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# A NEW CHARACTERIZATION OF COMPLETE HEYTING AND CO-HEYTING ALGEBRAS

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ABSTRACT. We give a new order-theoretic characterization of a complete Heyting and co-Heyting algebra $C$. This result provides an unexpected relationship with the field of Nash equilibria, being based on the so-called Veinott ordering relation on subcomplete sublattices of $C$, which is crucially used in Topkis' theorem for studying the order-theoretic stucture of Nash equilibria of supermodular games.

## INTRODUCTION

Complete Heyting algebras — also called frames, while locales is used for complete co-Heyting algebras — play a fundamental role as algebraic model of intuitionistic logic and in pointless topology [Johnstone 1982, Johnstone 1983]. To the best of our knowledge, no characterization of complete Heyting and co-Heyting algebras has been known. As reported in [Balbes and Dwinger 1974], a sufficient condition has been given in [Funayama 1959] while a necessary condition has been given by [Chang and Horn 1962].

We give here an order-theoretic characterization of complete Heyting and co-Heyting algebras that puts forward an unexected relationship with Nash equilibria. Topkis' theorem [Topkis 1998] is well known in the theory of supermodular games in mathematical economics. This result shows that the set of solutions of a supermodular game, *i.e.*, its set of pure-strategy Nash equilibria, is nonempty and contains a greatest element and a least one [Topkis 1978]. Topkis' theorem has been strengthned by [Zhou 1994], where it is proved that this set of Nash equilibria is indeed a complete lattice. These results rely on so-called Veinott's ordering relation (also called strong set relation). Let $\langle C, \leq, \wedge, \vee\rangle$ be a complete lattice. Then, the relation $\leq^v \subseteq \wp(C) \times \wp(C)$ on subsets of $C$, according to Topkis [Topkis 1978], has been introduced by Veinott [Topkis 1998, Veinott 1989]: for any $S, T \in \wp(C)$,

$$S \leq^v T \overset{\triangle}{\iff} \forall s \in S. \forall t \in T.\, s \wedge t \in S \;\&\; s \vee t \in T.$$

This relation $\leq^v$ is always transitive and antisymmetric, while reflexivity $S \leq^v S$ holds if and only if $S$ is a sublattice of $C$. If $\mathrm{SL}(C)$ denotes the set of nonempty subcomplete sublattices of $C$ then $\langle \mathrm{SL}(C), \leq^v\rangle$ is therefore a poset. The proof of Topkis' theorem is then based on the fixed points of a certain mapping defined on the poset $\langle \mathrm{SL}(C), \leq^v\rangle$.

*Key words and phrases:* Complete Heyting algebra, Veinott ordering.


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To the best of our knowledge, no result is available on the order-theoretic properties of the Veinott poset $\langle \mathrm{SL}(C), \leq^v \rangle$. When is this poset a lattice? And a complete lattice? Our efforts in investigating these questions led to the following main result: the Veinott poset $\mathrm{SL}(C)$ is a complete lattice if and only if $C$ is a complete Heyting and co-Heyting algebra. This finding therefore reveals an unexpected link between complete Heyting algebras and Nash equilibria of supermodular games. This characterization of the Veinott relation $\leq^v$ could be exploited for generalizing a recent approach based on abstract interpretation for approximating the Nash equilibria of supermodular games introduced by [Ranzato 2016].

## 1. Notation

If $\langle P, \leq \rangle$ is a poset and $S \subseteq P$ then $\mathrm{lb}(S)$ denotes the set of lower bounds of $S$, *i.e.*, $\mathrm{lb}(S) \triangleq \{x \in P \mid \forall s \in S.\, x \leq s\}$, while if $x \in P$ then $\downarrow x \triangleq \{y \in P \mid y \leq x\}$.
Let $\langle C, \leq, \wedge, \vee \rangle$ be a complete lattice. A nonempty subset $S \subseteq C$ is a subcomplete sublattice of $C$ if for all its nonempty subsets $X \subseteq S$, $\wedge X \in S$ and $\vee X \in S$, while $S$ is merely a sublattice of $C$ if this holds for all its nonempty and finite subsets $X \subseteq S$ only. If $S \subseteq C$ then the nonempty Moore closure of $S$ is defined as $\mathcal{M}^*(S) \triangleq \{\wedge X \in C \mid X \subseteq S, X \neq \varnothing\}$. Let us observe that $\mathcal{M}^*$ is an upper closure operator on the poset $\langle \wp(C), \subseteq \rangle$, meaning that: (1) $S \subseteq T \;\Rightarrow\; \mathcal{M}^*(S) \subseteq \mathcal{M}^*(T)$; (2) $S \subseteq \mathcal{M}^*(S)$; (3) $\mathcal{M}^*(\mathcal{M}^*(S)) = \mathcal{M}^*(S)$.
We define

$$\mathrm{SL}(C) \triangleq \{S \subseteq C \mid S \neq \varnothing,\ S \text{ subcomplete sublattice of } C\}.$$

Thus, if $\leq^v$ denotes the Veinott ordering defined in Section then $\langle \mathrm{SL}(C), \leq^v \rangle$ is a poset.
$C$ is a complete Heyting algebra (also called frame) if for any $x \in C$ and $Y \subseteq C$, $x \wedge (\bigvee Y) = \bigvee_{y \in Y} x \wedge y$, while it is a complete co-Heyting algebra (also called locale) if the dual equation $x \vee (\bigwedge Y) = \bigwedge_{y \in Y} x \vee y$ holds. Let us recall that these two notions are orthogonal, for example the complete lattice of open subsets of $\mathbb{R}$ ordered by $\subseteq$ is a complete Heyting algebra, but not a complete co-Heyting algebra. $C$ is (finitely) distributive if for any $x, y, z \in C$, $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$.
Let us also recall that $C$ is completely distributive if for any family $\{x_{j,k} \mid j \in J, k \in K(j)\} \subseteq C$, we have that

$$\bigwedge_{j \in J} \bigvee_{k \in K(j)} x_{j,k} = \bigvee_{f \in J \rightsquigarrow K} \bigwedge_{j \in J} x_{j,f(j)}$$

where $J$ and, for any $j \in J$, $K(j)$ are sets of indices and $J \rightsquigarrow K \triangleq \{f : J \to \cup_{j \in J} K(j) \mid \forall j \in J.\, f(j) \in K(j)\}$ denotes the set of choice functions. It turns out that the class of completely distributive complete lattices is strictly contained in the class of complete Heyting and co-Heyting algebras. Clearly, any completely distribuitive lattice is a complete Heyting and co-Heyting algebra. On the other hand, this containment turns out to be strict, as shown by the following counterexample.

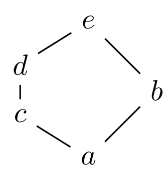
**Example 1.1.** Let us recall that a subset $S \subseteq [0,1]$ of real numbers is a regular open set if $S$ is open and $S$ coincides with the interior of the closure of $S$. For example, $(1/3, 2/3)$ and $(0, 1/3) \cup (2/3, 1)$ are both regular open sets, while $(1/3, 2/3) \cup (2/3, 1)$ is open but not regular. Let us consider $\mathcal{C} = \langle \{S \subseteq [0,1] \mid S \text{ is a regular open set}\}, \subseteq \rangle$. It is known that $\mathcal{C}$ is a complete Boolean algebra (see e.g. [Vladimirov 2002, Theorem 12, Section 2.5]). As a consequence, $\mathcal{C}$ is a complete Heyting and co-Heyting algebra (see e.g. [Vladimirov 2002, Theorem 3, Section 0.2.3]).
Recall that an element $a \in C$ in a complete lattice $C$ is an atom if $a$ is different from the least element $\bot_C$ of $C$ and for any $x \in C$, if $\bot_C < x \leq a$ then $x = a$, while $C$ is atomic if for any $x \in C \smallsetminus \{\bot_C\}$ there exists an atom $a \in C$ such that $a \leq x$. It turns out that $\mathcal{C}$ does not have atoms: in fact, any regular open set $S \in \mathcal{C}$ is a union of open sets, namely, $S = \cup\{U \subseteq [0,1] \mid U \text{ is open},\ U \subseteq S\}$

(see e.g. [Vladimirov 2002, Section 2.5]), so that no $S \in \mathcal{C} \smallsetminus \{\varnothing\}$ can be an atom of $\mathcal{C}$. In turn, this implies that $\mathcal{C}$ is a complete Boolean algebra which is not atomic. It known that a complete Boolean algebra is completely distributive if and only if it is atomic (see [Koppelberg 1989, Theorem 14.5, Chapter 5]). Hence, since $\mathcal{C}$ is not atomic, we obtain that $\mathcal{C}$ is not completely distributive. $\square$

## 2. The Sufficient Condition

To the best of our knowledge, no result is available on the order-theoretic properties of the Veinott poset $\langle \mathrm{SL}(C), \leq^v \rangle$. The following example shows that, in general, $\langle \mathrm{SL}(C), \leq^v \rangle$ is not a lattice.

**Example 2.1.** Consider the nondistributive pentagon lattice $N_5$, where, to use a compact notation, subsets of $N_5$ are denoted by strings of letters.




Consider $ed, abce \in \mathrm{SL}(N_5)$. It turns out that $\downarrow ed = \{a, c, d, ab, ac, ad, cd, ed, acd, ade, cde, abde, acde, abcde\}$ and $\downarrow abce = \{a, ab, ac, abce\}$. Thus, $\{a, ab, ac\}$ is the set of common lower bounds of $ed$ and $abce$. However, the set $\{a, ab, ac\}$ does not include a greatest element, since $a \leq^v ab$ and $a \leq^v ac$ while $ab$ and $ac$ are incomparable. Hence, $ab$ and $c$ are maximal lower bounds of $ed$ and $abce$, so that $\langle \mathrm{SL}(N_5), \leq^v \rangle$ is not a lattice. $\square$

Indeed, the following result shows that if $\mathrm{SL}(C)$ turns out to be a lattice then $C$ must necessarily be distributive.

**Lemma 2.2.** If $\langle \mathrm{SL}(C), \leq^v \rangle$ is a lattice then $C$ is distributive.

*Proof.* By the basic characterization of distributive lattices, we know that $C$ is not distributive iff either the pentagon $N_5$ is a sublattice of $C$ or the diamond $M_3$ is a sublattice of $C$. We consider separately these two possibilities.

$(N_5)$ Assume that $N_5$, as depicted by the diagram in Example 2.1, is a sublattice of $C$. Following Example 2.1, we consider the sublattices $ed, abce \in \langle \mathrm{SL}(C), \leq^v \rangle$ and we prove that their meet does not exist. By Example 2.1, $ab, ac \in \mathrm{lb}(\{ed, abce\})$. Consider any $X \in \mathrm{SL}(C)$ such that $X \in \mathrm{lb}(\{ed, abce\})$. Assume that $ab \leq^v X$. If $x \in X$ then, by $ab \leq^v X$, we have that $b \vee x \in X$. Moreover, by $X \leq^v abce$, $b \vee x \in \{a, b, c, e\}$. If $b \vee x = e$ then we would have that $e \in X$, and in turn, by $X \leq^v ed$, $d = e \wedge d \in X$, so that, by $X \leq^v abce$, we would get the contradiction $d = d \vee c \in \{a, b, c, e\}$. Also, if $b \vee x = c$ then we would have that $c \in X$, and in turn, by $ab \leq^v X$, $e = b \wedge c \in X$, so that, as in the previous case, we would get the contradiction $d = d \vee c \in \{a, b, c, e\}$. Thus, we necessarily have that $b \vee x \in \{a, b\}$. On the one hand, if $b \vee x = b$ then $x \leq b$ so that, by $ab \leq^v X$, $x = b \wedge x \in \{a, b\}$. On the other hand, if $b \vee x = a$ then $x \leq a$ so that, by $ab \leq^v X$, $x = a \wedge x \in \{a, b\}$. Hence, $X \subseteq \{a, b\}$. Since $X \neq \varnothing$, suppose that $a \in X$. Then, by $ab \leq^v X$, $b = b \vee a \in X$. If, instead, $b \in X$ then, by $X \leq^v abce$, $a = b \wedge a \in X$. We have therefore shown that $X = ab$. An analogous argument shows that if $ac \leq^v X$ then $X = ac$. If the meet of $ed$ and $abce$ would exist, call it $Z \in \mathrm{SL}(C)$, from $Z \in \mathrm{lb}(\{ed, abce\})$ and $ab, ac \leq^v Z$ we would get the contradiction $ab = Z = ac$.

$(M_3)$ Assume that the diamond $M_3$, as depicted by the following diagram, is a sublattice of $C$.

$$\begin{array}{ccccc} & & e & & \\ & \diagup & | & \diagdown & \\ b & & c & & d \\ & \diagdown & | & \diagup & \\ & & a & & \end{array}$$

In this case, we consider the sublattices $eb, ec \in \langle \mathrm{SL}(C), \leq^v \rangle$ and we prove that their meet does not exist. It turns out that $abce, abcde \in \mathrm{lb}(\{eb, ec\})$ while $abce$ and $abcde$ are incomparable. Consider any $X \in \mathrm{SL}(C)$ such that $X \in \mathrm{lb}(\{eb, ec\})$. Assume that $abcde \leq^v X$. If $x \in X$ then, by $X \leq^v eb, ec$, we have that $x \wedge b, x \wedge c \in X$, so that $x \wedge b \wedge c = x \wedge a \in X$. From $abcde \leq^v X$, we obtain that for any $y \in \{a, b, c, d, e\}$, $y = y \vee (x \wedge a) \in X$. Hence, $\{a, b, c, d, e\} \subseteq X$. From $X \leq^v eb$, we derive that $x \vee b \in \{e, b\}$, and, from $abcde \leq^v X$, we also have that $x \vee b \in X$. If $x \vee b = e$ then $x \leq e$, so that, from $abcde \leq^v X$, we obtain $x = e \wedge x \in \{a, b, c, d, e\}$. If, instead, $x \vee b = b$ then $x \leq b$, so that, from $abcde \leq^v X$, we derive $x = b \wedge x \in \{a, b, c, d, e\}$. In both cases, we have that $X \subseteq \{a, b, c, d, e\}$. We thus conclude that $X = abcde$. An analogous argument shows that if $abce \leq^v X$ then $X = abce$. Hence, similarly to the previous case ($N_5$), the meet of $eb$ and $ec$ does not exist. □

Moreover, we show that if we require $\mathrm{SL}(C)$ to be a complete lattice then the complete lattice $C$ must be a complete Heyting and co-Heyting algebra. Let us remark that this proof makes use of the axiom of choice.

**Theorem 2.3.** If $\langle \mathrm{SL}(C), \leq^v \rangle$ is a complete lattice then $C$ is a complete Heyting and co-Heyting algebra.

*Proof.* Assume that the complete lattice $C$ is not a complete co-Heyting algebra. If $C$ is not distributive, then, by Lemma 2.2, $\langle \mathrm{SL}(C), \leq^v \rangle$ is not a complete lattice. Thus, let us assume that $C$ is distributive. The (dual) characterization in [Gierz *et al.* 1980, Remark 4.3, p. 40] states that a complete lattice $C$ is a complete co-Heyting algebra iff $C$ is distributive and join-continuous (*i.e.*, the join distributes over arbitrary meets of directed subsets). Consequently, it turns out that $C$ is not join-continuous. Thus, by the result in [Bruns 1967] on directed sets and chains (see also [Gierz *et al.* 1980, Exercise 4.9, p. 42]), there exists an infinite descending chain $\{a_\beta\}_{\beta<\alpha} \subseteq C$, for some ordinal $\alpha \in \mathrm{Ord}$, such that if $\beta < \gamma < \alpha$ then $a_\beta > a_\gamma$, and an element $b \in C$ such that $\bigwedge_{\beta<\alpha} a_\beta \leq b < \bigwedge_{\beta<\alpha}(b \vee a_\beta)$. We observe the following facts:

(A) $\alpha$ must necessarily be a limit ordinal (so that $|\alpha| \geq |\mathbb{N}|$), otherwise if $\alpha$ is a successor ordinal then we would have that, for any $\beta < \alpha$, $a_{\alpha-1} \leq a_\beta$, so that $\bigwedge_{\beta<\alpha} a_\beta = a_{\alpha-1} \leq b$, and in turn we would obtain $\bigwedge_{\beta<\alpha}(b \vee a_\beta) = b \vee a_{\alpha-1} = b$, *i.e.*, a contradiction.

(B) We have that $\bigwedge_{\beta<\alpha} a_\beta < b$, otherwise $\bigwedge_{\beta<\alpha} a_\beta = b$ would imply that $b \leq a_\beta$ for any $\beta < \alpha$, so that $\bigwedge_{\beta<\alpha}(b \vee a_\beta) = \bigwedge_{\beta<\alpha} a_\beta = b$, which is a contradiction.

(C) Firstly, observe that $\{b \vee a_\beta\}_{\beta<\alpha}$ is an infinite descending chain in $C$. Let us consider a limit ordinal $\gamma < \alpha$. Without loss of generality, we assume that the glb's of the subchains $\{a_\rho\}_{\rho<\gamma}$ and $\{b \vee a_\rho\}_{\rho<\gamma}$ belong, respectively, to the chains $\{a_\beta\}_{\beta<\alpha}$ and $\{b \vee a_\beta\}_{\beta<\alpha}$. For our purposes, this is not a restriction because the elements $\bigwedge_{\rho<\gamma} a_\rho$ and $\bigwedge_{\rho<\gamma}(b \vee a_\rho)$ can be added to the respective chains $\{a_\beta\}_{\beta<\alpha}$ and $\{b \vee a_\beta\}_{\beta<\alpha}$ and these extensions would preserve both the glb's of the chains $\{a_\beta\}_{\beta<\alpha}$ and $\{b \vee a_\beta\}_{\beta<\alpha}$ and the inequalities $\bigwedge_{\beta<\alpha} a_\beta < b < \bigwedge_{\beta<\alpha}(b \vee a_\beta)$. Hence, by this nonrestrictive assumption, we have that for any limit ordinal $\gamma < \alpha$, $\bigwedge_{\rho<\gamma} a_\rho = a_\gamma$ and $\bigwedge_{\rho<\gamma}(b \vee a_\rho) = b \vee a_\gamma$ hold.

(D) Let us consider the set $S = \{a_\beta \mid \beta < \alpha,\ \forall\gamma \geq \beta.\ b \not\leq a_\gamma\}$. Then, $S$ must be nonempty, otherwise we would have that for any $\beta < \alpha$ there exists some $\gamma_\beta \geq \beta$ such that $b \leq a_{\gamma_\beta} \leq a_\beta$, and this would imply that for any $\beta < \alpha$, $b \vee a_\beta = a_\beta$, so that we would obtain $\bigwedge_{\beta<\alpha}(b \vee a_\beta) = \bigwedge_{\beta<\alpha} a_\beta$, which is a contradiction. Since any chain in (*i.e.*, subset of) $S$ has an upper bound in $S$, by Zorn's Lemma, $S$ contains the maximal element $a_{\bar\beta}$, for some $\bar\beta < \alpha$, such that for any $\gamma < \alpha$ and $\gamma \geq \bar\beta$, $b \not\leq a_\gamma$. We also observe that $\bigwedge_{\beta<\alpha} a_\beta = \bigwedge_{\bar\beta\leq\gamma<\alpha} a_\gamma$ and $\bigwedge_{\beta<\alpha}(b \vee a_\beta) = \bigwedge_{\bar\beta\leq\gamma<\alpha}(b \vee a_\gamma)$. Hence, without loss of generality, we assume that the chain $\{a_\beta\}_{\beta<\alpha}$ is such that, for any $\beta < \alpha$, $b \not\leq a_\beta$ holds.

For any ordinal $\beta < \alpha$ — therefore, we remark that the limit ordinal $\alpha$ is not included — we define, by transfinite induction, the following subsets $X_\beta \subseteq C$:

– $\beta = 0 \;\Rightarrow\; X_\beta \triangleq \{a_0,\, b \vee a_0\}$;
– $\beta > 0 \;\Rightarrow\; X_\beta \triangleq \bigcup_{\gamma<\beta} X_\gamma \cup \{b \vee a_\beta\} \cup \{(b \vee a_\beta) \wedge a_\delta \mid \delta \leq \beta\}$.

Observe that, for any $\beta > 0$, $(b \vee a_\beta) \wedge a_\beta = a_\beta$ and that the set $\{b \vee a_\beta\} \cup \{(b \vee a_\beta) \wedge a_\delta \mid \delta \leq \beta\}$ is indeed a chain. Moreover, if $\delta \leq \beta$ then, by distributivity, we have that $(b \vee a_\beta) \wedge a_\delta = (b \wedge a_\delta) \vee (a_\beta \wedge a_\delta) = (b \wedge a_\delta) \vee a_\beta$. Moreover, if $\gamma < \beta < \alpha$ then $X_\gamma \subseteq X_\beta$.

We show, by transfinite induction on $\beta$, that for any $\beta < \alpha$, $X_\beta \in \mathrm{SL}(C)$. Let $\delta \leq \beta$ and $\mu \leq \gamma < \beta$. We notice the following facts:

(1) $(b \vee a_\beta) \wedge (b \vee a_\gamma) = b \vee a_\beta \in X_\beta$
(2) $(b \vee a_\beta) \vee (b \vee a_\gamma) = b \vee a_\gamma \in X_\gamma \subseteq X_\beta$
(3) $(b \vee a_\beta) \wedge \big((b \vee a_\gamma) \wedge a_\mu\big) = (b \vee a_\beta) \wedge a_\mu \in X_\beta$
(4) $(b \vee a_\beta) \vee \big((b \vee a_\gamma) \wedge a_\mu\big) = (b \vee a_\beta) \vee (b \wedge a_\mu) \vee a_\gamma = b \vee a_\gamma \in X_\gamma \subseteq X_\beta$
(5) $\big((b \vee a_\beta) \wedge a_\delta\big) \wedge \big((b \vee a_\gamma) \wedge a_\mu\big) = (b \vee a_\beta) \wedge a_{\max(\delta,\mu)} \in X_\beta$
(6) $\big((b \vee a_\beta) \wedge a_\delta\big) \vee \big((b \vee a_\gamma) \wedge a_\mu\big) = \big((b \wedge a_\delta) \vee a_\beta\big) \vee \big((b \wedge a_\mu) \vee a_\gamma\big) = (b \wedge a_{\min(\delta,\mu)}) \vee a_\gamma = (b \vee a_\gamma) \wedge a_{\min(\delta,\mu)} \in X_\gamma \subseteq X_\beta$
(7) if $\beta$ is a limit ordinal then, by point (C) above, $\bigwedge_{\rho<\beta}(b \vee a_\rho) = b \vee a_\beta$ holds; therefore, $\bigwedge_{\rho<\beta}\big((b \vee a_\rho) \wedge a_\delta\big) = \big(\bigwedge_{\rho<\beta}(b \vee a_\rho)\big) \wedge a_\delta = (b \vee a_\beta) \wedge a_\delta \in X_\beta$; in turn, by taking the glb of these latter elements in $X_\beta$, we have that $\bigwedge_{\delta\leq\beta}\big((b \vee a_\beta) \wedge a_\delta\big) = (b \vee a_\beta) \wedge \big(\bigwedge_{\delta\leq\beta} a_\delta\big) = (b \vee a_\beta) \wedge a_\beta = a_\beta \in X_\beta$

Since $X_0 \in \mathrm{SL}(C)$ obviously holds, the points (1)-(7) above show, by transfinite induction, that for any $\beta < \alpha$, $X_\beta$ is closed under arbitrary lub's and glb's of nonempty subsets, *i.e.*, $X_\beta \in \mathrm{SL}(C)$. In the following, we prove that the glb of $\{X_\beta\}_{\beta<\alpha} \subseteq \mathrm{SL}(C)$ in $\langle \mathrm{SL}(C), \leq^v\rangle$ does not exist.

Recalling, by point (A) above, that $\alpha$ is a limit ordinal, we define $A \triangleq \mathcal{M}^*(\bigcup_{\beta<\alpha} X_\beta)$. By point (C) above, we observe that for any limit ordinal $\gamma < \alpha$, the $\bigcup_{\beta<\alpha} X_\beta$ already contains the glb's

$$\bigwedge_{\rho<\gamma}(b \vee a_\rho) = b \vee a_\gamma \in X_\gamma, \qquad \bigwedge_{\rho<\gamma} a_\rho = a_\gamma \in X_\gamma,$$

$$\{\big(\bigwedge_{\rho<\gamma}(b \vee a_\rho)\big) \wedge a_\delta \mid \delta < \gamma\} = \{(b \vee a_\gamma) \wedge a_\delta \mid \delta < \gamma\} \subseteq X_\gamma.$$

Hence, by taking the glb's of all the chains in $\bigcup_{\beta<\alpha} X_\beta$, $A$ turns out to be as follows:

$$A = \bigcup_{\beta<\alpha} X_\beta \cup \{\bigwedge_{\beta<\alpha}(b \vee a_\beta),\ \bigwedge_{\beta<\alpha} a_\beta\} \cup \{\big(\bigwedge_{\beta<\alpha}(b \vee a_\beta)\big) \wedge a_\delta \mid \delta < \alpha\}.$$

Let us show that $A \in \mathrm{SL}(C)$. First, we observe that $\bigcup_{\beta<\alpha} X_\beta$ is closed under arbitrary nonempty lub's. In fact, if $S \subseteq \bigcup_{\beta<\alpha} X_\beta$ then $S = \bigcup_{\beta<\alpha}(S \cap X_\beta)$, so that

$$\bigvee S = \bigvee \bigcup_{\beta<\alpha} (S \cap X_\beta) = \bigvee_{\beta<\alpha} \bigvee S \cap X_\beta.$$

Also, if $\gamma < \beta < \alpha$ then $S \cap X_\gamma \subseteq S \cap X_\beta$ and, in turn, $\bigvee S \cap X_\gamma \leq \bigvee S \cap X_\beta$, so that $\{\bigvee S \cap X_\beta\}_{\beta<\alpha}$ is an increasing chain. Hence, since $\bigcup_{\beta<\alpha} X_\beta$ does not contain infinite increasing chains, there exists some $\gamma < \alpha$ such that $\bigvee_{\beta<\alpha} \bigvee S \cap X_\beta = \bigvee S \cap X_\gamma \in X_\gamma$, and consequently $\bigvee S \in \bigcup_{\beta<\alpha} X_\beta$. Moreover, $\{\big(\bigwedge_{\beta<\alpha}(b \vee a_\beta)\big) \wedge a_\delta\}_{\delta<\alpha} \subseteq A$ is a chain whose lub is $\big(\bigwedge_{\beta<\alpha}(b \vee a_\beta)\big) \wedge a_0$ which belongs to the chain itself, while its glb is

$$\bigwedge_{\delta<\alpha} \big(\bigwedge_{\beta<\alpha}(b \vee a_\beta)\big) \wedge a_\delta = \big(\bigwedge_{\beta<\alpha}(b \vee a_\beta)\big) \wedge \bigwedge_{\delta<\alpha} a_\delta = \bigwedge_{\delta<\alpha} a_\delta \in A.$$

Finally, if $\delta \leq \gamma < \alpha$ then we have that:

(8) $\big(\bigwedge_{\beta<\alpha}(b \vee a_\beta)\big) \wedge (b \vee a_\gamma) = \bigwedge_{\beta<\alpha}(b \vee a_\beta) \in A$

(9) $\big(\bigwedge_{\beta<\alpha}(b \vee a_\beta)\big) \vee (b \vee a_\gamma) = b \vee a_\gamma \in X_\gamma \subseteq A$

(10) $\big(\bigwedge_{\beta<\alpha}(b \vee a_\beta)\big) \wedge \big((b \vee a_\gamma) \wedge a_\delta\big) = \big(\bigwedge_{\beta<\alpha}(b \vee a_\beta)\big) \wedge a_\delta \in A$

(11) We have that $\big(\bigwedge_{\beta<\alpha}(b \vee a_\beta)\big) \vee \big((b \vee a_\gamma) \wedge a_\delta\big) = \big(\bigwedge_{\beta<\alpha}(b \vee a_\beta)\big) \vee (b \wedge a_\delta) \vee a_\gamma = \big(\bigwedge_{\beta<\alpha}(b \vee a_\beta)\big) \vee a_\gamma$. Moreover, $b \vee a_\gamma \leq \big(\bigwedge_{\beta<\alpha}(b \vee a_\beta)\big) \vee a_\gamma \leq (b \vee a_\gamma) \vee a_\gamma = b \vee a_\gamma$; hence, $\big(\bigwedge_{\beta<\alpha}(b \vee a_\beta)\big) \vee \big((b \vee a_\gamma) \wedge a_\delta\big) = b \vee a_\gamma \in X_\gamma \subseteq A$.

Summing up, we have therefore shown that $A \in \mathrm{SL}(C)$.

We now prove that $A$ is a lower bound of $\{X_\beta\}_{\beta<\alpha}$, *i.e.*, we prove, by transfinite induction on $\beta$, that for any $\beta < \alpha$, $A \leq^v X_\beta$.

- $\big(A \leq^v X_0\big)$: this is a consequence of the following easy equalities, for any $\delta \leq \beta < \alpha$: $(b \vee a_\beta) \wedge a_0 \in X_\beta \subseteq A$; $(b \vee a_\beta) \vee a_0 = b \vee a_0 \in X_0$; $(b \vee a_\beta) \wedge (b \vee a_0) = b \vee a_\beta \in X_\beta \subseteq A$; $(b \vee a_\beta) \vee (b \vee a_0) = b \vee a_0 \in X_0$; $\big((b \vee a_\beta) \wedge a_\delta\big) \wedge a_0 = (b \vee a_\beta) \wedge a_\delta \in X_\beta \subseteq A$; $\big((b \vee a_\beta) \wedge a_\delta\big) \vee a_0 = a_0 \in X_0$; $\big((b \vee a_\beta) \wedge a_\delta\big) \wedge (b \vee a_0) = (b \vee a_\beta) \wedge a_\delta \in X_\beta \subseteq A$; $\big((b \vee a_\beta) \wedge a_\delta\big) \vee (b \vee a_0) = b \vee a_0 \in X_0$.
- $\big(A \leq^v X_\beta, \beta > 0\big)$: Let $a \in A$ and $x \in X_\beta$. If $x \in \bigcup_{\gamma<\beta} X_\gamma$ then $x \in X_\gamma$ for some $\gamma < \beta$, so that, since by inductive hypothesis $A \leq^v X_\gamma$, we have that $a \wedge x \in A$ and $a \vee x \in X_\gamma \subseteq X_\beta$. Thus, assume that $x \in X_\beta \smallsetminus \big(\bigcup_{\gamma<\beta} X_\gamma\big)$. If $a \in X_\beta$ then $a \wedge x \in X_\beta \subseteq A$ and $a \vee x \in X_\beta$. If $a \in X_\mu$, for some $\mu > \beta$, then $a \wedge x \in X_\mu \subseteq A$, while points (2), (4) and (6) above show that $a \vee x \in X_\beta$. If $a = \bigwedge_{\beta<\alpha}(b \vee a_\beta)$ then points (8)-(11) above show that $a \wedge x \in A$ and $a \vee x \in X_\beta$. If $a = \big(\bigwedge_{\gamma<\alpha}(b \vee a_\gamma)\big) \wedge a_\mu$, for some $\mu < \alpha$, and $\delta \leq \beta$ then we have that:

  (12) $\big(\big(\bigwedge_{\gamma<\alpha}(b \vee a_\gamma)\big) \wedge a_\mu\big) \wedge (b \vee a_\beta) = \big(\bigwedge_{\gamma<\alpha}(b \vee a_\gamma)\big) \wedge a_\mu \in A$

  (13) $\big(\big(\bigwedge_{\gamma<\alpha}(b \vee a_\gamma)\big) \wedge a_\mu\big) \vee (b \vee a_\beta) = \big(\big(\bigwedge_{\gamma<\alpha}(b \vee a_\gamma)\big) \vee (b \vee a_\beta)\big) \wedge (a_\mu \vee (b \vee a_\beta)) = (b \vee a_\beta) \wedge (b \vee a_{\min(\mu,\beta)}) = b \vee a_\beta \in X_\beta$

  (14) $\big(\big(\bigwedge_{\gamma<\alpha}(b \vee a_\gamma)\big) \wedge a_\mu\big) \wedge \big((b \vee a_\beta) \wedge a_\delta\big) = \big(\bigwedge_{\gamma<\alpha}(b \vee a_\gamma)\big) \wedge a_{\max(\mu,\delta)} \in A$

(15)
$$\begin{aligned}
\Big(\big(\bigwedge_{\gamma<\alpha}(b\vee a_\gamma)\big)\wedge a_\mu\Big)\vee\big((b\vee a_\beta)\wedge a_\delta\big) &=\\
\Big(\big(\bigwedge_{\gamma<\alpha}(b\vee a_\gamma)\big)\vee(b\vee a_\beta)\Big)\wedge\Big(\big(\bigwedge_{\gamma<\alpha}(b\vee a_\gamma)\big)\vee a_\delta\Big)\wedge\big(a_\mu\vee(b\vee a_\beta)\big)\wedge(a_\mu\vee a_\delta) &=\\
(b\vee a_\beta)\wedge(b\vee a_\delta)\wedge\big(b\vee a_{\min(\mu,\beta)}\big)\wedge a_{\min(\mu,\delta)} &=\\
(b\vee a_\beta)\wedge a_{\min(\mu,\delta)} &\in X_\beta
\end{aligned}$$

Finally, if $a=\bigwedge_{\gamma<\alpha}a_\gamma$ and $x\in X_\beta$ then $a\le x$ so that $a\wedge x=a\in A$ and $a\vee x=x\in X_\beta$. Summing up, we have shown that $A\le^v X_\beta$.

Let us now prove that $b\notin A$. Let us first observe that for any $\beta<\alpha$, we have that $a_\beta\not\le b$: in fact, if $a_\gamma\le b$, for some $\gamma<\alpha$ then, for any $\delta\le\gamma$, $b\vee a_\delta=b$, so that we would obtain $\bigwedge_{\beta<\alpha}(b\vee a_\beta)=b$, which is a contradiction. Hence, for any $\beta<\alpha$ and $\delta\le\beta$, it turns out that $b\neq b\vee a_\beta$ and $b\neq(b\wedge a_\delta)\vee a_\beta=(b\vee a_\beta)\wedge a_\delta$. Moreover, by point (B) above, $b\neq\bigwedge_{\beta<\alpha}(b\vee a_\beta)$, while, by hypothesis, $b\neq\bigwedge_{\beta<\alpha}a_\beta$. Finally, for any $\delta<\alpha$, if $b=\big(\bigwedge_{\beta<\alpha}(b\vee a_\beta)\big)\wedge a_\delta$ then we would derive that $b\le a_\delta$, which, by point (D) above, is a contradiction.

Now, we define $B\triangleq\mathcal{M}^*(A\cup\{b\})$, so that
$$B=A\cup\{b\}\cup\{b\wedge a_\delta\mid\delta<\alpha\}.$$
Observe that for any $a\in A$, with $a\neq\bigwedge_{\beta<\alpha}a_\beta$, and for any $\delta<\alpha$, we have that $b\wedge a_\delta\le a$, while $b\vee\Big(\big(\bigwedge_{\beta<\alpha}(b\vee a_\beta)\big)\wedge a_\delta\Big)=\Big(b\vee\big(\bigwedge_{\beta<\alpha}(b\vee a_\beta)\big)\Big)\wedge(b\vee a_\delta)=\big(\bigwedge_{\beta<\alpha}(b\vee a_\beta)\big)\wedge(b\vee a_\delta)=\bigwedge_{\beta<\alpha}(b\vee a_\beta)\in B$. Also, for any $\delta\le\beta<\alpha$, we have that $b\vee\big((b\vee a_\beta)\wedge a_\delta\big)=\big(b\vee(b\vee a_\beta)\big)\wedge(b\vee a_\delta)=b\vee a_\delta\in B$. Also, $b\vee\big(\bigwedge_{\beta<\alpha}(b\vee a_\beta)\big)=\bigwedge_{\beta<\alpha}(b\vee a_\beta)\in B$ and $b\vee\bigwedge_{\beta<\alpha}a_\beta=b\in B$. We have thus checked that $B$ is closed under lub's (of arbitrary nonempty subsets), *i.e.*, $B\in\mathrm{SL}(C)$. Let us check that $B$ is a lower bound of $\{X_\beta\}_{\beta<\alpha}$. Since we have already shown that $A$ is a lower bound, and since $b\wedge a_\delta\le b$, for any $\delta<\alpha$, it is enough to observe that for any $\beta<\alpha$ and $x\in X_\beta$, $b\wedge x\in B$ and $b\vee x\in X_\beta$. The only nontrivial case is for $x=(b\vee a_\beta)\wedge a_\delta$, for some $\delta\le\beta<\alpha$. On the one hand, $b\wedge\big((b\vee a_\beta)\wedge a_\delta\big)=b\wedge a_\delta\in B$, on the other hand, $b\vee\big((b\vee a_\beta)\wedge a_\delta\big)=b\vee\big((b\wedge a_\delta)\vee a_\beta\big)=b\vee a_\beta\in X_\beta$.

Let us now assume that there exists $Y\in\mathrm{SL}(C)$ such that $Y$ is the glb of $\{X_\beta\}_{\beta<\alpha}$ in $\langle\mathrm{SL}(C),\le^v\rangle$. Therefore, since we proved that $A$ is a lower bound, we have that $A\le^v Y$. Let us consider $y\in Y$. Since $b\vee a_0\in A$, we have that $b\vee a_0\vee y\in Y$. Since $Y\le^v X_0=\{a_0,b\vee a_0\}$, we have that $b\vee a_0\vee y\vee a_0=b\vee a_0\vee y\in\{a_0,b\vee a_0\}$. If $b\vee a_0\vee y=a_0$ then $b\le a_0$, which, by point (D), is a contradiction. Thus, we have that $b\vee a_0\vee y=b\vee a_0$, so that $y\le b\vee a_0$ and $b\vee a_0\in Y$. We know that if $x\in X_\beta$, for some $\beta<\alpha$, then $x\le b\vee a_0$, so that, from $Y\le^v X_\beta$, we obtain that $(b\vee a_0)\wedge x=x\in Y$, that is, $X_\beta\subseteq Y$. Thus, we have that $\bigcup_{\beta<\alpha}X_\beta\subseteq Y$, and, in turn, by subset monotonicity of $\mathcal{M}^*$, we get $A=\mathcal{M}^*(\bigcup_{\beta<\alpha}X_\beta)\subseteq\mathcal{M}^*(Y)=Y$. Moreover, from $y\le b\vee a_0$, since $A\le^v Y$ and $b\vee a_0\in A$, we obtain $(b\vee a_0)\wedge y=y\in A$, that is $Y\subseteq A$. We have therefore shown that $Y=A$. Since we proved that $B$ is a lower bound, $B\le^v Y=A$ must hold. However, it turns out that $B\le^v A$ is a contradiction: by considering $b\in B$ and $\bigwedge_{\beta<\alpha}a_\beta\in A$, we would have that $b\vee\big(\bigwedge_{\beta<\alpha}a_\beta\big)=b\in A$, while we have shown above that $b\notin A$. We have therefore shown that the glb of $\{X_\beta\}_{\beta<\alpha}$ in $\langle\mathrm{SL}(C),\le^v\rangle$ does not exist.

To close the proof, it is enough to observe that if $\langle C,\le\rangle$ is not a complete Heyting algebra then, by duality, $\langle\mathrm{SL}(C),\le^v\rangle$ does not have lub's. $\square$

## 3. THE NECESSARY CONDITION

It turns out that the property of being a complete lattice for the poset $\langle \mathrm{SL}(C), \leq^v \rangle$ is a necessary condition for a complete Heyting and co-Heyting algebra $C$.

**Theorem 3.1.** If $C$ is a complete Heyting and co-Heyting algebra then $\langle \mathrm{SL}(C), \leq^v \rangle$ is a complete lattice.

*Proof.* Let $\{A_i\}_{i \in I} \subseteq \mathrm{SL}(C)$, for some family of indices $I \neq \varnothing$. Let us define

$$G \triangleq \{x \in \mathcal{M}^*(\cup_{i \in I} A_i) \mid \forall k \in I.\ \mathcal{M}^*(\cup_{i \in I} A_i) \cap {\downarrow} x \leq^v A_k\}.$$

The following three points show that $G$ is the glb of $\{A_i\}_{i \in I}$ in $\langle \mathrm{SL}(C), \leq^v \rangle$.

$(1)$ We show that $G \in \mathrm{SL}(C)$. Let $\bot \triangleq \bigwedge_{i \in I} \bigwedge A_i$. First, $G$ is nonempty because it turns out that $\bot \in G$. Since, for any $i \in I$, $\bigwedge A_i \in A_i$ and $I \neq \varnothing$, we have that $\bot \in \mathcal{M}^*(\cup_i A_i)$. Let $y \in \mathcal{M}^*(\cup_i A_i) \cap {\downarrow} \bot$ and, for some $k \in I$, $a \in A_k$. On the one hand, we have that $y \wedge a \in \mathcal{M}^*(\cup_i A_i) \cap {\downarrow} \bot$ trivially holds. On the other hand, since $y \leq \bot \leq a$, we have that $y \vee a = a \in A_k$.

Let us now consider a set $\{x_j\}_{j \in J} \subseteq G$, for some family of indices $J \neq \varnothing$, so that, for any $j \in J$ and $k \in I$, $\mathcal{M}^*(\cup_i A_i) \cap {\downarrow} x_j \leq^v A_k$.

First, notice that $\bigwedge_{j \in J} x_j \in \mathcal{M}^*(\cup_i A_i)$ holds. Then, since ${\downarrow} (\bigwedge_{j \in J} x_j) = \bigcap_{j \in J} {\downarrow} x_j$ holds, we have that $\mathcal{M}^*(\cup_i A_i) \cap {\downarrow} (\bigwedge_{j \in J} x_j) = \mathcal{M}^*(\cup_i A_i) \cap (\bigcap_{j \in J} {\downarrow} x_j)$, so that, for any $k \in I$, $\mathcal{M}^*(\cup_i A_i) \cap {\downarrow} (\bigwedge_{j \in J} x_j) \leq^v A_k$, that is, $\bigwedge_{j \in J} x_j \in G$.

Let us now prove that $\bigvee_{j \in J} x_j \in \mathcal{M}^*(\cup_i A_i)$ holds. First, since any $x_j \in \mathcal{M}^*(\cup_{i \in I} A_i)$, we have that $x_j = \bigwedge_{i \in K(j)} a_{j,i}$, where, for any $j \in J$, $K(j) \subseteq I$ is a nonempty family of indices in $I$ such that for any $i \in K(j)$, $a_{j,i} \in A_i$. For any $i \in I$, we then define the family of indices $L(i) \subseteq J$ as follows: $L(i) \triangleq \{j \in J \mid i \in K(j)\}$. Observe that it may happen that $L(i) = \varnothing$. Since for any $i \in I$ such that $L(i) \neq \varnothing$, $\{a_{j,i}\}_{j \in L(i)} \subseteq A_i$ and $A_i$ is meet-closed, we have that if $L(i) \neq \varnothing$ then $\hat{a}_i \triangleq \bigwedge_{l \in L(i)} a_{l,i} \in A_i$. Since, given $k \in I$ such that $L(k) \neq \varnothing$, for any $j \in J$, $\mathcal{M}^*(\cup_{i \in I} A_i) \cap {\downarrow} x_j \leq^v A_k$, we have that for any $j \in J$, $x_j \vee \hat{a}_k \in A_k$. Since $A_k$ is join-closed, we obtain that $\bigvee_{j \in J} (x_j \vee \hat{a}_k) = (\bigvee_{j \in J} x_j) \vee \hat{a}_k \in A_k$. Consequently,

$$\bigwedge_{\substack{k \in I,\\ L(k) \neq \varnothing}} \big( (\bigvee_{j \in J} x_j) \vee \hat{a}_k \big) \in \mathcal{M}^*(\cup_{i \in I} A_i).$$

Since $C$ is a complete co-Heyting algebra,

$$\bigwedge_{\substack{k \in I,\\ L(k) \neq \varnothing}} \big( (\bigvee_{j \in J} x_j) \vee \hat{a}_k \big) = (\bigvee_{j \in J} x_j) \vee (\bigwedge_{\substack{k \in I,\\ L(k) \neq \varnothing}} \hat{a}_k).$$

Thus, since, for any $j \in J$,

$$\bigwedge_{\substack{k \in I,\\ L(k) \neq \varnothing}} \hat{a}_k = \bigwedge_{j \in J} \textstyle\bigwedge_{i \in K(j)} a_{j,i} \leq x_j,$$

we obtain that $(\bigvee_{j \in J} x_j) \vee (\bigwedge_{\substack{k \in I,\\ L(k) \neq \varnothing}} \hat{a}_k) = \bigvee_{j \in J} x_j$, so that $\bigvee_{j \in J} x_j \in \mathcal{M}^*(\cup_{i \in I} A_i)$.

Finally, in order to prove that $\bigvee_{j \in J} x_j \in G$, let us show that for any $k \in I$, $\mathcal{M}^*(\cup_i A_i) \cap {\downarrow} (\bigvee_{j \in J} x_j) \leq^v A_k$. Let $y \in \mathcal{M}^*(\cup_i A_i) \cap {\downarrow} (\bigvee_{j \in J} x_j)$ and $a \in A_k$. For any $j \in J$, $y \wedge x_j \in \mathcal{M}^*(\cup_i A_i) \cap {\downarrow} (\bigvee_{j \in J} x_j)$, so that $(y \wedge x_j) \vee a \in A_k$. Since $A_k$ is join-closed, we obtain that

$\bigvee_{j\in J}\big((y\wedge x_j)\vee a\big) = a\vee\big(\bigvee_{j\in J}(y\wedge x_j)\big)\in A_k$. Since $C$ is a complete Heyting algebra, $a\vee\big(\bigvee_{j\in J}(y\wedge x_j)\big) = a\vee\big(y\wedge(\bigvee_{j\in J}x_j)\big)$. Since $y\wedge(\bigvee_{j\in J}x_j)=y$, we derive that $y\vee a\in A_k$. On the other hand, $y\wedge a\in\mathcal{M}^*(\cup_i A_i)\cap{\downarrow}(\bigvee_{j\in J}x_j)$ trivially holds.

(2) We show that for any $k\in I$, $G\leq^v A_k$. Let $x\in G$ and $a\in A_k$. Hence, $x\in\mathcal{M}^*(\cup_i A_i)$ and for any $j\in I$, $\mathcal{M}^*(\cup_i A_i)\cap{\downarrow}x\leq^v A_j$. We first prove that $\mathcal{M}^*(\cup_i A_i)\cap{\downarrow}x\subseteq G$. Let $y\in\mathcal{M}^*(\cup_i A_i)\cap{\downarrow}x$, and let us check that for any $j\in I$, $\mathcal{M}^*(\cup_i A_i)\cap{\downarrow}y\leq^v A_j$: if $z\in\mathcal{M}^*(\cup_i A_i)\cap{\downarrow}y$ and $u\in A_j$ then $z\in\mathcal{M}^*(\cup_i A_i)\cap{\downarrow}x$ so that $z\vee u\in A_j$ follows, while $z\wedge u\in\mathcal{M}^*(\cup_i A_i)\cap{\downarrow}y$ trivially holds. Now, since $x\wedge a\in\mathcal{M}^*(\cup_i A_i)\cap{\downarrow}x$, we have that $x\wedge a\in G$. On the other hand, since $x\in\mathcal{M}^*(\cup_i A_i)\cap{\downarrow}x\leq^v A_k$, we also have that $x\vee a\in A_k$.

(3) We show that if $Z\in\mathrm{SL}(C)$ and, for any $i\in I$, $Z\leq^v A_i$ then $Z\leq^v G$. By point (1), $\bot=\bigwedge_{i\in I}\bigwedge A_i\in G$. We then define $Z^\bot\subseteq C$ as follows: $Z^\bot\triangleq\{x\vee\bot\mid x\in Z\}$. It turns out that $Z^\bot\subseteq\mathcal{M}^*(\cup_i A_i)$: in fact, since $C$ is a complete co-Heyting algebra, for any $x\in Z$, we have that $x\vee(\bigwedge_{i\in I}\bigwedge A_i)=\bigwedge_{i\in I}(x\vee\bigwedge A_i)$, and since $x\in Z$, for any $i\in I$, $\bigwedge A_i\in A_i$, and $Z\leq^v A_i$, we have that $x\vee\bigwedge A_i\in A_i$, so that $\bigwedge_{i\in I}(x\vee\bigwedge A_i)\in\mathcal{M}^*(\cup_i A_i)$. Also, it turns out that $Z^\bot\in\mathrm{SL}(C)$. If $Y\subseteq Z^\bot$ and $Y\neq\varnothing$ then $Y=\{x\vee\bot\}_{x\in X}$ for some $X\subseteq Z$ with $X\neq\varnothing$. Hence, $\bigvee Y=\bigvee_{x\in X}(x\vee\bot)=(\bigvee X)\vee\bot$, and since $\bigvee X\in Z$, we therefore have that $\bigvee Y\in Z^\bot$. On the other hand, $\bigwedge Y=\bigwedge_{x\in X}(x\vee\bot)$, and, as $C$ is a complete co-Heyting algebra, $\bigwedge_{x\in X}(x\vee\bot)=(\bigwedge X)\vee\bot$, and since $\bigwedge X\in Z$, we therefore obtain that $\bigwedge Y\in Z^\bot$. We also observe that $Z\leq^v Z^\bot$. In fact, if $x\in Z$ and $y\vee\bot\in Z^\bot$, for some $y\in Z$, then, clearly, $x\vee y\vee\bot\in Z^\bot$, while, by distributivity of $C$, $x\wedge(y\vee\bot)=(x\wedge y)\vee\bot\in Z^\bot$. Next, we show that for any $i\in I$, $Z^\bot\leq^v A_i$. Let $x\vee\bot\in Z^\bot$, for some $z\in Z^\bot$, and $a\in A_i$. Then, by distributivity of $C$, $(x\vee\bot)\wedge a=(x\wedge a)\vee(\bot\wedge a)=(x\wedge a)\vee\bot$, and since, by $Z\leq^v A_i$, we know that $x\wedge a\in Z$, we also have that $(x\wedge a)\vee\bot\in Z^\bot$. On the other hand, $(x\vee\bot)\vee a=(x\vee a)\vee\bot$, and since, by $Z\leq^v A_i$, we know that $\bot\leq x\vee a\in A_i$, we obtain that $(x\vee a)\vee\bot=x\vee a\in A_i$.

Summing up, we have therefore shown that for any $Z\in\mathrm{SL}(C)$ such that, for any $i\in I$, $Z\leq^v A_i$, there exists $Z^\bot\in\mathrm{SL}(C)$ such that $Z^\bot\subseteq\mathcal{M}^*(\cup_i A_i)$ and, for any $i\in I$, $Z^\bot\leq^v A_i$. We now prove that $Z^\bot\subseteq G$. Consider $w\in Z^\bot$, and let us check that for any $i\in I$, $\mathcal{M}^*(\cup_i A_i)\cap{\downarrow}w\leq^v A_i$. Hence, consider $y\in\mathcal{M}^*(\cup_i A_i)\cap{\downarrow}w$ and $a\in A_i$. Then, $y\wedge a\in\mathcal{M}^*(\cup_i A_i)\cap{\downarrow}w$ follows trivially. Moreover, since $y\in\mathcal{M}^*(\cup_i A_i)$, there exists a subset $K\subseteq I$, with $K\neq\varnothing$, such that for any $k\in K$ there exists $a_k\in A_k$ such that $y=\bigwedge_{k\in K}a_k$. Thus, since, for any $k\in K$, $z\wedge a_k\in\mathcal{M}^*(\cup_i A_i)\cap{\downarrow}z\leq^v A_i$, we obtain that $\{(z\wedge a_k)\vee a\}_{k\in K}\subseteq A_i$. Since $A_i$ is meet-closed, $\bigwedge_{k\in K}\big((w\wedge a_k)\vee a\big)\in A_i$. Since $C$ is a complete co-Heyting algebra, $\bigwedge_{k\in K}\big((w\wedge a_k)\vee a\big)= a\vee\big(\bigwedge_{k\in K}(w\wedge a_k)\big)=a\vee\big(w\wedge(\bigwedge_{k\in K}a_k)\big)=a\vee(w\wedge y)=a\vee y$, so that $a\vee y\in A_i$ follows.

To close the proof of point (3), we show that $Z^\bot\leq^v G$. Let $z\in Z^\bot$ and $x\in G$. On the one hand, since $Z^\bot\subseteq G$, we have that $z\in G$, and, in turn, as $G$ is join-closed, we obtain that $z\vee x\in G$. On the other hand, since $x\in\mathcal{M}^*(\cup_i A_i)$, there exists a subset $K\subseteq I$, with $K\neq\varnothing$, such that for any $k\in K$ there exists $a_k\in A_k$ such that $x=\bigwedge_{k\in K}a_k$. Thus, since $Z^\bot\leq^v A_k$, for any $k\in K$, we obtain that $z\wedge a_k\in Z^\bot$. Hence, since $Z^\bot$ is meet-closed, we have that $\bigwedge_{k\in K}(z\wedge a_k)=z\wedge\big(\bigwedge_{k\in K}a_k\big)=z\wedge x\in Z^\bot$.

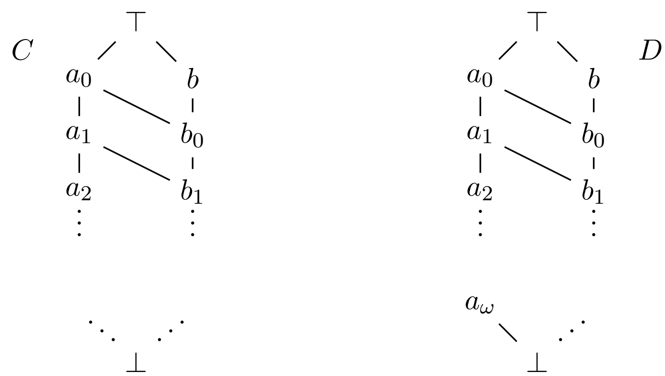
To conclude the proof, we notice that $\{\top_C\}\in\mathrm{SL}(C)$ is the greatest element in $\langle\mathrm{SL}(C),\leq^v\rangle$. Thus, since $\langle\mathrm{SL}(C),\leq^v\rangle$ has nonempty glb's and the greatest element, it turns out that it is a complete lattice. $\square$

We have thus shown the following characterization of complete Heyting and co-Heyting algebras.

**Corollary 3.2.** Let $C$ be a complete lattice. Then, $\langle \mathrm{SL}(C), \leq^v \rangle$ is a complete lattice if and only if $C$ is a complete Heyting and co-Heyting algebra.

To conclude, we provide an example showing that the property of being a complete lattice for the poset $\langle \mathrm{SL}(C), \leq^v \rangle$ cannot be a characterization for a complete Heyting (or co-Heyting) algebra $C$.

**Example 3.3.** Consider the complete lattice $C$ depicted on the left.




$C$ is distributive but not a complete co-Heyting algebra: $b \vee \big(\bigwedge_{i\geq 0} a_i\big) = b < \bigwedge_{i\geq 0}(b \vee a_i) = \top$. Let $X_0 \triangleq \{\top, a_0\}$ and, for any $i \geq 0$, $X_{i+1} \triangleq X_i \cup \{a_{i+1}\}$, so that $\{X_i\}_{i\geq 0} \subseteq \mathrm{SL}(C)$. Then, it turns out that the glb of $\{X_i\}_{i\geq 0}$ in $\langle \mathrm{SL}(C), \leq^v \rangle$ does not exist. This can be shown by mimicking the proof of Theorem 2.3. Let $A \triangleq \{\bot\} \cup \bigcup_{i\geq 0} X_i \in \mathrm{SL}(C)$. Let us observe that $A$ is a lower bound of $\{X_i\}_{i\geq 0}$. Hence, if we suppose that $Y \in \mathrm{SL}(C)$ is the glb of $\{X_i\}_{i\geq 0}$ then $A \leq^v Y$ must hold. Hence, if $y \in Y$ then $\top \wedge y = y \in A$, so that $Y \subseteq A$, and $\top \vee y \in Y$. Since, $Y \leq^v X_0$, we have that $\top \vee y \vee \top = \top \vee y \in X_0 = \{\top, a_0\}$, so that necessarily $\top \vee y = \top \in Y$. Hence, from $Y \leq^v X_i$, for any $i \geq 0$, we obtain that $\top \wedge a_i = a_i \in Y$. Hence, $Y = A$. The whole complete lattice $C$ is also a lower bound of $\{X_i\}_{i\geq 0}$, therefore $C \leq^v Y = A$ must hold: however, this is a contradiction because from $b \in C$ and $\bot \in A$ we obtain that $b \vee \bot = b \in A$.
It is worth noting that if we instead consider the complete lattice $D$ depicted on the right of the above figure, which includes a new glb $a_\omega$ of the chain $\{a_i\}_{i\geq 0}$, then $D$ becomes a complete Heyting and co-Heyting algebra, and in this case the glb of $\{X_i\}_{i\geq 0}$ in $\langle \mathrm{SL}(D), \leq^v \rangle$ turns out to be $\{\top\} \cup \{a_i\}_{i\geq 0} \cup \{a_\omega\}$. $\square$

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