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A PROOF OF STAVI'S THEOREM

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Abstract. Kamp's theorem established the expressive equivalence of the temporal logic with Until and Since and the First-Order Monadic Logic of Order (FOMLO) over the Dedekind-complete time flows. However, this temporal logic is not expressively complete for *FOMLO* over the rationals. Stavi introduced two additional modalities and proved that the temporal logic with Until, Since and Stavi's modalities is expressively equivalent to *FOMLO* over all linear orders. We present a simple proof of Stavi's theorem.

1. Introduction

Temporal Logic (*TL*) introduced to Computer Science by Pnueli in [Pnu77] is a convenient framework for reasoning about "reactive" systems. This has made temporal logics a popular subject in the Computer Science community, enjoying extensive research. In *TL* we describe basic system properties by *atomic propositions* that hold at some points in time, but not at others. More complex properties are conveyed by formulas built from the atoms using Boolean connectives and *Modalities* (temporal connectives): A k-place modality M transforms statements $\varphi_1, \ldots, \varphi_k$ possibly on 'past' or 'future' points to a statement $M(\varphi_1, \ldots, \varphi_k)$ on the 'present' point t_0. The rule to determine the truth of a statement $M(\varphi_1, \ldots, \varphi_k)$ at t_0 is called a *Truth Table*. The choice of particular modalities with their truth tables yields different temporal logics. A temporal logic with modalities $M_1, \ldots, M_k$ is denoted by $TL(M_1, \ldots, M_k)$.

The simplest example is the one place modality $\diamond P$ saying: "P holds some time in the future." Its truth table is formalized by $\varphi_\diamond(t_0, P) \equiv (\exists t > t_0)P(t)$. This is a formula of the First-Order Monadic Logic of Order (*FOMLO*) - a fundamental formalism in Mathematical Logic where formulas are built using atomic propositions $P(t)$, atomic relations between elements $t_1 = t_2$, $t_1 < t_2$, Boolean connectives and first-order quantifiers $\exists t$ and $\forall t$. Most modalities used in the literature are defined by such *FOMLO* truth tables, and as a result, every temporal formula translates directly into an equivalent *FOMLO* formula. Thus, different temporal logics may be considered as a convenient way to use fragments of *FOMLO*. *FOMLO* can also serve as a yardstick by which one is able to check the strength of temporal logics: A temporal logic is *expressively complete* for a fragment L of *FOMLO* if every formula of L with a single free variable t_0 is equivalent to a temporal formula.

Actually, the notion of expressive completeness refers to a temporal logic and to a model (or a class of models) since the question whether two formulas are equivalent depends on the

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domain over which they are evaluated. Any (partially) ordered set with monadic predicates is a model for *TL* and *FOMLO*, but the main, *canonical*, linear time intended models are the non-negative integers $\langle \mathbb{N}, <\rangle$ for discrete time and the non-negative reals $\langle \mathbb{R}^{\geq 0}, <\rangle$ for continuous time.

A major result concerning *TL* is Kamp's theorem [Kam68], which implies that the pair of modalities "*P Until Q*" and "*P Since Q*" is expressively complete for *FOMLO* over the above two linear time canonical models.

The temporal logic with the modalities *Until* and *Since* is not expressively complete for *FOMLO* over the rationals [GHR94].

Stavi introduced two additional modalities Until^s and Since^s (see Sect. 2.2.2) and proved that $TL(\mathsf{Until}, \mathsf{Since}, \mathsf{Until}^s, \mathsf{Since}^s)$ is expressively complete for *FOMLO* over all linear orders. There are only two published proofs of Stavi's theorem [GHR93, GHR94]; however, none is simple.

The objective of this paper is to present a simple proof of Stavi's theorem.

The rest of the paper is organized as follows: In Sect. 2 we recall the definitions of the monadic logic, the temporal logics and state Kamp's and Stavi's theorems. In Sect. 3 we introduce partition formulas which play an important role in our proof of Stavi's theorem. In Sect. 4 we prove Stavi's theorem. The proof of one proposition is postponed to Sect. 5. Sect. 6 comments on the previous proofs of Stavi's theorem.

2. Preliminaries

In this section we recall the definitions of linear orders, the first-order monadic logic of order, the temporal logics and state Kamp's and Stavi's theorems.

2.1. **Intervals and gaps in linear orders.** A subset I of a linear order $(T,<)$ is an *interval*, if for all $t_1 < t < t_2$ with $t_1, t_2 \in I$ also $t \in I$. For intervals with endpoints $a, b \in T$, whether open or closed on either end, we will use the standard notation, such as $[a,b) := \{t \in T \mid a \leq t < b\}$, $(a,b) := \{t \in T \mid a < t < b\}$, etc. For $a \in T$ let $[a,\infty) := \{t \mid t \geq a\}$ and, similarly, $(-\infty, a) := \{t \in \mid t < a\}$.

A *Dedekind cut* of a linearly ordered set $(T,<)$ is a downward closed non-empty set $C \subseteq T$ such that its complement is non-empty and if C has a least upper bound in $(T,<)$, then it is contained in C. A *proper cut* or a *gap* is a cut that has no least upper bound in $(T,<)$, i.e., one that has no maximal element.

A linear order is *Dedekind complete* if it has no gaps; equivalently, if for every non-empty subset S of T, if S has a lower bound in T, then it has a greatest lower bound, written $\inf(S)$, and if S has an upper bound in T, then it has a least upper bound, written $\sup(S)$.

For a gap g and an element $t \in T$ we write $t < g$ (respectively, $g < t$) if $t \in g$ (respectively, $t \notin g$). We also write (t, g) for the interval $\{a \in T \mid a > t \wedge a \in g\}$; similarly, (g, t) is $\{a \in T \mid a < t \wedge a \notin g\}$. Finally, for gaps g_1 and g_2 we write $g_1 \leq g_2$ if $g_1 \subseteq g_2$, and the interval (g_1, g_2) is defined as $\{a \in T \mid a \notin g_1 \wedge a \in g_2\}$.

2.2. **First-order Monadic Logic and Temporal Logics.** We present the basic definitions of First-Order Monadic Logic of Order (*FOMLO*) and Temporal Logic (*TL*), and well-known results concerning their expressive power. Fix a set Σ of *atoms*. We use $P, Q, R, \dots$ to denote members of Σ. The syntax and semantics of both logics are defined below with respect to such a Σ.

2.2.1. *First-Order Monadic Logic of Order.*

Syntax: In the context of *FOMLO* the atoms of Σ are referred to (and used) as ***unary predicate symbols***. Formulas are built using these symbols, plus two binary relation symbols: $<$ and $=$, and a set of first-order variables (denoted: $x, y, z, \dots$). Formulas are defined by the grammar:

$$atomic ::= \ x < y \mid x = y \mid P(x) \text{ (where } P \in \Sigma)$$

$$\varphi ::= \ atomic \mid \neg\varphi_1 \mid \varphi_1 \vee \varphi_2 \mid \varphi_1 \wedge \varphi_2 \mid \exists x \varphi_1 \mid \forall x \varphi_1$$

The notation $\varphi(x_1, \dots, x_n)$ implies that φ is a formula where the x_i are the only variables occurring free; writing $\varphi(x_1, \dots, x_n, P_1, \dots, P_k)$ additionally implies that the P_i are the only predicate symbols that occur in φ. We will also use the standard abbreviated notation for ***bounded quantifiers***, e.g.: $(\exists x)_{>z}(\dots)$ denotes $\exists x((x > z) \wedge (\dots))$ and $(\forall x)^{<z}_{>z_1}(\dots)$ denotes $\forall x((z_1 < x < z) \rightarrow (\dots))$, etc.

Semantics: Formulas are interpreted over *labeled linear orders* which are called *chains*. A Σ-***chain*** is a triplet $\mathcal{M} = (T, <, \mathcal{I})$ where T is a set - the ***domain*** of the chain, $<$ is a linear order relation on T, and $\mathcal{I} : \Sigma \rightarrow \mathcal{P}(T)$ is the ***interpretation*** of Σ (where $\mathcal{P}$ is the powerset notation). We use the standard notation $\mathcal{M}, t_1, t_2, \dots t_n \models \varphi(x_1, x_2, \dots x_n)$ to indicate that the formula φ with free variables among $x_1, \dots, x_n$ is satisfiable in $\mathcal{M}$ when x_i are interpreted as elements t_i of $\mathcal{M}$. For atomic $P(x)$ this is defined by: $\mathcal{M}, t \models P(x)$ iff $t \in \mathcal{I}(P)$; The semantics of $<, =, \neg, \wedge, \vee, \exists$ and $\forall$ is defined in a standard way.

2.2.2. *Temporal Logics.*

Syntax: In the context of *TL* the atoms of Σ are used as *atomic propositions* (also called *propositional atoms*). Formulas are built using these atoms and a set (finite or infinite) B of *modality names*, where an integer *arity*, denoted $|\mathsf{M}|$, is associated with each $\mathsf{M} \in B$. The syntax of *TL* with the ***basis*** B, denoted ***TL(B)***, is defined by the grammar:

$$F ::= \ P \mid \neg F_1 \mid F_1 \vee F_2 \mid F_1 \wedge F_2 \mid \mathsf{M}(F_1, F_2, \dots, F_n),$$

where $P \in \Sigma$ and $\mathsf{M} \in B$ an n-place modality (with arity $|\mathsf{M}| = n$). As usual, ***True*** denotes $P \vee \neg P$ and ***False*** denotes $P \wedge \neg P$; we will use infix notation for binary modalities, where $F_1 \ \mathsf{M} \ F_2$ is an alternative notation for $\mathsf{M}(F_1, F_2)$.

Semantics: Formulas are interpreted at *time-points* (or *moments*) in chains (elements of the domain). The domain T of $\mathcal{M} = (T, <, \mathcal{I})$ is called the *time domain*, and $(T, <)$ - the *time flow* of the chain. The semantics of each n-place modality $\mathsf{M} \in B$ is defined by a 'rule' specifying how the set of moments where $\mathsf{M}(F_1, \dots, F_n)$ holds (in a given structure) is determined by the n sets of moments where each of the formulas F_i holds. Such a 'rule' for M is formally specified by an operator $\mathcal{O}_{\mathsf{M}}$ on time flows, where given a time flow $\mathcal{F} = (T, <)$, $\mathcal{O}_{\mathsf{M}}(\mathcal{F})$ is an operator in $(\mathcal{P}(T))^n \longrightarrow \mathcal{P}(T)$.

The semantics of $TL(B)$ formulas is then defined inductively. Given a chain $\mathcal{M} = (T, <, \mathcal{I})$ and a moment $t \in \mathcal{M}$, define when a formula F *holds* in $\mathcal{M}$ at t - denoted $\mathcal{M}, t \models F$:

- $\mathcal{M}, t \models P$ iff $t \in \mathcal{I}(P)$, for any propositional atom P.

- $\mathcal{M}, t \models F \vee G$ iff $\mathcal{M}, t \models F$ or $\mathcal{M}, t \models G$; similarly for $\wedge$ and $\neg$.
- $\mathcal{M}, t \models \mathsf{M}(F_1, \dots, F_n)$ iff $t \in [\mathcal{O}_\mathsf{M}(T,<)](T_1, \dots, T_n)$ where $\mathsf{M} \in B$ is an n-place modality, $F_1, \dots, F_n$ are formulas and $T_i =_{def} \{s \in T : \mathcal{M}, s \models F_i\}$.

Truth tables: Practically most standard modalities studied in the literature can be specified in *FOMLO*: A *FOMLO* formula $\varphi(x, P_1, \dots, P_n)$ (with a single free first-order variable x and with n predicate symbols P_i) is called an ***n-place first-order truth table***. Such a truth table φ defines an n-ary modality M (whose semantics is given by an operator $\mathcal{O}_\mathsf{M}$) iff for any time flow $(T,<)$, for any $T_1, \dots, T_n \subseteq T$ and for any structure $\mathcal{M} = (T,<,\mathcal{I})$ where $\mathcal{I}(P_i) = T_i$:

$$[\mathcal{O}_M(T,<)](T_1, \dots, T_n) = \{t \in T : \mathcal{M}, t \models \varphi(x, P_1, \dots, P_n)\}$$

Example 2.1 . *Below are truth-table definitions for the (binary)* ***strict***-Until *and* ***strict***-Since *and the (unary)* $\Box$, $\overleftarrow{\Box}$, $\boldsymbol{K}^+$ *and* $\boldsymbol{K}^-$*:*

- P Until Q *is defined by :* $\varphi_{\mathsf{Until}}(x, P, Q) := (\exists x')_{>x}(Q(x') \wedge (\forall y)_{>x}^{<x'} P(y)).$
- P Since Q *is defined by:* $\varphi_{\mathsf{Since}}(x, P, Q) := (\exists x')^{<x}(Q(x') \wedge (\forall y)_{>x'}^{<x} P(y)).$
- $\Box(P)$ *(respectively,* $\overleftarrow{\Box}(P)$*) - "P holds everywhere after (respectively, before) the current moment":*

$$\varphi_{\Box}(x, P) := (\forall x')_{>x} P(x')$$
$$\varphi_{\overleftarrow{\Box}}(x, P) := (\forall x')_{<x} P(x')$$

- $\boldsymbol{K}^+$ *defined by:* $\varphi_{\boldsymbol{K}^+}(x, P) := (\forall x')_{>x}(\exists y)_{>x}^{<x'} P(y)).$
- $\boldsymbol{K}^-$ *defined by:* $\varphi_{\boldsymbol{K}^-}(x, P) := (\forall x')^{<x}(\exists y)_{>x'}^{<x} P(y)).$

Formula $\mathbf{K}^-(P)$ holds at a moment t iff $t = \sup(\{t' \mid t' < t \wedge P(t')\})$. Dually, $\mathbf{K}^+(P)$ holds at t iff $t = \inf(\{t' \mid t' > t \wedge P(t')\})$. Note that $\mathbf{K}^+(P)$ is equivalent to $\neg((\neg P)\mathsf{Until}\mathbf{True})$ and $\Box P$ is equivalent to $\neg(\mathbf{True}\mathsf{Until}(\neg P))$.

Let $\boldsymbol{\gamma}^+$ be a unary modality such that $\boldsymbol{\gamma}^+(P)$ holds at t if there is a gap $g > t$ in the order such that $g = \sup(\{t' \mid (\forall y)_{>t}^{<t'} P(y)\})$. We say that g is the gap left definable by P that succeeds t, or just that *g is P-gap that succeeds t*; P holds everywhere on the interval (t, g), and for every $t_1 > g$, there is $t' \in (g, t_1)$ such that $t' \notin P$.

A natural formalization of $\boldsymbol{\gamma}^+$ semantics uses a second-order quantifier - "there is a gap"; however, $\boldsymbol{\gamma}^+(P)$ is equivalent to the conjunction of the following formulas [GHR94]:

(1) $(P\mathsf{Until}P) \wedge \neg(P\mathsf{Until}\neg P)$.
(2) $\neg\Box P$ - "$\neg P$ holds somewhere in the future."
(3) $\neg(P\mathsf{Until}(P \wedge \mathbf{K}^+(\neg P)))$.

Since $\Box$ and $\mathbf{K}^+$ are equivalent to $TL(\mathsf{Until})$ formulas, $\boldsymbol{\gamma}^+(P)$ can be considered as an abbreviation of a $TL(\mathsf{Until})$ formula, and $\boldsymbol{\gamma}^+$ has a first-order truth table $\varphi_{\boldsymbol{\gamma}^+}(x, P)$.

$\boldsymbol{\gamma}^-$ is the mirror image of $\boldsymbol{\gamma}^+$, i.e., going into the past instead of into the future. $\boldsymbol{\gamma}^-(P)$ holds at t if there is a gap $g < t$ in the order such that P holds everywhere on the interval (g, t), and for every $t_1 < g$, there is $t' \in (t_1, g)$ such that $t' \notin P$. We say that g is the (right definable) P-gap that precedes t, or just that *g is P-gap that precedes t.*

The modalities Until^s and Since^s were introduced by Stavi. $P\mathsf{Until}^s Q$ holds at t if there is a gap $g > t$ such that:

- P is true on (t, g).
- In the future of the gap, P is false arbitrarily close to the gap, and

- Q is true from g into the future for some uninterrupted stretch of time.

Until^s has a first-order truth table $\varphi_{\mathsf{Until}^s}(x, P, Q)$ which is the conjunction of the following formulas:

(1) $\varphi_{\gamma^+}(x, P)$.

(2) $(\exists x_1)_{>x}\big(\neg P(x_1) \wedge (\forall y)^{<x_1}_{>x}[(\neg P(y)) \rightarrow (\forall z)^{<x_1}_{>y} Q(z)]\big)$.

Since^s is the mirror image of Until^s.

2.3. **Kamp's and Stavi's Theorems.** We are interested in the relative expressive power of *TL* (compared to *FOMLO*) over the class of ***linear structures***, where the time flow is an irreflexive linear order.

Equivalence between temporal and monadic formulas is naturally defined: F is equivalent to $\varphi(x)$ over a class $\mathcal{C}$ of structures iff for any $\mathcal{M} \in \mathcal{C}$ and $t \in \mathcal{M}$: $\mathcal{M}, t \models F \Leftrightarrow \mathcal{M}, t \models \varphi(x)$. If $\mathcal{C}$ is the class of all chains, we will say that F is equivalent to φ.

Expressive completeness/equivalence: A temporal language $TL(B)$ is *expressively complete* for *FOMLO* over a class $\mathcal{C}$ of structures iff for every *FOMLO* formula $\varphi(z)$ with one free variable there is a $\psi \in TL(B)$ such that φ is equivalent to ψ over $\mathcal{C}$. Similarly, one may speak of expressive completeness of *FOMLO* for some temporal language. If we have expressive completeness in both directions between two languages, then they are expressively equivalent.

If every modality in B has a *FOMLO* truth-table, then it is easy to translate every formula of $TL(B)$ to an equivalent *FOMLO* formula. Hence, in this case *FOMLO* is expressively complete for $TL(B)$.

The fundamental theorem of Kamp's states:

Theorem 2.2 ([Kam68])**.** $TL(\mathsf{Until}, \mathsf{Since})$ *is expressively equivalent to FOMLO over Dedekind complete chains.*

$TL(\mathsf{Until}, \mathsf{Since})$ is not expressively complete for *FOMLO* over the rationals [GHR94]. Stavi introduced two new modalities Until^s and Since^s (see Sect. 2.2.2) and proved:

Theorem 2.3. $TL(\mathsf{Until}, \mathsf{Since}, \mathsf{Until}^s, \mathsf{Since}^s)$ *is expressively equivalent to FOMLO over all chains.*

As Until, Since and Stavi's modalities are definable in *FOMLO*, it follows that *FOMLO* is expressively complete for $TL(\mathsf{Until}, \mathsf{Since}, \mathsf{Until}^s, \mathsf{Since}^s)$. The contribution of our paper is a proof that $TL(\mathsf{Until}, \mathsf{Since}, \mathsf{Until}^s, \mathsf{Since}^s)$ is expressively complete for *FOMLO*. Our proof is constructive. An algorithm which for every *FOMLO* formula $\varphi(x)$ constructs a $TL(\mathsf{Until}, \mathsf{Since}, \mathsf{Until}^s, \mathsf{Since}^s)$ formula which is equivalent to φ is easily extracted from our proof.

3. Partition Formulas

In this section we introduce partition formulas and state their properties. They will play an important role in our proof of Stavi's theorem. The basic partition formulas generalize the Decomposition formulas of [GPSS80].

Definition 3.1 (Partition expressions)**.** Let Σ be a set of monadic predicate names, and $\delta_1(x), \dots, \delta_n(x)$ are quantifier free first-order formulas over Σ with one free variable, and $O \subseteq \{1, \dots, n\}$. An expression $\mathtt{Part}(\langle \delta_1, \dots, \delta_n \rangle, O)$ is called a partition expression over Σ.

Semantics. Let I be an interval of a Σ-chain $\mathcal{M}$. A partition expression $\texttt{Part}(\langle\delta_1,\dots,\delta_n\rangle,O)$ holds on I in $\mathcal{M}$ (notation $\mathcal{M},I\models\texttt{Part}(\langle\delta_1,\dots,\delta_n\rangle,O)$ if I can be partitioned into n non-empty intervals $I_1,\dots,I_n$ such that δ_j holds on all points in I_j, and I_i precedes I_j for $i<j$, and I_j a one-point interval for $j\in O$. Note that we do not require that if I_j is a one-point interval, then $j\in O$. Observe that the semantics of partition expressions does not depend on the names of the variables that appear in δ_i.

For example, $\texttt{Part}(\langle P_1(x),P_2(x)\vee P_3(x)\rangle,\{1,2\})$ holds over I iff I is a two point interval and P_1 holds over its first point and P_2 or P_3 holds over its second point. $\texttt{Part}(\langle\mathbf{True},\mathbf{True}\rangle,\{1\})$ holds over I iff I has a minimal point and at least two points.

Definition 3.2 (Partition Formulas)**.** Let Σ be a set of monadic predicate names.

Basic Partition Formulas: A basic partition formula (over Σ) is an expression of one of the following forms:
(1) $z=y$ or $z<y$
(2) $\texttt{Part}(\langle\delta_1,\dots,\delta_n\rangle,O)[y,z]$
(3) $\texttt{Part}(\langle\delta_1,\dots,\delta_n\rangle,O)[z,\infty)$ or $\texttt{Part}(\langle\delta_1,\dots,\delta_n\rangle,O)(-\infty,z]$ or $\texttt{Part}(\langle\delta_1,\dots,\delta_n\rangle,O)(-\infty,\infty)$,

where $\texttt{Part}(\langle\delta_1,\dots,\delta_n\rangle,O)$ are partition expressions.

Partition Formulas: are constructed from the basic partition formulas by Boolean connectives and existential quantifier.

Simple Partition Formulas: are constructed from the basic partition formulas by conjunction and disjunction.

Normal Partition Formulas: A Normal Partition Formula is a partition formula of the form:

$$\begin{aligned}E(z_1,\dots,z_m) := \left(\bigwedge_{k=n+1}^{m} z_k=z_{i_k}\right) &\wedge (z_1<z_2<\dots<z_n)\\ \wedge\ &\bigwedge_{j=2}^{n} W_j[z_{j-1},z_j]\\ \wedge\ & W_{n+1}[z_n,\infty)\wedge W_1(-\infty,z_1]\\ \wedge\ & W_0(-\infty,\infty)\end{aligned}$$

where W_j are basic partition formulas, $n\le m$ and $i_{n+1},\dots,i_m\in\{1,\dots,n\}$.

The semantics of the partition formulas will not depend on the names of variables that occur in partition expressions. These occurrences of the variables are considered to be bound. For other occurrences of variables the definition whether occurrences are free or bound is standard.

Semantics. Partition formulas are interpreted over Σ-chains. Let $\mathcal{M}=(T,<,\mathcal{I})$ be a Σ-chain. We use the standard notation $\mathcal{M},t_1,t_2,\dots t_n\models\varphi(x_1,x_2,\dots x_n)$ to indicate that the formula φ with free variables among $x_1,\dots,x_n$ is satisfiable in $\mathcal{M}$ when x_i are interpreted as elements t_i of $\mathcal{M}$. For basic partition formulas this is defined by: $\mathcal{M},t\models\texttt{Part}(\langle\delta_1,\dots,\delta_n\rangle,O)[x,\infty)$ iff the partition expression $\texttt{Part}(\langle\delta_1,\dots,\delta_n\rangle,O)$ holds on the interval $[t,\infty)$ in $\mathcal{M}$; similarly, $\mathcal{M},t\models\texttt{Part}(\langle\delta_1,\dots,\delta_n\rangle,O)(-\infty,x]$ (respectively, $\mathcal{M}\models\texttt{Part}(\langle\delta_1,\dots,\delta_n\rangle,O)(-\infty,\infty)$) iff $\texttt{Part}(\langle\delta_1,\dots,\delta_n\rangle,O)$ holds on the interval $(-\infty,t]$

(respectively, the interval $(-\infty, \infty)$) in $\mathcal{M}$; and $\mathcal{M}, t_1, t_2 \models \texttt{Part}(\langle\delta_1, \ldots, \delta_n\rangle, O)[x_1, x_2]$ iff $\texttt{Part}(\langle\delta_1, \ldots, \delta_n\rangle, O)$ holds on the interval $[t_1, t_2]$ in $\mathcal{M}$; the semantics of $<, =, \neg, \wedge, \vee,$ and $\exists$ is defined in a standard way.

The following lemmas immediately follow from the definitions and standard logical equivalences.

Lemma 3.3. (1) *Every simple formula is equivalent to a disjunction of normal formulas.*
(2) *For every normal formula φ, the formula $\exists x\varphi$ is equivalent to a disjunction of normal formulas.*
(3) *For every simple formula φ, the formula $\exists x\varphi$ is equivalent to a simple formula.*

Lemma 3.4 (Closure properties)**.** *The set of simple formulas is (semantically) closed under disjunction, conjunction, and existential quantifier.*

The set of simple formulas is not closed under negation. However, we show later (see Proposition 4.2) that the negation of a simple formula is equivalent to a simple formula in the expansion of the chains by all $TL(\mathsf{Until}, \mathsf{Since}, \mathsf{Until}^s, \mathsf{Since}^s)$ definable predicates.

In the rest of this section we explain how to translate a simple partition formula with one free variable into an equivalent $TL(\mathsf{Until}, \mathsf{Since}, \mathsf{Until}^s, \mathsf{Since}^s)$ formula.

Let $\delta_1, \ldots, \delta_k$ be quantifier free first-order formulas with one free variable and $O \subseteq \{1, \ldots, k\}$. For $i = 1, \ldots k$, let D_i be a temporal formula equivalent to δ_i. Define:

$$F_k := D_k \tag{3.1}$$

$$F_{i-1} := D_{i-1} \wedge \begin{cases} \mathbf{False}\mathsf{Until}F_i & \text{if } i-1 \in O \text{ and } i \in O; \\ D_i\mathsf{Until}F_i & \text{if } i-1 \in O \text{ and } i \notin O; \\ D_{i-1}\mathsf{Until}F_i & \text{if } i-1 \notin O \text{ and } i \in O; \\ D_{i-1}\mathsf{Until}^*F_i & \text{if } i-1 \notin O \text{ and } i \notin O, \end{cases} \tag{3.2}$$

where $P\mathsf{Until}^*Q$ holds at t if there is $t' > t$ such that $\texttt{Part}(\langle P(x), Q(x)\rangle, \emptyset)$ holds on the interval $[t, t']$; $P\mathsf{Until}^*Q$ can be expressed as disjunction of the following formulas:

- $P \wedge \big((P\mathsf{Until}Q) \vee (Q\mathsf{Until}Q) \vee P\mathsf{Until}\big(P \wedge (Q\mathsf{Until}Q)\big)\big)$
- $P \wedge (P\mathsf{Until}^sQ)$

Lemma 3.5.

(1) *Assume that there is t and a partition of $[t_1, t]$ into non-empty intervals $I_1, \ldots, I_k$ such that δ_j holds on I_j and I_i precedes I_j for $i < j$, and I_i is a one-point interval for $i \in O$. Then F_{k-j} holds on I_{k-j}.*
(2) *if F_{k-j} holds at t_{k-j} then there is $t \geq t_{k-j}$ such that $\texttt{Part}(\langle\delta_{k-j}, \ldots, \delta_k\rangle, O_{k-j})$ holds on $[t_{k-j}, t]$, where $l \in O_{k-j}$ iff $l + k - 1 - j \in O$.*
(3) *F_1 holds at t_1 iff there is $t \geq t_1$ such that $\texttt{Part}(\langle\delta_1, \ldots, \delta_k\rangle, O)$ holds on $[t_1, t]$.*

Proof. (1) and (2) by induction on j. (3) immediately from (1) and (2). □

Let δ'_k be a quantifier-free first-order formula with one free variable and D'_k be a temporal formula equivalent to δ'_k. If $k \notin O$ and we set $D_k := D'_k \wedge \Box D'_k$ in equation (3.1), then F_1 holds at t_1 iff $\texttt{Part}(\langle\delta_1, \ldots, \delta_{k-1}, \delta'_k\rangle, O)$ holds on $[t_1, \infty)$; if $k \in O$ and we set $D_k := D'_k \wedge \Box\mathbf{False}$ in equation (3.1), then F_1 holds at t_1 iff $\texttt{Part}(\langle\delta_1, \ldots, \delta_{k-1}, \delta'_k\rangle, O)$ holds on $[t_1, \infty)$. Hence, we obtained:

Lemma 3.6. *For every $\delta_1, \ldots, \delta_k$ and $O \subseteq \{1, \ldots, k\}$ there is a $TL(\mathsf{Until}, \mathsf{Until}^s)$ formula F such that F holds at t iff $\texttt{Part}(\langle\delta_1, \ldots, \delta_k\rangle, O)$ holds on $[t, \infty)$.*

By Lemma 3.6 and standard logical equivalences we obtain:

Proposition 3.7 (From simple formulas to TL)**.** *Every simple formula with at most one free variable is equivalent to a* $TL(\mathsf{Until}, \mathsf{Since}, \mathsf{Until}^s, \mathsf{Since}^s)$ *formula.*

Proof. Note that every simple partition formula with at most one free variable z is equivalent to a boolean combination of basic partition formulas of the forms: $\mathtt{Part}(\langle\delta_1\rangle, O)[z, z]$, $\mathtt{Part}(\langle\delta_1, \dots, \delta_n\rangle, O)[z, \infty)$, $\mathtt{Part}(\langle\delta_1, \dots, \delta_n\rangle, O)(-\infty, z]$, or $\mathtt{Part}(\langle\delta_1, \dots, \delta_n\rangle, O)(-\infty, \infty)$. Let D_1 be a temporal formula equivalent to the first-order quantifier free formula δ_1. A formula of the form $\mathtt{Part}(\langle\delta_1\rangle, O)[z, z]$ is equivalent to D_1. By Lemma 3.6 and its mirror variant the formulas of the second and the third forms are equivalent to $TL(\mathsf{Until}, \mathsf{Since}, \mathsf{Until}^s, \mathsf{Since}^s)$ formulas. A formula of the form $\mathtt{Part}(\langle\delta_1, \dots, \delta_n\rangle, O)(-\infty, \infty)$ is equivalent to "$\mathtt{Part}(\langle\delta_1\rangle, O)(-\infty, z] \wedge \mathtt{Part}(\langle\delta_1, \dots, \delta_n\rangle, O)[z, \infty)$ for some z." Since each of the conjuncts is equivalent to a temporal formula, the conjunction is also equivalent to a temporal formula A, and $\mathtt{Part}(\langle\delta_1, \dots, \delta_n\rangle, O)(-\infty, \infty)$ is equivalent to $\neg(\Box\neg A \wedge \neg A \wedge \overleftarrow{\Box}\neg A)$. Hence, every simple formula with at most one free variable is equivalent to a $TL(\mathsf{Until}, \mathsf{Since}, \mathsf{Until}^s, \mathsf{Since}^s)$ formula. □

4. Proof of Stavi's Theorem

The next definition plays a major role in our proof of Stavi's theorem; a similar definition is used in the proof of Kamp's theorem [GPSS80].

Definition 4.1. Let $\mathcal{M}$ be a Σ chain. We denote by $\mathcal{E}[\Sigma]$ the set of unary predicate names $\Sigma \cup \{A \mid A \text{ is an } TL(\mathsf{Until}, \mathsf{Since}, \mathsf{Until}^s, \mathsf{Since}^s)\text{-formula over } \Sigma\ \}$. The canonical $TL(\mathsf{Until}, \mathsf{Since}, \mathsf{Until}^s, \mathsf{Since}^s)$-expansion of $\mathcal{M}$ is an expansion of $\mathcal{M}$ to an $\mathcal{E}[\Sigma]$-chain, where each predicate name $A \in \mathcal{E}[\Sigma]$ is interpreted as $\{a \in \mathcal{M} \mid \mathcal{M}, a \models A\}$[1].

Note that if A is a $TL(\mathsf{Until}, \mathsf{Since}, \mathsf{Until}^s, \mathsf{Since}^s)$ formula over $\mathcal{E}[\Sigma]$ predicates, then it is equivalent to a $TL(\mathsf{Until}, \mathsf{Since}, \mathsf{Until}^s, \mathsf{Since}^s)$ formula over Σ, and hence to an atomic formula in the canonical $TL(\mathsf{Until}, \mathsf{Since}, \mathsf{Until}^s, \mathsf{Since}^s)$-expansions.

From now on we say "formulas are equivalent in a chain $\mathcal{M}$" instead of "formulas are equivalent in the canonical $TL(\mathsf{Until}, \mathsf{Since}, \mathsf{Until}^s, \mathsf{Since}^s)$-expansion of $\mathcal{M}$." The partition formulas are defined as previously, but now they can use as atoms $TL(\mathsf{Until}, \mathsf{Since}, \mathsf{Until}^s, \mathsf{Since}^s)$ definable predicates.

It is clear that the results stated in Sect. 3 hold for this modified notion of partition formulas. In particular, every simple formula with at most one free variable is equivalent to a $TL(\mathsf{Until}, \mathsf{Since}, \mathsf{Until}^s, \mathsf{Since}^s)$ formula, and the set of simple formulas is closed under conjunction, disjunction and existential quantification. However, now the set of simple formulas is also closed under negation, due to the next proposition whose proof is postponed to Sect. 5.

Proposition 4.2 (Closure under Negation)**.** *The negation of every simple partition formula is equivalent to a simple partition formula.*

As a consequence we obtain:

Proposition 4.3. *Every first-order formula is equivalent to a simple formula.*

[1] We often use "$a \in \mathcal{M}$" instead of "a is an element of the domain of $\mathcal{M}$."

Proof. We proceed by structural induction.

Atomic: It is clear that every atomic formula is equivalent to a simple formula.
Negation: By Proposition 4.2.
$\exists$-quantifier and disjunction : This follows from Lemma 3.4. □

Proposition 4.3 and Proposition 3.7 immediately imply Stavi's Theorem:

Theorem 4.4. *Every FOMLO formula with one free variable is equivalent to a* $TL(\mathsf{Until}, \mathsf{Since}, \mathsf{Until}^s, \mathsf{Since}^s)$ *formula.*

This completes our proof of Stavi's theorem except for Proposition 4.2 which is proved in Sect. 5.

5. Proof of Proposition 4.2

Throughout our proof we will freely use that the following assertions and their negations are expressible by simple formulas:

(1) (z_0, z_1) contains a point in P.
(2) $suc(z_0, z_1)$ - z_1 is a successor of z_0.
(3) interval (z_0, z_1) contains exactly k points.
(4) interval (z_0, z_1) contains at most k points.

Let us introduce some helpful notations.

Notation 5.1. We use the abbreviated notations $\mathtt{Part}(\langle\delta_1, \dots, \delta_n\rangle, O)(z_0, z_1)$ for $\mathtt{Part}(\langle\mathbf{True}, \delta_1, \dots, \delta_n, \mathbf{True}\rangle, O')[z_0, z_1]$, where $O' := \{1, n+2\} \cup \{i+1 \mid i \in O\}$. Hence, $\mathcal{M}, t_0, t_1 \models \mathtt{Part}(\langle\delta_1, \dots, \delta_n\rangle, O)(z_0, z_1)$ iff $\mathtt{Part}(\langle\delta_1, \dots, \delta_n\rangle, O)$ holds on the open interval (t_0, t_1) in $\mathcal{M}$. Similarly, $\mathtt{Part}(\langle\delta_1, \dots, \delta_n\rangle, O)(z_0, z_1]$ stands for $\mathtt{Part}(\langle\mathbf{True}, \delta_1, \dots, \delta_n\rangle, O')[z_0, z_1]$, where $O' := \{1\} \cup \{i+1 \mid i \in O\}$; and $\mathtt{Part}(\langle\delta_1, \dots, \delta_n\rangle, O)[z_0, z_1)$ for $\mathtt{Part}(\langle\delta_1, \dots, \delta_n, \mathbf{True}\rangle, O')[z_0, z_1]$, where $O' := \{n+1\} \cup O$.

By Proposition 3.7 and standard logical equivalences we obtain:

Lemma 5.2. *If every formula of the form* $\neg\mathtt{Part}(\langle\delta_1, \dots, \delta_n\rangle, O)(z_0, z_1)$ *is equivalent to a simple formula, then the negation of every simple formula is equivalent to a simple formula.*

Proof.

(1) Every basic partition formula φ either (a) has at most one free variable and then φ and $\neg\varphi$ are equivalent to simple formulas by Proposition 3.7, or (b) is equivalent to a formula of the form $\mathtt{Part}(\langle\delta_1, \dots, \delta_k\rangle, O)[z_0, z_1]$.
(2) A formula of the form $\mathtt{Part}(\langle\delta_1, \dots, \delta_k\rangle, O)[z_0, z_1]$ is equivalent to a formula constructed by disjunction and conjunction from formulas of the forms: (a) $\mathtt{Part}(\langle\delta_1, \dots, \delta_n\rangle, O')(z_0, z_1)$ and (b) $suc(z_0, z_1)$, $z_0 < z_1$, $z_0 = z_1$, $\delta_1(z_0)$ and $\delta_k(z_1)$, where $\delta_i(z)$ is a quantifier-free first-order formula. Formulas of the form (b) and their negations are equivalent to simple formulas.

Hence, if every formula of the form $\neg\mathtt{Part}(\langle\delta_1, \dots, \delta_k\rangle, O)(z_0, z_1)$ is equivalent to a simple formula, by the definition of simple formulas, (1)-(2) and De Morgan's laws we obtain the conclusion of the Lemma. □

Lemma 5.2 and the next proposition immediately imply Proposition 4.2.

Proposition 5.3 (Closure under negation)**.** *Every formula of the form*

$$\neg \mathit{Part}(\langle \delta_1, \ldots, \delta_n \rangle, O)(z_0, z_1)$$

is equivalent to a simple formula.

Sect. 5.3 contains a proof of Proposition 5.3. In the next subsection we provide some useful temporal logic formalizations. A proof of the next proposition, which is very similar to the proof of Proposition 5.3 is presented in Sect. 5.2.

Proposition 5.4. *The formula*

$$\neg \exists x_1 \ldots \exists x_n \, (z_0 < x_1 < \cdots < x_n < z_1) \wedge \bigwedge_{i=1}^{n} P_i(x_i)$$

is equivalent to a simple formula.

5.1. **Some formalizations in** $TL(\mathsf{Until}, \mathsf{Since}, \mathsf{Until}^s, \mathsf{Since}^s)$**.** First, observe that there is a $TL(\mathsf{Until}, \mathsf{Until}^s)$ formula that holds at t if t succeeds by a (left definable) P_1-gap and until this gap $P_1 \wedge P_2$ holds. Indeed, the required formula is Until-gap$(P_1, P_2) := \boldsymbol{\gamma}^+(P_1) \wedge \boldsymbol{\gamma}^+(P_1 \wedge P_2) \wedge \neg((P_1 \wedge P_2)\mathsf{Until}^s P_1)$.

Let δ and $\delta'_1, \ldots, \delta'_k$ be quantifier free first-order formulas with one free variable. For $i = 1, \ldots k$, let D'_i be a temporal formula equivalent to δ'_i and let D be a temporal formula equivalent to δ.

If we set $D_k :=$ Until-gap(D, D'_k) in equation (3.1) (see page 7) and $D_i := D'_i \wedge D$ for $i = 1, \ldots, k-1$ in equation (3.2), then $F_j(t_j)$ holds iff there is a δ-gap g that succeeds t_j such that $\mathtt{Part}(\langle \delta'_j, \ldots, \delta'_k \rangle, O)$ holds on $[t_j, g)$. Hence, we obtained the following Lemma:

Lemma 5.5. *For every k-tuple $\langle \delta_1, \ldots, \delta_k \rangle$, $O \subseteq \{1, \ldots, k\}$ and δ there is a $TL(\mathsf{Until}, \mathsf{Until}^s)$ formula F such that F holds at t if and only if there is a δ-gap g that succeeds t such that $\mathit{Part}(\langle \delta_1, \ldots, \delta_k \rangle, O)$ holds on $[t, g)$.*

Lemma 5.6. *Suppose we are given $k \geq 1$ quantifier-free formulas $\delta_1, \ldots, \delta_k$ with one free variable, a set $O \subseteq \{1, \ldots, k\}$, and points a_1, d with $a_1 \leq d$. Let $F_1, \ldots, F_k$ be defined as in equations (3.1) and (3.2) on page 7. Then the following are equivalent:*

(1) *There are points $a_1 < a_2 < \cdots < a_k \leq d$ such that $\wedge_{i=1}^k F_i(a_i)$.*
(2) *There is $b \in [a_1, d]$ such that $\mathit{Part}(\langle \delta_1, \ldots, \delta_k \rangle, O)$ holds on $[a_1, b]$.*

Proof. $\Leftarrow$ direction. Let $I_1, \ldots, I_k$ be a partition of $[a_1, b]$ into non-empty intervals such that δ_j holds on all points in I_j and I_i precedes I_j for $i < j$, and I_i is a one-point interval for $i \in O$. Let us choose any $a_i \in I_i$ for $i = 2, \ldots, k$. Then $\wedge_{i=1}^k F_i(a_i)$ holds by Lemma 3.5(1).

$\Rightarrow$ direction. Let F_i for $i = 1, \ldots, k$ be as in the lemma. By induction on $l \leq k$ we prove that if there are points $a_1 < a_2 < \cdots < a_l$ such that $\wedge_{i=1}^l F_i(a_i)$ then there is $b \leq a_l$ such that $\mathtt{Part}(\langle \delta_1 \wedge F_1, \ldots, \delta_l \wedge F_l \rangle, O \cap \{1, \ldots, l\})$ holds on $[a_1, b]$.

The basis is immediate, take $b := a_1$.

Inductive step: $l \mapsto l + 1$.

By the inductive assumption there is $b' \leq a_l$ and a partition of $[a_1, b']$ into l non-empty intervals $I'_1, \ldots, I'_l$ such that $\delta_i \wedge F_i$ holds on I'_i for $i \leq l$ and I'_i is a one-point interval for every $i \in O \cap \{1, \ldots, l\}$.

In particular, $F_l(b')$ holds. Now, by inspecting the definition of F_l according to Equation (3.2) on page 7, it is easy to construct the required interval and its partition. In all four cases I_i is defined as I'_i for $i < l$ and we explain how I_l and I_{l+1} are defined.

If $l \in O$ and $l+1 \in O$, then $F_l := D_l \wedge \mathbf{False}\mathsf{Until}F_{l+1}$. Note that F_l, holds at b', therefore b' has a successor c and $c \leq a_{l+1}$ because $b' \leq a_l < a_{l+1}$. Define $I_l := I'_l$, $b := c$ and $I_{l+1} := \{b\}$. It is clear that $I_1, \dots, I_{l+1}$ is a required partition.

If $l \in O$ and $l+1 \notin O$, then $F_l := D_l \wedge D_{l+1}\mathsf{Until}F_{l+1}$; hence, there is $c > b'$ such that $F_{l+1}(c)$ and δ_{l+1} holds on $(b', c]$. Define $I_l := I'_l$. Define $b := \min(c, a_{l+1})$, and I_{l+1} as $(b', b]$. It is clear that $I_1, \dots, I_{l+1}$ is a required partition.

If $l \notin O$ and $l+1 \in O$, then $F_l := D_l \wedge D_l\mathsf{Until}F_{l+1}$; hence, there is $c > b'$ such that $F_{l+1}(c)$ and δ_l holds on $[b', c)$. Define $b := \min(c, a_{l+1})$. Define I_l as $I'_l \cup [b', b)$ and I_{l+1} as $\{b\}$.

If $l \notin O$ and $l+1 \notin O$, then $F_l := D_l \wedge D_l\mathsf{Until}^* F_{l+1}$. Since F_l holds at b' there is $c > b'$ and a partition of $[b', c]$ into two non-empty intervals J_1 and J_2 such that $J_1 < J_2$ and D_l holds at all points of J_1 and F_{l+1} holds at all points of J_2. If $c < a_{l+1}$ define $I_l := I'_l \cup J_1$ and $I_{l+1} := J_2$ and $b := c$. If $a_{l+1} \in J_2$ define $I_l := I'_l \cup J_1$, $I_{l+1} := J_2 \cap \{a \mid a \leq a_{l+1}\}$ and $b := a_{l+1}$. If $a_{l+1} \in J_1$, define $I_l := I'_l \cup (J_1 \cap \{a \mid a < a_{l+1}\})$, $I_{l+1} := \{a_{l+1}\}$ and $b := a_{l+1}$. It is clear that $b \leq a_{l+1}$ and $I_1, \dots, I_{l+1}$ is a required partition. □

5.2. **Proof of Proposition 5.4.** The proof of Proposition 5.4 is very similar to the proof of Proposition 5.3. Its Corollary 5.7 will be used in the proof of Proposition 5.3.

Let $A_n(P_1, \dots, P_n, z_0, z_1)$ be $\exists x_1 \dots \exists x_n\, (z_0 < x_1 < \dots < x_n < z_1) \wedge \bigwedge_{i=1}^n P_i(x_i)$. We have to prove that $\neg A_n$ is equivalent to a simple formula.

$\neg A_n$ is equivalent to the disjunction of $(z_0, z_1) = \emptyset$ and of $(z_0, z_1) \neq \emptyset \wedge \neg A_n$. The first disjunct is equivalent to a simple formula. Therefore, it is sufficient to prove that the second disjunct is equivalent to a simple formula.

Below we assume that (z_0, z_1) is non-empty, and prove by induction on n.

Basis: The case $n = 1$ is trivial.

Inductive step: $n \mapsto n+1$.

Since (z_0, z_1) is non-empty, then one of the following cases holds:

Case 1: There is no occurrence of P_1 in (z_0, z_1) or there is no occurrence of P_{n+1} in (z_0, z_1).
Case 2: $z_0 = \inf\{z \in (z_0, z_1) \mid P_1(z)\}$.
Case 2′: $z_1 = \sup\{z \in (z_0, z_1) \mid P_{n+1}(z)\}$. This case is dual to case 2.
Case 3: $\inf\{z \in (z_0, z_1) \mid P_1(z)\}$ is an element in (z_0, z_1).
Case 3′: $\sup\{z \in (z_0, z_1) \mid P_{n+1}(z)\}$ is an element in (z_0, z_1). This case is dual to case 3.
Case 4: (1) Both $c := \inf\{z \in (z_0, z_1) \mid P_1(z)\}$ and $d := \sup\{z \in (z_0, z_1) \mid P_{n+1}(z)\}$ are gaps in (z_0, z_1) and
(2) $c \geq d$.
Case 5: (1) Both $c := \inf\{z \in (z_0, z_1) \mid P_1(z)\}$ and $d := \sup\{z \in (z_0, z_1) \mid P_{n+1}(z)\}$ are gaps in (z_0, z_1) and
(2) $c < d$.

For each of these cases we construct a simple formula Cond_i which describes it (i.e., Case i holds iff Cond_i holds), and show that if Cond_i holds, then $\neg A_{n+1}$ is equivalent to a simple formula Form_i. Hence, $\neg A_{n+1}$ is equivalent to a simple formula $\vee_i[\mathrm{Cond}_i \wedge \mathit{Form}_i]$.

Case 1 This case holds iff $\texttt{Part}(\langle\neg P_1(x)\rangle,\emptyset)(z_0,z_1) \vee \texttt{Part}(\langle\neg P_{n+1}(x)\rangle,\emptyset)(z_0,z_1)$ In this case $\neg A_{n+1}$ is equivalent to **True**.

Case 2 Case 2 holds iff $\mathsf{K}^+(P_1)(z_0)$. In this case $\neg A_{n+1}$ iff $\neg A_n(P_2,\dots,P_{n+1},z_0,z_1)$ which is equivalent to a simple formula by the inductive assumption.

Case 2′ This case is dual to Case 2.

Case 3 This case holds iff there is (a unique) $r_0 \in (z_0,z_1)$ such that $\neg P_1$ holds along (z_0,r_0) and either $P_1(r_0)$ or $\mathsf{K}^+(P_1)(r_0)$.

This r_0 is definable by the following simple formula, i.e., r_0 is a unique z which satisfies it:

$$\begin{aligned} \mathit{INF}(P_1,z_0,z,z_1) := z_0 < z < z_1 \wedge \text{“no } P_1 \text{ in } (z_0,z)\text{”} \wedge \\ \wedge\,(P_1(z) \vee \mathsf{K}^+(P_1)(z)) \end{aligned}$$

Hence, this case is described by $(\exists z)^{<z_1}_{>z_0}\mathit{INF}(P_1,z_0,z,z_1)$ which is equivalent to a simple formula.

In this case $\neg A_{n+1}$ iff $(\exists z)^{<z_1}_{>z_0}\big(\mathit{INF}(P_1,z_0,z,z_1)\wedge\neg A_n(P_2,\dots,P_n,z,z_1)\big)$. The inductive assumption and Lemma 3.4 imply that this formula is equivalent to a simple formula.

Case 3′ This case is dual to Case 3.

Case 4 The first condition holds iff

- z_0 succeeded by $\neg P_1$ gap in (z_0,z_1), i.e. $\boldsymbol{\gamma}^+(\neg P_1)(z_0)$ and P_1 holds at some point in (z_0,z_1), and
- z_1 preceded by $\neg P_{n+1}$ gap in (z_0,z_1), i.e., $\boldsymbol{\gamma}^-(\neg P_{n+1})(z_1)$ and P_{n+1} holds at some point in (z_0,z_1).

(Modalities $\boldsymbol{\gamma}^+$ and $\boldsymbol{\gamma}^-$ were defined in Sect. 2.2.2.) Hence, the first condition is equivalent to a simple formula.

If the first condition holds, then the second condition holds iff in (z_0,z_1) no occurrence of P_1 precedes an occurrence of P_{n+1}, i.e., iff $\texttt{Part}(\langle\neg P_1,\neg P_{n+1}\rangle,\emptyset)(z_0,z_1)$. Hence, Case 4 is described by a simple formula.

In Case 4 $\neg A_{n+1}(P_1,\dots,P_{n+1},z_0,z_1)$ is equivalent to **True**.

Case 5 The first condition is the same as in Case 4. If the first condition holds, then z is between c and d iff z satisfies the formula:

$$\mathit{Between}(z_0,z,z_1) := (\exists x_1)^{<z}_{>z_0}P_1(x_1) \wedge (\exists x_{n+1})^{<z_1}_{>z}P_{n+1}(x_{n+1}).$$

Hence, this case can be described as the conjunction of the first condition and $\exists z\mathit{Between}(z_0,z,z_1)$ and this is equivalent to a simple formula.

Note that in this case $\exists x_1\dots\exists x_{n+1}\,(z_0<x_1<\dots<x_{n+1}<z_1)\wedge\ \bigwedge_{j=1}^{n+1}P_j(x_j)$ holds iff for every z between c and d one of the following $2n-1$ conditions holds: for $i=1,\dots,n$:

$$\exists x_1\dots\exists x_{n+1}\,(z_0<x_1<\dots<x_{n+1}<z_1)\wedge x_i<z<x_{i+1}\wedge\ \bigwedge_{j=1}^{n+1}P_j(x_j)$$

for $i=2,\dots,n$:

$$\exists x_1\dots\exists x_{n+1}\,(z_0<x_1<\dots<x_{n+1}<z_1)\wedge x_i=z\wedge\ \bigwedge_{j=1}^{n+1}P_j(x_j)$$

Hence, $\neg\exists x_1 \dots \exists x_{n+1} (z_0 < x_1 < \dots < x_{n+1} < z_1) \wedge \bigwedge_{j=1}^{n+1} P_j(x_j)$ is equivalent to

$$\exists z(\mathit{Between}(z) \wedge \bigwedge_{k=1}^{n} \left[\neg A_k(P_1,\dots,P_k,z_0,z) \vee \neg A_{n+1-k}(P_{k+1},\dots,P_{n+1},z,z_1)\right]$$

$$\wedge \bigwedge_{k=2}^{n} \left[\neg A_{k-1}(P_1,\dots,P_{k-1},z_0,z) \vee \neg P_k(z) \vee \neg A_{n+1-k}(P_{k+1},\dots,P_{n+1},z,z_1)\right]$$

By the inductive assumption $\neg A_k$ and $\neg A_{n+1-k}$ are simple for $k = 1,\dots,n$. Since *Between* is a simple formula, and the set of simple formulas is closed under conjunction, disjunction and existential quantifier, we obtain a formalization of this case by a simple formula. This completes the proof of Proposition 5.4.

By Proposition 5.4, Lemma 5.6 and standard logical equivalences we derive:

Corollary 5.7.
(1) $\neg(\exists z)^{<z_1}_{>z_0}\mathtt{Part}(\langle\delta'_1,\dots,\delta'_n\rangle, O')(z_0,z]$ *is equivalent to a simple formula.*
(2) $\neg(\exists z)^{<z_1}_{>z_0}\mathtt{Part}(\langle\delta'_1,\dots,\delta'_n\rangle, O')[z,z_1)$ *is equivalent to a simple formula.*

Proof.
(1) Set $k := n+1$, $\delta_1 :=$ **True**, $\delta_{i+1} := \delta'_i$ for $i = 1,\dots,n$ and $O := \{1\} \cup \{i+1 \mid i \in O'\}$. Observe: $\mathtt{Part}(\langle\delta'_1,\dots,\delta'_n\rangle, O')(z_0,z]$ iff $\mathtt{Part}(\langle\delta_1,\dots,\delta_k\rangle, O)[z_0,z]$.
Let F_i be defined as in Lemma 5.6. Then $\exists x_2 \dots \exists x_{k-1} z_0 < x_2 < \dots < x_{k-1} < x_k \wedge F_1(z_0) \wedge \bigwedge_{i=2}^{k} F_i(x_i)$ iff $\exists z(z_0 < z \le x_k \wedge \mathtt{Part}(\langle\delta_1,\dots,\delta_k\rangle, O)[z_0,z])$.
Hence, $\neg(\exists z)^{<z_1}_{>z_0}\mathtt{Part}(\langle\delta'_1,\dots,\delta'_n\rangle, O')(z_0,z]$ is equivalent to $\neg F_1(z_0) \vee \neg\, \exists x_2 \dots \exists x_k z_0 < x_2 < \dots < x_{k-1} < x_k < z_1 \wedge \bigwedge_{i=2}^{k} F_i(x_i)$. The first disjunct is an atom (in the canonical expansion) and the second disjunct is equivalent to a simple formula by Proposition 5.4. Therefore, $\neg(\exists z)^{<z_1}_{>z_0}\mathtt{Part}(\langle\delta'_1,\dots,\delta'_n\rangle, O')(z_0,z]$ is equivalent to a simple formula.
(2) is the mirror image of (1). □

5.3. **Proof of Proposition 5.3.**

Convention. We often will say "a formula is simple" instead of "a formula is equivalent to a simple formula." In all such cases equivalence to a simple formula is proved by Lemma 3.4 and by standard logical transformations and/or using the inductive hypotheses.

We proceed by induction on n.

Basis. The case $n = 1$ is immediate.

Inductive step $n \mapsto n+1$.

$\neg\mathtt{Part}(\langle\delta_1,\dots,\delta_{n+1}\rangle, O)(z_0,z_1)$ is equivalent to the disjunction of $(z_0,z_1) = \emptyset$ and of $(z_0,z_1) \neq \emptyset \wedge \neg\mathtt{Part}(\langle\delta_1,\dots,\delta_{n+1}\rangle, O)(z_0,z_1)$. The first disjunct is equivalent to a simple formula. Therefore, it is sufficient to prove that the second disjunct is equivalent to a simple formula.

From now on we assume that (z_0,z_1) is non-empty.

Observe that one of the following cases holds:

Case 1: δ_1 holds on all points in (z_0,z_1).
Case 1′: δ_{n+1} holds on all points in (z_0,z_1). This case is dual to case 1.
Case 2: $z_0 = \inf\{z \in (z_0,z_1) \mid \neg\delta_1(z)\}$ or $z_1 = \sup\{z \in (z_0,z_1) \mid \neg\delta_{n+1}(z)\}$.
Case 3: $\inf\{z \in (z_0,z_1) \mid \neg\delta_1(z)\}$ is an element in (z_0,z_1).
Case 3′: $\sup\{z \in (z_0,z_1) \mid \neg\delta_{n+1}(z)\}$ is an element in (z_0,z_1). This case is dual to case 3.

Case 4: Both $c := \inf\{z \in (z_0, z_1) \mid \neg\delta_1(z)\}$ and $d := \sup\{z \in (z_0, z_1) \mid \neg\delta_{n+1}(z)\}$ are gaps in (z_0, z_1) and $c > d$.
Case 5: Both $c := \inf\{z \in (z_0, z_1) \mid \neg\delta_1(z)\}$ and $d := \sup\{z \in (z_0, z_1) \mid \neg\delta_{n+1}(z)\}$ are gaps in (z_0, z_1) and $c < d$.
Case 6: Both $c := \inf\{z \in (z_0, z_1) \mid \neg\delta_1(z)\}$ and $d := \sup\{z \in (z_0, z_1) \mid \neg\delta_{n+1}(z)\}$ are gaps in (z_0, z_1) and $c = d$.

For each of these cases we construct a simple formula Cond_i which describes it (i.e., Case i holds iff Cond_i holds), and show that if Cond_i holds, then $\neg\mathtt{Part}(\langle\delta_1, \dots, \delta_{n+1}\rangle, O)(z_0, z_1)$ is equivalent to a simple formula $Form_i$. Hence, $\neg\mathtt{Part}(\langle\delta_1, \dots, \delta_{n+1}\rangle, O)(z_0, z_1)$ is equivalent to a simple formula $\vee_i[\mathrm{Cond}_i \wedge Form_i]$.

Case 1 is described by $\mathtt{Part}(\langle\delta_1\rangle, \emptyset)(z_0, z_1)$. In this case $\neg\mathtt{Part}(\langle\delta_1, \dots, \delta_{n+1}\rangle, O)(z_0, z_1)$ is equivalent to $\neg(\exists z)^{<z_1}_{>z_0}\mathtt{Part}(\langle\delta_1, \dots, \delta_{n+1}\rangle, O)[z, z_1)$, and by Corollary 5.7 this is a simple formula.

Case 1′ This case is dual to Case 1.

Case 2 This case is described by $\mathbf{K}^+(\neg\delta_1)(z_0) \vee \mathbf{K}^-(\neg\delta_{n+1})(z_1)$. In this case $\neg\mathtt{Part}(\langle\delta_1, \dots, \delta_{n+1}\rangle, O)(z_0, z_1)$ is equivalent to **True**.

Note that in the above cases we have not used the inductive assumption. Case 6 will be also proved directly. However, in cases 3-5 we will use the inductive assumption.

We introduce notations and state an observation which will be used several times.

For a set O of natural numbers and $i \in \mathbb{N}$, we denote by $O_{sh(i)}$ the set O shifted by i, i.e., $O_{sh(i)} := \{j \mid j > 0 \wedge j + i \in O\}$.

Define

$$C^{<i}(z_0, z) := \begin{cases} \text{“}z \text{ is the successor of } z_0\text{”} & \text{for } i = 1 \\ \mathtt{Part}(\langle\delta_1, \dots, \delta_{i-1}\rangle, O \cap \{1, \dots, i-1\})(z_0, z) & \text{for } i = 2, \dots, n+2 \end{cases}$$

$$C^{>i}(z, z_1) := \begin{cases} \text{“}z_1 \text{ is the successor of } z\text{”} & \text{for } i = n+1 \\ \mathtt{Part}(\langle\delta_{i+1}, \dots, \delta_{n+1}\rangle, O_{sh(i)})(z, z_1) & \text{for } i = 0, \dots, n \end{cases}$$

For $i = 1, \dots, n+1$ define

$$\begin{aligned} C^{\leq i}(z_0, z) &:= C^{<i}(z_0, z) \vee C^{<i+1}(z_0, z) \\ C^{\geq i}(z, z_1) &:= C^{>i}(z, z_1) \vee C^{>i-1}(z, z_1) \end{aligned}$$

$$A_i(z_0, z, z_1) := \begin{cases} C^{<i}(z_0, z) \wedge \delta_i(z) \wedge C^{>i}(z, z_1) & \text{if } i \in O \\ C^{\leq i}(z_0, z) \wedge \delta_i(z) \wedge C^{\geq i}(z, z_1) & \text{otherwise} \end{cases}$$

From these definitions we obtain the following equivalences:

$$\mathtt{Part}(\langle\delta_1, \dots, \delta_{n+1}\rangle, O)(z_0, z_1) \Leftrightarrow (\exists z)^{<z_1}_{>z_0} A_i \qquad \text{for } i \in 1, \dots, n+1 \tag{5.1}$$

and if $(z_0, z_1) \neq \emptyset$, then

$$\mathtt{Part}(\langle\delta_1, \dots, \delta_{n+1}\rangle, O)(z_0, z_1) \Leftrightarrow (\forall z)^{<z_1}_{>z_0} \big(\bigvee_i A_i\big) \tag{5.2}$$

Since, we assumed that (z_0, z_1) is non-empty, by (5.1)-(5.2) we have

$$\neg\mathtt{Part}(\langle\delta_1, \dots, \delta_{n+1}\rangle, O)(z_0, z_1)$$

is equivalent to

$$(\exists z)^{<z_1}_{>z_0}\big(\bigwedge_i \neg A_i\big)$$

and to

$$(\forall z)^{<z_1}_{>z_0}\big(\bigwedge_i \neg A_i\big)$$

Hence, for every $\varphi(z_0, z, z_1)$

$$(\exists z)^{<z_1}_{>z_0}\varphi(z) \wedge \neg \mathtt{Part}(\langle \delta_1, \dots, \delta_{n+1}\rangle, O)(z_0, z_1)$$

is equivalent to

$$(\exists z)^{<z_1}_{>z_0}\Big(\varphi(z) \wedge \big(\bigwedge_i \neg A_i\big)\Big)$$

is equivalent to

$$(\exists z)^{<z_1}_{>z_0}\Big(\big(\varphi(z) \wedge \bigwedge_{i\in\{2,\dots,n\}} \neg A_i\big) \wedge \big(\varphi(z) \wedge \neg A_1 \wedge \neg A_{n+1}\big)\Big)$$

By the inductive assumption, the definition of A_i, and Lemma 3.4, we obtain that $\neg A_i$ are simple formulas for $i \in \{2, \dots, n\}$. Similarly, if $1 \in O$ (respectively, $n+1 \in O$), then $\neg A_1$ (respectively, $\neg A_{n+1}$) is equivalent to a simple formula. The set of simple formulas is closed under $\wedge$, $\vee$ and $\exists$. Hence,

Observation 5.8. Assume that $\varphi(z)$ is equivalent to a simple formula, and if $1 \notin O$, then $\varphi(z) \wedge \neg A_1$ is equivalent to a simple formula, and if $n+1 \notin O$, then $\varphi(z) \wedge \neg A_{n+1}$ is equivalent to a simple formula. Then $(\exists z)^{<z_1}_{>z_0}\varphi(z) \wedge \neg \mathtt{Part}(\langle \delta_1, \dots, \delta_{n+1}\rangle, O)(z_0, z_1)$ is equivalent to a simple formula.

In cases 3-5 we will use this observation with some instances of φ.

Case 3 This case holds iff there is (a unique) $r_0 \in (z_0, z_1)$ such that δ_1 holds along (z_0, r_0) and $\neg\delta_1(r_0) \vee \mathbf{K}^+(\neg\delta_1)(r_0)$.

This r_0 is definable by the following simple formula, i.e., r_0 is a unique z which satisfies it:

$$\begin{aligned} INF_{\neg\delta_1}(z_0, z, z_1) := z_0 < z < z_1 \wedge (suc(z_0, z) \vee \ \mathtt{Part}(\langle \delta_1\rangle, \emptyset)(z_0, z)) \wedge \\ \wedge (\neg\delta_1(z) \vee \mathbf{K}^+(\neg\delta_1)(z)) \end{aligned}$$

Hence, this case is described by a simple formula $(\exists z)^{<z_1}_{>z_0} INF_{\neg\delta_1}(z_0, z, z_1)$.

By Observation 5.8 it is sufficient to prove that (1) if $1 \notin O$ then $INF_{\neg\delta_1} \wedge \neg A_1$ is equivalent to a simple formula, and (2) if $n+1 \notin O$, then $INF_{\neg\delta_1} \wedge \neg A_{n+1}$ is equivalent to a simple formula.

Note that $\neg\delta_1(z) \vee \mathbf{K}^+(\neg\delta_1)(z)$ implies $\neg\big(\delta_1(z) \wedge \mathtt{Part}(\langle \delta_1, \dots, \delta_{n+1}\rangle, O)(z, z_1)\big)$. Therefore, by the definition of A_1 for the case when $1 \notin O$, and standard logical transformations we obtain that $INF_{\neg\delta_1} \wedge \neg A_1$ is equivalent to $INF_{\neg\delta_1} \wedge \big(\neg C^{\leq 1} \vee \neg\delta_1(z) \vee \neg \mathtt{Part}(\langle \delta_2, \dots, \delta_{n+1}\rangle, O_{sh(1)})(z, z_1)\big)$. The last formula is equivalent to a simple formula by the inductive assumption and standard logical equivalences.

If $n+1 \notin O$, then $INF_{\neg\delta_1} \wedge \neg A_{n+1}$ is equivalent to

$$INF_{\neg\delta_1}(z_0, z, z_1) \wedge (\neg C^{\geq n+1}(z, z_1) \vee \neg\delta_{n+1}(z) \vee \neg C^{\leq n+1}(z_0, z)).$$

$\neg C^{\geq n+1}(z, z_1)$ is a simple formula by the induction basis. Note that $INF_{\neg\delta_1}(z_0, z, z_1)$ implies $suc(z_0, z)$ or "δ_1 holds along the interval (z_0, z)." By Case 1 the conjunction of "δ_1 holds along the interval (z_0, z)" and $\neg C^{\leq n+1}(z_0, z))$ is a simple formula. Therefore, $INF_{\neg\delta_1} \wedge \neg A_{n+1}$ is equivalent to a simple formula.

Case 3′ This case is dual to case 3.

Case 4 The conjunction of the following conditions expresses by a simple formula that z is in the interval (d, c):

- z_0 succeeded by δ_1 gap in (z_0, z_1) - $\boldsymbol{\gamma}^+(\delta_1)(z_0)$ and $\neg\delta_1$ holds at some point in (z_0, z_1).
- z_1 preceded by δ_{n+1} gap in (z_0, z_1) - $\boldsymbol{\gamma}^-(\delta_{n+1})(z_1)$ and $\neg\delta_{n+1}$ holds at some point in (z_0, z_1).
- δ_1 holds along (z_0, z) and δ_{n+1} holds along (z, z_1).

Let us denote this conjunction by $\mathrm{In}_{(d,c)}(z_0, z, z_1)$.

Hence, this case holds iff $(\exists z)^{<z_1}_{>z_0} \mathrm{In}_{(d,c)}(z_0, z, z_1)$.

By Observation 5.8 it is sufficient to show that (1) if $1 \notin O$, then $\mathrm{In}_{(d,c)}(z_0, z, z_1) \wedge \neg A_1$ is equivalent to a simple formula, and (2) if $n+1 \notin O$, then $\mathrm{In}_{(d,c)}(z_0, z, z_1) \wedge \neg A_{n+1}$ is equivalent to a simple formula.

if $1 \notin O$ then $\mathrm{In}_{(d,c)}(z_0, z, z_1) \wedge \neg A_1(z_0, z, z_1)$ is equivalent to

$$\mathrm{In}_{(d,c)}(z_0, z, z_1) \wedge \big(\neg\delta_1(z) \vee \neg C^{\leq 1}(z_0, z) \vee (\neg C^{>1}(z, z_1) \wedge \neg C^{>0}(z, z_1))\big).$$

$\mathrm{In}_{(d,c)}(z_0, z, z_1)$ implies that δ_{n+1} holds along (z, z_1), therefore, by Case 1′ both $\mathrm{In}_{(d,c)} \wedge \neg C^{>0}(z, z_1)$ and $\mathrm{In}_{(d,c)} \wedge \neg C^{>1}(z, z_1)$ are simple. By the basis of induction $\neg C^{\leq 1}$ is simple. Hence, $\mathrm{In}_{(d,c)}(z_0, z, z_1) \wedge \neg A_1(z_0, z, z_1)$ is simple.

Similar arguments show that if $n+1 \notin O$, then $\mathrm{In}_{(d,c)}(z_0, z, z_1) \wedge \neg A_{n+1}(z_0, z, z_1)$ is simple.

Case 5 It is easy to write a simple formula $Between(z_0, z, z_1)$ which expresses that z is in the interval (c, d). $Between(z_0, z, z_1)$ can be defined as the conjunction of $z_0 < z < z_1$ and of

- z_0 succeeded by δ_1 gap in (z_0, z) - $\boldsymbol{\gamma}^+(\delta_1)(z_0)$ and $\neg\delta_1$ holds at some point in (z_0, z).
- z_1 preceded by δ_{n+1} gap in (z, z_1) - $\boldsymbol{\gamma}^-(\delta_{n+1})(z_1)$ and $\neg\delta_{n+1}$ holds at some point in (z, z_1).

Hence, this case holds iff $(\exists z)^{<z_1}_{>z_0} Between(z_0, z, z_1)$.

By Observation 5.8 it is sufficient to show that (1) if $1 \notin O$, then $Between(z_0, z, z_1) \wedge \neg A_1(z_0, z, z_1)$ is equivalent to a simple formula, and (2) if $n+1 \notin O$, then $Between(z_0, z, z_1) \wedge \neg A_{n+1}(z_0, z, z_1)$ is equivalent to a simple formula. Since $Between$ implies $\neg C^{\leq 1}$ it follows that $Between \wedge \neg A_1$ is equivalent to $Between$. Since $Between$ implies $\neg C^{\geq n+1}$ it follows that $Between \wedge \neg A_{n+1}$ is equivalent to $Between$. Therefore, both $Between \wedge \neg A_1$ and $Between \wedge \neg A_{n+1}$ are simple.

Case 6 Both $c := \inf\{z \in (z_0, z_1) \mid \neg\delta_1(z)\}$ and $d := \sup\{z \in (z_0, z_1) \mid \neg\delta_{n+1}(z)\}$ are gaps in (z_0, z_1) and $c \geq d$ iff the conjunction of the following holds:

(1) z_0 succeeded by δ_1 gap in (z_0, z_1).
(2) z_1 preceded by δ_{n+1} gap in (z_0, z_1).
(3) $\mathtt{Part}(\langle\delta_1, \delta_{n+1}\rangle, \emptyset)(z_0, z_1)$.

If (1)-(3) holds, then $d < c$ iff $F(z_0)$ defined as $\delta_1 \mathsf{Until}(\delta_1 \wedge \text{Until-gap}(\delta_1, \delta_2))(z_0)$ holds, where Until-gap is defined on page 10.

Hence, this case can be described by the conjunction of (1)-(3) and $\neg F(z_0)$. (1) and (2) are expressed by simple formulas like in Case 4; (3) and $\neg F(z_0)$ are simple formulas. Therefore, this case is described by a simple formula.

In this case $\texttt{Part}(\langle \delta_1, \ldots, \delta_{n+1} \rangle, O)(z_0, z_1)$ holds iff there is i such that $\texttt{Part}(\langle \delta_1, \ldots, \delta_i \rangle, O)$ holds on (z_0, c) and $\texttt{Part}(\langle \delta_i, \ldots, \delta_{n+1} \rangle, O)$ or $\texttt{Part}(\langle \delta_{i+1}, \ldots, \delta_{n+1} \rangle, O)$ holds on (c, z_1). Applying Lemma 5.5 to the tuple $\langle \mathbf{True}, \delta_1, \ldots, \delta_i \rangle$, $O := \{1\} \cup \{j+1 \mid j \in O \wedge j \leq i\}$ and δ_1, we obtain a temporal formula F_i such that $F_i(z_0)$ iff $\texttt{Part}(\langle \delta_1, \ldots, \delta_i \rangle, O)$ holds on (z_0, c). By the mirror arguments there is a temporal formula H_i such that $H_i(z_1)$ iff $\texttt{Part}(\langle \delta_i, \ldots, \delta_{n+1} \rangle, O)$ holds on (c, z_1). Hence, in this case $\neg\texttt{Part}(\langle \delta_1, \ldots, \delta_{n+1} \rangle, O)(z_0, z_1)$ is equivalent to

$$\bigwedge_{i=1}^{n} \big(\neg F_i(z_0) \vee (\neg H_i(z_1) \wedge \neg H_{i+1}(z_1))\big).$$

6. Related Works

Our proof is very similar to the proof of Kamp's theorem in [Rab14]. The only novelty of our proof are partition formulas. Simple partition formulas generalize $\overrightarrow{\exists}\forall$-formulas which played a major role in the proof of Kamp's theorem [Rab14]. Roughly speaking an $\overrightarrow{\exists}\forall$-formula is a normal partition formula which uses only basic partition expressions $\texttt{Part}(\langle \delta_1, \ldots, \delta_n \rangle, O)$ with the following restriction: for $i < n$, if $i \notin O$ then $i+1 \in O$. This restriction implies that if a partition $I_1, \ldots, I_n$ witnesses that an interval $[t, t']$ of $\mathcal{M}$ satisfies $\texttt{Part}(\langle \delta_1, \ldots, \delta_n \rangle, O)$, then all intervals I_i have endpoints in $\mathcal{M}$. Over the Dedekind complete orders all intervals have end-points and every partition expression is equivalent to a disjunction of the restricted partition expressions; however, over general linear orders $\texttt{Part}(\langle P_1(x), P_2(x) \rangle, \emptyset)$ is not equivalent to a positive boolean combination of restricted partition expressions.

As far as we know, there are only two published proofs of Stavi's theorem. One is based on separation in Chapter 11 of [GHR94], and the other is based on games in [GHR93] (reproduced in Chapter 12 of [GHR94]). They are much more complicated than the proofs of Kamp's theorem in [GHR94].

A temporal logic has the *separation* property if its formulas can be equivalently rewritten as a boolean combination of formulas, each of which depends only on the past, present or future. The separation property was introduced by Gabbay [Gab81], and surprisingly, a temporal logic which can express $\Box$ and $\overleftarrow{\Box}$ has the separation property (over a class $\mathcal{C}$ of structures) iff it is expressively complete for *FOMLO* over $\mathcal{C}$.

In the proof based on separation, a special temporal language $L*$ is carefully designed. The formulas of $L*$ are evaluated over Dedekind-complete chains. For every chain $\mathcal{M}$ its completion $\mathcal{M}^c$ is defined. It is shown: (1) $L*$ has the separation property over the completions of chains; (2) for every $\varphi \in L*$ there is a formula $\psi \in TL(\mathsf{Until}, \mathsf{Since}, \mathsf{Until}^s, \mathsf{Since}^s)$ such that $\mathcal{M}, t \models \psi$ iff $\mathcal{M}^c, t \models \varphi$, and (3) for every formula $\xi(x) \in FOMLO$ there is $\varphi \in L*$ such that $\mathcal{M}, t \models \xi$ iff $\mathcal{M}^c, t \models \varphi$.

In the game-based proof for every chain $\mathcal{M}$ and $r \in \mathbb{N}$ a chain $\mathcal{M}_r$ is defined. $\mathcal{M}_r$ is the completion of $\mathcal{M}$ by the gaps definable by $TL(\mathsf{Until}, \mathsf{Since}, \mathsf{Until}^s, \mathsf{Since}^s)$ formulas of the nesting depth r. Then, special games on the temporal structures are considered. The game arguments are easier to grasp, then the separation ones, but they use complicated inductive assertions.

Our proof avoids completions and games and separates general logical equivalences and temporal arguments. The proof is similar to our proof of Kamps theorem [Rab14]; yet it is longer because it treats some additional cases related to gaps in time flows.

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