

ENVIRONMENTAL BISIMULATIONS FOR DELIMITED-CONTROL OPERATORS WITH DYNAMIC PROMPT GENERATION

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ABSTRACT. We present sound and complete environmental bisimilarities for a variant of Dybvig et al.’s calculus of multi-prompted delimited-control operators with dynamic prompt generation. The reasoning principles that we obtain generalize and advance the existing techniques for establishing program equivalence in calculi with single-prompted delimited control.

The basic theory that we develop is presented using Madiot et al.’s framework that allows for smooth integration and composition of up-to techniques facilitating bisimulation proofs. We also generalize the framework in order to express environmental bisimulations that support equivalence proofs of evaluation contexts representing continuations. This change leads to a novel and powerful up-to technique enhancing bisimulation proofs in the presence of control operators.

1. INTRODUCTION

Control operators for delimited continuations, introduced independently by Felleisen [17] and by Danvy and Filinski [14], allow the programmer to delimit the current context of computation and to abstract such a delimited context as a first-class value. It has been shown that all computational effects are expressible in terms of delimited continuations [19], and so there exists a large body of work devoted to this canonical control structure, including our work on a theory of program equivalence for the operators `shift` and `reset` [8, 9, 10, 11].

In their paper on type-directed partial evaluation for typed λ -calculus with sums, Balat et al. [4] have demonstrated that Gunter et al.’s delimited-control operators `set` and `cupto` [21], that support multiple prompts along with dynamic prompt generation, can have a

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practical advantage over single-prompted operators such as `shift` and `reset`. Delimited-control operators with dynamically-generated prompts are now available in several production programming languages such as OCaml [26] and Racket [20], and they have been given formal semantic treatment in the literature. In particular, Dybvig et al. [16] have proposed a calculus that extends the call-by-value λ -calculus with several primitives that allow for: fresh-prompt generation, delimiting computations with a prompt, abstracting control up to the corresponding prompt, and throwing to captured continuations. Dybvig et al.’s building blocks were shown to be able to naturally express most of other existing control operators and as such they form a general framework for studying delimited continuations. Reasoning about program equivalence in Dybvig et al.’s calculus is considerably more challenging than in single-prompted calculi: one needs to reconcile control effects with the intricacies introduced by fresh-prompt generation and local visibility of prompts.

In this article we investigate the behavioral theory of a slightly modified version of Dybvig et al.’s calculus that we call the $\lambda_{G\#}$ -calculus. One of the most natural notions of program equivalence in languages based on the λ -calculus is *contextual equivalence*: two terms are contextually equivalent if we cannot distinguish them when evaluated within any context. The quantification over contexts makes this relation hard to use in practice, so it is common to characterize it using simpler relations, like coinductively defined *bisimilarities*. As pointed out in [28], among the existing notions of bisimilarities, *environmental bisimilarity* [36] is the most appropriate candidate to characterize contextual equivalence in a calculus with generated resources, such as prompts in $\lambda_{G\#}$. Indeed, this bisimilarity features an environment which accumulates knowledge about the terms we compare. This is crucial in our case to remember the relationships between the prompts generated by the compared programs. We therefore define environmental bisimilarities for $\lambda_{G\#}$, as well as *up-to techniques*, which simplify the equivalence proof of two given programs. We do so using the recently developed framework of Madiot et al. [31, 32], where it is simpler to prove that a bisimilarity and its up-to techniques are *sound* (i.e., imply contextual equivalence).

After presenting the syntax, semantics, and contextual equivalence of the calculus in Section 2, in Section 3 we define a sound and complete environmental bisimilarity and its corresponding up-to techniques. In particular, we define a bisimulation *up to context*, which allows to forget about a common context when comparing two terms in a bisimulation proof. The bisimilarity we define is useful enough to prove, e.g., the folklore theorem about delimited control [7] expressing that the static delimited-control operators `shift` and `reset` [14] can be simulated by the dynamic control operators `control` and `prompt` [17]. The technique, however, in general requires a cumbersome analysis of terms of the form $E[e]$, where E is a captured evaluation context and e is any expression (not necessarily a value). We therefore define in Section 4 a refined bisimilarity, called \star -bisimilarity, and a more expressive bisimulation up to context, which allows to factor out a context built with captured continuations. Proving the soundness of these two relations requires us to extend Madiot et al.’s framework. We show how these new techniques can be applied to `shift` and `reset` in Section 5, improving over the existing results for these operators [10, 11]. Finally, we discuss related work and conclude in Section 6.

This article is an extended version of [1]. Compared to that paper, in Section 3.1 we discuss in more detail the intricacies of our treatment of public/private prompts in the definition of the LTS, whereas Section 5 is new. An accompanying research report [2] contains the omitted proofs from Sections 3, 4, and 5.

2. THE CALCULUS $\lambda_{\mathcal{G}\#}$

The calculus we consider, called $\lambda_{\mathcal{G}\#}$, extends the call-by-value λ -calculus with four building blocks for constructing delimited-control operators as first proposed by Dybvig et al. [16].¹

Syntax. We assume we have a countably infinite set of term variables, ranged over by x, y, z , and k , as well as a countably infinite set of prompts, ranged over by p, q . Given an entity denoted by a meta-variable m , we write \vec{m} for a (possibly empty) sequence of such entities. Expressions (e), values (v), and evaluation contexts (E) are defined as follows:

$$\begin{aligned} e &::= v \mid e e \mid \mathcal{P}x.e \mid \#_v e \mid \mathcal{G}_v x.e \mid v \triangleleft e && \text{(expressions)} \\ v &::= x \mid \lambda x.e \mid p \mid \ulcorner E \urcorner && \text{(values)} \\ E &::= \square \mid E e \mid v E \mid \#_p E && \text{(evaluation contexts)} \end{aligned}$$

Values include captured evaluation contexts $\ulcorner E \urcorner$, representing delimited continuations, as well as generated prompts p . Expressions include the four building blocks for delimited control: $\mathcal{P}x.e$ is a prompt-generating construct, where x represents a fresh prompt locally visible in e , $\#_v e$ is a control delimiter for e , $\mathcal{G}_v x.e$ is a continuation grabbing or capturing construct, and $v \triangleleft e$ is a throw construct.

Evaluation contexts, in addition to the standard call-by-value contexts, include delimited contexts of the form $\#_p E$, and they are interpreted outside-in. We use the standard notation $E[e]$ ($E[E']$) for plugging a context E with an expression e (with a context E'). Evaluation contexts are a special case of (general) contexts, understood as a term with a hole and ranged over by C .

The expressions $\lambda x.e$, $\mathcal{P}x.e$, and $\mathcal{G}_v x.e$ bind x ; we adopt the standard conventions concerning α -equivalence. If x does not occur in e , we write $\lambda_.e$, $\mathcal{P}_.e$, and $\mathcal{G}_v_.e$. The set of free variables of e is written $\text{fv}(e)$; a term e is called closed if $\text{fv}(e) = \emptyset$. We extend these notions to evaluation contexts. A variable is called *fresh* if it is free for all the entities under consideration. We write $\#(e)$ (or $\#(E)$) for the set of all prompts that occur in e (or E respectively). The set $\text{sp}(E)$ of surrounding prompts in E is the set of all prompts guarding the hole in E , defined as $\{p \mid \exists E_1, E_2, E = E_1[\#_p E_2]\}$.

Reduction semantics. The reduction semantics of $\lambda_{\mathcal{G}\#}$ is given by the following rules:

$$\begin{array}{lcl} (\lambda x.e) v & \rightarrow & e\{v/x\} \\ \#_p v & \rightarrow & v \\ \#_p E[\mathcal{G}_p x.e] & \rightarrow & e\{\ulcorner E \urcorner/x\} \quad p \notin \text{sp}(E) \\ \ulcorner E \urcorner \triangleleft e & \rightarrow & E[e] \\ \mathcal{P}x.e & \rightarrow & e\{p/x\} \quad p \notin \#(e) \end{array} \quad \begin{array}{l} \text{COMPATIBILITY} \\ \frac{e_1 \rightarrow e_2 \quad \text{fresh}(e_2, e_1, E)}{E[e_1] \rightarrow E[e_2]} \end{array}$$

The first rule is the standard β_v -reduction. The second rule signals that a computation has been completed for a given prompt. The third rule abstracts the evaluation context up to the dynamically nearest control delimiter matching the prompt of the grab operator. In the fourth rule, an expression is thrown (plugged, really) to the captured context. Note that,

¹Dybvig et al.'s control operators slightly differ from their counterparts considered in this work, but they can be straightforwardly macro-expressed in the $\lambda_{\mathcal{G}\#}$ -calculus.

like in Dybvig et al.'s calculus, the expression e is not evaluated before the throw operation takes place. In the last rule, a prompt p is generated under the condition that it is fresh for e .

The compatibility rule needs a side condition, simply because a prompt that is fresh for e may not be fresh for a surrounding evaluation context. Given three entities m_1, m_2, m_3 for which $\#$ is defined, we write $\text{fresh}(m_1, m_2, m_3)$ for the condition $(\#(m_1) \setminus \#(m_2)) \cap \#(m_3) = \emptyset$, so the side condition states that E must not mention prompts generated in the reduction step $e_1 \rightarrow e_2$. This approach differs from the previous work on bisimulations for resource-generating constructs [30, 29, 38, 39, 40, 5, 33], where configurations of the operational semantics contain explicit information about the resources, typically represented by a set. We find our way of proceeding less invasive to the semantics of the calculus.

When reasoning about reductions in the $\lambda_{\mathcal{G}\#}$ -calculus, we rely on the notion of *permutation* (a bijection on prompts), ranged over by σ , which allows to reshuffle the prompts of an expression to avoid potential collisions: e with prompts permuted by σ is written $e\sigma$. E.g., we can use the first item of the following lemma before applying the compatibility rule, to be sure that any prompt generated by $e_1 \rightarrow e_2$ is not in $\#(E)$.

Lemma 2.1. *Let σ be a permutation.*

- If $e_1 \rightarrow e_2$ then $e_1\sigma \rightarrow e_2\sigma$.
- For any entities m_1, m_2, m_3 , we have $\text{fresh}(m_1, m_2, m_3)$ iff $\text{fresh}(m_1\sigma, m_2\sigma, m_3\sigma)$.

A closed term e either uniquely, up to permutation of prompts, reduces to a term e' , or it is a normal form (i.e., there is no e'' such that $e \rightarrow e''$). In the latter case, we distinguish values, control-stuck terms $E[\mathcal{G}_p k.e]$ where $p \notin \text{sp}(E)$, and the remaining expressions that we call errors (e.g., $E[pv]$ or $E[\mathcal{G}_{\lambda x.e} k.e']$). We write $e_1 \rightarrow^* e_2$ if e_1 reduces to e_2 in many (possibly 0) steps, and we write $e \hat{\rightarrow}$ when a term e diverges (i.e., there exists an infinite sequence of reductions starting with e) or when it reduces (in many steps) to an error.

Example 2.2. Let us assume that u, v , and w are values, e is an expression, p and q are two different prompts with q not occurring in u, v, w and e . Then the following reduction sequence illustrates how fresh prompts are generated (1), how delimited continuations are captured (2 and 4), and how expressions are thrown to captured continuations (3):

$$\mathcal{P}x.\#_x u (\#_p v (\mathcal{G}_x k.w (k \triangleleft (\mathcal{G}_{p-} e)))) \rightarrow \quad (1)$$

$$\#_q u (\#_p v (\mathcal{G}_q k.w (k \triangleleft (\mathcal{G}_{p-} e)))) \rightarrow \quad (2)$$

$$w (\ulcorner u (\#_p v \square) \urcorner \triangleleft (\mathcal{G}_{p-} e)) \rightarrow \quad (3)$$

$$w (u (\#_p v (\mathcal{G}_{p-} e))) \rightarrow \quad (4)$$

$$w (u e)$$

If we throw $\mathcal{G}_x e$ to k in the initial term instead of $\mathcal{G}_{p-} e$, then the reduction sequence would terminate with a control-stuck term $w (u (\#_p v (\mathcal{G}_{q-} e)))$ that could not be unstuck by any evaluation context. Indeed, if we plug the initial expression modified as suggested above, e.g., in the context $\#_q \square$, the compatibility rule requires that in step (1) the generated prompt q be renamed into some other prompt r that does not occur in the terms under consideration, and the corresponding reduction sequence terminates with a control-stuck term $\#_q w (u (\#_p v (\mathcal{G}_{r-} e)))$.

When presenting more complex examples, we use the fixed-point operator `fix`, `let-construct`, `conditional if` along with boolean values `true` and `false`, and sequencing `”;`, all defined as in the call-by-value λ -calculus. We also use the diverging term $\Omega \stackrel{\text{def}}{=} (\lambda x.x x)(\lambda x.x x)$, and we define an operator $\stackrel{?}{=}$ to test the equality between prompts, as follows:

$$e_1 \stackrel{?}{=} e_2 \stackrel{\text{def}}{=} \text{let } x = e_1 \text{ in let } y = e_2 \text{ in } \#_x((\#_y \mathcal{G}_x.\text{false});\text{true})$$

If e_1 and e_2 evaluate to different prompts, then the grab operator captures the context up to the outermost prompt to throw it away, and `false` is returned; otherwise, `true` is returned.

Contextual equivalence. We now define formally what it takes for two terms to be considered equivalent in the $\lambda_{\mathcal{G}\#}$ -calculus. First, we characterize when two closed expressions have equivalent observable actions in the calculus, by defining the following relation \sim .

Definition 2.3. We say e_1 and e_2 have equivalent observable actions, noted $e_1 \sim e_2$, if

- (1) $e_1 \rightarrow^* v_1$ iff $e_2 \rightarrow^* v_2$,
- (2) $e_1 \rightarrow^* E_1[\mathcal{G}_{p_1}x.e'_1]$ iff $e_2 \rightarrow^* E_2[\mathcal{G}_{p_2}x.e'_2]$, where $p_1 \notin \text{sp}(E_1)$ and $p_2 \notin \text{sp}(E_2)$,
- (3) $e_1 \hat{\rightarrow}$ iff $e_2 \hat{\rightarrow}$.

We can see that errors and divergence are treated as equivalent, which is standard.

Based on \sim , we define *contextual equivalence* as follows.

Definition 2.4 (Contextual equivalence). Two closed expressions e_1 and e_2 are contextually equivalent, written, $e_1 \equiv_E e_2$, if for all E such that $\#(E) = \emptyset$, we have $E[e_1] \sim E[e_2]$.

Contextual equivalence can be extended to open terms in a standard way: if $\text{fv}(e_1) \cup \text{fv}(e_2) \subseteq \vec{x}$, then $e_1 \equiv_E e_2$ if $\lambda \vec{x}.e_1 \equiv_E \lambda \vec{x}.e_2$. We test terms using only promptless contexts, because the testing context should not use prompts that are private for the tested expressions. For example, the expressions $\lambda f.f p q$ and $\lambda f.f q p$ should be considered equivalent if nothing is known from the outside about p and q . Prompts occur in expressions because we use reduction semantics, also known as syntactic theory. But prompts, just as closures or continuations, are runtime entities, and in other semantics formats such as, e.g., abstract machines [16], they would not be a part of the syntax.

As common in calculi with resource generation [39, 38, 36], testing with evaluation contexts (as in \equiv_E) is not the same as testing with all contexts: we have $\mathcal{P}x.x \equiv_E p$, but these terms can be distinguished by

$$\text{let } f = \lambda x.\square \text{ in if } f \lambda x.x \stackrel{?}{=} f \lambda x.x \text{ then } \Omega \text{ else } \lambda x.x$$

In the rest of the article, we show how to characterize \equiv_E with environmental bisimilarities.²

Remark 2.5. Definition 2.3 distinguishes control-stuck terms from errors, as making the distinction allows comparisons with the previous work on `shift` and `reset` [11], where a similar choice is made. However, unlike in [11], the contextual equivalence of the present article cannot “unstuck” a control-stuck term in $\lambda_{\mathcal{G}\#}$, as we consider promptless contexts, so it can be natural to treat stuck terms as errors. We explain how making this latter choice impacts the definitions of our bisimilarities in Remark 3.10 and Remark 4.8.

²If \equiv_C is the contextual equivalence testing with all contexts, then we can prove that $e_1 \equiv_C e_2$ iff $\lambda x.e_1 \equiv_E \lambda x.e_2$, where x is any variable. We therefore obtain a proof method for \equiv_C as well.

3. ENVIRONMENTAL BISIMILARITY

In this section, we propose a first characterization of \equiv_E using an environmental bisimilarity. We express the bisimilarity in the style of [31], using a so called first-order labeled transition system (LTS), to factorize the soundness proofs of the bisimilarity and its up-to techniques. We start by defining the LTS and its corresponding bisimilarity.

3.1. Labeled Transition System and Bisimilarity. In the original formulation of environmental bisimulation [36], two expressions e_1 and e_2 are compared under some environment \mathcal{E} , which represents the knowledge of an external observer about e_1 and e_2 . The definition of the bisimulation enforces some conditions on e_1 and e_2 as well as on \mathcal{E} . In Madiot et al.'s framework [31, 32], the conditions on e_1 , e_2 , and \mathcal{E} are expressed using a LTS between *states* of the form (Γ, e_1) (and (Δ, e_2)), where Γ (and Δ) is a finite sequence of values corresponding to the first (and second) projection of the environment \mathcal{E} . Note that in (Γ, e_1) , e_1 may be a value, and therefore a state can be simply of the form Γ . Transitions from states of the form (Γ, e_1) (where e_1 is not a value) express conditions on e_1 , while transitions from states of the form Γ explain how we compare environments. In the rest of the paper we use Γ , Δ to range over finite sequences of values, and we write Γ_i , Δ_i for the i^{th} element of the sequence. We use Σ , Θ to range over states.

Figure 1 presents the LTS $\xrightarrow{\alpha}$, where α ranges over all the labels. We define $\#(\Gamma)$ as $\bigcup_i \#(\Gamma_i)$. The transition $\xrightarrow{\mathbb{E}}$ uses a relation $e \xrightarrow{\mathbb{E}} e'$, defined as follows: if $e \rightarrow e'$, then $e \xrightarrow{\mathbb{E}} e'$, and if e is a normal form, then $e \xrightarrow{\mathbb{E}} e$.³ To build expressions out of sequences of values, we use different kinds of *multi-hole contexts* defined as follows.

$$\begin{aligned} \mathbb{C} &::= \mathbb{C}_v \mid \mathbb{C} \mathbb{C} \mid \mathcal{P}x.\mathbb{C} \mid \#_{\mathbb{C}_v} \mathbb{C} \mid \mathcal{G}_{\mathbb{C}_v} x.\mathbb{C} \mid \mathbb{C}_v \triangleleft \mathbb{C} && \text{(contexts)} \\ \mathbb{C}_v &::= x \mid \lambda x.\mathbb{C} \mid \ulcorner \mathbb{E} \urcorner \mid \square_i && \text{(value contexts)} \\ \mathbb{E} &::= \square \mid \mathbb{E} \mathbb{C} \mid \mathbb{C}_v \mathbb{E} \mid \#_{\square_i} \mathbb{E} && \text{(evaluation contexts)} \end{aligned}$$

The holes of a multi-hole context are indexed, except for the special hole \square of an evaluation context \mathbb{E} , which is in evaluation position (that is, filling the other holes of \mathbb{E} with values gives a regular evaluation context E). We write $\mathbb{C}[\Gamma]$ (respectively $\mathbb{C}_v[\Gamma]$ and $\mathbb{E}[\Gamma]$) for the application of a context \mathbb{C} (respectively \mathbb{C}_v and \mathbb{E}) to a sequence Γ of values, which consists in replacing \square_i with Γ_i ; we assume that this application produces an expression (or an evaluation context in the case of \mathbb{E}), i.e., each hole index in the context is smaller or equal than the size of Γ , and for each $\#_{\square_i} \mathbb{E}$ construct, Γ_i is a prompt. We write $\mathbb{E}[e, \Gamma]$ as a shorthand for $E[e]$ where $E = \mathbb{E}[\Gamma]$, meaning that e is put in the non-indexed hole of \mathbb{E} (note that e may also be a value). Notice that prompts are not part of the syntax of \mathbb{C}_v , therefore a multi-hole context does not contain any prompt: if $\mathbb{C}[\Gamma]$, $\mathbb{C}_v[\Gamma]$, or $\mathbb{E}[e, \Gamma]$ contains a prompt, then it comes from Γ or e . Our multi-hole contexts are promptless because \equiv_E also tests with promptless contexts.

We now detail the rules of Figure 1, starting with the transitions that one can find in any call-by-value λ -calculus [31]. An internal action $(\Gamma, e_1) \xrightarrow{\tau} \Sigma$ corresponds to a reduction step, except we ensure that any generated prompt is fresh w.r.t. Γ . The transition $\Gamma \xrightarrow{\lambda, i, \mathbb{C}_v} \Sigma$

³The relation $\xrightarrow{\mathbb{E}}$ is not exactly the reflexive closure of \rightarrow , since an expression which is not a normal form *has* to reduce.

$e_1 \rightarrow e_2$	$\text{fresh}(e_2, e_1, \Gamma)$	$\Gamma_i = \lambda x.e$	$\Gamma_i = \ulcorner E \urcorner$
$(\Gamma, e_1) \xrightarrow{\tau} (\Gamma, e_2)$	$\Gamma \xrightarrow{\lambda, i, \mathbb{C}_v} (\Gamma, e\{\mathbb{C}_v[\Gamma]/x\})$	$\Gamma \xrightarrow{v} \Gamma$	$\Gamma \xrightarrow{\ulcorner \cdot \urcorner, i, \mathbb{C}} (\Gamma, E[\mathbb{C}[\Gamma]])$
$\Gamma_i = p$	$\Gamma_j = p$	$p \notin \#(\Gamma)$	$p \notin \text{sp}(E)$
$\Gamma \xrightarrow{\#, i, j} \Gamma$	$\Gamma \xrightarrow{\#} (\Gamma, p)$	$\mathbb{E}[E[\mathcal{G}_p x.e], \Gamma] \xrightarrow{\#} e'$	

Figure 1: Labeled Transition System for $\lambda_{\mathcal{G}\#}$

signals that Γ_i is a λ -abstraction, which can be tested by passing it an argument built from Γ with the context \mathbb{C}_v . The transition $\xrightarrow{\ulcorner \cdot \urcorner, i, \mathbb{C}}$ for testing continuations is built the same way, except we use a context \mathbb{C} , because any expression can be thrown to a captured context. Finally, the transition $\Gamma \xrightarrow{v} \Gamma$ means that the state Γ is composed only of values; it does not test anything on Γ , but this transition is useful for the soundness proofs of Section 3.2. When we have $\Gamma \mathcal{R} (\Delta, e)$ (where \mathcal{R} is, e.g., a bisimulation), then (Δ, e) has to match with $(\Delta, e) \xrightarrow{\tau}^* \xrightarrow{v} (\Delta, v)$ so that (Δ, v) is related to Γ . We can then continue the proofs with two related sequences of values. Such a transition has been suggested in [32, Remark 5.3.6] to simplify the proofs for a non-deterministic language, like $\lambda_{\mathcal{G}\#}$.

We now explain the rules involving prompts. When comparing two terms generating prompts, one can produce p and the other a different q , so we remember in Γ, Δ that p corresponds to q . But an observer can compare prompts using $\stackrel{?}{=}$, so p has to be related *only* to q . We check it with $\xrightarrow{\#, i, j}$: if $\Gamma \xrightarrow{\#, i, j} \Gamma$, then Δ has to match, meaning that $\Delta_i = \Delta_j$, and doing so for all j such that $\Gamma_i = \Gamma_j$ ensures that all copies of Γ_i are related only to Δ_i . The transition $\xrightarrow{\#, i, i}$ also signals that Γ_i is a prompt and should be related to a prompt. The other transition involving prompts is $\Gamma \xrightarrow{\#} (\Gamma, p)$, which encodes the possibility for an observer to generate fresh prompts to compare terms. If Γ is related to Δ , then Δ has to match by generating a prompt q , and we remember that p is related to q . For this rule to be automatically verified, we define the *prompt checking* rule for a relation \mathcal{R} as follows:

$$\frac{\Gamma \mathcal{R} \Delta \quad p \notin \#(\Gamma) \quad q \notin \#(\Delta)}{(\Gamma, p) \mathcal{R} (\Delta, q)} \text{ (\#-check)}$$

Henceforth, when we construct a bisimulation \mathcal{R} by giving a set of rules, we always include the (#-check) rule so that the $\xrightarrow{\#}$ transition is always verified.

Finally, the transition $\xrightarrow{\mathbb{E}}$ deals with stuck terms. An expression $E[\mathcal{G}_p x.e]$ is able to reduce if the surrounding context is able to provide a delimiter $\#_p$. However, it is possible only if p is available for the outside, and therefore is in Γ . If $p \notin \text{sp}(\mathbb{E}[\Gamma])$, then $\mathbb{E}[E[\mathcal{G}_p x.e], \Gamma]$ remains stuck, and we have $\mathbb{E}[E[\mathcal{G}_p x.e], \Gamma] \xrightarrow{\#} \mathbb{E}[E[\mathcal{G}_p x.e], \Gamma]$. Otherwise, it can reduce and we have $\mathbb{E}[E[\mathcal{G}_p x.e], \Gamma] \xrightarrow{\#} e'$, where e' is the result after the capture. The rule for $\xrightarrow{\mathbb{E}}$ may seem demanding, as it tests stuck terms with all contexts \mathbb{E} , but up-to techniques will alleviate this issue (see Example 3.8). Besides, we believe testing all contexts is necessary to be sound and complete w.r.t. contextual equivalence. Inspired by the previous work on shift and reset [10, 11], one could propose the following rule

$$\frac{p \notin \text{sp}(E) \quad \Gamma_i = p \quad \#_p \mathbb{E}[E[\mathcal{G}_p x.e], \Gamma] \rightarrow e'}{(\Gamma, E[\mathcal{G}_p x.e]) \xrightarrow{\#_{\square_i} \mathbb{E}} (\Gamma, e')} \quad (*)$$

which tests stuck terms with context of the form $\#_p \mathbb{E}$, and only if p is in Γ . This rule alone is not sound, as it would relate (\emptyset, Ω) and $(\emptyset, E[\mathcal{G}_p x.e])$, because p does not occur in the environment. We could retrieve soundness by simply adding a rule which tests if an expression is control-stuck, to deal with this kind of situation. However, the rule $(*)$ is also too discriminating and would break completeness, as we can see with the next two examples.

Example 3.1. Stuck terms may be equivalent, even though the prompts they use are not related in Γ, Δ . For example, consider $(p_1, \text{fix } x.\mathcal{G}_{p_1} y.x)$ and $(p_2, \mathcal{G}_q y.e)$, where $p_2 \neq q$ and e is any expression. Because we can use p_1 to build testing contexts, we can trigger the capture for the first term. By doing so, we make it reduce to itself, while the second term remains stuck in any context. We can prove them bisimilar with the rules of Figure 1. In contrast, $(p_2, \mathcal{G}_q y.e)$ cannot make a transition with rule $(*)$ (because $q \neq p_2$) while $(p_1, \text{fix } x.\mathcal{G}_{p_1} y.x)$ can, so rule $(*)$ would wrongfully distinguish these two expressions.

Example 3.2. Assuming $p \neq q$, the expression $e_1 \stackrel{\text{def}}{=} \mathcal{G}_q \text{--} \mathcal{G}_p \text{--} v$ aborts the current continuation up to the first enclosing delimiter $\#_p$ which is behind a delimiter $\#_q$, and then returns v . The term $e_2 \stackrel{\text{def}}{=} \text{fix } x.\mathcal{G}_p k.\text{if } q \in \text{sp}(k) \text{ then } v \text{ else } x$ has the same behavior: it decomposes the continuation piece by piece, repeatedly capturing k up to $\#_p$, until it finds $\#_q$ in k . Testing if $q \in \text{sp}(k)$ can be implemented in a similar way as testing prompt equality: $q \in \text{sp}(k) \stackrel{\text{def}}{=} \mathcal{P}x.\#_x \#_q((\#_x(k \triangleleft \mathcal{G}_q \text{--} \mathcal{G}_x \text{--} \text{false})); \text{true})$. Again, the rule $(*)$ wrongfully distinguishes (p, q, e_1) and (p, q, e_2) , because e_1 captures on q first while e_2 captures on p .

For weak transitions, we define \Rightarrow as $\xrightarrow{\tau^*}, \xRightarrow{\alpha}$ as \Rightarrow if $\alpha = \tau$ and as $\Rightarrow \xrightarrow{\alpha} \Rightarrow$ otherwise. We define bisimulation and bisimilarity using a more general notion of *progress*. Henceforth, we let \mathcal{R}, \mathcal{S} range over relations on states.

Definition 3.3. A relation \mathcal{R} progresses to \mathcal{S} , written $\mathcal{R} \succ \mathcal{S}$, if $\mathcal{R} \subseteq \mathcal{S}$ and $\Sigma \mathcal{R} \Theta$ implies

- if $\Sigma \xrightarrow{\alpha} \Sigma'$, then there exists Θ' such that $\Theta \xRightarrow{\alpha} \Theta'$ and $\Sigma' \mathcal{S} \Theta'$;
- the converse of the above condition on Θ .

A *bisimulation* is a relation \mathcal{R} such that $\mathcal{R} \succ \mathcal{R}$, and *bisimilarity* \approx is the union of all bisimulations.

3.2. Up-to Techniques, Soundness, and Completeness. Before defining the up-to techniques for $\lambda_{\mathcal{G}\#}$, we briefly recall the main definitions and results we use from [35, 31, 32]; see these works for more details. We use f, g to range over functions on relations on states. An *up-to technique* is a function f such that $\mathcal{R} \succ f(\mathcal{R})$ implies $\mathcal{R} \subseteq \approx$. However, this definition is difficult to use to prove that a given f is an up-to technique, so we rely on *compatibility* instead, which gives sufficient conditions for f to be an up-to technique.

We first define some auxiliary notions and notations. We write $f \subseteq g$ if $f(\mathcal{R}) \subseteq g(\mathcal{R})$ for all \mathcal{R} . We define $f \cup g$ argument-wise, i.e., $(f \cup g)(\mathcal{R}) = f(\mathcal{R}) \cup g(\mathcal{R})$, and given a set \mathfrak{F} of functions, we also write \mathfrak{F} for the function defined as $\bigcup_{f \in \mathfrak{F}} f$. We define f^ω as $\bigcup_{n \in \mathbb{N}} f^n$. We write id for the identity function on relations, and \widehat{f} for $f \cup \text{id}$. A function f is monotone if $\mathcal{R} \subseteq \mathcal{S}$ implies $f(\mathcal{R}) \subseteq f(\mathcal{S})$. We write $\mathcal{P}_{\text{fin}}(\mathcal{R})$ for the set of finite subsets

of \mathcal{R} , and we say f is continuous if it can be defined by its image on these finite subsets, i.e., if $f(\mathcal{R}) \subseteq \bigcup_{\mathcal{S} \in \mathcal{P}_{fin}(\mathcal{R})} f(\mathcal{S})$. The up-to techniques of the present paper are defined by inference rules with a finite number of premises, so they are trivially continuous. Continuous functions are interesting because of their properties:⁴

Lemma 3.4. *If f and g are continuous, then $f \circ g$ and $f \cup g$ are continuous.*

If f is continuous, then f is monotone, and $f \circ \widehat{f}^\omega \subseteq \widehat{f}^\omega$.

Definition 3.5. A function f evolves to g , written $f \rightsquigarrow g$, if for all $\mathcal{R} \rightsquigarrow \mathcal{S}$, we have $f(\mathcal{R}) \rightsquigarrow g(\mathcal{S})$. A set \mathfrak{F} of continuous functions is compatible if for all $f \in \mathfrak{F}$, $f \rightsquigarrow \widehat{\mathfrak{F}}^\omega$.

Lemma 3.6. *Let \mathfrak{F} be a compatible set, and $f \in \mathfrak{F}$; f is an up-to technique, and $f(\approx) \subseteq \approx$.*

Proving that f is in a compatible set \mathfrak{F} is easier than proving it is an up-to technique, because we just have to prove that it evolves towards a combination of functions in \mathfrak{F} . Besides, the second property of Lemma 3.6 can be used to prove that \approx is a congruence just by showing that bisimulation up to context is compatible.

The first technique we define allows to forget about prompt names; in a bisimulation relating (Γ, e_1) and (Δ, e_2) , we remember that $\Gamma_i = p$ is related to $\Delta_i = q$ by their position i , not by their names. Consequently, we can apply different permutations to the two states to rename the prompts without harm, and bisimulation up to permutations⁵ allows us to do so. It is reminiscent of bisimulation up to renaming [38], which operates on reference names. Given a relation \mathcal{R} , we define $\text{perm}(\mathcal{R})$ as $\Sigma \sigma_1 \text{perm}(\mathcal{R}) \Theta \sigma_2$, assuming $\Sigma \mathcal{R} \Theta$ and σ_1, σ_2 are any permutations.

We then allow to remove or add values from the states with, respectively, bisimulation up to weakening weak and bisimulation up to strengthening str , defined as follows

$$\frac{(\vec{v}, \Gamma, e_1) \mathcal{R} (\vec{w}, \Delta, e_2)}{(\Gamma, e_1) \text{weak}(\mathcal{R}) (\Delta, e_2)} \qquad \frac{(\Gamma, e_1) \mathcal{R} (\Delta, e_2)}{(\Gamma, \mathbb{C}_v[\Gamma], e_1) \text{str}(\mathcal{R}) (\Delta, \mathbb{C}_v[\Delta], e_2)}$$

Bisimulation up to weakening diminishes the testing power of states, since less values means less arguments to build from for the transitions $\xrightarrow{\lambda, i, \mathbb{C}_v}$, $\xrightarrow{\ulcorner \cdot \urcorner, i, \mathbb{C}}$, and $\xrightarrow{\mathbb{E}}$. This up-to technique is usual for environmental bisimulations, and is called “up to environment” in [36]. In contrast, str adds values to the state, but without affecting the testing power, since the added values are built from the ones already in Γ, Δ .

Finally, we define the well-known bisimulation up to context, which allows to factor out a common context when comparing terms. As usual for environmental bisimulations [36], we define two kinds of bisimulation up to context, depending whether we operate on values or any expressions. For values, we can factor out any common context \mathbb{C} , but for expressions that are not values, we can factor out only an evaluation context \mathbb{E} , since factoring out any context in that case would lead to an unsound up-to technique [32]. We define up to context for values ctx and for any expression ectx as follows:

⁴Unlike in [32], we use \widehat{f} instead of f in the last property of Lemma 3.4 (expressing idempotence of \widehat{f}^ω), as id has to be factored in somehow for the property to hold.

⁵Madiot defines a bisimulation “up to permutation” in [32] which reorders values in a state. Our bisimulation up to permutations operates on prompts.

$$\frac{\Gamma \mathcal{R} \Delta}{(\Gamma, \mathbb{C}[\Gamma]) \text{ctx}(\mathcal{R}) (\Delta, \mathbb{C}[\Delta])} \qquad \frac{(\Gamma, e_1) \mathcal{R} (\Delta, e_2)}{(\Gamma, \mathbb{E}[e_1, \Gamma]) \text{ectx}(\mathcal{R}) (\Delta, \mathbb{E}[e_2, \Delta])}$$

Lemma 3.7. *The set {perm, weak, str, ctx, ectx} is compatible.*

The function `ectx` particularly helps in dealing with stuck terms, as we can see below.

Example 3.8. Let $\Sigma \stackrel{\text{def}}{=} (\Gamma, \mathcal{G}_p x.e_1)$ and $\Theta \stackrel{\text{def}}{=} (\Delta, \mathcal{G}_q x.e_2)$ (for some e_1, e_2), so that $\Sigma \mathcal{R} \Theta$. If p and q are not in Γ, Δ , then the two expressions remain stuck, as we have $\Sigma \xrightarrow{\mathbb{E}} (\Gamma, \mathbb{E}[\mathcal{G}_p x.e_1, \Gamma])$ and similarly for Θ . We have directly $(\Gamma, \mathbb{E}[\mathcal{G}_p x.e_1, \Gamma]) \text{ectx}(\mathcal{R}) (\Delta, \mathbb{E}[\mathcal{G}_q x.e_2, \Delta])$. Otherwise, the capture can be triggered with a context \mathbb{E} of the form $\mathbb{E}_1[\#_{\square_i} \mathbb{E}_2]$, giving $\Sigma \xrightarrow{\mathbb{E}} (\Gamma, \mathbb{E}_1[e_1\{\ulcorner \mathbb{E}_2[\Gamma] \urcorner / x\}, \Gamma])$ and $\Theta \xrightarrow{\mathbb{E}} (\Delta, \mathbb{E}_1[e_2\{\ulcorner \mathbb{E}_2[\Delta] \urcorner / x\}, \Delta])$. Thanks to `ectx`, we can forget about \mathbb{E}_1 which does not play any role, and continue the bisimulation proof by focusing only on $(\Gamma, e_1\{\ulcorner \mathbb{E}_2[\Gamma] \urcorner / x\})$ and $(\Delta, e_2\{\ulcorner \mathbb{E}_2[\Delta] \urcorner / x\})$.

Because bisimulation up to context is compatible, Lemma 3.6 ensures that \approx is a congruence w.r.t. all contexts for values, and w.r.t. evaluation contexts for all expressions. As a corollary, we can deduce that \approx is sound w.r.t. \equiv_E ; we can also prove that it is complete w.r.t. \equiv_E , leading to the following full characterization result.

Theorem 3.9. $e_1 \equiv_E e_2$ iff $(\emptyset, e_1) \approx (\emptyset, e_2)$.

For completeness, we prove that $\{(\Gamma, e_1), (\Delta, e_2) \mid \forall \mathbb{E}, \mathbb{E}[e_1, \Gamma] \sim \mathbb{E}[e_2, \Delta]\}$ is a bisimulation up to permutation.

Remark 3.10. If we consider control-stuck terms as errors, as suggested in Remark 2.5, then a control-stuck term that cannot be unstuck can be related to a term that reduces to another kind of error or that diverges. To take this into account, we would change the transition $\xrightarrow{\mathbb{E}}$ as follows:

$$\frac{\mathbb{E}[e, \Gamma] \xrightarrow{\mathbb{E}}_G e'}{(\Gamma, e) \xrightarrow{\mathbb{E}} (\Gamma, e')}$$

where the relation $\xrightarrow{\mathbb{E}}_G$ differs from $\xrightarrow{\mathbb{E}}$ in that it enforces only the continuation-grabbing reduction step, if possible. Note that this transition is useless when comparing two states (Γ, e_1) and (Δ, e_2) where neither e_1 nor e_2 is stuck, but in that case, we obtain $(\Gamma, \mathbb{E}[e_1, \Gamma])$ and $(\Delta, \mathbb{E}[e_2, \Delta])$, which are directly related by `ectx`.

With such a change, the results of this section remain valid with respect to the notion of contextual equivalence defined in Remark 2.5. The proofs are almost the same, since the extra cases involving the $\xrightarrow{\mathbb{E}}$ transition applied to expressions that are not control-stuck can be dealt with using `ectx`, as explained above.

3.3. Example. As an example, we show a folklore theorem about delimited control [7], stating that the static operators `shift` and `reset` can be simulated by the dynamic operators `control` and `prompt`. In fact, what we prove is a more general and stronger result than the original theorem, since we demonstrate that this simulation still holds when multiple prompts are around.

Example 3.11 (Folklore theorem). We encode `shift`, `reset`, `control`, and `prompt` as follows

$$\begin{array}{ll} \text{shift}_p & \stackrel{\text{def}}{=} \lambda f. \mathcal{G}_p k. \#_p f(\lambda y. \#_p k \triangleleft y) & \text{control}_p & \stackrel{\text{def}}{=} \lambda f. \mathcal{G}_p k. \#_p f(\lambda y. k \triangleleft y) \\ \text{reset}_p & \stackrel{\text{def}}{=} \ulcorner \#_p \square \urcorner & \text{prompt}_p & \stackrel{\text{def}}{=} \ulcorner \#_p \square \urcorner \end{array}$$

Let $\text{shift}'_p \stackrel{\text{def}}{=} \lambda f. \text{control}_p (\lambda l. f (\lambda z. \text{prompt}_p \triangleleft l z))$; we prove that the pair $(\text{shift}_p, \text{reset}_p)$ (encoded as $\lambda f. f \text{shift}_p \text{reset}_p$) is bisimilar to $(\text{shift}'_p, \text{prompt}_p)$ (encoded as $\lambda f. f \text{shift}'_p \text{prompt}_p$).

Proof. We iteratively build a relation \mathcal{R} closed under ($\#$ -check) such that \mathcal{R} is a bisimulation up to context, starting with $(p, \text{shift}_p) \mathcal{R} (p, \text{shift}'_p)$. The transition $\xrightarrow{\#,1,1}$ is easy to check. For $\xrightarrow{\lambda,2,\mathbb{C}_v}$, we obtain states of the form (p, shift_p, e_1) , $(p, \text{shift}'_p, e_2)$ that we add to \mathcal{R} , where e_1 and e_2 are the terms below

$$\frac{\Gamma \mathcal{R} \Delta}{(\Gamma, \mathcal{G}_p k. \#_p \mathbb{C}_v[\Gamma] (\lambda y. \#_p k \triangleleft y)) \mathcal{R} (\Delta, \mathcal{G}_p k. \#_p (\lambda l. \mathbb{C}_v[\Delta] (\lambda z. \text{prompt}_p \triangleleft l z)) (\lambda y. k \triangleleft y))}$$

We use an inductive, more general rule, because we want $\xrightarrow{\lambda,2,\mathbb{C}_v}$ to be still verified after we extend (p, shift_p) and (p, shift'_p) . The terms e_1 and e_2 are stuck, so we test them with $\xrightarrow{\mathbb{E}}$. If \mathbb{E} does not trigger the capture, we obtain $\mathbb{E}[e_1, \Gamma]$ and $\mathbb{E}[e_2, \Delta]$, and we can use `ctx` to conclude. Otherwise, $\mathbb{E} = \mathbb{E}'[\#_{\square_1} \mathbb{E}']$ (where $\#_{\square_1}$ does not surround \square in \mathbb{E}''), and we get

$$\mathbb{E}'[\#_p \mathbb{C}_v[\Gamma] (\lambda y. \#_p \ulcorner \mathbb{E}''[\Gamma] \urcorner \triangleleft y), \Gamma] \text{ and } \mathbb{E}'[\#_p \mathbb{C}_v[\Delta] (\lambda z. \text{prompt}_p \triangleleft (\lambda y. \ulcorner \mathbb{E}''[\Delta] \urcorner \triangleleft y) z), \Delta]$$

We want to use `ctx` to remove the common context $\mathbb{E}'[\#_{\square_1} \mathbb{C}_v \square_i]$, which means that we have to add the following states in the definition of \mathcal{R} (again, inductively):

$$\frac{\Gamma \mathcal{R} \Delta}{(\Gamma, \lambda y. \#_p \ulcorner \mathbb{E}''[\Gamma] \urcorner \triangleleft y) \mathcal{R} (\Delta, \lambda z. \text{prompt}_p \triangleleft (\lambda y. \ulcorner \mathbb{E}''[\Delta] \urcorner \triangleleft y) z)}$$

Testing these functions with $\xrightarrow{\lambda,i,\mathbb{C}_v}$ gives on both sides states where $\#_{\square_1} \mathbb{E}''[\mathbb{C}_v]$ can be removed with `ctx`. Because $(\emptyset, \lambda f. f \text{shift}_p \text{reset}_p) \text{weak}(\text{ctx}(\mathcal{R})) (\emptyset, \lambda f. f \text{shift}'_p \text{prompt}_p)$, it is enough to conclude. Indeed, \mathcal{R} is a bisimulation up to context, so $\mathcal{R} \subseteq \approx$, which implies $\text{weak}(\text{ctx}(\mathcal{R})) \subseteq \text{weak}(\text{ctx}(\approx))$ (because `weak` and `ctx` are monotone), and $\text{weak}(\text{ctx}(\approx)) \subseteq \approx$ (by Lemma 3.6). Note that this reasoning works for any combination of monotone up-to techniques and any bisimulation (up-to). \square

What makes the proof of Example 3.11 quite simple is that we relate (p, shift_p) and (p, shift'_p) , meaning that p can be used by an outside observer. But the control operators $(\text{shift}_p, \text{reset}_p)$ and $(\text{shift}'_p, \text{prompt}_p)$ should be the only terms available for the outside, since p is used only to implement them. If we try to prove equivalent these two pairs directly, i.e., while keeping p private, then testing `reset` _{p} and `prompt` _{p} with $\xrightarrow{\ulcorner \cdot \urcorner, 2, \mathbb{C}}$ requires a cumbersome analysis of the behaviors of $\#_p \mathbb{C}[\Gamma]$ and $\#_p \mathbb{C}[\Delta]$. In the next section, we define a new kind of bisimilarity with a powerful up-to technique to make such proofs more tractable.

4. THE \star -BISIMILARITY

In this section we develop a refined version of bisimilarity along with a powerful up to context technique for the $\lambda_{\mathcal{G}\#}$ -calculus that relies on testing captured continuations with values only, instead of with arbitrary expressions. In order to account for such an enhancement we generalize Madiot's framework.

4.1. Motivation. Let us start with identifying some drawbacks of the existing environmental bisimulation techniques for control operators, such as the one of Section 3 and the ones of [10, 11], in the way captured contexts are tested and exploited.

Testing continuations. In Section 3, a continuation $\Gamma_i = \ulcorner E \urcorner$ is tested with $\Gamma \xrightarrow{\ulcorner \cdot \urcorner, i, \mathbb{C}} (\Gamma, E[\mathbb{C}[\Gamma]])$. We then have to study the behavior of $E[\mathbb{C}[\Gamma]]$, which depends primarily on how $\mathbb{C}[\Gamma]$ reduces; e.g., if $\mathbb{C}[\Gamma]$ diverges, then E does not play any role. Consequently, the transition $\xrightarrow{\ulcorner \cdot \urcorner, i, \mathbb{C}}$ does not really test the continuation directly, since we have to reduce $\mathbb{C}[\Gamma]$ first. To really exhibit the behavior of the continuation, we change the transition so that it uses a value context instead of a general one. We then have $\Gamma \xrightarrow{\ulcorner \cdot \urcorner, i, \mathbb{C}_v} (\Gamma, E[\mathbb{C}_v[\Gamma]])$, and the behavior of the term we obtain depends primarily on E . However, this is not equivalent to testing with \mathbb{C} , since $\mathbb{C}[\Gamma]$ may interact in other ways with E if $\mathbb{C}[\Gamma]$ is a stuck term. If E is of the form $E'[\#_p E'']$ with $p \notin \text{sp}(E'')$, and p is in Γ , then \mathbb{C} may capture E'' , since p can be used to build an expression of the form $\mathcal{G}_p x.e$. To take into account this possibility, we introduce a new transition $\Gamma \xrightarrow{\ulcorner \cdot \urcorner, i, j} (\Gamma, \ulcorner E' \urcorner, \ulcorner E'' \urcorner)$, which decomposes $\Gamma_i = E'[\#_p E'']$ into $\ulcorner E' \urcorner$ and $\ulcorner E'' \urcorner$, provided $\Gamma_j = p$. The stuck term $\mathbb{C}[\Gamma]$ may also capture E entirely, as part of a bigger context of the form $\mathbb{E}_1[E[\mathbb{E}_2]]$. To take this into account, we introduce a way to build such contexts using captured continuations. This is also useful to make bisimulation up to context more expressive, as we explain in the next paragraph.

A more expressive bisimulation up to context. As we already pointed out in [10, 11], bisimulation up to context is not very helpful in the presence of control operators. For example, suppose we prove the β_Ω axiom of [25], i.e., $(\lambda x.E[x]) e$ is equivalent to $E[e]$ if $x \notin \text{fv}(E)$ and $\text{sp}(E) = \emptyset$. If e is a stuck term $\mathcal{G}_p y.e_1$, we have to compare $e_1\{\ulcorner E_1[(\lambda x.E[x]) \square] \urcorner / y\}$ and $e_1\{\ulcorner E_1[E] \urcorner / y\}$ for some E_1 . If e_1 is of the form $y \triangleleft (y \triangleleft e_2)$, then we get respectively

$$E_1[(\lambda x.E[x]) E_1[(\lambda x.E[x]) e_2]] \text{ and } E_1[E[E_1[E[e_2]]]].$$

We can see that the two resulting expressions have the same shape, and yet we can only remove the outermost occurrence of E_1 with ectx . The problem is that bisimulation up to context can factor out only a *common* context. We want an up-to technique able to identify *related* contexts, i.e., contexts built out of related continuations. To do so, we modify the multi-hole contexts to include a construct $\star_i[\mathbb{C}]$ with a special hole \star_i , which can be filled only with $\ulcorner E \urcorner$ to produce a context $E[\mathbb{C}]$. As a result, if $\Gamma = (\ulcorner \lambda x.E[x] \urcorner \square \urcorner)$ and $\Delta = (\ulcorner E \urcorner)$, then $E_1[(\lambda x.E[x]) E_1[(\lambda x.E[x]) \square]]$ and $E_1[E[E_1[E[\square]]]]$ can be written $\mathbb{E}[\Gamma]$, $\mathbb{E}[\Delta]$ with $\mathbb{E} = E_1[\star_1[E_1[\star_1[\square]]]]$. We can then focus only on testing Γ and Δ .

However, such a bisimulation up to related context would be unsound if not restricted in some way. Indeed, let $\ulcorner E_1 \urcorner, \ulcorner E_2 \urcorner$ be any continuations, and let $\Gamma = (\ulcorner E_1 \urcorner)$, $\Delta = (\ulcorner E_2 \urcorner)$. Then the transitions $\Gamma \xrightarrow{\ulcorner \cdot \urcorner, 1, \mathbb{C}_v} (\Gamma, E_1[\mathbb{C}_v[\Gamma]])$ and $\Delta \xrightarrow{\ulcorner \cdot \urcorner, 1, \mathbb{C}_v} (\Delta, E_2[\mathbb{C}_v[\Delta]])$ produce states

of the form $(\Gamma, \mathbb{C}[\Gamma])$, $(\Delta, \mathbb{C}[\Delta])$ with $\mathbb{C} = \star_1[\mathbb{C}_v]$. If bisimulation up to related context was sound in that case, it would mean that $\ulcorner E_1 \urcorner$ and $\ulcorner E_2 \urcorner$ would be bisimilar for all E_1 and E_2 , which, of course, is wrong.⁶ To prevent this, we distinguish *passive* transitions (such as $\xrightarrow{\ulcorner \cdot \urcorner, i, \mathbb{C}_v}$) from the other ones (called *active*), so that only selected up-to techniques (referred to as *strong*) can be used after a passive transition. In contrast, any up-to technique (including this new bisimulation up to related context) can be used after an active transition. To formalize this idea, we have to extend Madiot et al.’s framework to allow such distinctions between transitions and between up-to techniques.

4.2. Labeled Transition System and Bisimilarity. First, we explain how we alter the LTS of Section 3.1 to implement the changes we sketched in Section 4.1. We extend the grammar of multi-hole contexts \mathbb{C} (resp. \mathbb{E}) as follows:

$$\begin{aligned} \mathbb{C} &::= \mathbb{C}_v \mid \mathbb{C}\mathbb{C} \mid \mathcal{P}x.\mathbb{C} \mid \#_{\mathbb{C}_v}\mathbb{C} \mid \mathcal{G}_{\mathbb{C}_v}x.\mathbb{C} \mid \mathbb{C}_v \triangleleft \mathbb{C} \mid \star_i[\mathbb{C}] && \text{(contexts)} \\ \mathbb{E} &::= \square \mid \mathbb{E}\mathbb{C} \mid \mathbb{C}_v\mathbb{E} \mid \#_{\square_i}\mathbb{E} \mid \star_i[\mathbb{E}] && \text{(evaluation contexts)} \end{aligned}$$

The grammar of value contexts \mathbb{C}_v is unchanged. The hole \star_i can be filled only with a continuation; when we write $(\star_i[\mathbb{C}])[\Gamma]$, we assume Γ_i is a continuation $\ulcorner E \urcorner$, and the result of the operation is $E[\mathbb{C}[\Gamma]]$ (and similarly for \mathbb{E}).

We also change the way we deal with captured contexts, by replacing the rule for $\xrightarrow{\ulcorner \cdot \urcorner, i, \mathbb{C}}$ with the two following rules—we otherwise keep unchanged the other transitions of Figure 1:

$$\frac{\Gamma_i = \ulcorner E \urcorner}{\Gamma \xrightarrow{\ulcorner \cdot \urcorner, i, \mathbb{C}_v} (\Gamma, E[\mathbb{C}_v[\Gamma]])} \qquad \frac{\Gamma_i = \ulcorner E_1[\#_p E_2] \urcorner \quad \Gamma_j = p \quad p \notin \text{sp}(E_2)}{\Gamma \xrightarrow{\ulcorner \cdot \urcorner, i, j} (\Gamma, \ulcorner E_1 \urcorner, \ulcorner E_2 \urcorner)}$$

The transition $\xrightarrow{\ulcorner \cdot \urcorner, i, \mathbb{C}_v}$ is the same as in Section 3, except that it tests with an argument built with a value context \mathbb{C}_v instead of a regular context \mathbb{C} . We also introduce the transition $\xrightarrow{\ulcorner \cdot \urcorner, i, j}$, which decomposes a captured context $\ulcorner E_1[\#_p E_2] \urcorner$ into sub-contexts $\ulcorner E_1 \urcorner$, $\ulcorner E_2 \urcorner$, provided that p is in Γ . This transition is necessary to take into account the possibility for an external observer to capture a part of a context, scenario which can no longer be tested with $\xrightarrow{\ulcorner \cdot \urcorner, i, \mathbb{C}_v}$, as explained in Section 4.1, and as illustrated with the next example.

Example 4.1. Let $\Gamma = (p, \ulcorner \#_p \square \urcorner)$, $\Delta = (q, \ulcorner \square \urcorner)$; then $\Gamma \xrightarrow{\ulcorner \cdot \urcorner, 2, \mathbb{C}_v} (\Gamma, \#_p \mathbb{C}_v[\Gamma]) \xrightarrow{\tau} (\Gamma, \mathbb{C}_v[\Gamma])$ and $\Delta \xrightarrow{\ulcorner \cdot \urcorner, 2, \mathbb{C}_v} (\Delta, \mathbb{C}_v[\Delta])$. Without the $\xrightarrow{\ulcorner \cdot \urcorner, i, j}$ transition, Γ and Δ would be bisimilar, which would not be sound (they are distinguished by the context $\square_2 \triangleleft \mathcal{G}_{\square_1} x.\Omega$).

The other rules are not modified, but their meaning is still affected by the change in the contexts grammars: the transitions $\xrightarrow{\lambda, i, \mathbb{C}_v}$ and $\xrightarrow{\mathbb{E}}$ can now test with more arguments. This is a consequence of the fact that an observer can build a bigger continuation from a captured context. For instance, if $\Gamma = (p, \ulcorner E \urcorner, \lambda x.x \triangleleft v)$, then with the LTS of Section 3, we have $\Gamma \xrightarrow{\ulcorner \cdot \urcorner, 2, \mathbb{E}_1[\mathcal{G}_{\square_1} x.x]} \xrightarrow{\#_{\square_1} \mathbb{E}_2} \xrightarrow{\lambda, 3, \square_4} (\Gamma, \ulcorner \mathbb{E}_1[E[\mathbb{E}_2[\Gamma]]], \Gamma \urcorner, \ulcorner \mathbb{E}_1[E[\mathbb{E}_2[\Gamma]]], \Gamma \urcorner \triangleleft v)$. In the new LTS, the first transition is no longer possible, but we can still test the λ -abstraction with the same argument using $\Gamma \xrightarrow{\lambda, 3, \mathbb{E}_1[\star_2[\mathbb{E}_2]]} (\Gamma, \ulcorner \mathbb{E}_1[E[\mathbb{E}_2[\Gamma]]], \Gamma \urcorner \triangleleft v)$.

⁶The problem is similar if we test continuations using contexts \mathbb{C} (as in Section 3) instead of \mathbb{C}_v .

As explained in Section 4.1, we want to prevent the use of some up-to techniques (like the bisimulation up to related context we introduce in Section 4.3) after some transitions, especially $\xrightarrow{\ulcorner \cdot \urcorner, i, \mathbb{C}_v}$. To do so, we distinguish the *passive* transitions $\xrightarrow{\ulcorner \cdot \urcorner, i, \mathbb{C}_v}, \xrightarrow{v}$ from the other ones, called *active*. A passive transition $\Sigma_1 \xrightarrow{\alpha} \Sigma_2$ can be inverted by an up-to technique, which is possible if no new information is generated between the states Σ_1 and Σ_2 . For example, the transition $\Gamma \xrightarrow{v} \Gamma$ is passive, as we already know that Γ is composed only of values. In contrast, the transition $\Gamma \xrightarrow{\# \cdot, i, j} \Gamma$ is active, as we gain some information: the prompts Γ_i and Γ_j are equal. The transition $\Gamma \xrightarrow{\ulcorner \cdot \urcorner, i, \mathbb{C}_v} (\Gamma, e)$ is passive as it simply recombines existing information in Γ to build e , without any reduction step taking place, and thus without generating new information. Some extra knowledge is produced only when (Γ, e) evolves (with active transitions), as it then tells us how the tested context Γ_i actually interacts with the value constructed from \mathbb{C}_v . Finally, $\xrightarrow{\lambda, i, \mathbb{C}_v}$ and $\xrightarrow{\mathbb{E}}$ correspond to reduction steps and are therefore active, and $\xrightarrow{\ulcorner \cdot \urcorner, i, j}$ is also active as it provides some information by telling us how to decompose a continuation.

With this distinction, we change the definition of progress, to allow a relation \mathcal{R} to progress towards different relations after passive and active transitions.

Definition 4.2. A relation \mathcal{R} diacritically progresses to \mathcal{S}, \mathcal{T} written $\mathcal{R} \rightsquigarrow \mathcal{S}, \mathcal{T}$, if $\mathcal{R} \subseteq \mathcal{S}$, $\mathcal{R} \subseteq \mathcal{T}$, and $\Sigma \mathcal{R} \Theta$ implies that

- if $\Sigma \xrightarrow{\alpha} \Sigma'$ and $\xrightarrow{\alpha}$ is passive, then there exists Θ' such that $\Theta \xrightarrow{\alpha} \Theta'$ and $\Sigma' \mathcal{S} \Theta'$;
- if $\Sigma \xrightarrow{\alpha} \Sigma'$ and $\xrightarrow{\alpha}$ is active, then there exists Θ' such that $\Theta \xrightarrow{\alpha} \Theta'$ and $\Sigma' \mathcal{T} \Theta'$;
- the converse of the above conditions on Θ .

A \star -bisimulation is a relation \mathcal{R} such that $\mathcal{R} \rightsquigarrow \mathcal{R}, \mathcal{R}$, and \star -bisimilarity $\overset{\star}{\approx}$ is the union of all \star -bisimulations.

With the same LTS, \rightsquigarrow and \rightsquigarrow would entail the same notions of bisimulation and bisimilarity; the distinction between active and passive transitions is interesting only when considering up-to techniques. We change the notation for the bisimilarity $\overset{\star}{\approx}$ to emphasize that we use a different LTS in this section.

4.3. Up-to Techniques, Soundness, and Completeness. We now discriminate up-to techniques, so that regular up-to techniques cannot be used after passive transitions, while *strong* ones can. An up-to technique (resp. *strong* up-to technique) is a function f such that $\mathcal{R} \rightsquigarrow \mathcal{R}, f(\mathcal{R})$ (resp. $\mathcal{R} \rightsquigarrow f(\mathcal{R}), f(\mathcal{R})$) implies $\mathcal{R} \subseteq \overset{\star}{\approx}$. We also adapt the notions of evolution and compatibility.

Definition 4.3. A function f evolves to g, h , written $f \rightsquigarrow g, h$, if for all $\mathcal{R} \rightsquigarrow \mathcal{R}, \mathcal{T}$, we have $f(\mathcal{R}) \rightsquigarrow g(\mathcal{R}), h(\mathcal{T})$.

A function f *strongly* evolves to g, h , written $f \rightsquigarrow_s g, h$, if for all $\mathcal{R} \rightsquigarrow \mathcal{S}, \mathcal{T}$, we have $f(\mathcal{R}) \rightsquigarrow g(\mathcal{S}), h(\mathcal{T})$.

Strong evolution is very general, as it uses any relation \mathcal{R} , while regular evolution is more restricted, as it relies on relations \mathcal{R} such that $\mathcal{R} \rightsquigarrow \mathcal{R}, \mathcal{T}$. But the definition of *diacritical compatibility* below still allows to use any combinations of strong up-to techniques after a passive transition, even for functions which are not themselves strong. In contrast, regular functions can only be used once after a passive transition of an other regular function.

Definition 4.4. A set \mathfrak{F} of continuous functions is *diacritically compatible* if there exists $\mathfrak{S} \subseteq \mathfrak{F}$ such that

- for all $f \in \mathfrak{S}$, we have $f \rightsquigarrow_s \widehat{\mathfrak{S}}^\omega, \widehat{\mathfrak{F}}^\omega$;
- for all $f \in \mathfrak{F}$, we have $f \rightsquigarrow \widehat{\mathfrak{S}}^\omega \circ \widehat{\mathfrak{F}} \circ \widehat{\mathfrak{S}}^\omega, \widehat{\mathfrak{F}}^\omega$.

If $(\mathfrak{S}_i)_{i \in I}$ is a family of subsets of \mathfrak{F} which verify the conditions of the definition, then $\bigcup_{i \in I} \mathfrak{S}_i$ also verifies them. We can therefore consider the largest of such subsets, written $\text{strong}(\mathfrak{F})$, which can be defined as the union of all subsets of \mathfrak{F} verifying the conditions of the definition. This (possibly empty) subset of \mathfrak{F} contains the strong up-to techniques of \mathfrak{F} .

Lemma 4.5. *Let \mathfrak{F} be a diacritically compatible set.*

- If $\mathcal{R} \rightsquigarrow \widehat{\text{strong}(\mathfrak{F})}^\omega(\mathcal{R}), \widehat{\mathfrak{F}}^\omega(\mathcal{R})$, then $\widehat{\mathfrak{F}}^\omega(\mathcal{R})$ is a \star -bisimulation.
- If $f \in \mathfrak{F}$, then f is an up-to technique. If $f \in \text{strong}(\mathfrak{F})$, then f is a strong up-to technique.
- For all $f \in \mathfrak{F}$, we have $f(\overset{\star}{\approx}) \subseteq \overset{\star}{\approx}$.

Proof. Let $\mathfrak{S} \stackrel{\text{def}}{=} \widehat{\text{strong}(\mathfrak{F})}$. For the first item, we prove that for all n

$$(\mathfrak{S} \circ \widehat{\mathfrak{F}} \circ \mathfrak{S}^\omega)^n(\mathcal{R}) \rightsquigarrow (\mathfrak{S} \circ \widehat{\mathfrak{F}} \circ \mathfrak{S}^\omega)^n(\mathfrak{S}^\omega(\mathcal{R})), \widehat{\mathfrak{F}}^\omega(\mathcal{R})$$

by induction on n . There is nothing to prove for $n = 0$. Suppose $n > 0$. We know that

$$(\mathfrak{S} \circ \widehat{\mathfrak{F}} \circ \mathfrak{S}^\omega)^{n-1}(\mathcal{R}) \rightsquigarrow (\mathfrak{S} \circ \widehat{\mathfrak{F}} \circ \mathfrak{S}^\omega)^{n-1}(\mathfrak{S}^\omega(\mathcal{R})), \widehat{\mathfrak{F}}^\omega(\mathcal{R}).$$

For all $f \in \mathfrak{S}$, we have

$$f((\mathfrak{S} \circ \widehat{\mathfrak{F}} \circ \mathfrak{S}^\omega)^{n-1}(\mathcal{R})) \rightsquigarrow \mathfrak{S}^\omega((\mathfrak{S} \circ \widehat{\mathfrak{F}} \circ \mathfrak{S}^\omega)^{n-1}(\mathfrak{S}^\omega(\mathcal{R}))), \widehat{\mathfrak{F}}^\omega(\widehat{\mathfrak{F}}^\omega(\mathcal{R})),$$

therefore we have

$$\mathfrak{S}^\omega(((\mathfrak{S} \circ \widehat{\mathfrak{F}} \circ \mathfrak{S}^\omega)^{n-1}(\mathcal{R}))) \rightsquigarrow \mathfrak{S}^\omega((\mathfrak{S} \circ \widehat{\mathfrak{F}} \circ \mathfrak{S}^\omega)^{n-1}(\mathfrak{S}^\omega(\mathcal{R}))), \widehat{\mathfrak{F}}^\omega(\mathcal{R}).$$

Because $\mathfrak{S}^\omega \circ (\mathfrak{S} \circ \widehat{\mathfrak{F}} \circ \mathfrak{S}^\omega)^{n-1} = \mathfrak{S}^\omega \circ (\mathfrak{S} \circ \widehat{\mathfrak{F}} \circ \mathfrak{S}^\omega)^{n-1} \circ \mathfrak{S}^\omega$, for all $f \in \widehat{\mathfrak{F}}$, we have

$$f(\mathfrak{S}^\omega(((\mathfrak{S} \circ \widehat{\mathfrak{F}} \circ \mathfrak{S}^\omega)^{n-1}(\mathcal{R})))) \rightsquigarrow (\mathfrak{S} \circ \widehat{\mathfrak{F}} \circ \mathfrak{S}^\omega)(\mathfrak{S}^\omega((\mathfrak{S} \circ \widehat{\mathfrak{F}} \circ \mathfrak{S}^\omega)^{n-1}(\mathfrak{S}^\omega(\mathcal{R}))), \widehat{\mathfrak{F}}^\omega(\widehat{\mathfrak{F}}^\omega(\mathcal{R})),$$

which implies $\widehat{\mathfrak{F}}(\mathfrak{S}^\omega(((\mathfrak{S} \circ \widehat{\mathfrak{F}} \circ \mathfrak{S}^\omega)^{n-1}(\mathcal{R})))) \rightsquigarrow (\mathfrak{S} \circ \widehat{\mathfrak{F}} \circ \mathfrak{S}^\omega)^n(\mathfrak{S}^\omega(\mathcal{R})), \widehat{\mathfrak{F}}^\omega(\mathcal{R})$. Finally, composing again with \mathfrak{S}^ω , we obtain

$$\mathfrak{S}^\omega(((\mathfrak{S} \circ \widehat{\mathfrak{F}} \circ \mathfrak{S}^\omega)^n(\mathcal{R}))) \rightsquigarrow \mathfrak{S}^\omega \circ (\mathfrak{S} \circ \widehat{\mathfrak{F}} \circ \mathfrak{S}^\omega)^n(\mathfrak{S}^\omega(\mathcal{R})), \widehat{\mathfrak{F}}^\omega(\mathcal{R}),$$

as wished.

Because $\widehat{\mathfrak{F}}^\omega = (\mathfrak{S} \circ \widehat{\mathfrak{F}} \circ \mathfrak{S}^\omega)^\omega$, we get that $\widehat{\mathfrak{F}}^\omega(\mathcal{R}) \rightsquigarrow \widehat{\mathfrak{F}}^\omega(\mathcal{R}), \widehat{\mathfrak{F}}^\omega(\mathcal{R})$, i.e., $\widehat{\mathfrak{F}}^\omega(\mathcal{R})$ is a \star -bisimulation.

For the second item, let $f \in \mathfrak{F}$ and $\mathcal{R} \rightsquigarrow \mathcal{R}, f(\mathcal{R})$. Then $\mathcal{R} \subseteq \widehat{\mathfrak{F}}^\omega(\mathcal{R})$ by definition of ω and $\widehat{\mathfrak{F}}^\omega(\mathcal{R}) \subseteq \overset{\star}{\approx}$ by the first item. Therefore we have $\mathcal{R} \subseteq \overset{\star}{\approx}$ and f is an up-to technique. Similarly, we can show that $f \in \text{strong}(\mathfrak{F})$ and $\mathcal{R} \rightsquigarrow f(\mathcal{R}), f(\mathcal{R})$ imply $\mathcal{R} \subseteq \overset{\star}{\approx}$, meaning that f is a strong up-to technique.

For the last item, for all $f \in \mathfrak{F}$, we have $f(\overset{\star}{\approx}) \subseteq \widehat{\mathfrak{F}}^\omega(\overset{\star}{\approx})$, and $\widehat{\mathfrak{F}}^\omega(\overset{\star}{\approx}) \subseteq \overset{\star}{\approx}$ by the first item, so we have $f(\overset{\star}{\approx}) \subseteq \overset{\star}{\approx}$ as wished. \square

We now use this framework to define up-to techniques for the \star -bisimulation. The definitions of `perm` and `weak` are unchanged. We define bisimulation up to related contexts for values `rctx` and for any term `rectx` as follows:

$$\frac{\Gamma \mathcal{R} \Delta}{(\Gamma, \vec{\mathcal{C}}_v[\Gamma], \mathbb{C}[\Gamma]) \text{rctx}(\mathcal{R}) (\Delta, \vec{\mathcal{C}}_v[\Delta], \mathbb{C}[\Delta])} \quad \frac{(\Gamma, e_1) \mathcal{R} (\Delta, e_2)}{(\Gamma, \vec{\mathcal{C}}_v[\Gamma], \mathbb{E}[e_1, \Gamma]) \text{rectx}(\mathcal{R}) (\Delta, \vec{\mathcal{C}}_v[\Delta], \mathbb{E}[e_2, \Delta])}$$

The definitions look similar to the ones of `ctx` and `ectx`, but the grammar of multi-hole contexts now include \star_i . Besides, we inline strengthening in the definitions of `rctx` and `rectx`, allowing Γ, Δ to be extended. This is necessary because, e.g., `str` and `rectx` cannot be composed after a passive transition (they are both not strong), so `rectx` have to include `str` directly. Note that the behavior of `str` can be recovered from `rectx` by taking $\mathbb{E} = \square$.

Lemma 4.6. $\mathfrak{F} \stackrel{\text{def}}{=} \{\text{perm}, \text{weak}, \text{rctx}, \text{rectx}\}$ is diacritically compatible, with $\text{strong}(\mathfrak{F}) = \{\text{perm}, \text{weak}\}$.

As a result, these functions are up-to techniques, and `weak` and `perm` can be used after a passive transition. Because of the last item of Lemma 4.5, $\overset{\star}{\approx}$ is also a congruence w.r.t. evaluation contexts, which means that $\overset{\star}{\approx}$ is sound w.r.t. \equiv_E . We can also prove it is complete the same way as for Theorem 3.9, leading again to full characterization.

Theorem 4.7. $e_1 \equiv_E e_2$ iff $(\emptyset, e_1) \overset{\star}{\approx} (\emptyset, e_2)$.

Remark 4.8. If we consider control-stuck terms as errors, as in Remark 2.5, then we can use the transition of Remark 3.10, considered as active, and the results of this section scale to such a version of the bisimilarity. While the compatibility proof for `rectx` does not change much, the one for `rctx` needs an extra case analysis to deal with the modified $\xrightarrow{\mathbb{E}}$ transition; see [2, Remark B.3] for further details.

4.4. Examples. We illustrate the use of $\overset{\star}{\approx}$, `rctx`, and `rectx` with two examples that would be much harder to prove with the techniques of Section 3.

Example 4.9 (β_Ω axiom). We prove $(\lambda x.E[x]) e \overset{\star}{\approx} E[e]$ if $x \notin \text{fv}(E)$ and $\text{sp}(E) = \emptyset$. Define \mathcal{R} starting with $(\ulcorner \lambda x.E[x] \urcorner \square \urcorner) \mathcal{R} (\ulcorner E \urcorner)$, and closing it under the ($\#$ -check) and the following rule:

$$\frac{\Gamma \mathcal{R} \Delta}{(\Gamma, (\lambda x.E[x]) \mathbb{C}_v[\Gamma]) \mathcal{R} (\Delta, E[\mathbb{C}_v[\Delta]])}$$

Then $(\emptyset, (\lambda x.E[x]) e) \text{weak}(\text{rctx}(\mathcal{R})) (\emptyset, E[e])$ and \mathcal{R} is a bisimulation up to context, since the sequence $\Gamma \xrightarrow{\ulcorner \cdot \urcorner, 1, \mathbb{C}_v} (\Gamma, (\lambda x.E[x]) \mathbb{C}_v[\Gamma]) \xrightarrow{\tau} (\Gamma, E[\mathbb{C}_v[\Gamma]])$ fits $\Delta \xrightarrow{\ulcorner \cdot \urcorner, 1, \mathbb{C}_v} (\Delta, E[\mathbb{C}_v[\Delta]]) \xrightarrow{\tau} (\Delta, E[\mathbb{C}_v[\Delta]])$, where the final states are in `rctx`. Notice we use `rctx` after $\xrightarrow{\tau}$, and not after the passive $\xrightarrow{\ulcorner \cdot \urcorner, 1, \mathbb{C}_v}$ transition.

Example 4.10 (Exceptions). A possible way of extending a calculus with exception handling is to add a construct `tryr e with v`, which evaluates `e` with a function raising an exception stored under the variable `r`. When `e` calls the function in `r` with some argument `v'`, even inside another `try` block, then the computation of `e` is aborted and replaced by `v v'`. We can implement this behavior directly in $\lambda_{G\#}$; more precisely, we write `tryr e with v`

as `handle` $(\lambda r.e) v$, where `handle` is a function expressed in the calculus. One possible implementation of `handle` in $\lambda_{\mathcal{G}\#}$ is very natural and heavily relies on fresh-prompt generation:

$$\text{handle} \stackrel{\text{def}}{=} \lambda f.\lambda h.\mathcal{P}x.\#_x f (\lambda z.\mathcal{G}_{x-}.h z)$$

The idea is to raise an exception by aborting the current continuation up to the corresponding prompt. The same function can be implemented using any comparable-resource generation and only one prompt p :

$$\begin{aligned} \text{handle}_p &\stackrel{\text{def}}{=} \lambda f.\lambda h.\mathcal{P}x.(\#_p \text{let } r = f \text{ raise}_{p,x} \text{ in } \lambda.\lambda.r) x h \\ \text{raise}_{p,x} &\stackrel{\text{def}}{=} \text{fix } r(z).\mathcal{G}_{p-}.\lambda y.\lambda h.\text{if } x \stackrel{?}{=} y \text{ then } h z \text{ else } r z \end{aligned}$$

Here the idea is to keep a freshly generated name x and a handler function h with the prompt corresponding to each call of `handlep`. The exception-raising function `raisep,x` iteratively aborts the current delimited continuation up to the nearest call of `handlep` and checks the name stored there in order to find the corresponding handler. Note that this implementation also uses prompt generation, since it is the only comparable resource that can be dynamically generated in $\lambda_{\mathcal{G}\#}$, but the implementation can be easily translated to, e.g., a calculus with single-prompted delimited-control operators and first-order store.

Proof. We prove that both versions of `handle` are \star -bisimilar. As in Example 3.11 we iteratively build a relation \mathcal{R} closed under the ($\#$ -check) rule, so that \mathcal{R} is a bisimulation up to context. We start with $(\text{handle}) \mathcal{R} (\text{handle}_p)$; to match the $\xrightarrow{\lambda,1,\mathbb{C}_v}$ transition, we extend \mathcal{R} as follows:

$$\frac{\Gamma \mathcal{R} \Delta}{(\Gamma, \lambda h.\mathcal{P}x.\#_x \mathbb{C}_v[\Gamma] (\lambda z.\mathcal{G}_{x-}.h z)) \mathcal{R} (\Delta, \lambda h.\mathcal{P}x.(\#_p \text{let } r = \mathbb{C}_v[\Delta] \text{ raise}_{p,x} \text{ in } \lambda.\lambda.r) x h)}$$

We obtain two functions which are in turn tested with $\xrightarrow{\lambda,n+1,\mathbb{C}'_v}$, and we obtain the states

$$(\Gamma, \#_{p_1} \mathbb{C}_v[\Gamma] (\lambda z.\mathcal{G}_{p_1-}.\mathbb{C}'_v[\Gamma] z)) \text{ and } (\Delta, (\#_{p_1} \text{let } r = \mathbb{C}_v[\Delta] \text{ raise}_{p,p_2} \text{ in } \lambda.\lambda.r) p_2 \mathbb{C}'_v[\Delta]).$$

Instead of adding them to \mathcal{R} directly, we decompose them into corresponding parts using up to context (with $\mathbb{C} = \star_{n+1}[\mathbb{C}_v \square_{n+2}]$), and we add these subterms to \mathcal{R} :

$$\frac{\Gamma \mathcal{R} \Delta \quad p_1 \notin \#(\Gamma) \quad p_2 \notin \#(\Delta)}{(\Gamma, \ulcorner \#_{p_1} \square \urcorner, \lambda z.\mathcal{G}_{p_1-}.\mathbb{C}'_v[\Gamma] z) \mathcal{R} (\Delta, \ulcorner (\#_{p_1} \text{let } r = \square \text{ in } \lambda.\lambda.r) p_2 \mathbb{C}'_v[\Delta] \urcorner, \text{raise}_{p,p_2})} \quad (**)$$

Testing the two captured contexts with $\xrightarrow{\ulcorner \cdot \urcorner, n+1, \mathbb{C}'_v}$ is easy, because they both evaluate to the thrown value. We now consider $\lambda z.\mathcal{G}_{p_1-}.\mathbb{C}'_v[\Gamma] z$ and `raisep,p2`; after the transition $\xrightarrow{\lambda, n+2, \mathbb{C}_v}$ we get the two control stuck terms

$$\mathcal{G}_{p_1-}.\mathbb{C}'_v[\Gamma] \mathbb{C}_v[\Gamma] \quad \text{and} \quad \mathcal{G}_{p-}.\lambda y.\lambda h.\text{if } p_2 \stackrel{?}{=} y \text{ then } h \mathbb{C}_v[\Delta] \text{ else } \text{raise}_{p,p_2} \mathbb{C}_v[\Delta]$$

Adding such terms to the relation will not be enough. The first one can be unstuck only using the corresponding context $\ulcorner \#_{p_1} \square \urcorner$, but the second one can be unstuck using any context added by rule (*), even for a different p_2 . In such a case, it will consume a part of the context and evaluate to itself. To be more general we add the following rule:

$$\frac{\Gamma \mathcal{R} \Delta \quad \mathbb{E}[\mathcal{G}_{p_1-}.\mathbb{C}'_v[\Gamma] \mathbb{C}_v[\Gamma], \Gamma] \text{ is control-stuck}}{(\Gamma, \mathbb{E}[\mathcal{G}_{p_1-}.\mathbb{C}'_v[\Gamma] \mathbb{C}_v[\Gamma], \Gamma]) \mathcal{R} (\Delta, \mathcal{G}_{p-}.\lambda y.\lambda h.\text{if } p_2 \stackrel{?}{=} y \text{ then } h \mathbb{C}_v[\Delta] \text{ else } \text{raise}_{p,p_2} \mathbb{C}_v[\Delta])}$$

The newly introduced stuck terms are tested with $\xrightarrow{\mathbb{E}'}$; if \mathbb{E}' does not have \star_i surrounding \square , they are still stuck, and we can use up to evaluation context to conclude. Assume $\mathbb{E}' = \mathbb{E}_1[\star_i[\mathbb{E}_2]]$ where \mathbb{E}_2 has not \star_j around \square . If i points to the evaluation context added by $(**)$ for the same p_2 , then they both evaluate to terms of the same shape, so we use up to context with $\mathbb{C} = \mathbb{E}_1[\mathbb{C}'_v \mathbb{C}_v]$. Otherwise, we know the second program compares two different prompts, so it evaluates to $\mathbb{E}_1[\mathcal{G}_{p..}\lambda y.\lambda h.\text{if } p_2 \stackrel{?}{=} y \text{ then } h \mathbb{C}_v[\Delta] \text{ else } \text{raise}_{p,p_2} \mathbb{C}_v[\Delta], \Delta]$ and we use `rectx` with the last rule. \square

5. SHIFT AND RESET

In this section, we show how \star -bisimilarity can be defined for $\lambda_{\mathcal{S}}$, a λ -calculus extended with `shift` and `reset`. These operators can be encoded in $\lambda_{\mathcal{G}\#}$ (see Example 3.11), but relying on this encoding would lead to a sound, but not complete bisimilarity for `shift` and `reset`. Indeed, there are terms equivalent in $\lambda_{\mathcal{S}}$, the encodings of which are no longer equivalent with the more expressive constructs of $\lambda_{\mathcal{G}\#}$: see Example 5.3. This is why we work with $\lambda_{\mathcal{S}}$ in this section, and not $\lambda_{\mathcal{G}\#}$.

We study several bisimilarities for $\lambda_{\mathcal{S}}$ in previous works [8, 9, 10, 11]. In particular, we define environmental ones in [10, 11], but without a relation equivalent to bisimulation up to related contexts, which makes the proof of the β_{Ω} axiom very difficult in these papers. The proof in Example 4.9 is as easy as the proof of the β_{Ω} axiom in [9], but the bisimilarity of [9] is not complete. Therefore, a sound and complete \star -bisimilarity for $\lambda_{\mathcal{S}}$ which allows for simple equivalence proofs thanks to up-to techniques improves over our previous work.

5.1. Syntax, Semantics, and Contextual Equivalence. The calculus $\lambda_{\mathcal{S}}$ is a single-prompted version of $\lambda_{\mathcal{G}\#}$, where the now unique delimiter $\langle \cdot \rangle$ is called `reset` and the capturing construct \mathcal{S} is called `shift`. The syntax of the different entities is as follows.

$$\begin{array}{ll}
e ::= v \mid ee \mid \langle e \rangle \mid \mathcal{S}x.e & \text{(expressions)} \\
v ::= x \mid \lambda x.e & \text{(values)} \\
E ::= \square \mid Ee \mid vE & \text{(pure contexts)} \\
F ::= \square \mid Fe \mid vF \mid \langle F \rangle & \text{(evaluation contexts)}
\end{array}$$

We distinguish two kinds of evaluation contexts: pure contexts, ranged over by E , can be captured by `shift`, while those represented by F are the regular evaluation contexts. Captured contexts are no longer part of the syntax, but are instead turned into λ -abstractions, as we can see in the following reduction rules.

$$\begin{array}{ll}
(\lambda x.e)v \rightarrow e\{v/x\} & \text{COMPATIBILITY} \\
\langle v \rangle \rightarrow v & \\
\langle E[\mathcal{S}x.e] \rangle \rightarrow \langle e\{\lambda y.\langle E[y] \rangle/x\} \rangle \quad y \text{ fresh} & \frac{e_1 \rightarrow e_2}{F[e_1] \rightarrow F[e_2]}
\end{array}$$

The operator \mathcal{S} captures a surrounding context E up to the first enclosing `reset`. This `reset` is left in place, but E remains delimited when captured in $\lambda y.\langle E[y] \rangle$.

The original semantics of `shift` and `reset` [6] applies these rules only to terms with an outermost `reset`; this requirement is often lifted in practical implementation [16, 19] or

studies of these operators [3, 24]. As in [10, 11], we define equivalences for the original and the relaxed semantics. The two semantics differ mainly in the normal forms they produce: an expression $\langle e \rangle$ cannot reduce to a control-stuck term $E[\mathcal{S}x.e']$ in the original semantics, while such a normal form can still be obtained with the relaxed semantics. As a result, we distinguish the observable actions for the original semantics \sim° from those for the relaxed semantics \sim^r . Unlike in $\lambda_{\mathcal{G}\#}$, both semantics cannot produce errors, so we simply write $e \uparrow$ when e diverges.

Definition 5.1. We write $e_1 \sim^\circ e_2$ if

- (1) $e_1 \rightarrow^* v_1$ iff $e_2 \rightarrow^* v_2$,
- (2) $e_1 \uparrow$ iff $e_2 \uparrow$.

We write $e_1 \sim^r e_2$ if

- (1) $e_1 \rightarrow^* v_1$ iff $e_2 \rightarrow^* v_2$,
- (2) $e_1 \rightarrow^* E_1[\mathcal{S}x.e'_1]$ iff $e_2 \rightarrow^* E_2[\mathcal{S}x.e'_2]$,
- (3) $e_1 \uparrow$ iff $e_2 \uparrow$.

Similarly, we define a contextual equivalence for each semantics.

Definition 5.2 (Contextual equivalence). Given two closed expressions e_1 and e_2 , we write $e_1 \equiv_E^\circ e_2$ if for all E , we have $\langle E[e_1] \rangle \sim^\circ \langle E[e_2] \rangle$, and we write $e_1 \equiv_E^r e_2$ if for all E , we have $E[e_1] \sim^r E[e_2]$.

Because we no longer have resource generation, note that testing with evaluation contexts F is equivalent to testing with any context C in $\lambda_{\mathcal{S}}$ [11].

Example 5.3. The expressions $\langle \langle e_1 \rangle (\langle e_2 \rangle \mathcal{S}x.\lambda y.y) \rangle$ and $\langle \langle e_2 \rangle (\langle e_1 \rangle \mathcal{S}x.\lambda y.y) \rangle$ are contextually equivalent in $\lambda_{\mathcal{S}}$ with either semantics, but their encodings are not bisimilar in $\lambda_{\mathcal{G}\#}$. In $\lambda_{\mathcal{S}}$, depending on whether $\langle e_1 \rangle$ or $\langle e_2 \rangle$ diverge or reduce to a value, the two above terms either diverge or reduce to $\lambda y.y$. In $\lambda_{\mathcal{G}\#}$, the encoding of $\langle e_1 \rangle$ can reduce to a control-stuck term, e.g., if $e_1 = \mathcal{P}x.\mathcal{G}_x y.y$, making $\langle \langle e_1 \rangle (\langle e_2 \rangle \mathcal{S}x.\lambda y.y) \rangle$ stuck as well, while e_2 may diverge, and a stuck term is not equivalent to a diverging one.

Remark 5.4. We can equivalently define $\lambda_{\mathcal{S}}$ with captured pure contexts as values and a throw construct $v \triangleleft t$, as in $\lambda_{\mathcal{G}\#}$, using the following reduction rules

$$\begin{aligned} \mathcal{S}x.e &\rightarrow e\{\ulcorner E^\triangleright/x\} \\ \ulcorner E^\triangleright \triangleleft v &\rightarrow E[v] \end{aligned}$$

and with $\ulcorner E^\triangleright \triangleleft F$ as an evaluation context and $\ulcorner E^\triangleright \triangleleft E'$ as a pure context. Only values are thrown to captured contexts, unlike in $\lambda_{\mathcal{G}\#}$. In this section, we stick to the syntax we use in [10, 11] to facilitate comparisons with these papers. We discuss how to adapt the LTS to the syntax with throw in Remark 5.5.

5.2. Bisimilarity and Up-to Techniques. For bisimulation up to related contexts to be useful, we want to be able to save evaluation context (not necessarily pure) in states. To do so, we let Ψ, Φ range over sequences of evaluation contexts, and we consider states of the form (Ψ, Γ, e) , where Γ is still a sequence of values. Multi-hole contexts, whose syntax is given below, are now filled with Ψ and Γ .

Rules common to both semantics:		
$\frac{e_1 \rightarrow e_2}{(\Psi, \Gamma, e_1) \xrightarrow{\tau} (\Psi, \Gamma, e_2)}$	$\frac{\Gamma_j = \lambda x.e}{(\Psi, \Gamma) \xrightarrow{\lambda, j, \mathbb{C}_v} (\Psi, \Gamma, e\{\mathbb{C}_v[\Psi, \Gamma]/x\})}$	$\frac{}{(\Psi, \Gamma) \xrightarrow{v} (\Psi, \Gamma)}$
$\frac{e \text{ is stuck} \quad \mathbb{F}[e, \Psi, \Gamma] \xrightarrow{\mathbb{F}} e'}{(\Psi, \Gamma, e) \xrightarrow{\mathbb{F}} (\Psi, \Gamma, e')}$	$\frac{}{(\Psi, \Gamma) \xrightarrow{\square, i, \mathbb{C}_v} (\Psi, \Gamma, \Psi_i[\mathbb{C}_v[\Psi, \Gamma]])}$	$\frac{\Psi_i = E}{(\Psi, \Gamma) \xrightarrow{\square, i} (\Psi, \Gamma)}$
$\frac{\Psi_i = F[\langle E \rangle]}{(\Psi, \Gamma) \xrightarrow{\langle \square \rangle, i} (\Psi, F[\langle \square \rangle], \langle E \rangle, \Gamma)}$		
Extra rule for the original semantics:		
$\frac{e \text{ is not stuck}}{(\Psi, \Gamma, e) \xrightarrow{\mathbb{F}} (\Psi, \Gamma, \mathbb{F}[e, \Psi, \Gamma])}$		
Up-to techniques for both semantics:		
$\frac{(\vec{F}, \Psi, \vec{v}, \Gamma, e_1) \mathcal{R} (\vec{F}', \Phi, \vec{w}, \Delta, e_2)}{(\Psi, \Gamma, e_1) \text{ weak}(\mathcal{R}) (\Phi, \Delta, e_2)}$		
$\frac{(\Psi, \Gamma) \mathcal{R} (\Phi, \Delta)}{(\Psi, \vec{\mathbb{F}}[\Psi, \Gamma], \Gamma, \vec{\mathbb{C}}_v[\Psi, \Gamma], \mathbb{C}[\Psi, \Gamma]) \text{ rctx}(\mathcal{R}) (\Phi, \vec{\mathbb{F}}[\Phi, \Delta], \Delta, \vec{\mathbb{C}}_v[\Phi, \Delta], \mathbb{C}[\Phi, \Delta])}$		
$\frac{(\Psi, \Gamma, e_1) \mathcal{R} (\Phi, \Delta, e_2)}{(\Psi, \vec{\mathbb{F}}[\Psi, \Gamma], \Gamma, \vec{\mathbb{C}}_v[\Psi, \Gamma], \mathbb{F}[e_1, \Psi, \Gamma]) \text{ rectx}(\mathcal{R}) (\Phi, \vec{\mathbb{F}}[\Phi, \Delta], \Delta, \vec{\mathbb{C}}_v[\Phi, \Delta], \mathbb{F}[e_2, \Phi, \Delta])}$		

Figure 2: LTS and up-to techniques for shift and reset

$\mathbb{C} ::= \mathbb{C}_v \mid \mathbb{C} \mathbb{C} \mid \langle \mathbb{C} \rangle \mid \mathcal{S}x.\mathbb{C} \mid \star_i [\mathbb{C}]$	(contexts)
$\mathbb{C}_v ::= x \mid \lambda x.\mathbb{C} \mid \square_i$	(value contexts)
$\mathbb{F} ::= \square \mid \mathbb{F} \mathbb{C} \mid \mathbb{C}_v \mathbb{F} \mid \langle \mathbb{F} \rangle \mid \star_i [\mathbb{F}]$	(evaluation contexts)

We write $\mathbb{C}[\Psi, \Gamma]$ to say that \star_i of \mathbb{C} is filled with the context Ψ_i , as in Section 4, and each hole \square_j is plugged with the value Γ_j . As before, it assumes that each index i of \star_i is smaller than the size of Ψ , and each j of \square_j is smaller than the size of Γ . Similarly, we write $\mathbb{F}[e, \Psi, \Gamma]$ for evaluation contexts, so that e goes into \square .

We present the LTS and up-to techniques for the two semantics of $\lambda_{\mathcal{G}\#}$ in Figure 2. In $\lambda_{\mathcal{G}\#}$, having \star holes in multi-hole contexts helps when testing captured contexts as well as for the up-to techniques. In contrast, in $\lambda_{\mathcal{S}}$, \star holes are useful only for the up-to techniques, and not for the bisimilarity itself, even if we consider the syntax with captured contexts (see Remark 5.5). As a result, some of the transitions are only for the bisimilarity, namely $\xrightarrow{\tau}$, $\xrightarrow{\lambda, j, \mathbb{C}_v}$, \xrightarrow{v} , and $\xrightarrow{\mathbb{F}}$, while the remaining three are for bisimulations up to context: they are used only if Ψ is not empty.

The transition $\xrightarrow{\square, i, \mathbb{C}_v}$ tests the evaluation context Ψ_i by passing it a value built from Ψ and Γ . A stuck term is able to distinguish a pure context from an impure one, and it can extract from $F[\langle E \rangle]$ the context up to the first enclosing reset $\langle E \rangle$. However, unlike in $\lambda_{\mathcal{G}\#}$, we cannot decompose F further, because the capture leaves the delimiter in place: we can distinguish \square from $\langle \square \rangle$, but not $\langle \square \rangle$ from $\langle \langle \square \rangle \rangle$. We use $\xrightarrow{\square, i}$ and $\xrightarrow{\langle \square \rangle, i}$ to perform these tests: $\xrightarrow{\square, i}$ simply states that Ψ_i is pure, while $\xrightarrow{\langle \square \rangle, i}$ decomposes $\Psi_i = F[\langle E \rangle]$ into $F[\langle \square \rangle]$ and $\langle E \rangle$. Because we leave a **reset** inside F , applying $\xrightarrow{\langle \square \rangle, i}$ to $F[\langle \square \rangle]$ does not decompose F further, but simply generates $F[\langle \square \rangle]$ again (and $\langle \square \rangle$), and duplicated contexts can then be ignored thanks to strengthening.

The transition $\xrightarrow{\mathbb{F}}$ compares stuck terms in the relaxed semantics. In the original semantics, we can also relate with the extra rule a stuck term with a regular term: we prove in Example 5.8 that $\mathcal{S}k.k e$ is equivalent to e in that semantics if $k \notin \text{fv}(e)$. When the extra rule is applied to two non stuck terms e_1 and e_2 , it generates expressions $\mathbb{F}[e_1, \Psi, \Gamma]$ and $\mathbb{F}[e_2, \Phi, \Delta]$ which are automatically related with up to contexts, so the extra rule does not produce additional testing for regular terms. The transition $\xrightarrow{\mathbb{F}}$ uses any evaluation context \mathbb{F} , and not simply a context of the form $\langle \mathbb{E} \rangle$ with \mathbb{E} a pure context, as we do in [10, 11]. We do so to take \star_i into account: a context $\star_i[\mathbb{E}]$ may also trigger a capture if Ψ_i is an impure context. Besides, if $(\Psi, \Gamma, e_1) \mathcal{R} (\Phi, \Delta, e_2)$ and Ψ_i is pure, then Φ_i may be impure if e_1 and e_2 contain infinite behavior (and thus, the transitions $\xrightarrow{\square, i}$ and $\xrightarrow{\langle \square \rangle, i}$ are never applied). For example, we have $(\square, \emptyset, \mathcal{S}k.\Omega) \xrightarrow{\star_1[\square]} (\square, \emptyset, \mathcal{S}k.\Omega)$ and $(\langle \square \rangle, \emptyset, \mathcal{S}k.\Omega) \xrightarrow{\star_1[\square]} (\langle \square \rangle, \emptyset, \langle \Omega \rangle)$; the two resulting states are distinguished in the relaxed semantics, but they are equated in the original one. However, what is beyond the first enclosing **reset** of a testing context $\mathbb{F}[\Psi, \Gamma]$, and therefore do not interact with the tested terms, can be ignored thanks to bisimulation up to related contexts, as in Example 3.8.

The transitions $\xrightarrow{\tau}$, $\xrightarrow{\lambda, j, \mathbb{C}_v}$, and $\xrightarrow{\mathbb{F}}$ are active because they correspond to reduction steps, and $\xrightarrow{\square, i}$ and $\xrightarrow{\langle \square \rangle, i}$ are active because they provide information on the tested contexts (being pure or not, and how to decompose contexts that are not pure). As before, $\xrightarrow{\vee}$ is passive because it informs about the nature of the tested states (composed only of values), and $\xrightarrow{\square, i, \mathbb{C}_v}$ is passive because it does not provide any information on the tested context nor does it correspond to a reduction step.

Remark 5.5. If captured contexts are considered values, as suggested in Remark 5.4, then they are stored in Γ and Δ , and therefore cannot be used to fill a \star hole in a multi-hole context. They are tested with the same rule as in $\lambda_{\mathcal{G}\#}$

$$\frac{\Gamma_i = \ulcorner E \urcorner}{(\Psi, \Gamma) \xrightarrow{\ulcorner \cdot \urcorner, i, \mathbb{C}_v} (\Psi, \Gamma, E[\mathbb{C}_v[\Psi, \Gamma]])}$$

except it would be an active transition in $\lambda_{\mathcal{S}}$, as testing with a value corresponds to the throw reduction rule. So unlike in $\lambda_{\mathcal{G}\#}$, we have two transitions to test contexts, in this version of $\lambda_{\mathcal{S}}$: one, active, to test a pure context in Γ , which is used for the bisimulation, and one, passive, to test any evaluation context in Ψ , which is useful only for up-to techniques.

The definitions of the up to techniques are as expected, with weakening and strengthening for contexts as well as for values. We write \approx° and \approx^r for the \star -bisimilarities obtained from

the transitions for respectively the original and relaxed semantics. For both semantics, the following lemma holds.

Lemma 5.6. $\mathfrak{F} \stackrel{\text{def}}{=} \{\text{weak}, \text{rctx}, \text{rectx}\}$ is diacritically compatible, with $\text{strong}(\mathfrak{F}) = \{\text{weak}\}$.

As before, this lemma implies that $\overset{\circ}{\approx}$ and $\overset{r}{\approx}$ are sound w.r.t. respectively \equiv_E° and \equiv_E^r , and completeness proofs are as usual.

Theorem 5.7. $e_1 \equiv_E^\circ e_2$ iff $(\emptyset, \emptyset, e_1) \overset{\circ}{\approx} (\emptyset, \emptyset, e_2)$, and $e_1 \equiv_E^r e_2$ iff $(\emptyset, \emptyset, e_1) \overset{r}{\approx} (\emptyset, \emptyset, e_2)$.

5.3. Examples. We give examples for the original semantics of equivalences proved in [10, 11], to show that the proofs are much easier here.

Example 5.8. If $k \notin \text{fv}(e)$, then $(\emptyset, \emptyset, \mathcal{S}k.k e) \overset{\circ}{\approx} (\emptyset, \emptyset, e)$. We show that the relation

$$\mathcal{R} \stackrel{\text{def}}{=} \{(\emptyset, \emptyset, \mathcal{S}k.k e), (\emptyset, \emptyset, e)\} \cup \{(\langle \lambda x. \langle E[x] \rangle \square), \langle \langle \square \rangle \rangle, \emptyset), (\langle E \rangle, \langle \square \rangle, \emptyset) \mid x \notin \text{fv}(E)\}$$

is a bisimulation up to related contexts. If e is not control-stuck, the transition $(\emptyset, \emptyset, \mathcal{S}k.k e) \xrightarrow{\mathbb{F}} (\emptyset, \emptyset, F[\langle \lambda x. \langle E[x] \rangle e \rangle])$ is matched by the transition $(\emptyset, \emptyset, e) \xrightarrow{\mathbb{F}} (\emptyset, \emptyset, F[\langle E[e] \rangle])$, assuming x is fresh and $\mathbb{F}[\emptyset, \emptyset] = F[\langle E \rangle]$ (the case $\mathbb{F}[\emptyset, \emptyset] = E$ is simple). If $e = E'[\mathcal{S}k'.e']$, then $(\emptyset, \emptyset, e) \xrightarrow{\mathbb{F}} (\emptyset, \emptyset, F[\langle e' \{ \lambda x. \langle E[E'[x]] \rangle / k' \} \rangle])$ is matched by the sequence $(\emptyset, \emptyset, \mathcal{S}k.k e) \xrightarrow{\mathbb{F}} \xrightarrow{\tau} (\emptyset, \emptyset, F[\langle e' \{ \lambda x. \langle (\lambda y. E[y]) E'[x] \rangle / k' \} \rangle])$, with x, y fresh and $\mathbb{F}[\emptyset, \emptyset] = F[\langle E \rangle]$. In both cases, the resulting states are in $\text{rctx}(\mathcal{R})$. Let $(\Psi, \emptyset) \stackrel{\text{def}}{=} (\langle \lambda x. \langle E[x] \rangle \square), \langle \langle \square \rangle \rangle, \emptyset)$ and $(\Phi, \emptyset) \stackrel{\text{def}}{=} (\langle E \rangle, \langle \square \rangle, \emptyset)$. Then the sequence $(\Psi, \emptyset) \xrightarrow{\square, 1, \mathbb{C}_v} \xrightarrow{\tau} (\Psi, \emptyset, \langle \langle E[\mathbb{C}_v[\Psi, \emptyset]] \rangle \rangle)$ is matched by $(\Psi, \emptyset) \xrightarrow{\square, 1, \mathbb{C}_v} (\Psi, \emptyset, \langle E[\mathbb{C}_v[\Psi, \emptyset]] \rangle)$, since the resulting states are in $\text{rctx}(\mathcal{R})$, and we use up to related contexts after a $\xrightarrow{\tau}$ transition. Finally, $(\Psi, \emptyset) \xrightarrow{\square, 2, \mathbb{C}_v} \xrightarrow{\tau} \xrightarrow{\tau} (\Psi, \emptyset, \mathbb{C}_v[\Psi, \emptyset])$ is matched by $(\Phi, \emptyset) \xrightarrow{\square, 2, \mathbb{C}_v} \xrightarrow{\tau} (\Phi, \emptyset, \mathbb{C}_v[\Phi, \emptyset])$, and the context splitting transitions $\xrightarrow{\langle \square \rangle, i}$ are easy to check for $i \in \{1, 2\}$.

Example 5.9. If $k \notin \text{fv}(e_2)$, then $(\emptyset, \emptyset, (\lambda x. \mathcal{S}k.e_1)e_2) \overset{\circ}{\approx} (\emptyset, \emptyset, \mathcal{S}k.((\lambda x.e_1)e_2))$. The relation

$$\mathcal{R} \stackrel{\text{def}}{=} \{(\emptyset, \emptyset, (\lambda x. \mathcal{S}k.e_1)e_2), (\emptyset, \emptyset, \mathcal{S}k.((\lambda x.e_1)e_2))\} \\ \cup \{(\langle E[(\lambda x. \mathcal{S}k.e_1) \square] \rangle, \emptyset), (\langle \lambda x.e_1 \{ \lambda y. \langle E[y] \rangle / k \} \square), \emptyset) \mid y \notin \text{fv}(E)\}$$

is a bisimulation up to related contexts. As in the previous example, a case analysis on whether e_1 is control-stuck or not shows that the $\xrightarrow{\mathbb{F}}$ transitions from $(\emptyset, \emptyset, (\lambda x. \mathcal{S}k.e_1)e_2)$ and $(\emptyset, \emptyset, \mathcal{S}k.((\lambda x.e_1)e_2))$ produce states in $\text{rctx}(\mathcal{R})$. If $(\Psi, \emptyset) \stackrel{\text{def}}{=} (\langle E[(\lambda x. \mathcal{S}k.e_1) \square] \rangle, \emptyset)$ and $(\Phi, \emptyset) \stackrel{\text{def}}{=} (\langle \lambda x.e_1 \{ \lambda y. \langle E[y] \rangle / k \} \square), \emptyset)$, then

$$(\Psi, \emptyset) \xrightarrow{\square, 1, \mathbb{C}_v} \xrightarrow{\tau} \xrightarrow{\tau} (\Psi, \emptyset, \langle e_1 \{ \mathbb{C}_v[\Psi, \emptyset] / x \} \{ \lambda y. \langle E[y] \rangle / k \} \rangle) \\ (\Phi, \emptyset) \xrightarrow{\square, 1, \mathbb{C}_v} \xrightarrow{\tau} (\Phi, \emptyset, \langle e_1 \{ \mathbb{C}_v[\Phi, \emptyset] / x \} \{ \lambda y. \langle E[y] \rangle / k \} \rangle)$$

The resulting states are in $\text{rctx}(\mathcal{R})$, as wished. A completely written proof of this result takes less than a page, while the proof of the same result in [11] requires several pages, because of the lack of useful up-to techniques.

6. RELATED WORK AND CONCLUSION

Related work. We discuss our previous work on `shift` and `reset` at the beginning of Section 5. In [42], the authors propose an environmental bisimilarity for a calculus with `call/cc`, an operator which captures the whole surrounding context. The difficulty in such a language is that reduction is not preserved by evaluation context: $e \rightarrow e'$ does not imply $E[e] \rightarrow E[e']$, as E may be captured by e . As a result, the environmental bisimilarity of [42] factors in these evaluation contexts when testing values. This relation is also not coinductive, making it closer to contextual equivalence than to a regular environmental bisimilarity. An accompanying bisimulation up to context is also defined, but it is barely used in the examples of [42]. The equivalence proofs of these examples are thus almost as difficult as with contextual equivalence. It is not clear if and how \star -bisimilarity can improve on these results; we plan to investigate further this question.

Environmental bisimilarity has been defined in several calculi with dynamic resource generation, like stores and references [30, 29, 38], information hiding constructs [39, 40], or name creation [5, 33]. In these works, an expression is paired with its generated resources, and behavioral equivalences are defined on these pairs. Our approach is different since we do not carry sets of generated prompts when manipulating expressions (e.g., in the semantic rules of Section 2); instead, we rely on side-conditions and permutations to avoid collisions between prompts. This is possible because all we need to know is if a prompt is known to an outside observer or not, and the correspondences between the public prompts of two related expressions; this can be done through the environment of the bisimilarity. This approach cannot be adapted to more complex generated resources, which are represented by a mapping (e.g., for stores or existential types), but we believe it can be used for name creation in π -calculus [33].

A line of work on program equivalence for which relating evaluation contexts is crucial, as in our work, are logical relations based on the notion of biorthogonality [34]. In particular, this concept has been successfully used to develop techniques for establishing program equivalence in ML-like languages with `call/cc` [15], and for proving the coherence of control-effect subtyping [12]. Hur et al. combine logical relations and behavioral equivalences in the definition of *parametric bisimulation* [22], where terms are reduced to normal forms that are then decomposed into subterms related by logical relations. This framework has been extended to abortive control in [23], where *stuttering* is used to allow terms not to reduce for a finite amount of time when comparing them in a bisimulation proof. This is reminiscent of our distinction between active and passive transitions, as passive transitions can be seen as “not reducing”, but there is still some testing involved in these transitions. Besides, the concern is different, since the active/passive distinction prevents the use of up-to techniques, while stuttering has been proposed to improve plain parametric bisimulations.

Conclusion and future work. We have developed a behavioral theory for Dybvig et al.’s calculus of multi-prompted delimited control, where the enabling technology for proving program equivalence are environmental bisimulations, presented in Madiot’s style. The obtained results generalize our previous work in that they account for multiple prompts and local visibility of dynamically generated prompts. Moreover, the results of Section 4 considerably enhance reasoning about captured contexts by treating them as first-class objects at the level of bisimulation proofs (thanks to the construct \star_i) and not only at

the level of terms. The resulting notion of bisimulation up to related contexts improves on the existing bisimulation up to context in the presence of control operators, as we can see when comparing Example 4.9 to the proof of the same result in [10, 11]. Moreover, as demonstrated in Section 5, the approach of Section 4 smoothly carries over to more traditional calculi with delimited-control operators, where, in contrast to $\lambda_{G\#}$, captured continuations are represented as functions.

We would like to see if this work scales to other formulations of control and continuations, such as symmetric calculi [18, 13, 41]. We believe bisimulation up to related contexts could be useful also for constructs akin to control operators, like passivation in π -calculus [33]. The soundness of this up-to technique has been proved in an extension of Madiot’s framework; we plan to investigate further this extension, to see how useful it could be in defining up-to techniques for other languages. Finally, it may be possible to apply the tools developed in this paper to [27], where a single-prompted calculus is translated into a multi-prompted one, but no operational correspondence is given to guarantee the soundness of the translation.

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