

COMPLEXITY THEORY FOR SPACES OF INTEGRABLE FUNCTIONS

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ABSTRACT. This paper investigates second-order representations in the sense of Kawamura and Cook for spaces of integrable functions that regularly show up in analysis. It builds upon prior work about the space of continuous functions on the unit interval: Kawamura and Cook introduced a representation inducing the right complexity classes and proved that it is the weakest second-order representation such that evaluation is polynomial-time computable.

The first part of this paper provides a similar representation for the space of integrable functions on a bounded subset of Euclidean space: The weakest representation rendering integration over boxes is polynomial-time computable. In contrast to the representation of continuous functions, however, this representation turns out to be discontinuous with respect to both the norm and the weak topology.

The second part modifies the representation to be continuous and generalizes it to L^p -spaces. The arising representations are proven to be computably equivalent to the standard representations of these spaces as metric spaces and to still render integration polynomial-time computable. The family is extended to cover Sobolev spaces on the unit interval, where less basic operations like differentiation and some Sobolev embeddings are shown to be polynomial-time computable.

Finally as a further justification quantitative versions of the Arzelà-Ascoli and Fréchet-Kolmogorov Theorems are presented and used to argue that these representations fulfill a minimality condition. To provide tight bounds for the Fréchet-Kolmogorov Theorem, a form of exponential time computability of the norm of L^p is proven.

1. INTRODUCTION

Classical computability and complexity theory are indispensable tools of theoretical computer science with numerous applications throughout computer science and discrete mathematics.

In many cases, however, it is desirable to also be able to consider computations over continuous structures. Engineers want to use computers to solve partial differential equations

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that describe their problems. The take of mathematics on this is Numerics. Usually implementations of algorithms from numerics are done using floating point arithmetics. From the point of view of a logician this leads to a gap between between the mathematical part and the implementation: Proofs of convergence of an algorithm in numerics regularly rely on properties of real numbers that are not reflected by the floating point numbers. This can lead to uncontrollable error propagation in some situations (see the introduction of [Sch02a] for examples). In other words: Floating point arithmetics are not the appropriate model of computation for Numerics. Furthermore, the finiteness of the set of machine numbers eludes a mathematically rigorous and comprehensible description. This inhibits the existence of proofs of incomputability and realistic and rigorous notions of resource consumption for computations using floating point arithmetics.

These problems are well aware to numerical scientists and have been addressed by the development of interval arithmetics, multiple precision arithmetics, etc. which in turn are only partial solutions of the problem from our point of view. Computable analysis provides a model that fulfills our requirements by replacing a finite description of an object by a function that delivers on demand information about the object. Computations on objects can then be modeled by programs that are allowed subroutine calls to some function describing the object. Computable Analysis reaches back to [Tur36, Lac55, Grz55], the established mathematically rigorous description are Weihrauch's representations and his type two theory of effectivity (TTE) [Wei00]. Many of the results from computable analysis meet the intuition of numerical scientists. For example the empirical experience that testing for equality is not a good idea and should be replaced by an epsilon test is reflected in undecidability of equality in computable analysis but decidability of the fuzzy, multivalued version.

A special case of complexity theory on continuous structures, namely a complexity theory for real functions, was developed by Ker-I Ko and Harvey Friedman [KF82]. With some technical efforts, their approach can be formulated within the TTE framework [Wei00]. However, complexity wise computations on real functions via the TTE are problematic. Thus, operators on real functions were investigated in a point wise manner instead. Important results were achieved: Ko and Friedman succeeded to prove that parametric maximization of continuous functions preserves polynomial-time computability point wise if and only if $\mathcal{P} = \mathcal{NP}$ [KF82]. Whether or not the latter is the case is one of the millenium problems. Friedman related integration to the even stronger \mathcal{FP} vs. $\#\mathcal{P}$ problem [Fri84]. These results remain true if the operators are restricted to smooth input, which deviates from the expectations of scientist from applied fields: Integration is regarded feasible, at least if smoothness assumptions are imposed, while maximization is considered more difficult.

Of course it is also desirable to do uniform complexity theory on function spaces. Unfortunately the TTE turns out to yield a too restrictive model of fast computations on many structures. This can only partially be fixed without changing the setting [Wei03, Sch04]. The problem is that the TTE only allows sequential access to the input. For instance for fast evaluation of a real function random access is necessary. An appropriate framework for complexity theory on more general spaces was more recently introduced by Kawamura and Cook [KC10]. It relies on second-order complexity theory, that is complexity theory for functionals on Baire space (cf. [Meh76, IRK01]), while maintaining the general idea of computable analysis to encode objects by integer functions. For this reason these encodings are called second-order representations. The framework is well accepted and investigated

[FGH14, KP14a, KO14]. Kawamura and Cook also specified a canonical second-order representation of the space of continuous functions on the unit interval, proving it to bring forth the same complexity classes and to be the minimal second-order representation to have polynomial-time evaluation up to polynomial-time reductions.

While continuous functions are an important starting point, more general functions are needed for many applications. The emphasis on evaluation seems misplaced: It requires the functions considered to be continuous and polynomial-time evaluation does not suffice to carry out other important operations, like integration, effectively. Furthermore, the sets of functions should be considered as spaces: A representation induces a topology and this topology should fit the natural topology on the function space. If the representation is continuous and open, computability of an operator is a refinement of continuity of this operator. In this case there is hope that results from numerical analysis can be lifted to the computability level. This paper encounters discontinuous representations and discards them for this reason.

Actually openness is sufficient but not necessary for the above hold. The appropriate notion from computable analysis would be the notion of admissibility of a representation. In the cases that turn up in this paper, admissibility is the same as equivalence to a Cauchy representation. To be able to hope for results from analysis to lead to algorithms that use bounded resources, a complexity theoretical equivalent of admissibility would be needed. Such a notion, however, does not yet exist. This paper very briefly mentions a representation that is the antithesis of having such a property at the end of Section 2. The representations considered in the later chapters, in contrast, are probable candidates for having such a property.

Content and organization of this paper. This paper specifies several second-order representations of spaces of integrable functions that appear in practice. All these representations provide oracle access to approximations of the integrals of over dyadic intervals or boxes. They differ, however, in the length a name of a function is given. On the one hand, this can be understood to modify the density of information and the time allotted for a computation on a function, on the other hand the length provides additional information and can be understood as ‘enrichment of data’[KM82].

Before we talk about the structure of the paper let us informally describe the representation for L^p in some more detail. The standard representation of the continuous functions on the unit interval establishes the following model of computation: A ‘program’ computing a continuous function f takes a rational number r and a rational precision requirement ε and returns an ε -approximation to $f(r)$ as well as a rational number δ such that all approximations stay valid whenever the input changed by less than δ (delta may only depend on ε , not on r). This paper claims that for functions from L^p the following is the right model of computation: A ‘program’ computes an L^p -function f if it takes a rational box $[r, s]$ and a rational precision requirement ε and returns an ε -approximation to $\int_r^s f d\lambda$ as well as a rational δ such that whenever the function is shifted by less than δ in the argument, it does not change more than ε in L^p -norm. The more straight forward version that δ is such that whenever the box only changes a little, the integrals do not change to much is also considered but disregarded as a discontinuous representation.

In the later chapters, the focus shifts to justification and general recipes for how to construct useful representations. Interestingly, classification results for compact sets are of importance for these constructions. Quantitative versions of such results for concrete spaces

can be connected to optimal running times of the metric with respect to any second-order representation. The kind of classification results that turn up have been investigated from different points of views: Approximation theory asked similar questions and comparable theorems turned up when constructive mathematicians tried to make analysis constructive [KT59, Tim94, BB85].

This paper is structured as follows: The remainder of the first section lists some of the facts from computable analysis and real complexity theory that are regularly used throughout the paper. In particular it introduces Cauchy representations and the standard representations of the continuous functions on the unit interval and recollects some of the properties. This is mostly for easy reference and to fix a notation where more than one is common.

Section 2 introduces the singular representation: The weakest representation of the integrable functions that allows for the computation of integrals over boxes in polynomial-time. First in one dimension (Definitions 2.1 and 2.4), and then in full generality for arbitrary dimensions (Definition 2.7). Theorem 2.8 proves that this representation is indeed minimal with respect to polynomial-time reduction. The singular representation is proven to be discontinuous in Theorem 2.10.

In Section 3 a family of representations of the spaces $L^p(\Omega)$ is defined and investigated. First, the L^p -modulus, a replacement for the modulus of continuity, is discussed (Definition 3.3). Then a representation of $L^p(\Omega)$ is defined (Definition 3.5) and Theorem 3.8 proves it to be computably equivalent to the Cauchy representation.

A straightforward extension to the Sobolev spaces $W^{m,p}([0, 1])$ (Definition 4.3) is presented and investigated in Section 4. The inclusions of the Sobolev spaces into the continuous and into the integrable functions are shown to be polynomial-time computable in Theorems 4.4 and 4.6 for one derivative, and in Theorems 4.10 and 4.11 for higher derivatives. Corollary 4.12 deduces that differentiation is polynomial-time computable.

Section 5 explores minimality properties of the representations at hand: It introduces the concept of metric entropy (Definition 5.3) and in Theorem 5.8 proves a result that connects the metric entropy of a compact space to the minimal running time of the metric. This theorem is used as motivation to examine quantitative versions of theorems classifying the compact subsets of function spaces. Two results of this form are presented: A version of the Arzelà-Ascoli Theorem 5.11 which is already known from approximation theory and a version of the Fréchet-Kolmogorov Theorem 5.13 that, to the knowledge of the author, has not been stated in this generality before. To provide a tight upper bound for the latter (Theorem 5.24), a slightly modified representation is introduced (Definition 5.20), and a strong form of exponential time computability of the norm on L^p with respect to this representation is proven in Theorem 5.23.

Sources and further readings. For the understanding of this paper a solid basic knowledge of computability and complexity theory is required. One of many excellent sources for read-up is [AB09]. The topology needed and basics about metric spaces can be found for instance in [Mun00]. For the understanding of some results measure theory is necessary (for instance [Wer00]) and for the latter chapters it is beneficial to know basics about L^p - and Sobolev-spaces [Bre11]. Furthermore, basics of real computability theory, in particular Weihrauch's type two theory of effectivity (TTE), are beneficial for understanding. All that is needed and more is described in detail in [Wei00]. For additional material on second-order complexity

theory see for example [Meh76] and [KC96]. For further information about the framework of Kawamura and Cook and how to apply this to computable analysis see [KC10].

The results presented here are from the authors PhD-Thesis [Ste16] and some were already mentioned in [KSZ16].

Basic notational conventions. Fix the finite alphabet $\Sigma := \{0, 1\}$. The following subsets of the set Σ^* of finite strings of zeros and ones are of relevance:

$\mathbb{N} = \{1, 10, 11, \dots\}$: the set of strictly positive **integers in binary** representation.

$\omega := \{\varepsilon, 1, 11, \dots\}$: the set of positive **integers in unary**, where ε denotes the empty string interpreted as zero.

We denote elements of Σ^* by $\mathbf{a}, \mathbf{b}, \dots$ and elements of \mathbb{N} and ω by n, m, \dots . If this leads to ambiguity we use 1^n with $n \in \mathbb{N}$ for the elements of ω . Let $|\cdot| : \Sigma^* \rightarrow \omega$ denote the **length function** replacing all 0s by 1. To compute on \mathbb{N} and ω we use the following **encodings** (i.e. **notations** in the sense of Weihrauch [Wei00]): For \mathbb{N} the function $\nu_{\mathbb{N}} : \Sigma^* \rightarrow \mathbb{N}$ that eliminates leading zeros. For ω the function $\nu_{\omega}(\mathbf{a}) := |\nu_{\mathbb{N}}(\mathbf{a})|$.

Computations on products are handled via pairing functions. Fix some **pairing function** $\langle \cdot, \cdot \rangle : \Sigma^* \times \Sigma^* \rightarrow \Sigma^*$ (that is: Some bijective, polynomial-time computable function with polynomial-time computable projections). Furthermore, the pairing function is required to be monotone in both arguments, i.e. whenever the length of one of the input strings is increased, the length of the output string will not decrease. The standard pairing functions fulfill all of these requirements. The corresponding **pairing of string functions** is defined as follows:

$$\langle \varphi, \psi \rangle(\mathbf{a}) := \langle \varphi(\mathbf{a}), \psi(\mathbf{a}) \rangle.$$

This function is bijective.

The set of **real numbers** is denoted by \mathbb{R} . For $x \in \mathbb{R}$ let $\lfloor x \rfloor$ resp. $\lceil x \rceil$ denote the largest integer smaller or equal resp. the least integer larger or equal to x . The binary logarithm of a number x is denoted by $\text{lb}(x)$. The following subsets of the real numbers are of importance to this work:

\mathbb{Z} : the set of **integers**.

\mathbb{D} : the set of numbers that can be written as $\frac{r}{2^n}$ with $r, n \in \mathbb{N}$ called **dyadic numbers**.

These sets are countable and can be handled by discrete computability and complexity theory via encodings. For the set \mathbb{Z} use the encoding $\nu_{\mathbb{Z}}(\mathbf{1a}) := \nu_{\mathbb{N}}(\mathbf{a})$ resp. $\nu_{\mathbb{Z}}(\mathbf{0a}) := -\nu_{\mathbb{N}}(\mathbf{a})$. For \mathbb{D} use the encoding $\llbracket \mathbf{c} \rrbracket := \frac{\nu_{\mathbb{Z}}(\mathbf{a})}{2^n}$ if $\mathbf{c} = a_1 1 a_2 1 \dots 1 a_m 0 a_{m+1} 1 \dots 1 a_{n+m} 0 \dots 0$, and $a_2 = 1$, i.e. the binary expansion with comma. This encoding is chosen such that it allows arbitrary long codes while approximations to the number can always be read from a short beginning segment.

For any dimension $d \in \mathbb{N}$ define an encoding of \mathbb{D}^d by $\llbracket \langle \mathbf{a}_1, \langle \mathbf{a}_2, \dots, \langle \mathbf{a}_{d-1}, \mathbf{a}_d \rangle \dots \rangle \rrbracket_d := (\llbracket \mathbf{a}_1 \rrbracket, \dots, \llbracket \mathbf{a}_d \rrbracket)$. Since the dimension d is usually fixed, it is often omitted. **Dyadic boxes**, i.e. boxes with dyadic vertex coordinates and edges parallel to the axes, are denoted as

$$[\mathbf{a}, \mathbf{b}] := [\llbracket \mathbf{a} \rrbracket_d, \llbracket \mathbf{b} \rrbracket_d] = \{x \in \mathbb{R}^d \mid \llbracket \mathbf{a} \rrbracket_d \leq x \leq \llbracket \mathbf{b} \rrbracket_d\},$$

where the inequalities have to be understood component wise.

For some $\Omega \subseteq \mathbb{R}^d$ denote the set of continuous functions from Ω to \mathbb{R} by $\mathcal{C}(\Omega)$.

1.1. Representations. Encodings allow computations on countable structures using discrete computability theory. Many of the spaces one would like to compute over, however, are uncountable. For instance the real numbers, or, to mention a compact one, the unit interval. Computable analysis overcomes this difficulty by encoding elements by infinite objects (infinite binary strings or string functions) instead of strings [Wei00]. The **Baire space** is the space of all string functions $(\Sigma^*)^{\Sigma^*}$ equipped with the product topology and denoted by \mathcal{B} .

Definition 1.1. A **representation** of a space X is a partial surjective mapping $\xi : \subseteq \mathcal{B} \rightarrow X$. The elements of $\xi^{-1}(x)$ are called the **names** of x .

A space with a fixed representation is called a represented space. Like for topological spaces the representation is only mentioned explicitly if necessary to avoid ambiguities. An element of a represented space is called **computable** if it has a computable name. It is said to lie within a complexity class if it has a name from that complexity class.

On one hand, any represented space carries a natural topology: The final topology of the representation. On the other hand, one often looks for a representation suitable for a topological space. It is reasonable to require such a representation to induce the topology the space is equipped with. For this, **continuity** is necessary but not sufficient. Continuity together with **openness** is sufficient but not necessary. A related concept from computable analysis is **admissibility** which for all representations this paper is concerned with is the same as continuous equivalence to Cauchy representations (to be introduced below) [Wei00]. It implies continuity but not openness (see [Sch02a, Sch02b] for admissibility and [BH02] for its connection to openness).

Recall from the introduction that $\mathbb{N} \subseteq \Sigma^*$.

Definition 1.2. Let $\mathcal{M} := (M, d, (x_m)_{m \in \mathbb{N}})$ be a triple such that (M, d) is a complete separable metric space and $(x_m)_{m \in \mathbb{N}}$ is a dense sequence. Define the **Cauchy representation** $\xi_{\mathcal{M}}$ of M : A string function $\varphi \in \mathcal{B}$ is a $\xi_{\mathcal{M}}$ -name of $x \in M$ if and only if

$$\forall n \in \mathbb{N} : d(x, x_{\varphi(n)}) < 2^{-n}.$$

Cauchy representations are continuous and open with respect to the metric topology.

Recall the pairing function $\langle \cdot, \cdot \rangle : \mathcal{B} \times \mathcal{B} \rightarrow \mathcal{B}$ on string functions from the introduction.

Definition 1.3. Let ξ_X and ξ_Y be representations of spaces X and Y . Define the **product representation** $\xi_{X \times Y}$ of the product $X \times Y$ by

$$\xi_{X \times Y}(\langle \varphi, \psi \rangle) := (\xi_X(\varphi), \xi_Y(\psi)), \text{ whenever } \varphi \in \text{dom}(\xi_X) \text{ and } \psi \in \text{dom}(\xi_Y).$$

This construction is used self-evidently throughout the paper.

1.2. Second-order complexity theory. Computing functions between represented spaces is done by operating on names and computing functions on Baire space:

Definition 1.4. Let ξ_X and ξ_Y representations of spaces X and Y . A partial function $F : \subseteq \mathcal{B} \rightarrow \mathcal{B}$ is called a **realizer** of a function $f : X \rightarrow Y$, if

$$\varphi \in \xi_X^{-1}(x) \Rightarrow F(\varphi) \in \xi_Y^{-1}(f(x)).$$

That is: A realizer translates names of x into names of $f(x)$. F being a realizer of a function f can be visualized by the diagram in Figure 1. However, the domain of F is allowed to be bigger than that of ξ_X . Therefore, F being a realizer of f does not translate to the diagram being commutative in the usual way.

On the Baire space there exists a well-established computability theory originating from [Kle52], see [Lon05] for an overview. A functional $F : \subseteq \mathcal{B} \rightarrow \mathcal{B}$ is called computable if there is an oracle Turing machine $M^?$ such that $M^\varphi(\mathbf{a}) = F(\varphi)(\mathbf{a})$ for all string functions φ from the domain of F . Or spelled out: The computation of $M^?$ with oracle φ and on input \mathbf{a} halts with the string $F(\varphi)(\mathbf{a})$ written on the output tape. A function between represented spaces is called **computable** if it has a computable realizer.

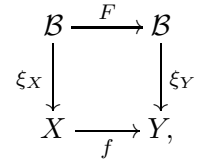


FIGURE 1

Complexity theory for functionals is called **second-order complexity theory**. It was originally introduced by Mehlhorn [Meh76]. This paper uses a characterization via resource bounded oracle Turing machines due to Kapron and Cook [KC96] as definition. The convention for time consumption of oracle queries is the following: When a query is asked, the answer is written on the answer tape within one time step, only reading it requires further time. Such a machine is granted time depending on the size of the input. The string functions are considered the input.

Definition 1.5. The **size** or **length** $|\varphi| : \omega \rightarrow \omega$ of a string function $\varphi \in \mathcal{B}$ is defined by

$$|\varphi|(1^n) := \max\{|\varphi(m)| \mid |m| \leq 1^n\}.$$

For instance: Each polynomial-time computable string function is of polynomial size.

A running time is a mapping that assigns to sizes of the inputs an allowed number of steps. Therefore, it is of type $\omega^\omega \times \omega \rightarrow \omega$. The subclass of running times that are considered polynomial, namely second-order polynomials, are recursively defined as follows:

- For p a positive integer polynomial $(l, n) \mapsto p(n)$ is a second-order polynomial.
- If P and Q are second-order polynomials, so are $P + Q$ and $P \cdot Q$.
- If P is a second-order polynomial, then so is $(l, n) \mapsto l(P(l, n))$.

An example for a second-order polynomial is the mapping $(l, n) \mapsto l(l(n^2+5)+l(l(n)^2))$. Second order polynomials have turned up independently from second-order complexity theory (compare for instance [Koh96]).

Definition 1.6. A functional $F : \subseteq \mathcal{B} \rightarrow \mathcal{B}$ is **polynomial-time computable**, if there is an oracle Turing machine $M^?$ and a second-order polynomial P such that for all string functions $\varphi \in \text{dom}(F)$ and strings \mathbf{a} the computation of $M^\varphi(\mathbf{a})$ terminates within at most $P(|\varphi|, |\mathbf{a}|)$ steps.

A function $f : X \rightarrow Y$ between represented spaces X and Y is called **polynomial-time computable** if it has a polynomial-time computable realizer (compare Definition 1.4).

An important special case is the following: If ξ and ξ' are representations of the same space X , then ξ is called **polynomial-time reducible** to ξ' if the identity from (X, ξ') to (X, ξ) is polynomial-time computable. A realizer of the identity is called a **translation** and we often say ξ' **is translatable to** ξ to express that ξ is reducible to ξ' , as this avoids confusion if the directions are important. The representations are **polynomial-time equivalent** if polynomial-time computable translations in both directions exist. If the translations are merely computable resp. continuous, one speaks of computable resp. continuous reduction and equivalence.

1.3. Second-order representations. The length $|\varphi|(1^n) = \max\{|\varphi(m)| \mid |m| \leq 1^n\}$ of a string function cannot be computed from the string function in polynomial-time: To find the maximum in the definition φ has to be queried an exponential number of times. For many applications, polynomial-time computability of the length of names is desirable.

Definition 1.7. A string function φ is called **length-monotone** if for all strings \mathbf{a} and \mathbf{b}

$$|\mathbf{a}| \leq |\mathbf{b}| \quad \Rightarrow \quad |\varphi(\mathbf{a})| \leq |\varphi(\mathbf{b})|.$$

The set of length-monotone string functions is denoted by Σ^{**} .

For a length-monotone string function it holds that $|\varphi|(|\mathbf{a}|) = |\varphi(\mathbf{a})|$, thus the length function restricted to Σ^{**} is polynomial-time computable.

Definition 1.8. A **second-order representation** is a representation whose domain is contained in Σ^{**} .

Equivalently: A second-order representation ξ of a space X is a partial surjective mapping $\xi : \Sigma^{**} \rightarrow X$ from the length-monotone string functions to the space. The prefix ‘second-order’ is for applicability of second-order complexity theory, and does not indicate the use of higher order objects than for regular representations.

The restriction of a representation to the length-monotone functions is usually still surjective and thus a second-order representation. All representations this paper is concerned with are second-order representations. For brevity ‘second-order’ is sometimes omitted.

Recall that we fixed an encoding $\llbracket \cdot \rrbracket$ of the dyadic numbers in the introduction.

Example 1.9 (The standard representation of reals). Define a second-order representation $\xi_{\mathbb{R}}$ of \mathbb{R} by letting a length-monotone string function φ be a name of x if for all $n \in \mathbb{N}$

$$\llbracket \varphi(1^n) \rrbracket - x < 2^{-n}.$$

A proof that this second-order representation induces the established notions of computability and polynomial-time computability of reals and real functions (i.e. the notions from [Ko91] or [Wei00]) can be found in [Lam06]. It is computably equivalent to the Cauchy representation of the reals from Definition 1.2 if the standard enumeration of the dyadic numbers is chosen as dense sequence. Polynomial-time equivalence fails since the input is encoded in unary, not in binary.

The pairing functions are carefully chosen such that the second-order representations are closed under the products from Definition 1.3. All the encodings from the introduction assign arbitrary big codes to each element. Since this paper only considers representations whose names return codes from one of these, an arbitrary name can always be padded to a length monotone one. Therefore, all representations this paper introduces restrict to Σ^{**} and are introduced as second-order representations right away.

1.4. The standard representation of $C([0, 1])$. Denote the supremum norm on \mathbb{R}^d by $|\cdot|_{\infty}$ and fix some bounded $\Omega \subseteq \mathbb{R}^d$.

Definition 1.10. A function $\mu : \omega \rightarrow \omega$ is called a **modulus of continuity** of $f \in C(\Omega)$ if for all $x, y \in \Omega$ and $n \in \omega$

$$|x - y|_{\infty} \leq 2^{-\mu(n)} \quad \Rightarrow \quad |f(x) - f(y)| < 2^{-n}$$

and $\mu(n) \neq 0 \Rightarrow \mu(n+1) > \mu(n)$, that is: μ is strictly increasing whenever non-zero.

This modulus should be called modulus of uniform continuity to distinguish it from a point-wise modulus of continuity. However, point-wise moduli are not mentioned in this work and we omit the ‘uniform’ for brevity.

Any continuous function on a compact set has a modulus of continuity. On a connected set a function is Hölder continuous if and only if it has a linear modulus of continuity, and Lipschitz continuous if and only if it has a modulus of the form $\mu(n) = n + C$. If the set is convex, 2^C is a Lipschitz constant.

Remark 1.11. The definition slightly differs from the most common one (compare [Ko91] or [Wei00]): Usually, there is no growth condition on the modulus. However, whenever μ is a modulus of continuity, Ω is convex and $\mu(n) - 1$ is not negative, it is a valid value for the modulus of continuity on $n - 1$. Thus, the condition of being strictly increasing when non-zero is a reasonable one and in particular fulfilled by the least modulus of continuity. Its importance becomes apparent in the proof of Theorem 5.13.

Recall that \mathbb{D} denotes the set of dyadic numbers and that computations on \mathbb{D} are carried out via the encoding $\llbracket \cdot \rrbracket$ fixed in the introduction. The following result is the starting point of many generalizations:

Theorem 1.12 ([Ko91]). *A function $f : [0, 1] \rightarrow \mathbb{R}$ is polynomial-time computable if and only if both of the following are fulfilled:*

- *There is a polynomial-time computable function $\varphi : \mathbb{D} \times \omega \rightarrow \mathbb{D}$ such that for any $r \in [0, 1] \cap \mathbb{D}$*

$$|\varphi(r, \mathbf{1}^n) - f(r)| < 2^{-n}.$$

- *The function allows a polynomial modulus of continuity.*

This theorem can be used to define complexity of functions between arbitrary effective metric spaces [LLM01]. Another application is to show that the following definition leads to the usual set of polynomial-time computable functions on the unit interval. Recall that the length $|\varphi|$ of a length-monotone string function is given by $|\varphi|(|\mathbf{a}|) = |\varphi(\mathbf{a})|$.

Definition 1.13. Define the **standard representation** ξ_C of $C([0, 1])$: A string function $\varphi \in \Sigma^{**}$ is a ξ_C -name of f if for all strings \mathbf{a} with $\llbracket \mathbf{a} \rrbracket \in [0, 1]$ and all $n \in \mathbb{N}$

$$|\llbracket \varphi(\langle \mathbf{a}, \mathbf{1}^n \rangle) \rrbracket - f(\llbracket \mathbf{a} \rrbracket)| < 2^{-n}$$

and $|\varphi|$ is a modulus of continuity of f .

$\mathcal{C}([0, 1])$ is a metric space and the Cauchy representation with respect to the standard enumeration of the rational polynomials as dense sequence (cf. Definition 1.2) induces the metric topology. The following is closely connected to the well-known computable Weierstraß approximation theorem (compare for instance [Ko91]):

Theorem 1.14. ξ_C is computably equivalent to the Cauchy representation of $\mathcal{C}([0, 1])$.

In particular, ξ_C is a continuous mapping. The strict inequality in the definition of the modulus of continuity guarantees that ξ_C is an open mapping. The standard representation has been characterized as the weakest representation that permits polynomial-time evaluation up to polynomial-time equivalence. Recall the evaluation operator given by

$$\text{eval} : C([0, 1]) \times [0, 1] \rightarrow \mathbb{R}, \quad (f, x) \mapsto f(x).$$

Here and in the following, both $[0, 1]$ and \mathbb{R} are equipped with the standard representation $\xi_{\mathbb{R}}$ of the real numbers from Example 1.9 and its range-restriction. For a subset $F \subseteq C([0, 1])$ the evaluation operator on F is the restriction of the above operator to $F \times [0, 1]$.

Theorem 1.15 (minimality). *For a second-order representation ξ of a subset $F \subseteq C([0, 1])$ the following are equivalent:*

- ξ renders the evaluation operator on F polynomial-time computable.
- ξ is polynomial-time translatable to the range-restriction of ξ_C to F .

A proof of this theorem for $F = C([0, 1])$ can be found in [KC10] and is easily seen to work for an arbitrary F as well. Note that the computable version of this theorem holds in a more general setting: $[0, 1]$ and \mathbb{R} can be replaced by arbitrary represented spaces X and Y and ξ_C by the function space representation of the continuously representable functions from X to Y . Under suitable assumptions about the represented spaces, this is a well behaved representation of the continuous functions from X to Y .

2. THE SINGULAR REPRESENTATION

Fix some bounded measurable set $\Omega \subseteq \mathbb{R}^d$. Recall that $L^1(\Omega)$ denotes the set of functions on Ω integrable with respect to the Lebesgue measure λ , where functions are identified if they coincide almost everywhere. Equipped with the norm $\|f\|_1 := \int_{\Omega} |f| d\lambda$ the space $L^1(\Omega)$ is a Banach space. This section specifies the weakest representation of $L^1(\Omega)$ that renders integration polynomial-time computable. More formally the following operator is supposed to be polynomial-time computable:

$$\text{int} : L^1(\Omega) \times \Omega^2 \rightarrow \mathbb{R}, \quad (f, x, y) \mapsto \int_{[x, y] \cap \Omega} f d\lambda, \quad (\text{INT})$$

where $[x, y]$ denotes the smallest box with edges parallel to the axis and corners x and y . Here, \mathbb{R}^d is equipped with the d -fold product of the standard representation of the real numbers and Ω with its range-restriction.

First consider the case $\Omega = [0, 1]$: Define an operator $\Phi : L^1([0, 1]) \rightarrow C([0, 1])$ by

$$\Phi(f)(x) := \int_0^x f(t) dt.$$

This defines a linear continuous operator between Banach spaces with $\|\Phi\| = 1$. The operator Φ translates the integration operator into the evaluation operator:

$$\text{eval}(\Phi(f), x) = \text{int}(f, 0, x), \quad \text{int}(f, x, y) = \text{eval}(\Phi(f), y) - \text{eval}(\Phi(f), x).$$

The image of Φ is the set $\mathcal{AC}_0([0, 1])$ of absolutely continuous functions that vanish in zero. Furthermore, Φ is injective and therefore invertible on its image.

From the above it follows that $\Phi^{-1} \circ \xi_C|_{\mathcal{AC}_0([0, 1])}$ is a minimal representation: Whenever ξ renders integration polynomial-time computable, $\Phi \circ \xi$ renders evaluation polynomial-time computable. Thus, the polynomial-time translation from $\xi_C|_{\mathcal{AC}_0([0, 1])}$ to $\Phi \circ \xi$ that exists by the minimality of ξ_C from Theorem 1.15 is also a polynomial-time translation from $\Phi^{-1} \circ \xi_C|_{\mathcal{AC}([0, 1])}$ to $\Phi^{-1} \circ \Phi \circ \xi = \xi$.

Since Φ^{-1} is a linear discontinuous operator between Banach spaces, this representation cannot be continuous: An abstract argument for this can be found in [Sch02a]. This

chapter specifies an alternative description of the above representation that allows for generalizations and proves that the representation and its multidimensional generalizations are discontinuous.

2.1. Singularity moduli. With the notation from the introduction of this section: For $f \in L^1(\Omega)$ a function μ is a modulus of continuity of $\Phi(f)$ if and only if

$$|x - y| \leq 2^{-\mu(n)} \quad \Rightarrow \quad \left| \int_x^y f d\lambda \right| < 2^{-n}$$

and it is strictly increasing whenever it is non-zero. This motivates the following definition. Let Ω be a measurable subset of \mathbb{R} (it is no longer assumed to be bounded). For $f \in L^1(\Omega)$ denote by \tilde{f} the extension of f to all of the real line by zero. The following modulus measures how bad the singularities of a function are:

Definition 2.1. A function $\mu : \omega \rightarrow \omega$ is called a **singularity modulus** of $f \in L^1(\Omega)$, if for any $n \in \omega$ and $x, y \in \mathbb{R}$

$$|x - y| \leq 2^{-\mu(n)} \quad \Rightarrow \quad \left| \int_{[x,y]} \tilde{f} d\lambda \right| < 2^{-n}.$$

Like any continuous function on the unit interval allows a modulus of continuity, any function from $L^1(\Omega)$ possesses a singularity modulus. For $\Omega = [0, 1]$ any modulus of continuity of the function $\Phi(f)$ from the introduction of this section may be chosen. It is possible to prove that any integrable function has a singularity modulus.

If Ω is bounded, the existence of a singularity modulus implies integrability. If the interior of Ω is unbounded the situation is more involved: On the one hand, there are non-integrable, but locally integrable functions that permit a singularity modulus. On the other hand not all locally integrable functions allow a singularity modulus. In the following, however, only bounded sets are considered.

The next proposition uses L^p -spaces, that are recollected in Section 3 in more detail. The case $p = \infty$, is understandable if one recalls that L^∞ are the essentially bounded functions and $\|\cdot\|_\infty$ the essential supremum norm.

Proposition 2.2 (small moduli). *For a function $f \in L^1([0, 1])$ and an integer $C \in \omega$ the following hold:*

- (1) *if $n \mapsto n + C$ is a singularity modulus of f , then $f \in L^\infty([0, 1])$ and $\text{lb}(\|f\|_\infty) \leq C$.*
- (2) *If $f \in L^p([0, 1])$ for some $1 < p \leq \infty$ and $C > \text{lb}(\|f\|_p)$ and $D \geq (1 - \frac{1}{p})^{-1}$ are integer constants, then $n \mapsto D(n + C)$ is a singularity modulus of f .*

For the proof recall the following theorem, a proof of which can be found in [Rud87].

Theorem 2.3 (Lebesgue Differentiation Theorem). *Let $f \in L^1(\mathbb{R})$. Then for any representative g of f and almost all $x \in \mathbb{R}$ it holds that*

$$g(x) = \lim_{m \rightarrow \infty} 2^m \int_{x-2^{-m-1}}^{x+2^{-m-1}} g d\lambda.$$

Proof of Proposition 2.2. First prove (1). For this assume that $n \mapsto n + C$ is a singularity modulus of f and let g be a representative of the function considered. By the Lebesgue

Differentiation Theorem 2.3 there exists a set $A \subseteq [0, 1]$ of measure one such that for any $x \in A$

$$|g(x)| = \lim_{m \rightarrow \infty} 2^m \left| \int_{x-2^{-m-1}}^{x+2^{-m-1}} g d\lambda \right| = \lim_{n \rightarrow \infty} 2^{n+C} \left| \int_{x-2^{-n+C-1}}^{x+2^{-n+C-1}} f d\lambda \right| < 2^C.$$

This proves that $\|f\|_\infty \leq 2^C$ and in particular that $f \in L^\infty([0, 1])$.

To prove (2) use Hölder's inequality (see Corollary 3.2) to deduce

$$\left| \int_x^{x+h} f(t) dt \right| \leq \int_x^{x+h} |f(t)| dt \leq \|f\|_p h^{1-\frac{1}{p}}.$$

From this it is easy to see that the assertion is true. It remains true for $p = \infty$ if the convention $\frac{1}{\infty} = 0$ is used. \square

In particular the functions with singularity modulus of form $n + C$ for some C are exactly the functions contained in L^∞ . The class of functions with linear modulus with slope $(1 - 1/p)^{-1}$ contains L^p , however, the inclusion is strict as can be seen by considering the function $x^{-1/p}$. The corresponding classes for the modulus of continuity are the Lipschitz and Hölder-continuous functions.

2.2. The singular representation in one dimension. Recall that the information about a continuous function can be divided into two parts by the characterization of polynomial-time computable functions from Theorem 1.12. The first part being approximations to the values on dyadic numbers and the second part being a modulus of continuity. Definition 1.13 uses this to introduce the standard representation of continuous functions.

The following definition carries this idea to the set of integrable functions, where integrals over dyadic intervals replace the point evaluations and the singularity modulus replaces the modulus of continuity. Recall that \mathbb{D} denotes the set numbers of the form $\frac{m}{2^n}$ for $m \in \mathbb{Z}$ and $n \in \omega$, that $[\cdot] : \Sigma^* \rightarrow \mathbb{D}$ is the encoding fixed in the introduction and that the length $|\varphi|$ of a length-monotone string function is given by $|\varphi|(|\mathbf{a}|) = |\varphi(\mathbf{a})|$.

Definition 2.4. Define the **singular representation** ξ_s of $L^1([0, 1])$: A length-monotone string function φ is a name of $f \in L^1([0, 1])$ if for all strings \mathbf{a}, \mathbf{b} with $[\mathbf{a}], [\mathbf{b}] \in [0, 1]$

$$\left| \int_{[\mathbf{a}]}^{[\mathbf{b}]} f d\lambda - [\varphi(\langle \mathbf{a}, \mathbf{b}, 1^n \rangle)] \right| < 2^{-n}$$

and $|\varphi|$ is a singularity modulus of f .

This definition is well posed: Firstly, for any distinct integrable functions there exists a dyadic interval such that their integrals over this interval differ. Thus, the above indeed defines a partial function. Secondly, any integrable function has a singularity modulus and therefore the mapping is surjective.

It can easily be verified that this representation renders the vector space operations of $L^1([0, 1])$ polynomial-time computable. The representation ξ_s is chosen such that it is polynomial-time equivalent to the representation from the introduction of this section. As a result, it possesses the same minimality property. We state this as a theorem, note however, that it is also covered by Theorem 2.8, which contains a more explicit statement and a direct proof.

Theorem 2.5 (minimality). ξ_s is a minimal representation of $L^1([0, 1])$ such that the integration operator is polynomial-time computable.

2.3. Higher dimensions. Definition 2.4 allows a straight forward generalization to higher dimensions. Fix some dimension d , let $\Omega \subseteq \mathbb{R}^d$ be a bounded measurable set and recall that \tilde{f} denotes the extension of a function to the whole space by zero.

Definition 2.6. A function $\mu : \omega \rightarrow \omega$ is called a **singularity modulus** of $f \in L^1(\Omega)$ if it is a singularity modulus (in the sense of Definition 2.1) for each of the functions

$$f_i(x) := \int_{\mathbb{R}^{d-1}} \tilde{f}(x_1, \dots, x_{i-1}, x, x_{i+1}, \dots, x_d) dx_1 \cdots dx_{i-1} dx_{i+1} \cdots dx_d.$$

(compare Figure 2).

Recall from the introduction that for $x, y \in \mathbb{R}^d$ the smallest box with corners x and y and edges parallel to the axis is denoted by $[x, y]$ and that the box $[[\mathbf{a}]_d, [\mathbf{b}]_d]$ is abbreviated as $[\mathbf{a}, \mathbf{b}]$.

Definition 2.7. Define the **singular representation** ξ_s of $L^1(\Omega)$: A length-monotone string function φ is a ξ_s -name of $f \in L^1(\Omega)$ if for all strings \mathbf{a}, \mathbf{b} with $[[\mathbf{a}]], [[\mathbf{b}]] \in \mathbb{D}^d$

$$\left| \int_{[\mathbf{a}, \mathbf{b}]} \tilde{f} d\lambda - [[\varphi(\langle \mathbf{a}, \mathbf{b}, 1^n \rangle))] \right| < 2^{-n}$$

and $|\varphi|$ is a singularity modulus of f .

Since no source for a multidimensional generalization of the minimality of the standard representation for continuous functions from Theorem 1.15 is known to the author, a direct proof of the minimality is given for the multidimensional case.

Theorem 2.8 (minimality of the singular representation). *For a second-order representation ξ of $L^1(\Omega)$ the following are equivalent:*

- ξ renders the integration operator from eq. (INT) polynomial-time computable.
- ξ is polynomial-time translatable to the singular representation ξ_s .

Proof. The proof is very similar to the proof of [KC10, Lemma 4.9], i.e. of the minimality of ξ_C from Theorem 1.15. First assume that ξ is a representation such that the integration operator from eq. (INT) polynomial-time computable. Describe an oracle Turing machine that whenever given a ξ -name φ of a function $f \in L^1(\Omega)$ returns correct values of a ξ_s -name of f : This machine simulates a machine computing the integration operator in polynomial-time to obtain approximations to the integrals of f from φ .

To obtain a singularity modulus of the input function let P be a second-order polynomial bounding the running time of the integration operator and p a polynomial such that any $(x, y) \in \Omega^2$ has a name of length p (This depends on the concrete encoding of dyadic numbers and products chosen, but exists for reasonable choices and bounded Ω). Then $\mu : \omega \rightarrow \omega$, $n \mapsto P(\langle |\varphi|, p \rangle, n + 1)$ is a singularity modulus of f : When queried for an approximation with quality 2^{-n-1} the machine computing the integration operator can at most take $\mu(n)$ steps. Therefore, it knows the boundaries a and b of the integral with precision at most

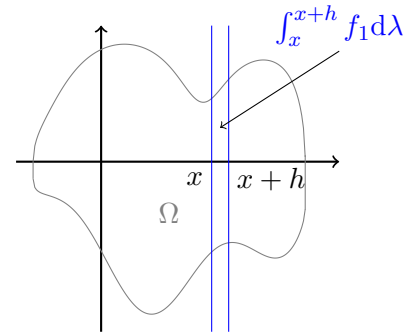


FIGURE 2. f_1 in two dimensions.

$2^{-\mu(n)}$. Recall the definition of the singularity modulus from Definition 2.6, in particular that f_i was f with all but the i -th variable integrated over. Since Ω is bounded, there are some \mathbf{c} and \mathbf{d} such that $\Omega \subseteq [\mathbf{c}, \mathbf{d}]^d$. Note that for any $x \in [\mathbf{c}, \mathbf{d}]$ and $h \in \mathbb{R}$ with $|h| \leq 2^{-\mu(n)}$ there is a dyadic vector $a = \llbracket \mathbf{a} \rrbracket$ which is a valid $2^{-\mu(n)}$ approximation for both x and $x+h$.

The argument works the same for any i . Set $i = d$ from now on to simplify notation. Define length-monotone string functions φ_a^+ and φ_a^- by

$$\varphi_a^+(\mathbf{b}) := \langle \mathbf{d}, \dots, \langle \mathbf{d}, \mathbf{a} \rangle \dots \rangle, \text{ resp. } \varphi_a^-(\mathbf{b}) := \langle \mathbf{c}, \dots, \langle \mathbf{c}, \mathbf{a} \rangle \dots \rangle.$$

Let q be the approximation encoded in the output of the machine computing int when handed φ as function name, $\langle \varphi_a^-, \varphi_a^+ \rangle$ as boundaries of the integral and 1^{n+1} as precision requirement.

Since a is an approximation to both x and $x+h$ and $[\mathbf{c}, \mathbf{d}]^{d-1} \times [a, a]$ is a set of Lebesgue measure zero

$$\left| \int_x^{x+h} f_i d\lambda \right| \leq \left| \int_{[\mathbf{c}, \mathbf{d}]^{d-1} \times [x, x+h]} \tilde{f} d\lambda - q \right| + \left| q - \int_{[\mathbf{c}, \mathbf{d}]^{d-1} \times [a, a]} \tilde{f} d\lambda \right| < 2^{-n}.$$

It is left to show that ξ_s renders the integration operator polynomial-time computable. Assume a ξ_s name φ of a function f , an oracle for a box and a precision requirement 1^n are given. Get approximations to the vertices of the box with precision $1^{|\varphi|(1^n) + \lceil \text{lb}(d) \rceil + 1}$ and query φ for a 2^{-n-1} approximation over this box. An easy triangle inequality argument shows that this is a valid approximation to the integral over the box. \square

The result includes null sets: In this case $L^1(\Omega)$ only contains one element and the integration operator is the constant zero function.

2.4. Discontinuity. Under reasonable assumptions, the singular representation is discontinuous. Since the proof of discontinuity is most naturally stated for the unit interval, we state a restricted version first:

Proposition 2.9 (discontinuity). *The singular representation ξ_s is not continuous with respect to the norm topology on $L^1([0, 1])$.*

Proof. Consider the sequence of functions on the unit interval defined by

$$f_m(x) := (-1)^{\min\{k \in \mathbb{N} \mid k2^{-m} \geq x\}}.$$

That is: Divide $[0, 1]$ into 2^m equally sized intervals and let the function values alternate between constantly being 1 and -1 respectively on these intervals (compare Figure 3). The functions f_m are

bounded by 1 in the norm $\|\cdot\|_\infty$ and thus allow the common singularity modulus $n \mapsto n+1$ by Proposition 2.2. Observe that the integrals of f_m over an interval is always smaller than the minimum of the length of the interval and 2^{-m} . Thus, since from $|\langle \mathbf{a}, \mathbf{b}, 1^k \rangle| < 1^m$ it follows that $k < m$, it is possible to choose a name φ_m of f_m such that upon this input a string \mathbf{c} of length more than $|\langle \mathbf{a}, \mathbf{b}, 1^k \rangle| + 1$ with $\llbracket \mathbf{c} \rrbracket = 0$ is returned. The sequence φ_m converges to a name of the zero function in Baire-space. However, $\xi_s(\varphi_m) = f_m$ has norm 1 for all m and therefore does not converge to the zero function in norm. This proves discontinuity of ξ_s . \square

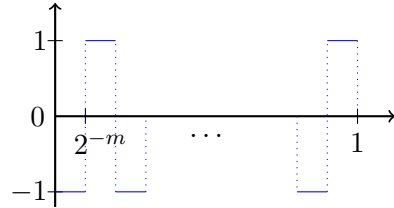


FIGURE 3. The function f_m

Theorem 2.10 (discontinuity). *Whenever $\Omega \subset \mathbb{R}^d$ is a bounded set with non-empty interior, the singular representation ξ_s of $L^1(\Omega)$ from Definition 2.4 is discontinuous.*

sketch of the proof. Since the interior of Ω is non-empty, there exists a small box with edges parallel to the axis and dyadic endpoints completely included in Ω . Lift the function sequence from the proof of Proposition 2.9 to a box by assuming the functions to be independent of the additional variables. Scale this box and the functions to fit inside of Ω . The arguments of the above proof still work for this new sequence and show discontinuity. \square

This result can be strengthened to prove discontinuity of the singular representation with respect to the weak topology on $L^1(\Omega)$. It is not known to the author if the final topology of ξ_s coincides with any topology of L^1 that has been considered before.

Using ξ_s it is possible to construct a weakest representation computably equivalent to the Cauchy representation, such that integration is polynomial time computable. However, this representation has very undesirable properties from a complexity theoretical point of view: While it renders many operations, like the metric, computable, only those that are already computable in bounded time with respect to the singular representation become computable in bounded time.

3. L^p -SPACES

The previous chapter introduced the weakest representation ξ_s of integrable functions such that the integration operator is polynomial-time computable and showed its discontinuity. However, the way the representation was introduced allows for straight forward generalizations: Like for the continuous functions the information was divided into a discrete part, the integrals over dyadic intervals, and a topological part: the singularity modulus. This section discusses a replacement for the singularity modulus which leads to a continuous representation. More precisely, for any p a modulus is defined that exists if and only if the function is an element of $L^p(\Omega)$.

First recall some basic facts about spaces of integrable functions: Let λ denote the Lebesgue measure of any dimension. In the following Ω denotes a bounded, measurable set. Recall that the space $L^p(\Omega)$ is the Banach space of equivalence classes of measurable functions up to equality almost everywhere such that

$$\|f\|_p = \|f\|_{p,\Omega} := \left(\int_{\Omega} |f|^p d\lambda \right)^{\frac{1}{p}} < \infty.$$

And that for the case $p = \infty$ the norm $\|\cdot\|_{\infty}$ is defined to be the essential supremum norm. If Ω is bounded with non-zero Lebesgue measure, then $\mathcal{C}(\Omega) \subsetneq L^p(\Omega) \subsetneq L^q(\Omega)$ whenever $1 \leq q < p \leq \infty$. The inclusions are continuous. This can be seen using the following well known result from analysis:

Theorem 3.1 (Hölder's Inequality). *For any measurable subset $\Omega \subseteq \mathbb{R}^d$, any measurable functions f, g on Ω and any $p \in [1, \infty]$ the inequality*

$$\|fg\|_1 \leq \|f\|_p \|g\|_q$$

holds. Where $q := \frac{1}{1-\frac{1}{p}}$ is the conjugate exponent of p and $q = \infty$ if $p = 1$.

A corollary from this is particularly often useful for our purposes:

Corollary 3.2. *For any measurable function f on a measurable set Ω it holds that*

$$\int_{\Omega} |f| d\lambda \leq \lambda(\Omega)^{1-\frac{1}{p}} \|f\|_p.$$

For $\Omega = [0, 1]$, $f := |g|^r$ and $p := \frac{s}{r}$ the above proves that $\|g\|_r \leq \|g\|_s$ whenever $r < s$.

3.1. The L^p -modulus. In the following let $1 \leq p < \infty$. Recall that \tilde{f} denotes the extension of f to all of \mathbb{R}^d by zero. For $h \in \mathbb{R}^d$ the shift operator τ_h is defined by $(\tau_h f)(x) := f(x+h)$.

Definition 3.3. A function $\mu : \omega \rightarrow \omega$ is called an L^p -**modulus** of $f \in L^p(\Omega)$ if

$$|h| \leq 2^{-\mu(n)} \quad \Rightarrow \quad \|\tilde{f} - \tau_h \tilde{f}\|_p < 2^{-n},$$

and $\mu(n) \neq 0 \Rightarrow \mu(n+1) > \mu(n)$ (i.e. it is strictly increasing whenever non-zero).

Due to the assumption $p < \infty$ any L^p -function has an L^p -modulus (see for instance [Bre11, Lemma 4.3]).

Whenever $1 \leq q \leq p < \infty$, an L^q -modulus can be obtained from an L^p -modulus by shifting with a constant. Let $f \in \mathcal{C}(\Omega)$ be such that its extension \tilde{f} is continuous. In this case a modulus of continuity μ of f is also a modulus of \tilde{f} and can be converted into an L^p -modulus of f : Whenever $|h| \leq 2^{-\mu(n)}$, then $|x - (x+h)| \leq 2^{-\mu(n)}$ and therefore

$$\|\tilde{f} - \tau_h \tilde{f}\|_p = \left(\int_{\mathbb{R}^d} |\tilde{f}(x) - \tilde{f}(x+h)|^p dx \right)^{\frac{1}{p}} < 2^{-n + \frac{1 + \text{lb}(\lambda(\Omega))}{p}}.$$

Thus, $n \mapsto \mu(n + \lceil (1 + \text{lb}(\lambda(\Omega))) / p \rceil)$ is an L^p -modulus of f . If f can not be continuously extended, additional information about the function and the domain is needed to obtain an L^p -modulus from a modulus of continuity (compare Lemma 4.5).

The modulus of continuity does not contain any information about the norm of a function as it does not change under shift with a constant function. In contrast to that a norm-bound can be deduced from an L^p -modulus. Recall the diameter of a set:

$$\text{diam}(\Omega) := \sup\{|x - y|_{\infty} \mid x, y \in \Omega\},$$

where $|\cdot|_{\infty}$ denotes the supremum norm on \mathbb{R}^d .

Lemma 3.4 (norm estimate). *Whenever μ is an L^p -modulus of $f \in L^p(\Omega)$, then*

$$\|f\|_p < \lceil \text{diam}(\Omega) \rceil 2^{\mu(0) - \frac{1}{p}}.$$

Proof. Fix some unit vector $e \in \mathbb{R}^d$. The intersection of $\Omega + \lceil \text{diam}(\Omega) \rceil e$ and Ω has zero Lebesgue measure. Thus:

$$2^{\frac{1}{p}} \|f\|_p \leq \sum_{i=1}^{\lceil \text{diam}(\Omega) \rceil 2^{\mu(0)}} \|\tilde{f} - \tau_{2^{-\mu(0)} e} \tilde{f}\|_p < \lceil \text{diam}(\Omega) \rceil 2^{\mu(0)},$$

which proves the assertion. □

3.2. Representing L^p . Let $\Omega \subseteq \mathbb{R}^d$ be a bounded measurable set of non-zero measure.

Definition 3.5. Define the **second-order representation ξ_p of $L^p(\Omega)$** : A length-monotone string function φ is a name of $f \in L^p(\Omega)$ if and only if for any strings $\mathbf{a}, \mathbf{b} \in \Sigma^*$ and $n \in \mathbb{N}$

$$\left| \int_{[\mathbf{a}, \mathbf{b}]} \tilde{f} d\lambda - \llbracket \varphi(\langle \mathbf{a}, \mathbf{b}, \mathbf{1}^n \rangle) \rrbracket \right| < 2^{-n},$$

and $|\varphi|$ is an L^p -modulus of f .

Again, the vector space operations of $L^p(\Omega)$ are easily shown to be polynomial-time computable with respect to this representation. Recall that for bounded Ω and $p \geq q$ it holds that $L^p(\Omega) \subseteq L^q(\Omega)$. Since an L^q -modulus can be obtained from an L^p -modulus by shifting with a constant, this inclusion is polynomial-time computable. The partial inverse of this inclusion is discontinuous and therefore not computable. To compare ξ_p to the singular representation from Definition 2.7 note:

Proposition 3.6. *Let μ be an L^p -modulus of a function. Then*

$$n \mapsto \mu(n + 1 + \text{lb}(\lambda(\Omega)) + (d - 1)\text{lb}(\text{diam}(\Omega)))$$

is a singularity modulus of the function.

Proof. Let $f \in L^p(\Omega)$ be a function and let μ be an L^p -modulus of f . Recall from Definition 2.6 that f_i denotes the function where all but the i -th variable has been integrated over. Apply the transformation rule and the version of Hölder's inequality from Corollary 3.2 to get

$$\begin{aligned} \left| \int_x^{x+h} f_i(t) dt \right| &\leq \left| \int_x^\infty f_i(t) dt - \int_{x+h}^\infty f_i(t) dt \right| \leq \int_{-\infty}^\infty |f_i - \tau_h f_i| d\lambda \\ &\leq (2\lambda(\Omega))^{1-\frac{1}{p}} \|f_i - \tau_h f_i\|_p \leq (2\lambda(\Omega))^{1-\frac{1}{p}} \text{diam}(\Omega)^{(d-1)-\frac{d-1}{p}} \|\tilde{f} - \tau_{he_i} \tilde{f}\|_p. \end{aligned}$$

Since $|h| = |he_i|_\infty$, the assertion follows from μ being an L^p -modulus of f . \square

This proposition implies that ξ_s is polynomial-time reducible to ξ_p . Thus:

Theorem 3.7 (efficiency of integration). ξ_p renders the restriction of the integration operator from eq. (INT) to $L^p(\Omega) \times \Omega^2$ polynomial-time computable.

Proof. Let N be a natural number that bounds $1 + \text{lb}(\lambda(\Omega)) + (d - 1)\text{lb}(\text{diam}(\Omega))$. By the previous proposition the mapping padding the length of a ξ_p -name φ of some function to have length $n \mapsto |\varphi|(n + N)$ is a polynomial-time translation to ξ_s . The assertion now follows from the minimality of the singular representation from Theorem 2.8. \square

3.3. Equivalence to the Cauchy representation. Recall from Definition 1.2 that to obtain a Cauchy representation of $L^p(\Omega)$ it is sufficient to fix a dense subsequence. The obvious choice for this sequence are the dyadic step functions: Call a function a dyadic step function, if it is a dyadic linear combination of characteristic functions of sets of the form $[\mathbf{a}, \mathbf{b}]$ with $\llbracket \mathbf{a} \rrbracket, \llbracket \mathbf{b} \rrbracket \in \mathbb{D}^d$. An enumeration of this set is cumbersome to write down, but there is one such that both the enumeration and its inverse are polynomial-time computable if a dyadic step function is identified with the list of the boxes on which it is non-zero and the corresponding values. The corresponding Cauchy representation is well established at least for investigating computability in $L^p([0, 1])$ (cf. [PER89, ZZ99, Zho99] and many more).

The goal of this section is to prove the following:

Theorem 3.8 (equivalence to the Cauchy representation). ξ_p is computably equivalent to the Cauchy representation of $L^p(\Omega)$.

One of the translations is easy and specified now. The other direction is more complicated and postponed until the end of the chapter.

*Proof that the Cauchy representation translates to ξ_p .*¹ An oracle Turing machine that translates a name φ of a function f in the Cauchy representation into a ξ_p -name can be specified as follows: Given φ as oracle and a string \mathbf{c} as input set $n := |\mathbf{c}|$. The machine obtains a valid value $\mu(n)$ of an L^p -modulus of f as follows: Let f_{n+2} be the function encoded by $\varphi(n+2)$. Since this function is encoded as a list of the boxes it does not vanish on and its values on these boxes, the machine can obtain a bound 2^k on the number of boxes, 2^l of their diameters and 2^m of the values. Note that a dyadic step function that is defined as a linear combination of 2^k characteristic functions on sets of size 2^l can, when shifted by y , at most differ from the original function on a set of size $d \cdot |y|_\infty \cdot 2^{(d-1)l+k}$. Thus, since the difference can be majorized by 2^{m+1} on the set where it is nonzero, get

$$\sup_{|y|_\infty \leq h} \|f_{n+2} - \tau_y f_{n+2}\|_p \leq 2^{m+1} \cdot \left(d \cdot h \cdot 2^{(d-1)l} 2^k\right)^{\frac{1}{p}}.$$

This means that $r := \lceil p \rceil(d-1)l + k + \lceil \log(d) \rceil + \lceil p \rceil(m+n+2)$ is a valid value of an L^p -modulus of f_{n+2} in $n+1$. Now, whenever $|y|_\infty \leq 2^{-r}$ then

$$\begin{aligned} \|f - \tau_y f\|_p &\leq \|f - f_{n+2} - \tau_y f + \tau_y f_{n+2}\|_p + \|f_{n+2} - \tau_y f_{n+2}\|_p \\ &\leq 2\|f - f_{n+2}\|_p + \|f_{n+2} - \tau_y f_{n+2}\|_p < 2^{-n}. \end{aligned}$$

Thus, r is indeed a candidate for a value of an L^p -modulus of f in n . By repeating the procedure for all values of n smaller than $|\mathbf{c}|$ and increasing r if necessary, the machine computes a value $\mu(|\mathbf{c}|)$ of an L^p -modulus of f .

Next, the machine checks if the input string is of the form $\mathbf{c} = \langle \mathbf{a}, \mathbf{b}, 1^n \rangle$. If it is, it computes approximations of the integrals by returning the integrals of a dyadic step function which approximates the function accurately enough in L^p . Before returning it, it pads the encoding of that approximation to length at least $\mu(|\mathbf{c}|)$. If the input string \mathbf{c} is not of the form $\langle \mathbf{a}, \mathbf{b}, 1^n \rangle$, the machine returns $1^{\mu(|\mathbf{c}|)}$ (in this case only the length is relevant). \square

The above proof can be checked to actually construct a polynomial-time reduction.

The basic idea for the other direction is to approximate the function from $L^p(\Omega)$ by step functions where the values are the integrals over boxes. For easier notation write

$$[x]_m := x + [-2^{-m-1}, 2^{-m-1}]^d.$$

Note, that the Lebesgue measure of these sets is given by $\lambda([x]_m) = 2^{-dm}$.

Definition 3.9. Let $f \in L^p$ be a function. Define the **sequence** $(f_m)_{m \in \mathbb{N}}$ of **continuous approximations** to f by

$$f_m(x) := 2^{dm} \int_{[x]_m} \tilde{f} d\lambda.$$

A modulus of continuity of f_m can be obtained from an L^p -modulus of f :

¹This proof was considerably simplified and strengthened thanks to a comment of an anonymous referee.

Lemma 3.10 (continuity). *Whenever μ is an L^p -modulus of $f \in L^p$, the function $n \mapsto \mu(n + \lceil \frac{d}{p} \rceil m)$ is a modulus of continuity of f_m .*

Proof. Use the version of Hölder's inequality from Corollary 3.2 to conclude

$$|f_m(x) - f_m(y)| \leq 2^{dm} \int_{[x]_m} |\tilde{f}(t) - \tilde{f}(t - (x - y))| dt \leq 2^{\frac{d}{p}m} \|\tilde{f} - \tau_{x-y}\tilde{f}\|_p.$$

From this the assertion is obvious. \square

How good an approximation f_m is to f can be read off from an L^p -modulus of f :

Lemma 3.11 (approximation). *Let μ be an L^p -modulus of f . Then $\|\tilde{f} - f_{\mu(n)}\|_p < 2^{-n}$.*

Proof. Using $\int_{[0]_m} 2^{dm} d\lambda = 1$ one sees that

$$\begin{aligned} \|\tilde{f} - f_m\|_p^p &\leq \int_{\mathbb{R}^d} \left| \tilde{f}(s) - 2^{dm} \int_{[0]_m} \tilde{f}(t+s) dt \right|^p ds \\ &\leq 2^{dmp} \int_{\mathbb{R}^d} \left(\int_{[0]_m} |\tilde{f}(s) - \tilde{f}(t+s)| dt \right)^p ds. \end{aligned}$$

Use the version of Hölder's inequality from Corollary 3.2 and Fubini to get

$$\|\tilde{f} - f_m\|_p^p \leq 2^{dm} \int_{[0]_m} \int_{\mathbb{R}^d} |\tilde{f}(s) - \tilde{f}(t+s)|^p ds dt.$$

Set $m := \mu(n)$ and use that μ is an L^p -modulus to see that

$$\|\tilde{f} - f_m\|_p < \left(2^{dm} \int_{[0]_m} 2^{-pn} dt \right)^{\frac{1}{p}} = 2^{-n},$$

which proves the assertion. \square

We are now prepared to prove the second half of the main theorem of this section:

Proposition 3.12. ξ_p computably translates to the Cauchy representation of $L^p(\Omega)$.

Proof (also of Theorem 3.8). Let φ be a ξ_p -name of $f \in L^p(\Omega)$. Set $C := \lceil \text{lb}(\lambda(\Omega)) \rceil$. For any $z \in \mathbb{D}^d$ fix some binary encoding $\mathbf{a}_z \in \Sigma^*$ (for example the unique canceled encoding). Furthermore, let e be the constant one vector $(1, \dots, 1)$ and set

$$d_{z,k,N} := 2^{dN} \left\| \varphi(\langle \mathbf{a}_{z-2^{-N-1}e}, \mathbf{a}_{z+2^{-N-1}e}, \mathbf{1}^k \rangle) \right\| \quad \text{and} \quad \mu := |\varphi|.$$

Thus, $d_{z,k,n}$ is a 2^{-k} -approximation to the integral of f over $[z]_N$ therefore also an approximation to the value of f_N in z .

Consider the step function

$$F_{k,N,M} := \sum_{z \in \mathbb{D}_M} d_{z,k,N} \chi_{[z]_M},$$

where \mathbb{D}_M denotes the set of $z \in \mathbb{D}^d$ such that each component is of the form $\frac{m}{2^M}$ and such that $[z]_M \cap \Omega \neq \emptyset$. Since Ω is bounded there is a constant D such that $\#\mathbb{D}_M \leq 2^{dM+D}$.

Obviously, the step function $F_{k,N,M}$ can be uniformly computed from the name φ and the constants k, N and M . To see how to choose k, N and M write

$$\|f - F_{k,N,M}\|_p \leq \|f - f_N\|_p + \|f_N - F_{k,N,M}\|_p \quad (3.1)$$

By the approximation property of f_N from Lemma 3.11 for the first summand to be smaller than 2^{-n} , N should be chosen $\mu(n+1)$. For the second summand, note that each $x \in \Omega$ is 2^{-M} close to some $z \in \mathbb{D}_M$ and that for these z

$$|f_N(z) - F_{k,N,M}(z)| < 2^{dN-k}.$$

Choosing $M := \mu(n + \lceil d(N+C)/p \rceil + 2)$ and $k := dN + n + 2$, using the modulus of continuity of f_N from Lemma 3.10 and that $F_{k,N,M}$ is piecewise constant obtain

$$\|f_N - F_{k,N,M}\|_p \leq \lambda(\Omega)^{\frac{1}{p}} \|f_N - F_{k,N,M}\|_\infty < 2^{-n-1}$$

Summing up, the result is smaller than 2^{-n} . \square

The Cauchy representation of L^p is continuous. Thus, the above proves that ξ_p is a continuous mapping. Whenever p is computable, the L^p -norm is computable with respect to the Cauchy representation, and therefore also with respect to ξ_p . The above translation does not run in polynomial time as it accesses the oracle an exponential number of times. That no polynomial-time reduction exists can be seen from the results of the last chapter: With respect to ξ_p the norm is not polynomial time computable.

It is not difficult to see that a minimality result like the ones for the representation of continuous functions (Theorem 1.15) and the singular representation (Theorem 2.8) cannot be proven for this representation. This remains true if only the continuous representations are required to polynomial-time reduce.

4. SOBOLEV SPACES

This chapter only considers the simplest domain $\Omega = [0, 1]$. To simplify notation the domain is often omitted. For a function $f \in L^1([0, 1])$ a function $f' \in L^1([0, 1])$ is called a **weak derivative**, if for any $g \in \mathcal{C}^\infty([0, 1])$ with $g(0) = 0 = g(1)$ it holds that

$$\int_{[0,1]} f g' d\lambda = - \int_{[0,1]} f' g d\lambda.$$

Recall that, if it exists, the weak derivative of a function is uniquely determined (as an element of L^1). Furthermore, if an element of L^1 allows a weak derivative f' , then there is a continuous representative f that fulfills

$$f(y) - f(x) = \int_{[x,y]} f' d\lambda.$$

While in higher dimensions weakly differentiable functions may have singularities, in one dimension they are continuous. The following refinement of the continuity of a weakly differentiable function follows directly from Proposition 3.6:

Lemma 4.1 (differentiability and moduli). *Whenever f is weakly differentiable and μ is an L^p -modulus of f' then $n \mapsto \mu(n+1)$ is a modulus of continuity of f .*

The Sobolev space $W^{1,p}$ is defined as the set of functions from L^p that have a weak derivative which is also an L^p -function. Sobolev spaces are of great importance in the theory of partial differential equations. It is well known that the Sobolev spaces can be characterized as spaces of functions with small L^p -moduli (compare for instance [Bre11, Proposition 8.5]). Since the named source uses different terminology and the result is stated for the whole space and not the unit interval we restate it and give a proof.

Lemma 4.2 (small moduli). *The following are equivalent for $f \in L^p$ with $1 < p < \infty$:*

- $f \in W^{1,p}$ and the continuous representative vanishes in 0 and 1.
- There is a $C \in \omega$ such that $n \mapsto n + C$ is an L^p -modulus of f .

Furthermore, the constant C can be chosen as any integer strictly larger than $\text{lb}(\|f'\|_p)$.

Proof. First assume that $f \in W^{1,p}$ and that the continuous representative vanishes at 0 and 1. In this case the extension \tilde{f} to the whole real line by zero is continuous and its weak derivative is the extension of the weak derivative by zero. Use the version of Hölder's inequality from Corollary 3.2 to conclude

$$\begin{aligned} \|\tilde{f} - \tau_h \tilde{f}\|_p &= \left(\int_{\mathbb{R}} \left| \int_x^{x+h} \tilde{f}'(t) dt \right|^p dx \right)^{\frac{1}{p}} \leq h \left(\int_{\mathbb{R}} \left(\int_0^1 |\tilde{f}'(x+sh)| ds \right)^p dx \right)^{\frac{1}{p}} \\ &\stackrel{3.2}{\leq} h \left(\int_{\mathbb{R}} \int_0^1 |\tilde{f}'(x+sh)|^p ds dx \right)^{\frac{1}{p}} = h \|f'\|_p. \end{aligned}$$

From this it is easy to see that $n + C$ is an L^p -modulus of f whenever C is strictly larger than $\text{lb}(\|f'\|_p)$.

For the other direction assume that $n + C$ is an L^p -modulus of f . Recall that [Bre11, Proposition 8.5] states that a function $g \in L^p(\mathbb{R})$ is an element of $W^{1,p}(\mathbb{R})$ if the inequality $\|g - \tau_h g\|_p \leq D|h|$ holds for all $h \in \mathbb{R}$. \tilde{f} fulfills this for $D := 2^{2C+1}$: Given h first check if there is a n such that $2^{-\mu(n+1)} \leq |h| < 2^{-\mu(n)}$. If so, then

$$\|\tilde{f} - \tau_h \tilde{f}\|_p < 2^{-n} = 2^{-n+\mu(n+1)-\mu(n+1)} \leq 2^{C+1} |h|$$

If there is no such n , then $2^{-\mu(0)} \leq |h|$ and using the norm bound from the L^p -modulus by Lemma 3.4 conclude

$$\|\tilde{f} - \tau_h \tilde{f}\|_p \leq 2 \|\tilde{f}\|_p < 2^{\mu(0)+1} \leq 2^{2C+1} |h|.$$

Thus, in any case

$$\|\tilde{f} - \tau_h \tilde{f}\|_p < 2^{2C+1} |h|. \tag{h}$$

It follows that the restriction of \tilde{f} to $[0, 1]$ is an element of the Sobolev space. Show that the continuous representative of f vanishes on the boundary by contradiction: Assume $f(0) \neq 0$, w.l.o.g. $f(0) > 0$. Then there exists some ε and some interval $[0, \delta]$ such that $f(x) \geq \varepsilon$ for any $x \in [0, \delta]$. Set $h := \min \{ \delta, (\varepsilon 2^{-2C-1})^{1/(1-1/p)} \}$, then

$$\|\tilde{f} - \tau_h \tilde{f}\|_p \geq \left(\int_0^h |f|^p d\lambda \right)^{\frac{1}{p}} \geq h^{\frac{1}{p}} \varepsilon \geq 2^{2C+1} h = 2^{2C+1} |h|,$$

which contradicts eq. (h). Therefore, f vanishes in zero. The argument for the other end of the interval is identical. \square

In the case $p = 1$ one of the directions of the result fails: Characteristic functions of intervals have $n+1$ as L^1 -modulus while not being weakly differentiable. The other direction still holds true.

In the remarks following Definition 1.10 of the modulus of continuity the corresponding class of functions was specified as the Lipschitz functions. In Proposition 2.2 the class for the singularity modulus was proven to be L^∞ .

4.1. Representing $W^{m,p}$. Denote the m times iterated weak derivative of a function f by $f^{(m)}$. The **Sobolev space** $W^{m,p}$ is the space of all functions $f \in L^p$ such that the weak derivatives $f', \dots, f^{(m)}$ exist and are L^p -functions. Equipped with the norm

$$\|f\|_{m,p} := \sqrt[p]{\|f\|_p^p + \|f^{(m)}\|_p^p}$$

this space is a Banach space, and for $p = 2$ a Hilbert space. In one dimension from $f^{(m)}$ being an L^p function it follows that $f^{(m-1)}$ is continuous.

Recall the encoding $\llbracket \cdot \rrbracket$ of the dyadic numbers from the introduction and that for a length monotone string function $|\varphi|(\llbracket \mathbf{a} \rrbracket) = |\varphi(\mathbf{a})|$.

Definition 4.3. Define the **second-order representation** $\xi_{m,p}$ of $W^{m,p}$: A length-monotone string function φ is a $\xi_{m,p}$ -name of $f \in W^{m,p}$ if for all strings \mathbf{a}, \mathbf{b} such that $\llbracket \mathbf{a} \rrbracket, \llbracket \mathbf{b} \rrbracket \in [0, 1]$ and all $n \in \mathbb{N}$

$$\left| \int_{\llbracket \mathbf{a} \rrbracket}^{\llbracket \mathbf{b} \rrbracket} f d\lambda - \llbracket \varphi(\langle \mathbf{a}, \mathbf{b}, \mathbf{1}^n \rangle) \rrbracket \right| < 2^{-n},$$

and $|\varphi|$ is an L^p -modulus (see Definition 3.3) of the highest derivative $f^{(m)}$ of f .

From now on always equip $W^{m,p}$ with the second-order representation $\xi_{m,p}$. The representations ξ_p from Definition 3.5 coincide with $\xi_{0,p}$, so no ambiguities arise. The space $\mathcal{C}([0, 1])$ is always equipped with $\xi_{\mathcal{C}}$.

4.2. The space $W^{1,p}$. Before investigating the space $W^{m,p}$ consider the simplest non-trivial case $m = 1$. As a set $W^{1,p}$ is contained in L^p . From the definition of the norm on $W^{1,p}$ it follows, that the inclusion mapping $W^{1,p} \hookrightarrow L^p$ is continuous.

Theorem 4.4 (Sobolev functions as L^p -functions). *The inclusion mapping $W^{1,p} \hookrightarrow L^p$ is polynomial-time computable.*

For the proof it is necessary to obtain an L^p -modulus of a function from a modulus of continuity and some extra information. The corresponding result is interesting on its own behalf. Therefore, we state it separately and in more generality than needed.

Lemma 4.5. *Let μ be a modulus of continuity of some function $f \in \mathcal{C}(\Omega)$ and let ν be an L^p -modulus of the characteristic function of Ω . Then an L^p -modulus of f is given by*

$$\eta(n) := \max \{ \mu(n + \lceil \text{lb}(\lambda(\Omega)) \rceil + 1), \nu(n + \lceil \text{lb}(\|f\|_\infty) \rceil + 1) \}.$$

Proof. for sets A and B denote the symmetric difference by $A\Delta B := (A \cup B) \setminus (A \cap B)$. A function ν is an L^p -modulus of the characteristic function of Ω if and only if from $|h| \leq 2^{-\nu(m)}$ it follows that $\lambda(\Omega\Delta(\Omega+h))^{1/p} < 2^{-m}$. Thus, for $|h| \leq 2^{-\eta(n)}$

$$\begin{aligned} \|f - \tau_h f\|_p &\leq \|\chi_{\Omega \setminus (\Omega+h) \cup \Omega \setminus (\Omega-h)} f\|_p + \|\chi_{\Omega \cap (\Omega+h)} (f - \tau_h f)\|_p \\ &\leq \|f\|_\infty \cdot \lambda(\Omega\Delta(\Omega+h))^{1/p} + \left(\int_{\Omega \cap (\Omega+h)} |f - \tau_h f|^p d\lambda \right)^{1/p} \\ &< 2^{\text{lb}(\|f\|_\infty)} 2^{-n - \lceil \text{lb}(\|f\|_\infty) \rceil - 1} + 2^{\frac{\text{lb}(\lambda(\Omega))}{p}} 2^{-n - \lceil \text{lb}(\lambda(\Omega)) \rceil - 1} \leq 2^{-n}. \end{aligned}$$

Which proves the assertion. \square

For $\Omega = [0, 1]$ the characteristic function has $n \mapsto n + 1$ as modulus and the previous result states that up to a bound on the norm, a modulus of continuity contains strictly more information about the function than an L^p -modulus.

Proof of Theorem 4.4. The following specifies an oracle Turing machine that transforms a $\xi_{1,p}$ -name φ of f into a $\xi_{0,p}$ -name of f : The approximations to the integrals for the $\xi_{0,p}$ -name can be read from φ . To find the right length of the output, access to an L^p -modulus of the function is needed. Since $|\varphi|$ is an L^p -modulus of f' , by Theorem 4.1 $\mu(n) := |\varphi|(n+1)$ is a modulus of continuity of f . Recall from Lemma 4.5 that to obtain an L^p -modulus of f from a modulus of continuity of f it suffices to have a bound on the supremum norm. By the mean value theorem for integration

$$\int_0^1 f d\lambda = f(y)$$

for some $y \in [0, 1]$. Let \mathbf{a} and \mathbf{b} be encodings of 0 and 1 as dyadic numbers. Then

$$|f(y)| \leq \left| f(y) - \int_0^1 f d\lambda \right| + \left| \int_0^1 f d\lambda - \llbracket \varphi(\langle \mathbf{a}, \mathbf{b}, \varepsilon \rangle) \rrbracket \right| + \|\llbracket \varphi(\langle \mathbf{a}, \mathbf{b}, \varepsilon \rangle) \rrbracket\| \leq \|\llbracket \varphi(\langle \mathbf{a}, \mathbf{b}, \varepsilon \rangle) \rrbracket\| + 1.$$

Choose some integer Q such that 2^Q is a bound for $\|\llbracket \varphi(\langle \mathbf{a}, \mathbf{b}, \varepsilon \rangle) \rrbracket\| + 1$. Bound the supremum norm of f by using the modulus of continuity and the triangle inequality: Fix some $x \in [0, 1]$ and set $x_i := x + (y-x)i2^{-\mu(0)}$, then $x_0 = x$, $x_{2^{\mu(0)}} = y$ and $|x_i - x_{i+1}| \leq 2^{-\mu(0)}$. Thus,

$$|f(x)| \leq \sum_{i=0}^{2^{\mu(0)}-1} |f(x_i) - f(x_{i+1})| + |f(y)| \leq 2^{\max\{\mu(0), Q\}+1}.$$

Taking the supremum on both sides gives $\|f\|_\infty \leq 2^{\max\{\mu(0), Q\}+1}$.

Lemma 4.5 now specifies an L^p -modulus that can be computed from φ in polynomial-time. Thus, the machine can pad the return values to an appropriate length. \square

In one dimension, the Sobolev spaces consist of continuous functions and the inclusion mapping $W^{1,p} \hookrightarrow \mathcal{C}([0, 1])$ is well known to be continuous (for $1 < p \leq \infty$ it is compact).

Theorem 4.6 (inclusion into continuous functions). *The inclusion mapping $W^{1,p} \hookrightarrow \mathcal{C}([0, 1])$ is polynomial-time computable.*

Proof. Let φ be a $\xi_{1,p}$ -name of a function $f \in W^{1,p}$. Describe an oracle Turing machine that transforms this name into a $\xi_{\mathcal{C}}$ -name of f : Assume the machine is given some input \mathbf{c} and provided φ as oracle. Note that by Theorem 4.1 the mapping $\mu(n) := |\varphi|(n+1)$ is a modulus of continuity of the continuous representative of f . Therefore the necessary length

of the return value is known. If the input is not of the form $\mathbf{c} = \langle \mathbf{a}, \mathbf{1}^n \rangle$, where \mathbf{a} is the encoding of some dyadic number $d \in [0, 1]$ return a sufficiently long sequence of zeros. If it is of that form an approximation to $f(d)$ can be obtained as follows: By the mean value theorem

$$2^{\mu(n+1)+1} \int_{d-2^{-\mu(n+1)}}^{d+2^{-\mu(n+1)}} f d\lambda = f(y)$$

for some $y \in [d - 2^{-\mu(n+1)}, d + 2^{-\mu(n+1)}]$ and therefore

$$\left| f(d) - 2^{\mu(n+1)+1} \int_{d-2^{-\mu(n+1)}}^{d+2^{-\mu(n+1)}} f d\lambda \right| < 2^{-n-1}.$$

Let \mathbf{b}^\pm denote encodings of $d \pm 2^{-\mu(n+1)}$. Such encodings are easily obtained from \mathbf{a} . Then $2^{\mu(n+1)+1} \llbracket \varphi(\langle \mathbf{b}^-, \mathbf{b}^+, \mathbf{1}^{\mu(n+1)+n+2} \rangle) \rrbracket$ is an approximation to $f(d)$ and (a sufficiently long encoding is) a valid return value. \square

Note that this result does not imply the previous Theorem 4.4: Polynomial time computability of the restriction of the integration operator from eq. (INT) is known to fail on $\mathcal{C}([0, 1])$ (for instance [KMRZ15, Example 6h]). On L^p on the other hand this operator is polynomial-time computable by Theorem 3.7. Thus, the inclusion mapping $\mathcal{C}([0, 1]) \hookrightarrow L^p$ is not polynomial-time computable.

Corollary 4.7 (differentiation). *The operator*

$$\frac{d}{dx} : W^{1,p} \rightarrow L^p, \quad f \mapsto f'$$

is polynomial-time computable.

Proof. A given $\xi_{1,p}$ -name φ of a function $f \in W^{1,p}$ can be transformed into a $\xi_{0,p}$ name of f' in polynomial-time as follows: An L^p -modulus is contained in the $\xi_{1,p}$ -name. It remains to compute the integrals. By the previous theorem it is possible to obtain approximations to the values of f on dyadic numbers. Using the formula

$$f(y) - f(x) = \int_x^y f' d\lambda$$

and the triangle inequality these can be converted to approximations of the integrals. \square

4.3. The space $W^{m,p}$. Recall from Definition 4.3 that a name of a $W^{m,p}$ function contains information about the integrals of the function over dyadic intervals and an L^p -modulus of the highest derivative of f . If $m > 1$ it is not so easy to combine information contained in the L^p -modulus and in the integrals of the function. The key is to iteratively apply the mean value theorem:

Lemma 4.8. *Whenever $f \in W^{m,p}$ and $(x_i)_{i \in \{1, \dots, 2^{m-1}\}} \subseteq [0, 1]$ are of pairwise distance at least 2^{-m} such that $|f(x_i)| \leq C$, then there exists a $z \in [0, 1]$ such that $f^{(m-1)}(z) \leq 2^{m^2-1}C$.*

Proof. Recursively for any $k < m$ construct a family of points $(x_i^k)_{i \in \{1, \dots, 2^{m-k-1}\}}$ of pairwise distance at least 2^{-m} such that $f^{(k)}(x_i^k) \leq 2^{k(m+1)}C$.

The case $k = 0$ is taken care of by the assumption. Now assume availability of a family x_i^{k-1} as needed. Since $f^{(k-1)}$ is a continuously differentiable function whenever $k < m$,

the mean value theorem states that for any $j \in \{1, \dots, 2^{m-k-1}\}$ there is some element $x_j^k \in [x_{2j-1}^{k-1}, x_{2j}^{k-1}]$ such that

$$f^{(k)}(x_j^k) = \frac{f^{(k-1)}(x_{2j-1}^{k-1}) - f^{(k-1)}(x_{2j}^{k-1})}{x_{2j-1}^{k-1} - x_{2j}^{k-1}}$$

and therefore

$$\left| f^{(k)}(x_j^k) \right| \leq 2 \cdot 2^{(k-1)(m+1)} C \cdot 2^m = 2^{k(m+1)} C.$$

Obviously, the distance of the points will not decrease.

Setting $k = m - 1$ proves the lemma. □

Proposition 4.9 (some Sobolev embeddings). *The inclusion mapping $W^{m,p} \hookrightarrow W^{m-1,p}$ is polynomial-time computable.*

Proof. Let φ be a $\xi_{m,p}$ -name of a function $f \in W^{m,p}$. Compute the value of a $\xi_{m-1,p}$ -name of f on a string \mathbf{a} as follows: To get an L^p -modulus of $f^{(m-1)}$ from the L^p -modulus of $f^{(m)}$ use the previous Lemma: By Theorem 4.1 the function $\mu(n) := |\varphi|(n+1)$ is a modulus of continuity of $f^{(m)}$. Use the mean value theorem for integrals like in the proof of Theorem 4.6 to produce a family of points and a constant C that fulfill the assumptions of Lemma 4.8. The lemma provides an explicit bound for the values of $f^{(m-1)}$. Combine this with the modulus of continuity like at the end the proof of Theorem 4.6 to get an integer bound Q on $\text{lb}(\|f^{(m-1)}\|_\infty)$. By Lemma 4.5

$$n \mapsto \max \{ \mu(n+1), n + Q + 1 \}$$

is an L^p -modulus of $f^{(m-1)}$. This function can be computed in polynomial-time and the padded return values of φ are valid return values. □

The algorithm specified in this proof accesses the oracle about 2^m times. This does not lead to exponential time consumption as m is fixed, however it might lead to large constants in the polynomials for the running time.

The following Theorems can be proven by induction, where Theorems 4.4 and 4.6 are the base cases and the previous proposition is the induction step.

Theorem 4.10. *The inclusion $W^{m,p} \hookrightarrow L^p$ is polynomial-time computable.*

Theorem 4.11. *The inclusion $W^{m,p} \hookrightarrow \mathcal{C}([0, 1])$ is polynomial-time computable.*

Finally consider the differentiation operator:

Corollary 4.12. *The k -wise differentiation operator*

$$\frac{d^k}{dx^k} : W^{m,p} \rightarrow W^{m-k,p}, \quad f \mapsto f^{(k)}$$

is polynomial-time computable for all $k \leq m$.

Proof. By the previous theorem obtain approximations to the values of f on dyadic elements. By means of

$$f(x) - f(y) = \int_y^x f' d\lambda$$

convert these into approximations of the integrals over f' . Iterate this process k -times to obtain approximations to the integrals over $f^{(k)}$. □

5. MOTIVATING THE USE OF THE L^p -MODULUS

This last chapter provides evidence that the L^p -modulus is far of from an arbitrary choice as the length parameter for a representation of L^p . The origin of the notion of an L^p -modulus as replacement for the modulus of continuity for L^p -spaces is a classification theorem of the compact subsets of L^p -spaces. Before considering the L^p case recall the following well known theorem from analysis:

Theorem 5.1 (Arzelà-Ascoli). *A subset of $\mathcal{C}([0, 1])$ is relatively compact if and only if it is bounded and equicontinuous.*

Equicontinuity of a subset of $\mathcal{C}([0, 1])$ is equivalent to the existence of a common modulus of continuity of all of the elements. Thus, this theorem provides a direct connection between compactness of a set of functions and their moduli of continuity.

A similar theorem is known for L^p -spaces, where equicontinuity is replaced by the existence of a common L^p -modulus.

Theorem 5.2 (Fréchet-Kolmogorov). *Let $1 \leq p < \infty$. A subset F of $L^p([0, 1])$ is relatively compact if and only if it is bounded and there is a function $\mu : \omega \rightarrow \omega$ that is an L^p -modulus of all of the functions from F .*

These statements are only qualitative. Quantitative refinements can be related optimality results for second-order representations of these spaces. For these refinements a notion of ‘size’ for compact sets is needed.

5.1. Metric entropies and spanning bounds. It is well known that in a complete metric space a subset is relatively compact if and only if it is totally bounded. The following notion is a straight forward quantification of total boundedness and can be used to measure the ‘size’ of compact subsets of metric spaces. It was first considered in [KT59], where many of the names we use originate. A comprehensive overview can be found in [Lor66]. These notions have been applied to computable analysis before [Wei03] and were also used in [Koh05]. For the following let M be a metric space and d its metric.

Definition 5.3. A function $\nu : \omega \rightarrow \omega$ is called **modulus of total boundedness** of a subset K of M , if for any $n \in \omega$ there are $2^{\nu(n)}$ balls of radius 2^{-n} that cover K . The smallest modulus of total boundedness is called the **metric entropy** or **size** of the set and denoted by $|K|$.

Thus

$$|K|(n) = \min\{k \in \omega \mid K \text{ can be covered by } 2^k \text{ balls of radius } 2^{-n}\}.$$

Like the smallest modulus of continuity of a function, the metric entropy of a set is usually hard to get hold of. Moduli of total boundedness as upper bounds can more often be chosen computable. In a complete metric space a closed set permits a metric entropy if and only if it is compact.

A modulus of total boundedness is an upper bound on the size of a compact set. For providing lower bounds, another notion is more convenient.

Definition 5.4. A function $\eta : \omega \rightarrow \omega$ is called a **spanning bound** of a subset $K \subseteq M$, if for any n there exist elements $x_1, \dots, x_{2^{\eta(n)}}$ such that

$$i \neq j \quad \Rightarrow \quad d(x_i, x_j) \geq 2^{-n+1}.$$

If there is a biggest spanning bound, it is called the **capacity** of K and denoted by $\text{cap}(K)$.

The condition on the x_i in the definition can be read as ‘the 2^{-n} -balls around the points are disjoint’. There is a biggest spanning bound if and only if the set K is relatively compact. The following is straight forward to verify:

Proposition 5.5. *Let $K \subseteq M$ be a subset, ν be a metric entropy of K and η a spanning bound. Then $\eta(n) \leq \nu(n)$, and furthermore $|K|(n) \leq \text{cap}(K)(n+1) + 1$.*

A proof can for instance be found in [KT59, Theorem IV], however, the result presented there is a little sharper since rounding to powers of two is avoided.

This implies comparability of the size and the capacity in the sense that

$$\text{cap}(K)(n) \leq |K|(n) \leq \text{cap}(K)(n+1) + 1.$$

This paper uses spanning bounds to provide lower bounds to the size of sets. The capacity is not mentioned again.

5.2. Connecting metric entropy and complexity. For sake of completeness this chapter states the result connecting the metric entropy to computational complexity. Some of the results presented in this chapter are in a slightly different form contained in [KSZ16]. There are several examples of similar observations [BK02, FGH14].

As already mentioned, neither the norm nor the metric of $\mathcal{C}([0, 1])$ or L^p are polynomial-time computable with respect to the representations considered in this paper. It is, however, possible to give resource bounds: The operations are polynomial space computable. Since space restricted computation in presence of oracles is tricky (compare [Bus88, KO14]), this paper considers the exponential-time computations instead.

Definition 5.6. An oracle Turing machine $M^?$ is said to run in **exponential-time** if there are constants $A, B, C \in \omega$ such that the computation $M^\psi(\mathbf{a})$ of $M^?$ with oracle φ on input \mathbf{a} terminates after at most $2^{A \cdot |\varphi|(|\mathbf{a}|+B)+C|\mathbf{a}|}$ steps.

Call a function between represented spaces **exponential-time computable** if it has an exponential-time computable realizer (compare Section 1.2). This notion of exponential-time computability is highly adapted for the concrete application at hand and quite restrictive: While an exponential running time is only bounded by a second-order polynomial if it is constant, not all second-order polynomials can be bounded by exponentials. As soon as there are iterations of the first order argument, no such bound exists.

Furthermore, the following notions are needed:

Definition 5.7. A function $l : \omega \rightarrow \omega$ is called a **length** of a second-order representation if each element of the represented space has a name of length at most l .

It can be proven that an open representation of a compact space always has a length. The standard representations of the real numbers, the continuous functions, L^p -spaces and integrable functions discussed in this paper so far do not have a length. But since all of them are open mappings, their range restrictions to compact subsets do have a length. The proof of Theorem 2.8 makes use of the finite length of the restriction of the standard representation of the reals to any bounded set.

Theorem 5.8. *Let M be a compact metric space of at least linear metric entropy.*

- (1) Assume that there exists a representation of M of length l such that the metric is computable in exponential-time. Then there exist some $A, B \in \omega$ such that

$$|M|(n) \leq 2^{Al(n)+B}.$$

- (2) Let $l : \omega \rightarrow \omega$ be monotone such that $|M|(n) \leq 2^{l(n)}$. Then there exists a representation ξ of M that has length l such that the metric is computable in exponential-time.

Sketch of the proof. To prove (1) use the folklore fact that a running time restricts the access a machine has to the oracles (compare for instance [BK02]). Make this a quantitative statement by bounding the number of possible communication sequences. From this obtain a bound on the number of pairs $\langle \varphi, \psi \rangle$ that can be distinguished in a computation of the norm up to precision 2^{-n} . This leads to a bound on the size of the set.

To prove (2) let l be such that $|M|(n) \leq 2^{l(n)}$. Then there exists a sequence such that the balls around the first $2^{2^{l(n)} + \lceil \text{lb}(n+1) \rceil}$ elements cover M . Consider the Cauchy representation of M with respect to this sequence (according to Definition 1.2). This representation has length $2^{l(n)} + \lceil \text{lb}(n+1) \rceil$. Add an oracle for the function $(n, m) \mapsto d(x_n, x_m)$ to each name, truncate the representation and flatten it to obtain one of constant length. Now pad the length of each name to be $l(n)$. This is the desired representation. \square

Example 5.9 (Lipschitz functions). Consider as K the set of Lipschitz functions that vanish in 0 and have a common Lipschitz constant 2^L . This set is compact by the Arzelà-Ascoli Theorem from Theorem 5.1. It is easy to verify that $|K|(n) = \lceil \text{lb}(3)2^{n+L} \rceil$ (see for instance [Wei03]). Let ξ be an open representation of this set such that the metric is computable in exponential time. As an open representation of a compact set, this representation has a length l . From the first item of the previous theorem it follows that there exists constants $A, B \in \omega$ such that

$$|K|(n) = \lceil \text{lb}(3)2^{n+L} \rceil \leq 2^{Al(n)+B}$$

and therefore $l(n) \geq (n + L - B)/A$, i.e. ξ has at least linear length.

The second item of the previous theorem specifies a representation of linear length of K that renders the metric exponential-time computable. In this special case we already knew that such a representation exists: The range restriction of the standard representation ξ_C of continuous functions to K has length $n \mapsto n + C$, where C depends on the Lipschitz constant, and it renders the metric exponential-time computable.

The rest of this chapter aims to generalize the above example by replacing the set of Lipschitz functions with more general compact sets. The compact subsets are fully classified by the Arzelà-Ascoli and Fréchet-Kolmogorov Theorems. By specifying the size of these compact sets it is possible to verify that ξ_C and ξ_p have the minimal length for any representation that renders the metric exponential-time computable on these. This justifies the modulus of continuity and the L^p -modulus as the right parameters for these function spaces.

5.3. Arzelà-Ascoli and Fréchet-Kolmogorov. The quantitative refinements of both the Arzelà-Ascoli and Fréchet-Kolmogorov Theorems have been investigated before in different contexts: There has been extensive work on these topics in approximation theory (for instance [KT59] or [Tim94]). These results can, however, not straight forwardly be transferred to the context of this paper. In approximation theory the notion of moduli considered differs

by convention. As a result, the theorems usually talk about the inverse modulus instead of the modulus itself. Furthermore, the results are often only stated or valid for small moduli. A very popular class is for instance the class corresponding to Hölder continuous functions for the continuous functions.

There have been some attempts to apply the results to computable metric spaces: Most prominently [Wei03]. However, the results seem rather restricted.

Definition 5.10. Define the family of **Arzelà-Ascoli-sets** $(K_{l,C}^\infty)_{l \in \omega^\omega, C \in \omega}$ in $\mathcal{C}([0, 1])$ by

$$K_{l,C}^\infty := \{f \in \mathcal{C}([0, 1]) \mid f \text{ has } l \text{ as modulus and } \|f\|_\infty \leq 2^C\}.$$

The classical Arzelà-Ascoli Theorem 5.1 states that a set of functions is relatively compact if and only if it is contained in some $K_{l,C}^\infty$. The quantitative refinement of that statement can be found in [Tim94]. The proof given there is very similar to the one here. A restricted version is also proven in [Wei03].

Theorem 5.11 (Arzelà-Ascoli). *A set $K \subseteq \mathcal{C}([0, 1])$ is relatively compact if and only if it is contained in $K_{l,C}^\infty$ for some l, C . Furthermore:*

$$2^{l(\max\{n-2,0\})+\min\{n-2,0\}} + n + C \leq |K_{l,C}^\infty| (n) \leq 2^{l(n)+1} + n + C + 2.$$

Proof. The first assertion follows from the classical Arzelà-Ascoli Theorem 5.1 and the proof is not repeated here. To provide the upper bound on the size of $K_{l,C}^\infty$ fix some $n \in \omega$. A collection of balls of size 2^{-n} that cover $K_{l,C}^\infty$ can be constructed as follows: Consider the index set

$$I := \{-2^{n+C}, \dots, 2^{n+C}\} \times \{0, 1, -1\}^{2^{l(n)}}.$$

For $\sigma = (\sigma_0, \sigma_1, \dots, \sigma_{2^{l(n)}}) \in I$ define a piecewise linear function $f_\sigma : [0, 1] \rightarrow \mathbb{R}$ by

$$f_\sigma(x) = \begin{cases} \sigma_0 2^{-n} & \text{if } x = 0 \\ 2^{-n} \left(\sum_{i=0}^{j-1} \sigma_i + \sigma_j (2^{l(n)} x - j) \right) & \text{if } x \in \left(\frac{j-1}{2^{l(n)}}, \frac{j}{2^{l(n)}} \right] \text{ for some } j \in \mathbb{N}. \end{cases}$$

The 2^{-n} -balls centered at f_σ cover $K_{l,C}^\infty$ and

$$\#I = (2^{n+C+1} + 1) 3^{2^{l(n)}} \leq 2^{2^{l(n)+1} + n + C + 2}.$$

Since n was arbitrary, the right hand side is an upper bound on the size of $K_{l,C}^\infty$.

To establish the lower bound replace f_σ with the function g_σ that may or may not have a bump of size 2^{-n-1} in the i -th interval of size $2^{-l(n)}$. The extra condition that a modulus of continuity has to be strictly increasing when non-zero implies that whenever the value of l on n allows the function to vary by 2^{-n} over an interval of length $2^{-l(n)}$ the subsequent values of l will not disallow this behavior. Thus $g_\sigma \in K_{l,C}^\infty$. For any two different elements σ and σ' of I it holds that

$\|g_\sigma - g_{\sigma'}\|_\infty \geq 2^{-n-1}$. Thus, $n + 2 \mapsto 2^{l(n)} + n + C + 2$ is a spanning bound in the sense of Definition 5.4. Since any spanning bound of a set has to be smaller than its size by Proposition 5.5, the lower bound on $|K_{l,C}^\infty|$ follows. \square

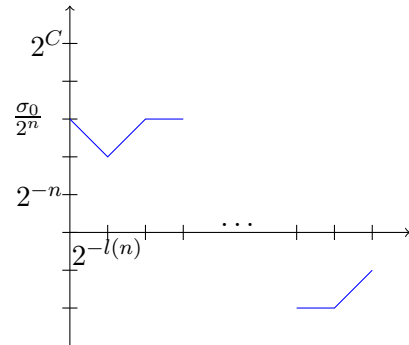


FIGURE 4. The function f_σ for $\sigma = (\sigma_0, -1, 1, 0, \dots, 0, 1)$.

For L^p -spaces replace the sets $K_{l,C}^\infty$ by the following sets:

Definition 5.12. Define the Family of **Fréchet-Kolmogorov-sets** $(K_l^p)_{l \in \omega}$ in $L^p([0, 1])$ by

$$K_l^p := \{f \in L^p([0, 1]) \mid f \text{ has } l \text{ as } L^p\text{-modulus}\}.$$

In this case there is no need to include an upper bound to the norm, since this bound can be extracted from an L^p -modulus by Lemma 3.4.

Theorem 5.13 (Fréchet-Kolmogorov). *A set $K \subseteq L^p([0, 1])$ is relatively compact if and only if it is contained in K_l^p for some l . There exists a second-order polynomial P such that*

$$|K_l^p|(n) \leq 2^{P(l,n)}.$$

Whenever $n \geq 3$ and $l(n-3) \geq 9$ it holds that $2^{l(n-3)-4} - 1 \leq |K_l^p|(n)$.

Proof of the upper bound. Using the lemmas from Section 3.3: By Lemma 3.11 any function $f \in K_l^p$ has a continuous function $f_{l(n+1)}$ in its 2^{-n-1} neighborhood. Lemma 3.10 guarantees $f_{l(n+1)} \in K_{l'}^\infty$ for $l'(m) := m + 1 + \lceil d/p \rceil l(n+1)$. The Arzelà-Ascoli Theorem 5.11 proves

$$|K_{l'}^\infty|(n) \leq 2^{l'(n)+1+n+\lceil \text{lb}(\|f_{l(n+1)}\|_\infty) \rceil + 2} \leq 2^{l(n+2+\lceil \frac{d}{p} \rceil l(n+1))+1+n+l} \left(1 + \left\lceil \frac{d}{p} \right\rceil l(1)\right) + 2.$$

The balls of radius 2^{-n-1} in supremum norm are included in the balls in L^p -norm. Therefore, the 2^{-n} balls in L^p around the same centers cover K_l^p . \square

To find a lower bound, we use the technique Lorentz used in [Lor66] for the prove of his Lemma 8. Namely we use the following lemma from coding theory:

Lemma 5.14. *For any natural number $N \geq 500$ and $M < \frac{N}{3}$ there exists a set I of binary strings of length N that differ pairwise in at least M places and such that*

$$\#I = \left\lfloor 2^{\frac{N}{16}-1} \right\rfloor.$$

Proof. Prove the stronger statement that there is a set I of strings of length $N \geq 500$ that differ in at least $M = \frac{N}{3}$ bits whenever

$$\#I \leq \left\lfloor 2^{\frac{N}{16}} \left(\frac{e}{\pi} 2^{-\frac{3}{2}} + 1 \right)^{-1} \right\rfloor.$$

Proceed by induction over the size $\#I$ of the set I . For $\#I = 2$ choose the constant zero string and the constant one string. Now assume that I is a set of strings that differ pairwise in at least M elements and that has strictly less elements than the number specified above. Use Stirling's Formula to estimate the number of strings that differ in less than M digits from one of the elements of I :

$$\begin{aligned} \#I \sum_{i=0}^M \binom{N}{i} &\stackrel{M \leq \frac{N}{2}}{\leq} \#I \left(M \binom{N}{M} + 1 \right) \stackrel{\text{Stirling}}{\leq} \#I \left(\frac{Me}{2\pi} \frac{N^{N+\frac{1}{2}}}{M^{M+\frac{1}{2}}(N-M)^{N-M+\frac{1}{2}}} + 1 \right) \\ &\stackrel{M = \frac{N}{3}}{=} \#I \left(\frac{\sqrt{N}e}{6\pi} 3^{N+1} 2^{-\frac{2}{3}N-\frac{1}{2}} + 1 \right) \stackrel{N \geq 500}{\leq} \#I \left(\frac{e}{2^{\frac{3}{2}}\pi} + 1 \right) 2^{-\frac{N}{16}} 2^N. \end{aligned}$$

By induction hypothesis the right hand side is strictly smaller than 2^N . Since the left hand side is an integer it is at most $2^N - 1$. Thus, at least one of the 2^N strings of length N does not lie in the union of these sets and can be added to I to increase its size by one. \square

Remark 5.15. From coding theory it is known that these bounds are not optimal. In particular the assumption $N \geq 500$ can be removed. See for instance [Sud01].

Proof of the lower bound in Theorem 5.13. Fix some $n \in \omega$. The assumption $l(n-3) \geq 9$ guarantees that Lemma 5.14 can be applied with $N := 2^{l(n-3)}$ and $M := 2^{l(n-3)-2} = \frac{N}{4}$ to find a set I of strings of length $2^{l(n-3)}$ such that the elements differ pairwise in at least $2^{l(n-3)-2}$ digits and $\#I = 2^{2^{l(n-3)-4}-1}$. Consider the functions

$$f : \mathbb{R} \rightarrow [0, 1], \quad x \mapsto \max \left\{ 0, 1 - 2 \left| x - \frac{1}{2} \right| \right\}, \quad w(n) := 2^{-l(n-3)} \text{ and } h(n) := (p+1)^{\frac{1}{p}} 2^{-n+1}.$$

For each $\sigma \in I$ define a function f_σ by

$$f_\sigma := h(n) \sum_{i=1}^{2^{l(n-3)}} \sigma_i f \left(\frac{x - iw(n)}{w(n)} \right).$$

That is: Divide $[0, 1]$ into intervals of width $w(n)$ and consider the set of functions that may or may not have a hat of height $h(n)$ in each of the intervals (see Figure 5). Since at most one hat is put in each interval for each string σ and $x \in [0, 1]$ it is true that almost everywhere $f_\sigma(x) < h(n)$ and therefore $\|f_\sigma\|_p < h(n)$. For the weak derivative of f_σ it holds that $\|f'_\sigma\|_\infty \leq 2h(n)/w(n)$.

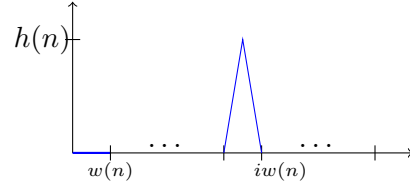


FIGURE 5. f_σ for σ with $\sigma_1 = 0$ and $\sigma_i = 1$.

To obtain the spanning bound prove that the f_σ are elements of K_l^p of pairwise distance more than 2^{-n} : To show that these functions are elements of K_l^p , claim that the smallest valid modulus function μ such that $\mu(n-3) = l(n-3)$ is an L^p -modulus of f_σ . Indeed: $h(n) \leq 2^{-n+2}$ since $x \mapsto (x+1)^{1/x}$ is decaying on the positive real line and takes value 2 in 1. Therefore, for an arbitrary shift y and any σ

$$\|f_\sigma - \tau_y f_\sigma\|_p \leq 2\|f_\sigma\|_p < 2h(n) \leq 2^{-n+3}.$$

Thus, for any $m < n-3$ zero is a valid value of an L^p -modulus of f_σ . To see the statement for $m \geq n-3$ use Lemma 4.2, which says that it suffices to estimate the L^p -norm of the weak derivative of f_σ :

$$\|f'_\sigma\|_p \leq \frac{2h(n)}{w(n)} = 2^{l(n-3)-n+2} < 2^{l(n-3)-n+3}.$$

Thus, $m \mapsto m + l(n-3) - n + 3$ is an L^p -modulus of f_σ .

Finally estimate the pairwise distance: The set I was chosen such that whenever $\sigma \neq \sigma'$, then σ and σ' differ in at least $M = 2^{l(n-3)-2}$ places. Thus

$$\|f_\sigma - f_{\sigma'}\|_p \geq 2h(n) \left(\frac{Mw(n)}{p+1} \right)^{\frac{1}{p}} = 2^{-n+2-\frac{2}{p}} \geq 2^{-n}.$$

This proves the assertion. □

5.4. Smoother approximations. The upper bound specified in Theorem 5.13 contains an iteration of the L^p -modulus while the lower bound does not. This leads to a huge gap between the upper and the lower bound for fast growing L^p -moduli. In this chapter the upper bound is improved by introducing another representation that uses a sequence of approximating functions with improved regularity.

Recall, that the convolution $h \star f$ integrable functions h, f is defined by

$$h \star f := \int_{\mathbb{R}^d} h(x-y)f(y)dy = \int_{\mathbb{R}^d} h(y)f(x-y)dy.$$

Recall the following well known result from the theory of convolution:

Proposition 5.16 (derivatives). *Whenever f is integrable and g is weakly differentiable, then $g \star f$ is weakly differentiable and*

$$\frac{\partial(g \star f)}{\partial x_i} = \frac{\partial g}{\partial x_i} \star f.$$

Moreover, recall the following formula for the L^p -norms of convoluted functions:

Proposition 5.17 (norm estimate). *Whenever $g \in L^p$ where $1 \leq p \leq \infty$ and $f \in L^1$, then $g \star f \in L^p$ and*

$$\|g \star f\|_p \leq \|g\|_p \|f\|_1.$$

The sequence of continuous approximations f_n from Definition 3.9 can be understood to arise from the function f by convoluting with the function sequence

$$g_n := 2^n \chi_{[-2^{-n-1}, 2^{-n-1}]}. \quad \text{I.e.} \quad f_n = g_n \star f.$$

From this point of view Definition 3.5 requires a ξ_p -name of a function f to fulfill

$$|\llbracket \varphi(\mathbf{a}, 1^n) \rrbracket - f_k(z)| < 2^{-n}$$

whenever \mathbf{a} is an encoding of $[z]_k$. Furthermore, it is possible to translate between an encoding of $[z]_k$ and an the pair $\langle z, 1^k \rangle$ in polynomial time.

Let $|\cdot|_\infty$ denote the supremum norm on \mathbb{R}^d . Replacing the sequence g_n with the following mollifier sequence lifts the approximations from being continuous to being weakly differentiable (thus the D):

Definition 5.18. Define the **mollifier sequence** $(g_m^D)_{m \in \mathbb{N}}$ of functions $g_m^D : \mathbb{R}^d \rightarrow \mathbb{R}$ by

$$g_0^D(x) := \max\{1 - |x|_\infty, 0\}$$

and

$$g_m^D(x) := d2^{d(m-1)}g_0^D(2^m x).$$

The function g_1^D is illustrated in Figure 6. One easily verifies that the support of g_m^D is the ball $[-2^{-m}, 2^{-m}]^d$ of radius 2^{-m} around zero in supremum norm, that for any m

$$\int_{\mathbb{R}^d} g_m^D d\lambda = \int_{[-2^{-m}, 2^{-m}]^d} g_m^D d\lambda = 1,$$

and that g_0^D is weakly (partially) differentiable and the weak derivatives are essentially

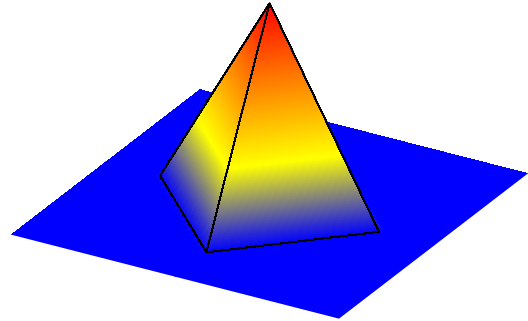


FIGURE 6. The function g_1^D for $d = 2$.

bounded with

$$\left\| \frac{dg_0^D}{dx_i} \right\|_{\infty} = 1.$$

Furthermore, if ∇g_0^D denotes the vector of partial derivatives and $\|\cdot\|_{\infty}$ the maximum of the supremum norms of the components of a vector, then for the gradient of g_m^D

$$\|\nabla g_m^D\|_{\infty} = d2^{d(m-1)+m}.$$

Definition 5.19. For a function $f \in L^p$ define the sequence $(f_m^D)_{m \in \mathbb{N}}$ of **differentiable approximations** $f_m^D : \mathbb{R}^d \rightarrow \mathbb{R}$ by

$$f_m^D := g_m \star \tilde{f} = \int_{\mathbb{R}^d} g_m^D(y) \tilde{f}(x-y) dy.$$

The following representation replaces the information about the continuous approximations provided by ξ_p by less information about smoother approximations.

Definition 5.20. Define a second-order representation ξ_p^D of $L^p(\Omega)$: A length-monotone string function φ is a ξ_p^D -name of a function $f \in L^p(\Omega)$ if and for all $n, k \in \mathbb{N}$ and each string \mathbf{a} encoding a dyadic number

$$\left| \left[\varphi(\langle \mathbf{a}, 1^k, 1^n \rangle) \right] - f_k^D(\llbracket \mathbf{a} \rrbracket) \right| < 2^{-n},$$

and $|\varphi|$ is an L^p -modulus of f .

It is possible to specify a bound on the supremum norm of the gradient of the ‘differentiable approximations’ f_n^D :

Lemma 5.21 (gradient estimate). *For any $f \in L^p$ the functions f_m^D are weakly differentiable with*

$$\|\nabla f_m^D\|_{\infty} \leq d2^{d(m-1)+m} \|f\|_1$$

Proof. From the properties of the mollifiers g_m following Definition 5.18 and the formulas for the convolution from Propositions 5.16 and 5.17 get

$$\left\| \frac{\partial f_m^D}{\partial x_i} \right\|_{\infty} = \left\| \frac{\partial g_m^D}{\partial x_i} \star f \right\|_{\infty} \leq \left\| \frac{\partial g_m^D}{\partial x_i} \right\|_{\infty} \|f\|_1 = d2^{d(m-1)+m} \|f\|_1.$$

Taking the supremum over i proves the assertion. \square

How good an approximation f_m^D is to f can be read from an L^p -modulus. The smoothness, however, comes at a price that depends on the dimension (compare to Lemma 3.11).

Lemma 5.22 (approximation). *Let μ be an L^p -modulus of f , then*

$$\|\tilde{f} - f_{\mu(n)}^D\|_p < 2^{-n+d}.$$

Proof. Like in the proof of 3.10 use $\int_{\mathbb{R}^d} g_m^D d\lambda = 1$ to see that

$$\|\tilde{f} - f_m^D\|_p^p \leq \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} |\tilde{f}(x) - \tilde{f}(x-y)| g_m^D(y) dy \right)^p dx.$$

Next note that from $g_0^D \leq 1$ and $p \geq 1$ it follows that

$$\begin{aligned} \left(g_m^D(y) \lambda([-2^{-m}, 2^{-m}]^d)^{1-\frac{1}{p}} \right)^p &= 2^{dm(p-1)} d^p 2^{dp(m-1)} g_0^D(2^m y)^p \\ &\leq d^p 2^{d(m-p)} g_0^D(2^m y) = 2^{d(p-1)} g_m^D(y). \end{aligned} \tag{5.1}$$

Using the version of Hölder's inequality from Corollary 3.2 conclude

$$\begin{aligned} \left(\int_{\mathbb{R}^d} |\tilde{f} - \tau_y \tilde{f}| g_m^D d\lambda \right)^p &\leq \left(\lambda([-2^{-m}, 2^{-m}]^d)^{\frac{1}{q}} \left\| (\tilde{f} - \tau_y \tilde{f}) g_m^D \right\|_p \right)^p \\ &\stackrel{(5.1)}{\leq} 2^{d(p-1)} \int_{[-2^{-m}, 2^{-m}]^d} |\tilde{f} - \tau_y \tilde{f}|^p g_m^D d\lambda \end{aligned}$$

Therefore applying Fubini leads to

$$\|f - f_m^D\|_p^p = 2^{d(p-1)} \int_{[-2^{-m}, 2^{-m}]^d} \|\tilde{f} - \tau_y \tilde{f}\|_p^p g_m^D(y) dy.$$

From this the assertion follows by setting $m := \mu(n)$ and using that μ is an L^p -modulus. \square

Now it is possible to prove the main result of this section:

Theorem 5.23. *With respect to ξ_p^D the norm of $L^p(\Omega)$ is exponential-time computable relative to p .*

Proof. Denote the constant one vector $(1, \dots, 1)$ by e . Let A be such that the distance of Ω to the complement of the box $K := [-2^A e, 2^A e]$ of radius 2^A around zero is bigger than one. Specify an oracle Turing machine that computes the norm as follows: Given a ξ_p^D -name φ of $f \in L^p(\Omega)$ as oracle and a precision requirement 1^n the machine first computes

$$N := \mu(n + d + 1), \quad M := d(N - 1) + N + A + \mu(0) + \lceil \text{lb}(d) \rceil$$

and $k := n + (d + 1)(\mu(N) + 2) + \mu(0)$. Let \mathbb{D}_M be the set of $z \in \mathbb{D}^d \cap K$ of the form $m2^{-M}$. For each $z \in \mathbb{D}_M$ the machine chooses some binary encoding $\mathbf{a}_z \in \Sigma^*$ (for example the unique canceled encoding such that the encoding of the enumerator has no leading zeros) and queries φ for

$$d_{z,n} := \left\lceil \left\lceil \varphi(\langle \mathbf{a}_z, 1^N, 1^k \rangle) \right\rceil \right\rceil.$$

After each query, it adds the p -th power of the result to the sum of the previous queries. In the end, the p -th root of the number is returned.

To see that this machine returns a correct approximation, note that it returns the L^p -norm of the function

$$F_n := \sum_{z \in \mathbb{D}_M} d_{z,n} \chi_{[z]_M}.$$

To see that F_n is a good approximation to f in the L^p norm, write

$$\|f - F_n\|_p \leq \|\tilde{f} - f_N^D\|_p + \|f_N^D - F_n\|_p \tag{5.2}$$

Each of the summands of the right hand side is smaller than 2^{-n-1} : The first term is taken care of by the choice of N together with the approximation property of the sequence f_N^D from Lemma 5.22. For the second summand, note that each $x \in \Omega$ is 2^{-M} close to some $z \in K$ and that for these z

$$|f_N^D(z) - F_n(z)| = |f_N^D(z) - d_{z,n}| < 2^{dN-k}.$$

By choice of M , the gradient estimate of f_N^D from Lemma 5.21 together with the bound on the L^p -norm from Lemma 3.4 and since F_n is piecewise constant it follows that

$$\|f_N^D - F_n\|_p \leq \lambda(\Omega)^{\frac{1}{p}} \|f_N^D - F_n\|_\infty < 2^{-n-1}.$$

Thus, from $|\|f\|_p - \|F_n\|_p| \leq \|f - F_n\|_p$ it follows that the return value is indeed a valid 2^{-n} approximation to the norm of f .

The assertion now follows from the fact that the machine carries out a loop that takes time linear in $|\varphi|(n + C)$ (provided that approximations to p are given) an exponential number of times. \square

Using the content of Section 5.2, the next result can be regarded as a corollary of the above. However, since it is also possible to give an independent proof without relying on that section we list it as a theorem. Recall the Fréchet-Kolmogorov sets from Definition 5.12.

Theorem 5.24. *There are constants $A, B, C \in \mathbb{N}$ such that*

$$|K_l^p|(n) \leq 2^{Al(n+d+1)+l(0)+B} + n + C.$$

Proof. We give two sketches: On the one hand it is possible to follow the proof of Theorem 5.13 and replace the lemmas from Section 3.3 by the lemmas of this section. On the other hand one can add an oracle for p to each L^p -name of a function and in this way construct a representation such that applying the first item of Theorem 5.8 proves the claim. \square

While the first sketch relies on the quantitative version of the Arzelà-Ascoli Theorem 5.11 and is therefore bound to the unit interval, the second sketch is not bound to such a restrictive setting: It remains correct for more general domains and higher dimensions.

CONCLUSION

A comprehensive summary of the content of the paper can be found in the introduction. Thus, this conclusion concentrates on high level comments and mentioning additional results or improvements of results and points out some loose ends. The remarks follow the general outline of the paper.

Like for the standard representation of continuous functions, the minimality property of the singular representation from Theorem 2.8 also applies to arbitrary range-restrictions. The discontinuity of the singular representation can be strengthened: Theorem 2.10 remains true if the norm topology is replaced by the weak topology. The singular representation seems to have a straight forward generalization to locally integrable functions. On locally integrable functions the usual norms do not make sense anymore and different topologies are considered. It would be interesting to find out whether the topology of the singular representation coincides with one previously considered by analysts.

Some might argue that the choice of an integration operator is too restrictive. At least in higher dimensions the restriction of the integral operator to only integrate over boxes seems very severe. This restrictive setting, however, seems unavoidable. Polynomial-time computability of many possible extensions is ruled out by hardness results proven in [Ko91]. The same holds for the approach to consider a function to be a functional on the continuous functions.

For the definition of the L^p -modulus in Definition 3.3 the function was extended to the whole space by zero before integrating. This is a convention: One could instead have integrated over the intersection of the domains of the function and the shifted function. However, in this case the property that exactly the L^p -functions allow a modulus is lost. If one uses this modification to define a representation technical difficulties are encountered when a proof of equivalence to the Cauchy representation is attempted. For complexity considerations it seems impossible to progress on this path without restricting the domains.

The first part of the proof of Theorem 3.8 can be seen to show the stronger statement of computable openness of the representation as introduced in [KP14b]. Furthermore, if exponential-time computability is introduced to allow a full second-order polynomial in the exponent (in contrast to Definition 5.6, where no function argument iteration is allowed in the exponent), the second part of the proof on page 19 shows exponential-time translatability to the Cauchy representation. From this a weaker form of exponential-time computability of the norm than that from Theorem 5.23 follows.

Recall from the introduction, that in practice maximization is considered difficult while integration is considered feasible. This is reflected in the second-order representations introduced in this paper: Neither the representation ξ_s nor the representations ξ_p allow in a straight-forward way to maximize a continuous or smooth function. Indeed², modifications of the smooth functions considered by Ko and Friedman in [KF82] show that the maximization operator will not preserve polynomial-time computability with respect to these representations unless $\mathcal{P} = \mathcal{NP}$.

While ordinary differential equations are a field of application for Sobolev spaces (compare for instance [Bre11, chapter 8.4]), partial differential equations are by far the most important application. However, many of the arguments from Section 4 cannot be translated in the most straight forward way to higher dimensions. For instance: Existence of a weak derivative does not imply continuity in higher dimensions. For the inclusions to make sense in higher dimensions further assumptions are necessary. Even if these assumptions are met, Theorem 4.1 cannot be straight forwardly replaced. Indeed, the argument from Proposition 3.6 cannot carry over to higher dimensions in the straight forward way, as it would only mention derivatives of first order and it is known that existence of the first weak partial derivatives does not imply continuity.

Partial differential equations have received increased interest in computable analysis in the last years. Compare for instance [Zho99, WZ07, WZ06, BY06]. There is a plethora of results for solving partial differential equations from numerical analysis. It seems reasonable to assume that formulating these algorithms in a rigorous framework and lifting results from the references above to a complexity theoretical level should be closely connected tasks. For instance many of the results from [BY06] are interesting in one dimension already.

All results from this paper that mention exponential-time computability can be improved to use polynomial-space computability instead. The model of space bounded computation in presence of oracles, however, is not completely straight forward: The right model of oracle access is a stack of finite depth (compare [KO14, Bus88]).

The representations ξ_p and the representation ξ_p^D from the last chapter can be combined to a representation featuring both polynomial-time computability of integrals and exponential-time computability of the norm. However, it does not seem reasonable to add the information provided by ξ_p^D : It increases the amount of information that has to be provided to specify a function for the sake of improving the runtime of an exponential-time computable (so not feasible) operation on input of big L^p -modulus. The first sketch of a proof of Theorem 5.24 suggests that convoluting with even smoother functions does not lead to further improvements in performance: The dominant term in the running time is independent of smoothness: The a supremum norm estimate obtained from Proposition 5.17.

Classification theorems for the compact subsets of function spaces are of interest to analysts and approximation theorists for reasons independent of those sketched in this paper.

²thanks to Akitoshi Kawamura for pointing this out.

They have been investigated for a long time and are well developed. The link between quantitative versions of these results and optimality results for running times provides a rich resource for finding interesting representations. Such results are in particular known for Banach space valued functions.

REFERENCES

- [AB09] Sanjeev Arora and Boaz Barak. *Computational complexity*. Cambridge University Press, Cambridge, 2009. A modern approach. doi:10.1017/CB09780511804090.
- [BB85] Errett Bishop and Douglas Bridges. *Constructive analysis*, volume 279 of *Grundlehren der Mathematischen Wissenschaften*. Springer-Verlag, Berlin, 1985. doi:10.1007/978-3-642-61667-9.
- [BH02] Vasco Brattka and Peter Hertling. Topological properties of real number representations. *Theoret. Comput. Sci.*, 284(2):241–257, 2002. Computability and complexity in analysis (Castle Dagstuhl, 1999). doi:10.1016/S0304-3975(01)00066-4.
- [BK02] Samuel R. Buss and Bruce M. Kapron. Resource-bounded continuity and sequentiality for type-two functionals. *ACM Trans. Comput. Log.*, 3(3):402–417, 2002. Special issue on logic in computer science (Santa Barbara, CA, 2000). doi:10.1145/507382.507387.
- [Bre11] Haim Brezis. *Functional analysis, Sobolev spaces and partial differential equations*. Universitext. Springer, New York, 2011. doi:10.1007/978-0-387-70914-7.
- [Bus88] Jonathan F. Buss. Relativized alternation and space-bounded computation. *J. Comput. System Sci.*, 36(3):351–378, 1988. Structure in Complexity Theory Conference (Berkeley, CA, 1986). doi:10.1016/0022-0000(88)90034-7.
- [BY06] Vasco Brattka and Atsushi Yoshikawa. Towards computability of elliptic boundary value problems in variational formulation. *J. Complexity*, 22(6):858–880, 2006. doi:10.1016/j.jco.2006.04.007.
- [FGH14] Hugo Férée, Walid Gomaa, and Mathieu Hoyrup. Analytical properties of resource-bounded real functionals. *J. Complexity*, 30(5):647–671, 2014. doi:10.1016/j.jco.2014.02.008.
- [Fri84] Harvey Friedman. The computational complexity of maximization and integration. *Adv. in Math.*, 53(1):80–98, 1984. doi:10.1016/0001-8708(84)90019-7.
- [Grz55] A. Grzegorzcyk. Computable functionals. *Fund. Math.*, 42:168–202, 1955.
- [IRK01] Robert J. Irwin, James S. Royer, and Bruce M. Kapron. On characterizations of the basic feasible functionals. I. *J. Funct. Programming*, 11(1):117–153, 2001. Special issue on functional programming and computational complexity (Baltimore, MD, 1998). doi:10.1017/S0956796800003841.
- [KC96] B. M. Kapron and S. A. Cook. A new characterization of type-2 feasibility. *SIAM J. Comput.*, 25(1):117–132, 1996. doi:10.1137/S0097539794263452.
- [KC10] Akitoshi Kawamura and Stephen Cook. Complexity theory for operators in analysis. In *STOC’10—Proceedings of the 2010 ACM International Symposium on Theory of Computing*, pages 495–502. ACM, New York, 2010. doi:10.1145/1806689.1806758.
- [KF82] Ker-I Ko and Harvey Friedman. Computational complexity of real functions. *Theoret. Comput. Sci.*, 20(3):323–352, 1982. doi:10.1016/S0304-3975(82)80003-0.
- [Kle52] Stephen Cole Kleene. *Introduction to metamathematics*. D. Van Nostrand Co., Inc., New York, N. Y., 1952.
- [KM82] G. Kreisel and A. Macintyre. Constructive logic versus algebraization. I. In *The L. E. J. Brouwer Centenary Symposium (Noordwijkerhout, 1981)*, volume 110 of *Stud. Logic Found. Math.*, pages 217–260. North-Holland, Amsterdam, 1982. doi:10.1016/S0049-237X(09)70130-2.
- [KMRZ15] Akitoshi Kawamura, Norbert Müller, Carsten Rösnick, and Martin Ziegler. Computational benefit of smoothness: Parameterized bit-complexity of numerical operators on analytic functions and Gevrey’s hierarchy. *J. Complexity*, 31(5):689–714, 2015. doi:10.1016/j.jco.2015.05.001.
- [Ko91] Ker-I Ko. *Complexity theory of real functions*. Progress in Theoretical Computer Science. Birkhäuser Boston, Inc., Boston, MA, 1991. doi:10.1007/978-1-4684-6802-1.
- [KO14] Akitoshi Kawamura and Hiroyuki Ota. Small complexity classes for computable analysis. In *Mathematical foundations of computer science 2014. Part II*, volume 8635 of *Lecture Notes in Comput. Sci.*, pages 432–444. Springer, Heidelberg, 2014. doi:10.1007/978-3-662-44465-8_37.

- [Koh96] Ulrich Kohlenbach. Mathematically strong subsystems of analysis with low rate of growth of provably recursive functionals. *Arch. Math. Logic*, 36(1):31–71, 1996. doi:10.1007/s001530050055.
- [Koh05] Ulrich Kohlenbach. Some computational aspects of metric fixed-point theory. *Nonlinear Anal.*, 61(5):823–837, 2005. doi:10.1016/j.na.2005.01.075.
- [KP14a] Akitoshi Kawamura and Arno Pauly. Function spaces for second-order polynomial time. In *Language, life, limits*, volume 8493 of *Lecture Notes in Comput. Sci.*, pages 245–254. Springer, Cham, 2014. doi:10.1007/978-3-319-08019-2_25.
- [KP14b] Takayuki Kihara and Arno Pauly. Point degree spectra of represented spaces. *arXiv preprint arXiv:1405.6866*, 2014.
- [KSZ16] Akitoshi Kawamura, Florian Steinberg, and Martin Ziegler. Towards computational complexity theory on advanced function spaces in analysis. In *Pursuit of the Universal: 12th Conference on Computability in Europe*, pages 142–152. Springer International Publishing, 2016. doi:10.1007/978-3-319-40189-8_15.
- [KT59] A. N. Kolmogorov and V. M. Tihomirov. ε -entropy and ε -capacity of sets in function spaces. *Uspehi Mat. Nauk*, 14(2 (86)):3–86, 1959.
- [Lac55] Daniel Lacombe. Extension de la notion de fonction récursive aux fonctions d’une ou plusieurs variables réelles. II, III. *C. R. Acad. Sci. Paris*, 241:13–14, 151–153, 1955.
- [Lam06] Branimir Lambov. The basic feasible functionals in computable analysis. *J. Complexity*, 22(6):909–917, 2006. doi:10.1016/j.jco.2006.06.005.
- [LLM01] S. Labhalla, H. Lombardi, and E. Moutai. Espaces métriques rationnellement présentés et complexité: le cas de l’espace des fonctions réelles uniformément continues sur un intervalle compact. *Theoret. Comput. Sci.*, 250(1-2):265–332, 2001. doi:10.1016/S0304-3975(99)00139-5.
- [Lon05] John R. Longley. Notions of computability at higher types. I. In *Logic Colloquium 2000*, volume 19 of *Lect. Notes Log.*, pages 32–142. Assoc. Symbol. Logic, Urbana, IL, 2005.
- [Lor66] G. G. Lorentz. Metric entropy and approximation. *Bull. Amer. Math. Soc.*, 72(6):903–937, 11 1966. doi:10.1090/S0002-9904-1966-11586-0.
- [Meh76] Kurt Mehlhorn. Polynomial and abstract subrecursive classes. *J. Comput. System Sci.*, 12(2):147–178, 1976. Sixth Annual ACM Symposium on the Theory of Computing (Seattle, Wash., 1974). doi:10.1145/800119.803890.
- [Mun00] J.R. Munkres. *Topology*. Featured Titles for Topology Series. Prentice Hall, Incorporated, Upper Saddle River, NJ 07458, 2000.
- [PER89] Marian B. Pour-El and J. Ian Richards. *Computability in analysis and physics*. Perspectives in Mathematical Logic. Springer-Verlag, Berlin, 1989.
- [Rud87] Walter Rudin. *Real and complex analysis*. McGraw-Hill Book Co., New York, third edition, 1987.
- [Sch02a] Matthias Schröder. *Admissible Representations for Continuous Computations*. PhD thesis, FernUniversität Hagen, 2002.
- [Sch02b] Matthias Schröder. Extended admissibility. *Theoret. Comput. Sci.*, 284(2):519–538, 2002. Computability and complexity in analysis (Castle Dagstuhl, 1999). doi:10.1016/S0304-3975(01)00109-8.
- [Sch04] Matthias Schröder. Spaces allowing type-2 complexity theory revisited. *MLQ Math. Log. Q.*, 50(4-5):443–459, 2004. doi:10.1002/malq.200310111.
- [Ste16] Florian Steinberg. *Computational Complexity Theory for Advanced Function Spaces in Analysis*. PhD thesis, Technische Universität Darmstadt, 2016.
- [Sud01] Madhu Sudan. Coding theory: tutorial & survey. In *42nd IEEE Symposium on Foundations of Computer Science (Las Vegas, NV, 2001)*, pages 36–53. IEEE Computer Soc., Los Alamitos, CA, 2001.
- [Tim94] A. F. Timan. *Theory of approximation of functions of a real variable*. Dover Publications, Inc., New York, 1994. Translated from the Russian by J. Berry, Translation edited and with a preface by J. Cossar, Reprint of the 1963 English translation.
- [Tur36] Alan Mathison Turing. On computable numbers, with an application to the entscheidungsproblem. *J. of Math*, 58(345-363):5, 1936. doi:10.2307/2268810.
- [Wei00] Klaus Weihrauch. *Computable analysis*. Texts in Theoretical Computer Science. An EATCS Series. Springer-Verlag, Berlin, 2000. An introduction. doi:10.1007/978-3-642-56999-9.
- [Wei03] Klaus Weihrauch. Computational complexity on computable metric spaces. *MLQ Math. Log. Q.*, 49(1):3–21, 2003. doi:10.1002/malq.200310001.

- [Wer00] Dirk Werner. *Funktionalanalysis*. Springer-Verlag, Berlin, extended edition, 2000.
- [WZ06] Klaus Weihrauch and Ning Zhong. Computing Schrödinger propagators on type-2 Turing machines. *J. Complexity*, 22(6):918–935, 2006. doi:10.1016/j.jco.2006.06.001.
- [WZ07] Klaus Weihrauch and Ning Zhong. Computable analysis of the abstract Cauchy problem in a Banach space and its applications. I. *MLQ Math. Log. Q.*, 53(4-5):511–531, 2007. doi:10.1002/malq.200710015.
- [Zho99] Ning Zhong. Computability structure of the Sobolev spaces and its applications. *Theoret. Comput. Sci.*, 219(1-2):487–510, 1999. Computability and complexity in analysis (Castle Dagstuhl, 1997). doi:10.1016/S0304-3975(98)00302-8.
- [ZZ99] Ning Zhong and Bing-Yu Zhang. L^p -computability. *MLQ Math. Log. Q.*, 45(4):449–456, 1999. doi:10.1002/malq.19990450403.