A NEW CHARACTERIZATION OF COMPLETE HEYTING AND CO-HEYTING ALGEBRAS

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ABSTRACT. We give a new order-theoretic characterization of a complete Heyting and co-Heyting algebra $C$. This result provides an unexpected relationship with the field of Nash equilibria, being based on the so-called Veinott ordering relation on subcomplete sublattices of $C$, which is crucially used in Topkis’ theorem for studying the order-theoretic structure of Nash equilibria of supermodular games.

INTRODUCTION

Complete Heyting algebras — also called frames, while locales is used for complete co-Heyting algebras — play a fundamental role as algebraic model of intuitionistic logic and in pointless topology [Johnstone 1982, Johnstone 1983]. To the best of our knowledge, no characterization of complete Heyting and co-Heyting algebras has been known. As reported in [Balbes and Dwinger 1974], a sufficient condition has been given in [Funayama 1959] while a necessary condition has been given by [Chang and Horn 1962].

We give here an order-theoretic characterization of complete Heyting and co-Heyting algebras that puts forward an unexpected relationship with Nash equilibria. Topkis’ theorem [Topkis 1998] is well known in the theory of supermodular games in mathematical economics. This result shows that the set of solutions of a supermodular game, i.e., its set of pure-strategy Nash equilibria, is nonempty and contains a greatest element and a least one [Topkis 1978]. Topkis’ theorem has been strengthen by [Zhou 1994], where it is proved that this set of Nash equilibria is indeed a complete lattice. These results rely on so-called Veinott’s ordering relation (also called strong set relation). Let $(C, \leq, \land, \lor)$ be a complete lattice. Then, the relation $\leq^v \subseteq \wp(C) \times \wp(C)$ on subsets of $C$, according to Topkis [Topkis 1978], has been introduced by Veinott [Topkis 1998, Veinott 1989]: for any $S, T \in \wp(C)$,

\[ S \leq^v T \iff \forall s \in S. \forall t \in T. s \land t \in S \land s \lor t \in T. \]

This relation $\leq^v$ is always transitive and antisymmetric, while reflexivity $S \leq^v S$ holds if and only if $S$ is a sublattice of $C$. If $\text{SL}(C)$ denotes the set of nonempty subcomplete sublattices of $C$ then $(\text{SL}(C), \leq^v)$ is therefore a poset. The proof of Topkis’ theorem is then based on the fixed points of a certain mapping defined on the poset $(\text{SL}(C), \leq^v)$.

Key words and phrases: Complete Heyting algebra, Veinott ordering.
To the best of our knowledge, no result is available on the order-theoretic properties of the Veinott poset \((\text{SL}(C), \leq^v)\). When is this poset a lattice? And a complete lattice? Our efforts in investigating these questions led to the following main result: the Veinott poset \(\text{SL}(C)\) is a complete lattice if and only if \(C\) is a complete Heyting and co-Heyting algebra. This finding therefore reveals an unexpected link between complete Heyting algebras and Nash equilibria of supermodular games. This characterization of the Veinott relation \(\leq^v\) could be exploited for generalizing a recent approach based on abstract interpretation for approximating the Nash equilibria of supermodular games introduced by [Ranzato 2016].

1. Notation

If \((P, \leq)\) is a poset and \(S \subseteq P\) then \(\text{lb}(S)\) denotes the set of lower bounds of \(S\), i.e., \(\text{lb}(S) \triangleq \{x \in P \mid \forall s \in S. x \leq s\}\), while if \(x \in P\) then \(\downarrow x \triangleq \{y \in P \mid y \leq x\}\).

Let \((C, \leq, \land, \lor)\) be a complete lattice. A nonempty subset \(S \subseteq C\) is a subcomplete sublattice of \(C\) if for all its nonempty subsets \(X \subseteq S, \land X \in S\) and \(\lor X \in S\), while \(S\) is merely a sublattice of \(C\) if this holds for all its nonempty and finite subsets \(X \subseteq S\) only. If \(S \subseteq C\) then the nonempty Moore closure of \(S\) is defined as \(\mathcal{M}^*(S) \triangleq \{X \in C \mid X \subseteq S, X \neq \emptyset\}\). Let us observe that \(\mathcal{M}^*\) is an upper closure operator on the poset \(\langle Y(C), \subseteq \rangle\), meaning that: (1) \(S \subseteq T \Rightarrow \mathcal{M}^*(S) \subseteq \mathcal{M}^*(T)\); (2) \(S \subseteq \mathcal{M}^*(S)\); (3) \(\mathcal{M}^*(\mathcal{M}^*(S)) = \mathcal{M}^*(S)\).

We define

\[
\text{SL}(C) \triangleq \{S \subseteq C \mid S \neq \emptyset, S \text{ subcomplete sublattice of } C\}.
\]

Thus, if \(\leq^v\) denotes the Veinott ordering defined in Section then \(\langle \text{SL}(C), \leq^v \rangle\) is a poset.

\(C\) is a complete Heyting algebra (also called frame) if for any \(x \in C\) and \(Y \subseteq C\), \(x \land (\lor Y) = \lor_{y \in Y} x \land y\), while it is a complete co-Heyting algebra (also called locale) if the dual equation \(x \lor (\land Y) = \land_{y \in Y} x \lor y\) holds. Let us recall that these two notions are orthogonal, for example the complete lattice of open subsets of \(\mathbb{R}\) ordered by \(\subseteq\) is a complete Heyting algebra, but not a complete co-Heyting algebra. \(C\) is (finitely) distributive if for any \(x, y, z \in C\), \((x \land y) \lor z = (x \land y) \lor z\).

Let us also recall that \(C\) is completely distributive if for any family \(\{x_{j,k} \mid j \in J, k \in K(j)\} \subseteq C\), we have that

\[
\bigwedge_{j \in J} \bigvee_{k \in K(j)} x_{j,k} = \bigvee_{f \in J \leadsto K} \bigwedge_{j \in J \setminus K(f)} x_{j,f(j)}
\]

where \(J\) and, for any \(j \in J, K(j)\) are sets of indices and \(J \leadsto K \triangleq \{f : J \to \cup_{j \in J} K(j) \mid \forall j \in J. f(j) \in K(j)\}\) denotes the set of choice functions. It turns out that the class of completely distributive complete lattices is strictly contained in the class of complete Heyting and co-Heyting algebras. Clearly, any completely distributive lattice is a complete Heyting and co-Heyting algebra.

On the other hand, this containment turns out to be strict, as shown by the following counterexample.

Example 1.1. Let us recall that a subset \(S \subseteq [0, 1]\) of real numbers is a regular open set if \(S\) is open and \(S\) coincides with the interior of the closure of \(S\). For example, \((1/3, 2/3)\) and \((0, 1/3) \cup (2/3, 1)\) are both regular open sets, while \((1/3, 2/3) \cup (2/3, 1)\) is open but not regular. Let us consider \(\mathcal{C} = \{\{S \subseteq [0, 1] \mid S \text{ is a regular open set}\}, \subseteq\\}\. It is known that \(\mathcal{C}\) is a complete Boolean algebra (see e.g. [Vladimirov 2002, Theorem 12, Section 2.5]). As a consequence, \(\mathcal{C}\) is a complete Heyting and co-Heyting algebra (see e.g. [Vladimirov 2002, Theorem 3, Section 0.2.3]).

Recall that an element \(a \in C\) in a complete lattice \(C\) is an atom if \(a\) is different from the least element \(\bot_C\) of \(C\) and for any \(x \in C\), if \(\bot_C < x < a\) then \(x = a\), while \(C\) is atomic if for any \(x \in C \setminus \{\bot_C\}\) there exists an atom \(a \in C\) such that \(a \leq x\). It turns out that \(\mathcal{C}\) does not have atoms: in fact, any regular open set \(S \subseteq \mathcal{C}\) is a union of open sets, namely, \(S = \cup U \subseteq [0, 1] \mid U\) is open, \(U \subseteq S\).
Thus, we necessarily have that $C$ is a complete Boolean algebra which is not atomic. It known that a complete Boolean algebra is completely distributive if and only if it is atomic (see [Koppelberg 1989, Theorem 14.5, Chapter 5]). Hence, since $C$ is not atomic, we obtain that $C$ is not completely distributive.

2. The Sufficient Condition

To the best of our knowledge, no result is available on the order-theoretic properties of the Veinott poset $\langle SL(C), \leq^v \rangle$. The following example shows that, in general, $\langle SL(C), \leq^v \rangle$ is not a lattice.

**Example 2.1.** Consider the nondistributive pentagon lattice $N_5$, where, to use a compact notation, subsets of $N_5$ are denoted by strings of letters.

```
      e
 b /   \
  d     c
    / \
 a   b
```

Consider $ed, abce \in SL(N_5)$. It turns out that $\downarrow ed = \{a, c, d, ab, ac, ad, cd, ed, acde, ade, cde, abde, acde, abcede\}$ and $\downarrow abce = \{a, ab, ac, abce\}$. Thus, $\{a, ab, ac\}$ is the set of common lower bounds of $ed$ and $abce$. However, the set $\{a, ab, ac\}$ does not include a greatest element, since $a \leq^v ab$ and $a \leq^v ac$ while $ab$ and $ac$ are incomparable. Hence, $ab$ and $c$ are maximal lower bounds of $ed$ and $abce$, so that $\langle SL(N_5), \leq^v \rangle$ is not a lattice.

Indeed, the following result shows that if $SL(C)$ turns out to be a lattice then $C$ must necessarily be distributive.

**Lemma 2.2.** If $\langle SL(C), \leq^v \rangle$ is a lattice then $C$ is distributive.

**Proof.** By the basic characterization of distributive lattices, we know that $C$ is not distributive iff either the pentagon $N_5$ is a sublattice of $C$ or the diamond $M_3$ is a sublattice of $C$. We consider separately these two possibilities.

$(N_5)$ Assume that $N_5$, as depicted by the diagram in Example 2.1, is a sublattice of $C$. Following Example 2.1, we consider the sublattices $ed, abce \in \langle SL(C), \leq^v \rangle$ and we prove that their meet does not exist. By Example 2.1, $ab, ac \in lb(\{ed, abce\})$. Consider any $X \in SL(C)$ such that $X \in lb(\{ed, abce\})$. Assume that $ab \leq^v X$. If $x \in X$ then, by $ab \leq^v X$, we have that $b \vee x \in X$. Moreover, by $X \leq^v abce$, $b \vee x \in \{a, b, c, e\}$. If $b \vee x = e$ then we would have that $e \in X$, and in turn, by $X \leq^v ed$, $d = e \wedge d \in X$, so that, by $X \leq^v abce$, we would get the condition $d = d \wedge c \in \{a, b, c, e\}$. Also, if $b \vee x = c$ then we would have that $c \in X$, and in turn, by $ab \leq^v X$, $e = b \wedge c \in X$, so that, as in the previous case, we would get the condition $d = d \wedge c \in \{a, b, c, e\}$. Thus, we necessarily have that $b \vee x \in \{a, b\}$. On the other hand, if $b \vee x = a$ then $x \leq a$ so that, by $ab \leq^v X$, $x = a \wedge x \in \{a, b\}$. Hence, $X \subseteq \{a, b\}$. Since $X \neq \varnothing$, suppose that $a \in X$. Then, by $ab \leq^v X$, $b = b \wedge a \in X$. If, instead, $b \in X$ then, by $X \leq^v abce$, $a = b \wedge a \in X$. We have therefore shown that $X = ab$. An analogous argument shows that if $ac \leq^v X$ then $X = ac$. If the meet of $ed$ and $abce$ would exist, call it $Z \in SL(C)$, from $Z \in lb(\{ed, abce\})$ and $ab, ac \leq^v Z$ we would get the contradiction $ab = Z = ac$.

$(M_3)$ Assume that the diamond $M_3$, as depicted by the following diagram, is a sublattice of $C$. 

```
      e
 b /   \
  d     c
    / \
 a   b
```

\[ a \cup b = c \Rightarrow a \cup c = b \]

Therefore, $\langle SL(C), \leq^v \rangle$ is not a lattice.
In this case, we consider the sublattices $eb, ec \in \langle SL(C), \leq^v \rangle$ and we prove that their meet does not exist. It turns out that $abce, abcede \in lb\{eb, ec\}$ while $abce$ and $abcede$ are incomparable. Consider any $X \in SL(C)$ such that $X \in lb\{eb, ec\}$. Assume that $abcede \leq^v X$. If $x \in X$ then, by $X \leq^v eb, ec$, we have that $x \wedge b, x \wedge c \in X$, so that $x \wedge b \wedge c = x \wedge a \in X$. From $abcede \leq^v X$, we obtain that for any $y \in \{a, b, c, d, e\}$, $y = y \vee (x \wedge a) \in X$. Hence, $\{a, b, c, d, e\} \subseteq X$. From $X \leq^v eb$, we derive that $x \vee b \in \{e, b\}$, and, from $abcede \leq^v X$, we also have that $x \vee b \in X$. If $x \vee b = e$ then $x \leq e$, so that, from $abcede \leq^v X$, we obtain $x = e \wedge x \in \{a, b, c, d, e\}$. If, instead, $x \vee b = b$ then $x \leq b$, so that, from $abcede \leq^v X$, we derive $x = b \wedge x \in \{a, b, c, d, e\}$. In both cases, we have that $X \subseteq \{a, b, c, d, e\}$. We thus conclude that $X = abcede$. An analogous argument shows that if $abce \leq^v X$ then $X = abce$. Hence, similarly to the previous case $(N_5)$, the meet of $eb$ and $ec$ does not exist.

Moreover, we show that if we require $SL(C)$ to be a complete lattice then the complete lattice $C$ must be a complete Heyting and co-Heyting algebra. Let us remark that this proof makes use of the axiom of choice.

**Theorem 2.3.** If $\langle SL(C), \leq^v \rangle$ is a complete lattice then $C$ is a complete Heyting and co-Heyting algebra.

**Proof.** Assume that the complete lattice $C$ is not a complete co-Heyting algebra. If $C$ is not distributive, then, by Lemma 2.2, $\langle SL(C), \leq^v \rangle$ is not a complete lattice. Thus, let us assume that $C$ is distributive. The (dual) characterization in [Gierz et al. 1980, Remark 4.3, p. 40] states that a complete lattice $C$ is a complete Heyting algebra if $C$ is distributive and join-continuous (i.e., the join distributes over arbitrary meets of directed subsets). Consequently, it turns out that $C$ is not join-continuous. Thus, by the result in [Bruns 1967] on directed sets and chains (see also [Gierz et al. 1980, Exercise 4.9, p. 42]), there exists an infinite descending chain $\{a_\beta\}_{\beta < \alpha} \subseteq C$, for some ordinal $\alpha \in Ord$, such that $\beta < \gamma < \alpha$ then $a_\beta > a_\gamma$, and an element $b \in C$ such that $\bigwedge_{\beta < \alpha} a_\beta \leq b < \bigwedge_{\beta < \alpha} (b \vee a_\beta)$. We observe the following facts:

(A) $\alpha$ must necessarily be a limit ordinal (so that $|\alpha| \geq |\mathbb{N}|$), otherwise if $\alpha$ is a successor ordinal then we would have that, for any $\beta < \alpha$, $a_{\alpha-1} \leq a_\beta$, so that $\bigwedge_{\beta < \alpha} a_\beta = a_{\alpha-1} \leq b$, and in turn we would obtain $\bigwedge_{\beta < \alpha} (b \vee a_\beta) = b \vee a_{\alpha-1} = b$, i.e., a contradiction.

(B) We have that $\bigwedge_{\beta < \alpha} a_\beta < b$, otherwise $\bigwedge_{\beta < \alpha} a_\beta = b$ would imply that $b \leq a_\beta$ for any $\beta < \alpha$, so that $\bigwedge_{\beta < \alpha} (b \vee a_\beta) = \bigwedge_{\beta < \alpha} a_\beta = b$, which is a contradiction.

(C) Firstly, observe that $\{b \vee a_\beta\}_{\beta < \alpha}$ is an infinite descending chain in $C$. Let us consider a limit ordinal $\gamma < \alpha$. Without loss of generality, we assume that the glb’s of the subchains $\{a_\beta\}_{\beta < \gamma}$ and $\{b \vee a_\beta\}_{\beta < \gamma}$ belong, respectively, to the chains $\{a_\beta\}_{\beta < \alpha}$ and $\{b \vee a_\beta\}_{\beta < \alpha}$. For our purposes, this is not a restriction because the elements $\bigwedge_{\beta < \gamma} a_\beta$ and $\bigwedge_{\beta < \gamma} (b \vee a_\beta)$ can be added to the respective chains $\{a_\beta\}_{\beta < \alpha}$ and $\{b \vee a_\beta\}_{\beta < \alpha}$ and these extensions would preserve both the glb’s of the chains $\{a_\beta\}_{\beta < \alpha}$ and $\{b \vee a_\beta\}_{\beta < \alpha}$ and the inequalities $\bigwedge_{\beta < \alpha} a_\beta < b < \bigwedge_{\beta < \alpha} (b \vee a_\beta)$. Hence, by this nonrestrictive assumption, we have that for any limit ordinal $\gamma < \alpha$, $\bigwedge_{\beta < \gamma} a_\beta = a_\gamma$ and $\bigwedge_{\beta < \gamma} (b \vee a_\beta) = b \vee a_\gamma$, hold.

(D) Let us consider the set $S = \{a_\beta \mid \beta < \alpha, \forall \gamma \geq \beta, b \not\leq a_\gamma\}$. Then, $S$ must be nonempty, otherwise we would have that for any $\beta < \alpha$ there exists some $\gamma_\beta \geq \beta$ such that $b \leq a_{\gamma_\beta} \leq a_\beta$,
and this would imply that for any $\beta < \alpha$, $b \lor a_\beta = a_\beta$, so that we would obtain $\bigwedge_{\beta < \alpha} (b \lor a_\beta) = \bigwedge_{\beta < \alpha} a_\beta$, which is a contradiction. Since any chain in (i.e., subset of) $S$ has an upper bound in $S$, by Zorn’s Lemma, $S$ contains the maximal element $a_\beta$, for some $\beta < \alpha$, such that for any $\gamma < \alpha$ and $\gamma \geq \beta$, $b \not\leq a_\gamma$. We also observe that $\bigwedge_{\beta < \alpha} a_\beta = \bigwedge_{\beta \leq \gamma < \alpha} a_\gamma$ and $\bigwedge_{\beta < \alpha} (b \lor a_\beta) = \bigwedge_{\beta \leq \gamma < \alpha} (b \lor a_\gamma)$. Hence, without loss of generality, we assume that the chain $\{a_\beta\}_{\beta < \alpha}$ is such that, for any $\beta < \alpha$, $b \not\leq a_\beta$ holds.

For any ordinal $\beta < \alpha$ — therefore, we remark that the limit ordinal $\alpha$ is not included — we define, by transfinite induction, the following subsets $X_\beta \subseteq C$:

- $\beta = 0 \Rightarrow X_0 \triangleq \{a_0, b \lor a_0\}$;
- $\beta > 0 \Rightarrow X_\beta \triangleq \bigcup_{\gamma < \beta} X_\gamma \cup \{b \lor a_\beta\} \cup \{(b \lor a_\beta) \land a_\delta | \delta \leq \beta\}$.

Observe that, for any $\beta > 0$, $(b \lor a_\beta) \land a_\beta = a_\beta$ and that the set $\{b \lor a_\beta\} \cup \{(b \lor a_\beta) \land a_\delta | \delta \leq \beta\}$ is indeed a chain. Moreover, if $\delta \leq \beta$ then, by distributivity, we have that $(b \lor a_\beta) \land a_\delta = (b \land a_\delta) \lor (a_\delta \land a_\beta) = (b \land a_\delta) \lor a_\beta$. Moreover, if $\gamma < \beta < \alpha$ then $X_\gamma \subseteq X_\beta$.

We show, by transfinite induction on $\beta$, that for any $\beta < \alpha$, $X_\beta \in \text{SL}(C)$. Let $\delta \leq \beta$ and $\mu \leq \gamma < \beta$. We notice the following facts:

1. $(b \lor a_\beta) \land (b \lor a_\gamma) = b \lor a_\beta \in X_\beta$
2. $(b \lor a_\beta) \lor (b \lor a_\gamma) = b \lor a_\gamma \in X_\gamma \subseteq X_\beta$
3. $(b \lor a_\beta) \land (b \lor a_\gamma) \land a_\mu = (b \lor a_\beta) \land a_\mu \in X_\beta$
4. $(b \lor a_\beta) \lor (b \lor a_\gamma) \lor a_\mu = (b \lor a_\beta) \lor a_\gamma \in X_\gamma \subseteq X_\beta$
5. $\left((b \lor a_\beta) \land a_\delta\right) \land \left((b \lor a_\gamma) \land a_\mu\right) = (b \lor a_\beta) \land (\max(\delta, \mu) \lor a_\gamma) \in X_\beta$
6. $\left((b \lor a_\beta) \lor a_\delta\right) \lor \left((b \lor a_\gamma) \lor a_\mu\right) = (b \lor a_\beta) \lor a_\gamma = (b \lor a_\gamma) \land a_\mu \in X_\gamma \subseteq X_\beta$
7. If $\beta$ is a limit ordinal then, by point (C) above, $\bigwedge_{\rho < \beta} (b \lor a_\rho) = b \lor a_\beta$ holds; therefore, $\bigwedge_{\rho < \beta} (b \lor a_\rho) \land a_\delta = \left(\bigwedge_{\rho < \beta} (b \lor a_\rho)\right) \land a_\delta = (b \lor a_\beta) \land a_\delta \in X_\beta$; in turn, by taking the glb of these latter elements in $X_\beta$, we have that $\bigwedge_{\delta \leq \beta} (b \lor a_\beta) \land a_\delta = (b \lor a_\beta) \land (\bigwedge_{\delta \leq \beta} a_\delta) = (b \lor a_\beta) \land a_\beta = a_\beta \in X_\beta$

Since $X_0 \in \text{SL}(C)$ obviously holds, the points (1)-(7) above show, by transfinite induction, that for any $\beta < \alpha$, $X_\beta$ is closed under arbitrary lub’s and glb’s of nonempty subsets, i.e., $X_\beta \in \text{SL}(C)$. In the following, we prove that the glb of $\{X_\beta\}_{\beta < \alpha} \subseteq \text{SL}(C)$ in $\text{SL}(C)$, let $A$ does not exist.

Recalling, by point (A) above, that $\alpha$ is a limit ordinal, we define $A \triangleq M^\ast(\bigcup_{\beta < \alpha} X_\beta)$. By point (C) above, we observe that for any limit ordinal $\gamma < \alpha$, the $\bigcup_{\beta < \alpha} X_\beta$ already contains the glb’s

$$\bigwedge_{\rho < \gamma} (b \lor a_\rho) = (b \lor a_\gamma) \in X_\gamma, \quad \bigwedge_{\rho < \gamma} a_\rho = a_\gamma \in X_\gamma,$$

$$\{\bigwedge_{\rho < \gamma} (b \lor a_\rho) \land a_\delta | \delta < \gamma\} = \{b \lor a_\gamma \land a_\delta | \delta < \gamma\} \subseteq X_\gamma.$$

Hence, by taking the glb’s of all the chains in $\bigcup_{\beta < \alpha} X_\beta$, $A$ turns out to be as follows:

$$A = \bigcup_{\beta < \alpha} X_\beta \cup \{\bigwedge_{\beta < \alpha} (b \lor a_\beta), \bigwedge_{\beta < \alpha} a_\beta\} \cup \{\bigwedge_{\beta < \alpha} (b \lor a_\beta) \land a_\delta | \delta < \alpha\}.$$
Let us show that $A \in \text{SL}(C)$. First, we observe that $\bigcup_{\beta < \alpha} X_\beta$ is closed under arbitrary nonempty lub’s. In fact, if $S \subseteq \bigcup_{\beta < \alpha} X_\beta$ then $S = \bigcup_{\beta < \alpha} (S \cap X_\beta)$, so that

$$\bigvee S = \bigvee \bigcup_{\beta < \alpha} (S \cap X_\beta) = \bigvee \bigvee S \cap X_\beta.$$  

Also, if $\gamma < \beta < \alpha$ then $S \cap X_\gamma \subseteq S \cap X_\beta$ and, in turn, $\bigvee S \cap X_\gamma \subseteq \bigvee S \cap X_\beta$, so that $\{\bigvee S \cap X_\beta\}_{\beta < \alpha}$ is an increasing chain. Hence, since $\bigcup_{\beta < \alpha} X_\beta$ does not contain infinite increasing chains, there exists some $\gamma < \alpha$ such that $\bigvee_{\beta < \alpha} \bigvee S \cap X_\gamma = \bigvee S \cap X_\gamma \in X_\gamma$, and consequently $\bigvee S \in \bigcup_{\beta < \alpha} X_\beta$.

Moreover, $\{\bigwedge_{\beta < \alpha} (b \lor a_\beta) \land a_\delta\}_{\delta < \alpha} \subseteq A$ is a chain whose lub is $\bigwedge_{\beta < \alpha} (b \lor a_\beta) \land a_0$ which belongs to the chain itself, while its glb is

$$\bigwedge_{\delta < \alpha} \bigwedge_{\beta < \alpha} (b \lor a_\beta) \land a_\delta = \bigwedge_{\beta < \alpha} (b \lor a_\beta) \land \bigwedge_{\delta < \alpha} a_\delta = \bigwedge_{\delta < \alpha} a_\delta \in A.$$  

Finally, if $\delta \leq \gamma < \alpha$ then we have that:

8) $\bigwedge_{\beta < \alpha} (b \lor a_\beta) \land (b \lor a_\gamma) = \bigwedge_{\beta < \alpha} (b \lor a_\beta) \in A$

9) $\bigwedge_{\beta < \alpha} (b \lor a_\beta) \lor (b \lor a_\gamma) = b \lor a_\gamma \in X_\gamma \subseteq A$

10) $\bigwedge_{\beta < \alpha} (b \lor a_\beta) \lor ((b \lor a_\gamma) \land a_\delta) = \bigwedge_{\beta < \alpha} (b \lor a_\beta) \land a_\delta \in A$

11) We have that $\bigwedge_{\beta < \alpha} (b \lor a_\beta) \lor ((b \lor a_\gamma) \land a_\delta) = \bigwedge_{\beta < \alpha} (b \lor a_\beta) \lor (b \lor a_\gamma) \lor a_\delta = \bigwedge_{\beta < \alpha} (b \lor a_\beta) \lor a_\delta$. Moreover, $b \lor a_\gamma \leq \bigwedge_{\beta < \alpha} (b \lor a_\beta) \lor a_\delta \leq (b \lor a_\gamma) \lor a_\delta = b \lor a_\gamma \in X_\gamma \subseteq A$.

Summing up, we have therefore shown that $A \in \text{SL}(C)$.

We now prove that $A$ is a lower bound of $\{X_\beta\}_{\beta < \alpha}$, i.e., we prove, by transfinite induction on $\beta$, that for any $\beta < \alpha$, $A \leq X_\beta$.

- $(A \leq X_0)$: this is a consequence of the following easy equalities, for any $\delta \leq \beta < \alpha$:

  (b \lor a_\beta) \land a_0 \in X_\beta \subseteq A; (b \lor a_\beta) \lor a_0 = b \lor a_0 \in X_0; (b \lor a_\beta) \land (b \lor a_0) = b \lor a_\beta \in X_\beta \subseteq A;

  (b \lor a_\beta) \lor (b \lor a_0) = b \lor a_0 \in X_0; (b \lor a_\beta) \land a_\delta \land a_0 = (b \lor a_\beta) \land a_\delta \in X_\beta \subseteq A;

  (b \lor a_\beta) \lor a_0 = a_0 \in X_0; (b \lor a_\beta) \land a_\delta \land (b \lor a_0) = (b \lor a_\beta) \land a_\delta \in X_\beta \subseteq A;

  (b \lor a_\beta) \land a_\delta \land (b \lor a_0) = b \lor a_0 \in X_0.

- $(A \leq X_\beta, \beta > 0)$: Let $a \in A$ and $x \in X_\beta$. If $x \in \bigcup_{\gamma < \beta} X_\gamma$ then $x \in X_\gamma$ for some $\gamma < \beta$, so that, since by inductive hypothesis $A \leq X_\gamma$, we have that $a \land x \in A$ and $a \lor x \in X_\gamma \subseteq X_\beta$.

Thus, assume that $x \in X_\beta \setminus \bigcup_{\gamma < \beta} X_\gamma$. If $a \in X_\beta$ then $a \land x \in X_\beta \subseteq A$ and $a \lor x \in X_\beta$. If $a \in X_\mu$, for some $\mu > \beta$, then $a \land x \in X_\mu \subseteq A$, while points (2), (4) and (6) above show that $a \lor x \in X_\beta$. If $a = \bigwedge_{\beta < \alpha} (b \lor a_\beta)$ then points (8)-(11) above show that $a \land x \in A$ and $a \lor x \in X_\beta$.

If $a = \bigwedge_{\gamma < \alpha} (b \lor a_\gamma) \land a_\mu$, for some $\mu < \alpha$, and $\delta \leq \beta$ then we have that:

12) $\bigwedge_{\gamma < \alpha} (b \lor a_\gamma) \land (b \lor a_\beta) = \bigwedge_{\gamma < \alpha} (b \lor a_\gamma) \land a_\mu \in A$

13) $\bigwedge_{\gamma < \alpha} (b \lor a_\gamma) \land a_\mu \land (b \lor a_\beta) = (\bigwedge_{\gamma < \alpha} (b \lor a_\gamma)) \land (b \lor a_\beta) \land (a_\mu \land (b \lor a_\beta)) = (b \lor a_\beta) \land (b \lor a_{\min(\mu, \beta)}) = b \lor a_\beta \in X_\beta$

14) $\bigwedge_{\gamma < \alpha} (b \lor a_\gamma) \land a_\mu \land (b \lor a_\beta) \land a_\delta = \bigwedge_{\gamma < \alpha} (b \lor a_\gamma) \land a_{\max(\mu, \delta)} \in A$
Finally, if $a = \bigwedge_{\gamma < \alpha} a_{\gamma}$ and $x \in X_\beta$ then $a \leq x$ so that $a \land x = a \in A$ and $a \lor x = x \in X_\beta$.

Summing up, we have shown that $A \subseteq Y X_\beta$.

Let us now prove that $b \notin A$. Let us first observe that for any $\beta < \alpha$, we have that $a_{\beta} \notin b$: in fact, if $a_{\gamma} \leq b$, for some $\gamma < \alpha$ then, for any $\delta \leq \gamma$, $b \lor a_{\delta} = b$, so that we would obtain $\bigwedge_{\beta < \alpha} (b \lor a_{\beta}) = b$, which is a contradiction. Hence, for any $\beta < \alpha$ and $\delta \leq \beta$, it turns out that $b \neq b \lor a_{\beta}$ and $b \neq (b \land a_{\delta}) \lor a_{\beta} = (b \lor a_{\beta}) \land a_{\delta}$. Moreover, by point (B) above, $b \neq \bigwedge_{\beta < \alpha} (b \lor a_{\beta})$, while, by hypothesis, $b \neq \bigwedge_{\beta < \alpha} a_{\beta}$. Finally, for any $\delta < \alpha$, if $b = \left( \bigwedge_{\beta < \alpha} (b \lor a_{\beta}) \right) \land a_{\delta}$ then we would derive that $b \leq a_{\delta}$, which, by point (D) above, is a contradiction.

Now, we define $B = \bigcup \{ b \} \cup \{ b \land a_{\delta} \mid \delta < \alpha \}$. Observe that for any $a \in A$, with $a \neq \bigwedge_{\beta < \alpha} a_{\beta}$, and for any $\delta < \alpha$, we have that $b \land a_{\delta} \leq a$, while $b \lor (\bigwedge_{\beta < \alpha} (b \lor a_{\beta}) \land a_{\delta}) = (b \lor (\bigwedge_{\beta < \alpha} (b \lor a_{\beta}))) \land (b \land a_{\delta}) = (\bigwedge_{\beta < \alpha} (b \lor a_{\beta})) \land (b \lor a_{\delta}) = \bigwedge_{\beta < \alpha} (b \lor a_{\beta}) \in B$. Also, for any $\delta \leq \beta < \alpha$, we have that $b \lor (b \land a_{\delta}) \land a_{\delta} = (b \lor (b \land a_{\delta})) \land (b \lor a_{\delta}) = b \lor a_{\delta} \in B$. Also, $b \lor (\bigwedge_{\beta < \alpha} (b \lor a_{\beta})) = \bigwedge_{\beta < \alpha} (b \lor a_{\beta}) \in B$ and $b \lor \bigwedge_{\beta < \alpha} a_{\beta} = b \in B$. We have thus checked that $B$ is closed under lub’s (of arbitrary nonempty subsets), i.e., $B \in SL(C)$. Let us check that $B$ is a lower bound of $\{ X_\beta \}_{\beta < \alpha}$. Since we have already shown that $A$ is a lower bound, and since $b \land a_{\delta} \leq b$, for any $\delta < \alpha$, it is enough to observe that for any $\beta < \alpha$ and $x \in X_\beta$, $b \lor x \in B$ and $b \land x \in X_\beta$. The only nontrivial case is for $x = (b \lor a_{\beta}) \land a_{\delta}$, for some $\delta \leq \beta < \alpha$. On the one hand, $b \lor ((b \lor a_{\beta}) \land a_{\delta}) = b \land a_{\delta} \in B$, on the other hand, $b \lor ((b \lor a_{\beta}) \land a_{\delta}) = b \lor ((b \land a_{\delta}) \lor a_{\beta}) = b \lor a_{\beta} \in X_\beta$.

Let us now assume that there exists $Y \in SL(C)$ such that $Y$ is the glb of $\{ X_\beta \}_{\beta < \alpha}$ in $\langle SL(C), \leq^v \rangle$. Therefore, since we proved that $A$ is a lower bound, we have proved that $A \leq^v Y$. Let us consider $y \in Y$. Since $b \lor a_{\alpha} \in A$, we have that $b \lor a_0 \lor y \in Y$. Since $Y \leq^v X_0 = \{ a_0, b \lor a_0 \}$, we have that $b \lor a_0 \lor y \lor a_0 = b \lor a_0 \lor y \in \{ a_0, b \lor a_0 \}$. If $b \lor a_0 \lor y = a_0$ then $b \leq a_0$, which, by point (D), is a contradiction. Thus, we have that $b \lor a_0 \lor y = b \lor a_0$, so that $y \leq b \lor a_0$ and $b \lor a_0 \in Y$. We know that if $x \in X_\beta$, for some $\beta < \alpha$, then $x \leq b \lor a_0$, so that, from $Y \leq^v X_\beta$, we obtain that $(b \lor a_0) \land x = x \in Y$, that is, $X_\beta \subseteq Y$. Thus, we have that $\bigcup_{\beta < \alpha} X_\beta \subseteq Y$, and, in turn, by subset monotonicity of $A^*$, we get $A = A^* (\bigcup_{\beta < \alpha} X_\beta) \subseteq M^* (Y) = Y$. Moreover, from $y \leq b \lor a_0$, since $A \leq^v Y$ and $b \lor a_0 \in A$, we obtain $(b \lor a_0) \land y = y \in A$, that is $Y \subseteq A$. We have therefore shown that $Y = A$. Since we proved that $B$ is a lower bound, $B \leq^v Y = A$ must hold. However, it turns out that $B \neq A$ is a contradiction: by considering $b \in B$ and $\bigwedge_{\beta < \alpha} a_{\beta} \in A$, we would have that $b \lor (\bigwedge_{\beta < \alpha} a_{\beta}) = b \in A$, while we have shown above that $b \notin A$. We have therefore shown that the glb of $\{ X_\beta \}_{\beta < \alpha}$ in $\langle SL(C), \leq^v \rangle$ does not exist.

To close the proof, it is enough to observe that if $\langle C, \leq \rangle$ is not a complete Heyting algebra then, by duality, $\langle SL(C), \leq^v \rangle$ does not have lub’s.
3. The Necessary Condition

It turns out that the property of being a complete lattice for the poset $\langle \text{SL}(C), \leq^v \rangle$ is a necessary condition for a complete Heyting and co-Heyting algebra $C$.

**Theorem 3.1.** If $C$ is a complete Heyting and co-Heyting algebra then $\langle \text{SL}(C), \leq^v \rangle$ is a complete lattice.

**Proof.** Let $\{A_i\}_{i \in I} \subseteq \text{SL}(C)$, for some family of indices $I \neq \emptyset$. Let us define

$$G \triangleq \{ x \in \mathcal{M}^*(\bigcup_{i \in I} A_i) \mid \forall k \in I, \mathcal{M}^*(\bigcup_{i \in I} A_i) \cap \downarrow x \leq^v A_k \}.$$ 

The following three points show that $G$ is the glb of $\{A_i\}_{i \in I}$ in $\langle \text{SL}(C), \leq^v \rangle$.

1. We show that $G \in \text{SL}(C)$. Let $\bot \triangleq \bigwedge_{i \in I} \bigwedge A_i$. First, $G$ is nonempty because it turns out that $\bot \in G$. Since, for any $i \in I$, $\bigwedge A_i \subseteq A_i$ and $I \neq \emptyset$, we have that $\bot \in \mathcal{M}^*(\bigcup_{i \in I} A_i)$. Let $y \in \mathcal{M}^*(\bigcup_{i \in I} A_i) \cap \downarrow \bot$ and, for some $k \in I$, $a \in A_k$. On the one hand, we have that $y \wedge a \in \mathcal{M}^*(\bigcup_{i \in I} A_i) \cap \downarrow \bot$ trivially holds. On the other hand, since $y \leq \bot \leq a$, we have that $y \lor a = a \in A_k$.

Let us now consider a set $\{x_j\}_{j \in J} \subseteq G$, for some family of indices $J \neq \emptyset$, so that, for any $j \in J$ and $k \in I$, $\mathcal{M}^*(\bigcup_{i \in I} A_i) \cap \downarrow x_j \leq^v A_k$.

First, notice that $\bigwedge_{j \in J} x_j \in \mathcal{M}^*(\bigcup_{i \in I} A_i)$ holds. Then, since $\downarrow (\bigwedge_{j \in J} x_j) = \bigcap_{j \in J} \downarrow x_j$ holds, we have that $\mathcal{M}^*(\bigcup_{i \in I} A_i) \cap \downarrow (\bigwedge_{j \in J} x_j) = \mathcal{M}^*(\bigcup_{i \in I} A_i) \cap (\bigcap_{j \in J} \downarrow x_j)$, so that, for any $k \in I$, $\mathcal{M}^*(\bigcup_{i \in I} A_i) \cap \downarrow (\bigwedge_{j \in J} x_j) \leq^v A_k$, that is, $\bigwedge_{j \in J} x_j \in G$.

Let us now prove that $\bigvee_{j \in J} x_j \in \mathcal{M}^*(\bigcup_{i \in I} A_i)$ holds. First, since any $x_j \in \mathcal{M}^*(\bigcup_{i \in I} A_i)$, we have that $x_j = \bigwedge_{i \in K(j)} a_{j,i}$, where, for any $j \in J$, $K(j) \subseteq I$ is a nonempty family of indices in $I$ such that for any $i \in K(j)$, $a_{j,i} \in A_i$. For any $i \in I$, we then define the family of indices $L(i) \subseteq J$ as follows: $L(i) \triangleq \{ j \in J \mid i \in K(j) \}$. Observe that it may happen that $L(i) = \emptyset$. Since for any $i \in I$ such that $L(i) \neq \emptyset$, $\{a_{j,i}\}_{j \in L(i)} \subseteq A_i$ and $A_i$ is meet-closed, we have that if $L(i) \neq \emptyset$ then $a_i \triangleq \bigwedge_{j \in L(i)} a_{j,i} \in A_i$. Since, given $k \in I$ such that $L(k) \neq \emptyset$, for any $j \in J$, $\mathcal{M}^*(\bigcup_{i \in I} A_i) \cap \downarrow x_j \leq^v A_k$, we have that for any $j \in J$, $x_j \lor \hat{a}_k \in A_k$. Since $A_k$ is join-closed, we obtain that $\bigvee_{j \in J} (x_j \lor \hat{a}_k) = (\bigvee_{j \in J} x_j) \lor \hat{a}_k \in A_k$. Consequently,

$$\bigwedge_{k \in I \setminus L(k) \neq \emptyset} \left( \bigvee_{j \in J} x_j \lor \hat{a}_k \right) \in \mathcal{M}^*(\bigcup_{i \in I} A_i).$$

Since $C$ is a complete co-Heyting algebra,

$$\bigwedge_{k \in I \setminus L(k) \neq \emptyset} \left( \bigvee_{j \in J} x_j \lor \hat{a}_k \right) = \left( \bigvee_{j \in J} x_j \right) \lor \left( \bigwedge_{k \in I \setminus L(k) \neq \emptyset} \hat{a}_k \right).$$

Thus, since, for any $j \in J$,

$$\bigwedge_{k \in I \setminus L(k) \neq \emptyset} \hat{a}_k = \bigwedge_{j \in J} \bigwedge_{i \in K(j)} a_{j,i} \leq x_j,$$

we obtain that $\left( \bigvee_{j \in J} x_j \right) \lor \left( \bigwedge_{k \in I \setminus L(k) \neq \emptyset} \hat{a}_k \right) = \left( \bigvee_{j \in J} x_j \right) \lor \left( \bigwedge_{k \in I \setminus L(k) \neq \emptyset} \hat{a}_k \right)$, so that $\bigvee_{j \in J} x_j \in \mathcal{M}^*(\bigcup_{i \in I} A_i)$.

Finally, in order to prove that $\bigvee_{j \in J} x_j \in G$, let us show that for any $k \in I$, $\mathcal{M}^*(\bigcup_{i \in I} A_i) \cap \downarrow (\bigvee_{j \in J} x_j) \leq^v A_k$. Let $y \in \mathcal{M}^*(\bigcup_{i \in I} A_i) \cap \downarrow (\bigvee_{j \in J} x_j)$ and $a \in A_k$. For any $j \in J$, $y \wedge x_j \in \mathcal{M}^*(\bigcup_{i \in I} A_i) \cap \downarrow (\bigvee_{j \in J} x_j)$, so that $(y \wedge x_j) \lor a \in A_k$. Since $A_k$ is join-closed, we obtain that
We show that if \( Z \) such that for any \( x \leq Z \), we have that \( y \lor a \in A_k \). On the other hand, \( y \land a \in M^*(\cup_i A_i) \cap \downarrow y \lor x \) trivially holds.

(2) We show that for any \( k \in I \), \( G \leq^v A_k \). Let \( x \in G \) and \( a \in A_k \). Hence, \( x \in M^*(\cup_i A_i) \) and for any \( j \in I \), \( M^*(\cup_i A_i) \cap \downarrow x \leq^v A_j \). We first prove that \( M^*(\cup_i A_i) \cap \downarrow x \subseteq G \). Let \( y \in M^*(\cup_i A_i) \cap \downarrow x \), and let us check that for any \( j \in I \), \( M^*(\cup_i A_i) \cap \downarrow y \leq^v A_j \). If \( z \in M^*(\cup_i A_i) \cap \downarrow y \) and \( u \in A_j \), then \( z \in M^*(\cup_i A_i) \cap \downarrow x \) so that \( z \lor u \in A_j \) follows, while \( z \land u \in M^*(\cup_i A_i) \cap \downarrow x \) trivially holds. Now, since \( x \land a \in M^*(\cup_i A_i) \cap \downarrow x \), we have that \( x \land a \in G \). On the other hand, since \( x \in M^*(\cup_i A_i) \cap \downarrow x \leq^v A_k \), we also have that \( x \lor a \in A_k \).

(3) We show that if \( Z \in SL(C) \) and, for any \( i \in I \), \( Z \leq^v A_i \) then \( Z \leq^v G \). By point (1), \( \bot = \bigwedge_{i \in I} \land A_i \in G \). We then define \( Z^\perp \subseteq C \) as follows: \( Z^\perp = \{ x \lor \bot \mid x \in Z \} \). It turns out that \( Z^\perp \subseteq M^*(\cup_i A_i) \): in fact, since \( C \) is a complete co-Heyting algebra, for any \( x \in Z \), we have that \( x \lor (\bigwedge_{i \in I} \land A_i) \leq \bigwedge_{i \in I} (x \lor \land A_i) \), and since \( x \in Z \), for any \( i \in I \), \( \land A_i \in A_i \), and \( Z \leq^v A_i \), we have that \( x \lor \land A_i \in A_i \), and so that \( \bigwedge_{i \in I} (x \lor \land A_i) \in M^*(\cup_i A_i) \). Also, it turns out that \( Z^\perp \subseteq SL(C) \). If \( Y \subseteq Z^\perp \) and \( Y \neq \emptyset \), then \( Y = \{ x \lor \bot \} \in X \) for some \( X \subseteq Z \) with \( X \neq \emptyset \). Hence, \( Y = \bigvee_{x \in X} (x \lor \bot) = (\bigvee X) \lor \bot \), and since \( \bigvee X \in Z \), we therefore have that \( \bigvee Y \in Z^\perp \). On the other hand, \( \bigwedge_Y = \bigwedge_{x \in X} (x \lor \bot) \), and, as \( C \) is a complete co-Heyting algebra, \( \bigwedge_{x \in X} (x \lor \bot) \lor \bot \), and since \( \bigwedge X \in Z \), we therefore obtain that \( \bigwedge Y \in Z^\perp \).

We also observe that \( Z \leq^v Z^\perp \). In fact, if \( x \in Z \) and \( y \lor \bot \in Z^\perp \), for some \( y \in Z \), then, clearly, \( x \lor y \lor \bot \in Z^\perp \), while, by distributivity of \( C \), \( x \lor (y \lor \bot) = (x \lor y) \lor \bot \in Z^\perp \). Next, we show that for any \( i \in I \), \( Z \leq^v A_i \). Let \( x \lor \bot \in Z^\perp \), for some \( z \in Z^\perp \), and \( a \in A_i \). Then, by distributivity of \( C \), \( (x \lor \bot) \land a = (x \land a) \lor (\bot \land a) = (x \land a) \lor \bot \), and since, by \( Z \leq^v A_i \), we know that \( x \land a \in Z \), we also have that \( (x \land a) \lor \bot \in Z^\perp \). On the other hand, \( (x \lor \bot) \lor a = (x \lor a) \lor \bot \), and since, by \( Z \leq^v A_i \), we know that \( \bot \leq x \lor a \in A_i \), we obtain that \( (x \lor a) \lor \bot = x \lor a \in A_i \).

Summing up, we have therefore shown that for any \( Z \in SL(C) \) such that, for any \( i \in I \), \( Z \leq^v A_i \), there exists \( Z^\perp \in SL(C) \) such that \( Z^\perp \subseteq M^*(\cup_i A_i) \) and, for any \( i \in I \), \( Z \leq^v A_i \). We now prove that \( Z^\perp \subseteq G \). Consider \( w \in Z^\perp \), and let us check that for any \( i \in I \), \( M^*(\cup_i A_i) \cap \downarrow w \leq^v A_i \). Hence, consider \( y \in M^*(\cup_i A_i) \cap \downarrow w \) and \( a \in A_i \). Then, \( y \land a \in M^*(\cup_i A_i) \cap \downarrow w \) follows trivially. Moreover, since \( y \in M^*(\cup_i A_i) \), it exists a subset \( K \subseteq I \), with \( K \neq \emptyset \), such that for any \( k \in K \) there exists \( a_k \in A_k \), such that \( y = \bigwedge_{k \in K} a_k \). Thus, since, for any \( k \in K \), \( z \lor a_k \in M^*(\cup_i A_i) \cap \downarrow z \leq^v A_i \), we obtain that \( \{ z \lor a_k \wedge a \}_{k \in K} \subseteq A_i \). Since \( A_i \) is meet-closed, \( \bigwedge_{k \in K} (w \lor a_k) \lor a = a \lor (\bigwedge_{k \in K} (w \lor a_k)) = a \lor (w \lor a_k) = a \lor y \), so that \( a \lor y \in A_i \) follows.

To close the proof of point (3), we show that \( Z^\perp \leq^v G \). Let \( z \in Z^\perp \) and \( x \in G \). On the one hand, since \( Z^\perp \subseteq G \), we have that \( z \in G \) and, in turn, as \( G \) is join-closed, we obtain that \( z \lor x \in G \). On the other hand, since \( x \in M^*(\cup_i A_i) \), there exists a subset \( K \subseteq I \), with \( K \neq \emptyset \), such that for any \( k \in K \) there exists \( a_k \in A_k \), such that \( x = \bigwedge_{k \in K} a_k \). Thus, since \( Z \leq^v A_k \), for any \( k \in K \), we obtain that \( z \lor a_k \in Z \). Hence, since \( Z^\perp \leq^v A_k \), we have that \( \bigwedge_{k \in K} (z \lor a_k) = z \lor \bigwedge_{k \in K} a_k = z \lor x \in Z^\perp \).

To conclude the proof, we notice that \( \{ T_C \} \subseteq SL(C) \) is the greatest element in \( \langle SL(C), \leq^v \rangle \). Thus, since \( \langle SL(C), \leq^v \rangle \) has nonempty glb’s and the greatest element, it turns out that it is a complete lattice.

We have thus shown the following characterization of complete Heyting and co-Heyting algebras.
**Corollary 3.2.** Let $C$ be a complete lattice. Then, $\langle \text{SL}(C), \leq^v \rangle$ is a complete lattice if and only if $C$ is a complete Heyting and co-Heyting algebra.

To conclude, we provide an example showing that the property of being a complete lattice for the poset $\langle \text{SL}(C), \leq^v \rangle$ cannot be a characterization for a complete Heyting (or co-Heyting) algebra $C$.

**Example 3.3.** Consider the complete lattice $C$ depicted on the left.

$C$ is distributive but not a complete co-Heyting algebra: $b \lor (\bigwedge_{i \geq 0} a_i) = b < \bigwedge_{i \geq 0} (b \lor a_i) = \top$. Let $X_0 \triangleq \{ \top, a_0 \}$ and, for any $i \geq 0$, $X_{i+1} \triangleq X_i \cup \{ a_{i+1} \}$, so that $\{ X_i \}_{i \geq 0} \subseteq \text{SL}(C)$. Then, it turns out that the glb of $\{ X_i \}_{i \geq 0}$ in $\langle \text{SL}(C), \leq^v \rangle$ does not exist. This can be shown by mimicking the proof of Theorem 2.3. Let $A \triangleq \{ \bot \} \cup \bigcup_{i \geq 0} X_i \in \text{SL}(C)$. Let us observe that $A$ is a lower bound of $\{ X_i \}_{i \geq 0}$. Hence, if we suppose that $Y \in \text{SL}(C)$ is the glb of $\{ X_i \}_{i \geq 0}$ then $A \leq^v Y$ must hold. Hence, if $y \in Y$ then $\top \land y = y \in A$, so that $Y \subseteq A$, and $\top \lor y \in Y$. Since, $Y \leq^v X_0$, we have that $\top \land y \lor \top = \top \lor y \in X_0 = \{ \top, a_0 \}$, so that necessarily $\top \lor y = \top \in Y$. Hence, from $Y \leq^v X_i$, for any $i \geq 0$, we obtain that $\top \land a_i = a_i \in Y$. Hence, $Y = A$. The whole complete lattice $C$ is also a lower bound of $\{ X_i \}_{i \geq 0}$, therefore $C \leq^v Y = A$ must hold: however, this is a contradiction because from $b \in C$ and $\bot \in A$ we obtain that $b \lor \bot = b \in A$.

It is worth noting that if we instead consider the complete lattice $D$ depicted on the right of the above figure, which includes a new glb $a_\omega$ of the chain $\{ a_i \}_{i \geq 0}$, then $D$ becomes a complete Heyting and co-Heyting algebra, and in this case the glb of $\{ X_i \}_{i \geq 0}$ in $\langle \text{SL}(D), \leq^v \rangle$ turns out to be $\{ \top \} \cup \{ a_i \}_{i \geq 0} \cup \{ a_\omega \}$.

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