

## HYPER NORMALISATION AND CONDITIONING FOR DISCRETE PROBABILITY DISTRIBUTIONS

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*Dedicated to Jiří Adámek on the occasion of his 70th birthday*

ABSTRACT. Normalisation in probability theory turns a subdistribution into a proper distribution. It is a partial operation, since it is undefined for the zero subdistribution. This partiality makes it hard to reason equationally about normalisation. A novel description of normalisation is given as a mathematically well-behaved total function. The output of this ‘hyper’ normalisation operation is a distribution of distributions. It improves reasoning about normalisation.

After developing the basics of this theory of (hyper) normalisation, it is put to use in a similarly new description of conditioning, producing a distribution of conditional distributions. This is used to give a clean abstract reformulation of refinement in quantitative information flow.

### 1. INTRODUCTION

We start with the RGB colour model to illustrate normalisation of distributions. This model describes each colour as an additive combination of the primary colours red (R), green (G) and blue (B). It is standardly used in colour screens and cameras. We can write a colour  $C$  for instance as sum:

$$C = \frac{1}{8}|R\rangle + \frac{1}{4}|G\rangle + \frac{1}{2}|B\rangle \quad \text{which is (colour) printed as } C.$$

The ‘ket’ notation  $| - \rangle$  is used as meaningless syntactic sugar in such formal sums. We see that the three weights add up to  $\frac{1}{8} + \frac{1}{4} + \frac{1}{2} = \frac{7}{8}$ . Normalisation, in its simplest form, re-scales these weights so that they add up to one. This is done via division by their sum, as in:

$$\text{nrm}(C) = \frac{1/8}{7/8}|R\rangle + \frac{1/4}{7/8}|G\rangle + \frac{1/2}{7/8}|B\rangle = \frac{1}{7}|R\rangle + \frac{2}{7}|G\rangle + \frac{4}{7}|B\rangle.$$

We see that in this normalised description  $\text{nrm}(C)$ , the relative weights of the values is the same, but their sum has been adjusted to one. We can understand  $\text{nrm}(C)$  as a formal *convex* sum of  $R, G, B$ , that is, as a probability distribution over the set  $\{R, G, B\}$ . The original colour  $C$  is called a subdistribution, since the sum of its values is below (sub) one.

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Normalisation of subdistributions (to distributions) is one of the fundamental operations in probability theory. It forms the basis of many other constructions, notably of conditioning, which is so important in calculating influences in Bayesian networks [2]. The problem with normalisation is that it is a partial operation: it is undefined for the zero subdistribution — of the form  $0|R\rangle + 0|G\rangle + 0|B\rangle$  in the context of the above colour example. This partiality makes it difficult to develop an equational system for normalisation.

The main contribution of this paper is a re-description of normalisation as a *total* operation that satisfies various equations. This new, mathematically civilised formulation makes use of ‘hyper’ distributions, that is, of distributions of distributions. Hence we often refer to the new formulation as ‘hyper’ normalisation, in order to distinguish it from traditional normalisation — illustrated in the earlier colour example. Our hyper normalisation operation  $\mathcal{N}$  takes the following form:

$$\mathcal{D}(n \cdot A) \xrightarrow{\mathcal{N}} \mathcal{D}(n \cdot \mathcal{D}(A))$$

The set  $A$  describes the sample space, and  $n$  is a natural number, used in the copower  $n \cdot A$ , which produces  $n$  copies of  $A$ . A distribution  $\omega \in \mathcal{D}(n \cdot A)$  over the copower  $n \cdot A$  consists of  $n$  subdistributions over  $A$ , over each of these copies of  $A$ . The normalisation  $\mathcal{N}(\omega)$  produces a distribution of normalised distributions, by normalising these subdistributions in parallel, each with weight proportional to the original subdistribution. How this works precisely is explained in Section 3, once the notions of distribution and copower are described in detail.

Applying hyper normalisation in conditioning yields what we call ‘hyper’ conditioning. It is again a total operation. The use of such hyper conditioning is briefly illustrated in a Bayesian reasoning example, and more extensively in a re-description of refinement in quantitative information flow. Since hyper normalisation satisfies various equations, for which see Section 3, it may be a useful operation in languages for probabilistic programming and reasoning; see *e.g.* [3, 27, 17, 1, 28, 16].

Actually, the whole idea of describing normalisation in ‘hyper’ form emerged from the study of the ‘denotation of a channel’ construction in [19, 24, 22, 20]. Normalisation is an implicit step in this construction, which is defined and characterised here as a separate, explicit operation. The original denotation construction in information flow then re-appears as hyper conditioning. We illustrate the close connection with a new, abstract proof of a known result from the area (see Theorem 18 below).

In addition, there are two clear points of influence for the current work.

- Quantum probability theory. The systematic and formal description of aspects of probability theory fits in a wider study of quantum foundations and probability theory [15, 9, 5, 16]. This influence becomes visible here in some of the notation, like the kets  $| - \rangle$ , and in some of the terminology, like tests and instruments. However, this quantum background is not needed to follow what happens here.
- Category theory. Several descriptions, constructions and calculations in probability theory can be greatly simplified by using the categorical notion of monad, concretely in the form of the distribution monad  $\mathcal{D}$  for discrete probability and the Giry monad  $\mathcal{G}$  for continuous probability, see [12] for more information. However, this categorical aspect is deliberately suppressed here in order to reach a wider audience: the main ideas and constructions of the paper are accessible, hopefully, to readers without such categorical background. But the categorical influence is manifest, for instance in the frequent use of diagrams to express equations.

This paper focuses on normalisation and conditioning in *discrete* probability. The question immediately arises: what about continuous or even quantum probability? This matter is postponed to future work.

## 2. MATHEMATICAL BACKGROUND

**2.1. Copowers.** For a number  $n$  and a set  $A$  one commonly writes  $A^n$  for the  $n$ -fold cartesian product  $A \times \cdots \times A$  of  $A$  with itself, consisting of all  $n$ -tuples of elements from  $A$ . Each function  $f: A \rightarrow B$  can be extended to  $f^n: A^n \rightarrow B^n$  by  $f^n(a_0, \dots, a_{n-1}) = (f(a_0), \dots, f(a_{n-1}))$ . More generally, for  $n$  different functions  $f_i: A \rightarrow B$  we can define a map  $A^n \rightarrow B^n$  that applies  $f_i$  to the  $i$ -th element in a tuple. This map is written as  $n$ -tuple  $\langle f_i \circ \pi_i \rangle_{i < n}: A^n \rightarrow B^n$ , where the maps  $\pi_i$  are *projections*  $A^n \rightarrow A$ . Finally, there is a diagonal map  $\Delta: A \rightarrow A^n$  sending an element  $a \in A$  to the diagonal  $n$ -tuple  $\langle a, \dots, a \rangle \in A^n$ .

For  $n = 0$ , the power  $A^n$  is the singleton set, commonly written as  $1$ . It contains only the empty tuple. For each set  $B$  there is a unique function  $B \rightarrow 1$ , which is written as  $!$ .

These sets  $A^n$  are called *powers* of  $A$ . There are also *copowers*  $n \cdot A$ , given by the cartesian product  $\{0, \dots, n-1\} \times A$ . Its elements are thus pairs  $(i, a)$  where  $0 \leq i \leq n-1$  and  $a \in A$ . We shall use ‘coprojection’ functions  $\kappa_i: A \rightarrow n \cdot A$ , given by  $\kappa_i(a) = (i, a)$ . As for powers, a function  $f: A \rightarrow B$  gives rise to a function  $n \cdot f: n \cdot A \rightarrow n \cdot B$ , given by  $\kappa_i a \mapsto \kappa_i f(a)$ . For different functions  $f_i: A \rightarrow B$  there is a map  $[\kappa_i \circ f_i]_{i < n}$  mapping  $\kappa_i a \in n \cdot A$  to  $\kappa_i f_i(a) \in n \cdot B$ . Notice that the empty copower is the empty set  $0$ . The analogue of the diagonal map  $\Delta: A \rightarrow A^n$  is the codiagonal  $\nabla: n \cdot A \rightarrow A$  sending each  $\kappa_i a$  to  $a$ . Clearly, it removes all the tags  $\kappa_i$ .

In line with these descriptions we write  $n$  not only for the natural number  $n \in \mathbb{N}$  but also for the  $n$ -element set  $\{0, 1, \dots, n-1\}$ . Notice that  $0$  is then the empty set,  $1$  is the singleton set  $\{0\}$ , and  $2 = \{0, 1\}$  is the set of Booleans. We have  $2 \cong 2 \cdot 1$ , and more generally  $n \cong n \cdot 1$ . When the copower  $n \cdot A$  is read as product  $n \times A$ , then  $\nabla$  is simply the second projection  $\pi_2: n \times A \rightarrow A$ . We freely switch between these descriptions.

(Categorically, the copower  $n \cdot A$  is the  $n$ -fold coproduct/sum  $A + \cdots + A$  of sets, just as the power  $A^n$  is the  $n$ -fold product  $A \times \cdots \times A$ . This coproduct description of copowers generalises to other categories. The coincidence of copowers  $n \cdot A$  with products  $n \times A$  for sets does not work in general categories.)

**2.2. Probability distributions.** A (discrete) *distribution* over a ‘sample’ set  $A$  is a weighted combination of elements of  $A$ , where the weights are probabilities from the unit interval  $[0, 1]$  that add up to 1. Here we only consider finite combinations and write them as:

$$\omega = r_1 |a_1\rangle + \cdots + r_n |a_n\rangle \quad \text{where} \quad \begin{cases} a_1, \dots, a_n \in A \\ r_1, \dots, r_n \in [0, 1] \text{ with } \sum_i r_i = 1. \end{cases} \quad (2.1)$$

The ‘ket’ notation  $|a\rangle$  is syntactic sugar, used to distinguish elements  $a \in A$  from their occurrence in such formal convex sums. For instance, the uniform distribution of  $n$ -elements  $a_1, \dots, a_n$  is described as  $\frac{1}{n} |a_1\rangle + \cdots + \frac{1}{n} |a_n\rangle$ , or more succinctly as  $\sum_i \frac{1}{n} |a_i\rangle$ .

We write  $\mathcal{D}(A)$  for the set of all (finite, discrete) distributions  $\sum_i r_i |a_i\rangle$  over  $A$  from (2.1). Distributions are also called *states*; they express knowledge, in terms of likelihoods of occurrence of elements of  $A$ . Notice that such a state  $\omega \in \mathcal{D}(A)$  can be identified with a ‘probability mass’ function  $\omega: A \rightarrow [0, 1]$  with finite support  $\text{supp}(\omega) = \{a \in A \mid \omega(a) \neq 0\}$

and with  $\sum_{a \in A} \omega(a) = 1$ . This function-description is often more convenient; we freely switch between this function-description and the formal convex sum description in (2.1).

In formal convex sums like  $\sum_i r_i |a_i\rangle$  in (2.1) we implicitly use equations such as:  $r|a\rangle + s|b\rangle = s|b\rangle + r|a\rangle$ , and:  $r|a\rangle + s|a\rangle = (r + s)|a\rangle$ . Further, terms  $0|a\rangle$  do not contribute to the sum and are omitted.

The elements of the set  $\mathcal{D}(n)$  can be identified with  $n$ -tuples of non-negative real numbers  $(r_1, \dots, r_n)$  with  $\sum_i r_i = 1$ . The set  $\mathcal{D}(n)$  is called the standard  $n - 1$  simplex in topology.

A *hyper distribution*, according to [19, 20, 22, 24], is a distribution of distributions, that is, an inhabitant of  $\mathcal{D}^2(A) = \mathcal{D}(\mathcal{D}(A))$ . There is ‘multiplication’ map  $\mu: \mathcal{D}^2(A) \rightarrow \mathcal{D}(A)$  turning a hyper distribution into an ordinary distribution, via:

$$\mu(\sum_i r_i |\omega_i\rangle) = \sum_a (\sum_i r_i \cdot \omega_i(a)) |a\rangle. \quad (2.2)$$

On the right hand side, the outer sum over  $a$  is a formal convex sum, whereas the inner sum over  $i$  is an actual sum, in the unit interval  $[0, 1]$ . In this equation (2.2), the formal convex sum and the function notation are mixed. We shall use the term ‘hyper distribution’ in ‘tagged’ form, as distribution on a copower  $n \cdot \mathcal{D}(A)$  of distributions, that is, as inhabitant of  $\mathcal{D}(n \cdot \mathcal{D}(A))$ .

The mapping  $A \mapsto \mathcal{D}(A)$  is *functorial*: it does not only work on sets, but also on functions. Each function  $f: A \rightarrow B$  gives rise to a function  $\mathcal{D}(A) \rightarrow \mathcal{D}(B)$ , for which we use the overloaded notation  $\mathcal{D}(f)$ . It is given in the obvious way, like map-list in functional programming:

$$\mathcal{D}(f)\left(\sum_i r_i |a_i\rangle\right) = \sum_i r_i |f(a_i)\rangle. \quad (2.3)$$

The result is sometimes called the *push-forward* distribution. The sum on the right hand side may involve fewer items than the original sum  $\sum_i r_i |a_i\rangle$ , when  $f(a_i) = f(a_j)$  for certain indices  $i \neq j$ . It is not hard to see that identity functions and compositions are preserved:  $\mathcal{D}(\text{id}) = \text{id}$  and  $\mathcal{D}(g \circ f) = \mathcal{D}(g) \circ \mathcal{D}(f)$ .

Marginalisation can be described via functoriality of  $\mathcal{D}$ . For a distribution  $\omega \in \mathcal{D}(A \times B)$  on a product set, the marginalisations of  $\omega$  are obtained as  $\mathcal{D}(\pi_1)(\omega) \in \mathcal{D}(A)$  and  $\mathcal{D}(\pi_2)(\omega) \in \mathcal{D}(B)$ , via the two projections  $A \xleftarrow{\pi_1} A \times B \xrightarrow{\pi_2} B$ . Explicitly:

$$\mathcal{D}(\pi_1)(\omega) = \sum_a \left(\sum_b \omega(a, b)\right) |a\rangle \quad \text{and} \quad \mathcal{D}(\pi_2)(\omega) = \sum_b \left(\sum_a \omega(a, b)\right) |b\rangle.$$

**2.3. Kleisli maps and Kleisli composition.** The mapping  $A \mapsto \mathcal{D}(A)$  is an instance of the categorical notion of monad. We shall suppress the categorical perspective, and stick to rather concrete descriptions. It is not hard to see that a map of the form  $n \rightarrow \mathcal{D}(m)$  corresponds to an  $m \times n$  stochastic matrix, with  $n$  columns of  $m$  entries adding up to 1. Matrix composition corresponds to a special form of function composition, which we shall write as  $\bullet$ . We often call  $\bullet$  Kleisli composition, since it is composition in the so-called Kleisli category associated with  $\mathcal{D}$ , as monad.

We shall write  $f: A \dashrightarrow B$  to express that  $f$  is a function  $A \rightarrow \mathcal{D}(B)$ . Such a map is sometimes called a *conditional distribution*, or just a *conditional*, since one can understand  $f(a)(b) \in [0, 1]$  as the conditional probability  $P(b | a)$ . The point of the notation  $A \dashrightarrow B$  is that the letter ‘ $\mathcal{D}$ ’ can be suppressed in the codomain. A bit formally, we can write a state  $\omega \in \mathcal{D}(A)$  as a Kleisli map  $\omega: 1 \dashrightarrow A$ , where  $1 = \{0\}$  is the singleton set, as above. This arrow formulation is useful in diagrams.

If we have two such Kleisli maps  $A \multimap B$  and  $B \multimap C$ , given by functions  $f: A \rightarrow \mathcal{D}(B)$  and  $g: B \multimap \mathcal{D}(C)$ , then we write  $g \bullet f = g_* \circ f: A \multimap C$ , where  $\circ$  is ordinary composition, and  $g_*: \mathcal{D}(B) \rightarrow \mathcal{D}(C)$  is the ‘Kleisli lifting’ function defined by:

$$g_*\left(\sum_i r_i | b_i \rangle\right) = \sum_{c \in C} \left(\sum_i r_i \cdot g(b_i)(c)\right) | c \rangle. \quad (2.4)$$

Abstractly, we can write  $g_* = \mu \circ \mathcal{D}(g)$ .

When  $f, g$  are seen as matrices, then  $\bullet$  is matrix composition. It is not hard to see that Kleisli composition  $\bullet$  is associative. Its unit is the ‘Dirac’ map  $\eta: A \rightarrow \mathcal{D}(A)$  given by ‘point’ distributions  $\eta(a) = 1 | a \rangle$ . In various calculations we shall use the following basic equations about Kleisli extension  $(-)_*$ ; they hold for monads in general. Proofs are left to the interested reader.

**Lemma 1.** The above definition (2.4) satisfies:

- (1)  $g_* \circ \eta = g$ ;
- (2)  $\eta_* = \text{id}$ ;
- (3)  $(\eta \circ f)_* = \mathcal{D}(f)$ ;
- (4)  $\mathcal{D}(h) \circ g_* = (\mathcal{D}(h) \circ g)_*$ ;
- (5)  $g_* \circ f_* = (g \bullet f)_*$ .

□

The following special maps play an important role in the sequel.

**Definition 2.** Let  $A, B$  be arbitrary sets. There are two *strength* functions:

$$\begin{array}{ccc} \mathcal{D}(A) \times B & \xrightarrow{\text{st}_1} & \mathcal{D}(A \times B) & & A \times \mathcal{D}(B) & \xrightarrow{\text{st}_2} & \mathcal{D}(A \times B) \\ \text{st}_1(\sum_i r_i | a_i \rangle, b) & \longmapsto & \sum_i r_i | a_i, b \rangle & & \text{st}_2(a, \sum_i r_i | b_i \rangle) & \longmapsto & \sum_i r_i | a, b_i \rangle \end{array} \quad (2.5)$$

For a function  $f: A \rightarrow \mathcal{D}(B)$  there is a *graph* function:

$$A \xrightarrow{\text{gr}(f)} \mathcal{D}(B \times A) \quad \text{via} \quad \text{gr}(f)(a) = \sum_b f(a)(b) | b, a \rangle. \quad (2.6)$$

If we write  $\text{tw} = \langle \pi_2, \pi_1 \rangle: A \times B \xrightarrow{\cong} B \times A$  for the ‘twist’ map, then we see that the two strength maps are related via  $\text{st}_2 \circ \text{tw} = \mathcal{D}(\text{tw}) \circ \text{st}_1$ . These strength functions  $\text{st}_i$  make  $\mathcal{D}$  a ‘strong monad’, a basic notion in functional programming. The graph map can be defined abstractly as  $\text{gr}(f) = \text{st}_1 \circ \langle f, \text{id} \rangle$ , as in [7].

We need some basic results about how strength and graph interact with marginalisation, as succinctly expressed in the following diagrams.

$$\begin{array}{ccc} \mathcal{D}(A) & \xleftarrow{\mathcal{D}(\pi_1)} & \mathcal{D}(A \times B) & \xrightarrow{\mathcal{D}(\pi_2)} & \mathcal{D}(B) \\ & \swarrow \pi_1 & \uparrow \text{st}_1 & & \uparrow \eta \\ & & \mathcal{D}(A) \times B & \xrightarrow{\pi_2} & B \end{array} \quad \begin{array}{ccc} \mathcal{D}(B) & \xleftarrow{\mathcal{D}(\pi_1)} & \mathcal{D}(B \times A) & \xrightarrow{\mathcal{D}(\pi_2)} & \mathcal{D}(A) \\ & \swarrow f & \uparrow \text{gr}(f) & & \uparrow \eta \\ & & A & & \end{array} \quad (2.7)$$

These results are easily verified. On an abstract level, the rectangle on the left follows from the fact that  $\mathcal{D}$  is a ‘strongly affine’ monad; see [10, 12].

**2.4. Normalisation, traditionally.** In (2.1) we have seen that in a distribution  $\sum_i r_i |a_i\rangle$  the weights  $r_i \in [0, 1]$  add up to one. We speak of a *subdistribution* when the sum is below one, that is, when  $\sum_i r_i \leq 1$ . What we call *normalisation*, in the traditional sense, is the process of turning a subdistribution into a proper distribution by adjusting the weights so that they add up to one — as illustrated in the RGB example at the very beginning of this article. Normalisation is a partial operation that can be described as follows. If  $\omega = \sum_i r_i |a_i\rangle$  is a subdistribution we first take the sum  $r = \sum_i r_i$  of all weights; if  $r \neq 0$ , then we can readjust the original weights to form a proper distribution:

$$\text{nrm}(\omega) = \sum_i \frac{r_i}{r} |a_i\rangle. \quad (2.8)$$

By construction  $\text{nrm}(\omega)$  is a distribution since its weights add up to one:  $\sum_i \frac{r_i}{r} = \frac{r}{r} = 1$ .

Via the graph construction in Definition 2 one can produce a joint distribution on a set  $B \times A$  from a Kleisli map (conditional)  $A \rightarrow \mathcal{D}(B)$ . The reverse process is sometimes called *disintegration*. We shall concentrate on the special case of joint distributions on copowers  $n \cdot A = n \times A$ .

If have a ‘joint’ distribution  $\Omega \in \mathcal{D}(n \cdot A)$  on a copower  $n \cdot A$  we obtain for each element  $a \in A$  a subdistribution on  $n$ , namely:

$$\Omega_a = \sum_i \Omega(\kappa_i a | i). \quad (2.9)$$

Normalisation of these subdistributions is what we call *pointwise* normalisation. It is crucial in the following result showing how a *conditional*  $A \rightarrow \mathcal{D}(n)$  can be associated with a joint distribution on  $n \cdot A$ . It can be seen as a discrete version of *e.g.* [6, Prop. 3.3] and [25, Prop. 6.7]. The existence of such ‘regular conditional probability’ in (continuous) measure theory is a consequence of the Radon-Nikodym Theorem. Here, in the discrete setting, things are much simpler.

**Proposition 3.** There is a bijective correspondence between  $\Omega$  above the double lines and pairs  $(f, \omega)$  below, in:

$$\frac{\Omega \in \mathcal{D}(n \cdot A) \quad \text{with} \quad \text{supp}(\mathcal{D}(\pi_2)(\Omega)) = A}{A \xrightarrow{f} \mathcal{D}(n) \quad \text{and} \quad \omega \in \mathcal{D}(A) \quad \text{with} \quad \text{supp}(\omega) = A}$$

The side-condition  $r \neq 0$  in normalisation (2.8) translates in this pointwise formalisation into the requirement that the support of the relevant distributions is the whole set  $A$ .

**Proof** In the upward direction we define  $\Omega(\kappa_i(a)) = \omega(a) \cdot f(a)(i)$ . More formally, we first take the graph  $\text{gr}(f): A \rightarrow \mathcal{D}(n \cdot A)$  from Definition 2 and then obtain a joint distribution by applying its Kleisli extension to  $\omega \in \mathcal{D}(A)$ , as in:

$$\Omega = \text{gr}(f)_*(\omega) = \sum_{i,a} \omega(a) \cdot f(a)(i) | \kappa_i a \rangle \in \mathcal{D}(n \cdot A). \quad (2.10)$$

We show that  $\mathcal{D}(\pi_2)(\Omega) = \omega$  in two ways. First we reason with distributions:

$$\begin{aligned} \mathcal{D}(\pi_2)(\Omega) &= \sum_a \left( \sum_i \omega(a) \cdot f(a)(i) \right) | a \rangle \\ &= \sum_a \left( \omega(a) \cdot \sum_i f(a)(i) \right) | a \rangle \\ &= \sum_a \left( \omega(a) \cdot 1 \right) | a \rangle \quad \text{since } f(a) \in \mathcal{D}(n) \\ &= \omega. \end{aligned}$$

A more abstract proof uses Lemma 1 and Diagram (2.7):

$$\mathcal{D}(\pi_2)(\Omega) = (\mathcal{D}(\pi_2) \circ \text{gr}(f)_*)(\omega) = (\mathcal{D}(\pi_2) \circ \text{gr}(f))_*(\omega) = \eta_*(\omega) = \omega.$$

Hence  $\text{supp}(\mathcal{D}(\pi_2)(\Omega)) = \text{supp}(\omega) = A$ .

In the other direction, given  $\Omega \in \mathcal{D}(n \cdot A)$  we take  $\omega = \mathcal{D}(\pi_2)(\Omega) = \mathcal{D}(\nabla)(\Omega) \in \mathcal{D}(A)$  and use the subdistribution (2.9) to define a function  $f: A \rightarrow \mathcal{D}(n)$  via normalisation:

$$f(a) = \text{nrm}(\Omega_a) = \sum_i \frac{\Omega(\kappa_i a)}{\sum_i \Omega(\kappa_i a)} |i\rangle = \sum_i \frac{\Omega(\kappa_i a)}{\omega(a)} |i\rangle. \quad (2.11)$$

This is well-defined since  $\text{supp}(\omega) = A$ , so that  $\omega(a) = \sum_i \Omega(\kappa_i a) \neq 0$ , for each  $a \in A$ .

We show that  $\Omega$  re-appears via the formula (2.10):

$$\text{gr}(f)_*(\omega) = \sum_{i,a} \omega(a) \cdot f(a)(i) |\kappa_i a\rangle = \sum_{i,a} \Omega(\kappa_i a) |\kappa_i a\rangle = \Omega.$$

We leave it to the interested reader to show that first applying (2.10) to  $f, \omega$  and then (2.11) yields the original pair  $f, \omega$ .  $\square$

Since distributions in the current setting always have finite support, the assumptions  $\text{supp}(\mathcal{D}(\pi_2)(\Omega)) = A$  and  $\text{supp}(\omega) = A$  in Proposition 3 imply that  $A$  must be a finite set. Hence we could identify  $A$  with a finite set  $m$ .

### 3. HYPER NORMALISATION

Having seen these preliminary definitions and results, we can turn to our new description of normalisation in ‘hyper’ form. It will be a function  $\mathcal{N}$  of the following type.

$$\mathcal{D}(n \cdot A) \xrightarrow{\mathcal{N}} \mathcal{D}(n \cdot \mathcal{D}(A))$$

This normalisation map  $\mathcal{N}$  thus sends a distribution over a copower of a set  $A$  to a distribution over a copower of distributions over  $A$ . Before defining the map  $\mathcal{N}$  in full generality we give an illustration of how it works.

Consider a finite set  $A = \{a, b, c, d\}$  and number  $n = 3$ . Let’s start from the distribution  $\omega \in \mathcal{D}(3 \cdot A)$  given by:

$$\omega = \frac{1}{8} |\kappa_0 a\rangle + \frac{1}{4} |\kappa_0 b\rangle + \frac{1}{2} |\kappa_1 c\rangle + \frac{1}{8} |\kappa_1 d\rangle.$$

This distribution contains elements  $a, b \in A$  from the first sum component in the copower  $3 \cdot A = A + A + A$ , and elements  $c, d \in A$  from the second component, and nothing from the third component. There are associated subdistributions  $\omega_i$ , for  $i \in 3$ , are given by:

$$\omega_0 = \frac{1}{8} |a\rangle + \frac{1}{4} |b\rangle \quad \omega_1 = \frac{1}{2} |c\rangle + \frac{1}{8} |d\rangle \quad \omega_2 = 0.$$

We see that these subdistributions  $\omega_i$  aggregate the items in  $\omega$  from the same component — *i.e.* with the same coprojection  $\kappa_i$ . Normalisation turns these subdistributions  $\omega_i$  into proper ‘inner’ distributions in  $\mathcal{N}(\omega)$  via normalisation as in (2.8), while keeping track of their origin. That is,  $\mathcal{N}(\omega) \in \mathcal{D}(3 \cdot \mathcal{D}(A)) = \mathcal{D}(\mathcal{D}(A) + \mathcal{D}(A) + \mathcal{D}(A))$  is given by:

$$\begin{aligned} \mathcal{N}(\omega) &= \frac{3}{8} \left| \kappa_0(\text{nrm}(\omega_0)) \right\rangle + \frac{5}{8} \left| \kappa_1(\text{nrm}(\omega_1)) \right\rangle + 0 \left| \kappa_1(\text{nrm}(\omega_2)) \right\rangle \\ &= \frac{3}{8} \left| \kappa_0\left(\frac{1/8}{3/8} |a\rangle + \frac{1/4}{3/8} |b\rangle\right) \right\rangle + \frac{5}{8} \left| \kappa_1\left(\frac{1/2}{5/8} |c\rangle + \frac{1/8}{5/8} |d\rangle\right) \right\rangle \\ &= \frac{3}{8} \left| \kappa_0\left(\frac{1}{3} |a\rangle + \frac{2}{3} |b\rangle\right) \right\rangle + \frac{5}{8} \left| \kappa_1\left(\frac{4}{5} |c\rangle + \frac{1}{5} |d\rangle\right) \right\rangle. \end{aligned}$$

The outer distribution is a convex combination  $\frac{3}{8}|-\rangle + \frac{5}{8}|-\rangle + 0|-\rangle$  of inner distributions, where the weights  $\frac{3}{8} = \frac{1}{8} + \frac{1}{4}$  and  $\frac{5}{8} = \frac{1}{2} + \frac{1}{8}$  and 0 are the normalisation factors for  $\omega_0$  and  $\omega_1$  and  $\omega_2$ . Notice that the third term  $0|\kappa_1(\text{nrn}(\omega_2))\rangle$  in the above first line of  $\mathcal{N}(\omega)$  disappears because of the weight 0 upfront. This is good news, because normalisation of the zero subdistribution  $\omega_2$  is not defined. Hence the hyper formulation deals with undefinedness in a natural way: it disappears automatically.

We are now ready for the general description of hyper normalisation.

**Definition 4.** Let  $A$  be a set, and  $n$  be a natural number. The hyper normalisation map  $\mathcal{N}: \mathcal{D}(n \cdot A) \rightarrow \mathcal{D}(n \cdot \mathcal{D}(A))$  is defined as:

$$\mathcal{N}(\omega) = \sum_{\substack{0 \leq i \leq n-1 \\ \omega[i] \neq 0}} \omega[i] \left| \kappa_i \left( \sum_{a \in A} \frac{\omega(\kappa_i a)}{\omega[i]} |a\rangle \right) \right\rangle \quad (3.1)$$

where:

$$\omega[i] = \sum_a \omega(\kappa_i a) \quad \text{so that} \quad \sum_i \omega[i] = 1.$$

Notice that each inner distribution  $\sum_{a \in A} \frac{\omega(\kappa_i a)}{\omega[i]} |a\rangle$  in  $\mathcal{D}(A)$  is the normalisation (2.8) of the subdistribution  $\omega_i = \sum_a \omega(\kappa_i a) |a\rangle$ . It is well-defined, since  $\omega[i] = \sum_a \omega(\kappa_i a) \neq 0$  in the above formal convex sum (3.1) and:

$$\sum_a \frac{\omega(\kappa_i a)}{\omega[i]} = \frac{\sum_a \omega(\kappa_i a)}{\omega[i]} = \frac{\omega[i]}{\omega[i]} = 1.$$

For  $n \leq 1$  the map  $\mathcal{N}: \mathcal{D}(n \cdot A) \rightarrow \mathcal{D}(n \cdot \mathcal{D}(A))$  is trivial: if  $n = 0$ , then  $n \cdot A = 0 = n \cdot \mathcal{D}(A)$ , so that  $\mathcal{N}$  is the identity map on the empty set  $0 = \mathcal{D}(0)$ . For  $n = 1$  we have  $1 \cdot A \cong A$  and  $1 \cdot \mathcal{D}(A) \cong \mathcal{D}(A)$ , so that the map  $\mathcal{N}: \mathcal{D}(1 \cdot A) \rightarrow \mathcal{D}(1 \cdot \mathcal{D}(A))$  can be identified with the unit / Dirac map  $\eta: \mathcal{D}(A) \rightarrow \mathcal{D}(\mathcal{D}(A))$ , sending  $\omega$  to  $1|\omega\rangle$ . We prefer not to exclude these trivial border cases, to avoid unnecessary side conditions.

One can call a distribution  $\omega \in \mathcal{D}(n \cdot A)$  *normalised* if each  $\kappa_i$  occurs at most once in  $\omega$ . More formally, this can be expressed as  $\mathcal{N}(\omega) = \mathcal{D}(n \cdot \eta)(\omega)$ , so that  $\mathcal{N}(\omega)$  consists of point distributions  $r_i |\kappa_i \eta(a)\rangle$ , for subexpressions  $r_i |\kappa_i a\rangle$  in  $\omega$ . The fact that  $\mathcal{N}(\omega)$  is itself normalised occurs in point (3) below.

The hyper normalisation map  $\mathcal{N}$  is mathematically quite civilised: it satisfies some basic equations, listed below. These equations are formulated — in categorical style — in terms of commuting diagrams, so that the relevant types are clearly visible.

**Lemma 5.** The hyper normalisation map  $\mathcal{N}$  from Definition 4 makes the diagrams below commute.

(1) Normalising trivial input gives trivial output:

$$\begin{array}{ccc} \mathcal{D}(A) & \xrightarrow{\mathcal{D}(\kappa_i)} & \mathcal{D}(n \cdot A) & & \mathcal{D}(n) \times A & \xrightarrow{\text{st}_1} & \mathcal{D}(n \cdot A) \\ \kappa_i \downarrow & & \downarrow \mathcal{N} & & \text{id} \times \eta \downarrow & & \downarrow \mathcal{N} \\ n \cdot \mathcal{D}(A) & \xrightarrow{\eta} & \mathcal{D}(n \cdot \mathcal{D}(A)) & & \mathcal{D}(n) \times \mathcal{D}(A) & \xrightarrow{\text{st}_1} & \mathcal{D}(n \cdot \mathcal{D}(A)) \end{array} \quad (3.2)$$



(2) Destroying the output structure destroys normalisation:

$$\begin{array}{ccc}
 \mathcal{D}(n \cdot A) \xrightarrow{\mathcal{N}} \mathcal{D}(n \cdot \mathcal{D}(A)) & \mathcal{D}(n \cdot A) \xrightarrow{\mathcal{N}} \mathcal{D}(n \cdot \mathcal{D}(A)) \xrightarrow{\mathcal{D}(\nabla)} \mathcal{D}(\mathcal{D}(A)) & \\
 \mathcal{D}(n \cdot !)\downarrow \mathcal{D}(\pi_1) & \mathcal{D}(n \cdot !)\downarrow \mathcal{D}(\pi_1) & \mathcal{D}(\nabla)\downarrow \mathcal{D}(\pi_2) \\
 \mathcal{D}(n) \equiv \mathcal{D}(n) & \mathcal{D}(A) \equiv \mathcal{D}(A) & \downarrow \mu
 \end{array} \quad (3.3)$$

(3) Normalisation is idempotent:

$$\begin{array}{ccc}
 \mathcal{D}(n \cdot A) \xrightarrow{\mathcal{N}} \mathcal{D}(n \cdot \mathcal{D}(A)) & & \\
 \mathcal{N} \downarrow & & \downarrow \mathcal{N} \\
 \mathcal{D}(n \cdot \mathcal{D}(A)) \xrightarrow{\mathcal{D}(n \cdot \eta)} \mathcal{D}(n \cdot \mathcal{D}^2(A)) & & 
 \end{array} \quad (3.4)$$

And thus:

$$\begin{array}{ccc}
 \mathcal{D}(n \cdot A) \xrightarrow{\mathcal{N}} \mathcal{D}(n \cdot \mathcal{D}(A)) \xrightarrow{\mathcal{N}} \mathcal{D}(n \cdot \mathcal{D}^2(A)) & & \\
 \mathcal{N} \downarrow & & \downarrow \mathcal{D}(n \cdot \mu) \\
 \mathcal{D}(n \cdot \mathcal{D}(A)) \equiv \mathcal{D}(n \cdot \mathcal{D}(A)) & & 
 \end{array} \quad (3.5)$$

(4) Normalisation can be undone: it has a left inverse (is a split mono):

$$\begin{array}{ccc}
 \mathcal{D}(n \cdot A) \xrightarrow{\mathcal{N}} \mathcal{D}(n \cdot \mathcal{D}(A)) & & \\
 \parallel & & \downarrow (\text{st}_2)_* \\
 \mathcal{D}(n \cdot A) & & 
 \end{array} \quad (3.6)$$

(5) Normalisation is natural both for ordinary functions and for Kleisli maps: for all functions  $f: A \rightarrow B$  and  $g: A \rightarrow \mathcal{D}(B)$  the following diagram commutes.

$$\begin{array}{ccc}
 \mathcal{D}(n \cdot A) \xrightarrow{\mathcal{N}} \mathcal{D}(n \cdot \mathcal{D}(A)) & \mathcal{D}(n \cdot A) \xrightarrow{\mathcal{N}} \mathcal{D}(n \cdot \mathcal{D}(A)) & \\
 \mathcal{D}(n \cdot f) \downarrow & \downarrow \mathcal{D}(n \cdot \mathcal{D}(f)) & (\text{st}_2)_* \downarrow \\
 \mathcal{D}(n \cdot B) \xrightarrow{\mathcal{N}} \mathcal{D}(n \cdot \mathcal{D}(B)) & \mathcal{D}(n \cdot B) \xrightarrow{\mathcal{N}} \mathcal{D}(n \cdot \mathcal{D}(B)) & \downarrow \mathcal{D}(n \cdot g_*)
 \end{array} \quad (3.7)$$

We write  $n \cdot g: n \cdot A \rightarrow \mathcal{D}(n \cdot B)$  for the function  $\kappa_i a \mapsto \sum_b g(a)(b) | \kappa_i b \rangle$ , that is,  $n \cdot g = \text{st}_2 \circ (\text{id}_n \times g)$ . Commutation of the first rectangle in (3.7) follows from commutation of the second one, for  $g = \eta \circ f$ . But we prefer to make this special (first) case explicit.

**Proof** (1) For the first diagram in (3.2) we have for  $\varphi \in \mathcal{D}(A)$ ,

$$\begin{aligned}
 (\mathcal{N} \circ \mathcal{D}(\kappa_i))(\varphi) &= \mathcal{N} \left( \mathcal{D}(\kappa_i) \left( \sum_a \varphi(a) | a \rangle \right) \right) \\
 &= \mathcal{N} \left( \sum_a \varphi(a) | \kappa_i a \rangle \right) \\
 &= 1 | \kappa_i \left( \sum_a \varphi(a) | a \rangle \right) \rangle \\
 &= 1 | \kappa_i \varphi \rangle \\
 &= (\eta \circ \kappa_i)(\varphi).
 \end{aligned}$$

Commutation of the second diagram is obtained via:

$$\begin{aligned}
(\mathcal{N} \circ \text{st}_1)(\sum_i r_i | i \rangle, a) &= \mathcal{N}(\sum_i r_i | \kappa_i a \rangle) \\
&= \sum_i r_i | \kappa_i(1|a) \rangle \rangle \\
&= \text{st}_1(\sum_i r_i | i \rangle, 1|a) \\
&= (\text{st}_1 \circ (\text{id} \times \eta))(\sum_i r_i | i \rangle, a).
\end{aligned}$$

(2) For the first diagram in (3.3) we first note that:

$$\begin{aligned}
\mathcal{D}(\pi_1)(\omega) &= \mathcal{D}(\pi_1)(\sum_{i,a} \omega(\kappa_i a) | \kappa_i a \rangle) \\
&= \sum_{i,a} \omega(\kappa_i a) | \pi_1(\kappa_i a) \rangle \\
&= \sum_{i,a} \omega(\kappa_i a) | i \rangle \\
&= \sum_i (\sum_a \omega(\kappa_i a)) | i \rangle \\
&= \sum_i \omega[i] | i \rangle.
\end{aligned}$$

Via a similar but simpler calculation one also gets  $\mathcal{D}(\pi_1)(\mathcal{N}(\omega)) = \sum_i \omega[i] | i \rangle$ .

For the second diagram in (3.3) we have:

$$\begin{aligned}
(\mu \circ \mathcal{D}(\pi_2) \circ \mathcal{N})(\omega) &= (\mu \circ \mathcal{D}(\pi_2))(\sum_i \omega[i] | \kappa_i(\sum_{a \in A} \frac{\omega(\kappa_i a)}{\omega[i]} | a \rangle) \rangle) \\
&= \mu(\sum_i \omega[i] | \sum_{a \in A} \frac{\omega(\kappa_i a)}{\omega[i]} | a \rangle \rangle) \\
&\stackrel{(2.2)}{=} \sum_a (\sum_i \omega[i] \cdot \frac{\omega(\kappa_i a)}{\omega[i]}) | a \rangle \\
&= \sum_a (\sum_i \omega(\kappa_i a)) | a \rangle \\
&= \mathcal{D}(\pi_2)(\sum_{a,i} \omega(\kappa_i a) | \kappa_i a \rangle) \\
&= \mathcal{D}(\pi_2)(\omega).
\end{aligned}$$

(3) Normalisation is idempotent since for  $\omega \in \mathcal{D}(n \cdot A)$ ,

$$\begin{aligned}
(\mathcal{N} \circ \mathcal{N})(\omega) &= \mathcal{N}(\sum_i \omega[i] | \kappa_i(\sum_a \frac{\omega(\kappa_i a)}{\omega[i]} | a \rangle) \rangle) \\
&= \sum_i \omega[i] | \kappa_i(1 | \sum_a \frac{\omega(\kappa_i a)}{\omega[i]} | a \rangle) \rangle \rangle \\
&= \mathcal{D}(n \cdot \eta)(\sum_i \omega[i] | \kappa_i(\sum_a \frac{\omega(\kappa_i a)}{\omega[i]} | a \rangle) \rangle) \\
&= (\mathcal{D}(n \cdot \eta) \circ \mathcal{N})(\omega).
\end{aligned}$$

Commutation of (3.5) is now easy:

$$\begin{aligned}
\mathcal{D}(n \cdot \mu) \circ \mathcal{N} \circ \mathcal{N} &\stackrel{(3.4)}{=} \mathcal{D}(n \cdot \mu) \circ \mathcal{D}(n \cdot \eta) \circ \mathcal{N} \\
&= \mathcal{D}(n \cdot (\mu \circ \eta)) \circ \mathcal{N} \\
&= \mathcal{N}
\end{aligned}$$

(4) Recall from Definition 2 that  $\text{st}_2(\kappa_i\varphi) = \sum_a \varphi(a) | \kappa_i a \rangle$ . Hence:

$$\begin{aligned}
 ((\text{st}_2)_* \circ \mathcal{N})(\omega) &= (\mu \circ \mathcal{D}(\text{st}_2)) \left( \sum_i \omega[i] \left| \kappa_i \left( \sum_a \frac{\omega(\kappa_i a)}{\omega[i]} | a \rangle \right) \right. \right) \\
 &= \mu \left( \sum_i \omega[i] \left| \sum_a \frac{\omega(\kappa_i a)}{\omega[i]} | \kappa_i a \rangle \right. \right) \\
 &= \sum_{i,a} (\omega[i] \cdot \frac{\omega(\kappa_i a)}{\omega[i]}) | \kappa_i a \rangle \\
 &= \sum_{i,a} \omega(\kappa_i a) | \kappa_i a \rangle \\
 &= \omega.
 \end{aligned}$$

(5) We only (need to) prove commutation of the diagram on the right in (3.7). So let  $g: A \rightarrow \mathcal{D}(B)$  be given. Then, for  $\omega \in \mathcal{D}(n \cdot A)$ ,

$$\begin{aligned}
 (\mathcal{N} \circ (n \cdot g)_*)(\omega) &= \mathcal{N} \left( \sum_{i,b} (\sum_a \omega(\kappa_i a) \cdot g(a)(b)) | \kappa_i b \rangle \right) \\
 &= \sum_i \omega[i] \left| \kappa_i \left( \sum_b \frac{\sum_a \omega(\kappa_i a) \cdot g(a)(b)}{\omega[i]} | b \rangle \right) \right. \\
 &\quad \text{since } \sum_{b,a} \omega(\kappa_i a) \cdot g(a)(b) \\
 &\quad = \sum_a \omega(\kappa_i a) \cdot (\sum_b g(a)(b)) \\
 &\quad = \sum_a \omega(\kappa_i a) \\
 &\quad = \omega[i] \\
 &= \sum_i \omega[i] \left| \kappa_i (g_* (\sum_a \frac{\omega(\kappa_i a)}{\omega[i]} | a \rangle)) \right. \\
 &= \mathcal{D}(n \cdot g_*) \left( \sum_i \omega[i] \left| \kappa_i \left( \sum_a \frac{\omega(\kappa_i a)}{\omega[i]} | a \rangle \right) \right. \right) \\
 &= (\mathcal{D}(n \cdot g_*) \circ \mathcal{N})(\omega). \quad \square
 \end{aligned}$$

We need the following auxiliary map for the subsequent next result about hyper normalisation.

**Definition 6.** Let  $A$  be a set and  $n \in \mathbb{N}$  an arbitrary number. We define a ‘sprinkle’ function

$$\mathcal{D}(n) \times \mathcal{D}(A)^n \xrightarrow{\text{spr}} \mathcal{D}(A)$$

by:

$$\text{spr}((r_1, \dots, r_n), (\varphi_1, \dots, \varphi_n)) = \sum_i r_i \varphi_i = \sum_a (\sum_i r_i \cdot \varphi_i(a)) | a \rangle.$$

This function  $\text{spr}$  thus sprinkles the convex  $n$ -tuple  $r_1, \dots, r_n$  of probabilities over the  $n$ -tuple of distributions  $\varphi_1, \dots, \varphi_n \in \mathcal{D}(A)$ , and produces a new distribution over  $A$ , namely the convex sum of the  $\varphi_i$ . This works because the set  $\mathcal{D}(A)$  is a convex set, in which such convex sums  $\sum_i r_i \varphi_i$  exist. More abstractly, the sprinkle map can be obtained from strength followed by evaluation and multiplication:  $\mathcal{D}(n) \times \mathcal{D}(A)^n \rightarrow \mathcal{D}(n \times \mathcal{D}(A)^n) \rightarrow \mathcal{D}(\mathcal{D}(A)) \rightarrow \mathcal{D}(A)$ .

Our next result about normalisation is an equational characterisation. It says that  $\mathcal{N}$  is the unique function satisfying  $\mathcal{N}(\sum_i r_i \mathcal{D}(\kappa_i)(\varphi_i)) = \sum_i r_i | \kappa_i(\varphi_i) \rangle$ . As before, this equation is expressed in diagrammatic form.

**Theorem 7.** For each set  $A$  and number  $n$ , the normalisation map  $\mathcal{N}$  is the unique map  $h: \mathcal{D}(n \cdot A) \rightarrow \mathcal{D}(n \cdot \mathcal{D}(A))$  making the following diagram commute.

$$\begin{array}{ccc}
\mathcal{D}(n) \times \mathcal{D}(A)^n & \xrightarrow{\text{id} \times \langle \mathcal{D}(\kappa_i) \circ \pi_i \rangle_i} & \mathcal{D}(n) \times \mathcal{D}(n \cdot A)^n & \xrightarrow{\text{spr}} & \mathcal{D}(n \cdot A) \\
\text{id} \times \langle \kappa_i \circ \pi_i \rangle_i \downarrow & & & & \downarrow h \\
\mathcal{D}(n) \times (n \cdot \mathcal{D}(A))^n & \xrightarrow{\text{id} \times \eta^n} & \mathcal{D}(n) \times \mathcal{D}((n \cdot \mathcal{D}(A))^n) & \xrightarrow{\text{spr}} & \mathcal{D}(n \cdot \mathcal{D}(A))
\end{array}$$

where  $\text{spr}$  is the sprinkle map from Definition 6.

**Proof** We first show that the map  $\mathcal{N}$  as introduced in Definition 4 makes the above rectangle commute.

$$\begin{aligned}
& (\mathcal{N} \circ \text{spr} \circ (\text{id} \times \langle \mathcal{D}(\kappa_i) \circ \pi_i \rangle_i))((r_1, \dots, r_n), (\varphi_1, \dots, \varphi_n)) \\
&= (\mathcal{N} \circ \text{spr})((r_1, \dots, r_n), (\mathcal{D}(\kappa_1)(\varphi_1), \dots, \mathcal{D}(\kappa_n)(\varphi_n))) \\
&= \mathcal{N}(\sum_i r_i (\sum_a \varphi_i(a) | \kappa_i a \rangle)) \\
&= \mathcal{N}(\sum_{i,a} r_i \cdot \varphi_i(a) | \kappa_i a \rangle) \\
&= \sum_i r_i | \kappa_i (\sum_a \varphi_i(a) | a \rangle) \rangle \\
&= \sum_i r_i | \kappa_i (\varphi_i) \rangle \\
&= \text{spr}((r_1, \dots, r_n), (1 | \kappa_1(\varphi_1) \rangle, \dots, 1 | \kappa_n(\varphi_n) \rangle)) \\
&= (\text{spr} \circ (\text{id} \times \eta^n) \circ (\text{id} \times \langle \kappa_i \circ \pi_i \rangle_i))((r_1, \dots, r_n), (\varphi_1, \dots, \varphi_n)).
\end{aligned}$$

Next, let  $h: \mathcal{D}(n \cdot A) \rightarrow \mathcal{D}(n \cdot \mathcal{D}(A))$  make the above diagram commute. Then, for  $\omega \in \mathcal{D}(n \cdot A)$ ,

$$\begin{aligned}
h(\omega) &= h(\sum_{a,i} \omega(\kappa_i a) | \kappa_i a \rangle) \\
&= h(\sum_{i, \omega[i] \neq 0} \omega[i] (\sum_a \frac{\omega(\kappa_i a)}{\omega[i]} | \kappa_i a \rangle)) \\
&= h(\sum_{i, \omega[i] \neq 0} \omega[i] \mathcal{D}(\kappa_i) (\sum_a \frac{\omega(\kappa_i a)}{\omega[i]} | a \rangle)) \\
&= (h \circ \text{spr} \circ (\text{id} \times \langle \mathcal{D}(\kappa_i) \circ \pi_i \rangle_i))((\omega[1], \dots, \omega[n]), (\varphi_1, \dots, \varphi_n)) \\
&\quad \text{where } \varphi_i = \begin{cases} \sum_a \frac{\omega(\kappa_i a)}{\omega[i]} | a \rangle & \text{if } \omega[i] \neq 0 \\ \text{arbitrary} & \text{otherwise} \end{cases} \\
&= (\text{spr} \circ (\text{id} \times \eta) \circ (\text{id} \times \langle \kappa_i \circ \pi_i \rangle_i))((\omega[1], \dots, \omega[n]), (\varphi_1, \dots, \varphi_n)) \\
&= \text{spr}((\omega[1], \dots, \omega[n]), (1 | \kappa_1(\varphi_1) \rangle, \dots, 1 | \kappa_n(\varphi_n) \rangle)) \\
&= \sum_i \omega[i] | \kappa_i \varphi_i \rangle \\
&= \sum_i \omega[i] | \kappa_i (\sum_a \frac{\omega(\kappa_i a)}{\omega[i]} | a \rangle) \rangle \\
&= \mathcal{N}(\omega). \quad \square
\end{aligned}$$

In Lemma 5 (5) we have seen naturality of the normalisation map  $\mathcal{N}: \mathcal{D}(n \cdot A) \rightarrow \mathcal{D}(n \cdot \mathcal{D}(A))$  in the parameter  $A$ . But what about naturality in the other parameter  $n$ ? This also exists, but in more complicated form.

**Lemma 8.** For a Kleisli map  $h: n \rightarrow \mathcal{D}(m)$  write  $h \cdot A = \text{st}_1 \circ (h \times \text{id}): n \cdot A \rightarrow \mathcal{D}(m \cdot A)$  for the map  $\kappa_i a \mapsto \sum_j h(i)(j) | \kappa_j a \rangle$ . The following diagram then commutes.

$$\begin{array}{ccc}
 \mathcal{D}(n \cdot A) & \xrightarrow{\mathcal{N}} \mathcal{D}(n \cdot \mathcal{D}(A)) & \xrightarrow{(h \cdot \mathcal{D}(A))_*} \mathcal{D}(m \cdot \mathcal{D}(A)) & \xrightarrow{\mathcal{N}} \mathcal{D}(m \cdot \mathcal{D}^2(A)) \\
 (h \cdot A)_* \downarrow & & & \downarrow \mathcal{D}(m \cdot \mu) \\
 \mathcal{D}(m \cdot A) & \xrightarrow{\mathcal{N}} & \mathcal{D}(m \cdot \mathcal{D}(A)) & 
 \end{array} \quad (3.8)$$

**Proof** For  $\omega \in \mathcal{D}(n \cdot A)$  we compute:

$$\begin{aligned}
 & (\mathcal{D}(m \cdot \mu) \circ \mathcal{N} \circ (h \cdot \mathcal{D}(A))_* \circ \mathcal{N})(\omega) \\
 &= (\mathcal{D}(m \cdot \mu) \circ \mathcal{N} \circ (h \cdot \mathcal{D}(A))_*) \left( \sum_i \omega[i] | \kappa_i \left( \sum_a \frac{\omega(\kappa_i a)}{\omega[i]} | a \rangle \right) \rangle \right) \\
 &= (\mathcal{D}(m \cdot \mu) \circ \mathcal{N}) \left( \sum_{i,j} h(i)(j) \cdot \omega[i] | \kappa_j \left( \sum_a \frac{\omega(\kappa_i a)}{\omega[i]} | a \rangle \right) \rangle \right) \\
 &= \mathcal{D}(m \cdot \mu) \left( \sum_j \left( \sum_i h(i)(j) \cdot \omega[i] \right) | \kappa_j \left( \sum_i \frac{h(i)(j) \cdot \omega[i]}{\sum_i h(i)(j) \cdot \omega[i]} \cdot \left( \sum_a \frac{\omega(\kappa_i a)}{\omega[i]} | a \rangle \right) \right) \rangle \right) \\
 &= \sum_j \left( \sum_i h(i)(j) \cdot \omega[i] \right) | \kappa_j \left( \sum_a \left( \sum_i \frac{h(i)(j) \cdot \omega[i]}{\sum_i h(i)(j) \cdot \omega[i]} \cdot \frac{\omega(\kappa_i a)}{\omega[i]} \right) | a \rangle \right) \rangle \\
 &= \sum_j \left( \sum_i h(i)(j) \cdot \omega[i] \right) | \kappa_j \left( \sum_a \frac{\sum_i h(i)(j) \cdot \omega(\kappa_i a)}{\sum_i h(i)(j) \cdot \omega[i]} | a \rangle \right) \rangle \\
 &= \sum_j \left( \sum_{i,a} h(i)(j) \cdot \omega(\kappa_i a) \right) | \kappa_j \left( \sum_a \frac{\sum_i h(i)(j) \cdot \omega(\kappa_i a)}{\sum_{i,a} h(i)(j) \cdot \omega(\kappa_i a)} | a \rangle \right) \rangle \\
 &= \mathcal{N} \left( \sum_{j,a} \left( \sum_i h(i)(j) \cdot \omega(\kappa_i a) \right) | \kappa_j a \rangle \right) \\
 &= (\mathcal{N} \circ (h \cdot A)_*)(\omega). \quad \square
 \end{aligned}$$

**Remark 9.** Normalisation  $\mathcal{N}: \mathcal{D}(n \cdot A) \rightarrow \mathcal{D}(n \cdot \mathcal{D}(A))$  is *not* an affine map, that is, it does not preserve convex combinations. We describe a simple counterexample, for  $A = \{a, b\}$  and  $n = 2$ .

$$\begin{aligned}
 \frac{1}{4} \mathcal{N}(1 | \kappa_0 a \rangle) + \frac{3}{4} \mathcal{N}(1 | \kappa_0 b \rangle) &= \frac{1}{4} (1 | \kappa_0 (1 | a \rangle) \rangle) + \frac{3}{4} (1 | \kappa_0 (1 | b \rangle) \rangle) \\
 &= \frac{1}{4} | \kappa_0 (1 | a \rangle) \rangle + \frac{3}{4} | \kappa_0 (1 | b \rangle) \rangle \\
 \mathcal{N} \left( \frac{1}{4} (1 | \kappa_0 a \rangle) + \frac{3}{4} (1 | \kappa_0 b \rangle) \right) &= \mathcal{N} \left( \frac{1}{4} | \kappa_0 a \rangle + \frac{3}{4} | \kappa_0 b \rangle \right) \\
 &= 1 | \kappa_0 \left( \frac{1}{4} | a \rangle + \frac{3}{4} | b \rangle \right) \rangle.
 \end{aligned}$$

One may ask how pointwise normalisation from Proposition 3 and hyper normalisation are related. This requires some preparatory work, where we use a ‘twisted’ version of Proposition 3, using the twist map  $\text{tw} = \langle \pi_2, \pi_1 \rangle$ . For a distribution  $\omega \in \mathcal{D}(n \cdot A)$  we write  $\omega_{\text{tw}} = \mathcal{D}(\text{tw})(\omega) \in \mathcal{D}(A \times n)$  and assume that  $\omega_1 = \mathcal{D}(\pi_1)(\omega) = \mathcal{D}(\pi_2)(\omega_{\text{tw}}) \in \mathcal{D}(n)$  satisfies  $\text{supp}(\omega_1) = n$ . Notice that  $\omega_1(i) = \sum_a \omega(\kappa_i a) = \omega[i]$ , as introduced in Definition 4. Via Proposition 3 we can write  $\omega_{\text{tw}} = \text{gr}(f)_*(\omega_1)$ , for the unique conditional  $f: n \rightarrow \mathcal{D}(A)$ . This map  $f$  is, basically as described in (2.11):

$$f(i) = \text{nrn}(\omega_i) = \sum_a \frac{\omega(\kappa_i a)}{\sum_a \omega(\kappa_i a)} | a \rangle = \sum_a \frac{\omega(\kappa_i a)}{\omega[i]} | a \rangle \quad (3.9)$$

We see that these  $f(i)$ ’s are the normalised ‘inner’ distributions occurring in the formula for  $\mathcal{N}(\omega)$  in Definition 4. The next result describes this situation in a precise manner.

**Proposition 10.** Let  $\omega \in \mathcal{D}(n \cdot A)$  be a distribution whose twisted version  $\omega_{\text{tw}} = \mathcal{D}(\text{tw})(\omega) \in \mathcal{D}(A \times n)$  has conditional  $f: n \rightarrow \mathcal{D}(A)$ , so that:

$$\omega_{\text{tw}} = \text{gr}(f)_*(\omega_1) \quad \text{where} \quad \omega_1 = \mathcal{D}(\pi_1)(\omega).$$

The hyper normalisation  $\mathcal{N}(\omega) \in \mathcal{D}(n \cdot \mathcal{D}(A))$  of  $\omega$  can then be described via the adapted conditional  $\eta \circ f: n \rightarrow \mathcal{D}(\mathcal{D}(A))$  as:

$$\mathcal{N}(\omega)_{\text{tw}} = \text{gr}(\eta \circ f)_*(\omega_1) \in \mathcal{D}(\mathcal{D}(A) \times n).$$

**Proof** We first notice that the graph function  $\text{gr}(\eta \circ f): n \rightarrow \mathcal{D}(\mathcal{D}(A) \cdot n)$  is given by  $\text{gr}(\eta \circ f)(i) = 1|f(i), i\rangle$ , see Definition 2. Then:

$$\begin{aligned} \text{gr}(\eta \circ f)_*(\omega_1) &\stackrel{(2.4)}{=} \sum_{i, \varphi} (\sum_j \omega_1(j) \cdot \text{gr}(\eta \circ f)(j)(\varphi, i)) | \kappa_i \varphi \rangle \\ &= \sum_{i, \varphi} \omega[i] | f(i), i \rangle \\ &\stackrel{(3.9)}{=} \sum_i \omega[i] | \text{tw}(\kappa_i(\sum_a \frac{\omega(\kappa_i a)}{\omega[i]} | a)) \rangle \\ &\stackrel{(3.1)}{=} \mathcal{D}(\text{tw})(\mathcal{N}(\omega)) \\ &= \mathcal{N}(\omega)_{\text{tw}}. \end{aligned} \quad \square$$

**3.1. Comparison to other formulations of normalisation.** We briefly compare our ‘hyper’ approach to normalisation to other approaches. First, in [15] normalisation is defined for non-zero subdistributions. A subdistribution on  $A$  is a subconvex combination  $\sum_i r_i |a_i\rangle$  with  $\sum_i r_i \leq 1$ . It may be identified with a distribution  $\omega \in \mathcal{D}(A+1)$  on  $A+1$ , of the form  $\sum_{i \leq n} r_i | \kappa_1 a_i \rangle + r_{n+1} | \kappa_2 0 \rangle$ , where  $r_{n+1} = 1 - (\sum_i r_i)$  is the ‘one-deficit’ capturing the probability of non-termination; see also [23, 21]. This  $\omega$  is non-zero if  $r_{n+1} \neq 1$ . In that case we can normalise it to  $\sum_i \frac{r_i}{1-r_{n+1}} |a_i\rangle$ . This process is described abstractly in [15].

We sketch how it fits in the current setting. We first map  $\omega \in \mathcal{D}(A+1)$  to the distribution  $\omega' \in \mathcal{D}((A+1) + (A+1))$ , given by  $\omega' = \mathcal{D}(\kappa_1 + \kappa_2)(\omega)$ . Applying our hyper normalisation operation  $\mathcal{N}: \mathcal{D}((A+1) + (A+1)) \rightarrow \mathcal{D}(\mathcal{D}(A+1) + \mathcal{D}(A+1))$  yields  $\mathcal{N}(\omega')$  of the form:

$$(1 - r_{n+1}) \left| \kappa_1 \left( \frac{r_1}{1-r_{n+1}} | \kappa_1 a_1 \rangle + \cdots + \frac{r_n}{1-r_{n+1}} | \kappa_1 a_n \rangle \right) \right\rangle + r_{n+1} \left| \kappa_2 (1 | \kappa_2 0) \right\rangle.$$

The normalised distribution now appears as the first inner component.

Second, in [28] normalisation is defined wrt. a ‘score’. We slightly adapt its description, so that it fits in the current setting. Normalisation like in [28] can then be described as a partial function  $\mathcal{D}([0, 1] \times A) \rightarrow \mathcal{D}(A)$ . The number in  $[0, 1]$  in the input type  $\mathcal{D}([0, 1] \times A)$  is called the score. It is a non-negative real number in [28], but here we restrict it to the unit interval. It allows us to massage the input type via a strength map, so that it becomes a subdistribution that can be normalised, as above. We use that  $[0, 1] \cong \mathcal{D}(2)$  in:

$$\begin{aligned} &\mathcal{D}([0, 1] \times A) \\ &\quad \parallel \mathcal{D}(\text{st}_1) \\ &\mathcal{D}(\mathcal{D}(2) \times A) \xrightarrow{\quad} \mathcal{D}(\mathcal{D}(2 \times A)) \\ &\quad \parallel \mathcal{D}(\text{id}+) \\ &\mathcal{D}(\mathcal{D}(A + A)) \xrightarrow{\quad \mu \quad} \mathcal{D}(A + A) \xrightarrow{\quad} \mathcal{D}(A + 1) \end{aligned}$$

This form of normalisation sends a distribution  $\sum_i r_i | (s_i, a_i) \rangle \in \mathcal{D}([0, 1] \times A)$ , with scores  $s_i$ , to  $\sum_i \frac{r_i \cdot s_i}{\sum_i r_i \cdot s_i} |a_i\rangle \in \mathcal{D}(A)$ . It is only defined if  $\sum_i r_i \cdot s_i \neq 0$ .

## 4. NORMALISATION AS DISTRIBUTIVE LAW

This section is meant for the categorically proficient reader, knowing about (co)monads and distributive laws — see *e.g.* [11] for more information. It can be skipped safely, since it presents only a categorical curiosity. This section shows that hyper normalisation  $\mathcal{N}$  forms a distributive law, between a comonad and a functor. It is not a distributive law between two comonads, since one of the counit laws fails to hold — whereas the corresponding comultiplication law does hold.

The standard adjunction  $\mathcal{Kl}(\mathcal{D}) \rightleftarrows \mathbf{Sets}$  between a Kleisli category and its underlying category induces a comonad on  $\mathcal{Kl}(\mathcal{D})$ , which shall write as  $\overline{\mathcal{D}}$ . On objects it is given by  $X \mapsto \mathcal{D}(X)$ . It sends a map  $f: X \rightarrow \mathcal{D}(Y)$  to  $\overline{\mathcal{D}}(f) = \eta \circ f_* = \eta \circ \mu \circ \mathcal{D}(f): \mathcal{D}(X) \rightarrow \mathcal{D}^2(Y)$ . The counit  $\varepsilon: \overline{\mathcal{D}}(X) \rightarrow X$  is the identity map  $\mathcal{D}(X) \rightarrow \mathcal{D}(X)$ . The comultiplication  $\delta: \overline{\mathcal{D}}(X) \rightarrow \overline{\mathcal{D}}^2(X)$  is  $\eta \circ \eta = \mathcal{D}(\eta) \circ \eta$ .

For each  $n \in \mathbb{N}$  the  $n$ -fold copower  $n \cdot (-)$  is a comonad on a category with finite coproducts. This is also the case on the Kleisli category  $\mathcal{Kl}(\mathcal{D})$ . In this case we describe it with a star  $n * (-)$ , to distinguish it from  $n \cdot (-)$  on  $\mathbf{Sets}$ . For a map  $f: X \rightarrow \mathcal{D}(Y)$  we get  $n * f: n * X \rightarrow \mathcal{D}(n * Y)$  given by  $\text{st}_2 \circ (\text{id}_n \times f)$ . Explicitly,  $(n * f)(\kappa_i x) = \sum_y f(x)(y) | \kappa_i y \rangle$ . The counit  $\varepsilon: n * X \rightarrow X$  is  $\eta \circ \nabla$ . The comultiplication  $\delta: n * X \rightarrow n * (n * X)$  is the map  $\eta \circ (\kappa_1 + \dots + \kappa_n) = [\eta \circ \kappa_i \circ \kappa_i]_i$ .

Thus we are looking at a situation:

$$\begin{array}{ccc}
 \overline{\mathcal{D}} \left( \begin{array}{c} \curvearrowright \\ \mathcal{Kl}(\mathcal{D}) \\ \curvearrowleft \end{array} \right) & \xrightarrow{n * (-)} & \\
 \downarrow \scriptstyle{-1} & & \\
 \mathcal{D} \left( \begin{array}{c} \curvearrowright \\ \mathbf{Sets} \\ \curvearrowleft \end{array} \right) & & 
 \end{array}
 \quad \text{where} \quad
 \overline{\mathcal{D}}(n * (-)) \xrightarrow{\mathcal{N}} n * \overline{\mathcal{D}}(-) \quad (4.1)$$

The normalisation operation  $\mathcal{N}$  is a map  $\mathcal{N}_A: \overline{\mathcal{D}}(n * A) \rightarrow n * \overline{\mathcal{D}}(A)$  in  $\mathcal{Kl}(\mathcal{D})$ . It is natural by (3.7), since for a map  $f: A \rightarrow B$  in  $\mathcal{Kl}(\mathcal{D})$ ,

$$\begin{aligned}
 (n * \overline{\mathcal{D}})(f) \bullet \mathcal{N}_A &= \mu \circ \mathcal{D}(\text{st}_2 \circ (\text{id} \times \overline{\mathcal{D}}(f))) \circ \mathcal{N} \\
 &= \mu \circ \mathcal{D}(\text{st}_2 \circ (\text{id} \times \eta) \circ (\text{id} \times f_*)) \circ \mathcal{N} \\
 &= \mu \circ \mathcal{D}(\eta \circ (\text{id} \times f_*)) \circ \mathcal{N} \\
 &= \mathcal{D}(\text{id} \times f_*) \circ \mathcal{N} \\
 &\stackrel{(3.7)}{=} \mathcal{N}_B \circ (n * f)_* \\
 &= \mu \circ \eta \circ \mathcal{N}_B \circ (n * f)_* \\
 &= \mu \circ \mathcal{D}(\mathcal{N}_B) \circ \eta \circ (n * f)_* \\
 &= \mathcal{N}_B \bullet \overline{\mathcal{D}}(n * f).
 \end{aligned}$$

The normalisation map  $\mathcal{N}$  commutes appropriately with the comultiplication maps of the two comonads, as expressed in the two rectangles:

$$\begin{array}{ccc}
\overline{\mathcal{D}}(n * A) & \xrightarrow{\delta_{n * A}} & \overline{\mathcal{D}}^2(n * A) \\
\downarrow \mathcal{N}_A & & \downarrow \overline{\mathcal{D}}(\mathcal{N}_A) \\
\mathcal{N}_A & & \overline{\mathcal{D}}(n * \overline{\mathcal{D}}(A)) \\
\downarrow & & \downarrow \mathcal{N}_{\overline{\mathcal{D}}(A)} \\
n * \overline{\mathcal{D}}(A) & \xrightarrow{n * \delta_A} & n * \overline{\mathcal{D}}^2(A)
\end{array}
\qquad
\begin{array}{ccc}
\overline{\mathcal{D}}(n * A) & \xrightarrow{\overline{\mathcal{D}}(\delta_A)} & \overline{\mathcal{D}}(n * (n * A)) \\
\downarrow \mathcal{N}_A & & \downarrow \mathcal{N}_{n * A} \\
\mathcal{N}_A & & n * \overline{\mathcal{D}}(n * A) \\
\downarrow & & \downarrow n * \mathcal{N}_A \\
n * \overline{\mathcal{D}}(A) & \xrightarrow{\delta_{\overline{\mathcal{D}}(A)}} & n * (n * \overline{\mathcal{D}}(A))
\end{array}$$

Commutation of the diagram on the left follows from (3.4):

$$\begin{aligned}
& \mathcal{N}_{\overline{\mathcal{D}}(A)} \bullet \overline{\mathcal{D}}(\mathcal{N}_A) \bullet \delta_{n * A} \\
&= \mu \circ \mathcal{D}(\mathcal{N}) \circ \mu \circ \mathcal{D}(\eta \circ \mu \circ \mathcal{D}(\mathcal{N})) \circ \eta \circ \eta \\
&= \mu \circ \mathcal{D}(\mathcal{N} \circ \mu \circ \mathcal{D}(\mathcal{N})) \circ \eta \circ \eta \\
&= \mu \circ \eta \circ \mathcal{N} \circ \mu \circ \mathcal{D}(\mathcal{N}) \circ \eta \\
&= \mathcal{N} \circ \mu \circ \eta \circ \mathcal{N} \\
&= \mathcal{N} \circ \mathcal{N} \\
&\stackrel{(3.4)}{=} \mathcal{D}(\text{id} \times \eta) \circ \mathcal{N} \\
&= \mu \circ \mathcal{D}(\eta \circ (\text{id} \times \eta)) \circ \mathcal{N} \\
&= \mu \circ \mathcal{D}(\text{st}_2 \circ (\text{id} \times (\eta \circ \eta))) \circ \mathcal{N} \\
&= (n * \delta_A) \bullet \mathcal{N}_A.
\end{aligned}$$

The above diagram on the right requires more work:

$$\begin{aligned}
& (n * \mathcal{N}_A \bullet \mathcal{N}_{n * A} \bullet \overline{\mathcal{D}}(\delta_A)) \\
&= \mu \circ \mathcal{D}(n * \mathcal{N}_A) \circ \mu \circ \mathcal{D}(\mathcal{N}_{n * A}) \circ \eta \circ \mu \circ \mathcal{D}(\eta \circ [\kappa_i \circ \kappa_i]_i) \\
&= \mu \circ \mathcal{D}(n * \mathcal{N}_A) \circ \mu \circ \eta \circ \mathcal{N}_{n * A} \circ \mathcal{D}([\kappa_i \circ \kappa_i]_i) \\
&= \mu \circ \mathcal{D}(n * \mathcal{N}_A) \circ \mathcal{N}_{n * A} \circ \mathcal{D}([\kappa_i \circ \kappa_i]_i) \\
&\stackrel{(*)}{=} \mathcal{D}([\kappa_i \circ \kappa_i]_i) \circ \mathcal{N}_A \\
&= \mu \circ \mathcal{D}([\eta \circ \kappa_i \circ \kappa_i]_i) \circ \mathcal{N}_A \\
&= \delta_{\overline{\mathcal{D}}(A)} \bullet \mathcal{N}_A
\end{aligned}$$



We explicitly prove the marked equation:

$$\begin{aligned}
 & (\mu \circ \mathcal{D}(n * \mathcal{N}_A) \circ \mathcal{N}_{n * A} \circ \mathcal{D}([\kappa_i \circ \kappa_i]_i))(\omega) \\
 &= (\mu \circ \mathcal{D}(n * \mathcal{N}_A) \circ \mathcal{N}_{n * A}) \left( \sum_{i,a} \omega(\kappa_i a) | \kappa_i \kappa_i a \right) \\
 &= (\mu \circ \mathcal{D}(n * \mathcal{N}_A)) \left( \sum_i \omega[i] | \kappa_i \left( \sum_a \frac{\omega(\kappa_i a)}{\omega[i]} | \kappa_i a \right) \right) \rangle \rangle \\
 &= \mu \left( \sum_i \omega[i] | \sum_{\varphi} \mathcal{N}_A \left( \sum_a \frac{\omega(\kappa_i a)}{\omega[i]} | \kappa_i a \right) (\varphi) | \kappa_i \varphi \right) \rangle \rangle \\
 &= \mu \left( \sum_i \omega[i] | 1 | \kappa_i \kappa_i \left( \sum_a \frac{\omega(\kappa_i a)}{\omega[i]} | a \right) \right) \rangle \rangle \\
 &= \sum_i \omega[i] | \kappa_i \kappa_i \left( \sum_a \frac{\omega(\kappa_i a)}{\omega[i]} | a \right) \rangle \\
 &= \mathcal{D}([\kappa_i \circ \kappa_i]_i) \left( \sum_i \omega[i] | \kappa_i \left( \sum_a \frac{\omega(\kappa_i a)}{\omega[i]} | a \right) \right) \rangle \rangle \\
 &= (\mathcal{D}([\kappa_i \circ \kappa_i]_i) \circ \mathcal{N}_A)(\omega).
 \end{aligned}$$

Commutation of  $\mathcal{N}$  with the two counits is expressed in the diagrams:

$$\begin{array}{ccc}
 \overline{\mathcal{D}}(n * A) \xrightarrow{\mathcal{N}} n * \overline{\mathcal{D}}(A) & & \overline{\mathcal{D}}(n * A) \xrightarrow{\mathcal{N}} n * \overline{\mathcal{D}}(A) \\
 \varepsilon_{n * A} \downarrow & & \downarrow \varepsilon_{\overline{\mathcal{D}}(A)} \\
 n * A & \xlongequal{\quad} & n * A \\
 \varepsilon_{n * A} \downarrow & & \downarrow \varepsilon_{\overline{\mathcal{D}}(A)} \\
 n * A & \xlongequal{\quad} & n * A
 \end{array}$$

The diagram on the left commutes:

$$\begin{aligned}
 n * \varepsilon_A \bullet \mathcal{N} &= \mu \circ \mathcal{D}(\text{st}_2 \circ (\text{id} \times \text{id})) \circ \mathcal{N} \\
 &= (\text{st}_2)_*(\mathcal{N}) \\
 &\stackrel{(3.6)}{=} \text{id} \\
 &= \varepsilon_{n * A}.
 \end{aligned}$$

Somewhat surprisingly, the above rectangle on the right does not commute, despite (3.3). The latter diagram translates into the following diagram in  $\mathcal{Kl}(\mathcal{D})$ .

$$\begin{array}{ccc}
 \overline{\mathcal{D}}(n * A) \xrightarrow{\mathcal{N}} n * \overline{\mathcal{D}}(A) & & \\
 \downarrow \varepsilon_{\overline{\mathcal{D}}(A)} & & \downarrow \varepsilon_{\overline{\mathcal{D}}(A)} \\
 \overline{\mathcal{D}}(A) & \xlongequal{\quad} & \overline{\mathcal{D}}(A) \\
 \varepsilon_A \downarrow & & \downarrow \varepsilon_A \\
 A & \xlongequal{\quad} & A
 \end{array}$$

This non-standard diagram does commute in  $\mathcal{Kl}(\mathcal{D})$ , because:

$$\begin{aligned}
 \varepsilon_A \bullet \varepsilon_{\overline{\mathcal{D}}(A)} \bullet \mathcal{N} &= \mu \circ \mathcal{D}(\text{id}) \circ \mu \circ \mathcal{D}(\eta \circ \nabla) \circ \mathcal{N} \\
 &= \mu \circ \mathcal{D}(\nabla) \circ \mathcal{N} \\
 &\stackrel{(3.3)}{=} \mathcal{D}(\nabla) \\
 &= \mu \circ \mathcal{D}(\eta \circ \nabla) \\
 &= (\eta \circ \nabla)_* \\
 &= \mu \circ \mathcal{D}(\text{id}) \circ \eta \circ (\varepsilon_A)_* \\
 &= \varepsilon_A \bullet \overline{\mathcal{D}}(\varepsilon_A)
 \end{aligned}$$

We summarise what we have described above.

**Proposition 11.** In the situation (4.1) the hyper normalisation map  $\mathcal{N}$  is a distributive law of the functor  $n * (-)$  over the comonad  $\overline{\mathcal{D}}$ . It commutes with the  $n * (-)$ -comultiplication, but not with the  $n * (-)$ -counit.  $\square$

## 5. PREDICATES

We continue the main line of our story by using the new ‘hyper’ form of normalisation to describe conditioning. Our description of conditioning makes crucial uses of predicates. Hence we first have to explain what predicates in a (discrete) probabilistic setting are, and how they are used as ‘evidence’. The current section provides the required background information on predicates, which is used in the next section to describe ‘hyper’ conditioning.

**5.1. Events and predicates.** Let  $A$  be an arbitrary set, seen as ‘sample space’. An *event* is a subset  $E \subseteq A$  of the sample space. These events are traditionally used as predicates on  $A$ . We need to use a more general ‘fuzzy’ kind of predicate, namely functions  $p: A \rightarrow [0, 1]$ , where  $[0, 1] \subseteq \mathbb{R}$  is the unit interval. An event  $E \subseteq A$  can be identified with a ‘sharp’ predicate  $A \rightarrow \{0, 1\} \subseteq [0, 1]$ , taking values in the subset  $\{0, 1\}$  of Booleans. For an event  $E$  we write  $\mathbf{1}_E \in [0, 1]^A$  for the associated sharp predicate, given by the indicator function  $\mathbf{1}_E$ , defined by  $\mathbf{1}_E(a) = 1$  if  $a \in E$  and  $\mathbf{1}_E(a) = 0$  if  $a \notin E$ .

Sharp predicates (subsets) on  $A$  form a Boolean algebra. The set  $[0, 1]^A$  of (non-sharp, fuzzy) predicates over  $A$  however is an ‘effect module’, see [14, 9, 5]. We briefly describe the relevant structure, without going into the details of what an effect module precisely is. There are truth and falsity predicates  $\mathbf{1}, \mathbf{0} \in [0, 1]^A$  which map each element  $a \in A$  to 1, or to 0 respectively. Given two predicates  $p, q \in [0, 1]^A$  we say that they are orthogonal, written as  $p \perp q$ , if  $p(a) + q(a) \leq 1$ , for all  $a \in A$ . In that case we write  $p \oplus q \in [0, 1]^A$  for the pointwise sum:  $(p \oplus q)(a) = p(a) + q(a)$ . These  $(\oplus, \mathbf{0})$  make  $[0, 1]^A$  a partially commutative monoid. There is also a ‘negation’, usually written as orthosupplement  $p^\perp$ , with  $(p^\perp)(a) = 1 - p(a)$ . Notice that  $p^{\perp\perp} = p$  and  $p \oplus p^\perp = \mathbf{1}$ . Moreover,  $(\mathbf{1}_E)^\perp = \mathbf{1}_{\neg E}$ , where  $\neg E = \{a \in A \mid a \notin E\}$ . Finally, for a scalar  $s \in [0, 1]$  and a predicate  $p \in [0, 1]^A$  there is a ‘scaled’ predicate  $s \cdot p \in [0, 1]^A$  given by  $(s \cdot p)(a) = s \cdot p(a)$ .

An  $n$ -tuple of predicates  $p_1, \dots, p_n \in [0, 1]^A$  is called a *test* — or an  $n$ -test, to be more specific — if  $p_1 \oplus \dots \oplus p_n = \mathbf{1}$ . This terminology comes from quantum theory, see *e.g.* [9, 5]. This means that these predicates  $p_i$  add up to one, pointwise. When we write such sum expressions, we implicitly assume that the relevant predicates are orthogonal.

Notice that  $\mathcal{D}(1) \cong 1$  and  $\mathcal{D}(2) \cong [0, 1]$ . Hence we can identify predicates on  $A$  with maps  $A \rightarrow \mathcal{D}(2)$ . It takes a bit more effort to see that  $n$ -tests on  $A$  can be identified with functions  $p: A \rightarrow \mathcal{D}(n)$ , that is, with Kleisli maps  $A \dashrightarrow n$ . Indeed, the  $i$ -th predicate  $p_i \in [0, 1]^A$  can be extracted from  $p$  as  $p_i(a) = p(a)(i)$ , using the functional notation for distributions. We thus see that a test can be seen as a probabilistic partition. After all, an ordinary, partition of a set  $A$  into  $n$ -parts can be identified with a function  $A \rightarrow n$ , see also Section 7. A predicate  $p \in [0, 1]^A$  can be identified with a 2-test, consisting of  $p$  itself and and its orthosupplement  $p^\perp$ .

For a Kleisli map  $f: A \multimap B$  and a predicate  $q \in [0, 1]^B$  there is a (weakest precondition) predicate  $f^*(q)$  on  $A$  defined by:

$$f^*(q)(a) = \sum_{b \in B} f(a)(b) \cdot q(b). \quad (5.1)$$

It is not hard to see that this map  $f^*: [0, 1]^B \rightarrow [0, 1]^A$  preserves the effect module structure described in Subsection 5.1. In this way an  $n$ -test  $q_1, \dots, q_n$  can be turned into an  $n$ -test  $f^*(q_1), \dots, f^*(q_n)$  on  $A$ .

**5.2. Validity.** Given an event  $E \subseteq A$  on a sample space  $A$  we often like to know its probability wrt. a distribution on  $A$ . This probability is commonly written as  $P(E)$ . In order to make the underlying distribution  $\omega \in \mathcal{D}(A)$  explicit we prefer to write  $P_\omega(E)$  instead of just  $P(E)$ . This probability is defined as  $P_\omega(E) = \sum_{a \in E} \omega(a)$ . Notice that this is a finite sum, in  $[0, 1]$ , since the distribution  $\omega$  has finite support.

More generally, for a not necessarily sharp predicate  $p \in [0, 1]^A$  and a distribution  $\omega \in \mathcal{D}(A)$  we define the validity (expected value)  $\omega \models p$  in  $[0, 1]$  as:

$$\omega \models p \stackrel{\text{def}}{=} \sum_{a \in A} \omega(a) \cdot p(a) \quad \text{so that} \quad \omega \models \mathbf{1}_E = P_\omega(E). \quad (5.2)$$

It is easy to see that  $(\omega \models \mathbf{1}) = 1$  and  $(\omega \models \mathbf{0}) = 0$ . Moreover,  $(\omega \models p^\perp) = 1 - (\omega \models p)$  and  $(\omega \models p \otimes q) = (\omega \models p) + (\omega \models q)$ .

**5.3. Conditionals, traditionally.** For a predicate  $p \in [0, 1]^A$  and a distribution  $\omega \in \mathcal{D}(A)$  with  $\omega \models p \neq 0$  we describe a *conditional distribution*  $\omega|_p \in \mathcal{D}(A)$ , pronounced as “ $\omega$  given  $p$ ”, and defined as:

$$\omega|_p = \sum_{a \in A} \frac{\omega(a) \cdot p(a)}{\omega \models p} |a\rangle. \quad (5.3)$$

The big nuisance with these ‘traditional’ conditionals  $\omega|_p$  is that they are not always defined: they involve division by the validity  $\omega \models p \in [0, 1]$ , which should thus be non-zero. The key improvement in our novel ‘hyper’ description of conditioning (in the next section) is that it is a total operation which does not require such side-conditions — like hyper normalisation.

As illustration of validity and conditioning, consider a distribution  $\omega = \frac{1}{4}|a\rangle + \frac{1}{3}|b\rangle + \frac{5}{12}|c\rangle$  on a set  $A = \{a, b, c\}$ , an event  $E = \{a, c\} \subseteq A$  and a predicate  $p \in [0, 1]^A$  with  $p(a) = \frac{1}{2}, p(b) = \frac{1}{4}, p(c) = 1$ . Then:

$$\begin{array}{lll} \omega \models p = \frac{5}{8} & \omega|_p = \frac{1}{5}|a\rangle + \frac{2}{15}|b\rangle + \frac{2}{3}|c\rangle & P_\omega(E) = \omega \models \mathbf{1}_E = \frac{2}{3} \\ \omega \models p^\perp = \frac{3}{8} & \omega|_{p^\perp} = \frac{1}{3}|a\rangle + \frac{2}{3}|b\rangle & \omega|_{\mathbf{1}_E} = \frac{3}{8}|a\rangle + \frac{5}{8}|c\rangle. \end{array}$$

## 6. HYPER CONDITIONING

We shall formulate our characterisation of conditionals for tests and not for predicates. As noted before, predicates are subsumed by tests, as 2-tests. Tests not only provide greater generality, but also better capture the underlying idea. They lead to what may be called ‘parallel’ conditioning. Therefor we use the sign  $\parallel$ , commonly used for a parallel processes in concurrency theory.

**Definition 12.** Let  $t: A \rightarrow \mathcal{D}(n)$  be an  $n$ -test on a set  $A$ , and  $\omega \in \mathcal{D}(A)$  be distribution on  $A$ . The ‘hyper’ conditional  $\omega \parallel_t \in \mathcal{D}(n \cdot \mathcal{D}(A))$  is defined as:

$$\omega \parallel_t \stackrel{\text{def}}{=} \mathcal{N}\left(\text{gr}(t)_*(\omega)\right).$$

Thus, the hyper conditioning is a function:

$$(-) \parallel_t = \left( \mathcal{D}(A) \xrightarrow{\text{gr}(t)_*} \mathcal{D}(n \cdot A) \xrightarrow{\mathcal{N}} \mathcal{D}(n \cdot \mathcal{D}(A)) \right) \quad (6.1)$$

The map (6.1) is a Kleisli map  $\mathcal{D}(A) \dashrightarrow n \cdot \mathcal{D}(A)$ . It is called an ‘abstract channel’ in [22, 24], where it is claimed that such abstract channels capture the essence of leakages in quantitative information flow. This will be elaborated in Section 7.

The map  $\text{gr}(t): A \rightarrow \mathcal{D}(n \cdot A)$  is called the *instrument* associated with the test  $t: A \rightarrow n$ , in the sense of [9, 10, 12].

This definition of  $\omega \parallel_t$  is quite abstract, so we give a more concrete illustration. We re-use the example from the end of Subsection 5.3, with  $A = \{a, b, c\}$ , and  $p \in [0, 1]^A$  given by  $p(a) = \frac{1}{2}, p(b) = \frac{1}{4}, p(c) = 1$ . We identify the predicate  $p$  with the 2-test  $t = (p, p^\perp)$ , giving a graph map  $\text{gr}(t): A \rightarrow \mathcal{D}(2 \cdot A)$  defined by  $\text{gr}(t)(a) = p(x) \mid \kappa_0 x \rangle + p^\perp(x) \mid \kappa_1 x \rangle$ .

The distribution  $\omega = \frac{1}{4} \mid a \rangle + \frac{1}{3} \mid b \rangle + \frac{5}{12} \mid c \rangle$  on  $A$  gives rise to:

$$\begin{aligned} \text{gr}(t)_*(\omega) &\stackrel{(2.4)}{=} \sum_{z \in A+A} \left( \sum_{x \in A} \omega(x) \cdot \text{gr}(t)(x)(z) \right) \mid z \rangle \\ &= \omega(a) \cdot p(a) \mid \kappa_0 a \rangle + \omega(b) \cdot p(b) \mid \kappa_0 b \rangle + \omega(c) \cdot p(c) \mid \kappa_0 c \rangle \\ &\quad \omega(a) \cdot p^\perp(a) \mid \kappa_1 a \rangle + \omega(b) \cdot p^\perp(b) \mid \kappa_1 b \rangle + \omega(c) \cdot p^\perp(c) \mid \kappa_1 c \rangle \\ &= \frac{1}{8} \mid \kappa_0 a \rangle + \frac{1}{12} \mid \kappa_0 b \rangle + \frac{5}{12} \mid \kappa_0 c \rangle + \frac{1}{8} \mid \kappa_1 a \rangle + \frac{1}{4} \mid \kappa_1 b \rangle. \end{aligned}$$

Let’s use the short name  $\rho = \text{gr}(t)_*(\omega)$  for the latter distribution. Then, according to Definition 4,

$$\rho[0] = \sum_{x \in A} \rho(\kappa_0 x) = \frac{1}{8} + \frac{1}{12} + \frac{5}{12} = \frac{5}{8} = \omega \models p.$$

Similarly, we have  $\rho[1] = \frac{3}{8} = \omega \models p^\perp$ . We can now describe the hyper conditional more concretely:

$$\begin{aligned} \omega \parallel_t &= \mathcal{N}(\rho) \quad \text{since we abbreviate } \rho = \text{gr}(t)_*(\omega) \\ &\stackrel{(3.1)}{=} \rho[0] \mid \kappa_0 \left( \sum_x \frac{\rho(\kappa_0 x)}{\rho[0]} \mid x \rangle \right) \rangle + \rho[1] \mid \kappa_1 \left( \sum_x \frac{\rho(\kappa_1 x)}{\rho[1]} \mid x \rangle \right) \rangle \\ &= (\omega \models p) \mid \kappa_0 \left( \sum_x \frac{\omega(x) \cdot p(x)}{\omega \models p} \mid x \rangle \right) \rangle + (\omega \models p^\perp) \mid \kappa_1 \left( \sum_x \frac{\omega(x) \cdot p^\perp(x)}{\omega \models p^\perp} \mid x \rangle \right) \rangle \\ &= (\omega \models p) \mid \kappa_0(\omega|_p) \rangle + (\omega \models p^\perp) \mid \kappa_1(\omega|_{p^\perp}) \rangle \\ &= \frac{5}{8} \mid \kappa_0 \left( \frac{1}{5} \mid a \rangle + \frac{2}{15} \mid b \rangle + \frac{2}{3} \mid c \rangle \right) \rangle + \frac{3}{8} \mid \kappa_1 \left( \frac{1}{3} \mid a \rangle + \frac{2}{3} \mid b \rangle \right) \rangle. \end{aligned}$$

Generalising this example we get the following formulation of hyper conditioning in terms of traditional conditioning.

**Lemma 13.** For an  $n$ -test  $t = (p_1, \dots, p_n)$  of predicates  $p_i$  and a state  $\omega$  we have:

$$\omega \parallel_t = \sum_{\substack{1 \leq i \leq n \\ \omega \models p_i \neq 0}} (\omega \models p_i) \mid \kappa_i(\omega|_{p_i}) \rangle. \quad \square$$

Notice that the problem that traditional conditionals  $\omega|_{p_i}$  are not defined if  $\omega \models p_i = 0$  (again) disappears in this ‘hyper’ formulation, since the entries with  $\omega \models p_i = 0$  do not show up in the above formal convex sum.

It turns out that under the distribution  $\omega$  and test  $t$  can be recovered from a hyper conditional  $\Omega = \omega \parallel_t$ , via the bijective correspondence of Proposition 3.

**Proposition 14.** Let  $\omega \in \mathcal{D}(A)$  be a distribution with  $\text{supp}(\omega) = A$ , and let  $t: A \rightarrow \mathcal{D}(n)$  be an  $n$ -test. Then both  $\omega$  and  $t$  can be recovered from the hyper conditional  $\omega \parallel_t \in \mathcal{D}(n \cdot \mathcal{D}(A))$ , namely via:

- $\omega = (\pi_2)_*(\omega \parallel_t)$ ;
- $t: A \rightarrow \mathcal{D}(n)$  is the map determined by the distribution  $\Omega = (\text{st}_2)_*(\omega \parallel_t) \in \mathcal{D}(n \cdot A)$ , as in Proposition 3.

**Proof** The first bullet point is easy:

$$(\pi_2)_*(\omega \parallel_t) \stackrel{(3.3)}{=} \mathcal{D}(\pi_2)(\text{gr}(t)_*(\omega)) = (\mathcal{D}(\pi_2) \circ \text{gr}(t))_*(\omega) \stackrel{(2.7)}{=} \eta_*(\omega) = \omega.$$

For the second bullet point, we write  $\Omega = (\text{st}_2)_*(\omega \parallel_t) \in \mathcal{D}(n \cdot A)$ . We first show that  $\mathcal{D}(\pi_2)(\Omega) = \mathcal{D}(\nabla)(\Omega) = \omega$ , using what we have just proven:

$$\mathcal{D}(\pi_2)(\Omega) = (\mathcal{D}(\pi_2) \circ (\text{st}_2)_*)(\omega \parallel_t) = (\mathcal{D}(\pi_2) \circ \text{st}_2)_*(\omega \parallel_t) \stackrel{(2.7)}{=} (\pi_2)_*(\omega \parallel_t) = \omega.$$

Hence the side-condition in the bijective correspondence of Proposition 3 is satisfied for  $\Omega$ . Thus, we can write  $\Omega = \text{gr}(f)_*(\omega)$ , as in (2.10), for a unique map  $f: A \rightarrow \mathcal{D}(n)$ . We have to show that  $f = t$ , the original test. But this follows from:

$$\Omega = (\text{st}_2)_*(\omega \parallel_t) = ((\text{st}_2)_* \circ \mathcal{N})(\text{gr}(t)_*(\omega)) \stackrel{(3.6)}{=} \text{gr}(t)_*(\omega). \quad \square$$

**Example 15.** We illustrate how the distribution and test can be recovered for a hyper distribution  $\Phi \in \mathcal{D}(2 \cdot \mathcal{D}(\{H, T\}))$  over the 2-element set  $\{H, T\}$  of ‘head’ and ‘tail’ outcomes.

$$\Phi = \frac{1}{2} | \kappa_0(\frac{2}{3} | H \rangle + \frac{1}{3} | T \rangle) \rangle + \frac{1}{2} | \kappa_1(\frac{1}{3} | H \rangle + \frac{2}{3} | T \rangle) \rangle$$

The first bullet in Proposition 14 says that we can obtain the underlying distribution  $\omega \in \mathcal{D}(\{H, T\})$  as:

$$\begin{aligned} \omega &= (\pi_2)_*(\Phi) = \mu(\mathcal{D}(\pi_2)(\frac{1}{2} | \kappa_0(\frac{2}{3} | H \rangle + \frac{1}{3} | T \rangle) \rangle + \frac{1}{2} | \kappa_1(\frac{1}{3} | H \rangle + \frac{2}{3} | T \rangle) \rangle)) \\ &= \mu(\frac{1}{2} | \frac{2}{3} | H \rangle + \frac{1}{3} | T \rangle \rangle + \frac{1}{2} | \frac{1}{3} | H \rangle + \frac{2}{3} | T \rangle \rangle) \\ &= \frac{1}{2} \cdot \frac{2}{3} | H \rangle + \frac{1}{2} \cdot \frac{1}{3} | T \rangle + \frac{1}{2} \cdot \frac{1}{3} | H \rangle + \frac{1}{2} \cdot \frac{2}{3} | T \rangle \\ &= \frac{1}{2} | H \rangle + \frac{1}{2} | T \rangle. \end{aligned}$$

For the second bullet we compute:

$$\begin{aligned} (\text{st}_2)_*(\Psi) &= \mu(\mathcal{D}(\text{st}_2)(\frac{1}{2} | \kappa_0(\frac{2}{3} | H \rangle + \frac{1}{3} | T \rangle) \rangle + \frac{1}{2} | \kappa_1(\frac{1}{3} | H \rangle + \frac{2}{3} | T \rangle) \rangle)) \\ &= \mu(\frac{1}{2} | \frac{2}{3} | \kappa_0 H \rangle + \frac{1}{3} | \kappa_0 T \rangle \rangle + \frac{1}{2} | \frac{1}{3} | \kappa_1 H \rangle + \frac{2}{3} | \kappa_1 T \rangle \rangle) \\ &= \frac{1}{3} | \kappa_0 H \rangle + \frac{1}{6} | \kappa_0 T \rangle + \frac{1}{6} | \kappa_1 H \rangle + \frac{1}{3} | \kappa_1 T \rangle. \end{aligned}$$

The recipe (2.10) now gives a test function  $s: \{H, T\} \rightarrow \mathcal{D}(2)$ , namely:

$$s(H) = \frac{1/3}{1/2} | 0 \rangle + \frac{1/6}{1/2} | 1 \rangle = \frac{2}{3} | 0 \rangle + \frac{1}{3} | 1 \rangle \quad s(T) = \frac{1/6}{1/2} | 0 \rangle + \frac{1/3}{1/2} | 1 \rangle = \frac{1}{3} | 0 \rangle + \frac{2}{3} | 1 \rangle.$$

Then indeed,  $\Phi = \omega \parallel_s$ , as can be checked easily.

The reader may wish to do a similar computation for the hyper distribution  $\Psi \in \mathcal{D}(3 \cdot \mathcal{D}(\{H, T\}))$  given by:

$$\Psi = \frac{1}{3} |\kappa_0(\frac{2}{3}|H\rangle + \frac{1}{3}|T\rangle)\rangle + \frac{1}{3} |\kappa_1(\frac{1}{2}|H\rangle + \frac{1}{2}|T\rangle)\rangle + \frac{1}{3} |\kappa_2(\frac{1}{3}|H\rangle + \frac{2}{3}|T\rangle)\rangle$$

The answer appears in Example 19.

We illustrate how the ‘hyper’ approach works in Bayesian reasoning, for a standard medical examination example copied from [16].

**Example 16.** Write  $2_D = \{d, d^\perp\}$  and  $2_T = \{t, t^\perp\}$  for two 2-element sets, where  $d$  represents ‘disease’ and  $d^\perp$  represents ‘no disease’. Similarly, the element  $t$  represents a positive test (examination outcome), and  $t^\perp$  a negative outcome. Consider the following simple Bayesian network, described as Kleisli maps (as in [16]):

$$1 \xrightarrow{\omega} 2_D \xrightarrow{s} 2_T \quad \text{with} \quad \begin{cases} \omega = \frac{1}{100}|d\rangle + \frac{99}{100}|d^\perp\rangle \\ s(d) = \frac{9}{10}|t\rangle + \frac{1}{10}|t^\perp\rangle \\ s(d^\perp) = \frac{1}{20}|t\rangle + \frac{19}{20}|t^\perp\rangle. \end{cases}$$

The state  $\omega$  captures the prior probability of 1% of having the disease. The function  $s: 2_D \rightarrow \mathcal{D}(2_T)$  describes the sensitivity of the test.

We write  $T?: 2_T \rightarrow [0, 1]$  for the (sharp) predicate given by  $T?(t) = 1$  and  $T?(t^\perp) = 0$ . Together with  $T?^\perp$  it forms a 2-test  $T! = (T?, T?^\perp)$  on  $2_T$ . It gives rise to a 2-test  $s^*(T!) = (s^*(T?), s^*(T?^\perp))$  on  $2_D$  via (5.1), given by:

$$s^*(T?)(d) = \frac{9}{10} \quad s^*(T?^\perp)(d) = \frac{1}{10} \quad s^*(T?)(d^\perp) = \frac{1}{20} \quad s^*(T?^\perp)(d^\perp) = \frac{19}{20}.$$

The associated instrument map  $\text{gr}(s^*(T!)): 2_D \rightarrow \mathcal{D}(2 \cdot 2_D)$  is:

$$\text{gr}(s^*(T!))(d) = \frac{9}{10} |\kappa_0 d\rangle + \frac{1}{10} |\kappa_1 d\rangle \quad \text{gr}(s^*(T!))(d^\perp) = \frac{1}{20} |\kappa_0 d^\perp\rangle + \frac{19}{20} |\kappa_1 d^\perp\rangle$$

When applied to the (prior) state  $\omega$  it gives:

$$\text{gr}(s^*(T!))_*(\omega) = \frac{9}{1000} |\kappa_0 d\rangle + \frac{1}{1000} |\kappa_1 d\rangle + \frac{99}{2000} |\kappa_0 d^\perp\rangle + \frac{1881}{2000} |\kappa_1 d^\perp\rangle$$

The resulting hyper conditional  $\omega \|_{s^*(T!)} = \mathcal{N}(\text{gr}(s^*(T!))_*(\omega))$  is then:

$$\omega \|_{s^*(T!)} = \frac{117}{2000} \left| \kappa_0 \left( \frac{18}{117} |d\rangle + \frac{99}{117} |d^\perp\rangle \right) \right\rangle + \frac{1883}{2000} \left| \kappa_1 \left( \frac{2}{1883} |d\rangle + \frac{1881}{1883} |d^\perp\rangle \right) \right\rangle$$

This hyper distribution  $\omega \|_{s^*(T!)} \in \mathcal{D}(2 \cdot \mathcal{D}(2_D))$  is obtained by backward learning, from the 2-test  $T!$ . It is given by a convex combination of two conditional (normalised) inner distributions. The left inner distribution describes the probability  $\frac{18}{117} \sim 15\%$  of having the disease after a positive test outcome  $T?$ , whereas the right inner distribution gives the probability  $\frac{2}{1883} \sim 0.1\%$  of having the disease after a negative outcome  $T?^\perp$ . One could say that the parallel conditioning that happens in a hyper conditional  $\omega \|_t$  corresponds to a many worlds view — as is sometimes used, for instance, in counterfactual reasoning [26].

The hyper approach does not give direct access to these inner distributions. But further calculations can be done with this hyper distribution. If one is not interested in the second inner distribution it can be removed via a  $!$  map to the final (singleton) set 1, leading to a distribution:

$$\frac{117}{2000} \left| \kappa_0 \left( \frac{18}{117} |d\rangle + \frac{99}{117} |d^\perp\rangle \right) \right\rangle + \frac{1883}{2000} \left| \kappa_1 0 \right\rangle \in \mathcal{D}(\mathcal{D}(2_D) + 1).$$

Via multiplication it can be further reduced to a distribution in  $\mathcal{D}(2_D + 1)$ , but then one loses the conditional, as in:

$$\frac{18}{2000}|\kappa_0 d\rangle + \frac{99}{2000}|\kappa_0 d^\perp\rangle + \frac{1883}{2000}|\kappa_1 0\rangle \in \mathcal{D}(2_D + 1).$$

## 7. APPLICATIONS IN QUANTITATIVE INFORMATION FLOW

The hyper conditional construction  $\omega \parallel_t$  that we use here — see Definition 12 — is inspired by a ‘denotation of a channel’ construction in quantitative information flow, see [19, 24, 22, 20]. This will be sketched first. Subsequently we describe how tests and hyper distributions are ordered, and how these orders are related.

An abstract channel in [19, 24, 22, 20] from a set  $X$  to set  $Y$  is what we call a Kleisli map  $c: X \multimap Y$ , that is, a function  $c: X \rightarrow \mathcal{D}(Y)$ . The sets  $X, Y$  used in this context are finite, so we can replace them by numbers, and write a channel as Kleisli map  $n \multimap m$ . As noted in Subsection 2.3 such a channel gives an  $m$ -test on  $n$ .

The *denotation* of a channel  $c: n \multimap m$  is defined in [24, 22] as a function  $\llbracket c \rrbracket: \mathcal{D}(n) \rightarrow \mathcal{D}^2(n)$ . It uses conditional distributions, via normalisation. We redescribe this denotation via the notation from this paper. Let channel  $c: n \multimap m$  correspond to  $m$ -test  $c_i \in [0, 1]^n$  given by  $c_i(j) = c(j)(i)$ . The denotation  $\llbracket c \rrbracket(\omega) \in \mathcal{D}(\mathcal{D}(n))$  is defined for  $\omega \in \mathcal{D}(n)$  as:

$$\llbracket c \rrbracket(\omega) = \sum_{\substack{0 \leq i \leq m-1 \\ \omega \models_{c_i} \neq 0}} (\omega \models c_i) \Big| \omega \Big|_{c_i} \rangle. \quad (7.1)$$

There is an obvious similarity with ‘our’ formula for hyper conditioning in Lemma 13. The difference is that we use an inner copower  $\mathcal{D}(m \cdot \mathcal{D}(n))$  instead of  $\mathcal{D}(\mathcal{D}(n))$ , with corresponding coprojections  $\kappa_j$ , to keep the inner conditional distributions  $\omega \Big|_{c_i}$  separate.

It is not hard to see that the above formulation (7.1) can be obtained from ours as  $\llbracket c \rrbracket(\omega) = \mathcal{D}(\nabla)(\omega \parallel_c)$ , by removing the coprojections, via the codiagonal  $\nabla: m \cdot \mathcal{D}(n) \rightarrow \mathcal{D}(n)$ . In [22] it is observed<sup>1</sup> that applying multiplication  $\mu$  to  $\llbracket c \rrbracket(\omega)$  yields the original distribution  $\omega$ . In our case this follows directly from the first bullet in Proposition 14.

Denotations  $\llbracket c \rrbracket: \mathcal{D}(n) \rightarrow \mathcal{D}^2(n)$  are instances of Hidden Markov Models in [22], whose action on ‘uncertainty measures’ is characterised in terms of uncertainty transformers. Here we zoom in on the order theoretic aspects.

**7.1. Refinements.** We continue with refinements of partitions (tests), and start with the ordinary (non-probabilistic) case. Let  $(S_i)_{i \in n}$  be a partition of a set  $A$ . That means  $S_i \subseteq A$  with  $\bigcup_i S_i = A$ , and  $S_i \cap S_{i'} = \emptyset$  for  $i \neq i'$ . Thus, each element  $a \in A$  can be mapped to a unique element  $i \in n$  with  $a \in S_i$ . Hence the partition  $(S_i)$  can be identified with a function  $s: A \rightarrow n$ , where  $S_i = s^{-1}(i)$ .

If we have two partitions  $(S_i)_{i \in n}$  and  $(T_j)_{j \in m}$  of the same set  $A$  we can say that  $(S_i) \sqsubseteq (T_j)$  if for each  $i \in n$  there is a  $j \in m$  with  $S_i \subseteq T_j$ . This means that the  $S$ -partition is more refined than the  $T$ -partition, since each subsets  $S_i$  fits in some  $T_j$ .

<sup>1</sup>See after Defn. 7 in [22], where multiplication  $\mu$  is called average. We add that the construction of  $\llbracket c \rrbracket(\omega)$  in [24, 22] requires some *ad hoc* ‘removal’ and ‘renaming’ of redundant data that happens automatically in the current situation by the formal convex sum formalism from Subsection 2.2.

There is particularly simple way to express this refinement relation when we switch to the description in terms of functions. Let  $s: A \rightarrow n$  and  $t: A \rightarrow m$  be the functions corresponding to the partitions  $(S_i)$  and  $(T_j)$ . Then it is not hard to see:

$$(S_i) \sqsubseteq (T_j) \iff \text{there is a } h: n \rightarrow m \text{ with } \begin{array}{ccc} & & n \\ & \nearrow s & \downarrow h \\ A & & m \\ & \searrow t & \end{array}$$

Here we have to assume that  $S_i \neq \emptyset$ , for each  $i \in n$ . Then we can define  $h(i) = j$  iff  $S_i \subseteq T_j$ . This yields what is sometimes called the lattice of information [18].

This functional description of refinement can be translated very easily to a probabilistic setting, simply by using Kleisli maps instead of ordinary functions. This done in the first item below. The second item gives an alternative formulation of refinement on hyper distributions, used in quantitative information flow, see *e.g.* [19, 24, 22, 20]. We slightly adapt it to the current setting.

**Definition 17.** Let  $A$  be a set and  $n, m$  be natural numbers.

(1) For two tests  $s: A \rightarrow \mathcal{D}(n)$  and  $t: A \rightarrow \mathcal{D}(m)$  on  $A$  one defines:

$$s \sqsubseteq t \quad \text{iff} \quad \text{there is a function } h: n \rightarrow \mathcal{D}(m) \text{ with } h \bullet s = t, \text{ as in: } \begin{array}{ccc} & & n \\ & \nearrow s \bullet & \downarrow h \\ A & & m \\ & \searrow t & \end{array}$$

(2) For two hyper distributions  $\Phi \in \mathcal{D}(n \cdot \mathcal{D}(A))$  and  $\Psi \in \mathcal{D}(m \cdot \mathcal{D}(A))$  we put:

$$\Phi \sqsubseteq \Psi \quad \text{iff} \quad \begin{cases} \text{there is an } \Omega \in \mathcal{D}(m \cdot \mathcal{D}(n \cdot \mathcal{D}(A))) \text{ with} \\ (\pi_2)_*(\Omega) = \Phi \text{ and } \mathcal{D}(m \cdot (\pi_2)_*)(\Omega) = \Psi. \end{cases}$$

The theorem below is a basic result in quantitative information flow, see [19, 24, 22, 20]. Our aim is to illustrate how our approach to normalisation and conditioning can be used, by giving abstract proof constructions.

**Theorem 18.** In the situation of Definition 17,

- (1) if  $s \sqsubseteq t$  then  $\omega \parallel_s \sqsubseteq \omega \parallel_t$  for each  $\omega \in \mathcal{D}(A)$ ;
- (2) if  $\omega \parallel_s \sqsubseteq \omega \parallel_t$  for some  $\omega \in \mathcal{D}(A)$  with  $\text{supp}(\omega) = A$  and  $\text{supp}(s_*(\omega)) = n$ , then  $s \sqsubseteq t$ .

**Proof** Let  $s \sqsubseteq t$  via Kleisli map  $h: n \rightarrow \mathcal{D}(m)$ , so that  $h \bullet s = t$ . We write  $h_1 = h \circ \pi_1: n \cdot \mathcal{D}(A) \rightarrow \mathcal{D}(m)$ , with associated graph map  $\text{gr}(h_1): n \cdot \mathcal{D}(A) \rightarrow \mathcal{D}(m \cdot (n \cdot \mathcal{D}(A)))$ . For an arbitrary distribution  $\omega \in \mathcal{D}(A)$  we take:

$$\Omega \stackrel{\text{def}}{=} (\omega \parallel_s) \parallel_{h_1} = \mathcal{N}\left(\text{gr}(h_1)_*(\omega \parallel_s)\right) \in \mathcal{D}\left(m \cdot \mathcal{D}(n \cdot \mathcal{D}(A))\right). \quad (7.2)$$

By construction,  $(\pi_2)_*(\Omega) = \omega \parallel_s$ , see Proposition 14. The proofs of the following two auxiliary equations are easily obtained.

$$\mathcal{D}(m \cdot \pi_2) \circ \text{gr}(h_1) = h \cdot \mathcal{D}(A) \quad (h \cdot A) \bullet \text{gr}(s) = \text{gr}(h \bullet s) \quad (7.3)$$



Then  $\omega \parallel_s \sqsubseteq \omega \parallel_t$  via  $\Omega$  follows from:

$$\begin{aligned}
 \mathcal{D}(m \cdot (\pi_2)_*)(\Omega) &= (\mathcal{D}(m \cdot \mu) \circ \mathcal{D}(m \cdot \mathcal{D}(\pi_2)) \circ \mathcal{N} \circ \text{gr}(h_1)_* \circ \mathcal{N} \circ \text{gr}(s)_*)(\omega) \\
 &\stackrel{(3.7)}{=} (\mathcal{D}(m \cdot \mu) \circ \mathcal{N} \circ \mathcal{D}(m \cdot \pi_2) \circ \text{gr}(h_1)_* \circ \mathcal{N} \circ \text{gr}(s)_*)(\omega) \\
 &= (\mathcal{D}(m \cdot \mu) \circ \mathcal{N} \circ (\mathcal{D}(m \cdot \pi_2) \circ \text{gr}(h_1))_* \circ \mathcal{N} \circ \text{gr}(s)_*)(\omega) \\
 &\stackrel{(7.3)}{=} (\mathcal{D}(m \cdot \mu) \circ \mathcal{N} \circ (h \cdot \mathcal{D}(A))_* \circ \mathcal{N} \circ \text{gr}(s)_*)(\omega) \\
 &\stackrel{(3.8)}{=} (\mathcal{N} \circ (h \cdot A)_* \circ \text{gr}(s)_*)(\omega) \\
 &= (\mathcal{N} \circ ((h \cdot A) \bullet \text{gr}(s))_*)(\omega) \\
 &\stackrel{(7.3)}{=} (\mathcal{N} \circ \text{gr}(h \bullet s)_*)(\omega) \\
 &= (\mathcal{N} \circ \text{gr}(t)_*)(\omega) \\
 &= \omega \parallel_t.
 \end{aligned}$$

In the other direction, let  $\omega \parallel_s \sqsubseteq \omega \parallel_t$  via  $\Omega \in \mathcal{D}(m \cdot \mathcal{D}(n \cdot \mathcal{D}(A)))$ , so that  $(\pi_2)_*(\Omega) = \omega \parallel_s$  and  $\mathcal{D}(m \cdot (\pi_2)_*)(\Omega) = \omega \parallel_t$ , where  $\omega \in \mathcal{D}(A)$  satisfies  $\text{supp}(s_*(\omega)) = n$  and  $\text{supp}(\omega) = A$ . We need to find a map  $h: n \rightarrow \mathcal{D}(m)$  with  $h \bullet s = t$ . Consider the distribution:

$$\Theta = ((\text{st}_2)_* \circ \mathcal{D}(m \cdot \mathcal{D}(\pi_1)))(\Omega) \in \mathcal{D}(m \cdot n). \quad (7.4)$$

It is not hard to see that the second marginal  $\mathcal{D}(\pi_2)(\Theta) \in \mathcal{D}(n)$  equals  $s_*(\omega)$ . Since the support of the latter distribution is  $n$ , by assumption, we may use Proposition 3. Hence there is a unique map  $h: n \rightarrow \mathcal{D}(m)$  with  $\Theta = \text{gr}(h)_*(s_*(\omega))$ .

Our aim is to prove  $s \sqsubseteq t$  via  $h \bullet s = t$ . We shall switch to a more concrete level. Since the distributions  $\omega \parallel_s \in \mathcal{D}(n \cdot \mathcal{D}(A))$  and  $\omega \parallel_t \in \mathcal{D}(m \cdot \mathcal{D}(A))$  are normalised, we can write them as formal convex combinations:

$$\omega \parallel_s = \sum_{i \in n} u_i |\kappa_i \varphi_i\rangle \quad \text{and} \quad \omega \parallel_t = \sum_{j \in m} v_j |\kappa_j \psi_j\rangle, \quad (7.5)$$

for  $\varphi_i, \psi_j \in \mathcal{D}(A)$  and  $u_i, v_j \in [0, 1]$  with  $\sum_i u_i = 1 = \sum_j v_j$ . The equation  $\mathcal{D}(m \cdot (\pi_2)_*)(\Omega) = \omega \parallel_t$  means that we can write:

$$\Omega = \sum_{j \in m} v_j |\kappa_j \rho_j\rangle \quad \text{for } \rho_j \in \mathcal{D}(n \cdot \mathcal{D}(A)) \text{ with } (\pi_2)_*(\rho_j) = \psi_j. \quad (7.6)$$

The other equation about  $\Omega$  gives:

$$\omega \parallel_s = (\pi_2)_*(\Omega) = \mu(\mathcal{D}(\pi_2)(\Omega)) = \mu(\sum_j v_j |\rho_j\rangle) = \sum_{i, \chi} (\sum_j v_j \cdot \rho_j(\kappa_i \chi)) |\kappa_i \chi\rangle.$$

But since this  $\omega \parallel_s$  is normalised, as described in (7.5), the only possible distributions  $\chi \in \mathcal{D}(A)$  are  $\varphi_i$ . Hence we can write:

$$\rho_j = \sum_i \rho_j(\kappa_i \varphi_i) |\kappa_i \varphi_i\rangle \quad \text{with} \quad u_i = \sum_j v_j \cdot \rho_j(\kappa_i \varphi_i). \quad (7.7)$$

The second equation in (7.6) can now be unfolded to:

$$\begin{aligned}
 \psi_j &= (\pi_2)_*(\rho_j) \stackrel{(7.7)}{=} \mu(\mathcal{D}(\pi_2)(\sum_i \rho_j(\kappa_i \varphi_i) |\kappa_i \varphi_i\rangle)) \\
 &= \mu(\sum_i \rho_j(\kappa_i \varphi_i) |\varphi_i\rangle) \\
 &= \sum_a (\sum_i \rho_j(\kappa_i \varphi_i) \cdot \varphi_i(a)) |a\rangle.
 \end{aligned} \quad (7.8)$$

We can now express the distribution  $\Theta \in \mathcal{D}(m \cdot n)$  from (7.4) as:

$$\begin{aligned}
\Theta &= ((\text{st}_2)_* \circ \mathcal{D}(m \cdot \mathcal{D}(\pi_1)))(\Omega) \\
&\stackrel{(7.6)}{=} (\text{st}_2)_* \left( \sum_j v_j | \kappa_j \mathcal{D}(\pi_1)(\rho_j) \rangle \right) \\
&\stackrel{(7.7)}{=} \mu \left( \mathcal{D}(\text{st}_2) \left( \sum_j v_j | \kappa_j (\sum_i \rho_j(\kappa_i \varphi_i) | i \rangle) \rangle \right) \right) \\
&= \mu \left( \sum_j v_j | \sum_i \rho_j(\kappa_i \varphi_i) | \kappa_j i \rangle \right) \\
&= \sum_{j,i} v_j \cdot \rho_j(\kappa_i \varphi_i) | \kappa_j i \rangle.
\end{aligned}$$

According to (2.11), the function  $h: n \rightarrow \mathcal{D}(m)$  corresponding to this  $\Theta \in \mathcal{D}(m \cdot n)$  is given by:

$$h(i) = \sum_j \frac{\Theta(\kappa_j i)}{\sum_j \Theta(\kappa_j i)} | j \rangle = \sum_j \frac{v_j \cdot \rho_j(\kappa_i \varphi_i)}{\sum_j v_j \cdot \rho_j(\kappa_i \varphi_i)} | j \rangle \stackrel{(7.7)}{=} \sum_j \frac{v_j \cdot \rho_j(\kappa_i \varphi_i)}{u_i} | j \rangle. \quad (7.9)$$

We now prove the following equality of distributions in  $\mathcal{D}(m \cdot A)$ .

$$\begin{aligned}
\text{gr}(h \bullet s)_*(\omega) &\stackrel{(7.3)}{=} ((h \cdot A) \bullet \text{gr}(s))_*(\omega) \\
&\stackrel{(3.6)}{=} ((h \cdot A)_* \circ (\text{st}_2)_*)(\omega \|_s) \\
&\stackrel{(7.5)}{=} ((h \cdot A)_* \circ \mu \circ \mathcal{D}(\text{st}_2))(\sum_i u_i | \kappa_i \varphi_i \rangle) \\
&= ((h \cdot A)_* \circ \mu)(\sum_i u_i | \sum_a \varphi_i(a) | \kappa_i a \rangle) \\
&= (\mu \circ \mathcal{D}(h \cdot A))(\sum_{i,a} u_i \cdot \varphi_i(a) | \kappa_i a \rangle) \\
&= \mu(\sum_{i,a} u_i \cdot \varphi_i(a) | \sum_j h(i)(j) | \kappa_j a \rangle) \\
&= \sum_{j,i,a} u_i \cdot \varphi_i(a) \cdot h(i)(j) | \kappa_j a \rangle \\
&\stackrel{(7.9)}{=} \sum_{j,i,a} \varphi_i(a) \cdot v_j \cdot \rho_j(\kappa_i \varphi_i) | \kappa_j a \rangle \\
&= \sum_{j,a} v_j \cdot (\sum_i \varphi_i(a) \cdot \rho_j(\kappa_i \varphi_i)) | \kappa_j a \rangle \\
&\stackrel{(7.8)}{=} \sum_{j,a} v_j \cdot \psi_j(a) | \kappa_j a \rangle \\
&= \mu(\sum_j v_j | \sum_a \psi_j(a) | \kappa_j a \rangle) \\
&= \mu(\mathcal{D}(\text{st}_2)(\sum_j v_j | \kappa_j \psi_j \rangle)) \\
&\stackrel{(7.5)}{=} (\text{st}_2)_*(\omega \|_t) \\
&\stackrel{(3.6)}{=} \text{gr}(t)_*(\omega).
\end{aligned}$$

We have  $\mathcal{D}(\pi_2)(\text{gr}(t)_*(\omega)) = \omega$ , and  $\text{supp}(\omega) = A$  by assumption. Hence we can use uniqueness from Proposition 3 to obtain the required conclusion  $h \bullet s = t$ .  $\square$

We conclude with an example of this refinement theorem, taken from the unpublished extended version<sup>2</sup> of [22], building on Example 15.

**Example 19.** We recall the two hyper distributions in  $\mathcal{D}(2 \cdot \mathcal{D}(\{H, T\}))$  and  $\mathcal{D}(3 \cdot \mathcal{D}(\{H, T\}))$  from Example 15:

$$\begin{aligned}
\Phi &= \frac{1}{2} | \kappa_0(\frac{2}{3} | H \rangle + \frac{1}{3} | T \rangle) \rangle + \frac{1}{2} | \kappa_1(\frac{1}{3} | H \rangle + \frac{2}{3} | T \rangle) \rangle \\
\Psi &= \frac{1}{3} | \kappa_0(\frac{2}{3} | H \rangle + \frac{1}{3} | T \rangle) \rangle + \frac{1}{3} | \kappa_1(\frac{1}{2} | H \rangle + \frac{1}{2} | T \rangle) \rangle + \frac{1}{3} | \kappa_2(\frac{1}{3} | H \rangle + \frac{2}{3} | T \rangle) \rangle
\end{aligned}$$

<sup>2</sup>Available from <http://www.cse.unsw.edu.au/~carrollm/probs/Papers/LiCS15.pdf>

They satisfy  $(\pi_2)_*(\Phi) = \omega = (\pi_2)_*(\Psi)$ , for  $\omega = \frac{1}{2}|H\rangle + \frac{1}{2}|T\rangle$ . Moreover, they can be written as  $\Phi = \omega \parallel_s$  and  $\Psi = \omega \parallel_t$  for tests  $s: \{H, T\} \rightarrow \mathcal{D}(2)$  and  $t: \{H, T\} \rightarrow \mathcal{D}(3)$  given by:

$$\begin{cases} s(H) = \frac{2}{3}|0\rangle + \frac{1}{3}|1\rangle \\ s(T) = \frac{1}{3}|0\rangle + \frac{2}{3}|1\rangle \end{cases} \quad \text{and} \quad \begin{cases} t(H) = \frac{4}{9}|0\rangle + \frac{1}{3}|1\rangle + \frac{2}{9}|2\rangle \\ t(T) = \frac{2}{9}|0\rangle + \frac{1}{3}|1\rangle + \frac{4}{9}|2\rangle \end{cases}$$

We claim  $\Phi = \omega \parallel_s \sqsubseteq \omega \parallel_t = \Psi$ , via the distribution  $\Omega \in \mathcal{D}(3 \cdot \mathcal{D}(2 \cdot \mathcal{D}(\{H, T\})))$  given by:

$$\begin{aligned} \Omega &= \frac{1}{3} \left| \kappa_0(1 | \kappa_0(\frac{2}{3}|H\rangle + \frac{1}{3}|T\rangle)) \right\rangle + \\ &\quad \frac{1}{3} \left| \kappa_1(\frac{1}{2} | \kappa_0(\frac{2}{3}|H\rangle + \frac{1}{3}|T\rangle)) + \frac{1}{2} | \kappa_1(\frac{1}{3}|H\rangle + \frac{2}{3}|T\rangle) \right\rangle + \\ &\quad \frac{1}{3} \left| \kappa_2(1 | \kappa_1(\frac{1}{3}|H\rangle + \frac{2}{3}|T\rangle)) \right\rangle \end{aligned}$$

This  $\Omega$  proves the refinement  $\Phi \sqsubseteq \Psi$  as in Definition 17, since:

$$\begin{aligned} (\pi_2)_*(\Omega) &= (\mu \circ \mathcal{D}(\pi_2))(\Omega) = \frac{1}{3} \left| \kappa_0(\frac{2}{3}|H\rangle + \frac{1}{3}|T\rangle) \right\rangle + \\ &\quad \frac{1}{6} \left| \kappa_0(\frac{2}{3}|H\rangle + \frac{1}{3}|T\rangle) + \frac{1}{6} | \kappa_1(\frac{1}{3}|H\rangle + \frac{2}{3}|T\rangle) \right\rangle + \\ &\quad \frac{1}{3} \left| \kappa_1(\frac{1}{3}|H\rangle + \frac{2}{3}|T\rangle) \right\rangle \\ &= \frac{1}{2} \left| \kappa_0(\frac{2}{3}|H\rangle + \frac{1}{3}|T\rangle) \right\rangle + \frac{1}{2} \left| \kappa_1(\frac{1}{3}|H\rangle + \frac{2}{3}|T\rangle) \right\rangle \\ &= \Phi \end{aligned}$$

$$\begin{aligned} \mathcal{D}(3 \cdot (\pi_2)_*)(\Omega) &= \mathcal{D}(3 \cdot (\mu \circ \mathcal{D}(\pi_2)))(\Omega) = \frac{1}{3} \left| \kappa_0(\frac{2}{3}|H\rangle + \frac{1}{3}|T\rangle) \right\rangle + \\ &\quad \frac{1}{3} \left| \kappa_1(\frac{2}{6}|H\rangle + \frac{1}{6}|T\rangle + \frac{1}{6}|H\rangle + \frac{2}{6}|T\rangle) \right\rangle + \\ &\quad \frac{1}{3} \left| \kappa_2(\frac{1}{3}|H\rangle + \frac{2}{3}|T\rangle) \right\rangle \\ &= \frac{1}{3} \left| \kappa_0(\frac{2}{3}|H\rangle + \frac{1}{3}|T\rangle) \right\rangle + \\ &\quad \frac{1}{3} \left| \kappa_1(\frac{1}{2}|H\rangle + \frac{1}{2}|T\rangle) \right\rangle + \\ &\quad \frac{1}{3} \left| \kappa_2(\frac{1}{3}|H\rangle + \frac{2}{3}|T\rangle) \right\rangle \\ &= \Psi. \end{aligned}$$

We illustrate how to obtain from  $\Omega$  the map  $h: 2 \rightarrow \mathcal{D}(3)$  that proves the refinement  $s \sqsubseteq t$ , as in the proof of Theorem 18, via the distribution  $\Theta$  in (7.4):

$$\begin{aligned} \Theta &= ((\text{st}_2)_* \circ \mathcal{D}(3 \cdot \mathcal{D}(\pi_1)))(\Omega) \\ &= \mu \left( \mathcal{D}(\text{st}_2) \left( \frac{1}{3} \left| \kappa_0(1|0\rangle) \right\rangle + \frac{1}{3} \left| \kappa_1(\frac{1}{2}|0\rangle + \frac{1}{2}|1\rangle) \right\rangle + \frac{1}{3} \left| \kappa_2(1|1\rangle) \right\rangle \right) \right) \\ &= \mu \left( \frac{1}{3} (1 | \kappa_0 0) + \frac{1}{3} (\frac{1}{2} | \kappa_1 0) + \frac{1}{2} | \kappa_1 1 + \frac{1}{3} (1 | \kappa_2 1) \right) \\ &= \frac{1}{3} | \kappa_0 0 + \frac{1}{6} | \kappa_1 0 + \frac{1}{6} | \kappa_1 1 + \frac{1}{3} | \kappa_2 1 \end{aligned}$$

From this  $\Theta$  we obtain the function  $h: 2 \rightarrow \mathcal{D}(3)$  by pointwise normalisation (2.11):

$$h(0) = \frac{1/3}{1/2}|0\rangle + \frac{1/6}{1/2}|1\rangle = \frac{2}{3}|0\rangle + \frac{1}{3}|1\rangle \quad h(1) = \frac{1/6}{1/2}|1\rangle + \frac{1/3}{1/2}|2\rangle = \frac{1}{3}|1\rangle + \frac{2}{3}|2\rangle$$

There is a refinement  $s \sqsubseteq t$ , as in Definition 17, since we have  $h \bullet s = t$ .

In the other direction, given this function  $h: 2 \rightarrow \mathcal{D}(3)$ , one may check that the formula (7.2) gives the distribution  $\Omega$  that we used above.

## 8. CONCLUDING REMARKS

This paper provides a novel perspective on normalisation of discrete probability distributions, by presenting it in ‘hyper’ form as a map  $\mathcal{N}: \mathcal{D}(n \cdot A) \rightarrow \mathcal{D}(n \cdot \mathcal{D}(A))$  that satisfies various nice properties. The associated hyper conditioning operation  $\omega \parallel_t$  performs conditioning for all the predicates incorporated in the test  $t$  in parallel, and is a total operation too. It has been implemented in the EfProb tool [4], see especially the manual [13].

Since we deal with finite discrete probability distributions, using this copower is  $n \cdot A$  is quite natural. But one could have described normalisation also using a cartesian product  $B \times A$ , for an arbitrary not necessarily finite set  $B$ , or as an indexed coproduct  $\coprod_{i \in I} A_i$ , as in:

$$\mathcal{D}(B \times A) \xrightarrow{\mathcal{N}} \mathcal{D}(B \times \mathcal{D}(A)) \quad \text{or as} \quad \mathcal{D}(\coprod_{i \in I} A_i) \xrightarrow{\mathcal{N}} \mathcal{D}(\coprod_{i \in I} \mathcal{D}(A_i))$$

This does not fundamentally change the theory.

A different dimension of change is to consider other functors than distribution  $\mathcal{D}$ . First, one could use the multiset functor  $\mathcal{M}$  over the non-negative real number, given by:

$$\mathcal{M}(X) = \{\varphi: X \rightarrow \mathbb{R}_{\geq 0} \mid \text{supp}(\varphi) \text{ is finite}\}.$$

Then one can generalise normalisation from subdistributions to such multisets (or ‘scores’, as in [28]) via a map:

$$\mathcal{M}(n \cdot A) \longrightarrow \mathcal{M}(n \cdot \mathcal{D}(A))$$

One then normalises non-negative real numbers to 1.

A more drastic step is the move from discrete probability to continuous probability, by replacing the distribution monad  $\mathcal{D}$  on sets with the Giry monad  $\mathcal{G}$  on measurable spaces [8, 25]. How to best do this will be explored in later work.

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## REFERENCES

- [1] R. Adams and B. Jacobs. A type theory for probabilistic and Bayesian reasoning. To appear in: Proceedings of TYPES’15, see [arxiv.org/abs/1511.09230](https://arxiv.org/abs/1511.09230), 2017.
- [2] D. Barber. *Bayesian Reasoning and Machine Learning*. Cambridge Univ. Press, 2012.
- [3] J. Borgström, A.D. Gordon, M. Greenberg, J. Margetson, and J. Van Gael. Measure transformer semantics for Bayesian machine learning. *Logical Methods in Comp. Sci.*, 9(3):1–39, 2013.
- [4] K. Cho and B. Jacobs. The EfProb library for probabilistic calculations. In F. Bonchi and B. König, editors, *Conference on Algebra and Coalgebra in Computer Science (CALCO 2017)*, volume 72 of *LIPICs*. Schloss Dagstuhl, 2017.
- [5] K. Cho, B. Jacobs, A. Westerbaan, and B. Westerbaan. An introduction to effectus theory. see [arxiv.org/abs/1512.05813](https://arxiv.org/abs/1512.05813), 2015.
- [6] B. Fong. Causal theories: A categorical perspective on Bayesian networks. Master’s thesis, Univ. of Oxford, 2012. see [arxiv.org/abs/1301.6201](https://arxiv.org/abs/1301.6201).
- [7] R. Furber and B. Jacobs. Towards a categorical account of conditional probability. In C. Heunen, P. Selinger, and J. Vicary, editors, *Quantum Physics and Logic (QPL) 2015*, number 195 in *Elect. Proc. in Theor. Comp. Sci.*, pages 179–195, 2015.
- [8] M. Giry. A categorical approach to probability theory. In B. Banaschewski, editor, *Categorical Aspects of Topology and Analysis*, number 915 in *Lect. Notes Math.*, pages 68–85. Springer, Berlin, 1982.
- [9] B. Jacobs. New directions in categorical logic, for classical, probabilistic and quantum logic. *Logical Methods in Comp. Sci.*, 11(3):1–76, 2015.

- [10] B. Jacobs. Affine monads and side-effect-freeness. In I. Hasuo, editor, *Coalgebraic Methods in Computer Science (CMCS 2016)*, number 9608 in Lect. Notes Comp. Sci., pages 53–72. Springer, Berlin, 2016.
- [11] B. Jacobs. Introduction to coalgebra. Towards mathematics of states and observations. Cambridge Univ. Press, to appear, 2016.
- [12] B. Jacobs. From probability monads to commutative effectuses. *Journ. of Logical and Algebraic Methods in Programming*, 156, 2017. See <http://dx.doi.org/10.1016/j.jlamp.2016.11.006>.
- [13] B. Jacobs and K. Cho. EffProb user manual. See [efprob.cs.ru.nl](http://efprob.cs.ru.nl), 2017.
- [14] B. Jacobs and J. Mandemaker. The expectation monad in quantum foundations. In B. Jacobs, P. Selinger, and B. Spitters, editors, *Quantum Physics and Logic (QPL) 2011*, number 95 in Elect. Proc. in Theor. Comp. Sci., pages 143–182, 2012.
- [15] B. Jacobs, B. Westerbaan, and A. Westerbaan. States of convex sets. In A. Pitts, editor, *Foundations of Software Science and Computation Structures*, number 9034 in Lect. Notes Comp. Sci., pages 87–101. Springer, Berlin, 2015.
- [16] B. Jacobs and F. Zanasi. A predicate/state transformer semantics for Bayesian learning. In L. Birkedal, editor, *Math. Found. of Programming Semantics*, number 325 in Elect. Notes in Theor. Comp. Sci., pages 185–200. Elsevier, Amsterdam, 2016.
- [17] N. Jansen, B. Lucien Kaminski, J-P. Katoen, F. Olmedo, F. Gretz, and A. McIver. Conditioning in probabilistic programming. In D. Ghica, editor, *Math. Found. of Programming Semantics*, number 319 in Elect. Notes in Theor. Comp. Sci., pages 199–216. Elsevier, Amsterdam, 2015.
- [18] J. Landauer and T. Redmond. A lattice of information. In *6th IEEE Computer Security Foundations Workshop (CSFW'93)*, pages 65–70, 1993.
- [19] A. McIver, L. Meinicke, and C. Morgan. Compositional closure for Bayes risk in probabilistic noninterference. In S. Abramsky, C. Gavaille, C. Kirchner, F. Meyer auf der Heide, and P. Spirakis, editors, *Int. Coll. on Automata, Languages and Programming*, number 6199 in LNCS, pages 223–235. Springer, Berlin, 2010.
- [20] A. McIver, L. Meinicke, and C. Morgan. Hidden-Markov program algebra with iteration. *Math. Struct. in Comp. Sci.*, 25(2):320–360, 2015.
- [21] A. McIver and C. Morgan. *Abstraction, refinement and proof for probabilistic systems*. Monographs in Comp. Sci. Springer, 2004.
- [22] A. McIver, C. Morgan, and T. Rabehaja. Abstract hidden Markov models: A monadic account of quantitative information flow. In *Logic in Computer Science*, pages 597–608. IEEE, Computer Science Press, 2015.
- [23] A. McIver, C. Morgan, and K. Seidel. Probabilistic predicate transformers. *ACM Trans. on Progr. Lang. and Systems*, 18(3):325–353, 1996.
- [24] A. McIver, C. Morgan, G. Smith, B. Espinoza, and L. Meinicke. Abstract channels and their robust information-leakage ordering. In M. Abadi and S. Kremer, editors, *Princ. of Security and Trust*, number 8414 in Lect. Notes Comp. Sci., pages 83–102. Springer, Berlin, 2014.
- [25] P. Panangaden. *Labelled Markov Processes*. Imperial College Press, London, 2009.
- [26] J. Pearl. *Causality. Models, Reasoning, and Inference*. Cambridge Univ. Press, 2<sup>nd</sup> ed. edition, 2009.
- [27] A. Ścibior, Z. Ghahramani, and A. Gordon. Practical probabilistic programming with monads. In *Proc. 2015 ACM SIGPLAN Symp. on Haskell*, pages 165–176. ACM, 2015.
- [28] S. Staton, H. Yang, C. Heunen, O. Kammar, and F. Wood. Semantics for probabilistic programming: higher-order functions, continuous distributions, and soft constraints. *Logic in Computer Science*, 2016.