H(\gamma)$ is a limit ordinal then $H(\gamma + 1) = 0$.

The **input** of the machine is $T(0, \cdot)$. The machine **halts** if there is some $\gamma < \alpha$ such that $S(\gamma) = h$, and in this case, $T(\gamma, \cdot)$ is the **output** of the machine (which is well-defined by the first condition).

We will refer to an (α, α)-ITTM as simply an α-ITTM.

We now show how to perform such computations using feedback.

Lemma 2.15. *There is a feedback machine* **ittm** *such that if*

$$\mathcal{M}_{\alpha,\beta,X,C}:=\langle(\alpha,\in_\alpha),(\beta,\in_\beta),X,C\rangle$$

where $X\colon \beta\to\{0,1\}$ *and* C *is a lookup table, then*

$$\langle\mathbf{ittm}\rangle^{\mathcal{M}_{\alpha,\beta,X,C}}\cong\mathcal{N}_{\alpha,\beta,X,C}$$

where $\mathcal{N}_{\alpha,\beta,X,C} = \langle(\alpha,\in_\alpha),(\beta,\in_\beta),T_X,H_X,S_X,C\rangle$, *with* (T_X,H_X,S_X,C) *the (unique)* (α,β)-*ITTM with* $T_X(0,\cdot)=X$ *and code* C.

Proof. Note that given (α,β), a code C, and initial values X, the definition of an (α,β)-ITTM uniquely determines the functions T_X, H_X, and S_X by a transfinite recursion along α that is uniform in α, β, and X. Given a representation of an ordinal, we can feedback computably identify if an element of that representation corresponds to a limit ordinal (and if not find its successor). Hence from any isomorphic copy of $\mathcal{M}_{\alpha,\beta,X,C}$ we can feedback compute $\mathcal{N}_{\alpha,\beta,X,C}$. $\square$

Henceforth all (α,β)-ITTMs will have $\alpha=\beta$. As is standard in this situation, we will imagine that there is an input tape, an output tape, and an extra parameter tape (in which all but finitely many values are 0). This can be encoded into the ITTM in the standard way by interleaving these three tapes.

Definition 2.16. A function $f\colon \mathcal{P}_{<\omega}(\alpha)\to\mathcal{P}_{<\omega}(\alpha)$ is α-**ITTM computable** if there is an α-ITTM with a fixed finite extra parameter set such that when the input tape is the characteristic function of A for some finite sequence A of elements of α, then the α-ITTM halts with the characteristic function of $f(A)$ on the output tape. The notion of α-ITTM computability naturally extends to functions $f\colon \alpha^n\to\alpha^m$ for $n,m\in\mathbb{N}$.

Lemma 2.17. *Let* α *and* γ *be ordinals. Suppose* $(\alpha,\lhd)$ *is well-ordered with order type* γ *where* $\lhd$ *is* α-*ITTM computable. Then for all finite* $B\subseteq\gamma$ *there is a feedback machine* e *such that* $\langle e\rangle^{(\alpha,\in_\alpha,A)}\asymp(\gamma,\in_\gamma,B)$ *for some finite* $A\subseteq\alpha$.

Proof. For any finite lookup table C and finite $A^*\in\mathcal{P}_{<\omega}(\alpha)$, from $(\alpha,\in_\alpha,A^*)$ we can feedback compute $\mathcal{M}_{\alpha,\alpha,A^*,C}$ (via a feedback machine that intrinsically encodes C). Hence for any such C and A^*, the structure $\mathcal{N}_{\alpha,\alpha,A^*,C}$ is feedback computable from $(\alpha,\in_\alpha,A^*)$ by Lemma 2.15. By assumption, there is some C and some finite $A\in\mathcal{P}_{<\omega}(\alpha)$ such that from $\mathcal{N}_{\alpha,\alpha,A,C}$ we can feedback compute $(\gamma,\in_\gamma,B)$. Hence for some $A\in\mathcal{P}_{<\omega}(\alpha)$, the structure $(\gamma,\in_\gamma,B)$ can be feedback computed from $(\alpha,\in_\alpha,A)$. $\square$

2.5. **Feedback computation from ordinals.** For the remainder of this section, we consider an extended example, feedback computability relative to a countable admissible ordinal.

Proposition 2.18. *Let* γ *be an ordinal, let* A *be a finite subset of* γ, *let* $U\subseteq\gamma$, *and suppose that* e *is such that the function* $\langle e\rangle^{(\gamma,\in_\gamma,A)}$ *feedback computes* $(\gamma,\in_\gamma,U)$. *Then* $U\in L(\gamma^+)$.

Proof. First note we can assume without loss of generality that γ is countable, as if it isn't we can move to a forcing extension where γ is countable. Next note that the partial ordering $(\gamma^{<\omega},\preceq)\in L(\gamma^+)$ where $a\preceq b$ if a is an initial segment of b. Let G be a generic for $(\gamma^{<\omega},\preceq)$. Then G is a surjection from ω onto γ. By [Ers90, Theorem 1], the set $L(\gamma^+)[G]$ is

admissible. From the surjection G it is easy to find a real $G^* \in 2^\omega$ encoding $(\gamma, \in_\gamma)$ such that $G^* \in L(\gamma^+)[G]$.

Now let $M \in L(\gamma^+)[G]$ be a real encoding the structure $(\gamma, \in_\gamma, A)$ and let M_U be the corresponding real encoding $(\gamma, \in_\gamma, A, U)$. Then M_U is feedback computable from M (by assumption). Therefore, by [AFL15, Proposition 16], the structure M_U is in any admissible set containing M and in particular is in $L(\gamma^+)[G]$. In particular this implies that $U \in L(\gamma^+)[G]$. But as G was an arbitrary generic for $(\gamma^{<\omega}, \preceq)$, we must have that $U \in L(\gamma^+)$, as desired. $\qquad\square$

In general, even for a countable admissible α, an ordinal $\gamma < \alpha^+$, and a finite subset A of γ, there are not necessarily reals in $L(\alpha^+)$ encoding $(\alpha, \in_\alpha)$ or $(\gamma, \in_\gamma, A)$. However, Proposition 2.18 still yields an upper bound on how complicated a function can be that is feedback computable from an arbitrary ordinal γ.

It is an interesting open question to pin down exactly how complicated the sets feedback computable from α can be. This is a question that we completely answer when α is a Gandy ordinal. It is worth pointing out that in Proposition 2.18 we cannot simply absorb the finite set A into the code of the program, as the specific natural numbers representing the elements of A depend on the particular representation of $(\gamma, \in_\gamma)$.

Proposition 2.19. *There is an e such that $\langle e \rangle^{(\gamma, \in_\gamma)} \asymp (L(\gamma), \in_{L(\gamma)})$. Further, e is independent of γ.*

Proof. Without loss of generality we can assume γ is countable. For a first-order formula φ in the language of set theory, an ordinal $\alpha \in_\gamma \gamma$, a tuple A of ordinals in α of length one less than the number of free variables in φ, define $\mathcal{S}_{\varphi, A, \alpha} := \{x \in L(\alpha) : L(\alpha) \models \varphi(x, A)\}$.

Note that from $(\gamma, \in_\gamma)$ we can feedback compute the set of all such $\mathcal{S}_{\varphi, A, \alpha}$. Further we can feedback compute the relation $\mathcal{S}_{\varphi_0, A_0, \alpha_0} \in_{L(\gamma)} \mathcal{S}_{\varphi_1, A_1, \alpha_1}$ by induction on α_0 and α_1. Next, by induction on α we can compute an equivalence relation $\equiv$ where $\mathcal{S}_{\varphi_0, A_0, \alpha_0} \equiv \mathcal{S}_{\varphi_1, A_1, \alpha_1}$ if and only if they contain the same elements.

Let i be a surjection from first-order formulas in the language of set theory to $\mathbb{N}$. For each $\equiv$-class, choose as a distinguished representative the $\mathcal{S}_{\varphi, A, \alpha}$ where $(i(\varphi), A, \alpha)$ is lexicographically minimal. Finally, observe that $(L(\gamma), \in_{L(\gamma)})$ is isomorphic to the resulting collection of representatives under $\equiv$. $\qquad\square$

We then have the following corollary.

Corollary 2.20. *If there is an α-computable well-ordering of α of height γ, then for any finite $B \subseteq \gamma$ there is an $e \in \mathbb{N}$ and a finite subset $A \subseteq \alpha$ such that $\langle e \rangle^{(\alpha, \in_\alpha, A)} \asymp (\gamma, \in_\gamma, B)$.*

Proof. We can feedback compute $L(\alpha + 1)$ along with a well-ordering $\sqsubseteq_{L(\alpha+1)}$ from $(\alpha, \in_\alpha)$, and so by Proposition 2.19 we can feedback compute $L(\alpha + 1)$ from α. But every α-computable well-ordering of α is in $L(\alpha + 1)$, and so there must be some triple (φ, A^*, α) such that $\mathcal{S}_{\varphi, A^*, \alpha}$ is an α-computable well-ordering of α of order type γ. Therefore for some finite A encoding φ, A^*, and B, we can compute $(\gamma, \in_\gamma, B)$. $\qquad\square$

This suggests the following definition.

Definition 2.21. An admissible ordinal α is defined to be a **feedback Gandy** ordinal if for all $\gamma < \alpha^+$ there is a well-ordering $(\alpha, \lhd)$ of order type γ which is feedback computable from $(\alpha, \in_\alpha, A)$ for some finite $A \subseteq \alpha$.

In particular, Corollary 2.20 shows that all Gandy ordinals are feedback Gandy ordinals. Whereas it has been established that there are admissible ordinals that are not Gandy ordinals [Gos79], it is an open question whether or not every admissible ordinal is a feedback Gandy ordinal.

The following corollary is then immediate.

Corollary 2.22. *If α is a feedback Gandy ordinal and $\gamma < \alpha^+$, then for all finite $B \subseteq \gamma$ there is a finite $A \subseteq \alpha$ such that $(\gamma, \in_\gamma, B)$ is feedback computable from $(\alpha, \in_\alpha, A)$.*

Proposition 2.23. *Let $\alpha < \gamma$ be ordinals, let A be a finite subset of α, and let $e \in \mathbb{N}$. Suppose that $U \in L(\gamma)$ is such that $U \subseteq \alpha$, and suppose that $\langle e \rangle^{(\alpha, \in_\alpha, A)} \asymp (\gamma, \in_\gamma)$. Then there is some finite $A^* \subseteq \alpha$ and $e^* \in \mathbb{N}$ such that $\langle e^* \rangle^{(\alpha, \in_\alpha, A^*)} \asymp^+ (\alpha, \in_\alpha, U)$.*

Proof. This follows immediately from Proposition 2.19 by letting $A^* := A \cup \{a\}$, where a is a code for U in a definable bijection from γ to $L(\gamma)$. $\qquad\square$

Combining Propositions 2.18 and 2.23 and Corollary 2.22, we obtain the following.

Theorem 2.24. *If α is a feedback Gandy ordinal then the following are equivalent for $U \subseteq \alpha$.*
- *$U \in L(\alpha^+)$.*
- *U is feedback computable from $(\alpha, \in_\alpha, A)$ for some $A \in \mathcal{P}_{<\omega}(\alpha)$.*

3. Open questions

We end with several open questions. For each of these questions, let α be an ordinal and A a finite subset of α.
- For what β is there some set $U \in L(\beta + 1) \setminus L(\beta)$ that is feedback computable from $(\alpha, \in_\alpha, A)$?
- If there is some $U \in L(\beta + 1) \setminus L(\beta)$ that is feedback computable from $(\alpha, \in_\alpha, A)$, must $L(\beta + 1)$ be feedback computable from $(\alpha, \in_\alpha, A^*)$ for some finite $A^* \subseteq \alpha$?
- Which ordinals are feedback Gandy? In particular, are there feedback Gandy ordinals that are not Gandy ordinals? Indeed, are there any admissible ordinals that are not feedback Gandy ordinals?

Acknowledgement

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