

## PETRI AUTOMATA

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**ABSTRACT.** Kleene algebra axioms are complete with respect to both language models and binary relation models. In particular, two regular expressions recognise the same language if and only if they are universally equivalent in the model of binary relations.

We consider Kleene allegories, i.e., Kleene algebras with two additional operations and a constant which are natural in binary relation models: intersection, converse, and the full relation. While regular languages are closed under those operations, the above characterisation breaks. Putting together a few results from the literature, we give a characterisation in terms of languages of directed and labelled graphs.

By taking inspiration from Petri nets, we design a finite automata model, Petri automata, allowing to recognise such graphs. We prove a Kleene theorem for this automata model: the sets of graphs recognisable by Petri automata are precisely the sets of graphs definable through the extended regular expressions we consider.

Petri automata allow us to obtain decidability of identity-free relational Kleene lattices, i.e., the equational theory generated by binary relations on the signature of regular expressions with intersection, but where one forbids unit. This restriction is used to ensure that the corresponding graphs are acyclic. We actually show that this decision problem is EXPSpace-complete.

### 1. INTRODUCTION

Consider binary relations and the operations of union ( $\cup$ ), intersection ( $\cap$ ), composition ( $\cdot$ ), converse ( $^{-\smile}$ ), transitive closure ( $^{-+}$ ), reflexive-transitive closure ( $^{-*}$ ), and the constants identity ( $1$ ), empty relation ( $0$ ) and universal relation ( $\top$ ). These objects give rise to an (in)equational theory over the following signature.

$$\Sigma \triangleq \langle \cup_2, \cap_2, \cdot_2, -^{\smile}_1, -^+{}_1, -^*{}_1, 0_0, 1_0, \top_0 \rangle$$

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A pair of terms  $e, f$  built from those operations and some variables  $a, b, \dots$  is a *valid equation*, denoted  $\mathcal{R}el \models e = f$ , if the corresponding equality holds universally. Similarly, an inequation  $\mathcal{R}el \models e \leq f$  is valid when the corresponding containment holds universally. Here are valid equations and inequations: they hold whatever the relations we assign to variables  $a, b$ , and  $c$ .

$$\mathcal{R}el \models (a \cup b)^* \cdot b \cdot (a \cup b)^* = (a^* \cdot b \cdot a^*)^+ \quad (1.1)$$

$$\mathcal{R}el \models a^* \leq 1 \cup a \cdot a^\smile \cdot a^+ \quad (1.2)$$

$$\mathcal{R}el \models a \cdot b \cap c \leq a \cdot (b \cap a^\smile \cdot c) \quad (1.3)$$

$$\mathcal{R}el \models a^+ \cap 1 \leq (a \cdot a)^+ \quad (1.4)$$

$$\mathcal{R}el \models \top \cdot a \cdot \top \cdot b \cdot \top = \top \cdot b \cdot \top \cdot a \cdot \top \quad (1.5)$$

Various fragments of this theory have been studied in the literature:

- *Kleene algebra* [11], where one removes intersection, converse, and  $\top$ , so that terms are plain regular expressions. The equational theory is decidable [18], and actually PSPACE-complete [25]. The equational theory is not finitely based [31], but finite quasi-equational axiomatisations exist [21, 19, 4]. Equation (1.1) lies in this fragment, and one can notice that the two expressions recognise the same language.
- *Kleene algebra with converse*, where one only removes intersection and  $\top$ , is also a decidable fragment [3]. It remains PSPACE [8, 9], and can be axiomatised relatively to Kleene algebra [13]. Inequation (1.2) belongs to this fragment.
- (representable, distributive) *allegories* [14], sometimes called positive relation algebras, where transitive and reflexive-transitive closures and the full relation are not allowed. They are decidable [14, page 208], and not finitely based. Inequation (1.3) is known as the *modularity law* in this setting.
- (representable) *identity-free Kleene lattice* [1], where converse is removed, as well as the full relation, the identity relations, and reflexive-transitive closure (transitive closure is kept: reflexive-transitive closure is removed just because it hides the identity constant, we have  $0^* = 1$ ).

Accordingly we call the whole (in)equational theory that of *representable Kleene allegories*<sup>1</sup>. We obtain several important steps towards decidability of this (in)equational theory.

- (1) we characterise it in terms of *graph languages*;
- (2) we design a new automata model, called *Petri automata*, to recognise such graph languages, and we prove a Kleene like theorem: the languages definable through Kleene allegory expressions are precisely those recognisable by Petri automata;
- (3) we prove that the fragment where converse, full relation, and identity are forbidden is EXPSPACE-complete.

The latter fragment was studied by Andr eka, Mikul as and N emeti [1]. They proved that on the corresponding signature, the equational theory generated by binary relations coincides with that generated by formal languages:

$$\mathcal{R}el \models e = f \quad \Leftrightarrow \quad \mathcal{L}ang \models e = f$$

<sup>1</sup>An algebra is representable when it is isomorphic to an algebra of binary relation. The class of Kleene algebra is usually defined as a finitely based quasi-variety, and its equational theory coincides with that generated by binary relations (i.e., by representable algebras); in contrast, allegories are defined by a set of equations which is not complete with respect to representable algebra: there are some allegories which are not representable [14]; similarly for Kleene lattices.

(where  $\mathcal{Lang} \models e = f$  stands for universal validity in all  $\Sigma$ -algebras of formal languages with the standard interpretation of the operations—in particular,  $\_^\smile$  is language reversal and  $\top$  is the language of all words). It is important to understand that such a property highly depends on the considered fragment:

- It does not hold when  $1$ ,  $\cap$  and  $\cdot$  are all available: we have  $\mathcal{Lang} \models (a \cap 1) \cdot b = b \cdot (a \cap 1)$  but this equation is not valid in  $\mathcal{Rel}$ .
- It does hold in Kleene algebra (*i.e.*, on the usual signature of regular expressions).
- It does not hold when both  $\top$  and  $\cdot$  are available: we have  $\mathcal{Rel} \models \top \cdot a \cdot \top \cdot b \cdot \top = \top \cdot b \cdot \top \cdot a \cdot \top$  while this equation fails in  $\mathcal{Lang}$ .
- It does not hold when both  $\_^\smile$  and  $\cdot$  are available: we have  $\mathcal{Rel} \models a \leq a \cdot a^\smile \cdot a$  while this inequation fails in  $\mathcal{Lang}$ . Ésik et al. actually proved that this is the only missing law for the signature of Kleene algebra with converse, and that the two equational theories are decidable [3, 13].

**Note.** This paper is a follow-up to the paper we presented at LiCS’15 [10]. It contains more details and proofs, but also a full Kleene theorem for Petri automata (we only have the easiest half of it in [10]), a comparison of our automata model with the branching automata of Lodaya and Weil [23], EXPSPACE-hardness of identity-free Kleene lattices (we only have it for the underlying automata problem in [10]), and the handling of the constant  $\top$ .

We continue this introductory section by an informal description of the characterisation based on graph languages, and of our automata model.

**1.1. Languages.** In the simple case of Kleene algebra, *i.e.*, without converse and intersection, the (in)equational theory generated by relations (or languages) can be characterised by using regular languages. Write  $\mathcal{L}(e)$  for the language denoted by a regular expression  $e$ ; for all regular expressions  $e, f$ , we have

$$\mathcal{Rel} \models e \leq f \text{ if and only if } \mathcal{L}(e) \subseteq \mathcal{L}(f). \tag{1.6}$$

(This result is easy and folklore; proving that this is also equivalent to provability using Kleene algebra axioms [21, 19, 4] is much harder.)

While regular languages are closed under intersection, the above characterisation does not scale. Indeed, consider two distinct variables  $a$  and  $b$ . The extended regular expressions  $a \cap b$  and  $0$  both recognise the empty language, while  $\mathcal{Rel} \not\models a \cap b = 0$ : one can interpret  $a$  and  $b$  by relations with a non-empty intersection.

**1.2. Graphs.** Freyd and Scedrov’ decision procedure for representable allegories [14, page 208] relies on a notion of directed, labelled, 2-pointed graph. The same notion was proposed independently by Andr eka and Bredikhin [2], in a more comprehensive way.

Call *terms* the terms built out of composition, intersection, converse, constants  $1$  and  $\top$ , and variables  $a, b, \dots$ . A term  $u$  can be represented as a labelled directed graph  $\mathcal{G}(u)$  with two distinguished vertices called the *input* and the *output*. We give some examples in Figure 1, see Section 3 for a precise definition.

This algebra of two-pointed graphs can be used—and was proposed by Freyd, Scedrov, Andr eka and Bredikhin, for the full syntax of allegories. All results from this section extend to such a setting, in particular characterisation (1.8) below. In the present case, where we

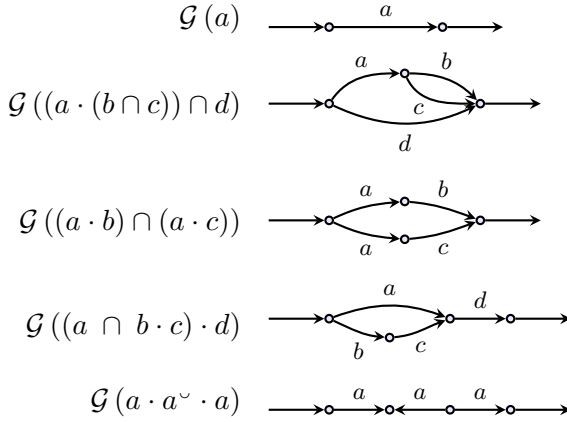


Figure 1: Graphs associated with some terms.

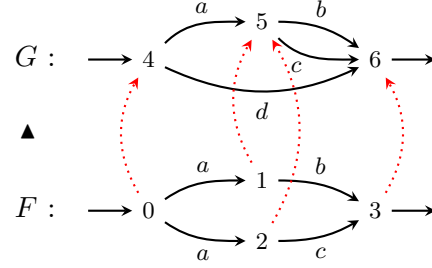


Figure 2: A graph homomorphism.

forbid converse and the constants identity and full relation, an important property is that the considered graphs are acyclic, and in fact, series-parallel.

Graphs can be endowed with a preorder relation: we write  $G \blacktriangleleft F$  when there exists a graph homomorphism from  $F$  to  $G$  preserving labels, inputs, and outputs. For instance the graph corresponding to  $(a \cdot (b \cap c)) \cap d$  is less than the graph of  $(a \cdot b) \cap (a \cdot c)$ , thanks to the homomorphism depicted in Figure 2 using dotted arrows. Notice that the homomorphism needs not be injective or surjective, so that this preorder has nothing to do with the respective sizes of the graphs: a graph may very well be smaller than another in the sense of  $\blacktriangleleft$ , while having more vertices or edges (and vice versa).

The key result from Freyd and Scedrov [14, page 208], or Andr eka and Bredikhin [2, Theorem 1], is that for all terms  $u, v$ , we have

$$\mathcal{R}el \models u \leq v \text{ if and only if } \mathcal{G}(u) \blacktriangleleft \mathcal{G}(v). \quad (1.7)$$

The graphs are finite so that one can search exhaustively for a homomorphism, whence the decidability result.

**1.3. Graph languages.** To extend the above graph-theoretical characterisation to identity-free Kleene lattices, we need to handle union, zero, and transitive closure. It suffices for that to consider sets of graphs: to each expression  $e$ , we associate a set of graphs  $\mathcal{G}(e)$ . This set is infinite whenever the expression  $e$  contains transitive closures.

Writing  $\blacktriangleleft X$  for the downward closure of a set of graphs  $X$  by the preorder  $\blacktriangleleft$  on graphs, we obtain the following generalisation of both (1.6) and (1.7): for all expressions  $e, f$ ,

$$\mathcal{R}el \models e \leq f \text{ if and only if } \blacktriangleleft \mathcal{G}(e) \subseteq \blacktriangleleft \mathcal{G}(f). \quad (1.8)$$

This is Theorem 3.9 in the sequel, and this result is almost there in the work by Andr eka et al. [2, 1]. To the best of our knowledge this explicit formulation is new, as well as its use towards decidability results.

When  $e$  and  $f$  are terms, we recover the characterisation (1.7) for representable allegories:  $\mathcal{G}(e)$  and  $\mathcal{G}(f)$  are singleton sets in this case. For plain regular expressions, the graphs are just words and the preorder  $\blacktriangleleft$  reduces to isomorphism. We thus recover the characterisation (1.6) for Kleene algebra. This result also generalises the characterisation provided by  sik et al. [3] for Kleene algebra with converse: graphs of expressions without intersection are just words

over a duplicated alphabet, and the corresponding restriction of the preorder  $\blacktriangleleft$  precisely corresponds to the word rewriting system they use.

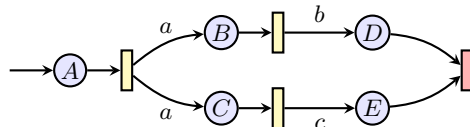
**1.4. Petri automata.** In order to exploit the above characterisation and obtain decidability results, one has first to represent graph languages in a finitary way. We propose for that a new finite automata model, largely based on Petri nets [30, 29, 28]. We describe this model below, ignoring converse, 1, and  $\top$  for the sake of clarity.

Recall that a Petri net consists of

- a finite set of *places*, denoted with circles;
- a set of *transitions*, denoted with rectangles;
- for each transition, a set of input places and a set of output places, denoted with arrows;
- an *initial place*, denoted by an entrant arrow;
- a set of *final markings*.

The execution model is the following: start by putting a token on the initial place; choose a transition whose input places all contain a token, remove those tokens and put new tokens in the output places of the transition; repeat this process until a final marking is reached. The obtained sequence of transitions is called an *accepting run*. (We actually restrict ourselves to *safe* Petri nets, to ensure that there is always at most one token in a given place when playing this game.)

A *Petri automaton* is just a safe Petri net with variables labelling the outputs of each transition, and with a single final marking consisting in the empty set of tokens. The automaton depicted below is the automaton we will construct for the term  $a \cdot b \cap a \cdot c$ . A run must start by firing the left-most transition, reaching the marking  $\{B, C\}$ ; then we have the choice of firing the upper transition first, reaching the marking  $\{D, C\}$ , or the lower one, reaching the marking  $\{B, E\}$ . In both cases we reach the marking  $\{D, E\}$  by firing the remaining transition. We complete the run by firing the last transition, which leads to the final marking  $\emptyset$ .



To read a graph in such an automaton, we try to find an accepting run that matches the graph up to homomorphism (Definitions 8.1 and 8.3). We do that by using a sequence of functions from the successive configurations of the run to the vertices of the graph. We start with the function mapping the unique token in the initial place, to the input vertex of the graph. To fire a transition, we must check that all its input tokens are mapped to the same vertex in the graph, and that this vertex has several outgoing edges, labelled according to the outputs of the transition. If this is the case, we update the function by removing the mappings corresponding to the deleted tokens, and by adding new mappings for each of the created tokens (using the target vertices of the aforementioned outgoing edges, according to the labels). If a transition has no output, before firing it we additionally require that each of its input tokens are mapped to the output vertex of the graph. The graph is accepted by the Petri automaton if we can reach the final marking  $\emptyset$ .

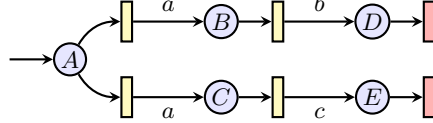
For instance, the previous automaton accepts the graph of  $a \cdot b \cap a \cdot c$  ( $F$  in Figure 2). We start with the function  $\{A \mapsto 0\}$ . We can fire the first transition, updating the function into  $\{B \mapsto 1, C \mapsto 2\}$  (We could also choose to update the function into  $\{B \mapsto 2, C \mapsto 1\}$ ,

or  $\{B, C \mapsto 1\}$ , or  $\{B, C \mapsto 2\}$  but this would lead to a dead-end). Then we can fire the upper transition, evolving the function into  $\{D \mapsto 3, C \mapsto 2\}$ , the lower transition leading to  $\{D, E \mapsto 3\}$  and we finish by firing the remaining transition, thus getting the function with domain  $\emptyset$ .

We call *language* of  $\mathcal{A}$  the set of graphs  $\mathcal{L}(\mathcal{A})$  accepted by a Petri automaton  $\mathcal{A}$ . This language is downward-closed:  $\mathcal{L}(\mathcal{A}) = \blacktriangleleft \mathcal{L}(\mathcal{A})$ . For instance, the previous automaton also accepts the graph  $G$  from Figure 2, which is smaller than  $F$ . Indeed, when we fire the first transition, we can associate the two newly created tokens (in places  $B$  and  $C$ ) with the same vertex (5). This actually corresponds to composing the functions used to accept  $F$  with the homomorphism depicted with dotted arrows.

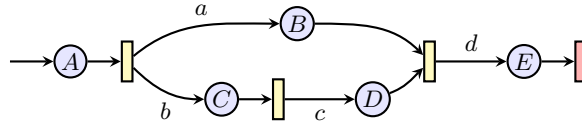
This automata model is expressive enough for Kleene allegories: for every expression  $e$ , we can construct a Petri automaton  $\mathcal{A}(e)$  such that  $\mathcal{L}(\mathcal{A}(e)) = \blacktriangleleft \mathcal{G}(e)$  (Sections 6 and 8). We give three other examples of Petri automata to give more intuition on their behaviour.

The first transition in the previous Petri automaton splits the initial token into two tokens, which are moved concurrently in the remainder of the run. This corresponds to an intersection in the considered expression. This is to be contrasted with the behaviour of the following automaton, which we would construct for the expression  $a \cdot b \cup a \cdot c$ . This automaton has two accepting runs:  $\{A\}, \{B\}, \{D\}, \emptyset$  and  $\{A\}, \{C\}, \{E\}, \emptyset$ , which can be used to accept the (graphs of the) terms  $a \cdot b$  and  $a \cdot c$ .

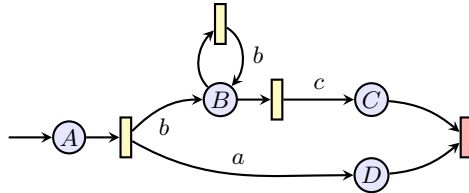


In a sense, two transitions competing for the same tokens represent a non-deterministic choice, *i.e.*, a union in an expression.

Still in the first example, the two tokens created by the first transition are later collected in the final transition. Tokens may also be collected and merged by a non-final transition. Consider for instance the following automaton for  $(a \cap b \cdot c) \cdot d$ . It has only one accepting run,  $\{A\}, \{B, C\}, \{B, D\}, \{E\}, \emptyset$ , and this run can be used to read the fourth graph from Figure 1.



As a last example, consider the following automaton for the expression  $a \cap b^+ \cdot c$ . The upper transition introduces a loop, so that there are infinitely many accepting runs. For all  $n > 0$ , the graph of the term  $a \cap b^n \cdot c$  is accepted by this automaton.



CONTENTS

1. Introduction	1
2. Terms and expressions	7
3. Graphs	9
4. Series-parallel graphs	11
5. Petri automata	13
6. From expressions to Petri automata	16
7. From Petri automata to expressions	21
8. Reading graphs modulo homomorphism	33
9. Comparing Petri automata modulo homomorphism	36
10. Complexity	43
11. Relationship with standard Petri net notions	44
12. Branching Automata	45
References	49

In Section 2 we define the various sets of expressions we consider in the paper as well as a few auxiliary syntactic functions. We define graphs in Section 3, and we use them to characterise the equational theory of representable Kleene allegories. Series-parallel graphs play a central role, they are defined and related to fragments of expressions in Section 4.

Petri automata and the sets of graphs they produce are defined in Section 5. The next two sections are devoted to the Kleene theorem for Petri automata: in Section 6 we translate expressions into automata, and we obtain the converse translation in Section 7. The latter translation is technically involved and requires a large number of preliminary definitions. The results and definitions in this long section are not required in the sequel.

Section 8 is devoted to the reading of graphs in Petri automata modulo homomorphism, in a local and incremental way, as illustrated in the Introduction. This leads us to Section 9, where we show how to compare Petri automata modulo homomorphism, using an appropriate notion of simulation. We provide complexity bounds in Section 10.

We relate the notion of graph produced by a Petri automaton to the standard notion of *pomset-trace* of a Petri net in Section 11. We discuss other works on regular sets of graphs and we compare Petri automata with Lodaya and Weil’s *branching automata* in Section 12.

2. TERMS AND EXPRESSIONS

We let  $a, b$  range over a set  $X$  of *variables*. We consider the following syntax of *expressions*:

$$e, f ::= e \cup f \mid e \cdot f \mid e \cap f \mid e^+ \mid e^\vee \mid 0 \mid 1 \mid \top \mid a \quad (a \in X)$$

We denote their set by  $\text{Exp}_X$ . We write  $\text{Trm}_X$  for the subset of *terms*: those expressions that do not contain  $0, \cup$  or  $_+^+$ .

$$u, v ::= u \cdot v \mid u \cap v \mid u^\vee \mid 1 \mid \top \mid a \quad (a \in X)$$

If  $\sigma : X \rightarrow \mathcal{Rel} \langle S \rangle$  is an interpretation of the variables into some algebra of relations, we write  $\hat{\sigma}$  for the unique homomorphism extending  $\sigma$  into a function from  $\text{Exp}_X$  to  $\mathcal{Rel} \langle S \rangle$ .

**Definition 2.1** (Validity of an (in)equation). An inequation between two expressions  $e$  and  $f$  is *valid*, written  $\mathcal{R}el \models e \leq f$ , if for every relational interpretation  $\sigma$  we have  $\widehat{\sigma}(e) \subseteq \widehat{\sigma}(f)$ . Similarly, we write  $\mathcal{R}el \models e = f$ , if for every relational interpretation  $\sigma$  we have  $\widehat{\sigma}(e) = \widehat{\sigma}(f)$ .

Expressions actually denote sets of terms.

**Definition 2.2** (Terms of an expression). The *set of terms* of an expression  $e \in \text{Exp}_X$ , written  $\llbracket e \rrbracket$ , is the set of terms defined inductively as follows.

$$\begin{aligned} \llbracket e \cdot f \rrbracket &\triangleq \{u \cdot v \mid u \in \llbracket e \rrbracket \text{ and } v \in \llbracket f \rrbracket\} & \llbracket 1 \rrbracket &\triangleq \{1\} \\ \llbracket e \cap f \rrbracket &\triangleq \{u \cap v \mid u \in \llbracket e \rrbracket \text{ and } v \in \llbracket f \rrbracket\} & \llbracket \top \rrbracket &\triangleq \{\top\} \\ \llbracket e^\vee \rrbracket &\triangleq \{u^\vee \mid u \in \llbracket e \rrbracket\} \\ \llbracket e \cup f \rrbracket &\triangleq \llbracket e \rrbracket \cup \llbracket f \rrbracket & \llbracket 0 \rrbracket &\triangleq \emptyset \\ \llbracket e^+ \rrbracket &\triangleq \bigcup_{n>0} \{u_1 \cdots u_n \mid \forall i, u_i \in \llbracket e \rrbracket\} & \llbracket a \rrbracket &\triangleq \{a\} \end{aligned}$$

As observed by Andr eka, Mikul as, and N emeti the (relational) semantics of an expression only depends on the above set of terms.

**Lemma 2.3** ([1, Lemma 2.1]). *For every expression  $e \in \text{Exp}_X$ , every set  $S$  and every relational interpretation  $\sigma : X \rightarrow \mathcal{R}el \langle S \rangle$ , we have*

$$\widehat{\sigma}(e) = \bigcup_{u \in \llbracket e \rrbracket} \widehat{\sigma}(u).$$

We will consider the following subsets of terms and expressions when we focus on identity-free Kleene lattices. *Simple terms* (resp. *simple expressions*) are those terms (resp. expressions) that do not contain converse, 1 or  $\top$ ; their set is denoted by  $\text{Trm}_{\bar{X}}$  (resp.  $\text{Exp}_{\bar{X}}$ ).

Let  $X_\bullet \triangleq X \cup \{a' \mid a \in X\} \cup \{1, \top\}$ . The following function  $\llbracket \_ \rrbracket$  on the left-hand side associates a simple expression over  $X_\bullet$  to every expression over  $X$ . It is defined recursively together with the auxiliary function on the right-hand side, it basically consists in pushing converse to the leaves.

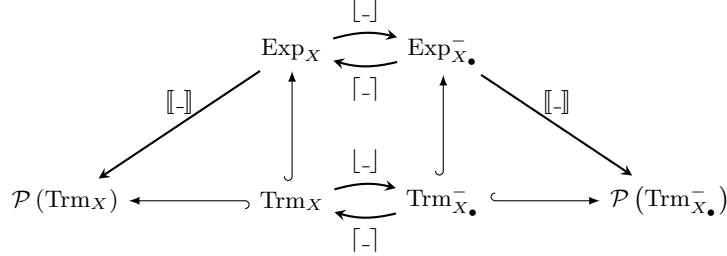
$$\begin{array}{ll} \llbracket \_ \rrbracket : \text{Exp}_X \rightarrow \text{Exp}_{\bar{X}_\bullet} & \llbracket \_ \rrbracket' : \text{Exp}_X \rightarrow \text{Exp}_{\bar{X}_\bullet} \\ e \cdot f \mapsto \llbracket e \rrbracket \cdot \llbracket f \rrbracket & e \cdot f \mapsto \llbracket f \rrbracket' \cdot \llbracket e \rrbracket' \\ e \cap f \mapsto \llbracket e \rrbracket \cap \llbracket f \rrbracket & e \cap f \mapsto \llbracket e \rrbracket' \cap \llbracket f \rrbracket' \\ e \cup f \mapsto \llbracket e \rrbracket \cup \llbracket f \rrbracket & e \cup f \mapsto \llbracket e \rrbracket' \cup \llbracket f \rrbracket' \\ e^+ \mapsto \llbracket e \rrbracket^+ & e^+ \mapsto \llbracket e \rrbracket'^+ \\ e^\vee \mapsto \llbracket e \rrbracket' & e^\vee \mapsto \llbracket e \rrbracket \\ 1 \mapsto 1 \in X_\bullet & 1 \mapsto 1 \in X_\bullet \\ \top \mapsto \top \in X_\bullet & \top \mapsto \top \in X_\bullet \\ 0 \mapsto 0 & 0 \mapsto 0 \\ a \in X \mapsto a \in X_\bullet & a \in X \mapsto a' \in X_\bullet \end{array}$$

Conversely, we write  $\lceil \_ \rceil : \text{Exp}_{\bar{X}_\bullet} \rightarrow \text{Exp}_X$  for the unique homomorphism such that  $\lceil a \rceil = a$ ,  $\lceil a' \rceil = a^\vee$ ,  $\lceil 1 \rceil = 1$ , and  $\lceil \top \rceil = \top$ . It is straightforward to check that for every simple expression  $e \in \text{Exp}_{\bar{X}_\bullet}$ , we have  $\llbracket \lceil e \rceil \rrbracket = e$ . In the other direction, the equality is not syntactic: for every expression  $e \in \text{Exp}_X$ , we have  $\mathcal{R}el \models \llbracket \lceil e \rceil \rrbracket = e$ . (This equation also



holds in language algebras, where converse also commutes with the other operations in the appropriate way.) Those two functions naturally restrict to terms and simple terms.

The diagram below summarises the sets and functions we defined so far.



### 3. GRAPHS

We let  $G, H$  range over 2-pointed labelled directed graphs, which we simply call *graphs* in the sequel. Those are tuples  $\langle V, E, \iota, o \rangle$  with  $V$  a finite set of vertices,  $E \subseteq V \times X \times V$  a set of edges labelled with  $X$ , and  $\iota, o \in V$  two distinguished vertices, respectively called *input* and *output*. We write  $\text{Gph}_X$  for the set of such graphs. They were introduced independently by Freyd and Scedrov [14, page 208], and Andr eka and Bredikhin [2], for allegories; they form an algebra for the sub-signature of terms:

- $G \cdot H$  is the *series composition* of the two graphs  $G$  and  $H$ , obtained by merging the output of  $G$  with the input of  $H$ ;
- $G \sqcap H$  is the *parallel composition* of the two graphs  $G$  and  $H$ , obtained by merging their inputs and their outputs;
- $G^\vee$  is the graph obtained from  $G$  by swapping input and output;
- $1$  is the graph with a single vertex (both input and output) and no edge;
- $\top$  is the disconnected graph with two vertices (the input and the output) and no edge.

See Figure 3 for a graphical description of these operations.

**Definition 3.1** (Graph of a term). The *graph*  $\mathcal{G}(u)$  of a term  $u$  is obtained by using the unique homomorphism  $\mathcal{G}$  mapping a variable  $a$  to the graph with two vertices (the input and the output), and a single edge labelled  $a$  from the input to the output.

Figure 1 in the Introduction contains some examples; graphs of non-simple terms are displayed in Figure 4.

A significant number of the results presented in [2, 1] rely on the following technical lemma.

**Lemma 3.2** ([2, Lemma 3]). *Let  $v \in \text{Trm}_X$  be a term and write  $\mathcal{G}(v) = \langle V_v, E_v, \iota_v, o_v \rangle$ . Let  $S$  be a set and let  $\sigma: X \rightarrow \text{Rel}\langle S \rangle$ . For all  $i, j \in S$ , we have  $\langle i, j \rangle \in \widehat{\sigma}(v)$  if and only if there exists a function  $\varphi: V_v \rightarrow S$  such that*

- (i)  $\varphi(\iota_v) = i$ ,
- (ii)  $\varphi(o_v) = j$ , and
- (iii) if  $\langle p, a, q \rangle \in E_v$  then  $\langle \varphi(p), \varphi(q) \rangle \in \sigma(a)$ .

This lemma actually is the bridge between relational models and the following natural definition of graph homomorphism.

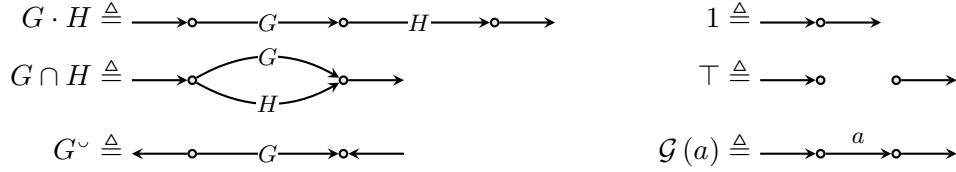


Figure 3: The algebra of graphs and the graph of a variable.

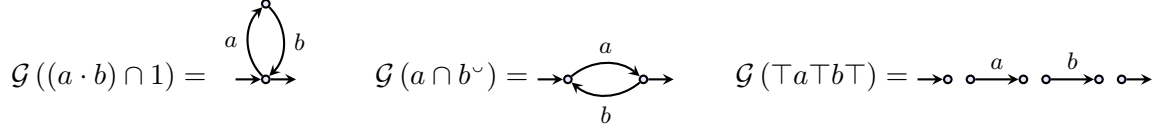


Figure 4: Graphs of non-simple terms.

**Definition 3.3** (Graph homomorphism). A *graph homomorphism* from  $\langle V_1, E_1, \iota_1, o_1 \rangle$  to  $\langle V_2, E_2, \iota_2, o_2 \rangle$  is a map  $\varphi: V_1 \rightarrow V_2$  such that

- (i)  $\varphi(\iota_1) = \iota_2$ ,
- (ii)  $\varphi(o_1) = o_2$ , and
- (iii)  $\langle p, a, q \rangle \in E_1$  entails  $\langle \varphi(p), a, \varphi(q) \rangle \in E_2$ .

**Definition 3.4** (Preorders  $\blacktriangleleft$  and  $\triangleleft$ ). We denote by  $\blacktriangleleft$  the relation on graphs defined by  $G \blacktriangleleft G'$  if there exists a graph homomorphism from  $G'$  to  $G$ . This relation gives rise to a relation on terms, written  $\triangleleft$  and defined by  $u \triangleleft v$  if  $\mathcal{G}(u) \blacktriangleleft \mathcal{G}(v)$ .

Those two relations are preorders: they are reflexive and transitive. As explained in the introduction, they precisely characterise inclusion under arbitrary relational interpretations:

**Theorem 3.5** ([2, Theorem 1], [14, page 208]). *For all terms  $u, v$ , we have*

$$\mathcal{R}el \models u \leq v \text{ if and only if } u \triangleleft v.$$

To extend this result to the whole syntax of expressions, we introduce the following generalisation of the language of a regular expression. Sets of words become sets of graphs.

**Definition 3.6** (Graphs of an expression). The *set of graphs* of an expression  $e \in \text{Exp}_X$  is the set  $\mathcal{G}(e) \triangleq \{\mathcal{G}(u) \mid u \in \llbracket e \rrbracket\}$

If  $e \in \text{Exp}_X$  is a regular expression (meaning it never uses the intersection and converse operators, nor the constant  $\top$ ), then  $\mathcal{G}(e)$  is isomorphic to the rational language of words usually associated with  $e$ : terms in  $\llbracket e \rrbracket$  are monoid expressions, so that graphs in  $\mathcal{G}(e)$  can be identified with words. We define accordingly a notion of regular set of graphs.

**Definition 3.7** (Regular set of graphs). A set  $S \subseteq \text{Gph}_X$  of graphs is *regular* if there exists an expression  $e \in \text{Exp}_X$  such that  $\mathcal{G}(e) = S$ .

Coming back to representable Kleene allegories, we need a slight refinement of Lemma 2.3 to obtain the characterisation announced in the Introduction:

**Lemma 3.8.** *For every expression  $e \in \text{Exp}_X$ , every set  $S$  and every relational interpretation  $\sigma: X \rightarrow \mathcal{R}el \langle S \rangle$ , we have*

$$\widehat{\sigma}(e) = \bigcup_{u \in \triangleleft \llbracket e \rrbracket} \widehat{\sigma}(u).$$

*Proof.* We use Lemma 2.3 and the fact that  $\hat{\sigma}(w) \subseteq \hat{\sigma}(u)$  whenever  $w \triangleleft u$ , thanks to Theorem 3.5.  $\square$

Given a set  $S$  of graphs, we write  $\blacktriangleleft S$  for its downward closure:  $\blacktriangleleft S \triangleq \{G \mid G \blacktriangleleft G', G' \in S\}$ . Similarly, we write  $\triangleleft S$  for the downward closure of a set of terms w.r.t.  $\triangleleft$ .

**Theorem 3.9.** *The following properties are equivalent, for all expressions  $e, f \in \text{Exp}_X$ :*

- (i)  $\mathcal{R}el \models e \leq f$ ,
- (ii)  $\llbracket e \rrbracket \subseteq \triangleleft \llbracket f \rrbracket$ ,
- (iii)  $\mathcal{G}(e) \subseteq \blacktriangleleft \mathcal{G}(f)$ .

(Note that this theorem amounts to the characterisation announced in Section 1 (1.8): for all sets  $X, Y$ , we have  $\blacktriangleleft X \subseteq \blacktriangleleft Y$  if and only if  $X \subseteq \blacktriangleleft Y$ .) In other words, deciding the (in)equational theory of representable Kleene allegories reduces to comparing regular sets graphs, modulo homomorphism.

*Proof.* The implication (ii)  $\Rightarrow$  (i) follows easily from Lemma 3.8, and (iii)  $\Rightarrow$  (ii) is a matter of unfolding definitions. For (i)  $\Rightarrow$  (iii), we mainly use Lemma 3.2:

Let  $e, f$  be two expressions such that  $\mathcal{R}el \models e \leq f$ , and  $u \in \llbracket e \rrbracket$  such that  $\mathcal{G}(u) = \langle V_u, E_u, \iota_u, o_u \rangle$ ; we can build an interpretation  $\sigma: X \rightarrow \mathcal{R}el \langle V_u \rangle$  by specifying:

$$\sigma(a) \triangleq \{ \langle p, q \rangle \mid \langle p, a, q \rangle \in E_u \}.$$

It is simple to check that  $\hat{\sigma}(u) = \{ \langle \iota_u, o_u \rangle \}$ . By Lemma 2.3 and  $\mathcal{R}el \models e \leq f$ , we know that

$$\hat{\sigma}(u) \subseteq \hat{\sigma}(f) = \bigcup_{v \in \llbracket f \rrbracket} \hat{\sigma}(v).$$

Thus there is some  $v \in \llbracket f \rrbracket$  such that  $\langle \iota_u, o_u \rangle \in \hat{\sigma}(v)$ . By Lemma 3.2 we get that there is a map  $\varphi: V_v \rightarrow V_u$  such that  $\varphi(\iota_v) = \iota_u$ ;  $\varphi(o_v) = o_u$  and

$$\langle p, a, q \rangle \in E_v \Rightarrow \langle \varphi(p), \varphi(q) \rangle \in \sigma(a).$$

Using the definition of  $\sigma$ , we rewrite this last condition as

$$\langle p, a, q \rangle \in E_v \Rightarrow \langle \varphi(p), a, \varphi(q) \rangle \in E_u.$$

Thus  $\varphi$  is a graph homomorphism from  $\mathcal{G}(v)$  to  $\mathcal{G}(u)$ , proving that  $\mathcal{G}(u) \blacktriangleleft \mathcal{G}(v)$ , hence  $\mathcal{G}(u) \in \blacktriangleleft \mathcal{G}(f)$ .  $\square$

#### 4. SERIES-PARALLEL GRAPHS

When focusing on identity-free Kleene lattice, we can forget the converse operation and the constants 1 and  $\top$ , and work with simple terms and expressions. In such a situation, the manipulated graphs are *series-parallel*.

More precisely, consider Valdes et al. *SP-rewriting system* [34]. This system, extended to edge-labelled graphs, is recalled in Figure 5. The second rule can only be applied if the intermediate vertex on the left-hand side is not the input and the output, and has no other adjacent edge. Valdes et al. [34] showed that that system has the Church-Rosser property. This property can be extended to labelled graphs without difficulty, modulo the congruence generated by the associativity of  $\cdot$  and the associativity and commutativity of  $\cap$ .

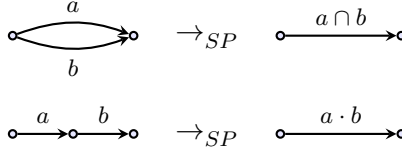
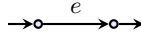


Figure 5: The SP-rewriting system

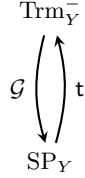
A graph  $G$  is *series-parallel* if it SP-reduces in some number of steps to a graph of the following shape.



In such a case,  $e$  is a simple term called a *term representation* of  $G$ . This term is not unique but we can choose one which we write  $t(G)$ . We write  $\text{SP}_Y$  for the set of series-parallel graphs labelled in a set  $Y$ . Every series-parallel graph has a single source, its input and a single sink, its output. Also note the series-parallel graphs are acyclic.

**Lemma 4.1.** *A graph is series-parallel if and only if it is the graph of a simple term.*

In other words, we have the following situation.



It is thus natural to consider only simple expressions for defining the notion of regular set of series-parallel graphs.

**Definition 4.2** (Regular set of series-parallel graphs). A set  $S \subseteq \text{SP}_X$  of series-parallel graphs is *regular* if there exists a simple expression  $e \in \text{Exp}_{\bar{X}}$  such that  $\mathcal{G}(e) = S$ .

Recall the functions  $\lfloor \_ \rfloor : \text{Trm}_X \rightarrow \text{Trm}_{\bar{X}_\bullet}$  and  $\lceil \_ \rceil : \text{Trm}_{\bar{X}_\bullet} \rightarrow \text{Trm}_X$  from the previous section. A counterpart to the latter can be defined at the level of graphs.

**Definition 4.3** ( $\lceil G \rceil$ ). Let  $G = \langle V, E, \iota, o \rangle \in \text{Gph}_{X_\bullet}$  be a graph labelled with  $X_\bullet$ . Let  $\equiv$  be the smallest equivalence relation on  $V$  containing all pairs  $\langle i, j \rangle$  such that  $\langle i, 1, j \rangle \in E$ , and  $[i]$  be the equivalence class of  $i$ . We associate with  $G$  the following graph labelled in  $X$ .

$$\lceil G \rceil \triangleq \langle V/\equiv, E', [\iota], [o] \rangle, \text{ where}$$

$$E' \triangleq \{ \langle [i], x, [j] \rangle \mid x \in X, \exists \langle k, l \rangle \in [i] \times [j] : \langle k, x, l \rangle \in E \text{ or } \langle l, x', k \rangle \in E \}.$$

Informally,  $\lceil G \rceil$  is obtained by merging all vertices related by an edge labelled with 1, removing all edges labelled with  $\top$ , and reversing all edges labelled with some  $a' \in X_\bullet$ . This operation is defined on arbitrary graphs labelled in  $X_\bullet$ , but we shall use it only on series-parallel graphs.

**Proposition 4.4.** *For every term  $u \in \text{Trm}_X$ , we have  $\mathcal{G}(u) = \lceil \mathcal{G}(\lfloor u \rfloor) \rceil$ . For every simple term  $u \in \text{Trm}_{\bar{X}_\bullet}$ , we have  $\mathcal{G}(\lceil u \rceil) = \lceil \mathcal{G}(u) \rceil$ . In other words, the following diagrams commute.*

$$\begin{array}{ccc}
 Trm_X & \xrightarrow{[-]} & Trm_{X_\bullet}^- \\
 \mathcal{G} \downarrow & & \downarrow \mathcal{G} \\
 Gph_X & \xleftarrow{[-]} & SP_{X_\bullet}
 \end{array}
 \qquad
 \begin{array}{ccc}
 Trm_X & \xleftarrow{[-]} & Trm_{X_\bullet}^- \\
 \mathcal{G} \downarrow & & \downarrow \mathcal{G} \\
 Gph_X & \xleftarrow{[-]} & SP_{X_\bullet}
 \end{array}$$

This result can be extended to expressions and sets of graphs, which we can depict using the following commutative diagrams.

$$\begin{array}{ccc}
 Exp_X & \xrightarrow{[-]} & Exp_{X_\bullet}^- \\
 \mathcal{G} \downarrow & & \downarrow \mathcal{G} \\
 \mathcal{P}(Gph_X) & \xleftarrow{[-]} & \mathcal{P}(SP_{X_\bullet})
 \end{array}
 \qquad
 \begin{array}{ccc}
 Exp_X & \xleftarrow{[-]} & Exp_{X_\bullet}^- \\
 \mathcal{G} \downarrow & & \downarrow \mathcal{G} \\
 \mathcal{P}(Gph_X) & \xleftarrow{[-]} & \mathcal{P}(SP_{X_\bullet})
 \end{array}$$

## 5. PETRI AUTOMATA

We fix an alphabet  $Y$ , which we will later instantiate either with the set  $X$  of variables used in expressions, or with its extended version  $X_\bullet$ .

A Petri automaton is essentially a safe Petri net where the arcs coming out of transitions are labelled by letters from  $Y$ .

**Definition 5.1** (Petri Automaton). A *Petri automaton*  $\mathcal{A}$  over the alphabet  $Y$  is a tuple  $\langle P, \mathcal{T}, \iota \rangle$  where:

- $P$  is a finite set of *places*,
- $\mathcal{T} \subseteq \mathcal{P}(P) \times \mathcal{P}(Y \times P)$  is a set of *transitions*,
- $\iota \in P$  is the *initial place* of the automaton.

For each transition  $t = \langle {}^a t, t^\flat \rangle \in \mathcal{T}$ ,  ${}^a t$  is assumed to be non-empty;  ${}^a t \subseteq P$  is the *input* of  $t$ ; and  $t^\flat \subseteq Y \times P$  is the *output* of  $t$ . Transitions with empty outputs are called *final*, and transitions with input  $\{\iota\}$  are called *initial*. We write  $\text{PA}_Y$  for the set of Petri automata over the alphabet  $Y$ .

We write  $\pi_2(t^\flat) \triangleq \{p \mid \exists a, \langle a, p \rangle \in t^\flat\}$  for the set of places appearing in the output of  $t$ . We will add two constraints on this definition along the way (Constraints 5.3 and 5.5), but we need more definitions to state them. An example of such an automaton is described in Figure 6. The graphical representation used here draws round vertices for places and rectangular vertices for transitions, with the incoming and outgoing arcs to and from the transition corresponding respectively to the inputs and outputs of said transition. The initial place is denoted by an unmarked incoming arc.

**5.1. Runs and reachable states.** We define the operational semantics of Petri automata. Let us fix a Petri automaton  $\mathcal{A} = \langle P, \mathcal{T}, \iota \rangle$  for the remainder of this section. A *state* of this automaton is a set of places (i.e., a configuration in the Petri nets terminology). In a given state  $S \subseteq P$ , a transition  $t = \langle {}^a t, t^\flat \rangle$  is *enabled* if  ${}^a t \subseteq S$ . In that case, we may fire  $t$ , leading to a new state  $S' = (S \setminus {}^a t) \cup \pi_2(t^\flat)$ . This will be denoted in the following by  $\mathcal{A} \vdash t : S \rightarrow S'$ . We extend this notation to sequences of transitions in the natural way:

$$\frac{\mathcal{A} \vdash t_1 : S_0 \rightarrow S_1, \quad \mathcal{A} \vdash t_2; \dots; t_n : S_1 \rightarrow S_n}{\mathcal{A} \vdash t_1; t_2; \dots; t_n : S_0 \rightarrow S_n}$$

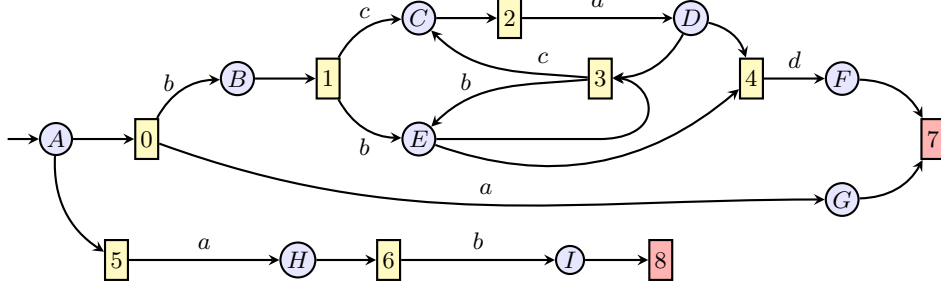


Figure 6: A Petri automaton.

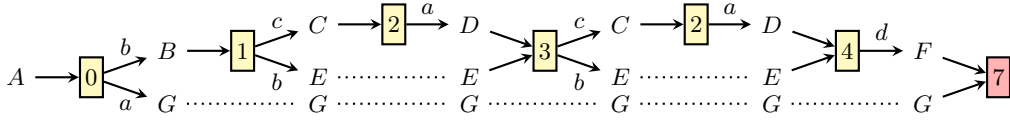


Figure 7: An accepting run in the automaton from Figure 6.

In that case we say that  $\langle S_0, t_1; t_2; \dots; t_n, S_n \rangle$  is a *valid run*, or simply *run*, from  $S_0$  to  $S_n$ . If  $S_0 = \{\iota\}$  then the run is *initial* and if  $S_n = \emptyset$  it is *final*. A run that is both initial and final is called *accepting*. A state  $S$  is *reachable* in  $\mathcal{A}$  if there is an initial run leading to  $S$ .

We write  $\text{Run}_{\mathcal{A}}$  for the set of runs of an automaton  $\mathcal{A}$ , and  $\text{Run}_{\mathcal{A}}(A, B)$  for the set of runs from state  $A$  to state  $B$  in  $\mathcal{A}$ . The set of its accepting runs is then equal to  $\text{Run}_{\mathcal{A}}(\{\iota\}, \emptyset)$ , and written  $\text{Run}_{\mathcal{A}}^{\text{acc}}$ .

**Example 5.2** (Accepting run). The triple  $\langle \{A\}, 0; 1; 2; 3; 2; 4; 7, \emptyset \rangle$  is a valid run in the automaton from Figure 6. It is easy enough to check that:

$$\begin{aligned} \mathcal{A} \vdash 0 : \{A\} &\rightarrow \{B, G\}; & \mathcal{A} \vdash 1 : \{B, G\} &\rightarrow \{C, E, G\}; \\ \mathcal{A} \vdash 2 : \{C, E, G\} &\rightarrow \{D, E, G\}; & \mathcal{A} \vdash 3 : \{D, E, G\} &\rightarrow \{C, E, G\}; \\ \mathcal{A} \vdash 4 : \{D, E, G\} &\rightarrow \{F, G\}; & \mathcal{A} \vdash 7 : \{F, G\} &\rightarrow \emptyset. \end{aligned}$$

Thus we can prove that  $\mathcal{A} \vdash 0; 1; 2; 3; 2; 4; 7 : \{A\} \rightarrow \emptyset$ . Furthermore, as  $A$  is the initial place, this run is accepting. It can be represented graphically as in Figure 7.

We may now state the first constraint we impose on Petri automata.

**Constraint 5.3** (Safety). *For every reachable state  $S$  of a Petri automaton  $\mathcal{A}$ , if for some transition  $t$  we have  $\mathcal{A} \vdash t : S \rightarrow S'$ , then*

$$(S \setminus {}^{\circ}t) \cap \pi_2(t^{\circ}) = \emptyset.$$

This constraint corresponds to the classic Petri net property of safety (also called one-boundedness): at any time, there is at most one token in a given place, thus justifying or use of sets of places rather than multi-sets of places for configurations. Checking this constraint is feasible: the set of transitions is finite, and there are finitely many reachable states (those are subsets of a fixed finite set). The net in Figure 8 is not safe.

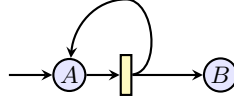


Figure 8: Example of non-safe Petri Net.

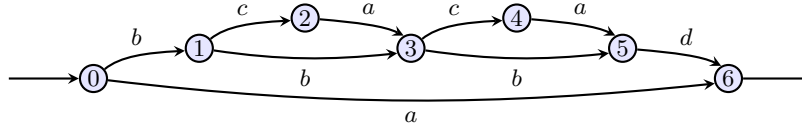


Figure 9: Trace of the run from Example 5.2.

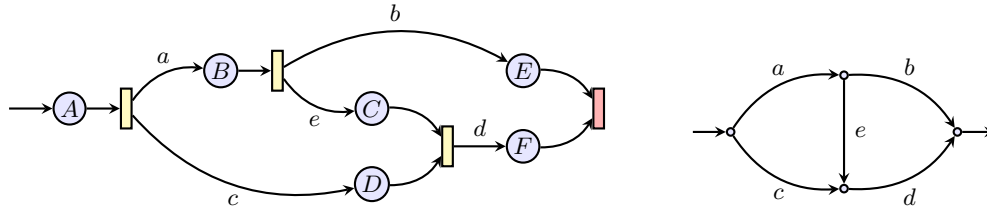


Figure 10: An automaton yielding a non series-parallel trace.

**5.2. Graphs produced by a Petri automaton.** Petri automata produce series-parallel graphs: to each accepting run we associate such a graph, called its *trace*. Consider an accepting run  $\langle \{t\}, t_0; \dots; t_n, \emptyset \rangle$ . Its trace is constructed by creating a vertex  $k$  for each transition  $t_k = \langle {}^a t_k, t_k^p \rangle$  of the run. We add an edge  $\langle k, a, l \rangle$  whenever there is some place  $q$  such that  $\langle a, q \rangle \in t_k^p$ , and  $t_l$  is the first transition after  $t_k$  in the run with  $q$  among its inputs. The definition we give here is a generalisation for arbitrary valid runs, that coincides with the informal presentation we just gave on accepting runs.

**Definition 5.4** (Trace of a run). Let  $R = \langle S, t_0; \dots; t_n, S' \rangle$  be a run in  $\mathcal{A}$ . For every  $k$  and  $p \in \pi_2(t_k^p)$ , we define

$$\nu(k, p) = \{l \mid l > k \text{ and } p \in {}^a t_l\}.$$

The *trace* of  $R$ , denoted by  $\mathcal{G}(R)$ , is the graph with vertices  $\{0, \dots, n\} \cup S'$  and the set of edges defined by:

$$E_R = \{\langle k, a, l \rangle \mid \langle a, p \rangle \in t_k^p \text{ and } (p = l \wedge \nu(k, p) = \emptyset) \vee (l = \min(\nu(k, p)))\}.$$

The trace of the run given in Example 5.2 is presented in Figure 9. Unfortunately, the trace of an accepting run is not necessarily series-parallel. Consider for instance the net in Figure 10; the trace of its only accepting run is the graph drawn on the right-hand side, which is not series-parallel. This leads us to the second constraint we impose on Petri automata.

**Constraint 5.5** (Series-parallel). *Every graph  $G \in \mathcal{G}(\mathcal{A})$  is series-parallel.*

Despite its infinitary formulation, this property is decidable. In fact, the procedure we describe in Section 7 to extract a simple expression out of a Petri automaton fails in finite time if it is given an automaton violating this constraint, and succeeds otherwise. Thanks to this constraint, we can restrict our attention to *proper* runs:

**Definition 5.6** (Proper run). A run  $R = \langle S_0, t_1; \dots; t_n, S_n \rangle \in \text{Run}_{\mathcal{A}}$  is *proper* if for all  $1 \leq i \leq n$  we have  $t_i^{\circ} = \emptyset \Rightarrow (i = n \wedge S_n = \emptyset)$ .

Sub-runs of a proper run are proper by definition, and we have

**Lemma 5.7.** *All accepting runs of a Petri automaton (satisfying Constraints 5.3 and 5.5) are proper runs.*

*Proof.* Let  $R = \langle \{\iota\}, t_0; \dots; t_n, \emptyset \rangle$  be an accepting run (which is series-parallel by Constraint 5.5). Since the last state is empty we know that the run must contain a final transition. Hence we may reformulate the definition of proper as  $t_n^{\circ} = \emptyset$  and  $\forall 0 \leq i < n$ ,  $t_i^{\circ} \neq \emptyset$ .

The proof relies on the following observation: the node  $i$  in  $\mathcal{G}(R)$  is a sink if and only if  $t_i^{\circ} = \emptyset$ . To prove this, remember that the edges coming out of  $i$  in  $\mathcal{G}(R)$  are:

$$\{\langle i, a, \min\{l \mid l > i \text{ and } p \in {}^a t_l\} \rangle \mid \forall \langle a, p \rangle \in t_i^{\circ}\}.$$

Because the run ends in state  $\emptyset$ , we know that  $\forall \langle a, p \rangle \in t_i^{\circ}$  there is a later transition  $t_j$  consuming the token at place  $p$  (otherwise  $p$  would remain in the last state). This entails that if  $t_i$  is not final, then  $i$  has a successor in  $\mathcal{G}(R)$ . On the other hand, if  $t_i^{\circ}$  is empty, then there cannot be a transition coming out of node  $i$ .

As a series-parallel graph has a single sink, it can only use a single final transition. Lastly, notice that  $\langle i, a, j \rangle \in E_R$  entails  $i < j$ . This means that the sink can only be the maximal node (for the ordering of natural numbers).  $\square$

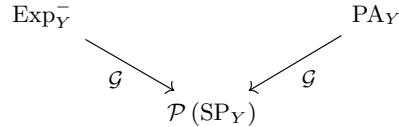
We finally define the set of graphs produced by a Petri automaton.

**Definition 5.8.** The *set of (series-parallel) graphs produced by a Petri automaton  $\mathcal{A}$*  is the set of traces of accepting (proper) runs of  $\mathcal{A}$ :

$$\mathcal{G}(\mathcal{A}) \triangleq \{\mathcal{G}(R) \mid R \in \text{Run}_{\mathcal{A}}^{\text{acc}}\}.$$

## 6. FROM EXPRESSIONS TO PETRI AUTOMATA

So far we have seen two ways of defining sets of series-parallel graphs: through simple expressions and through Petri automata.



Now we show how to associate with every simple expression  $e \in \text{Exp}_{\bar{Y}}$  a Petri automaton  $\mathcal{A}(e)$  labelled over  $Y$  such that:  $\mathcal{G}(e) = \mathcal{G}(\mathcal{A}(e))$ . It will follow that regular sets of series-parallel graphs are recognisable (Theorem 6.9).

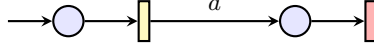
The construction goes by structural induction on the simple expressions  $e$ : we provide Petri automata constructions for each syntactic entry, thus proving that recognisable languages are closed under union, iteration and series and parallel compositions.



*Petri automaton for 0.* We need the automaton for 0 to have no accepting run. The following automaton  $\underline{0}$  has this property.



*Petri automaton for a letter.* For  $a \in Y$ , we want an automaton whose set of traces is simply the graph  $\mathcal{G}(a)$ . The following automaton  $\underline{a}$  works.



*Union of Petri automata.* Let  $\mathcal{A}_1 = \langle P_1, \mathcal{T}_1, \iota_1 \rangle$  and  $\mathcal{A}_2 = \langle P_2, \mathcal{T}_2, \iota_2 \rangle$  be two Petri automata with disjoint sets of places. To get an automation for  $\mathcal{G}(\mathcal{A}_1) \cup \mathcal{G}(\mathcal{A}_2)$ , we simply put the two automata side by side, we add a new initial place  $\iota$ , and for every initial transition  $\langle \{\iota_j\}, t^\flat \rangle$  we add a new transition  $\langle \{\iota\}, t^\flat \rangle$ . Formally:

$$\mathcal{A}_1 \cup \mathcal{A}_2 \triangleq \langle P_1 \cup P_2 \cup \{\iota\}, \mathcal{T}_1 \cup \mathcal{T}_2 \cup \{ \langle \{\iota\}, t^\flat \rangle \mid \langle \{\iota_j\}, t^\flat \rangle \in \mathcal{T}_j, j \in \{1, 2\} \}, \iota \rangle.$$

Equivalently the transitions in this automaton follow from the following rules:

$$\frac{\mathcal{A}_j \vdash t : S \rightarrow T}{\mathcal{A}_1 \cup \mathcal{A}_2 \vdash t : S \rightarrow T} \quad j \in 1, 2 \qquad \frac{\mathcal{A}_j \vdash \langle \{\iota_j\}, S \rangle : \{\iota_j\} \rightarrow S}{\mathcal{A}_1 \cup \mathcal{A}_2 \vdash \langle \{\iota\}, S \rangle : \{\iota\} \rightarrow S} \quad j \in 1, 2$$

**Lemma 6.1.** *We have  $\mathcal{G}(\mathcal{A}_1 \cup \mathcal{A}_2) = \mathcal{G}(\mathcal{A}_1) \cup \mathcal{G}(\mathcal{A}_2)$ .*

*Proof.* Let  $G \in \mathcal{G}(\mathcal{A}_1) \cup \mathcal{G}(\mathcal{A}_2)$ . There is an accepting run  $R = \langle \{\iota_j\}, t_0; t_1 \dots t_n; \emptyset \rangle$  in  $\mathcal{A}_j$ , for either  $j = 1$  or  $j = 2$ , such that  $G = \mathcal{G}(R)$ . By definition of a valid run, we know that there is a state  $S$  such that  $\mathcal{A}_j \vdash t_0 : \{\iota_j\} \rightarrow S$  and  $\mathcal{A}_j \vdash t_1; \dots; t_n : S \rightarrow \emptyset$ . Because  $\mathcal{A}_1 \cup \mathcal{A}_2$  contains in particular all places and transitions of  $\mathcal{A}_j$ , we can deduce that  $\mathcal{A}_1 \cup \mathcal{A}_2 \vdash t_1; \dots; t_n : S \rightarrow \emptyset$ . Furthermore  $\mathcal{A}_j \vdash t_0 : \{\iota_j\} \rightarrow S$  entails  $t_0 = \langle \{\iota_j\}, t_0^\flat \rangle$ , thus in  $\mathcal{A}_1 \cup \mathcal{A}_2$  we have the transition  $t'_0 = \langle \{\iota\}, t_0^\flat \rangle$ . Hence:  $\mathcal{A}_1 \cup \mathcal{A}_2 \vdash t'_0 : \{\iota\} \rightarrow S$ . This proves that  $R' = \langle \{\iota\}, t'_0; t_1 \dots t_n; \emptyset \rangle$  is an accepting run in  $\mathcal{A}_1 \cup \mathcal{A}_2$ . As  $\mathcal{G}(R') = \mathcal{G}(R)$ , we have proved that  $G \in \mathcal{G}(\mathcal{A}_1 \cup \mathcal{A}_2)$ .

Now take  $G \in \mathcal{G}(\mathcal{A}_1 \cup \mathcal{A}_2)$ , and  $R = \langle \{\iota\}, t_0; t_1 \dots t_n; \emptyset \rangle$  such that  $R = \mathcal{G}(G)$ . Necessarily,  $t_0$  is of the shape  $\langle \{\iota\}, t_0^\flat \rangle$ , and thus was produced from  $t'_0 = \langle \{\iota_j\}, t_0^\flat \rangle \in \mathcal{T}_j$ , for either  $j = 1$  or  $j = 2$ . If  $S_0$  is the state reached after the first transition, meaning  $\mathcal{A}_1 \cup \mathcal{A}_2 \vdash t_0 : \{\iota_j\} \rightarrow S_0$ , it follows that  $S_0 \subseteq P_j$ . Because we assumed  $P_1 \cap P_2 = \emptyset$ , it is straightforward that  $\mathcal{A}_1 \cup \mathcal{A}_2 \vdash t : S \rightarrow T$  and  $S \subseteq P_j$  entails  $\mathcal{A}_j \vdash t : S \rightarrow T$  and  $T \subseteq P_j$ . This result extends to sequences of transitions, allowing us to check that because  $\mathcal{A}_1 \cup \mathcal{A}_2 \vdash t_1; \dots; t_n : S_0 \rightarrow \emptyset$  and  $S_0 \subseteq P_j$  we have  $\mathcal{A}_j \vdash t_1; \dots; t_n : S_0 \rightarrow \emptyset$ . Thus  $R' = \langle \{\iota_j\}, t'_0; t_1 \dots t_n; \emptyset \rangle$  is an accepting run in  $\mathcal{A}_j$ , and as  $\mathcal{G}(R) = \mathcal{G}(R')$ , we get  $G \in \mathcal{G}(\mathcal{A}_j) \subseteq \mathcal{G}(\mathcal{A}_1) \cup \mathcal{G}(\mathcal{A}_2)$ .  $\square$

*Series composition of Petri automata.* Let  $\mathcal{A}_1 = \langle P_1, \mathcal{T}_1, \iota_1 \rangle$  and  $\mathcal{A}_2 = \langle P_2, \mathcal{T}_2, \iota_2 \rangle$  be two Petri automata with disjoint sets of places. To obtain an automaton for  $\mathcal{G}(\mathcal{A}_1) \cdot \mathcal{G}(\mathcal{A}_2)$ , we need that every accepting run  $R_1$  in  $\mathcal{A}_1$  be followed by every accepting run  $R_2$  in  $\mathcal{A}_2$ . Because of Lemma 5.7, we know that  $R_1$  must end with a final transition, and the definition of accepting run imposes that  $R_2$  starts with an initial transition. Thus it suffices to put  $\mathcal{A}_1$  in front of  $\mathcal{A}_2$ , declare  $\iota_1$  initial, remove the final transitions  $\langle {}^a t, \emptyset \rangle$  in  $\mathcal{A}_1$  and replace them with  $\langle {}^a t, t^\flat \rangle$ , for every initial transition  $\langle \{\iota_2\}, t^\flat \rangle$  in  $\mathcal{A}_2$ . This yields the following definition:

$$\mathcal{A}_1 \cdot \mathcal{A}_2 \triangleq \langle P_1 \cup P_2, \mathcal{T}, \iota_1 \rangle,$$

where  $\mathcal{T} \triangleq (\mathcal{T}_1 \setminus \mathcal{P}(P_1) \times \{\emptyset\}) \cup \mathcal{T}_2 \cup \{\langle {}^a t, t^\flat \rangle \mid \langle \langle {}^a t, \emptyset \rangle, \langle \{\iota_2\}, t^\flat \rangle \rangle \in \mathcal{T}_1 \times \mathcal{T}_2\}$ . As before, the following inference rules may give more intuition:

$$\frac{\mathcal{A}_1 \vdash t : S \rightarrow T, \quad T \neq \emptyset}{\mathcal{A}_1 \cdot \mathcal{A}_2 \vdash t : S \rightarrow T} \qquad \frac{\mathcal{A}_2 \vdash t : S \rightarrow T}{\mathcal{A}_1 \cdot \mathcal{A}_2 \vdash t : S \rightarrow T}$$

$$\frac{\mathcal{A}_1 \vdash \langle {}^a t, \emptyset \rangle : S \rightarrow \emptyset, \quad \mathcal{A}_2 \vdash \langle \{\iota_2\}, t^\flat \rangle : \{\iota_2\} \rightarrow T}{\mathcal{A}_1 \cdot \mathcal{A}_2 \vdash \langle {}^a t, t^\flat \rangle : S \rightarrow T}$$

**Remark 6.2.** The inference system above is not entirely faithful to the above definition, in the sense that the last rule should be:

$$\frac{\mathcal{A}_1 \vdash \langle {}^a t, \emptyset \rangle : S \rightarrow S', \quad \mathcal{A}_2 \vdash \langle \{\iota_2\}, t^\flat \rangle : \{\iota_2\} \rightarrow T}{\mathcal{A}_1 \cdot \mathcal{A}_2 \vdash \langle {}^a t, t^\flat \rangle : S \rightarrow S' \cup T}.$$

However, the application of this rule in the case where  $S' \neq \emptyset$  cannot yield an accepting run without  $\mathcal{A}_1$  violating Constraint 5.5. Indeed if there was an accepting run in  $\mathcal{A}_1 \cdot \mathcal{A}_2$  using this rule, this run would need at least another occurrence of the same rule to consume the remaining places in  $P_1$ , thus allowing one to build an accepting run in  $\mathcal{A}_1$  with several final transitions. This contradicts Lemma 5.7 by violating Constraint 5.5. The same phenomenon occurs below in the constructions we give for  $\mathcal{A}^+$  and  $\mathcal{A}_1 \cap \mathcal{A}_2$ , thus explaining the slight discrepancies between the inference systems we give to drive the intuition, and the formal definitions.

**Lemma 6.3.**  $\mathcal{G}(\mathcal{A}_1 \cdot \mathcal{A}_2) = \mathcal{G}(\mathcal{A}_1) \cdot \mathcal{G}(\mathcal{A}_2)$ .

*Proof.* Let  $G \in \mathcal{G}(\mathcal{A}_1) \cdot \mathcal{G}(\mathcal{A}_2)$ . There are two accepting runs  $R_j = \langle \{\iota_j\}, t_{j,0}; \dots; t_{j,n_j}, \emptyset \rangle$  (with  $j \in \{1, 2\}$ ) such that  $G = \mathcal{G}(R_1) \cdot \mathcal{G}(R_2)$ . Let us name some intermediary states: we call  $S_1$  and  $S_2$  the states such that:

$$\begin{aligned} \mathcal{A}_1 \vdash t_{1,0}; \dots; t_{1,n_1-1} : \{\iota_1\} \rightarrow S_1; & \quad \mathcal{A}_1 \vdash t_{1,n_1} : S_1 \rightarrow \emptyset; \\ \mathcal{A}_2 \vdash t_{2,0} : \{\iota_2\} \rightarrow S_2; & \quad \mathcal{A}_2 \vdash t_{2,1}; \dots; t_{2,n_2} : S_2 \rightarrow \emptyset. \end{aligned}$$

Because of Lemma 5.7, we know that in  $R_1$ , only  $t_{1,n_1}$  is a final transition. Using the inference rules above, we obtain:

$$\begin{aligned} \mathcal{A}_1 \cdot \mathcal{A}_2 \vdash t_{1,0}; \dots; t_{1,n_1-1} : \{\iota_1\} \rightarrow S_1; & \quad \mathcal{A}_1 \cdot \mathcal{A}_2 \vdash t_{2,1}; \dots; t_{2,n_2} : S_2 \rightarrow \emptyset; \\ \mathcal{A}_1 \cdot \mathcal{A}_2 \vdash \langle {}^a t_{1,n_1}, t_{2,0}^\flat \rangle : S_1 \rightarrow S_2. & \end{aligned}$$

Thus we obtain  $R = \langle \{\iota_1\}, t_{1,0}; \dots; t_{1,n_1-1}; \langle {}^a t_{1,n_1}, t_{2,0}^\flat \rangle; t_{2,1}; \dots; t_{2,n_2}, \emptyset \rangle$  which is accepting in  $\mathcal{A}_1 \cdot \mathcal{A}_2$  and satisfies  $\mathcal{G}(R) = \mathcal{G}(R_1) \cdot \mathcal{G}(R_2) = G$ .

For the other direction, let  $G \in \mathcal{G}(\mathcal{A}_1 \cdot \mathcal{A}_2)$  be a graph, and  $R = \langle \{\iota_1\}, t_0; \dots; t_n, \emptyset \rangle$  be the corresponding accepting run. We can extract from this run two runs  $R_1$  and  $R_2$  respectively in  $\mathcal{A}_1$  and  $\mathcal{A}_2$  as follows:

$$\begin{aligned} R_1^0 \triangleq_{\varepsilon} & & R_1^{k+1} \triangleq & \begin{cases} R_1^k; t_k & (\text{if } t_k \in \mathcal{T}_1) \\ R_1^k; \langle {}^a t_k, \emptyset \rangle & (\text{if } \langle \langle {}^a t_k, \emptyset \rangle, \langle \{\iota_2\}, t_k^\triangleright \rangle \rangle \in \mathcal{T}_1 \times \mathcal{T}_2) \\ R_1^k & (\text{otherwise}) \end{cases} \\ R_2^0 \triangleq_{\varepsilon} & & R_2^{k+1} \triangleq & \begin{cases} R_2^k; t_k & (\text{if } t_k \in \mathcal{T}_2) \\ R_2^k; \langle \{\iota_2\}, t_k^\triangleright \rangle & (\text{if } \langle \langle {}^a t_k, \emptyset \rangle, \langle \{\iota_2\}, t_k^\triangleright \rangle \rangle \in \mathcal{T}_1 \times \mathcal{T}_2) \\ R_2^k & (\text{otherwise}) \end{cases} \\ R_1 \triangleq_{R_1^{n+1}} & & R_2 \triangleq_{R_2^{n+1}} & \end{aligned}$$

Using these definitions, we may prove that  $\forall k \leq n$ ,  $\mathcal{A}_1 \cdot \mathcal{A}_2 \vdash t_0; \dots; t_k : \{\iota_1\} \rightarrow S$  entails  $\mathcal{A}_1 \vdash R_1^{k+1} : \{\iota_1\} \rightarrow (S \setminus P_2)$ . This means  $R_1$  is an accepting run in  $\mathcal{A}_1$ . By Lemma 5.7 this entails there is a single final transition in  $R_1$ , hence a single transition  $t_j$  in  $R$  such that  $\langle \langle {}^a t_j, \emptyset \rangle, \langle \{\iota_2\}, t_j^\triangleright \rangle \rangle \in \mathcal{T}_1 \times \mathcal{T}_2$ . Again by Lemma 5.7 we know that  $t_j$  marks the end of  $R_1$ , and clearly  $R_2$  cannot begin before  $t_j$  has been fired (no place in  $P_2$  can appear before that). From this we can deduce that  $R$  has almost the shape  $R_1; R_2$  (but with the two transitions in the middle merged into  $t_j$ ). We can also check that  $\forall k \geq j$ ,  $\mathcal{A}_1 \cdot \mathcal{A}_2 \vdash t_0; \dots; t_k : \{\iota_1\} \rightarrow S$  entails  $\mathcal{A}_2 \vdash R_2^{k+1} : \{\iota_2\} \rightarrow (S \setminus P_1)$ , thus proving that  $R_2$  is an accepting run in  $\mathcal{A}_2$ . Finally we get that  $\mathcal{G}(R) = \mathcal{G}(R_1) \cdot \mathcal{G}(R_2) \in \mathcal{G}(\mathcal{A}_1) \cdot \mathcal{G}(\mathcal{A}_2)$ .  $\square$

*Strict iteration of a Petri automata.* Then strict iteration of a Petri automaton  $\mathcal{A} = \langle P, \mathcal{T}, \iota \rangle$  can be done by using the previous construction on the automaton itself: we keep the places and transitions of the automaton, but simply add a transition  $\langle {}^a t, t^\triangleright \rangle$  for every pair of an initial transition  $\langle \{\iota\}, t^\triangleright \rangle$  and a final transition  $\langle {}^a t, \emptyset \rangle$  in  $\mathcal{T}$ .

$$\mathcal{A}^+ \triangleq \langle P, \mathcal{T} \cup \{ \langle {}^a t, t^\triangleright \rangle \mid \langle \langle {}^a t, \emptyset \rangle, \langle \{\iota\}, t^\triangleright \rangle \rangle \in \mathcal{T} \times \mathcal{T} \}, \iota \rangle.$$

As an inference system, we get:

$$\frac{\mathcal{A} \vdash t : S \rightarrow T}{\mathcal{A}^+ \vdash t : S \rightarrow T} \quad \frac{\mathcal{A} \vdash \langle {}^a t, \emptyset \rangle : S \rightarrow \emptyset, \quad \mathcal{A} \vdash \langle \{\iota\}, t^\triangleright \rangle : \{\iota\} \rightarrow T}{\mathcal{A}^+ \vdash t : S \rightarrow T}$$

**Lemma 6.4.** *We have  $\mathcal{G}(\mathcal{A}^+) = \mathcal{G}(\mathcal{A})^+$ .*

*Proof.* The proof follows the same scheme as the previous one.  $\square$

*Parallel composition of Petri automata.* Let  $\mathcal{A}_1 = \langle P_1, \mathcal{T}_1, \iota_1 \rangle$  and  $\mathcal{A}_2 = \langle P_2, \mathcal{T}_2, \iota_2 \rangle$  be two Petri automata with disjoint sets of places. To obtain a an automaton for  $\mathcal{G}(\mathcal{A}_1) \cap \mathcal{G}(\mathcal{A}_2)$ , we merge the initial transitions of the two automata, and then we merge their final transitions. This yields the following automaton:

$$\mathcal{A}_1 \cap \mathcal{A}_2 \triangleq \left\langle P_1 \cup P_2 \cup \{\iota\}, (\mathcal{T}_1 \setminus \mathcal{P}(P_1) \times \{\emptyset\}) \cup (\mathcal{T}_1 \setminus \mathcal{P}(P_2) \times \{\emptyset\}) \cup \mathcal{T}^i \cup \mathcal{T}^f, \iota \right\rangle,$$

$$\text{where } \mathcal{T}^i \triangleq \{ \langle \{\iota\}, t_1^\triangleright \cup t_2^\triangleright \rangle \mid \langle \langle \{\iota_1\}, t_1^\triangleright \rangle, \langle \{\iota_2\}, t_2^\triangleright \rangle \rangle \in \mathcal{T}_1 \times \mathcal{T}_2 \},$$

$$\text{and } \mathcal{T}^f \triangleq \{ \langle {}^a t_1 \cup {}^a t_2, \emptyset \rangle \mid \langle \langle {}^a t_1, \emptyset \rangle, \langle {}^a t_2, \emptyset \rangle \rangle \in \mathcal{T}_1 \times \mathcal{T}_2 \}.$$

Notice that because  $P_1$  and  $P_2$  are of empty intersection, the set of states of this automaton (i.e.,  $\mathcal{P}(P_1 \cup P_2 \cup \{\iota\})$ ) is isomorphic to  $\mathcal{P}(P_1) \times \mathcal{P}(P_2) \times \mathcal{P}(\{\iota\})$ . For clarity, we use the later notation in the sequel.

$$\frac{\mathcal{A}_1 \vdash t : S \rightarrow T, \quad t^\triangleright \neq \emptyset}{\mathcal{A}_1 \cap \mathcal{A}_2 \vdash t : \langle S, S', \emptyset \rangle \rightarrow \langle T, S', \emptyset \rangle} \quad \frac{\mathcal{A}_2 \vdash t : S \rightarrow T, \quad t^\triangleright \neq \emptyset}{\mathcal{A}_1 \cap \mathcal{A}_2 \vdash t : \langle S', S, \emptyset \rangle \rightarrow \langle S', T, \emptyset \rangle}$$

$$\frac{\mathcal{A}_1 \vdash \langle \{\iota_1\}, t^\triangleright \setminus (Y \times P_2) \rangle : \{\iota_1\} \rightarrow S_1, \quad \mathcal{A}_2 \vdash \langle \{\iota_2\}, t^\triangleright \setminus (Y \times P_1) \rangle : \{\iota_2\} \rightarrow S_2}{\mathcal{A}_1 \cap \mathcal{A}_2 \vdash t : \langle \emptyset, \emptyset, \{\iota\} \rangle \rightarrow \langle S_1, S_2, \emptyset \rangle}$$

$$\frac{\mathcal{A}_1 \vdash \langle {}^a t \setminus P_2, \emptyset \rangle : S_1 \rightarrow \emptyset, \quad \mathcal{A}_2 \vdash \langle {}^a t \setminus P_1, \emptyset \rangle : S_2 \rightarrow \emptyset, \quad t^\triangleright = \emptyset}{\mathcal{A}_1 \cap \mathcal{A}_2 \vdash t : \langle S_1, S_2, \emptyset \rangle \rightarrow \langle \emptyset, \emptyset, \emptyset \rangle}$$

**Lemma 6.5.** *We have  $\mathcal{G}(\mathcal{A}_1 \cap \mathcal{A}_2) = \mathcal{G}(\mathcal{A}_1) \cap \mathcal{G}(\mathcal{A}_2)$ .*

*Proof.* Let  $G \in \mathcal{G}(\mathcal{A}_1) \cap \mathcal{G}(\mathcal{A}_2)$ . There are two accepting runs in  $\mathcal{A}_1$  and  $\mathcal{A}_2$ , that we call  $R_i = \langle \{\iota_i\}, t_{i,0}; \dots; t_{i,n_i}, \emptyset \rangle$  (with  $i \in \{1, 2\}$ ), such that  $G = \mathcal{G}(R_1) \cap \mathcal{G}(R_2)$ . We build the following run:

$$R \triangleq \left\langle \{\iota\}, \left\langle \{\iota\}, t_{1,0}^\triangleright \cup t_{2,0}^\triangleright \right\rangle; t_{1,1}; \dots; t_{1,n_1-1}; t_{2,1}; \dots; t_{2,n_2-1}; \left\langle {}^a t_{1,n_1} \cup {}^a t_{2,n_2}, \emptyset \right\rangle, \emptyset \right\rangle.$$

It is a simple exercise to check that indeed  $R$  is an accepting run in  $\mathcal{A}_1 \cap \mathcal{A}_2$ , and that its trace does satisfy  $\mathcal{G}(R) = G$ .

For the other direction, the presentation as an inference system simplifies the reasoning: by projecting an accepting run in  $\mathcal{A}_1 \cap \mathcal{A}_2$  on the first (respectively second) component, we get an accepting run in  $\mathcal{A}_1$  (resp.  $\mathcal{A}_2$ ). From that remark, one can deduce that the parallel product of the traces of these two projected runs is isomorphic to the trace of the whole run.  $\square$

*Conclusion.* Now we have all ingredients required to associate a Petri automaton  $\mathcal{A}(e)$  with every expression  $e$ , by induction on  $e$ :

$$\begin{array}{ll} \mathcal{A}(e \cup f) \triangleq \mathcal{A}(e) \cup \mathcal{A}(f) & \mathcal{A}(0) \triangleq \underline{0} \\ \mathcal{A}(e \cdot f) \triangleq \mathcal{A}(e) \cdot \mathcal{A}(f) & \mathcal{A}(a) \triangleq \underline{a} \\ \mathcal{A}(e \cap f) \triangleq \mathcal{A}(e) \cap \mathcal{A}(f) & \mathcal{A}(e^+) \triangleq \mathcal{A}(e)^+ \end{array}$$

Lemmas 6.1, 6.3, 6.4 and 6.5 show that this construction is correct, whence the first half of the Kleene theorem for Petri automata.

**Proposition 6.6.** *For every simple expression  $e \in \text{Exp}_Y^-$ , we have  $\mathcal{G}(\mathcal{A}(e)) = \mathcal{G}(e)$ .*

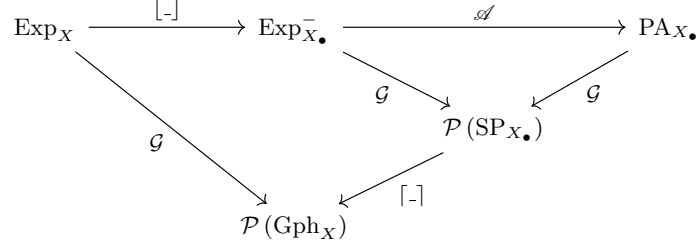
$$\begin{array}{ccc} \text{Exp}_Y^- & \xrightarrow{\mathcal{A}} & \text{PA}_Y \\ & \searrow \mathcal{G} & \swarrow \mathcal{G} \\ & \mathcal{P}(\text{SP}_Y) & \end{array}$$

**Theorem 6.7.** *Regular sets of series-parallel graphs are recognisable.*

When a Petri automaton is labelled in  $X_\bullet$ , it can be used to produce graphs labelled in  $X$  rather than series-parallel graphs labelled in  $X_\bullet$ , using the function  $[-]$ . Thus we define:

**Definition 6.8.** A set  $S \subseteq \text{Gph}_X$  of graphs is *recognisable* if there exists a Petri automaton  $\mathcal{A} \in \text{PA}_{X_\bullet}$  such that  $S = \lceil \mathcal{G}(\mathcal{A}) \rceil$ .

We can combine Propositions 4.4 and 6.6 to obtain the following commutative diagram.



Whence

**Theorem 6.9.** *Regular sets of graphs are recognisable.*

**Remark 6.10.** If  $e$  is an expression without intersection, then the transitions in  $\mathcal{A}(\lfloor e \rfloor)$  are all of the form  $\langle \{p\}, \{\langle x, q \rangle\} \rangle$ , with only one input, one output and a label in  $X \cup \{1\}$ . As a consequence, the accessible configurations are singletons, and the resulting Petri automaton has the structure of a non-deterministic finite-state automaton with epsilon transitions (NFA). Actually, in that case, the construction we described above is just a variation on Thompson’s construction [33].

## 7. FROM PETRI AUTOMATA TO EXPRESSIONS

Now we prove the converse implications of Theorems 6.7 and 6.9, thus resulting in full Kleene theorems for Petri automata and expressions. Like previously, most of the work is done with simple expressions and series-parallel graphs; the extension to arbitrary expressions and graphs will follow from Proposition 4.4.

This section is technically more involved than the other parts of the paper. The following sections (8 and beyond) do not depend on the results and notions introduced here.

**7.1. More notions on graphs.** Given an oriented (but not necessarily labelled and pointed) graph  $G$ , its *sources*  $\min G$  (resp. *sinks*  $\max G$ ) are the vertices with no incoming (resp. outgoing) edge. A vertex  $v$  is *reachable* (respectively *co-reachable*) from another vertex  $v'$  if there is a path from  $v'$  to  $v$  (resp. from  $v$  to  $v'$ ). A graph is *connected* if there is a non-directed path between any two vertices. A subgraph is called a *connected component* if it is maximal amongst the connected subgraphs.

A (*rooted*) *tree* is an acyclic graph  $T$  such that either  $\min T$  or  $\max T$  contains a single node, called the root, and for any two nodes  $x, y \in V$  there exists at most one path from  $x$  to  $y$ . If  $\min T$  is a singleton, then  $T$  is a *top-down tree*, otherwise it is a *bottom-up tree*. A tree is a *proper tree* if its root has degree one and if it does not contain a node with exactly one incoming edge and one outgoing edge. There is only a finite number of proper trees with leaves chosen from a finite set.

The following technical lemma will be instrumental later on. We illustrate it in Figure 11.

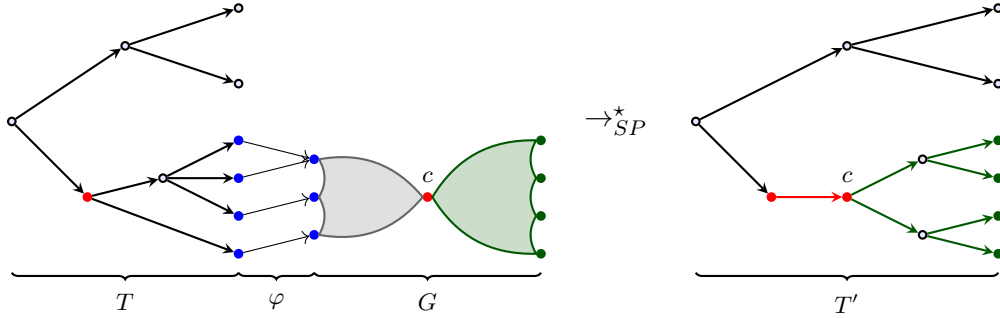


Figure 11: Illustration of Lemma 7.1

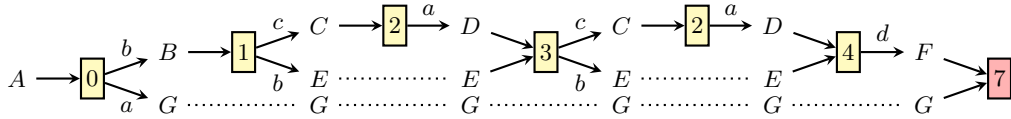
**Lemma 7.1.** *Let  $T = \langle V_T, E_T \rangle$  be a proper unlabelled top down tree with root  $r$  and set of leaves  $F \subseteq V_T$ , and  $G = \langle V, E \rangle$  be a connected DAG. Let  $\varphi : F \rightarrow V$  be a partial function defined on  $F' \subseteq F$ . The gluing of  $T$  and  $G$  along  $\varphi$  is the graph*

$$T \cdot_{\varphi} G \triangleq \langle V_T \cup V, E_T \cup E \cup \{ \langle f, \varphi(f) \rangle \} \mid f \in F' \rangle.$$

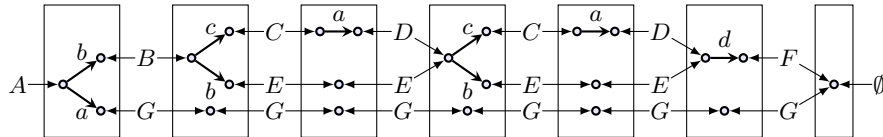
*If  $T \cdot_{\varphi} G \rightarrow_{SP}^* T'$  and if  $T'$  is a tree, then there is a node  $c$  in  $G$  such that for every  $\langle f, M \rangle \in F' \times \max G$  every path from  $\varphi(f)$  to  $M$  in  $G$  visits  $c$ .*

*Proof.* The maximal elements of  $G$  remain in  $T'$  as leaves, and because  $G$  is connected there will be a subtree of  $T'$  whose leaves are exactly  $\max G$ . The root of this tree must be a node accessible from  $F'$ , hence coming from  $G$ . Because paths are preserved during SP-rewriting, this vertex has the desired property.  $\square$

**7.2. Boxes.** In the standard Kleene theorem for words, an important aspect is that a partial execution in an automaton always denotes a word, even if the path is not initial and final. This is not true with graphs and Petri automata, where partial executions denote slices of a graph. To this end, we will use *boxes*. Recall the following run, that we used as an example of run from the automaton displayed in Figure 6:



We will abstract each transition by a box, and the run itself by the sequential composition of those boxes:



These boxes allow us to represent every subrun as a single box. For instance, the subrun firing transitions 2 and 3 in sequence, and looping from state  $\{C, E, G\}$  back to itself can be

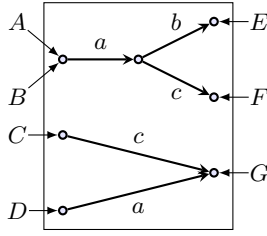
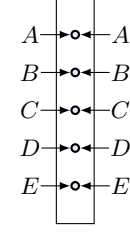
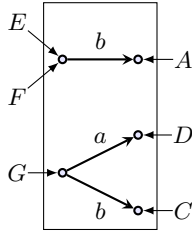
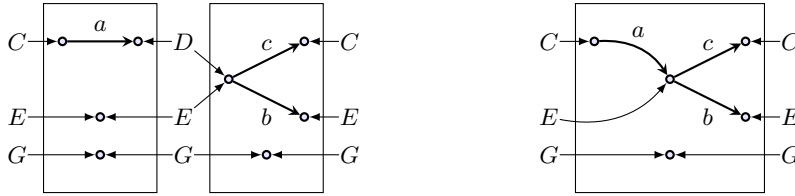


Figure 12: Examples of boxes


 Figure 13:  $\text{id}_\sigma$ .

represented as the box:



We start by defining those boxes formally, and we study some of their properties.

7.2.1. *The category of boxes.* We fix a finite set  $P$  of ports, which will be instantiated in the next section with the set of places of some Petri automaton.

**Definition 7.2** (Box). Let  $S, S' \subseteq P$  be two sets of ports. A *box* labelled over  $Y$  from  $S$  to  $S'$  is a triple  $\langle \vec{\mathfrak{p}}, G, \overleftarrow{\mathfrak{p}} \rangle$  where  $G$  is a directed acyclic graph (DAG) with edges labelled in  $Y$ ,  $\vec{\mathfrak{p}}$  is a map from  $S$  to the vertices of  $G$ , and  $\overleftarrow{\mathfrak{p}}$  is a bijective map from  $S'$  to  $\max G$ . The set of boxes labelled by  $Y$  is written  $\mathcal{B}_Y$ , and the boxes from  $S$  to  $S'$  are denoted by  $\mathcal{B}_Y[S, S']$ , or simply  $\mathcal{B}[S, S']$  if the set of labels is clear from the context.

As for graphs, we consider boxes up to renaming of internal nodes. We represent boxes graphically as in Figure 12. Inside the rectangle is the DAG, with the input ports on the left-hand side and the output ports on the right-hand side. The maps  $\vec{\mathfrak{p}}$  and  $\overleftarrow{\mathfrak{p}}$  are represented by the arrows going from the ports to vertices inside the rectangle.

**Definition 7.3** (Identity box). If  $S = \{p_1, \dots, p_n\} \subseteq P$  is a set of ports, the *identity box* on  $S$  is defined as  $\text{id}_S \triangleq \langle [p_i \mapsto i], \langle \{1, \dots, n\}, \emptyset \rangle, [p_i \mapsto i] \rangle$ .

For instance, the box  $\text{id}_{\{A, B, C, D, E\}}$  is represented in Figure 13.

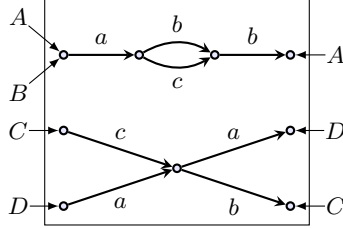
Now we define how to compose boxes sequentially. Intuitively, if the set  $S$  of output ports of  $\beta_1$  is equal to the set of input ports of  $\beta_2$ , we may compose them by putting the graph of  $\beta_1$  to the left of the graph of  $\beta_2$ , and for every port  $p \in S$ , we identify the node  $\overleftarrow{\mathfrak{p}}_1(p)$  with the node  $\vec{\mathfrak{p}}_2(p)$ .

**Definition 7.4** (Composition of boxes). Let  $S_1, S_2, S_3 \subseteq P$  and for  $i \in \{1, 2\}$ , let  $\beta_i$  be a box from  $S_i$  to  $S_{i+1}$ , with  $\beta_i = \langle \vec{\mathfrak{p}}_i, \langle V_i, E_i \rangle, \overleftarrow{\mathfrak{p}}_i \rangle$ , such that  $V_1 \cap V_2 = \emptyset$ . The *composition* of  $\beta_1$  and  $\beta_2$ , written  $\beta_1 \odot \beta_2$  is defined as  $\langle \vec{\mathfrak{p}}, \langle V'_1 \cup V_2, E'_1 \cup E \cup E_2 \rangle, \overleftarrow{\mathfrak{p}} \rangle$  with:

- $V'_1 \triangleq V_1 \setminus \overleftarrow{\mathfrak{p}}_1(S_2)$ , and  $E'_1 \triangleq E_1 \cap (V'_1 \times Y \times V'_1)$ ;
- $E \triangleq \{ \langle x, a, \vec{\mathfrak{p}}_2(p) \rangle \mid \langle x, a, y \rangle \in E_1, y = \overleftarrow{\mathfrak{p}}_1(p) \}$ ;

- $\overleftarrow{p} \triangleq \overleftarrow{p}_2$  and  $\overrightarrow{p}(p) \triangleq \begin{cases} \overrightarrow{p}_2(q), & \text{if } \overrightarrow{p}_1(p) = \overleftarrow{p}_1(q), \\ \overrightarrow{p}_1(p), & \text{otherwise.} \end{cases}$

For instance the two boxes in Figure 12 can be composed, and yield the following box:



We obtain a category of sets of ports and boxes:

**Proposition 7.5.** *We have a category  $\text{Box}_P^Y$  whose objects are subsets of  $P$ , and whose morphisms between  $S$  and  $S'$  are the boxes from  $S$  to  $S'$ , i.e.,  $\mathcal{B}_Y[S, S']$ .*

*Proof.* To give a high level proof of the associativity of box composition, notice that the computation of  $\beta \odot \gamma$  may be split in two steps:

- (1) compute an intermediate box  $\beta \xrightarrow{\varepsilon} \gamma$  (with  $\varepsilon$  being a fresh label), build by keeping the input port map of  $\beta$ , the output map of  $\gamma$ , the disjoint union of their vertices and edges, and simply adding edges  $x \xrightarrow{\varepsilon} y$  whenever  $\langle x, y \rangle \in \max \beta \times \gamma$  such that  $\overleftarrow{p}_\beta(x) = \overrightarrow{p}_\gamma(y)$ .
- (2) collapse all edges labelled by  $\varepsilon$  by identifying their source and target vertices. The effect of this will be the destruction of all the nodes from  $\max \beta$ , and the redirection of their incoming arrows to the corresponding vertices in  $\gamma$ .

Suppose we have three boxes  $\beta, \gamma, \delta$  with the appropriate input and output sets of ports (the input ports of  $\gamma$  should be exactly the output ports of  $\beta$ ...). We may now describe the two ways in which to compose them by the following diagram:

$$\begin{array}{ccccc}
 \beta, \gamma, \delta & \longrightarrow & \beta, \gamma \xrightarrow{\varepsilon_2} \delta & \longrightarrow & \beta, \gamma \odot \delta \\
 \downarrow & (1) & \downarrow & (2) & \downarrow \\
 \beta \xrightarrow{\varepsilon_1} \gamma, \delta & \longrightarrow & \beta \xrightarrow{\varepsilon_1} \gamma \xrightarrow{\varepsilon_2} \delta & \longrightarrow & \beta \xrightarrow{\varepsilon_1} \gamma \odot \delta \\
 \downarrow & (3) & \downarrow & (4) & \downarrow \\
 \beta \odot \gamma, \delta & \longrightarrow & \beta \odot \gamma \xrightarrow{\varepsilon_2} \delta & \longrightarrow & \beta \odot \gamma \odot \delta
 \end{array}$$

The commutation of square (1) is clear: the box  $\beta \xrightarrow{\varepsilon_1} \gamma \xrightarrow{\varepsilon_2} \delta$  could even be described in one step (just put edges with  $\varepsilon_1$  between  $\max \beta$  and  $\gamma$  and with  $\varepsilon_2$  between  $\max \gamma$  and  $\delta$ ).

Squares (2) and (3) commute as well. For instance for square (3), notice that collapsing edges  $\varepsilon_1$  in  $\beta \xrightarrow{\varepsilon_1} \gamma$  doesn't affect the set  $\max \beta \xrightarrow{\varepsilon_1} \gamma = \max \gamma$ . Hence this has no bearing on the computation of  $\beta \odot \gamma \xrightarrow{\varepsilon_2} \delta$ .

Finally, for square (4), an additional step could be added: replacing both labels  $\varepsilon_1$  and  $\varepsilon_2$  by a common label  $\varepsilon$ , before collapsing the resulting graph along  $\varepsilon$ -edges. This clearly produces the same result. The point is then to show that the collapsing operation is confluent. This is true, because one could compute the normal form from the beginning, as the set of equivalence classes of the smallest equivalence relation on vertices containing all pairs  $x, y$  of vertices linked by an  $\varepsilon$ -edge.



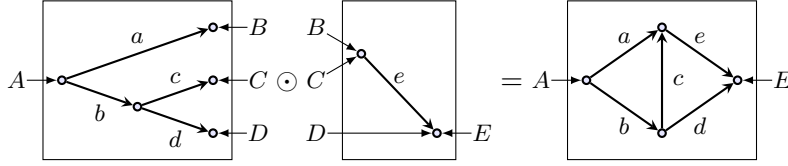
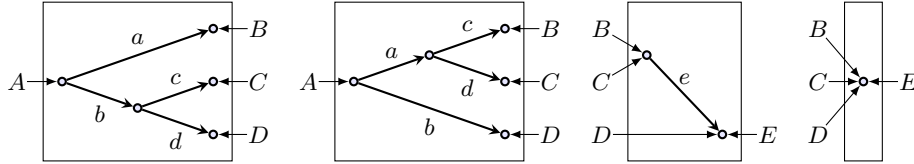


Figure 14: An example of “bad” composition

The fact that for every box  $\beta = \langle \vec{p}, \langle V, E \rangle, \overleftarrow{p} \rangle \in \mathcal{B}[A, B]$ , we have  $\text{id}_A \odot \beta = \beta$  is straightforward from the definitions. The vertices of the composite box are exactly those of  $\beta$ , because the image of  $A$  through the output map of  $\text{id}_A$  is the set of vertices of  $\text{id}_A$  itself. As  $\text{id}_A$  has no edge, the edges of  $\text{id}_A \odot \beta$  are again simply those of  $\beta$ . By definition the output map is that of  $\beta$ , but so is the input map, because for every port  $p_i \in A$ , the images of  $p_i$  via the input and output maps of  $\text{id}_A$  are both equal to  $i$ . The fact that  $\beta \odot \text{id}_B$  is also equal to  $\beta$  follows from a similar argument.  $\square$

7.2.2. *The category of typed boxes.* Since we want to represent slices of series-parallel graphs only, we actually need to enforce a stronger typing discipline on boxes. Indeed there are boxes in the above category that are slices of series parallel graphs, but whose composition cannot be a part of a series-parallel graph. For instance consider the four boxes below, called  $\beta_1$  through  $\beta_4$ :



According to their interfaces,  $\beta_1$  and  $\beta_2$  can both compose with  $\beta_3$  and  $\beta_4$ . Indeed  $\beta_1 \odot \beta_4$ ,  $\beta_2 \odot \beta_3$  and  $\beta_2 \odot \beta_4$  all yield boxes containing series-parallel graphs. However the composition  $\beta_1 \odot \beta_3$  produces the box shown in Figure 14. The graph in this box is not series-parallel, and is in fact the forbidden subgraph of the class of series-parallel graphs [34]. To prevent this situation statically, we introduce a stronger notion of typing. The types are trees with leaves labelled with ports:

**Definition 7.6** (Type). A *type* over  $S \subseteq P$  is a triple  $\tau = \langle V, E, \lambda \rangle$  such that  $\langle V, E \rangle$  is a proper unlabelled top-down tree, and  $\lambda$  is a bijective function from  $S$  to  $\max(V, E)$ . The set of types over subsets of  $P$  is written  $\mathbb{T}_P$ .

As before, types are considered up to bijective renaming of internal vertices. It is then a simple observation to notice that  $\mathbb{T}_P$  is finite (recall that  $P$  was assumed to be finite as well). We present two examples of such types in Figure 15.

We may forget about the label information in a box: the  $a$ -erasing of a box  $\beta$  is the box  $[\beta]_a$  where all labels are replaced by an arbitrary letter  $a \in Y$ . When the labels are not relevant (or, in the next section when we label edges with expressions rather than variables), we can define SP-reductions of boxes. Given a box  $\beta = \langle \vec{p}, G, \overleftarrow{p} \rangle$ , if  $G \rightarrow_{SP} G'$ , and if no vertex in the image of  $\vec{p}$  was erased in the reduction, we write  $\beta \rightarrow_{SP} \langle \vec{p}, G', \overleftarrow{p} \rangle$ . Composition commutes with SP-reductions: if  $\beta \rightarrow_{SP} \beta'$ , then  $\beta \odot \gamma \rightarrow_{SP} \beta' \odot \gamma$  and  $\gamma \odot \beta \rightarrow_{SP} \gamma \odot \beta'$ .

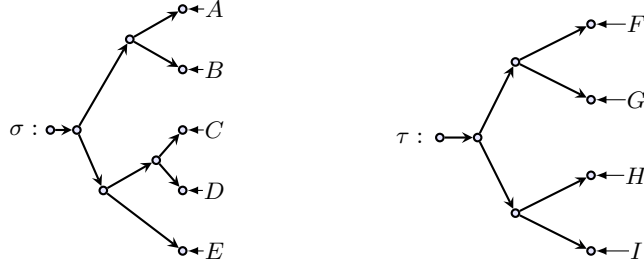


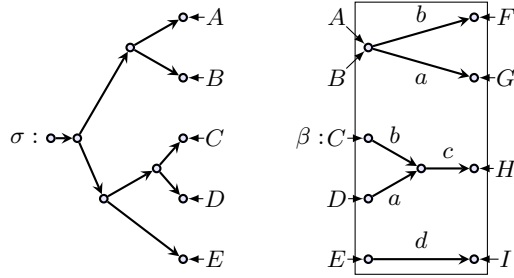
Figure 15: Examples of types

A type  $\tau = \langle V, E, \lambda \rangle$  over  $S$  may be seen as a box from  $\{\iota\}$  to  $S$ : if  $r$  is the root of  $\tau$  and  $a \in Y$ , we can build the box  $\boxed{\tau}_a$  from  $\{\iota\}$  to  $S$  as  $\langle [\iota \mapsto r], \langle V, E' \rangle, \lambda \rangle$ , with  $E' = \{\langle x, a, y \rangle \mid \langle x, y \rangle \in E\}$ . For every type  $\tau$ ,  $\boxed{\tau}_a = \llbracket \boxed{\tau}_a \rrbracket_a$ ; for all boxes  $\beta \in \mathcal{B}[S_1, S_2]$  and  $\gamma \in \mathcal{B}[S_2, S_3]$ , we have  $\llbracket \beta \odot \gamma \rrbracket_a = \llbracket \beta \rrbracket_a \odot \llbracket \gamma \rrbracket_a$ .

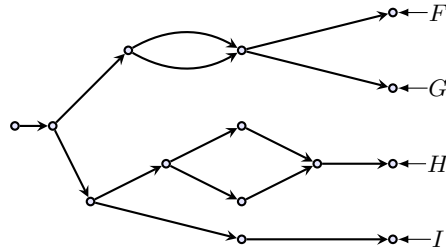
Now we can define typed boxes.

**Definition 7.7** (Typed boxes). Let  $\beta \in \mathcal{B}[S, S']$  be a box,  $\sigma$  and  $\tau$  be types respectively over  $S$  and  $S'$ .  $\beta$  has the type  $\sigma \rightarrow \tau$  if  $\llbracket \sigma \rrbracket_a \odot \beta \rrbracket_a \rightarrow_{SP}^* \llbracket \tau \rrbracket_a$ . We write  $\mathfrak{B}_Y[\sigma, \tau]$  for the set of boxes over  $Y$  of type  $\sigma \rightarrow \tau$ , and  $\mathfrak{B}_Y$  for the set of typed boxes over  $Y$ .

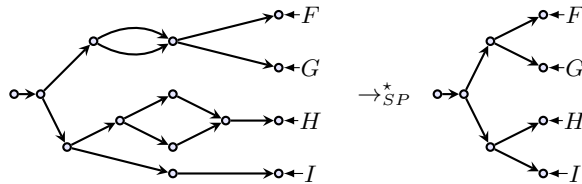
**Example 7.8.** Consider the following type  $\sigma$  and box  $\beta$ :



First, we glue them together, and erase the labels. This yields the graph:



Then, we apply as many reductions steps as much as possible:



The last graph is the box corresponding to a tree  $\tau$ . Hence we can state that  $\beta$  has the type  $\sigma \rightarrow \tau$ .

**Remark 7.9.** Notice that the type of a box is not unique: a single box may have multiple types. However, given an input type  $\sigma$  and a box  $\beta$ , there is at most one output type  $\tau$  such that  $\beta \in \mathfrak{B}[\sigma, \tau]$ .

When  $\sigma$  is a type over  $S$ , we set  $\text{id}_\sigma \triangleq \text{id}_S$ .

**Proposition 7.10.** *We have a category  $\mathcal{TBox}_P^Y$  of typed boxes, with  $\mathbb{T}_P$  as the set of objects, and  $\mathfrak{B}_Y[\tau, \sigma]$  as the set of morphisms from  $\tau$  to  $\sigma$ .*

*Proof.* It suffices to show that the identity box can be typed:  $[\square_\sigma]_a \odot \text{id}_S = [\square_\sigma]_a = \square_\sigma$ ; and that composition of typed boxes is a typed box. Let  $\beta, \gamma \in \mathfrak{B}[\sigma, \tau] \times \mathfrak{B}[\tau, \theta]$ , we show that  $\beta \odot \gamma \in \mathfrak{B}[\sigma, \theta]$ :

$$[\square_\sigma]_a \odot (\beta \odot \gamma) = [\square_\sigma]_a \odot \beta \odot [\gamma]_a \xrightarrow{*}_{SP} [\tau]_a \odot [\gamma]_a = [\tau]_a \odot [\gamma]_a \xrightarrow{*}_{SP} [\theta]_a$$

□

**Remark 7.11.** Given a category  $\mathcal{C}$ , one can form a Kleene category  $\mathcal{K}$  (also called typed Kleene algebra [20, 27]) whose types are the objects of the category, and where the morphisms from  $a$  to  $b$  are subsets of the set  $\mathcal{C}[a, b]$  of morphisms from  $a$  to  $b$  in  $\mathcal{C}$ . The product in this algebra is the pointwise composition of morphisms, and sum is union. In particular,  $\mathcal{P}(\mathcal{B})$  and  $\mathcal{P}(\mathfrak{B})$  are Kleene categories.

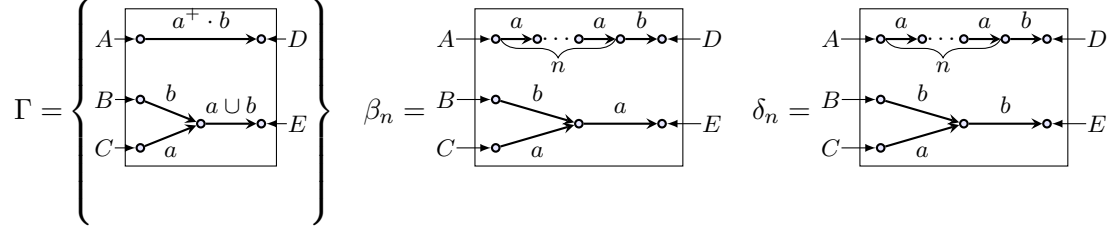
**7.2.3. Templates.** Another important notion we need for the second direction of our Kleene theorem is that of template. In the proof of the classical Kleene theorem, one moves from automata to generalised automata, which are labelled with regular expressions rather than with letters. This allows for the label of a single transition to represent many paths in the original automaton. Finite templates will serve this function in our proof. A finite template is a finite set of boxes sharing the same input and output ports, and labelled with simple expressions.

**Definition 7.12** ((Finite) Template). Let  $S$  and  $S'$  be sets of ports. A *template* (respectively *finite template*) from  $S$  to  $S'$  is a set (resp. finite set)  $\Gamma \subseteq \mathcal{B}_{\text{Exp}_Y}[S, S']$  of boxes from  $S$  to  $S'$  labelled with simple expressions. If furthermore  $\sigma$  is a type over  $S$ ,  $\tau$  is a type over  $S'$  and  $\Gamma \subseteq \mathfrak{B}_{\text{Exp}_Y}[\sigma, \tau]$ , we say that  $\Gamma$  has type  $\sigma \rightarrow \tau$ . We write  $\mathcal{BT}[S, S']$  for the set of finite templates from  $S$  to  $S'$ , and  $\mathfrak{BT}[\sigma, \tau]$  for finite templates of type  $\sigma \rightarrow \tau$ .

One extracts boxes out of templates as follows.

**Definition 7.13** (Boxes generated by a template). Let  $\Gamma \in \mathcal{BT}[S, S']$  be a template from  $S$  to  $S'$ . Then  $\beta \in \mathcal{B}_Y[S, S']$  is *generated* by  $\Gamma$  if it can be obtained from a box  $\langle \overrightarrow{\mathfrak{p}}, G, \overleftarrow{\mathfrak{p}} \rangle \in \Gamma$  by replacing each edge  $x \xrightarrow{e} y$  by a series-parallel graph  $G' \in \mathcal{G}(e)$  with input vertex  $x$  and output vertex  $y$ . We write  $(\Gamma)$  for the set of boxes generated by  $\Gamma$ .

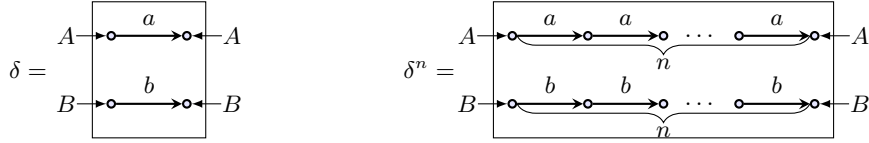
For instance in the example below the template  $\Gamma$  generates all boxes of the shapes  $\beta_n$  and  $\delta_n$ , for  $n > 0$ :



**Fact 7.14.** If  $\Gamma$  has type  $\sigma \rightarrow \tau$ , then  $\langle \Gamma \rangle \subseteq \mathfrak{B}_Y[\sigma, \tau]$ .

According to Remark 7.11, templates and typed templates form Kleene categories. The function  $\langle \_ \rangle$  used to extract boxes from templates is a functor from the Kleene category  $\mathcal{BT}$  to the Kleene category  $\mathcal{P}(\mathcal{B})$ , and it preserves the regular operations.

If  $\Gamma, \Delta$  are finite templates of appropriate sort, then  $\Gamma \cup \Delta$  and  $\Gamma \cdot \Delta$  are also finite templates. However the Kleene star of a finite set of boxes yields an infinite set of boxes. Even worse, this set of boxes cannot always be generated by a finite template. For instance, the star of the box  $\delta$  below yields all boxes  $\delta^n$ , for  $n \in \mathbb{N}$ :



This set of boxes cannot be represented by a finite number of boxes, as the two branches have no way to synchronise. (This means that we cannot ensure there will be exactly the same number of iterations on both branches.) Fortunately, it is possible to define Kleene star for a rich enough class of templates, namely for *atomic* templates.

#### 7.2.4. Atomic boxes and templates.

**Definition 7.15** (Support). The *support* of a box  $\beta \in \mathcal{B}[S, S]$  is the following set

$$\text{support}(\beta) \triangleq \{p \mid \vec{\mathfrak{p}}(p) \neq \overleftarrow{\mathfrak{p}}(p)\}$$

The support of a template  $\Gamma \in \mathcal{BT}[S, S]$  is defined as  $\bigcup_{\beta \in \Gamma} \text{support}(\beta)$ .

Intuitively, the support constitutes the irreflexive part of a box. In particular,  $\text{support}(\text{id}_\sigma)$  is always empty.

**Fact 7.16.** If  $\Gamma, \Gamma'$  are templates of type  $\sigma \rightarrow \sigma$  with disjoint support, then  $\Gamma \odot \Gamma' = \Gamma' \odot \Gamma$ .

**Definition 7.17** (Atomic box, atomic template). A box  $\alpha = \langle \vec{\mathfrak{p}}, G, \overleftarrow{\mathfrak{p}} \rangle \in \mathcal{B}[S, S]$  is *atomic* if its graph has a single non-trivial connected component  $C$ , and if for every vertex  $v$  outside  $C$ , there is a unique port  $p \in S$  such that  $\vec{\mathfrak{p}}(p) = \overleftarrow{\mathfrak{p}}(p) = v$ . An *atomic template* is a finite template exclusively composed of atomic boxes.

Atomic templates with singleton support can be easily iterated. Indeed, the non-trivial connected component of an atomic box with singleton support is series-parallel, and may thus be SP-reduced to a graph with only two vertices, joined by a single edge labelled with some simple expression  $e$ . If we then replace  $e$  with  $e^+$ , and put the resulting box  $\alpha'$  in a template  $\alpha^* \triangleq \{\text{id}_\sigma, \alpha'\}$ , we easily get  $\langle \alpha^* \rangle = \langle \alpha \rangle^*$ . Now if  $\Gamma = \{\alpha_1, \dots, \alpha_n\}$  is an

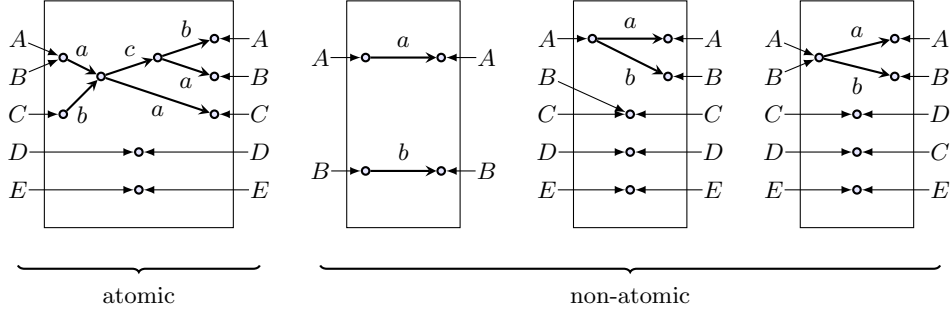


Figure 16: Atomic and non-atomic boxes

atomic template with singleton support, it must be that every  $\alpha_i$  has the same singleton support. We may thus reduce their graphs, yielding expressions  $e_1, \dots, e_i$ , and use the label  $(e_1 \cup \dots \cup e_i)^+$  to do the same construction.

**Lemma 7.18.** *The non-trivial connected component of an atomic box of type  $\sigma \rightarrow \sigma$  always contains a vertex  $c$ , such that for every port  $p$  mapped inside that component, all paths from  $\vec{p}(p)$  to a maximal vertex visit  $c$ .*

*Proof.* This is a direct consequence of Lemma 7.1 □

This lemma allows us to split an atomic box into the product  $\alpha = \alpha^1 \odot \alpha^2$  of two typed boxes such that  $\alpha^2 \odot \alpha^1$  has singleton support. Furthermore if  $\text{support}(\Gamma) \subseteq \text{support}(\alpha)$  then  $\{\alpha^2\} \odot \Gamma \odot \{\alpha^1\}$  has singleton support.

Another important property is that the supports of atomic boxes of the same type are either disjoint or comparable:

**Lemma 7.19.** *For all atomic boxes  $\beta, \gamma \in \mathfrak{B}[\sigma, \sigma]$ , we have either*

- $\text{support}(\beta) \subseteq \text{support}(\gamma)$ , or
- $\text{support}(\gamma) \subseteq \text{support}(\beta)$ , or
- $\text{support}(\beta) \cap \text{support}(\gamma) = \emptyset$ .

*Proof.* It suffices to observe that the support of every atomic box  $\beta \in \mathfrak{B}[\sigma, \sigma]$  has to be the set of leaves reachable from some vertex in  $\sigma$ . □

**Proposition 7.20.** *For every atomic template  $\Gamma$  with type  $\sigma \rightarrow \sigma$ , there is a finite template  $\Gamma^*$  with the same type such that  $\langle\langle \Gamma \rangle\rangle^* = \langle\langle \Gamma^* \rangle\rangle^*$ .*

*Proof.* Let  $\Gamma = \{\alpha_1, \dots, \alpha_n\}$  be an atomic template of type  $\sigma \rightarrow \sigma$ , indexed in such a way that  $i < j$  entails either  $\text{support}(\alpha_i) \subseteq \text{support}(\alpha_j)$  or  $\text{support}(\alpha_i) \cap \text{support}(\alpha_j) = \emptyset$ . We define  $\emptyset^* = 1_\sigma$ . Then for every  $k \leq n$ , we split  $\{\alpha_1, \dots, \alpha_{k-1}\}$  into  $\Gamma_1$  and  $\Gamma_2$ , such that  $\text{support}(\Gamma_1) \subseteq \text{support}(\alpha_k)$  and  $\text{support}(\Gamma_2) \cap \text{support}(\alpha_k) = \emptyset$ . We obtain:

$$\begin{aligned}
 \langle\langle \{\alpha_1, \dots, \alpha_{k-1}, \alpha_k\} \rangle\rangle^* &= \langle\langle (\Gamma_1 \cup \alpha_k) \cup \Gamma_2 \rangle\rangle^* && \text{(commutativity)} \\
 &= (\langle\langle \Gamma_1 \rangle\rangle^* \cup \langle\langle \alpha_k \rangle\rangle^*)^* \odot \langle\langle \Gamma_2 \rangle\rangle^* && (\text{support}(\Gamma_2) \cap \text{support}(\Gamma_1 \cup \alpha_k) = \emptyset) \\
 &= \langle\langle \Gamma_1^* \rangle\rangle^* \langle\langle \alpha_k \cdot \Gamma_1^* \rangle\rangle^* \odot \langle\langle \Gamma_2^* \rangle\rangle^* && \text{(regular laws)} \\
 &= \langle\langle \Gamma_1^* \rangle\rangle^* ((1_\sigma) \cup \langle\langle \alpha_k \Gamma_1^* \rangle\rangle^* \langle\langle \alpha_k \Gamma_1^* \rangle\rangle^*) \odot \langle\langle \Gamma_2^* \rangle\rangle^* && \text{(regular laws)} \\
 &= \langle\langle \Gamma_1^* \rangle\rangle^* ((1_\sigma) \cup \langle\langle \alpha_k^1 \alpha_k^2 \Gamma_1^* \rangle\rangle^* \langle\langle \alpha_k^1 \alpha_k^2 \Gamma_1^* \rangle\rangle^*) \odot \langle\langle \Gamma_2^* \rangle\rangle^* && (\alpha_k = \alpha_k^1 \alpha_k^2) \\
 &= \langle\langle \Gamma_1^* \rangle\rangle^* ((1_\sigma) \cup \langle\langle \alpha_k^1 \rangle\rangle^* \langle\langle \alpha_k^2 \Gamma_1^* \alpha_k^1 \rangle\rangle^* \langle\langle \alpha_k^2 \Gamma_1^* \rangle\rangle^*) \odot \langle\langle \Gamma_2^* \rangle\rangle^* && \text{(regular laws)}
 \end{aligned}$$

As we noticed earlier, because we have  $\text{support}(\Gamma_1) \subseteq \text{support}(\alpha_k)$  the template  $\alpha_k^2 \Gamma_1^* \alpha_k^1$  has a singleton support. This means we can compute its star, thus reduce the last expression into a single finite template.  $\square$

**7.3. Extracting simple expressions from Petri Automata.** Now we have enough material to embark in the proof that recognisable languages of series-parallel graphs are regular.

Let us fix an automaton  $\mathcal{A} = \langle P, \mathcal{T}, \iota \rangle$ . Assume that  $\iota$  never appears in the output of a transition and that  $P$  contains a place  $f$  not connected to any transition. (It is easy to modify  $\mathcal{A}$  to enforce these two properties).

We start by building a finite state automaton whose states are types, and transitions are typed boxes. Then, using a procedure similar to the proof of the classical Kleene's Theorem, we reduce it into a single box template from which we can extract a simple expression.

**7.3.1. A regular language of runs.** We can associate a box with every transition of a proper run, as illustrated in Example 7.23.

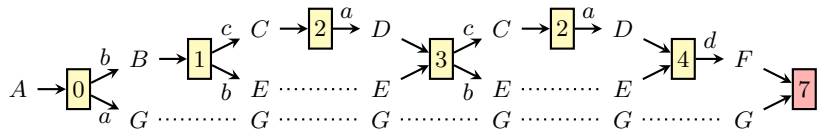
**Definition 7.21** (Box of a transition). Suppose  $t \in \mathcal{T}$  is a non-final transition in  $\mathcal{A}$ , and  $S, S' \subseteq P$  are two states such that  $\mathcal{A} \vdash t : S \rightarrow S'$ . The *box of  $t$  from  $S$*  (written  $\text{box}(t, S)$ ) is built as follows. Its set of input ports (respectively output ports) is  $S$  (resp.  $S'$ ). The vertices of its graph are the places in  $S'$ , together with an additional node  $*$ . Places in  ${}^{\leftarrow}t$  are mapped by  $\vec{\mathfrak{p}}$  to  $*$ , and the others are mapped to themselves.  $\overleftarrow{\mathfrak{p}}$  sends every place in  $S'$  (seen as a port) to itself (seen as a vertex). Finally, we put edges  $\langle *, a, q \rangle$  whenever  $\langle a, q \rangle \in \overleftarrow{\mathfrak{p}}$ . For final transitions  $t = \langle {}^{\leftarrow}t, \emptyset \rangle$  we adapt the construction from state  ${}^{\leftarrow}t$  to reach the state  $\{f\}$ , by defining  $\text{box}(t, {}^{\leftarrow}t)$  to be  $\langle [- \mapsto *], \langle \{*\}, \emptyset \rangle, [f \mapsto *] \rangle$ .

We extend this construction to runs in a straightforward way: if  $S_0, \dots, S_n$  are states and  $R = \langle S_0, t_1; \dots; t_n, S_n \rangle$  is a proper run in  $\mathcal{A}$  we define  $\text{box}(R) \triangleq \text{box}(t_1, S_0) \odot \text{box}(t_2, S_1) \odot \dots \odot \text{box}(t_n, S_{n-1})$ . With this definition, accepting runs yield boxes in  $\mathcal{B}[\{\iota\}, \{f\}]$ . This actually amounts to computing the trace of  $R$ :

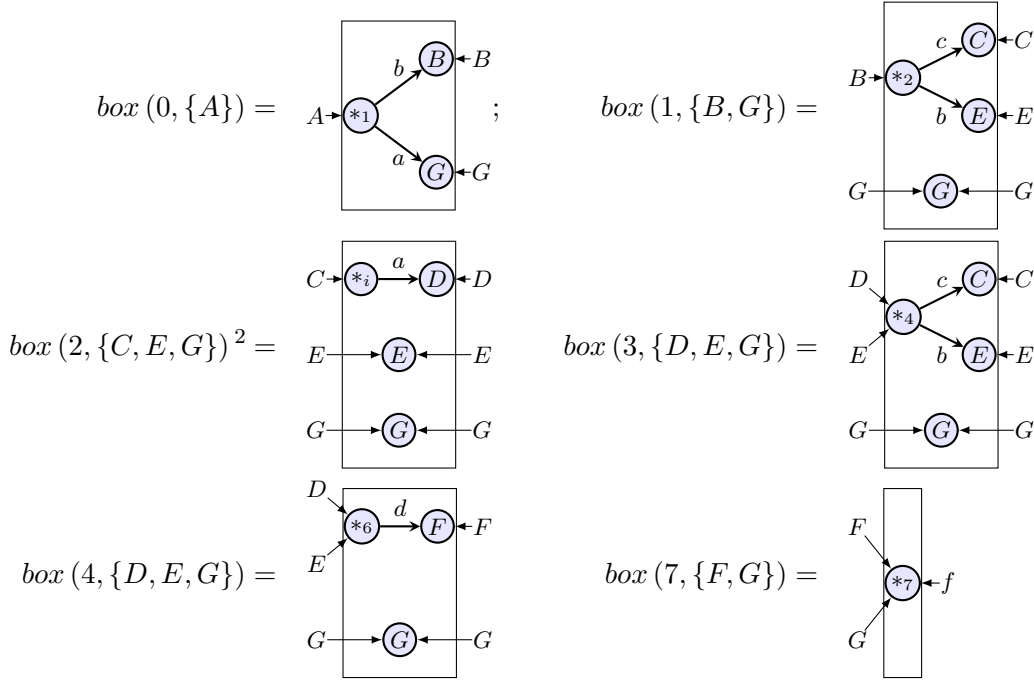
**Lemma 7.22.** *If  $\text{box}(R) = \langle \vec{\mathfrak{p}}, G, \overleftarrow{\mathfrak{p}} \rangle$ , then  $G$  is isomorphic to  $\mathcal{G}(R)$ .*

*Proof.* The vertex  $k$  in  $\mathcal{G}(R)$  (produced by  $t_k$ ) is equivalent to the vertex  $*$  coming from the same transition.  $\square$

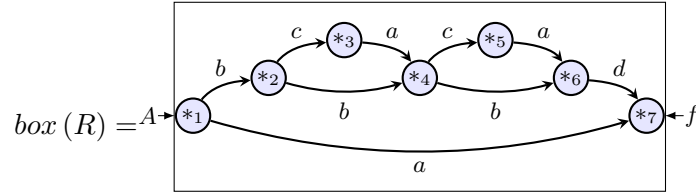
**Example 7.23.** Recall the accepting run from Example 5.2:



The transitions of this run yield the following boxes:



Accordingly, the box of the run is the one below.



The graph in this box is exactly the trace of the run (Figure 9). Also notice that the only vertices in the composite box are the  $*_i$  vertices, one for each transition.

This equivalence with traces yields another property: because of Constraint 5.5, we know that every trace in  $\mathcal{A}$  is series-parallel. This means that the box of every run can be typed as going from  $\tau_\iota = \circ \longrightarrow \circ \longleftarrow \iota$  to type  $\tau_f = \circ \longrightarrow \circ \longleftarrow f$ .

**Lemma 7.24.** *Let  $R = \langle \{\iota\}, t_1; \dots; t_n, \emptyset \rangle$  be an accepting run, with intermediary states  $\{\iota\} = S_0, S_1, \dots, S_{n-1}, S_n = \emptyset$ . There exists a sequence  $\tau_0, \dots, \tau_n$  of types over  $P$  such that  $\tau_0 = \tau_\iota$ ,  $\tau_n = \tau_f$  and  $\forall 1 \leq i \leq n$ ,  $box(t_i, S_{i-1}) \in \mathfrak{B}[\tau_{i-1}, \tau_i]$*

*Proof.* Simply put, since  $\mathcal{G}(R)$  is series-parallel, then for every  $k$  it must be the case that  $\mathcal{G}(\langle \{\iota\}, t_1; \dots; t_k, S_k \rangle)$  SP-reduces to a tree  $\tau_k$ . Using Lemma 7.22 we can use these trees to type the boxes of every transition in  $R$ .

The fact that  $\tau_n = \tau_f = \circ \longrightarrow \circ \longleftarrow f$  is straightforward, as this is the only type over  $\{f\}$ .  $\square$

<sup>2</sup>Two instances of this box have to be used, one with  $i = 3$  and one with  $i = 5$ .

Consider the finite state automaton  $Aut$  with states  $\mathbb{T}_P$  (the set of types over  $P$ ), initial state  $\tau_\iota$ , a single final state  $\tau_f$  and two kinds of transitions. For every non-final transition  $t$ , states  $S, S'$  and types  $\tau, \sigma$  such that  $\mathcal{A} \vdash t : S \rightarrow S'$  and  $box(t, S) \in \mathfrak{B}[\tau, \sigma]$ , there is a transition  $\langle \tau, box(t, S), \sigma \rangle$  in  $Aut$ . There are also transitions  $\langle \tau, box(t, {}^a t), \tau_f \rangle$  for every final transition  $\langle {}^a t, \emptyset \rangle$  and every type  $\tau$  over  ${}^a t$ .

This automaton captures exactly the accepting runs of  $\mathcal{A}$ . Indeed, Lemma 7.24 assures us that for every accepting run in  $\mathcal{A}$  we can find a corresponding accepting run in  $Aut$ . On the other hand every accepting run in  $Aut$  stems by construction from an accepting run in  $\mathcal{A}$ .

This means we can extract a regular expression  $e$  with letters in  $\mathfrak{B}$  from this automaton using the standard Kleene theorem. We can also get the language of  $\mathcal{A}$  by extracting the graphs of boxes  $\beta_1 \odot \dots \odot \beta_n$ , whenever  $\beta_1 \dots \beta_n$  is a word in the language of  $e$ . However, this is not yet what we are looking for, that is, a simple expression over  $Y$ . To get this expression, we replay the standard proof of Kleene's theorem, with some modifications to suit our needs.

**Remark 7.25.** The automaton above is constructed from the state  $\tau_\iota$ , by exploring all initial runs. This construction fails if either Constraint 5.3 or Constraint 5.5 are not satisfied. This shows that the two constraints we imposed are decidable.

**7.3.2. Computing the expression.** A standard way of proving Kleene's theorem consists in moving from automata to *generalised automata*, that is automata with regular expressions labelling transitions. We perform a similar step here, by replacing boxes with box templates in transition labels. The transformation is straightforward: for every pair of states  $\sigma, \tau$ , we put in the generalised automaton a transition labelled with  $\{\beta \mid \langle \sigma, \beta, \tau \rangle \in Aut\}$ . This ensures that there is exactly one transition between every pair of states. Notice that in the resulting generalised automaton, if the transition from  $\sigma$  to  $\tau$  is labelled with  $\Gamma$ , all boxes in  $\Gamma$  have type  $\sigma \rightarrow \tau$ , therefore  $\Gamma$  itself has this type.

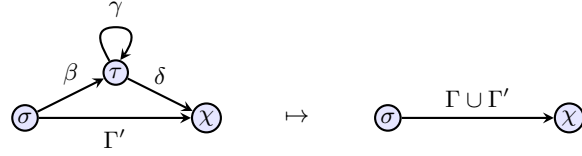
Then, we proceed to remove non-initial non-final states, in an arbitrary order. Notice that the automaton  $Aut$  we are considering has a single initial state and a single final state, respectively  $\tau_\iota$  and  $\tau_f$ . These states are distinct, there is no outgoing transition from  $\tau_f$ , nor is there an incoming transition in  $\tau_\iota$  (remember that  $\iota$  does not appear in the output of transitions). At the end of this procedure, the only remaining states will thus be  $\tau_\iota$  and  $\tau_f$ . There will be exactly one transition from  $\tau_\iota$  to  $\tau_f$  of the shape  $\langle \tau_\iota, \{\beta_1, \dots, \beta_n\}, \tau_f \rangle$ . As every  $\beta_i$  has type  $\tau_\iota \rightarrow \tau_f$ , its graph  $G_i$  must be series-parallel, thus we will get that  $e \triangleq \mathfrak{t}(G_1) \cup \dots \cup \mathfrak{t}(G_n)$  is a simple expression such that  $(\{\beta_1, \dots, \beta_n\}) = \mathcal{G}(e)$ .

We maintain an invariant throughout the construction: the boxes generated by templates labelling each transition of the automaton should stem from runs in  $\mathcal{A}$ . More precisely, we require every template labelling a transition to be  $\mathcal{A}$ -valid:

**Definition 7.26** ( $\mathcal{A}$ -validity). A template  $\gamma \in \mathfrak{B}\mathfrak{E}[\tau, \sigma]$  is  $\mathcal{A}$ -valid if for every  $\beta \in (\gamma)$  there exists a run  $R$  in  $\mathcal{A}$  such that  $box(R) = \beta$ .

Now we only need to show how to remove one state, preserving the language of the automaton. The idea when removing a state  $\tau$  is to add transitions to replace every run going through  $\tau$ . For every pair of states  $\sigma, \chi$  with transitions labelled with  $\beta, \delta, \gamma$  and  $\Gamma'$  as below, we will define a template  $\Gamma$ . We will then replace  $\Gamma'$  with  $\Gamma \cup \Gamma'$  on the transition going from  $\sigma$  to  $\chi$ .





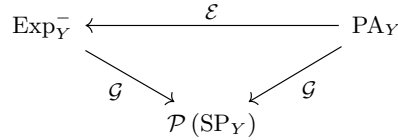
We would like  $\Gamma$  to be  $\beta \cdot \gamma^* \cdot \delta$ . However, remember that we can only compute the star of atomic templates. But in this case, we can approximate  $\gamma$  with a good-enough atomic template:

**Lemma 7.27.** *For every  $\mathcal{A}$ -valid template  $\gamma \in \mathfrak{B}\mathfrak{E}[\tau, \tau]$  there exists an  $\mathcal{A}$ -valid atomic template  $At(\gamma) \in \mathfrak{B}\mathfrak{E}[\tau, \tau]$  such that  $\langle \gamma \rangle \subseteq \langle At(\gamma)^* \rangle$ .*

*Proof.* From each connected component of the graph of each box in  $\gamma$  stems an  $\mathcal{A}$ -valid atomic box. Every box in  $\gamma$  is equal to the product (in any order) of the boxes corresponding to its connected components. We then take  $At(\gamma)$  to be the set of all these atomic boxes.  $\square$

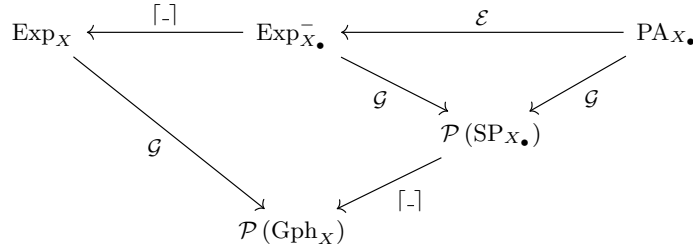
We then define  $\Gamma$  to be  $\beta \cdot At(\gamma)^* \cdot \delta$ . Hence we get  $\langle \beta \rangle \odot \langle \gamma \rangle^* \odot \langle \delta \rangle \subseteq \langle \Gamma \rangle$ . The lemma also ensures that for every graph produced by a run in the automaton where  $\tau$  is removed, there is a run in  $\mathcal{A}$  yielding the same graph. Both these properties allow us to conclude that this step is valid, preserving both the invariant and the language of the automaton.

**Proposition 7.28.** *There exists a function  $\mathcal{E}: PA_Y \rightarrow Exp_Y^-$  such that for every Petri automaton  $\mathcal{A}$ , we have  $\mathcal{G}(\mathcal{E}(\mathcal{A})) = \mathcal{G}(\mathcal{A})$ .*



**Theorem 7.29.** *Recognisable sets of series-parallel graphs are regular.*

As in Section 6, we can use Proposition 4.4 to extend this result to sets of arbitrary graphs: the following diagram commutes.



Whence

**Theorem 7.30.** *Recognisable sets of graphs are regular.*

## 8. READING GRAPHS MODULO HOMOMORPHISM

In Section 6 we have shown how to associate a Petri automaton  $\mathcal{A}(e)$  with every expression  $e$ , such that  $\mathcal{G}(\mathcal{A}(e)) = \mathcal{G}(e)$ . However, remember from Section 3 that the set of graphs we are interested in for representable Kleene algebras is not directly  $\mathcal{G}(e)$ , but its downward closure  $\blacktriangleleft \mathcal{G}(e)$  w.r.t. homomorphism (Theorem 3.9). Given a Petri automaton  $\mathcal{A} \in PA_{X_\bullet}$ , we show how to read the graphs in  $\blacktriangleleft [\mathcal{G}(\mathcal{A})]$  in a local and incremental way.

**Definition 8.1** (Reading, language of a run). A *reading* of  $G = \langle V, E, \iota, o \rangle$  along a run  $R = \langle S_0, t_0; \dots; t_n, S_{n+1} \rangle$  (with intermediary states  $S_1, \dots$ ) is a sequence  $(\rho_k)_{0 \leq k \leq n+1}$  such that for all  $k$ ,  $\rho_k$  is a map from  $S_k$  to  $V$ ,  $\rho_0(S_0) = \{\iota\}$ , and  $\forall 0 \leq k \leq n$ , the following holds:

- all tokens in the input of the transition are mapped to the same vertex in the graph:

$$\forall p, q \in {}^{\triangleleft}t_k, \rho_k(p) = \rho_k(q);$$

- a final transition may only be fired from the output of a graph:

$$t_k^{\triangleright} = \emptyset \Rightarrow \forall p \in {}^{\triangleleft}t_k, \rho_k(p) = o;$$

- the images of tokens in  $S_{k+1}$  that are not in the input of the transition are unchanged:

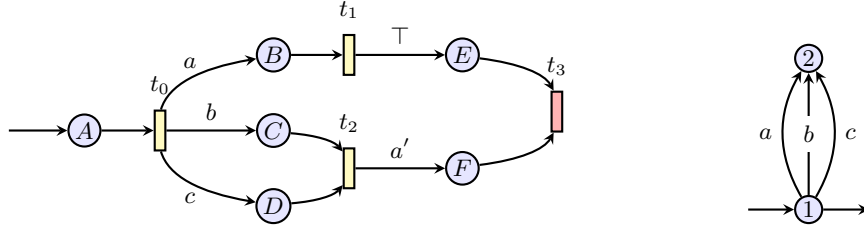
$$\forall p \in S_{k+1} \setminus {}^{\triangleleft}t_k, \rho_k(p) = \rho_{k+1}(p);$$

- each pair in the output of the transition can be validated by the graph:

$$\forall p \in {}^{\triangleleft}t_k, \forall \langle x, q \rangle \in t_k^{\triangleright}, \langle \rho_k(p), x, \rho_{k+1}(q) \rangle \in E.$$

The *language of a run*  $R$ , denoted by  $\mathcal{L}(R)$ , is the set of graphs that can be read along  $R$ .

**Example 8.2.** The following automaton on the left-hand side can read the graph on the right-hand side.



Let  $R$  be the run  $\langle \{A\}, t_0; t_1; t_2; t_3, \emptyset \rangle$ . To read the graph above along  $R$ , one should use the following reading:

$$[A \mapsto 1]; \left[ \begin{array}{l} B \mapsto 2 \\ C \mapsto 2 \\ D \mapsto 2 \end{array} \right]; \left[ \begin{array}{l} E \mapsto 1 \\ C \mapsto 2 \\ D \mapsto 2 \end{array} \right]; \left[ \begin{array}{l} E \mapsto 1 \\ F \mapsto 1 \end{array} \right]; \square.$$

One can easily check that this is indeed a valid reading along  $R$ .

The language of a Petri automaton is finally obtained by considering all accepting runs.

**Definition 8.3** (Language of a Petri automaton). The *language of*  $\mathcal{A}$ , written  $\mathcal{L}(\mathcal{A})$ , is the set of graphs readable from accepting runs:

$$\mathcal{L}(\mathcal{A}) \triangleq \bigcup_{R \in \text{Run}_{\mathcal{A}}^{\text{acc}}} \mathcal{L}(R).$$

To avoid confusions between the languages  $\mathcal{L}(\mathcal{A})$  and  $\mathcal{G}(\mathcal{A})$ , we write “ $G$  is produced by  $\mathcal{A}$ ” when  $G \in \mathcal{G}(\mathcal{A})$ , reserving language theoretic terminology like “ $G$  is accepted by  $\mathcal{A}$ ” to cases where we mean  $G \in \mathcal{L}(\mathcal{A})$ .

**Lemma 8.4.** *For every accepting run  $R$ , we have  $G \in \mathcal{L}(R)$  if and only if  $G \blacktriangleleft [\mathcal{G}(R)]$ .*

*Proof.* Let us fix a Petri automaton  $\mathcal{A} = \langle P, \mathcal{T}, \iota_{\mathcal{A}} \rangle$ . Let  $G = \langle V, E, \iota, o \rangle$  and  $R = \langle \{\iota\}, t_0; \dots; t_n, \emptyset \rangle$ , with intermediary states  $S_1, \dots$ .

Remember that for every  $k$  and  $p \in \pi_2(t_k^{\triangleright})$ , we defined (Definition 5.4):

$$\nu(k, p) = \{l \mid l > k \text{ and } p \in {}^{\triangleleft}t_l\}.$$

$\mathcal{G}(R)$ , is then the graph with vertices  $\{0, \dots, n\}$  and the set of edges defined by:

$$E_R = \{\langle k, a, l \rangle \mid \langle a, p \rangle \in t_k^{\triangleright} \text{ and } l = \min \nu(k, p)\}.$$

Finally, we get  $[\mathcal{G}(R)] = \langle \{[i] \mid 0 \leq i \leq n\}, E', [0], [n] \rangle$ . (It is easy to check that the source of  $\mathcal{G}(R)$  is the vertex 0, and that its sink is  $n$ .)

It will prove convenient in the following to use the notation  $\mathcal{N}(k, p) = \min \nu(k-1, p) = \min \{l \mid l \geq k \text{ and } p \in {}^{\triangleleft}t_l\}$ . Notice that  $\forall k, p \in S_k, k \leq \mathcal{N}(k, p) \leq n$ , and that whenever  $p \in {}^{\triangleleft}t_k$ , we have  $\mathcal{N}(k, p) = k$ .

Suppose there exists a graph homomorphism  $\varphi$  from  $[\mathcal{G}(R)]$  to  $G$ . We build a reading  $(\rho_k)_k$  of  $G$  along  $R$  by letting  $\rho_k(p) \triangleq \varphi([\mathcal{N}(k, p)])$  for  $0 \leq k \leq n$  and  $p \in S_k$ . We now have to check that  $\rho$  is truly a reading of  $G$  in  $\mathcal{A}$ :

- for the initialisation of the reading:

$$\begin{aligned} \rho_0(\iota_{\mathcal{A}}) &= \varphi([\mathcal{N}(0, \iota_{\mathcal{A}})]) && \text{(by definition)} \\ &= \varphi([0]) = \{\iota\} && (\varphi \text{ is a homomorphism}) \end{aligned}$$

- for the final transition:

$$\begin{aligned} p \in S_n, \rho_n(p) &= \varphi([\mathcal{N}(n, p)]) \\ &= \varphi([n]) = \{o\}. && (\varphi \text{ is a homomorphism}) \end{aligned}$$

- for all  $p \in {}^{\triangleleft}t_k, \rho_k(p) = \varphi([\mathcal{N}(k, p)]) = \varphi([k])$  which does not depend on  $p$ .
- for all  $p \in S_{k+1} \setminus {}^{\triangleleft}t_k$ , we have  $\mathcal{N}(k, p) = \mathcal{N}(k+1, p)$  (since  $p \notin {}^{\triangleleft}t_k$ ). Hence

$$\begin{aligned} \rho_k(p) &= \varphi([\mathcal{N}(k, p)]) = \varphi([\mathcal{N}(k+1, p)]) \\ &= \rho_{k+1}(p). \end{aligned}$$

- for all  $p \in {}^{\triangleleft}t_k$  and  $\langle x, q \rangle \in t_k^{\triangleright}$ , we know that  $\rho_k(p) = \varphi([k])$  and that  $\langle k, x, \mathcal{N}(k+1, q) \rangle \in E_R$ .
  - If  $x \in X$ , we also have  $\langle [k], x, [\mathcal{N}(k+1, q)] \rangle \in E'$ . Because  $\varphi$  is a homomorphism we can deduce that:

$$\langle \varphi([k]), x, \varphi([\mathcal{N}(k+1, q)]) \rangle \in E,$$

which can be rewritten  $\langle \rho_k(p), x, \rho_{k+1}(q) \rangle \in E$ .

- If  $x = y', y \in X$ , we also have  $\langle [\mathcal{N}(k+1, q)], y, [k] \rangle \in E'$ . Because  $\varphi$  is a homomorphism we can get like before  $\langle \rho_{k+1}(q), y, \rho_k(p) \rangle \in E$ .
- If finally  $x = 1$ , then we know that  $k \equiv \mathcal{N}(k+1, q)$ , thus proving that

$$\rho_k(p) = \varphi([k]) = \varphi([\mathcal{N}(k+1, q)]) = \rho_{k+1}(q).$$

If on the other hand we have a reading  $(\rho_k)_{0 \leq k \leq n}$  of  $G$ , we define  $\varphi : \{0, \dots, n\} \rightarrow V$  by  $\varphi([k]) \triangleq \rho_k(p)$  for all  $p \in {}^{\triangleleft}t_k$ . As  $(\rho_k)_k$  is a reading,  $\varphi$  is well defined<sup>3</sup>. Let us check that  $\varphi$  is a homomorphism from  $[\mathcal{G}(R)]$  to  $G$ :

- $\varphi([0]) = \rho_0(\iota_{\mathcal{A}}) = \iota$ ;
- $\varphi([n]) = \rho_n(p)$  with  $p \in S_n$ , and since  $(\rho_k)_k$  is a reading and  $t_n$  is final,  $\rho_n(p) = o$ .

<sup>3</sup>It is not difficult to check that  $k \equiv l \Rightarrow \forall \langle p, q \rangle \in {}^{\triangleleft}t_k \times {}^{\triangleleft}t_l, \rho_k(p) = \rho_l(q)$ .

- if  $\langle [k], x, [l] \rangle \in E'$  is an edge of  $[\mathcal{G}(R)]$ , then it was produced from some edge  $(i, y, j) \in E_R$ , with either  $x = y$  and  $\langle i, j \rangle \in [k] \times [l]$  or  $y = x'$  and  $\langle i, j \rangle \in [l] \times [k]$ . There is some  $p \in {}^a t_i$  and  $q$  such that  $\langle y, q \rangle \in t_j^*$  and  $j = \mathcal{N}(i+1, q)$ .

By definition of  $\mathcal{N}$  we know that  $\forall i+1 \leq m < j, q \notin {}^a t_m$ . Thus, because  $(\rho_k)_k$  is a reading,  $\rho_{i+1}(q) = \rho_j(q)$  and either  $\langle \rho_i(p), x, \rho_{i+1}(q) \rangle \in E$  or  $\langle \rho_{i+1}(q), x, \rho_i(p) \rangle \in E$ , and thus  $\langle \varphi([k]), x, \varphi([l]) \rangle \in E$ .  $\square$

As an immediate consequence, we obtain the following characterisations of the language of a Petri automaton.

**Theorem 8.5.** *For every Petri automaton  $\mathcal{A} \in PA_{X_\bullet}$ , we have  $\mathcal{L}(\mathcal{A}) = \blacktriangleleft[\mathcal{G}(\mathcal{A})]$ .*

**Theorem 8.6.** *For every Petri automaton  $\mathcal{A} \in PA_X$ , we have  $\mathcal{L}(\mathcal{A}) = \blacktriangleleft\mathcal{G}(\mathcal{A})$ .*

The left-hand side language is defined through readings along accepting runs, which is a local and incremental notion and which allows us to define *simulations* in the following section. By contrast, the right-hand side language is defined globally.

## 9. COMPARING PETRI AUTOMATA MODULO HOMOMORPHISM

While we can use Petri automata over  $X_\bullet$  to read arbitrary graphs, and thus reason about arbitrary expressions, we do not know how to compare the languages of these automata in general. We thus restrict to Petri automata over  $X$  and to simple expressions. At the algebraic level, this means we consider identity-free Kleene lattices. We prove in this section that the containment problem is decidable.

Assembling the results from Sections 3, 6 and 8, we have the following proposition.

**Proposition 9.1.** *For all simple expressions  $e, f \in \text{Exp}_X^-$ , the following are equivalent:*

- (i)  $\text{Rel} \models e \leq f$ ,
- (ii)  $\mathcal{G}(\mathcal{A}(e)) \subseteq \mathcal{L}(\mathcal{A}(f))$ .

*For all Petri automata  $\mathcal{A}, \mathcal{B} \in PA_X$ , the following are equivalent:*

- (i)  $\mathcal{L}(\mathcal{A}) \subseteq \mathcal{L}(\mathcal{B})$ ,
- (ii)  $\mathcal{G}(\mathcal{A}) \subseteq \mathcal{L}(\mathcal{B})$ .

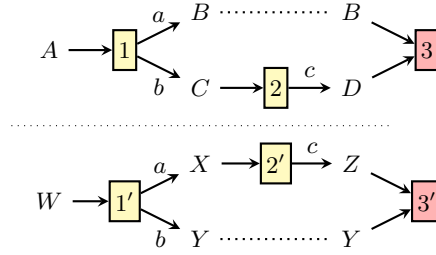
*Proof.* We have

$$\begin{aligned}
& \text{Rel} \models e \leq f \\
& \Leftrightarrow \mathcal{G}(e) \subseteq \blacktriangleleft\mathcal{G}(f) && \text{(Theorem 3.9)} \\
& \Leftrightarrow \mathcal{G}(\mathcal{A}(e)) \subseteq \blacktriangleleft\mathcal{G}(\mathcal{A}(f)) && \text{(Proposition 6.6)} \\
& \Leftrightarrow \mathcal{G}(\mathcal{A}(e)) \subseteq \mathcal{L}(\mathcal{A}(f)) && \text{(Theorem 8.6)}
\end{aligned}$$

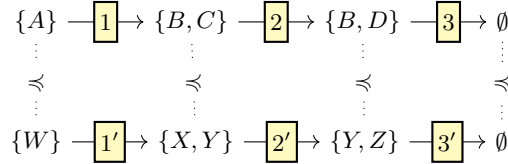
The second point follows from Theorem 8.6 and the fact that  $S \subseteq \blacktriangleleft T$  iff  $\blacktriangleleft S \subseteq \blacktriangleleft T$ .  $\square$

As a consequence, to decide the (in)equational theory of identity-free Kleene lattices, or to compare two Petri automata, it suffices to find a way to decide whether  $\mathcal{G}(\mathcal{A}) \subseteq \mathcal{L}(\mathcal{B})$ , given two Petri automata  $\mathcal{A}$  and  $\mathcal{B}$ .

**9.1. Intuition.** In this section, we show how the notion of simulation relation, that allows one to compare NFA, can be adapted to handle simple Petri automata. Consider two automata  $\mathcal{A}_1 = \langle P_1, \mathcal{T}_1, \iota_1 \rangle$  and  $\mathcal{A}_2 = \langle P_2, \mathcal{T}_2, \iota_2 \rangle$ , we try to show that for every accepting run  $R$  in  $\mathcal{A}_1$ ,  $\mathcal{G}(R)$  is recognised by some accepting run  $R'$  in  $\mathcal{A}_2$ . Leaving non-determinism aside, the first idea that comes to mind is to find a relation between the states in  $\mathcal{A}_1$  and the states in  $\mathcal{A}_2$ , that satisfy some conditions on the initial and final states, and such that if  $S_k \preceq S'_k$  and  $\mathcal{A}_1 \vdash t : S_k \rightarrow S_{k+1}$ , then there is a state  $S'_{k+1}$  in  $\mathcal{A}_2$  such that  $S_{k+1} \preceq S'_{k+1}$ ,  $\mathcal{A}_2 \vdash t' : S'_k \rightarrow S'_{k+1}$ , and these transition steps are compatible in some sense. However, such a definition will not give us the result we are looking for. Consider these two runs:

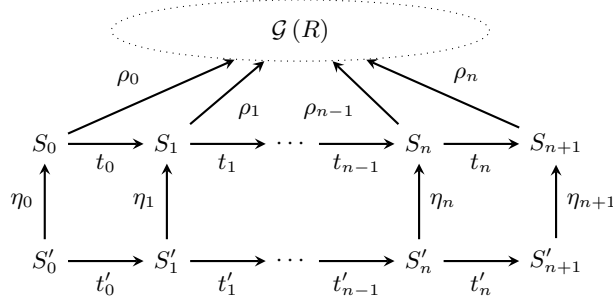


The graphs produced by the first and the second runs correspond respectively to the simple terms  $a \cap (b \cdot c)$  and  $(a \cdot c) \cap b$ . These two terms are incomparable, but the relation  $\preceq$  depicted below satisfies the previously stated conditions.



The problem here is that in Petri automata, runs are token firing games. To adequately compare two runs, we need to closely track the tokens. For this reason, we will relate a state  $S_k$  in  $\mathcal{A}_1$  not only to a state  $S'_k$  in  $\mathcal{A}_2$ , but to a map  $\eta_k$  from  $S'_k$  to  $S_k$ . This will enable us to associate with each token situated on some place in  $P_2$  another token placed on  $\mathcal{A}_1$ .

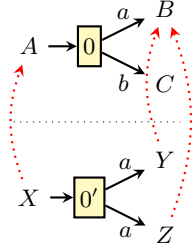
We want to find a reading of  $\mathcal{G}(R)$  in  $\mathcal{A}_2$ , *i.e.*, a run in  $\mathcal{A}_2$  together with a sequence of maps associating places in  $\mathcal{A}_2$  with vertices in  $\mathcal{G}(R)$ . Consider the picture below. Since we already have a reading of  $\mathcal{G}(R)$  along  $R$  (by defining  $\rho_k(p) = \mathcal{N}(k, p)$ , as in the proof of Lemma 8.4), it suffices to find maps from the places in  $\mathcal{A}_2$  to the places in  $\mathcal{A}_1$  (the maps  $\eta_k$ ): the reading of  $\mathcal{G}(R)$  in  $\mathcal{A}_2$  will be obtained by composing  $\eta_k$  with  $\rho_k$ .



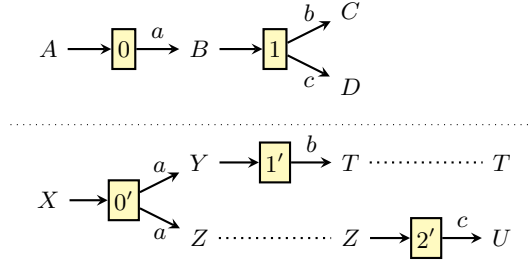
We need to impose some constraints on the maps  $(\eta_k)$  to ensure that  $(\rho_k \circ \eta_k)_{0 \leq k \leq n}$  is indeed a correct reading in  $\mathcal{A}_2$ . First, we need to ascertain that a transition  $t'_k$  in  $\mathcal{A}_2$  may be fired from the reading state  $\rho_k \circ \eta_k$  to reach the reading state  $\rho_{k+1} \circ \eta_{k+1}$ . Furthermore,

as for NFA, we want transitions  $t_k$  and  $t'_k$  to be related: specifically, we require  $t'_k$  to be included (via the homomorphisms  $\eta_k$  and  $\eta_{k+1}$ ) in the transition  $t_k$ . This is meaningful because transition  $t_k$  contains a lot of information about the vertex  $k$  of  $\mathcal{G}(R)$  and about  $\rho$ : the labels of the outgoing edges from  $k$  are the labels on the output of  $t_k$ , and the only places that will ever be mapped to  $k$  in the reading  $\rho$  are exactly the places in the input of  $t_k$ .

This already shows an important difference between the simulations for NFA and Petri automata. For NFA, we relate a transition  $p \xrightarrow{a} p'$  to a transition  $q \xrightarrow{a} q'$  with the same label  $a$ . Here the transitions  $\mathcal{A}_1 \vdash t_k : S_k \rightarrow S_{k+1}$  and  $\mathcal{A}_2 \vdash t'_k : S'_k \rightarrow S'_{k+1}$  may have different labels. Consider the step represented below, corresponding to a square in the above diagram. The output of transition 0 has a label  $b$  that does not appear in  $0'$ , and  $0'$  has two outputs labelled by  $a$ . Nevertheless this satisfies the conditions informally stated above, indeed,  $a \cap b \leq a \cap a$  holds.



However this definition is not yet satisfactory. Consider the two runs below:



Their produced graphs correspond respectively to the simple terms  $a \cdot (b \cap c)$  and  $(a \cdot b) \cap (a \cdot c)$ . The problem is that  $a \cdot (b \cap c) \leq (a \cdot b) \cap (a \cdot c)$ , but with the previous definition, we cannot relate these runs: they do not have the same length. The solution here consists in grouping the transitions  $1'$  and  $2'$  together, and considering these two steps as a single step in a *parallel run*. This last modification gives us a notion of simulation that suits our needs.

**9.2. Simulations.** Before getting to the notion of simulation, we need to define what is a parallel run, and a parallel reading.

A set of transitions  $T \subseteq \mathcal{T}$  is *compatible* if their inputs are pairwise disjoint. If furthermore all transitions in  $T$  are enabled in a state  $S$ , one can observe that the state  $S'$  reached after firing them successively does not depend on the order in which they are fired. In that case we write  $\mathcal{A} \vdash T : S \rightarrow S'$ .

A *parallel run* is a sequence  $\mathcal{R} = \langle S_0, T_0; \dots; T_n, S_{n+1} \rangle$ , where the  $T_k \subseteq \mathcal{T}$  are compatible sets of transitions such that  $\mathcal{A} \vdash T_k : S_k \rightarrow S_{k+1}$ . We define a *parallel reading*  $\rho$  along some parallel run  $\mathcal{R} = \langle S_0, T_0; \dots; T_n, \emptyset \rangle$  by requiring that:  $\rho_0(S_0) = \{\iota\}$ ,  $\rho_n(S_n) = \{o\}$ , and  $\forall k \leq n$  the following holds:

- $\forall p \in S_k \setminus \bigcup_{t \in T_k} {}^a t, \rho_{k+1}(p) = \rho_k(p)$ ;

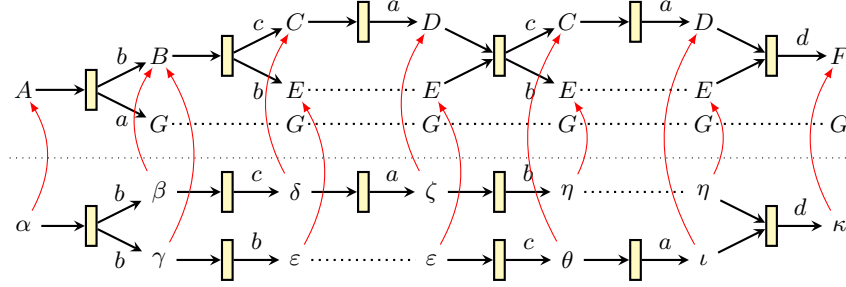


Figure 17: Embedding of a parallel run into the run from Figure 7.

- $\forall t \in T_k, \forall p, q \in {}^a t, \rho_k(p) = \rho_k(q)$ ;
- $\forall t \in T_k, \forall p \in {}^a t, \forall \langle x, q \rangle \in t^\flat, \langle \rho_k(p), x, \rho_{k+1}(q) \rangle \in E$

**Definition 9.2** (Simulation). A relation  $\preceq \subseteq \mathcal{P}(P_1) \times \mathcal{P}(P_2 \rightarrow P_1)$  between the states of  $\mathcal{A}_1$  and the partial maps from the places of  $\mathcal{A}_2$  to the places of  $\mathcal{A}_1$  is called a *simulation* between  $\mathcal{A}_1$  and  $\mathcal{A}_2$  if:

- if  $S \preceq E$  and  $\eta \in E$  then the range of  $\eta$  must be included in  $S$ ;
- $\{\iota_1\} \preceq \{[\iota_2 \mapsto \iota_1]\}$ ;
- if  $S \preceq E$  and  $\mathcal{A}_1 \vdash t : S \rightarrow S'$ , then  $S' \preceq E'$  where  $E'$  is the set of all  $\eta'$  such that there is some  $\eta \in E$  and a compatible set of transitions  $T \subseteq \mathcal{T}_2$  such that:
  - $\mathcal{A}_2 \vdash T : \text{dom}(\eta) \rightarrow \text{dom}(\eta')$ ;
  - $\forall t' \in T, \eta({}^a t') \subseteq {}^a t$  and  $\forall \langle x, q \rangle \in t'^\flat, \langle x, \eta'(q) \rangle \in t^\flat$ ;
  - $\forall p \in \text{dom}(\eta), (\forall t' \in T, p \notin {}^a t') \Rightarrow \eta(p) = \eta'(p)$ .
- if  $S \preceq E$  and  $S = \emptyset$ , then there must be some  $\eta \in E$  such that  $\text{dom}(\eta) = \emptyset$ .

We will now prove that the language of  $\mathcal{A}_1$  is contained in the language of  $\mathcal{A}_2$  if and only if there exists such a simulation. We first introduce the following notion of embedding.

**Definition 9.3** (Embedding). Let  $R = \langle S_0, t_0; \dots; t_{n-1}, S_n \rangle$  be a run in  $\mathcal{A}_1$ , and let  $\mathcal{R} = \langle S'_0, T_0; \dots; T_{n-1}, S'_n \rangle$  be a parallel run in  $\mathcal{A}_2$ . An *embedding* of  $\mathcal{R}$  into  $R$  is a sequence  $(\eta_i)_{0 \leq i \leq n}$  of maps such that for all  $i < n$ , we have:

- $\eta_i$  is a map from  $S'_i$  to  $S_i$ ;
- the image of  $T_i$  by  $\eta_i$  is included in  $t_i$ , meaning that for all  $t \in T_i$ , for all  $p \in {}^a t$  and  $\langle x, q \rangle \in t^\flat$ ,  $\eta_i(p)$  is contained in the input of  $t_i$  and  $\langle x, \eta_{i+1}(q) \rangle$  is in the output of  $t_i$ ;
- the image of the tokens in  $S_i$  that do not appear in the input of  $T_i$  are preserved ( $\eta_i(p) = \eta_{i+1}(p)$ ) and their image is not in the input of  $t_i$ .

$$\begin{array}{ccc}
 S_i & \xrightarrow{t_i} & S_{i+1} \\
 \eta_i \uparrow & & \uparrow \eta_{i+1} \\
 S'_i & \xrightarrow{T_i} & S'_{i+1}
 \end{array}$$

Figure 17 illustrates the embedding of some parallel run, into the run presented in Figure 7. Notice that it is necessary to have a parallel run instead of a simple one: to find something that matches the second transition in the upper run, we need to fire two transitions in parallel in the lower run.

There is a close relationship between simulations and embeddings:

**Lemma 9.4.** Let  $\mathcal{A}_1$  and  $\mathcal{A}_2$  be two Petri automata, the following are equivalent:

- there exists a simulation  $\preceq$  between  $\mathcal{A}_1$  and  $\mathcal{A}_2$ ;
- for every accepting run  $R$  in  $\mathcal{A}_1$ , there is an accepting parallel run  $\mathcal{R}$  in  $\mathcal{A}_2$  that can be embedded into  $R$ .

*Proof.* If we have a simulation  $\preceq$ , let  $R = \langle S_0, t_0; \dots; t_n, \emptyset \rangle$  be an accepting run in  $\mathcal{A}_1$ . By the definition of simulation, we can find a sequence of sets of maps  $(E_k)_{0 \leq k \leq n+1}$  such that  $E_0 = \{\{\iota_2 \mapsto \iota_1\}\}$  and  $\forall k, S_k \preceq E_k$ . Furthermore, we can extract from this a sequence of maps  $(\eta_k)_{0 \leq k \leq n+1}$  and a sequence of parallel transitions  $(T_k)_{0 \leq k \leq n}$  such that:

- $\forall k, \mathcal{A}_2 \vdash T_k : \text{dom}(\eta_k) \rightarrow \text{dom}(\eta_{k+1})$
- the parallel run  $\langle \text{dom}(\eta_0), T_0; \dots; T_n; \text{dom}(\eta_{n+1}) \rangle$  is accepting, and can be embedded into  $R$  using  $(\eta_k)$ .

This follows directly from the definitions of embedding and simulation.

On the other hand, if we have property ((ii)), then we can define a relation  $\preceq$  by saying that  $S \preceq E$  if there is an accepting run  $R = \langle S_0, t_0; \dots; t_n, S_{n+1} \rangle$  in  $\mathcal{A}_1$  such that there is an index  $k_0$ :  $S = S_{k_0}$ ; and the following holds:  $\eta \in E$  if there is an accepting parallel run  $\mathcal{R}$  in  $\mathcal{A}_2$  visiting successively the states  $S'_0, \dots, S'_{n+1}$  and  $(\eta'_k)_{0 \leq k \leq n+1}$  an embedding of  $\mathcal{R}$  into  $R$  such that  $\eta = \eta'_{k_0}$ . It is then immediate to check that  $\preceq$  is indeed a simulation.  $\square$

If  $\eta$  is an embedding of  $\mathcal{R}$  into  $R$ , and  $\rho$  is a reading of  $\mathcal{G}(R)$  along  $R$ , then we can easily check that  $(\rho_i \circ \eta_i)_{0 \leq i \leq n}$  is a parallel reading of  $\mathcal{G}(R)$  along  $\mathcal{R}$  in  $\mathcal{A}_2$ . Thus, it is clear that once we have such a run  $\mathcal{R}$  with the sequence of maps  $\eta$ , we have that  $\mathcal{G}(R)$  is indeed in the language of  $\mathcal{A}_2$ . The more difficult question is the completeness of this approach: if  $\mathcal{G}(R)$  is recognised by  $\mathcal{A}_2$ , is it always the case that we can find a run  $\mathcal{R}$  that may be embedded into  $R$ ? The answer is affirmative, thanks to Lemma 9.5 below. If  $(\rho_j)_{0 \leq j \leq n+1}$  is a reading of  $G$  along  $R = \langle S_0, t_0; \dots; t_n, S_{n+1} \rangle$ , we write  $\text{active}(j)$  for the only position in  $\rho_j({}^a t_j)$ <sup>4</sup>. A binary relation  $\sqsubseteq$  is a *topological ordering* on  $G = \langle V, E, \iota, \circ \rangle$  if  $\langle V, \sqsubseteq \rangle$  is a linear order and  $(p, x, q) \in E$  entails  $p \sqsubseteq q$ .

**Lemma 9.5.** *Let  $G \in \mathcal{L}(\mathcal{A}_2)$  and let  $\sqsubseteq$  be a topological ordering on  $G$ . Then there exists a run  $R$  and a reading  $(\rho_j)_{0 \leq j \leq n+1}$  of  $G$  along  $R$  such that  $\forall k, \text{active}(k) \sqsubseteq \text{active}(k+1)$ .*

The proof of this result is achieved by taking a run  $R$  accepting  $G$ , and then exchanging transitions in  $R$  according to  $\sqsubseteq$ , while preserving the existence of a reading. However we need to introduce some lemmas first.

Let us fix  $\mathcal{A} = \langle P, \mathcal{T}, \iota \rangle$  a Petri automaton, and  $R = \langle S_0, t_0; \dots; t_n, S_{n+1} \rangle$  a run of  $\mathcal{A}$ .

**Definition 9.6** (Exchangeable transitions). Two transitions  $t_k$  and  $t_{k+1}$  are *exchangeable* in  $R$  if for all  $p \in S_k$ ,  $p$  is in  ${}^a t_{k+1}$  implies that there is no  $x \in X$  such that  $(x, p) \in t_k^\circ$ .

As the name might suggest, two exchangeable transitions may be exchanged in a run, and every graph read along the initial run can still be read along the permuted run.

**Lemma 9.7.** *Suppose  $t_k$  and  $t_{k+1}$  are exchangeable for some  $0 \leq k < n$ . We write  $C' = S_k \setminus {}^a t_{k+1} \cup \pi_2 \left( t_{k+1}^\circ \right)$ . Then  $\mathcal{A} \vdash t_{k+1} : S_k \rightarrow C'$  and  $\mathcal{A} \vdash t_k : C' \rightarrow S_{k+2}$ . Furthermore, for every graph  $G$ , if  $G \in \mathcal{L}(R)$ , then  $G \in \mathcal{L}(R[k \leftrightarrow k+1])$ , where:*

$$R[k \leftrightarrow k+1] \triangleq \langle S_0, t_0; \dots; t_{k+1}; t_k; \dots; t_n, S_n \rangle.$$

*Proof.* The fact that  $S_k \xrightarrow{t_{k+1}}_{\mathcal{A}} C'$  and  $C' \xrightarrow{t_k}_{\mathcal{A}} S_{k+2}$  is trivial to check, with the definition of exchangeable.

<sup>4</sup>Recall that if  $(\rho_j)_{0 \leq j \leq n+1}$  is a reading along  $R$  then for all  $p, q \in {}^a t_j$ , we have  $\rho_j(p) = \rho_j(q)$ .



Let  $(\rho_j)_{0 \leq j \leq n+1}$  be a reading of  $G$  along  $R$ . If  $(\rho'_j)_{0 \leq j \leq n+1}$  is defined by:

$$\rho'_j(p) = \begin{cases} \rho_j(p) & \text{if } j \neq k+1, \\ \rho_{k+2}(p) & \text{if } j = k+1 \text{ and } (x, p) \in t_{k+1}^\triangleright \text{ for some } x, \\ \rho_k(p) & \text{otherwise.} \end{cases}$$

Then  $(\rho'_j)_{0 \leq j \leq n+1}$  is a reading of  $G$  along  $R[k \leftrightarrow k+1]$ .  $\square$

Recall that if  $(\rho_j)_{0 \leq j \leq n}$  is a reading of  $G$  along  $\xi$  we write  $active(j)$  for the only position in  $\rho_j({}^a t_j)$ .

**Lemma 9.8.** *Let  $\sqsubseteq$  be a topological ordering on  $G$ . If  $(\rho_j)_{0 \leq j \leq n+1}$  is a reading of  $G$  along  $R$ , and if  $active(k+1) \sqsubseteq active(k)$  for some  $k$ , then  $t_k$  and  $t_{k+1}$  are exchangeable.*

*Proof.* Let  $G = \langle V, E, \iota, o \rangle$ . since  $(\rho_i)_{0 \leq i \leq n+1}$  is a reading, for every  $\langle x, p \rangle \in t_k^\triangleright$ , we have  $\langle active(k), x, \rho_{k+1}(p) \rangle \in E$ , and thus

$$active(k) \sqsubset \rho_{k+1}(p).$$

We know that  $active(k+1) \sqsubseteq active(k)$ , meaning by transitivity that  $active(k+1) \sqsubset \rho_{k+1}(p)$ . Hence

$$active(k+1) \neq \rho_{k+1}(p)$$

and because  $(\rho_i)_{0 \leq i \leq n+1}$  is a reading we can infer that  $p \notin {}^a t_{k+1}$ , thus proving that  $t_k$  and  $t_{k+1}$  are exchangeable.  $\square$

*Proof of Lemma 9.5.* Because  $G$  is in  $\mathcal{L}(\mathcal{A})$ , we can find a reading  $\rho'$  along some run  $R$ . If that reading is not in the correct order, then by Section 9.8 we can exchange two transitions and Lemma 9.7 ensures that we can find a corresponding reading. We repeat this process until we get a reading in the correct order.  $\square$

(Notice that if  $G$  contains cycles, this lemma cannot apply because of the lack of a topological ordering.)

Lemma 9.5 enables us to build an embedding from every reading of  $\mathcal{G}(R)$  in  $\mathcal{A}_2$ .

**Lemma 9.9.** *Let  $R$  be an accepting run of  $\mathcal{A}_1$ . Then  $\mathcal{G}(R) \in \mathcal{L}(\mathcal{A}_2)$  if and only if there is an accepting parallel run in  $\mathcal{A}_2$  that can be embedded into  $R$ .*

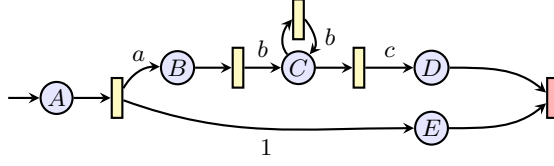
*Proof.* Let  $R = \langle S_0, t_0; \dots; t_n, S_{n+1} \rangle$  be a run. For all indexes  $k$  and places  $p \in S_k$ , define  $\rho_k^1(p) \triangleq \mathcal{N}(k, p) = \min \{l \mid l \geq k \text{ and } p \in {}^a t_l\}$ . We have that  $(\rho_k^1)_{0 \leq k \leq n+1}$  is a reading of  $\mathcal{G}(R)$  along  $R$ .

- Assume an embedding  $(\eta_k)_{0 \leq k \leq n+1}$  of an accepting parallel run  $\mathcal{R}$  into  $R$ . We define a parallel reading  $(\rho_k^2)$  of  $\mathcal{G}(R)$  in  $\mathcal{A}_2$  by letting  $\rho_k^2(p) \triangleq \rho_k^1(\eta_k(p))$ .
- On the other hand, notice that the natural ordering on  $\mathbb{N}$  is a topological ordering on  $\mathcal{G}(R)$ , and that  $\forall 0 \leq k \leq n, \rho_k^1({}^a t_k) = \{k\}$ . By Lemma 9.5 we gather that  $\mathcal{G}(R)$  is in  $\mathcal{L}(\mathcal{A}_2)$  if and only if there exists a reading  $(\rho_j^2)_{0 \leq j \leq n'}$  of  $\mathcal{G}(R)$  along some run  $R' = \langle S'_0, t'_0; \dots; t'_{n'}, S'_{n'+1} \rangle$  such that  $\forall j, active(j) \leq active(j+1)$  (with  $active(j)$  the only position in  $\rho_j^2({}^a t_j)$ ).

Now, suppose we have such a reading; we can build an embedding  $(\eta_k)_{0 \leq k \leq n+1}$  as follows.

For  $k \leq n$ , define  $T_k \triangleq \{t'_j \mid active(j) = k\}$ . We describe the construction incrementally:

–  $\eta_0 = [\iota_2 \mapsto \iota_1]$ .

Figure 18: A Petri automaton for  $1 \cap a \cdot b^+ \cdot c$ .

- For all  $p \in \text{dom}(\eta_k) \setminus \bigcup_{t \in T_k} {}^a t$  we simply set  $\eta_k(p) = \eta_{k-1}(p)$ .
- Otherwise,  $\forall t'_j \in T_k$ , let  $q \in {}^a t'_j$ . Then, for all  $\langle x, p \rangle$  in  $t'_j$ , because  $\rho^2$  is a reading and by construction of  $\mathcal{G}(R)$  we also know that there is some  $p' \in S_{k+1}$  that satisfies  $\langle x, p' \rangle \in t'_k$  and  $\rho_{k+1}^1(p') = \rho_{j+1}^2(p)$ . That  $p'$  is a good choice for  $p$ , hence we define  $\eta_k(p) = p'$ .

It is then routine to check that  $(\eta_k)_{0 \leq k \leq n+1}$  is indeed an embedding.

For the if direction, we build a parallel reading from the embedding, as explained above. For the other direction, we consider a reading of  $\mathcal{G}(R)$  in  $\mathcal{A}_2$  along some run  $R'$ . Notice that the natural ordering on  $\mathbb{N}$  is a topological ordering on  $\mathcal{G}(R)$ ; we may thus change the order of the transitions in  $R'$  (using Lemma 9.5) and group them adequately to obtain a parallel reading  $\mathcal{R}$  that embeds in  $R$ .  $\square$

So we know that the existence of embeddings is equivalent to the inclusion of languages, and we previously established that it is also equivalent to the existence of a simulation relation. Hence, the following characterisation holds:

**Proposition 9.10.** *Let  $\mathcal{A}, \mathcal{B} \in PA_X$  be two Petri automata. We have  $\mathcal{G}(\mathcal{A}) \subseteq \mathcal{L}(\mathcal{B})$  if and only if there exists a simulation relation  $\preceq$  between  $\mathcal{A}$  and  $\mathcal{B}$ .*

*Proof.* By Lemmas 9.4 and 9.9.  $\square$

As Petri automata are finite, there are finitely many relations in  $\mathcal{P}(\mathcal{P}(P_1) \times \mathcal{P}(P_2 \rightarrow P_1))$ . The existence of a simulation thus is decidable, allowing us to prove the main result:

**Theorem 9.11.** *Given two Petri automata  $\mathcal{A}, \mathcal{B} \in PA_X$ , testing whether  $\mathcal{G}(\mathcal{A}) \subseteq \mathcal{L}(\mathcal{B})$  is decidable.*

In practice, we can build the simulation on-the-fly, starting from the pair  $\{\{\iota_1\}, \{\{\iota_2 \mapsto \iota_1\}\}\}$  and progressing from there. We have implemented this algorithm in OCAML [7]. Even though its theoretical worst case time complexity is huge<sup>5</sup>, we get a result almost instantaneously on simple one-line examples.

By Proposition 9.1, the equational theory of identity-free Kleene lattices is thus decidable, and so is language inclusion of Petri automata.

**9.3. The problems with converse, 1, and  $\top$ .** The previous algorithm is not complete in presence of converse, 1, or  $\top$ . More precisely, Lemma 9.9 does not hold for general automata. Indeed, it is not possible to compare two runs just by relating the tokens at each step, and checking each transition independently. Consider the automaton from Figure 18. This automaton has in particular an accepting run recognising  $1 \cap abc$ . Let us try to test if this

<sup>5</sup>A quick analysis gives a  $\mathcal{O}(2^{n+(n+1)^m})$  complexity bound, where  $n$  and  $m$  are the numbers of places of the automata.

is smaller than the following runs from another automaton (we represent the transitions simply as arrows, because they only have a single input and a single output):

$$\begin{array}{cccccccccccccccc} x_0 & \xrightarrow{a} & x_1 & \xrightarrow{b} & x_2 & \xrightarrow{c} & x_3 & \xrightarrow{a} & x_4 & \xrightarrow{b} & x_5 & \xrightarrow{c} & x_6 \\ y_0 & \xrightarrow{a} & y_1 & \xrightarrow{b} & y_2 & \xrightarrow{c} & y_3 & \xrightarrow{a} & y_4 & \xrightarrow{b} & y_5 & \xrightarrow{b} & y_6 & \xrightarrow{c} & y_7 \end{array}$$

It stands to reason that we would reach a point where:

- for the first run:  $\{D, E\} \preceq \{[x_3 \mapsto D]\}$ ;
- for the second run:  $\{D, E\} \preceq \{[y_3 \mapsto D]\}$ .

So if it were possible to relate the end of the runs just with this information, they should both be bigger than  $1 \cap abc$  or both smaller or incomparable. But in fact the first run (recognising  $abcabc$ ) is bigger than  $1 \cap abc$  but the second (recognising  $abcabbc$ ) is not. This highlights the need for having some memory of previously fired transition when trying to compare runs of general Petri automata, thus preventing our local approach to bear fruits. The same kind of example can be found with the converse operation or  $\top$  instead of  $1$ .

## 10. COMPLEXITY

The notion of simulation from the previous section actually allows us to decide language inclusion of Petri automata in  $\text{EXPSPACE}$ . We will eventually show that this problem is  $\text{EXPSPACE}$ -complete.

**Proposition 10.1.** *Comparing Petri automata is in  $\text{EXPSPACE}$ .*

*Proof.* Our measure for the size of an automaton here is its number of places (the number of transitions is at most exponential in this number). Here is a non-deterministic semi-algorithm that tries to refute the existence of a simulation relation between  $\mathcal{A}_1$  and  $\mathcal{A}_2$ .

- 1: start with  $S \triangleq \{\iota_1\}$  and  $E \triangleq \{[\iota_2 \mapsto \iota_1]\}$ ;
- 2: if  $S = \emptyset$ , check if there is some  $\eta \in E$  such that  $\text{dom}(\eta) = \emptyset$ , if not return **FALSE**;
- 3: choose non-deterministically a transition  $t \in \mathcal{T}_1$  such that  ${}^a t \subseteq S$ ;
- 4: fire  $t$ , which means that  $S \triangleq S \setminus {}^a t \cup \pi_2(t^b)$ ;
- 5: have  $E$  progress along  $t$  as well, according to the conditions from Definition 9.2.
- 6: go to step 2:.

All these computations can be done in exponential space. In particular as  $S$  is a set of places in  $P_1$ , it can be stored in space  $|P_1| \times \log(|P_1|)$ . Similarly,  $E$ , being a set of partial functions from  $P_2$  to  $P_1$ , each of which of size  $|P_2| \times \log(|P_1| + 1)$ , can be stored in space  $|P_1 + 1|^{|P_2|} \times |P_2| \times \log(|P_1| + 1)$ . This non-deterministic  $\text{EXPSPACE}$  semi-algorithm can then be turned into an  $\text{EXPSPACE}$  algorithm by Savitch' theorem [32]. □

**Proposition 10.2.** *The (in)equational theory of representable identity-free Kleene lattices is  $\text{EXPSPACE}$ -hard.*

*Proof.* We perform a reduction from the problem of universality of regular expressions with intersection. Fürer showed that given a regular expression with intersection  $e$  over the alphabet  $\Sigma$ , checking whether the language denoted by  $e$  is equal to  $\Sigma^*$  requires exponential space [15]. In fact, by slightly modifying Fürer's proof, one can strengthen the result and use expressions without  $1$ , *i.e.*, over the syntax of  $\text{Exp}_\Sigma^-$ . We write  $L(e)$  for the language (set of strings) denoted by such an expression.

Inputs	Problem	Lower bound	Upper bound
$e, f \in \text{Exp}_X^-$	$\mathcal{R}el \models e = f$	EXPSpace	EXPSpace
$\mathcal{A}, \mathcal{B} \in \text{PA}_X$	$\mathcal{L}(\mathcal{A}) = \mathcal{L}(\mathcal{B})$	EXPSpace	EXPSpace
	$\mathcal{G}(\mathcal{A}) = \mathcal{G}(\mathcal{B})$	PSPACE	EXPSpace
$e, f \in \text{Exp}_X$	$\mathcal{R}el \models e = f$	EXPSpace	co-r.e.
$\mathcal{A}, \mathcal{B} \in \text{PA}_X \bullet$	$\mathcal{L}(\mathcal{A}) = \mathcal{L}(\mathcal{B})$	EXPSpace	co-r.e.
	$[\mathcal{G}(\mathcal{A})] = [\mathcal{G}(\mathcal{B})]$	PSPACE	-

Table 1: Summary of the complexity results

The reduction relies on the fact that for every expression  $e$ ,  $[\Sigma^+] \subseteq \triangleleft[e]$  if and only if  $L(\Sigma^+) \subseteq L(e)$ . First notice that that  $\Sigma^+ = L(\Sigma^+) = [\Sigma^+]$ . Then, we can show by induction on  $e$  that for every word  $w \in \Sigma^+$  we have  $w \in L(e)$  if and only if  $w \in \triangleleft[e]$ .

This means that to answer the question of universality of the regular expression with intersection  $e$ , we can ask instead whether  $[\Sigma^+] \subseteq \triangleleft[e]$  which is equivalent by Theorem 3.9 to  $\mathcal{R}el \models \Sigma^+ \leq e$ .  $\square$

Consider the automaton  $\mathcal{A}(e)$  we associate with an expression  $e$ . The number of places in  $\mathcal{A}(e)$  is linear in the size of the simple expression  $e$ . (And the exponential upper-bound on the number of transitions is asymptotically reached, consider for instance the automaton for  $(a_1 \cup b_1) \cap (a_2 \cup b_2) \cap \dots \cap (a_n \cup b_n)$ .)

Therefore, Propositions 10.1 and 10.2 finally lead to

**Theorem 10.3.** *The problem of comparing Petri automata, as well as the (in)equational theory of representable identity-free Kleene lattices are EXPSpace-complete.*

Table 1 summarises the various complexity results. The first two lines correspond to Theorem 10.3. The upper-bound in the third line is Theorem 11.1 from the following section, it is a corollary from a result by Jategaonkar and Meyer [17]; the lower-bound in this line follows from the Kleene algebra fragment, which is known to be PSPACE-complete [26]. The fourth and fifth lines are equivalent by Theorem 3.9; the lower bound follows from Proposition 10.2, for the upper-bound it suffices to enumerate all graphs: one can easily decide whether a given graph is accepted by a Petri automaton. For the last line, the problem also contains Kleene algebra, whence the lower bound; it is unclear however whether this last problem is co-recursively enumerable: given a graph  $G$  and a Petri automaton, we need to decide whether  $G = [H]$  for some series-parallel graph  $H$  produced by the automaton. A bound on the size of such a graph  $H$  w.r.t.  $G$  would suffice; this seems plausible but we do not have a proof.

The table is presented with equations and equalities; the same table holds for inequations and inclusions. Also remember that  $\mathcal{L}(\mathcal{A}) \subseteq \mathcal{L}(\mathcal{B})$  is equivalent to  $\mathcal{G}(\mathcal{A}) \subseteq \mathcal{L}(\mathcal{B})$ , but not to  $\mathcal{G}(\mathcal{A}) \subseteq \mathcal{G}(\mathcal{B})$ .

## 11. RELATIONSHIP WITH STANDARD PETRI NET NOTIONS

Our notion of Petri automaton is really close to the standard notion of labelled (safe) Petri net, where the transitions themselves are labelled, rather than their outputs. We motivate this design choice, and we relate some of the notions we introduced to the standard ones [28].

A Petri automaton can be translated into a safe Petri net whose transitions are labelled by  $X \uplus \{\tau\}$ , the additional label  $\tau$  standing for silent actions. For each automaton transition

$\langle \{p_1, \dots, p_n\}, \{ \langle x_1, q_1 \rangle, \dots, \langle x_m, q_m \rangle \} \rangle$  with  $m > 1$ , we introduce  $m$  fresh places  $r_1, \dots, r_m$  and  $m + 1$  transitions:

- a silent transition  $t_0$  with preset  $\{p_1, \dots, p_n\}$  and postset  $\{r_1, \dots, r_m\}$ ;
- and for each  $1 \leq k \leq m$  a transition  $t_k$  labelled by  $x_k$ , with preset  $\{r_k\}$  and postset  $\{q_k\}$ .

The inductive construction from Section 6 is actually simpler to write using such labelled Petri nets, as one can freely use  $\tau$ -labelled transitions to assemble automata into larger ones, one does not need to perform the  $\tau$ -elimination steps on the fly.

On the other hand, we could not define an appropriate notion of simulation for Petri nets: we need to fire several transitions at once in the small net, to provide enough information for the larger net to answer; delimiting which transitions to group and which to separate is non-trivial; similarly, defining a notion of parallel step is delicate in presence of  $\tau$ -transitions. By switching to our notion of Petri automata, we impose strong constraints about how those  $\tau$ -transitions should be used, resulting in a more fitted model.

To describe a run in a Petri net  $N$ , one may use a *process*  $p: K \rightarrow N$ , where  $K$  is an *occurrence net* (a partially ordered Petri net) [16]. The graphical representation (Figures 7 and 17) we used to describe runs in an automaton are in fact a mere adaptation of this notion to our setting (with labels on arcs rather than on transitions).

Our notion  $\mathcal{G}(R)$  of trace of a run actually corresponds to the standard notion of *pomset-trace* [17], via dualisation. Let  $R$  be a run in a Petri automaton, and let  $R'$  be the corresponding run in the corresponding labelled Petri net. Let  $\mathcal{G}(R) = \langle V, E, \iota, o \rangle$ . It is not difficult to check that the pomset-trace of  $R'$  is isomorphic to  $\langle E, <_E \rangle$ , where  $<_E$  is the transitive closure of the relation  $<$  defined on  $E$  by  $\forall x, y, z \in V, a, b \in X, \langle x, a, y \rangle < \langle y, b, z \rangle$ .

The correspondence is even stronger: two graphs produced by accepting runs in (possibly different) automata  $\mathcal{G}(R_1) = \langle V_1, E_1, \iota_1, o_1 \rangle$  and  $\mathcal{G}(R_2) = \langle V_2, E_2, \iota_2, o_2 \rangle$  are isomorphic if and only if their pomset-traces  $(E_1, <_{E_1})$  and  $(E_2, <_{E_2})$  are isomorphic. The proof of this relies on the fact that accepting runs produce graphs satisfying the following properties:

$$\begin{aligned} \forall \langle x, a, y \rangle \in E, x \neq \iota &\Leftrightarrow (\exists \langle z, b \rangle \in V \times X : \langle z, b, x \rangle \in E), \\ \forall \langle x, a, y \rangle \in E, y \neq o &\Leftrightarrow (\exists \langle z, b \rangle \in V \times X : \langle y, b, z \rangle \in E). \end{aligned}$$

Hence, pomset-trace language equivalence corresponds exactly to equivalence of the sets of produced graphs in our setting (up to isomorphism).

Jategaonkar and Meyer showed that the pomset-trace equivalence problem for safe Petri nets is EXPSPACE-complete [17]. As a consequence, comparing the sets of series-parallel graphs produced by Petri automata is in EXPSPACE:

**Theorem 11.1.** *Deciding whether  $\mathcal{G}(\mathcal{A}) = \mathcal{G}(\mathcal{B})$  for two Petri automata  $\mathcal{A}, \mathcal{B} \in PA_X$  is in EXPSPACE.*

As explained before the corresponding equivalence does not coincide with the one discussed in the present paper, where we compare the sets of graphs modulo homomorphism (see, e.g., Proposition 9.1). Also note that the above result is for series-parallel graphs only: it does not explain how to compare the sets of graphs  $[\mathcal{G}(\mathcal{A})]$  and  $[\mathcal{G}(\mathcal{B})]$  when given two Petri automata  $\mathcal{A}, \mathcal{B} \in PA_{X_\bullet}$  labelled in  $X_\bullet$ . This latter question remains open.

## 12. BRANCHING AUTOMATA

There are several notions of regular/recognisable/definable sets of graphs in the literature, and several Kleene-like theorems. See for instance the whole line of research that followed

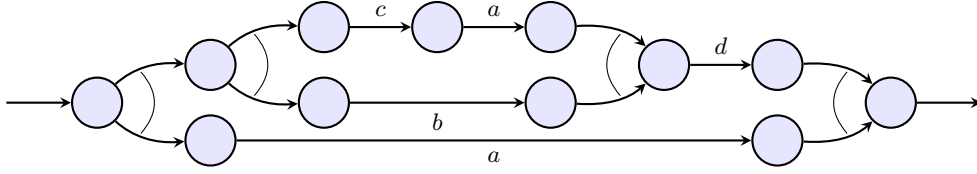


Figure 19: Example of a branching automaton

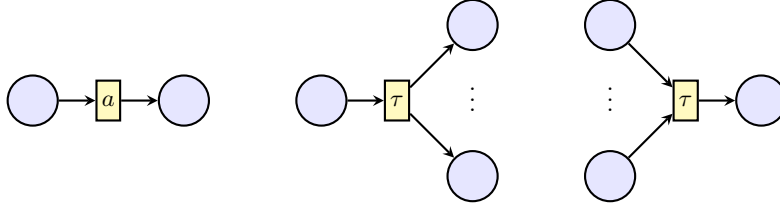


Figure 20: Prescribed transitions of branching automata

from Courcelle’s work [12], where monadic second order logic plays a central role. Courcelle’s conjecture that recognisability coincides with definability on classes of graphs of bounded treewidth has been proved recently [5]—whether it also coincides with the notions of recognisability and regularity we use here is unclear. In particular, since we mainly restrict to series-parallel graphs, our syntax for expressions greatly departs from Courcelle’s one, and we can work with a simpler automata model. The work of Bossut et al. [6] on planar directed acyclic graphs is also worth mentioning, but again the syntax and automata they use are really different from ours: their graphs are hypergraphs and they are not necessarily series-parallel.

Much closer to the present work is that of Lodaya and Weil [22, 23, 24], who introduced another kind of automata to recognise series-parallel graphs, called “branching automata”. They obtained a Kleene Theorem for this model, using the same notion of regularity as the one we use here (Definition 4.2). In this section we recall the definition of branching automata and describe precisely the relationship between their result and our own.

### 12.1. Definitions and Kleene Theorem.

**Definition 12.1** (Branching automaton). A *branching automaton* over the alphabet  $X$  is a tuple  $\langle Q, T_{seq}, T_{fork}, T_{join}, I, F \rangle$ , where  $Q$  is a set of states,  $I$  and  $F$  are subsets of  $Q$ , respectively the input and output states, and the transitions are split in three sets:

- $T_{seq} \subseteq Q \times X \times Q$  is the set of sequential transitions;
- $T_{fork} \subseteq Q \times \mathcal{M}_{ns}(Q)$  is the set of opening transitions;
- $T_{join} \subseteq \mathcal{M}_{ns}(Q) \times Q$  is the set of closing transitions.

(Here  $\mathcal{M}_{ns}(Q)$  represents the set of multisets over  $Q$  with cardinality at least 2.)

An example of such an automaton is given in Figure 19. These automata can be seen as labelled Petri nets of a particular shape: transitions are restricted to the three types described on Figure 20.

Lodaya and Weil formally use these automata on simple terms quotiented by associativity of  $\cdot$  and  $\cap$ , and commutativity of  $\cap$ . This is equivalent to working with series-parallel graphs modulo isomorphism.

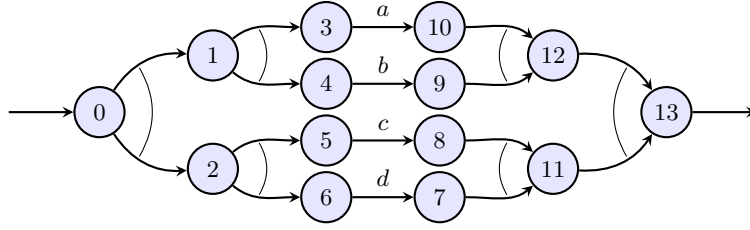


Figure 21: Example of branching automaton

**Definition 12.2** (Runs and language of a branching automaton). Let  $t$  be a simple term, there is a *run* on  $t$  from state  $p$  to state  $q$  if:

- $t = a \in X$  and  $\langle p, a, q \rangle \in T_{seq}$ ;
- $t = t_1 \cap \dots \cap t_n$  with  $n \geq 2$ , there are two transitions  $\langle p, [p_1, \dots, p_n] \rangle \in T_{fork}$  and  $\langle [q_1, \dots, q_n], q \rangle \in T_{join}$ , and for every  $1 \leq i \leq n$  there is a run on  $t_i$  from  $p_i$  to  $q_i$ ;
- $t = t_1 \cdot \dots \cdot t_n$  with  $n \geq 2$ , there are states  $p = p_0, p_1, \dots, p_n = q$ , and for every  $0 \leq i < n$  there is a run on  $t_i$  from  $p_i$  to  $p_{i+1}$ .

The *language* of a branching automaton  $\mathcal{B}$ , written  $\mathcal{G}(\mathcal{B})$  is defined as the set of series-parallel graphs  $\mathcal{G}(t)$  such that there exists a pair of states  $\langle q_i, q_f \rangle \in I \times F$ , and a run on  $t$  from  $q_i$  to  $q_f$ .

Lodaya and Weil impose restrictions on the runs of the automaton, that correspond to a safety constraint over the underlying Petri net, much like Constraint 5.3. Here we only consider branching automata implicitly satisfying those constraints.

**Theorem 12.3** (Kleene Theorem for branching automata [23]). *For every set  $S$  of series-parallel graphs the following are equivalent:*

- (i) *there is a simple expression  $e \in Exp_X^-$  such that  $S = \mathcal{G}(e)$ ;*
- (ii) *there is a branching automaton  $\mathcal{B}$  such that  $S = \mathcal{G}(\mathcal{B})$ .*

**12.2. Comparison with Petri automata.** At first glance the two Kleene theorems and the fact that both branching automata and Petri automata are Petri net-based seem to mean they are completely equivalent. Indeed the same set of regular-like expressions may be used to describe their semantics. However they still exhibit some deep differences.

The first difference comes from the runs in the two models. In some sense, the runs in a branching automaton require a “global” view of the term being read. Consider the branching automaton in Figure 21, and the term  $t = b \cap c \cap a \cap d$ .  $t$  is accepted by this automaton, but in order to reach that conclusion, one must: 1) refactor  $t$  as  $(a \cap b) \cap (c \cap d)$ ; 2) “match” the opening transition  $\langle 0, [1, 2] \rangle$  with the closing transition  $\langle [11, 12], 13 \rangle$ ; 3) read the subterms  $(a \cap b)$  and  $(c \cap d)$  respectively from state 1 to state 12 and from 2 to 11. This means that the run is built as a nesting of runs (rather than a sequential process), and that it needs to manipulate the term as a whole (rather than using a partial, local view of it).

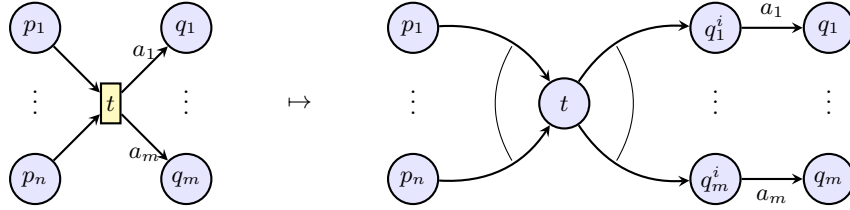
By contrast, to fire a transition in a Petri automaton, one simply needs to see one vertex of the graph and the edges coming out of it. Furthermore, the matching of transitions is somewhat automatic in our model, and knowledge of it is not needed to compute a run. For these reasons, our notion of language of a run could be defined using standard Petri net

notions (namely *pomset-traces*, see Section 11), whereas runs in a branching automaton rely crucially on the term representation.

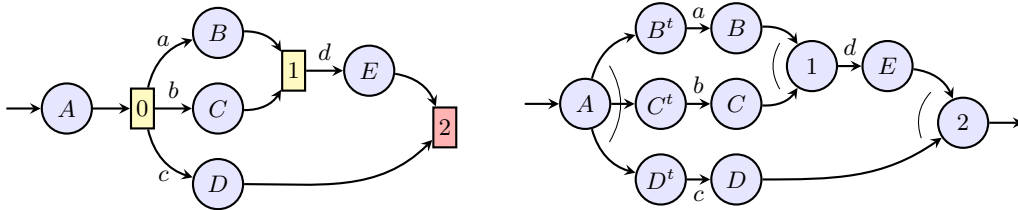
Another difference stems from the way automata are labelled. Branching automata do not label their opening and closing transitions, making these “silent” transitions. This highly complicates the task of comparing automata using simulation-based methods. In our case instead, every transition is labelled, and this helps us to obtain an algorithm.

These differences make the task of converting from one model to the other rather subtle. To translate a branching automaton into a Petri automaton, one would need some kind of epsilon elimination procedure. We believe such a procedure could be devised using boxes from Section 7 to keep track of the order in which opening and closing transitions are combined. However we do not have a precise formulation of this translation yet.

The other direction is slightly easier. Starting from a Petri automaton, first modify the transitions as sketched below.

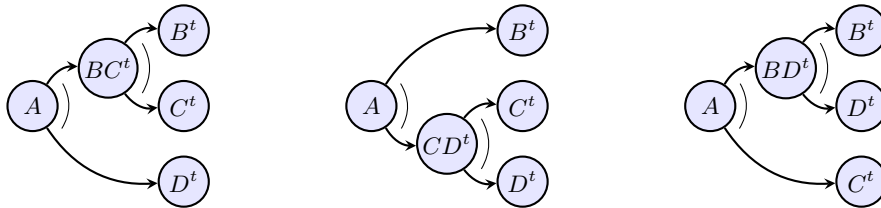


This yields an automaton whose pomset-trace language (when seen as a Petri net) is the language of the original Petri automaton. However, its branching automaton language is not the same, as illustrated by the following example:



The pomset-trace language of the branching automaton corresponds to the singleton set  $\{(a \cap b) \cdot d \cap c\}$ , but its language is empty, as no factorisation of that term has three parallel subterms. To solve this problem, one needs to saturate the automaton by splitting opening and closing transitions to allow for every factorisation. Formally, it means that for an opening transition  $\langle p, [q_1, \dots, q_n] \rangle$ , for every tree<sup>6</sup> with  $n$  leaves labelled with  $[q_1, \dots, q_n]$  there should be a sequence of opening transitions of that shape.

In the case of the above example, one would add three states  $BC^t$ ,  $BD^t$  and  $CD^t$ , and six transitions:



<sup>6</sup>We consider here trees where no vertex has out-degree one.



After this transformation, the pomset-trace and the branching automata languages coincide: we obtained a branching automaton recognising the same language as the original Petri automaton.

This translation gives rise to a completely different proof of Theorem 7.29: first translate the Petri automaton into an equivalent branching automaton, then use Theorem 12.3 to obtain an expression describing its language. Conversely, one could obtain Theorem 12.3 through a translation from branching automata to Petri automata and Theorem 7.29.

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