Logical Methods in Computer Science
Vol. 13(4:18)2017, pp. 1–34
https://lmcs.episciences.org/

Submitted Sep. 06, 2016
Published Nov. 27, 2017

A FEW NOTES ON FORMAL BALLS

JEAN GOUBAULT-LARRECQ AND KOK MIN NG

LSV, ENS Paris-Saclay, CNRS, Université Paris-Saclay, 94230 Cachan, France
e-mail address: goubault@lsv.fr

National Institute of Education, Nanyang Technological University, Singapore
e-mail address: ngkokmin@gmail.com

Abstract. Using the notion of formal ball, we present a few new results in the theory of quasi-metric spaces. With no specific order: every continuous Yoneda-complete quasi-metric space is sober and convergence Choquet-complete hence Baire in its d-Scott topology; for standard quasi-metric spaces, algebraicity is equivalent to having enough center points; on a standard quasi-metric space, every lower semicontinuous $\overline{\mathbb{R}}_+$-valued function is the supremum of a chain of Lipschitz Yoneda-continuous maps; the continuous Yoneda-complete quasi-metric spaces are exactly the retracts of algebraic Yoneda-complete quasi-metric spaces; every continuous Yoneda-complete quasi-metric space has a so-called quasi-ideal model, generalizing a construction due to K. Martin. The point is that all those results reduce to domain-theoretic constructions on posets of formal balls.

Introduction

In his gem of a paper on how to write Mathematics [Hal70, Section 2], Paul Halmos recommends to "say something". He then comments on books and papers that violate this principle by either saying nothing or saying too many things. The present paper may appear to say too many, relatively random, things. On the contrary, let us stress that the unique idea of the present paper can be summarized by the motto: "Formal balls are the essence of quasi-metric spaces".

We will explain all terms in Section 1.

The first author has been convinced of the truth of that motto while writing the book [Gou13], and most of its Chapter 7 arises from that conviction. Several papers had already been based on that premise [Vic05, Vic09, AAHPR09, RV10, KW10]. The first author has given a few talks on the topic, in particular at the Domains XII conference in Cork, Ireland, 2015. However, most of it has already been published, and only a few crumbs remain to offer the reader. We hope those are interesting crumbs.

We outline a notion of *standard* quasi-metric space in Section 2. This is a natural notion, and all Yoneda-complete quasi-metric spaces, as well as all metric spaces, are standard. We give a simple characterization of Waszkiewicz's *continuous* Yoneda-complete quasi-metric spaces in Section 3, and this suggests a definition of continuous, not necessarily complete,

Key words and phrases: Quasi-metric, formal balls.

DOI:10.23638/LMCS-13(4:18)2017

quasi-metric spaces: namely, the standard quasi-metric spaces whose poset of formal balls is continuous. All metric spaces are continuous in this sense, in particular.

The above crumbs can be considered as additional basic facts on quasi-metric spaces, complementing Section 1. Those facts have not appeared earlier in the literature, as far as we know. In Section 4, we grab low-hanging fruit and show that every continuous Yoneda-complete quasi-metric space is sober, and also convergence Choquet-complete hence Baire. In Section 5, we characterize those standard quasi-metric spaces that are *algebraic*, as those that have *enough center points*. The latter is a simple condition on the Scott topology on the poset of formal balls. In Section 6, we look at morphisms, and first show that every lower semicontinuous map from a standard quasi-metric space to $\overline{\mathbb{R}}_+$ is a pointwise supremum of a chain of Lipschitz Yoneda-continuous maps. This generalizes a standard construction on metric spaces, and involves defining the appropriate variant of the distance of a point to a closed set. Then, in Section 7, we show that the continuous Yoneda-complete quasi-metric spaces are exactly the retracts of algebraic Yoneda-complete quasi-metric spaces, generalizing a similar result in the theory of dcpos.

A final crumb, in Section 8, explores the notion of quasi-ideal domains: algebraic dcpos whose finite elements are below all non-finite elements. Using a variant of a construction due to K. Martin, we show that every continuous Yoneda-complete quasi-metric space has a quasi-ideal model; and that the spaces that have an ω-quasi-ideal model are exactly M. de Brecht's quasi-Polish spaces.

We conclude in Section 9.

1. Basics on Quasi-Metric Spaces

Let $\overline{\mathbb{R}}_+$ be the set of extended non-negative reals. A *quasi-metric* on a set X is a map $d\colon X \times X \to \overline{\mathbb{R}}_+$ satisfying: $d(x,x) = 0$; $d(x,z) \leq d(x,y) + d(y,z)$ (triangular inequality); $d(x,y) = d(y,x) = 0$ implies $x = y$.

The relation $\leq^d$ defined by $x \leq^d y$ if and only if $d(x,y) = 0$ is then an ordering, which turns out to be the specialization ordering of the open ball topology. The latter is generated by the open balls $B^d_{x,<r}$, $x \in X$, $r > 0$, as basic open sets.

Example 1.1. On any subset of $\mathbb{R} \cup \{+\infty\}$, define the quasi-metric $\mathsf{d}_\mathbb{R}$ by $\mathsf{d}_\mathbb{R}(x,y) = 0$ if $x \leq y$, $\mathsf{d}_\mathbb{R}(+\infty, y) = +\infty$ if $y \neq +\infty$, $\mathsf{d}_\mathbb{R}(x,y) = x - y$ if $x > y$ and $x \neq +\infty$. This is a quasi-metric with specialization ordering $\leq^{\mathsf{d}_\mathbb{R}}$ equal to the usual ordering $\leq$. $\Box$

A *formal ball* is a pair (x,r) where $x \in X$ and $r \in \mathbb{R}_+$. This is just syntax for an actual ball: x is the *center*, and r is the *radius*. Formal balls are ordered by $(x,r) \leq^{d^+} (y,s)$ if and only if $d(x,y) \leq r - s$. Note that this implies $r \geq s$, in particular.

The poset of formal balls $\mathbf{B}(X,d)$ has many serendipitous properties. Most of them are described and proved in [Gou13], and we will recapitulate the most fundamental ones here. In the remaining sections, we will proceed to state a few new results that stem from the study of $\mathbf{B}(X,d)$.

Although the book [Gou13] is a good source of information, we would not like to insinuate that the first author of the present paper is the author of the theory of formal balls. Formal balls were introduced by Klaus Weihrauch and Ulrich Schreiber [WS81]. Reinhold Heckmann and Abbas Edalat showed why they were so important in the metric case [EH98]. In the general, quasi-metric case, we would like to stress the import of Mateusz Kostanek and Paweł Waszkiewicz [KW10], who showed that X, d is Yoneda-complete if and only if

$\mathbf{B}(X, d)$ is a dcpo — such an important characterization that we will actually take it as a *definition* of Yoneda-completeness, although we will recapitulate the standard definition in Section 5. (Kostanek and Waszkiewicz consider the more general case of quasi-metrics with values in a $\mathcal{Q}$-category instead of $\overline{\mathbb{R}}_+$, a natural extension if we start from Lawvere's view of metric spaces as enriched categories [Law02].) The supremum of a directed family of formal balls $(x_i, r_i)_{i \in I}$ is then equal to (x, r) where x is the so-called *d-limit* of the Cauchy net $(x_i)_{i \in I, \leq}$—here $i \leq j$ iff $(x_i, r_i) \leq^{d^+} (x_j, r_j)$. We will again define what a d-limit is, and what Cauchy nets are, in Section 5. The point is that the d-limit is independent of the radii r_i. Also, we have the innocuous-looking equality $r = \inf_{i \in I} r_i$, which will be a crucial property of standard quasi-metric spaces (Section 2).

Salvador Romaguera and Oscar Valero proved an important theorem of the same kind, stating that X, d is Smyth-complete if and only if $\mathbf{B}(X, d)$ is a continuous dcpo and its way-below relation is the relation $\prec$, defined by $(x, r) \prec (y, s)$ if and only if $d(x, y) < r - s$ [RV10]. Smyth-completeness is a stronger property than Yoneda-completeness, and means that every (forward) Cauchy net in X, d has a limit in the metric space X, d^{sym} (taking $d^{sym}(x, y) = \max(d(x, y), d(y, x))$).

It turns out that $\mathbf{B}(X, d)$ can also be given a natural quasi-metric d^+, defined by $d^+((x, r), (y, s)) = \max(d(x, y) - r + s, 0)$. The associated ordering is the ordering $\leq^{d^+}$ we have already defined: $(x, r) \leq^{d^+} (y, s)$ if and only if $d(x, y) \leq r - s$.

With the open ball topology of d^+, $\mathbf{B}(X, d)$ is a *C-space*, as defined by Marcel Erné [Ern91]. In general, a C-space is a T_0 topological space Y that has the following strong form of local compactness: for every $y \in Y$ and every open neighborhood V of y, there is a point $z \in V$ such that y is in the interior of the compact set $\uparrow z$. (We write $\uparrow z$ for the upward closure of z.) This is particularly easy in the case of $\mathbf{B}(X, d)$: for every open neighborhood U (in the open ball topology of d^+) of a formal ball (x, r), U contains an open ball $B^{d^+}_{(x,r),<\epsilon}$, and we can take $z = (x, r + \epsilon/2)$; (x, r) is in the interior of $\uparrow z$ since (x, r) is in the open ball $B^{d^+}_{(x,r),<\epsilon/2}$, which is included in $\uparrow z$.

Writing $z \prec y$ for "y is in the interior of $\uparrow z$", that makes every C-space an abstract basis, and conversely, any abstract basis yields a C-space in a canonical way. A sober C-space is exactly the same as a continuous dcpo, the topology then being the Scott topology, and $\prec$ being the way-below relation $\ll$. In the case of $\mathbf{B}(X, d)$, $\prec$ is the relation we have already defined: $(x, r) \prec (y, s)$ if and only if $d(x, y) < r - s$. It follows that the Romaguera-Valero can be restated in the following synthetic form: X, d is Smyth-complete if and only if $\mathbf{B}(X, d)$ is *sober* in the open ball topology of d^+ [Gou13, Theorem 8.3.40]. Sobriety is a fundamental notion, and we refer the reader to [Gou13, Chapter 8] for more information on the topic.

Example 1.2. With the quasi-metric $\mathsf{d}_{\mathbb{R}}$, every closed interval $[a, b]$ is Smyth-complete. In fact, $[a, b]$ is even symcompact, namely, compact in the open ball topology of the symmetrized metric $\mathsf{d}^{sym}_{\mathbb{R}}$, and every symcompact space is Smyth-complete [Gou13, Lemma 7.2.21]. □

Example 1.3. $\mathbb{R}, \mathsf{d}_{\mathbb{R}}$ is not Smyth-complete, not even Yoneda-complete, and similarly for $\mathbb{R}_+, \mathsf{d}_{\mathbb{R}}$. Indeed, the chain of formal balls $(n, 0)$, $n \in \mathbb{N}$, does not have a supremum. □

Example 1.4. $\overline{\mathbb{R}}_+, \mathsf{d}_{\mathbb{R}}$, on the contrary, is Yoneda-complete. Noting that for $(x, r), (y, s) \in (\overline{\mathbb{R}}_+, \mathsf{d}_{\mathbb{R}})$, $(x, r) \leq^{\mathsf{d}^+_{\mathbb{R}}} (y, s)$ if and only if $r \geq s$ and $x - r \leq y - s$, the map $(x, r) \mapsto (x - r, -r)$ defines an order isomorphism from $\mathbf{B}(\overline{\mathbb{R}}_+, \mathsf{d}_{\mathbb{R}})$ onto $C = \{(a, b) \in (\mathbb{R} \cup \{+\infty\}) \times (-\infty, 0] \mid a - b \geq 0\}$. Since C is a Scott-closed subset of a continuous dcpo, it is itself a continuous dcpo. We shall see in Example 3.8 that this can be used to immediately conclude that $\mathbb{R}, \mathsf{d}_{\mathbb{R}}$

is a so-called *continuous* Yoneda-complete quasi-metric space. We can in fact say something stronger: $\overline{\mathbb{R}}_+, \mathsf{d}_\mathbb{R}$ is algebraic [Gou13, Exercise 7.4.64]. Since $(a, b) \ll (a', b')$ in C if and only if $a < a'$ and $b < b'$, we obtain that $(x, r) \ll (y, s)$ in $\mathbf{B}(\overline{\mathbb{R}}_+, \mathsf{d}_\mathbb{R})$ if and only if $x - r < y - s$ and $r > s$, if and only if $x \neq +\infty$ and $(x, r) \prec (y, s)$. Hence the relations $\ll$ and $\prec$ differ (although by a narrow margin), showing that $\overline{\mathbb{R}}_+, \mathsf{d}_\mathbb{R}$ is not Smyth-complete. □

Example 1.5. The *Sorgenfrey line* $\mathbb{R}_\ell$ is $\mathbb{R}, \mathsf{d}_\ell$, where $\mathsf{d}_\ell(x, y)$ is equal to $+\infty$ if $x > y$, and to $y - x$ if $x \leq y$. Its specialization ordering $\leq^{\mathsf{d}_\ell}$ is equality. Its open ball topology is the topology generated by the half-open intervals $[x, x + r)$, and is a well-known counterexample in topology. On formal balls, $(x, r) \leq^{\mathsf{d}_\ell^+} (y, s)$ if and only if $x \leq y$ and $x + r \geq y + s$. The map $(x, r) \mapsto (-x - r, x)$ therefore defines an order isomorphism from $\mathbf{B}(\mathbb{R}, \mathsf{d}_\ell)$ to $C_\ell = \{(a, b) \in \mathbb{R}^2 \mid a + b \leq 0\}$. Although $\mathbb{R}^2$ is not a dcpo, C_ℓ is, as one can see by realizing that C_ℓ is the Scott-closed subset of the continuous dcpo $(\mathbb{R} \cup \{+\infty\})^2$ consisting of the pairs (a, b) such that $a + b \leq 0$. As such, C is even a continuous dcpo, with $(a, b) \ll (a', b')$ if and only if $a < a'$ and $b < b'$. The way-below relation on $\mathbf{B}(\mathbb{R}, \mathsf{d}_\ell)$ is given by $(x, r) \ll (y, s)$ if and only if $x + r > y + s$ and $x < y$, if and only if $(x, r) \prec (y, s)$ and $x < y$. Since $\ll$ and $\prec$ differ, $\mathbb{R}_\ell$ is not Smyth-complete. One can show that it is even a non-algebraic Yoneda-complete quasi-metric space, see [KW10] or [Gou13, Exercise 7.4.73]. This example is due to Kostanek and Waszkiewicz, who also show that it is continuous Yoneda-complete, a fact we shall retrieve in Example 3.9. □

Example 1.6. Every poset X can be seen as a quasi-metric space by letting $d_\leq(x, y) = 0$ if $x \leq y$, $d(x, y) = +\infty$ otherwise. On formal balls, $(x, r) \leq^{d_\leq^+} (y, s)$ if and only if $x \leq y$ and $r \geq s$, so $(x, r) \mapsto (x, -r)$ defines an order isomorphism of $\mathbf{B}(X, d_\leq)$ onto $X \times (-\infty, 0]$. It follows that X is Yoneda-complete, as a quasi-metric space, if and only if X is a dcpo.

Note that $(x, r) \ll (y, s)$ in $\mathbf{B}(X, d)$ if and only if $x \ll y$ in X and $r > s$, if and only if $(x, r) \prec (y, s)$ and $x \ll y$. In particular, $\ll$ and $\prec$ only coincide on formal balls when $\leq$ and $\ll$ coincide on X, that is, X is Smyth-complete as a quasi-metric space if and only if it is an algebraic dcpo whose elements are all finite. Such posets are exactly those that have the *ascending chain condition*: every chain $x_1 \leq x_2 \leq \cdots \leq x_n \leq \cdots$ is finite. □

It has been argued that the proper topology one should take on a quasi-metric space X, d is not its open ball topology, but its *generalized Scott topology* [BvBR98]. We will use another one, which is arguably simpler to understand, and coincides with the latter in many cases.

The map $x \mapsto (x, 0)$ is an order embedding of $X, \leq^{d^+}$ into its poset of formal balls $\mathbf{B}(X, d)$. Accordingly, we shall consider X as a subset of $\mathbf{B}(X, d)$. The latter has a natural topology, the Scott topology, and this induces a topology on its subspace X:

Definition 1.7 (d-Scott Topology)**.** Let X, d be a quasi-metric space. The *d-Scott topology* on X is the induced topology from $\mathbf{B}(X, d)$, namely: the d-Scott opens subsets U of X are those such that there is a Scott-open subset V of $\mathbf{B}(X, d)$ such that $U = \{x \mid (x, 0) \in V\}$.

In general, the generalized Scott topology is finer than the d-Scott topology on X [Gou13, Exercise 7.4.51]. The two topologies coincide when X is algebraic Yoneda-complete [Gou13, Exercise 7.4.69].

The reader might be worried at this point that we are relinquishing the open ball topology in favor of a more exotic topology. However, the d-Scott topology is the *same* as the open ball topology if X, d is *metric* [Gou13, Proposition 7.4.46]. The two topologies also coincide if X, d is Smyth-complete [Gou13, Proposition 7.4.47], and in fact as soon as all the points of X are d-finite (comment after Proposition 7.4.48, op. cit.).

Example 1.8. For a poset X, seen as a quasi-metric space with the quasi-metric $d_\leq$ (Example 1.6), the $d_\leq$-Scott topology is just the same as the Scott topology. To show this, use the order isomorphism of $\mathbf{B}(X, d_\leq)$ onto $X \times (-\infty, 0]$. The d-Scott topology on X is then the topology induced by the embedding $x \mapsto (x, 0)$ of X into $X \times (-\infty, 0]$. We consider X as a subset of $X \times (-\infty, 0]$ through that embedding. For a Scott-open subset U of X, $U \times (-\infty, 0]$ is Scott-open in $X \times (-\infty, 0]$, and its intersection with X is U, so U is d-Scott open. Conversely, any d-Scott open U is of the form $V \cap X$ for some Scott open subset V of $X \times (-\infty, 0]$. It is an easy exercise to show that U is Scott-open, and this concludes the argument. □

Example 1.9. There are cases where the d-Scott topology and the open ball topology are very different. The most immediate examples are given by posets X, in which the $d_\leq$-Scott topology is the Scott topology, and the open ball topology is the Alexandroff topology, whose open sets are all the upwards-closed subsets. □

Example 1.10. For another example, consider the Sorgenfrey line $\mathbb{R}_\ell$. We have seen that its open ball topology has a basis of half-open intervals $[x, x + r)$. This is a bizarre topology: it is paracompact Hausdorff hence T_4 but the topological product of $\mathbb{R}_\ell$ with itself is not normal; it is first-countable but not countably-based; it is zero-dimensional, and not locally compact. The d_ℓ-Scott topology is tamer: it has a basis of opens of the form $\mathbb{R}_\ell \cap \mathrel{\rlap{\uparrow}\raise.4ex\hbox{$\uparrow$}}(x, r)$, namely the open intervals $(x, x + r)$, hence it is just the usual topology on $\mathbb{R}$. (We write $\mathrel{\rlap{\uparrow}\raise.4ex\hbox{$\uparrow$}}a$, in general, for $\{b \mid a \ll b\}$.) □

2. Standard Quasi-Metric Spaces

The Scott topology, algebraicity and continuity, are mostly studied on dcpos, not posets. However, those notions do make sense on general posets. Similarly, research on quasi-metric spaces, their spaces of formal balls, their generalized Scott topology, and so on, mostly focused on Yoneda-complete spaces, but those should have a meaning even in non-complete spaces.

However, non-complete spaces may exhibit a few pathologies that we would like to exclude. Here is how. We shall use the notions and results in subsequent sections.

Definition 2.1 (Standard Quasi-Metric Space)**.** A quasi-metric space X, d is *standard* if and only if, for every directed family of formal balls $(x_i, r_i)_{i \in I}$, for every $s \in \mathbb{R}_+$, $(x_i, r_i)_{i \in I}$ has a supremum in $\mathbf{B}(X, d)$ if and only if $(x_i, r_i + s)_{i \in I}$ has a supremum in $\mathbf{B}(X, d)$.

Many quasi-metric spaces are standard, as we observe now. Here and in the sequel, a *net* is a family $(z_i)_{i \in I, \sqsubseteq}$ of points z_i indexed by a set I equipped with a quasi-ordering $\sqsubseteq$ that makes I directed. A limit of that net is any point z such that every open neighborhood U of z contains z_i for i large enough.

Proposition 2.2. *Every metric space is standard. Every Yoneda-complete quasi-metric space is standard. Every poset is standard.*

Proof. When X, d is metric, we need to observe that: $(*)$ (x, r) is the supremum of the directed family $(x_i, r_i)_{i \in I}$ if and only if x is the limit of the net $(x_i)_{i \in I, \sqsubseteq}$ (in X with its open ball topology) and $r = \inf_{i \in I} r_i$ [EH98, Theorem 5]. The quasi-ordering $\sqsubseteq$ is defined by $i \sqsubseteq j$ if and only if $(x_i, r_i) \leq^{d^+} (x_j, r_j)$, and since that is directed, $(x_i)_{i \in I, \sqsubseteq}$ is a net. A consequence of $(*)$ is that $(x_i, r_i + s)_{i \in I}$ has a supremum (x', r') in $\mathbf{B}(X, d)$ if and only if x' is the limit of $(x_i)_{i \in I, \sqsubseteq}$ and $r' = \inf_{i \in I} r_i + s$. It follows that the existence of a supremum is equivalent for $(x_i, r_i)_{i \in I}$ and for $(x_i, r_i + s)_{i \in I}$, both being equivalent to the existence of a limit of the net $(x_i)_{i \in I, \sqsubseteq}$.

When X, d is Yoneda-complete, we use Lemma 7.4.25 and Lemma 7.4.26 of [Gou13], which together state the similar result that (x, r) is the supremum of the directed family $(x_i, r_i)_{i \in I}$ if and only if x is the d-limit of $(x_i)_{i \in I, \sqsubseteq}$ and $r = \inf_{i \in I} r_i$. The notion of d-limit is irrelevant for our purposes here. (We shall need it later, and we shall define what it is there.) The important point is that, as above, the existence of a supremum is equivalent for $(x_i, r_i)_{i \in I}$ and for $(x_i, r_i + s)_{i \in I}$, both being equivalent to the existence of a d-limit of the net $(x_i)_{i \in I, \sqsubseteq}$.

If X is a poset, we have seen in Example 1.6 that $(x, r) \mapsto (x, -r)$ defines an order-isomorphism from $\mathbf{B}(X, d_\leq)$ onto $X \times (-\infty, 0]$. Since suprema in the latter are taken componentwise, and since all suprema exist in $(-\infty, 0]$, the last claim is clear. $\square$

Remark 2.3. Not every quasi-metric space is standard. For a counterexample, let $X = [0, 1]$ with the quasi-metric defined by $d(x, y) = |x - y|$ if $x, y \neq 0$, $d(0, x) = a$ for every $x \neq 0$, and $d(x, 0) = 0$ for every x, where $a \geq 1$ is a fixed constant. (We need $a \geq 1$ to make sure the triangular inequality holds, notably $d(x, y) \leq d(x, 0) + d(0, y)$ for all x, y.)

The directed family of formal balls $(1/2^m, 1/2^m)$, $m \in \mathbb{N}$, has $(0, 0)$ has least upper bound, which one can show by verifying that $(0, 0)$ is in fast its sole upper bound. Indeed, if (x, r) is an upper bound, and $x \neq 0$, then $|1/2^m - x| \leq 1/2^m - r$ for every m, and as m tends to $+\infty$, this forces $x = r = 0$, a contradiction. If $x = 0$, then $d(1/2^m, x) = 0 \leq 1/2^m - r$, which forces $r = 0$.

For $s > 0$, the upper bounds of the directed family $(1/2^m, 1/2^m + s)$, $m \in \mathbb{N}$, are those formal balls (x, r) such that $x + r \leq s$. This proceeds in the same way as above. If (x, r) is an upper bound, and $x \neq 0$, then $|1/2^m - x| \leq 1/2^m + s - r$ for every m, and as m tends to $+\infty$, $x \leq s - r$. Conversely, if $x \leq s - r$, then $|1/2^m - x| \leq 1/2^m + x \leq 1/2^m + s - r$, so (x, r) is an upper bound of the family. If $x = 0$, then $d(1/2^m, x) = 0 \leq 1/2^m + s - r$ for every m implies $s \geq r$, which is again equivalent to $x + r \leq s$. Conversely, if $s \geq r$ then $d(1/2^m, x) = 0 \leq 1/2^m + s - r$ for every m.

For example, $(s/3, 2s/3)$ is such an upper bound. We shall see in Proposition 2.4 (2) that, if X, d were standard, then the least such upper bound would be $(0, s)$. Now pick s so that $0 < s < 3$: then $(0, s) \not\leq (s/3, 2s/3)$, since $d(0, s/3) = a \not\leq s - 2s/3$. This shows that $(0, s)$ is not least among all upper bounds of the family, hence that X, d cannot be standard.

Proposition 2.4. *In a standard quasi-metric space X, d, the following hold:*

(1) *for every directed family of formal balls $(x_i, r_i)_{i \in I}$ with supremum (x, r), $r = \inf_{i \in I} r_i$;*
(2) *for every directed family of formal balls $(x_i, r_i)_{i \in I}$ with supremum (x, r), for every $s \in \mathbb{R}$ such that $s \geq -r$, the supremum of $(x_i, r_i + s)_{i \in I}$ exists and is equal to $(x, r + s)$;*
(3) *the* radius map *$(x, r) \mapsto r$ is Scott-continuous from $\mathbf{B}(X, d)$ to $\mathbb{R}_+^{op}$ (the set of non-negative real numbers with the opposite ordering $\geq$);*
(4) *the map $\_ + s \colon (x, r) \mapsto (x, r + s)$ is Scott-continuous from $\mathbf{B}(X, d)$ to itself.*

Proof. (1) Let $r_\infty = \inf_{i \in I} r_i$. Observe that, for all formal balls, $(y, s) \leq^{d^+} (z, t)$ implies $s \geq t$. Since $(x_i, r_i) \leq^{d^+} (x, r)$, $r_i \geq r$ for every $i \in I$, so $r_\infty \geq r$. By Definition 2.1 with

$s = r_\infty - r$, the family $(x_i, r_i - r_\infty)_{i\in I}$ also has a supremum (x', r'), and by similar reasoning $r_i - r_\infty \geq r'$ for every i. This implies $0 = \inf_{i\in I} r_i - r_\infty \geq r'$, so $r' = 0$. In particular, $(x', 0)$ is an upper bound of $(x_i, r_i - r_\infty)_{i\in I}$, namely, $d(x_i, x') \leq r_i - r_\infty$ for every $i \in I$. Equivalently, $(x_i, r_i) \leq^{d^+} (x', r_\infty)$. It follows that (x', r_∞) is an upper bound of $(x_i, r_i)_{i\in I}$, hence is above the least one, (x, r). In particular, $r \geq r_\infty$. We have already proved the converse inequality, so $r = r_\infty$.

(2) By (1), $r = \inf_{i\in I} r_i$. By Definition 2.1, $(x_i, r_i + s)_{i\in I}$ has a supremum (x', r'), and by (1) again, $r' = \inf_{i\in I} r_i + s = r + s$. We now use the fact that $(x_i, r_i) \leq^{d^+} (x, r)$ for every $i \in I$, i.e., $d(x_i, x) \leq r_i - r = (r_i + s) - (r + s)$, from which it follows $(x_i, r_i + s) \leq^{d^+} (x, r + s)$ for every $i \in I$. The formal ball $(x, r + s)$ is an upper bound of $(x_i, r_i + s)_{i\in I}$, hence is above its least upper bound $(x', r + s)$: $d(x', x) \leq (r + s) - (r + s) = 0$. Working in the converse direction, the formal ball $(x', r) = (x', r + s + (-s))$ is an upper bound of $(x_i, r_i + s + (-s))_{i\in I} = (x_i, r_i)_{i\in I}$, hence is above (x, r), so that $d(x, x') \leq r - r = 0$. Because X, d is quasi-metric, and $d(x, x') = d(x', x) = 0$, $x = x'$.

(3) The radius map is monotonic, namely $(x, r) \leq^{d^+} (y, s)$ implies $r \geq s$ (recall $\mathbb{R}_+^{op}$ has the opposite ordering $\geq$), and what remains to be shown is (1).

(4) If $(x, r) \leq^{d^+} (x', r')$, then $d(x, x') \leq r - r' = (r + s) - (r' + s)$, so $(x, r + s) \leq^{d^+} (x', r')$. This shows that $\_ + s$ is monotone. Scott-continuity per se follows from (2). ☐

The mapping $x \mapsto (x, 0)$ allows us to see X (with the d-Scott topology) as a topological subspace of $\mathbf{B}(X, d)$ (with the Scott topology). If $(x, 0) \leq^{d^+} (y, s)$, then $0 \geq s$, so $s = 0$. It follows:

Fact 2.5. Let X, d be a quasi-metric space. Then X embeds as an upwards-closed subset of $\mathbf{B}(X, d)$.

When X, d is standard, we can say more.

Proposition 2.6. *Let X, d be a standard quasi-metric space. Then X embeds as a G_δ subset of $\mathbf{B}(X, d)$.*

Proof. Let $U_n = \{(x, r) \in \mathbf{B}(X, d) \mid r < 1/2^n\}$. This is the inverse image of $[0, 1/2^n)$ by the radius map. Since the former is open in the Scott topology of $\mathbb{R}_+^{op}$, and using Proposition 2.4 (3), U_n is open. Clearly, $X = \bigcap_{n\in\mathbb{N}} U_n$. ☐

3. Continuous Quasi-Metric Spaces

In domain theory, there are dcpos, continuous dcpos, and algebraic dcpos. There are quasi-metric analogies of each notion, and we have described them, except for continuous dcpos.

The definition of a *continuous* Yoneda-complete quasi-metric space stems from enriched category-theoretic considerations, and is pretty complicated. The first author claimed in [Gou13, Definition 7.4.72] that a quasi-metric space X, d is continuous Yoneda-complete, in the sense of Kostanek and Waszkiewicz, if and only $\mathbf{B}(X, d)$ is a continuous dcpo. That happens to be true, as we shall see, but no proof is given of that claim there. We repair this omission, and also deal not only with Yoneda-complete quasi-metric spaces, but with standard quasi-metric spaces.

In order to define continuous Yoneda-complete quasi-metric spaces, in principle, we need to first define the way-below $\mathcal{Q}$-relation $\mathbf{w}\colon X \times X \to \overline{\mathbb{R}}_+$. This is defined by an

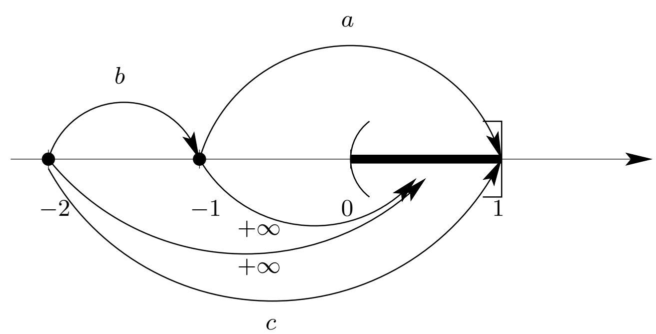

Figure 1: A standard quasi-metric space with a non-standard way-below relation on formal balls

enriched category-theoretic analogue of the definition of the usual way-below relation on posets. Fortunately, we will not need the actual definition. Kostanek and Waszkiewicz show that [KW10, Theorem 9.1]:

- If X, d is continuous Yoneda-complete then $\mathbf{B}(X, d)$ is a continuous dcpo with way-below relation given by $(x, r) \ll (y, s)$ if and only if $r > \mathbf{w}(x, y) + s$.
- If, for some map $v\colon X \times X \to \overline{\mathbb{R}}_+$, $\mathbf{B}(X, d)$ is a continuous dcpo and its way-below relation is characterized by $(x, r) \ll (y, s)$ if and only if $r > v(x, y) + s$, then X, d is a continuous Yoneda-complete quasi-metric space.

We propose a simpler characterization, as a first step towards our final simplification. The key notion is a new twist on standardness.

Definition 3.1. A binary relation R on $\mathbf{B}(X, d)$ is *standard* if and only if, for every $a \in \mathbb{R}_+$, $(x, r)\ R\ (y, s)$ if and only if $(x, r + a)\ R\ (y, s + a)$.

This can be generalized to relations of any arity, including infinite arities. This way, this new notion of standardness also encompasses Definition 2.1. Note that $\leq^{d^+}$ is always standard.

We are interested in the cases where $\ll$ is standard. Half of the equivalence defining standardness is automatic:

Lemma 3.2. *Let X, d be a standard quasi-metric space. For every $a \in \mathbb{R}_+$, if $(x, r + a) \ll (y, s + a)$ then $(x, r) \ll (y, s)$.*

Proof. Let $(z_i, t_i)_{i \in I}$ be a directed family of formal balls with a supremum (z, t) above (y, s). By Proposition 2.4 (2), $(z_i, t_i + a)_{i \in I}$ is a directed family, and its supremum $(z, t + a)$ is above $(y, s + a)$. Hence there is an index i such that $(y_i, s_i + a) \leq^{d^+} (z, t + a)$. It follows that $(y_i, s_i) \leq^{d^+} (z, t)$. □

The converse implication, namely that $(x, r) \ll (y, s)$ implies $(x, r + a) \ll (y, s + a)$, is wrong in general, as the following example shows.

Example 3.3. Let X be the disjoint union of $(0, 1]$ with the Sorgenfrey quasi-metric d_ℓ, with two extra points which we shall name -2 and -1. We extend d_ℓ to a quasi-metric d on X by letting $d(x, y) = +\infty$ if $x > y$, and when $x \leq y$ we let: $d(x, y) = y - x$ if $x, y \in (0, 1]$, $d(-1, y) = +\infty$ if $y \in (0, 1)$, $d(-1, 1) = a$, $d(-2, -1) = b$, $d(-2, y) = +\infty$ if $y \in (0, 1)$, $d(-2, 1) = c$, where a, b and c are elements of $\mathbb{R}_+$ such that $c \leq a + b$ and $a, b > 0$. See

Figure 1, where only the distances from x to y with $x < y$ are depicted. To check that it is a quasi-metric, note that $d(x, z) \leq d(x, y) + d(y, z)$ is trivial whenever the right-hand side is equal to $+\infty$, or when any two points from x, y, z are equal; so the only cases we have to check are those where $x < y < z$, and when $x = -2$, $y = -1$, $z = 1$, this requires the inequality $c \leq a + b$.

Given any directed family $(x_i, r_i)_{i \in I}$ of formal balls, observe that $(x_i, r_i) \leq^{d^+} (x_j, r_j)$ forces $x_i \leq x_j$, since otherwise $d(x_i, x_j) = +\infty \not\leq r_i - r_j$. Then, either some x_i is in $(0, 1]$, and we see that the cofinal family of those elements (x_i, r_i) with $x_i \in (0, 1]$ has a supremum which is given as in $\mathbb{R}_\ell$, namely (x, r) where $r = \inf_{i \in I} r_i$ and $x = \sup_{i \in} x_i$; or $(x_i, r_i)_{i \in I}$ is included in $\{-2, -1\}$ and some x_i equals -1, in which case there is a cofinal family where each x_i is equal to -1, so the supremum is $(-1, \inf_{i \in I,} r_i)$; or every x_i is equal to -2, and the supremum is $(-2, \inf_{i \in I} r_i)$. In any case, the family has a supremum, so $\mathbf{B}(X, d)$ is a dcpo, in other words X, d is Yoneda-complete. In particular, it is standard.

However, $\ll$ is not standard. To this end, we fix some arbitrary real number $b' > b$, and we check that $(-2, b') \ll (-1, 0)$. Assume a monotone net $(x_i, r_i)_{i \in I, \sqsubseteq}$ whose supremum (x, r) is above $(-1, 0)$. Since $d(-1, x) \leq 0 - r$, we must have $r = 0$, and then a case analysis on x shows that $x = -1$. (The inequality $a > 0$ serves to show that $x = 1$ is impossible.) Since $r = \inf_{i \in I} r_i$, r_i must be strictly less than $b > 0$ for i large enough, and then the inequality $(x_i, r_i) \leq^{d^+} (x, r) = (-1, 0)$ implies $x_i = -1$. Then $d(-2, -1) = b \leq b' - r_i$ for i large enough, which shows that $(-2, b') \leq^{d^+} (-1, r_i) = (x_i, r_i)$ for i large enough.

It remains to check that $(-2, b' + a)$ is not way-below $(-1, a)$. Let $(x_n, r_n) = (1 - 1/2^{n+1}, 1/2^{n+1})$, a directed family in X included in $(0, 1]$. Its supremum is $(x, r) = (1, 0)$, which is above $(-1, a)$ since $d(-1, 1) = a$. But no (x_n, r_n) is above $(-2, b' + a)$, since that would mean that $d(-2, 1 - 1/2^{n+1}) = +\infty$ would be at most $b' + a - 1/2^{n+1}$, which is impossible. $\square$

We need the following easy lemma for the next theorem, and for later results as well.

Lemma 3.4. *Let X, d be a quasi-metric space. For every $(y, s) \in \mathbf{B}(X, d)$, (y, s) is the supremum in $\mathbf{B}(X, d)$ of the chain $(y, s + 1/2^n)$, $n \in \mathbb{N}$.*

Proof. Clearly $(y, s + 1/2^n) \leq^{d^+} (y, s)$ for every $n \in \mathbb{N}$. For every upper bound (z, t) of the elements $(y, s + 1/2^n)$, $n \in \mathbb{N}$, $d(y, z) \leq s + 1/2^n - t$ for every $n \in \mathbb{N}$, so $d(y, z) \leq s - t$, showing that $(y, s) \leq^{d^+} (z, t)$. $\square$

Proposition 3.5. *A quasi-metric space X, d is a continuous Yoneda-complete quasi-metric space if and only if $\mathbf{B}(X, d)$ is a continuous dcpo with a standard way-below relation.*

Proof. If X, d is continuous Yoneda-complete, then $(x, r) \ll (y, s)$ if and only if $r < \mathbf{w}(x, y) + s$. This is clearly a standard relation.

Conversely, assume that $\mathbf{B}(X, d)$ is a continuous dcpo and that $\ll$ is standard. Let $v(x, y) = \inf\{r - s \mid (x, r) \ll (y, s)\}$, the infimum being equal to $+\infty$ if the right-hand set is empty. We claim that $(x, r) \ll (y, s)$ if and only if $r > v(x, y) + s$.

In one direction, assume $(x, r) \ll (y, s)$. By interpolation, find a formal ball (z, t) such that $(x, r) \ll (z, t) \ll (y, s)$. Using Lemma 3.4, $(z, t) \leq^{d^+} (y, s + 1/2^n)$ for some $n \in \mathbb{N}$, hence $(x, r) \ll (y, s + 1/2^n)$. By definition of v, $v(x, y) \leq r - s - 1/2^n$, so that $v(x, y) < r - s$.

Conversely, if $r > v(x, y) + s$, then by definition of v, there are numbers $r', s' \in \mathbb{R}^+$ such that $(x, r') \ll (y, s')$ and $r > r' - s' + s$. By Lemma 3.2, $(x, r' - s') \ll (y, 0)$, and since $\ll$ is standard, $(x, r' - s' + s) \ll (y, s)$. Using $r > r' - s' + s$, we obtain that $(x, r) \leq^{d^+} (x, r' - s' + s) \ll (y, s)$. $\square$

We now refine this by showing that the continuity of $\mathbf{B}(X,d)$ (for standard X,d) is enough to ensure that pathologies such as Example 3.3 do not actually happen.

Proposition 3.6. *Let X,d be a standard quasi-metric space. If $\mathbf{B}(X,d)$ is a continuous poset, then it has a standard way-below relation.*

Proof. Let $(x,r) \ll (y,s)$, and fix $a \in \mathbb{R}_+$. The family $\mathord{\downarrow\!\!\downarrow}(y, s+a)$ is directed, has $(y, s+a)$ as supremum, and consists of elements whose radius is at least $s + a \geq a$. Write that family $(z_i, t_i + a)_{i \in I}$. Note that $(z_i, t_i + a) \leq^{d^+} (z_j, t_j + a)$ if and only if $(z_i, t_i) \leq^{d^+} (z_j, t_j)$, because both are equivalent to $d(z_i, z_j) \leq t_i - t_j$. Therefore $(z_i, t_i)_{i \in I}$ is also a directed family. By Proposition 2.4 (2), it admits (y,s) as supremum. Since $(x,r) \ll (y,s)$, $(x,r) \leq^{d^+} (z_i, t_i)$ for some $i \in I$. It is easy to see that $(x, r+a) \leq^{d^+} (z_i, t_i + a)$. Since $(z_i, t_i + a) \in \mathord{\downarrow\!\!\downarrow}(y, s+a)$, $(x, r+a) \ll (y, s+a)$. ☐

Together with Proposition 3.5, we therefore obtain:

Theorem 3.7. *A quasi-metric space X,d is a continuous Yoneda-complete quasi-metric space if and only if $\mathbf{B}(X,d)$ is a continuous dcpo.* ☐

Example 3.8. Following up on Example 1.4, $\overline{\mathbb{R}}_+, \mathsf{d}_{\mathbb{R}}$ is continuous Yoneda-complete. It is even algebraic Yoneda-complete, as we shall see in Example 5.12. ☐

Example 3.9. $\mathbb{R}_\ell$ is continuous Yoneda-complete (see Example 1.5). ☐

Much as continuous dcpos can be generalized to continuous posets, this allows us to define continuous quasi-metric spaces without Yoneda continuity.

Definition 3.10. A standard quasi-metric space X,d is *continuous* if and only if $\mathbf{B}(X,d)$ is a continuous poset.

Example 3.11. Edalat and Heckmann noticed that, when X,d is a metric space, $\mathbf{B}(X,d)$ is always a continuous poset, with $(x,r) \ll (y,s)$ if and only if $d(x,y) < r - s$ [EH98]. It follows that all metric spaces are continuous. ☐

Example 3.12. Let us return to the case of posets (Example 1.6). For a poset X, recall that $\mathbf{B}(X, d_{\leq})$ is order-isomorphic to $X \times (-\infty, 0]$. The latter is continuous if and only if X is continuous. Hence a poset is continuous qua quasi-metric space if and only if it is continuous in the usual sense. ☐

4. Continuous Yoneda-complete Spaces are Sober, Choquet-complete

We now observe that continuous Yoneda-complete quasi-metric spaces have a number of desirable properties: they are sober, and they are Choquet-complete, in particular they are Baire.

Proposition 4.1. *Every continuous Yoneda-complete quasi-metric space X,d is sober in its d-Scott topology.*

Proof. Since $\mathbf{B}(X,d)$ is a continuous dcpo, it is sober in its Scott topology, see e.g. [AJ94, Proposition 7.2.27], [GHK+03, Corollary II-1.12], or [Gou13, Proposition 8.2.12 (b)]. Consider the diagram:

$$\mathbf{B}(X,d) \underset{0}{\overset{rad}{\rightrightarrows}} \mathbb{R}_+^{op}$$

where rad is the radius map and 0 is the constant 0 map. Both rad and 0 are continuous, using Proposition 2.4 (3). We claim that the map $\eta\colon x \in X \mapsto (x, 0)$ is an equalizer of that diagram in the category of topological spaces. Consider any continuous map $f\colon Y \to \mathbf{B}(X, d)$ such that $rad \circ f = 0 \circ f$. For every $y \in Y$, $f(y)$ is an element of the form $(x, 0)$, and we define $f'(y)$ as x. Clearly, $f = \eta \circ f'$. Moreover, f' is continuous: for every open subset U of X, by definition of the d-Scott topology there is a Scott-open subset V of $\mathbf{B}(X, d)$ such that $U = \eta^{-1}(V)$, and then ${f'}^{-1}(U) = f^{-1}(V)$ is open. The result follows because spaces obtained as equalizers of continuous maps from a sober space to a topological space are sober; see, e.g., [Gou13, Lemma 8.4.12]. □

Let us turn to Choquet completeness.

Given a topological space X, the *strong Choquet game* on X is played as follows. There are two players, α and β, who alternate turns. Player β starts and chooses a non-empty open subset V_0 of X, and a point $x_0 \in V_0$. Then α plays an open subset U_0 of V_0 containing x_0. Player β finds a non-empty open subset V_1 of U_0, and picks a point $x_1 \in V_1$, then α produces an open subset U_1 of V_1 containing x_1, and so on. Clearly, $\bigcap_{n\in\mathbb{N}} U_n = \bigcap_{n\in\mathbb{N}} V_n$, and we say that α wins the game if and only if that set is non-empty. A strategy for α is a map from *histories* $x_0, V_0, U_0, x_1, V_1, U_1, \cdots, x_n, V_n$ (namely, $n \in \mathbb{N}$, all U_i and V_i are open, $V_0 \supseteq U_0 \supseteq \cdots \supseteq U_{n-1} \supseteq V_n$, $x_0 \in U_0$, $\ldots$, $x_{n-1} \in U_{n-1}$, $x_n \in V_n$) to opens U_n such that $x_n \in U_n \subseteq V_n$. X is *Choquet-complete* if and only if α has a winning strategy. See [Gou13, Section 7.6].

Every continuous dcpo is Choquet-complete in its Scott topology, an observation due to K. Martin [Mar99]. Player α's winning strategy can even be chosen to be *stationary* [Gou13, Lemma 7.6.3], i.e., so that U_n depends only on β's last move x_n, V_n; and *convergent* [DM10], i.e., so that $(U_n)_{n\in\mathbb{N}}$, or equivalently $(V_n)_{n\in\mathbb{N}}$, is a neighborhood base of some element y. (We call a space *convergence Choquet-complete* if and only if α has a convergent winning strategy.) The argument is simple: given β's last move x_n, V_n, α picks an element $y_n \ll x_n$ such that $y_n \in V_n$, and plays $U_n = \mathop{\Uparrow} y_n$; then $y = \sup_{n\in\mathbb{N}} y_n$.

Every G_δ subset of a Choquet-complete space is Choquet complete. This is mentioned as Theorem 2.30 (*iv*) in [Mar99], and a proof can be found in [HKL90, Proposition 2.1 (*iii*)]. The same proof shows:

Lemma 4.2. *Every G_δ subset of a convergence Choquet-complete space is a convergence Choquet-complete subspace.*

Proof. Let G be a G_δ subset of a convergence Choquet-complete space X, and write G as the intersection of a decreasing sequence of opens $G_0 \supseteq G_1 \supseteq \cdots \supseteq G_n \supseteq \cdots$. For convenience, we assume $G_0 = X$. Every open subset V of G can be written as $V' \cap G$ for some open subset V' of X. Picking the largest such open for V', we can ensure that the assignment $V \mapsto V'$ is monotonic.

Assume a convergent winning strategy σ for α on X. We obtain a winning strategy for α on G as follows: on the history $x_0, V_0, U_0, x_1, V_1, U_1, \cdots, x_n, V_n$ (inside G), α computes $U_n = G \cap W_n$, where $W_n = G_n \cap \sigma(x_0, V'_0, U'_0 \cap G_0, x_1, V'_1, U'_1 \cap G_1, \cdots, x_n, V'_n)$, therefore simulating a play on X with strategy σ. (Note that each x_i, $1 \leq i \leq n-1$, is indeed in $U'_{i-1} \cap G_{i-1}$, since $x_i \in U_i = U'_i \cap G$, and $x_n \in V'_n$, $x_n \in G_n$. Note also that $W_n = U'_n \cap G_n$.) By assumption, $(V'_n)_{n\in\mathbb{N}}$ is a neighborhood base of some point $y \in X$. By construction, $y \in \bigcap_{n\in\mathbb{N}} W_n \subseteq \bigcap_{n\in\mathbb{N}} G_n = G$. Every open neighborhood V of y in G is such that $V'_n \subseteq V'$ for some $n \in \mathbb{N}$, so $V_n = V'_n \cap G \subseteq V' \cap G = V$, showing that $(V_n)_{n\in\mathbb{N}}$ is a neighborhood base of y in G. □

Recalling Proposition 2.6 and the fact that every Yoneda-complete space is standard, we obtain the following.

Theorem 4.3. *Every continuous Yoneda-complete quasi-metric space is convergence Choquet-complete in its d-Scott topology.*

Remark 4.4. Theorem 4.3 generalizes the fact that every Smyth-complete space is convergence Choquet-complete in its open ball topology [Gou13, Exercise 7.6.5]. Indeed, every Smyth-complete space is continuous Yoneda-complete, and its open ball topology coincides with its d-Scott topology.

Recall that every Choquet-complete space is Baire (see, e.g., [Gou13, Theorem 7.6.8]), namely, the intersection of countably many dense open subsets is dense.

Corollary 4.5. *Every continuous Yoneda-complete quasi-metric space is Baire in its d-Scott topology.*

5. Algebraic Quasi-Metric Spaces

The original definition of *algebraic* Yoneda-complete spaces is pretty complicated, and the point we would like to make here is that there is a simpler one, which extends naturally to non-complete spaces as well.

This is the point where we have to recapitulate the standard definitions. Fix a quasi-metric space X, d. A net $(x_i)_{i \in I, \sqsubseteq}$ is *Cauchy* if and only if for every $\epsilon > 0$, there is an $i_0 \in I$ such that for all $i, j \in I$ with $i_0 \sqsubseteq i \sqsubseteq j$, $d(x_i, x_j) < \epsilon$. A point $x \in X$ is the *d-limit* of the Cauchy net $(x_i)_{i \in I, \sqsubseteq}$ if and only if, for every $y \in X$, $d(x, y) = \limsup_{i \in I, \sqsubseteq} d(x_i, y)$. A Yoneda-complete space is a quasi-metric space where every Cauchy net has a d-limit.

The relation to formal balls is as follows. Observe that, if $(x_i, r_i)_{i \in I, \sqsubseteq}$ is any monotone net of formal balls such that $\inf_{i \in I} r_i = 0$, then $(x_i)_{i \in I}$ is Cauchy [Gou13, Lemma 7.2.7]. Such a monotone net is called a *Cauchy-weighted* net, as the numbers r_i act as weights that witness the fact that $(x_i)_{i \in I}$ is Cauchy. A net $(x_i)_{i \in I, \sqsubseteq}$ is *Cauchy-weightable* if and only if one can find weights r_i that make $(x_i, r_i)_{i \in I, \sqsubseteq}$ Cauchy-weighted. So every Cauchy-weightable net is Cauchy. The converse fails [Gou13, Exercise 7.2.12], but every Cauchy net has a Cauchy-weightable subnet [Gou13, Lemma 7.2.8], and they behave similarly as far as d-limits are concerned: if a Cauchy net has a d-limit, then all its subnets are Cauchy and have the same d-limit [Gou13, Exercise 7.4.7], and conversely, if a Cauchy subnet of a Cauchy net has a d-limit, then this is also a d-limit of the Cauchy net [Gou13, Lemma 7.4.6]. In particular, an equivalent definition of Yoneda-completeness is: every Cauchy-*weightable* net has a d-limit. This trick finds its roots in [EH98, Section 2.2].

Remark 5.1. While d-limits of Cauchy nets are defined through a limit superior, d-limits of Cauchy-weightable nets can be characterized by a simpler formula: given a Cauchy-weighted net $(x_i, r_i)_{i \in I, \sqsubseteq}$, x is the d-limit of $(x_i)_{i \in I, \sqsubseteq}$ if and only if, for every $y \in X$, $d(x, y) = \sup_{i \in I}(d(x_i, y) - r_i)$ [Gou13, Lemma 7.4.9]. Moreover, the latter is a directed supremum.

Example 5.2. In $\overline{\mathbb{R}}_+, \mathsf{d}_{\mathbb{R}}$, every net $(x_i)_{i \in I, \sqsubseteq}$ (even not Cauchy) has a $\mathsf{d}_{\mathbb{R}}$-limit, which is its limit superior [Gou13, Exercise 7.1.16]. Given a Cauchy-weighted net $(x_i, r_i)_{i \in I, \sqsubseteq}$, the $\mathsf{d}_{\mathbb{R}}$-limit of $(x_i)_{i \in I, \sqsubseteq}$ can be expressed as the simpler, directed supremum $\sup_{i \in I}(x_i - r_i)$.

More generally, the supremum (x, r) of any directed family $(x_i, r_i)_{i \in I}$ of formal balls in $\mathbf{B}(\overline{\mathbb{R}}_+, \mathsf{d}_\mathbb{R})$ is given by $r = \inf_{i \in I} r_i$, and x is the directed supremum $\sup_{i \in I}(x_i + r - r_i)$. □

Example 5.3. Look at the case of posets, seen as quasi-metric spaces. A net $(x_i)_{i \in I, \sqsubseteq}$ is Cauchy if and only if it is an eventually monotone net, that is, for i, j large enough, $i \sqsubseteq j$ implies $x_i \leq x_j$. It is Cauchy-weightable if and only if it is a monotone net. The notion of $d_\leq$-limit of Cauchy-weightable nets coincides with the notion of directed supremum. □

A point x of a quasi-metric space X, d is called *d-finite* if and only if, for every directed family $(y_i, s_i)_{i \in I}$ of open balls with a supremum of the form $(y, 0)$, $d(x, y)$ is the infimum of the filtered family $(d(x, y_i) + s_i)_{i \in I}$ of elements of $\overline{\mathbb{R}}_+$ [Gou13, Lemma 7.4.56]. This is not the standard definition, which involves limits inferiors and Cauchy nets, but it is closer to our needs.

A quasi-metric space X, d is *algebraic* if and only if every point is a d-limit of some Cauchy net of d-finite points. By the same argument as above, it is equivalent to require that every point be a d-limit of some Cauchy-*weightable* net of d-finite points.

We would like to offer a simpler view of those notions—at least on standard quasi-metric spaces—based on formal balls. The starting point is the following. We have two topologies on $\mathbf{B}(X, d)$, the open ball topology of d^+, and the Scott topology of $\leq^{d^+}$. They have the same specialization ordering, $\leq^{d^+}$, and the former is finer than the latter, as we see now.

Lemma 5.4. *Let X, d be a quasi-metric space. The open ball topology on $\mathbf{B}(X, d), d^+$ is finer than the Scott topology.*

Proof. Let U be a Scott-open subset of $\mathbf{B}(X, d)$, and (y, s) be a point in U. By Lemma 3.4, $(y, s+1/2^n)$ is in U for some $n \in \mathbb{N}$. We check easily that (y, s) is in the open ball $B^{d^+}_{(y,s),<1/2^n}$ and that the latter open ball is included in $\Uparrow(y, s + 1/2^n)$, which is included in U. Hence we have found an open neighborhood of (y, s) for the open ball topology that is included in U. It follows that U is open in the open ball topology. □

Conversely, one may wonder when the open ball topology coincides with the Scott topology on $\mathbf{B}(X, d)$. We shall solve this question below.

In the meantime, observe that we may separate the open balls in $\mathbf{B}(X, d), d^+$ into the nice open balls, those which are Scott-open, and those that are not nice. We may then consider a point $x \in X$ as nice if and only if all the open balls centered at $(x, 0)$ are Scott-open. Let us give a name to that notion.

Definition 5.5 (Center point)**.** In a quasi-metric space X, d, an element $x \in X$ is a *center point* if and only if, for every $\epsilon > 0$, the open ball $B^{d^+}_{(x,0),<\epsilon}$ is Scott-open in $\mathbf{B}(X, d)$.

The intersection of $B^{d^+}_{(x,0),<\epsilon}$ with X is just $B^d_{x,<\epsilon}$, making the following immediate.

Remark 5.6. For every center point x of a hemi-metric space X, d, every open ball $B^d_{x,<\epsilon}$ centered at x is open in the d-Scott topology.

The following shows that we can replace the complex definition of "d-finite" by the more synthetic notion of center point, in standard spaces. The proof also shows that every center point is d-finite, even without assuming standardness.

Lemma 5.7. *Let X, d be a standard quasi-metric space. The following are equivalent, for every $x \in X$:*

(1) *x is d-finite;*
(2) *for all $r \in \mathbb{R}_+$ and $\epsilon > 0$, the open ball $B^{d^+}_{(x,r),<\epsilon}$ is Scott-open in $\mathbf{B}(X,d)$;*
(3) *x is a center point.*

A related result was given in [KW10, Theorem 8.3], who instead show that all points are d-finite if and only if all points are center points.

Proof. (3) $\Rightarrow$ (1). Let $(y_i, s_i)_{i \in I}$ be a directed family of open balls with supremum $(y, 0)$. Since $(y_i, s_i) \leq^{d^+} (y, 0)$ for every $i \in I$, $d(y_i, y) \leq s_i$, so $d(x, y) \leq d(x, y_i) + s_i$, by the triangular inequality. Assume the inequality $d(x, y) \leq \inf_{i \in I}(d(x, y_i) + s_i)$ were strict: $d(x, y) < \epsilon \leq \inf_{i \in I}(d(x, y_i) + s_i)$ for some $\epsilon > 0$. Then $(y, 0)$ would be in $B^{d^+}_{(x,0),<\epsilon}$, so some (y_i, s_i) would be in $B^{d^+}_{(x,0),<\epsilon}$, which is Scott-open by assumption. That is, $d(x, y_i) + s_i < \epsilon$. However, that would contradict the inequality $\epsilon \leq \inf_{i \in I}(d(x, y_i) + s_i)$. Therefore $d(x, y) = \inf_{i \in I}(d(x, y_i) + s_i)$, showing that x is d-finite.

(1) $\Rightarrow$ (2), assuming X, d standard. Let x be a d-finite point of X. The open ball $B^{d^+}_{(x,r),<\epsilon}$ is the set of formal balls (y, s) such that $d^+((x, r), (y, s)) < \epsilon$, i.e., such that $d(x, y) - r + s < \epsilon$. This is upwards-closed with respect to $\leq^{d^+}$, by the triangular inequality. Let (y, s) be the supremum of some directed family $(y_i, s_i)_{i \in I}$ of formal balls, and assume that (y, s) is in $B^{d^+}_{(x,r),<\epsilon}$. By Proposition 2.4 (2), $(y, 0) = \sup_{i \in I}(y_i, s_i - s)$, and the fact that x is d-finite then implies that $d(x, y) = \inf_{i \in I}(d(x, y_i) + s_i - s)$. Using the inequality $d(x, y) - r + s < \epsilon$, there must exist an $i \in I$ such that $d(x, y_i) + s_i - r < \epsilon$, i.e., such that (y_i, s_i) is in $B^{d^+}_{(x,r),<\epsilon}$. In other words, $B^{d^+}_{(x,r),<\epsilon}$ is Scott-open.

(2) $\Rightarrow$ (3) is trivial: take $r = 0$. □

Further characterizations follow, assuming X, d continuous.

Lemma 5.8. *Let X, d be a continuous quasi-metric space. The following are equivalent, for every $x \in X$:*
(1) *x is a center point;*
(2) *$v(x, \_) = d(x, \_)$, where $v(x, y)$ is defined as $\inf\{r - s \mid (x, r) \ll (y, s)\}$;*
(3) *for every $\epsilon > 0$, $B^{d^+}_{(x,0),<\epsilon} = \mathrel{\rlap{\uparrow}{\raise.4ex\hbox{$\uparrow$}}}(x, \epsilon)$.*

(The map v was already used in the proof of Proposition 3.5.)

Proof. (1) $\Rightarrow$ 2. If $(x, r) \ll (y, s)$, then $(x, r) \leq^{d^+} (y, s)$, so $d(x, y) \leq r - s$. Taking infima, $d(x, y) \leq v(x, y)$. If the inequality were strict, there would be a positive real ϵ such that $d(x, y) < \epsilon \leq v(x, y)$. In particular, $(y, 0) \in B^{d^+}_{(x,0),<\epsilon}$. Notice that $B^{d^+}_{(x,0),<\epsilon} \subseteq \uparrow(x, \epsilon)$: every element (z, t) of $B^{d^+}_{(x,0),<\epsilon}$ is such that $d(x, z) + t < \epsilon$, so $(x, \epsilon) \leq^{d^+} (z, t)$. Since x is a center point, $B^{d^+}_{(x,0),<\epsilon}$ is included in the Scott interior of $\uparrow(x, \epsilon)$, and that is $\mathrel{\rlap{\uparrow}{\raise.4ex\hbox{$\uparrow$}}}(x, \epsilon)$: that the Scott interior of $\uparrow a$ is $\mathrel{\rlap{\uparrow}{\raise.4ex\hbox{$\uparrow$}}} a$ is true in every continuous poset [Gou13, Proposition 5.1.35]. We have therefore shown that $(y, 0) \in \mathrel{\rlap{\uparrow}{\raise.4ex\hbox{$\uparrow$}}}(x, \epsilon)$, that is, $(x, \epsilon) \ll (y, 0)$. It follows that $v(x, y) \leq \epsilon$, a contradiction.

(2) $\Rightarrow$ (3). For every $(y, s) \in B^{d^+}_{(x,0),<\epsilon}$, $d(x, y) + s < \epsilon$, and using (2), $v(x, y) + s < \epsilon$. That implies $\epsilon > s$, and the existence of $r', s' \in \mathbb{R}_+$ such that $(x, r') \ll (y, s')$ and $r' - s' < \epsilon - s$. Since $\ll$ is standard by Proposition 3.6, $(x, \epsilon) \ll (y, s' + \epsilon - r')$. Since $r' - s' < \epsilon - s$, $s' + \epsilon - r' > s$ and in particular the formal ball $(y, s' + \epsilon - r')$ makes sense; further,

$(y, s' + \epsilon - r') \leq^{d^+} (y, s)$, so $(x, \epsilon) \ll (y, s)$. It follows that $B^{d^+}_{(x,0),<\epsilon} \subseteq \mathord{\Uparrow}(x, \epsilon)$. The converse direction is simpler. For every $(y, s) \in \mathord{\Uparrow}(x, \epsilon)$, there is an $n \in \mathbb{N}$ such that $(x, \epsilon) \leq^{d^+} (y, s + 1/2^n)$ by Lemma 3.4. That implies $d(x, y) \leq \epsilon - s - 1/2^n < \epsilon - s$, whence $(y, s) \in B^{d^+}_{(x,0),<\epsilon}$.

(3) $\Rightarrow$ (1) is obvious. □

Example 5.9. The $\mathsf{d}_\mathbb{R}$-finite points of $\overline{\mathbb{R}}_+$ are known to be those that are different from $+\infty$. Equivalently, let us check that they are the center points. We already know from Example 3.8 that $v(x, \_) = \mathsf{d}_\mathbb{R}(x, \_)$ if and only if $x \neq +\infty$. It is now enough to apply Lemma 5.8. □

Example 5.10. *No* point of $\mathbb{R}_\ell$ is d_ℓ-finite [Gou13, Exercise 7.4.73]. We check this here. Recalling Example 3.9, $\mathbb{R}_\ell$ is continuous Yoneda-complete, and we see that $v(x, \_)$ is never equal to the map $\mathsf{d}_\ell(x, \_)$, since $v(x, x) = +\infty$ but $\mathsf{d}_\ell(x, x) = 0$. □

To capture algebraicity, we define the following notion.

Definition 5.11 (Enough center points)**.** A quasi-metric space X, d *has enough center points* if and only if the open balls $B^{d^+}_{(x,0),<\epsilon}$, with x a center point and $\epsilon > 0$, form a base of the Scott topology on $\mathbf{B}(X, d)$.

In concrete terms, X, d has enough center points if and only if, for every Scott-open subset $\mathcal{U}$ of $\mathbf{B}(X, d)$, for every $(y, s) \in \mathcal{U}$, there is a center point x and an $\epsilon > 0$ such that $(y, s) \in B^{d^+}_{(x,0),<\epsilon} \subseteq \mathcal{U}$. This is what we shall use in proofs.

Example 5.12. $\overline{\mathbb{R}}_+, \mathsf{d}_\mathbb{R}$ has enough center points. Indeed, recall that $\mathbf{B}(\overline{\mathbb{R}}_+, \mathsf{d}_\mathbb{R})$ is isomorphic to the closed set $C = \{(a, b) \in (\mathbb{R} \cup \{+\infty\}) \times (-\infty, 0] \mid a - b \geq 0\}$ of Example 1.4. The latter is a continuous dcpo with $(a, b) \ll (a', b')$ if and only if $a < a'$ and $b < b'$, and therefore the points $(a, b) \in C$ with $a \neq +\infty$ form a basis. Using the isomorphism, a basis of $\mathbf{B}(\overline{\mathbb{R}}_+, \mathsf{d}_\mathbb{R})$ consists of all the formal balls (x, r) with $x \neq +\infty$, hence a base of the Scott topology is given by the sets $\mathord{\Uparrow}(x, r)$, $x \neq +\infty$. Using Lemma 5.8, the latter are the open balls $B^{d^+}_{(x,0),<r}$, and x is a center point, as we have seen in Example 5.9. □

Example 5.13. Since no point of $\mathbb{R}_\ell$ is finite, $\mathbb{R}_\ell$ is far from having enough center points. We shall see that having enough center points is equivalent to being algebraic, and thus we retrieve the fact that $\mathbb{R}_\ell$ is not algebraic ([KW10], see also [Gou13, Exercise 7.4.73]). □

Proposition 5.14. *Let X, d be a quasi-metric space with enough center points. The open balls $B^d_{x,<\epsilon}$ with x a center point and $\epsilon > 0$ form a base of the d-Scott topology on X. If X, d is standard, then we can additional require that all the involved radii ϵ are less than some prescribed upper bound $b > 0$.*

Proof. Given $x \in X$ and an open neighborhood U of x, let $\mathcal{U}$ be some Scott-open subset of $\mathbf{B}(X, d)$ whose intersection with X equals U. By definition, there is an open ball $B^{d^+}_{(a,0),<\epsilon}$ that contains $(x, 0)$ and is included in $\mathcal{U}$, where a is a center point and $\epsilon > 0$. Taking intersections with X, $B^d_{a,<\epsilon}$ is an open neighborhood of x that is included in U.

If X, d is standard, then for $b \in \mathbb{R}_+$, $b > 0$, the set $V_b = \{(y, s) \mid s < b\}$ is open, as the inverse image of $[0, b)$ by the radius map, which is continuous by Proposition 2.4 (3). Then $\mathcal{U} \cap V_b$ is also Scott-open and contains $(x, 0)$, so there is an open ball $B^{d^+}_{(a,0),<\epsilon}$, where a is a center point, that contains $(x, 0)$ and is included in $\mathcal{U} \cap V_b$. As above, we conclude that $B^d_{a,<\epsilon}$

is an open neighborhood of x that is included in U. Additionally, since $(a, s) \in B^{d^+}_{(a,0),<\epsilon} \subseteq V_b$ for every $s < \epsilon$, ϵ is less than or equal to b. $\square$

We have already announced the following result, as part of the Kostanek-Waszkiewicz theorem, assuming X, d Yoneda-complete. We now show that this holds in all standard quasi-metric spaces, even not Yoneda-complete.

Lemma 5.15. *Let X, d be a standard quasi-metric space. For every Cauchy-weighted net $(x_i, r_i)_{i \in I, \sqsubseteq}$, a point $x \in X$ is a d-limit of $(x_i)_{i \in I, \sqsubseteq}$ if and only if $(x, 0)$ is the supremum of the directed family $(x_i, r_i)_{i \in I}$ in $\mathbf{B}(X, d)$.*

Proof. Assume x is a d-limit of $(x_i)_{i \in I, \sqsubseteq}$. For every formal ball (y, s), (y, s) is an upper bound of $(x_i, r_i)_{i \in I}$ if and only if $d(x_i, y) \leq r_i - s$ for every $i \in I$, if and only if $s = 0$ (using $\inf_{i \in I} r_i = 0$) and $d(x_i, y) - r_i \leq 0$ for every $i \in I$. Since, by Remark 5.1, $d(x, y) = \sup_{i \in I}(d(x_i, y) - r_i)$, (y, s) is an upper bound of $(x_i, r_i)_{i \in I}$ if and only if $s = 0$ and $d(x, y) = 0$, and that is equivalent to $d(x, y) \leq 0 - s$, i.e., to $(x, 0) \leq^{d^+} (y, s)$. The least such formal ball (y, s) is then $(x, 0)$.

Conversely, assume that $(x, 0)$ is the least upper bound of $(x_i, r_i)_{i \in I}$.

The inequality $d(x, y) \geq \sup_{i \in I}(d(x_i, y) - r_i)$ is automatic: since $(x_i, r_i) \leq^{d^+} (x, 0)$, $d(x_i, x) \leq r_i$ for every $i \in I$, so $d(x_i, y) \leq d(x_i, x) + d(x, y) \leq d(x, y) + r_i$, which entails that $d(x_i, y) - r_i \leq d(x, y)$ for every $i \in I$. If the inequality were strict, there would be an $r \in \mathbb{R}_+$ such that $d(x, y) > r \geq \sup_{i \in I}(d(x_i, y) - r_i)$. We use Proposition 2.2 (2) and note that (x, r) is the supremum of $(x_i, r_i + r)_{i \in I}$. For every $i \in I$, $d(x_i, y) - r_i \leq r$, so $(x_i, r_i + r) \leq^{d^+} (y, 0)$, and therefore $(y, 0)$ is an upper bound of the family $(x_i, r_i)_{i \in I}$. Since (x, r) is the least one, $(x, r) \leq^{d^+} (y, 0)$, that is, $d(x, y) \leq r$. That contradicts $d(x, y) > r$, so the inequality is an equality, showing that x is a d-limit of $(x_i)_{i \in I, \sqsubseteq}$. $\square$

Theorem 5.16. *A standard quasi-metric space X, d has enough center points if and only if it is algebraic.*

Proof. Assume X, d is algebraic, let $\mathcal{U}$ be a Scott-open subset of $\mathbf{B}(X, d)$ and (y, s) be a formal ball in $\mathcal{U}$. By definition, y is the d-limit of some Cauchy net consisting of d-finite points, and we have seen that we could replace that Cauchy net with a Cauchy-weightable subnet. By Lemma 5.15, we can therefore express $(y, 0)$ as the supremum of some directed family $(x_i, r_i)_{i \in I}$, where each x_i is d-finite. By Proposition 2.4 (2), (y, s) is the supremum of the directed family $(x_i, r_i + s)_{i \in I}$. Therefore $(x_i, r_i + s)$ is in $\mathcal{U}$ for some $i \in I$. Using Lemma 3.4, there is even an $\epsilon > 0$ such that $(x_i, r_i + s + \epsilon)$ is in $\mathcal{U}$. Since $(x_i, r_i + s) \leq^{d^+} (y, s)$, $d(x_i, y) \leq (r_i + s) - s < (r_i + s) - s + \epsilon$, so (y, s) is in $B^{d^+}_{(x_i,0),<r_i+s+\epsilon}$. Every formal ball (z, t) in $B^{d^+}_{(x_i,0),<r_i+s+\epsilon}$ is such that $d(x_i, z) < r_i + s - t + \epsilon$, and that implies $(x_i, r_i + s + \epsilon) \leq^{d^+} (z, t)$. Since $(x_i, r_i + s + \epsilon)$ is in $\mathcal{U}$, and open subsets are upwards-closed, (z, t) is in $\mathcal{U}$, too. We sum up: $(y, s) \in B^{d^+}_{(x_i,0),<r_i+s+\epsilon} \subseteq \mathcal{U}$; since x_i is d-finite (equivalently, a center point, see Lemma 5.7), X, d has enough center points.

Conversely, assume that X, d has enough center points. Fix $x \in X$. Let I be the family of all non-empty finite sets of open neighborhoods of $(x, 0)$ in $\mathbf{B}(X, d)$, ordered by set inclusion. To stress it, an element i of I is a finite set $\{\mathcal{U}_1, \cdots, \mathcal{U}_n\}$ where $n \geq 1$ and each $\mathcal{U}_i$ is a Scott-open set of formal balls containing $(x, 0)$. By induction on the cardinality n of $i = \{\mathcal{U}_1, \cdots, \mathcal{U}_n\}$, we build a formal ball (x_i, r_i), where x_i is a center point, in such a way that $(x, 0) \in B^{d^+}_{(x_i,0),<r_i} \subseteq \mathcal{U}_1 \cap \cdots \cap \mathcal{U}_n \cap \bigcap_{j \subseteq i, j \neq \emptyset, j \neq i} B^{d^+}_{(x_j,0),<r_j}$. To do so, just observe that

X, d has enough center points, and that, by induction hypothesis each $B^{d^+}_{(x_j,0),<r_j}$ is an open neighborhood of $(x, 0)$.

If $j \subseteq i$, then $B^{d^+}_{(x_i,0),<r_i} \subseteq B^{d^+}_{(x_j,0),<r_j}$. For every $\epsilon > 0$ such that $\epsilon \leq r_i$, $(x_i, r_i - \epsilon)$ is in $B^{d^+}_{(x_i,0),<r_i}$, hence in $B^{d^+}_{(x_j,0),<r_j}$, so $d(x_j, x_i) + r_i - \epsilon < r_j$. As ϵ tends to 0, $d(x_j, x_i) \leq r_j - r_i$, and this shows that the net $(x_i, r_i)_{i \in I, \subseteq}$ is a monotone net. By construction, any upper bound (z, t) of that net is in every open neighborhood $\mathcal{U}$ of $(x, 0)$, hence must be above $(x, 0)$. By construction again, for every $i \in I$, $(x, 0) \in B^{d^+}_{(x_i,0),<r_i}$, so $d(x_i, x) < r_i$, which implies $(x_i, r_i) \leq^{d^+} (x, 0)$, therefore showing that $(x, 0)$ is an upper bound of the net. We have seen that any upper bound (z, t) would be above $(x, 0)$, so $(x, 0)$ is the supremum of the directed family of formal balls (x_i, r_i). Since every x_i is a center point, namely, a d-finite point, we conclude. □

As an application, recall that a quasi-metric space X, d is Smyth-complete if and only if $\mathbf{B}(X, d)$ is sober in its open ball topology. If that is the case, then $\mathbf{B}(X, d)$ is a monotone convergence space, and in particular *every* open ball of $\mathbf{B}(X, d), d^+$ is Scott-open. By definition, this implies that *every* point of X is a center point.

Conversely, if every point of X is a center point, then the open ball topology on $\mathbf{B}(X, d), d^+$ is coarser than the Scott topology, and by Lemma 5.4, the two topologies coincide. Since $\mathbf{B}(X, d)$ is a C-space in its open ball topology, it is also a C-space in its Scott topology. But the posets that are C-spaces in their Scott topology are exactly the continuous posets [Ern05, Proposition 4]. If X, d is also Yoneda-complete, then $\mathbf{B}(X, d)$ is a continuous dcpo, hence its Scott topology is sober. The open ball topology on $\mathbf{B}(X, d)$ is then sober as well, hence X, d is Smyth-complete by the Romaguera-Valero theorem.

We have therefore obtained a proof of the following. The equivalence between (1) and (2) is, modulo some details, due to Ali-Akbari, Honarii, Pourmahdian, and Rezaii [AAHPR09]. What they state is that X, d is Smyth-complete if and only if it is Yoneda-complete and all its points are d-finite. This is equivalent, since Yoneda-complete spaces are standard, and d-finite points coincide with center points in standard spaces.

Proposition 5.17. *For a quasi-metric space X, d, the following are equivalent:*
(1) *X, d is Smyth-complete;*
(2) *X, d is Yoneda-complete and all its points are center points;*
(3) *$\mathbf{B}(X, d)$ is a dcpo, and the open ball topology on $\mathbf{B}(X, d), d^+$ coincides with the Scott topology.*

As another application, the following generalizes the fact that every algebraic Yoneda-complete quasi-metric space is continuous.

Proposition 5.18. *Every (standard) algebraic quasi-metric space X, d is continuous. When z is a center point, $(z, t) \ll (y, s)$ if and only if $(z, t) \prec (y, s)$, if and only if $d(z, y) < t - s$. In general, $(x, r) \ll (y, s)$ if and only there is a center point z and some $t \in \mathbb{R}_+$ such that $(x, r) \leq^{d^+} (z, t) \prec (y, s)$.*

Proof. For every center point z, for every $t \in \mathbb{R}_+$, $B^{d^+}_{(z,0),<t} = \{(y, s) \mid d(z, y) < t - s\}$ is Scott-open by definition, and is included in $\Uparrow(z, t)$. Therefore $B^{d^+}_{(z,0),<t}$ is included in the interior of $\Uparrow(z, t)$. Conversely, if (y, s) is in the interior of $\Uparrow(z, t)$, then $(y, s + 1/2^n)$ is in $\Uparrow(z, t)$ by Lemma 3.4, so $d(z, y) \leq t - s - 1/2^n < t - s$.

It follows that $(z,t) \ll (y,s)$ if and only if (y,s) is in the interior of $\uparrow\!(z,t)$, if and only if $(y,s) \in B^{d^+}_{(z,0),<t}$, if and only if $d(z,y) < t - s$.

Fix a formal ball (y,s). Since X,d is algebraic, y is the d-limit of some Cauchy-weightable net of d-finite points. That is, $(y,0)$ is the supremum of some directed family $(z_i,t_i)_{i\in I}$ where each z_i is finite (equivalently, a center point). Since X,d is standard, (y,s) is the supremum of the directed family $(z_i,t_i+s)_{i\in I}$.

The family $(z_i,t_i+s+1/2^n)_{i\in I,n\in\mathbb{N}}$ is again directed: given $i,j \in I$ and $m,n \in \mathbb{N}$, find $k \in I$ such that $(z_i,t_i+s),(z_j,t_j+s) \leq^{d^+} (z_k,t_k+s)$, and let $p = \min(m,n)$, then $(z_i,t_i+s+1/2^m),(z_j,t_j+s+1/2^n) \leq^{d^+} (z_k,t_k+s+1/2^p)$. The upper bounds of $(z_i,t_i+s+1/2^n)_{i\in I,n\in\mathbb{N}}$ are exactly those of $(z_i,t_i+s)_{i\in I}$, so $(z_i,t_i+s+1/2^n)_{i\in I,n\in\mathbb{N}}$ admits (y,s) as upper bound. Moreover, since $(z_i,t_i+s) \leq^{d^+} (y,s)$, $d(z_i,y) \leq t_i < t_i + 1/2^n$, hence $(z_i,t_i+s+1/2^n) \ll (y,s)$. This allows us to conclude that $\mathbf{B}(X,d)$ is continuous, hence that X,d is continuous.

Finally, for general formal balls (x,r) and (y,s), we show that $(x,r) \ll (y,s)$ if and only if there is a formal ball (z,t) such that $(x,r) \leq^{d^+} (z,t)$ and $d(z,y) < t - s$. If $(x,r) \ll (y,s)$, since (y,s) is the directed supremum of a family of formal balls (z,t) (z center point) way-below (y,s), $(x,r) \leq^{d^+} (z,t) \ll (y,s)$ for some such formal ball. The converse direction is obvious. ☐

6. Continuous and Lipschitz Real-Valued Maps

It is time we talked about morphisms.

Given $\alpha \in \mathbb{R}_+$, and two quasi-metric spaces X,d and Y,∂, a map $f\colon X \to Y$ is α-*Lipschitz* if and only if $\partial(f(x),f(x')) \leq \alpha\ d(x,x')$ for all $x,x' \in X$. (When $\alpha = 0$ and $d(x,x') = +\infty$, we take the convention that $0.+\infty = +\infty$.) It is *Lipschitz* if and only if it is α-Lipschitz for some $\alpha \in \mathbb{R}_+$.

Example 6.1. If X and Y are posets, considered as quasi-metric spaces, the α-Lipschitz maps for $\alpha > 0$ are the monotonic maps. ☐

We are particularly interested in the case $\alpha = 1$; 1-Lipschitz maps are sometimes called *non-expansive*. Quasi-metric spaces and 1-Lipschitz maps form a category **QMet**.

Remark 6.2. We have repeatedly taken the example of posets as specific quasi-metric spaces. One can see the category of posets and monotonic maps as a full subcategory of **QMet**, consisting of those quasi-metric spaces whose quasi-metric takes the values 0 and $+\infty$ only. It is a remarkable fact that the posets are exactly the exponentiable objects in **QMet** [Gou13, Exercise 6.6.18].

Given $f\colon X \to Y$, and $\alpha \in \mathbb{R}_+$, we can lift f to spaces of formal balls by defining $\mathbf{B}^\alpha(f)\colon \mathbf{B}(X,d) \to \mathbf{B}(Y,\partial)$ as: $\mathbf{B}^\alpha(f)(x,r) = (f(x),\alpha r)$. (In particular, $\mathbf{B}^1(f)(x,r) = (f(x),r)$.) It is easy to verify that f is α-Lipschitz if and only if $\mathbf{B}^\alpha(f)$ is monotonic.

All Lipschitz maps are continuous with respect to the underlying open ball topologies, but not so for d-Scott topologies. Bonsangue *et al.* defined the following notion of Yoneda-continuity [BvBR98]. Given a Lipschitz map f as above, f is *Yoneda-continuous* if and only if f maps d-limits of Cauchy nets in X to ∂-limits in Y. The notion applies, in more generality, not just to Lipschitz maps, but to all uniformly continuous maps (op. cit.), but we will not consider that case here. Once again, formal balls offer a reduction of the quasi-metric concept to a domain-theoretic notion: when X, d and Y, ∂ are both Yoneda-complete, and given an α-Lipschitz map $f: X \to Y$, f is Yoneda-continuous if and only if $\mathbf{B}^\alpha(f)$ is Scott-continuous [Gou13, Proposition 7.4.38]. That extends to standard quasi-metric spaces:

Lemma 6.3. *Let X, d and Y, ∂ be two standard quasi-metric spaces. For every map $f: X \to Y$, f is α-Lipschitz Yoneda-continuous if and only if $\mathbf{B}^\alpha(f)$ is Scott-continuous.*

Proof. If f is α-Lipschitz Yoneda-continuous, then $\mathbf{B}^\alpha(f)$ is monotonic. Given a formal ball (x, r) that is the supremum of a directed family $(x_i, r_i)_{i \in I}$, $(x, 0)$ is the supremum of the directed family $(x_i, r_i - r)_{i \in I}$, where $r = \inf_{i \in I} r_i$, using the fact that X, d is standard. Hence x is the d-limit of the Cauchy net $(x_i)_{i \in I, \sqsubseteq}$, where $i \sqsubseteq j$ if and only if $(x_i, r_i) \leq^{d^+} (x_j, r_j)$. By assumption, $f(x)$ is the ∂-limit of the Cauchy net $(f(x_i))_{i \in I, \sqsubseteq}$. Since Y, ∂ is standard, we can use Lemma 5.15 and conclude that $(f(x), 0)$ is the supremum of the directed family $(f(x_i), \alpha r_i - \alpha r)_{i \in I}$. Using standardness again, $(f(x), \alpha r)$ is the supremum of $(f(x_i), \alpha r_i)_{i \in I}$. Therefore $\mathbf{B}^\alpha(f)$ is Scott-continuous.

Conversely, if $\mathbf{B}^\alpha(f)$ is Scott-continuous, then it is monotonic, so f is α-Lipschitz. For every Cauchy net $(x_i)_{i \in I, \sqsubseteq}$ in X with a d-limit x, extract a Cauchy-weightable subnet [Gou13, Lemma 7.2.8]. We shall show that the image of that subnet by f has a ∂-limit, hence the whole Cauchy net $(f(x_i))_{i \in I, \sqsubseteq}$ will have the same ∂-limit [Gou13, Lemma 7.4.6]. Note also that the subnet we have taken still has x as its d-limit [Gou13, Exercise 7.4.7]. (We have already used these tricks in Section 5.) Hence, without loss of generality, assume that $(x_i, r_i)_{i \in I, \sqsubseteq}$ if a Cauchy-weighted net. Since x is a d-limit of our original net, $(x, 0) = \sup_{i \in I}(x_i, r_i)$. By Scott-continuity, $(f(x), 0) = \sup_{i \in I}(f(x_i), \alpha r_i)$. Since Y, ∂ is standard, we can use Lemma 5.15 and conclude that f is the ∂-limit of $(f(x_i))_{i \in I, \sqsubseteq}$, showing that f is Yoneda-continuous. $\square$

In turn, the above result can be simplified as follows when $Y = \overline{\mathbb{R}}_+$.

Lemma 6.4. *Let X, d be a standard quasi-metric space. A map $f: X \to \overline{\mathbb{R}}_+$ is α-Lipschitz Yoneda-continuous from X, d to $\overline{\mathbb{R}}_+, \mathsf{d}_\mathbb{R}$ if and only if the map $f': \mathbf{B}(X, d) \to \mathbb{R} \cup \{+\infty\}$, defined by $f'(x, r) = f(x) - \alpha r$, is Scott-continuous.*

Proof. If f is α-Lipschitz Yoneda-continuous, then $\mathbf{B}^\alpha(f)$ is Scott-continuous, and using the isomorphism of Example 1.4, the map $(x, r) \mapsto (f(x) - \alpha r, -\alpha r) = (f'(x, r), -\alpha r)$ is Scott-continuous from $\mathbf{B}(X, d)$ to a Scott-closed subset of $(\mathbb{R} \cup \{+\infty\}) \times (-\infty, 0]$. Taking first components, f' is Scott-continuous.

Conversely, if f' is Scott-continuous, then the map $(x, r) \mapsto (f'(x, r), -\alpha r)$ is Scott-continuous, too. To show this, we only need to show that the map $(x, r) \mapsto -\alpha r$ is Scott-continuous, and that is a consequence of Proposition 2.4 (3). Using the same isomorphism as above, this implies that $\mathbf{B}^\alpha(f)$ is Scott-continuous. $\square$

Fact 6.5. The Yoneda-complete quasi-metric spaces, together with the 1-Lipschitz Yoneda-continuous maps, form a category **YQMet**.

Example 6.6. If X and Y are posets, considered as quasi-metric spaces, the α-Lipschitz Yoneda-continuous maps from X to Y, for $\alpha > 0$, are the Scott-continuous maps. $\square$

We shall require the following easy facts.

Proposition 6.7. *Let X, d be a standard quasi-metric space, $\alpha, \beta \in \mathbb{R}_+$, and f, g be maps from X, d to $\overline{\mathbb{R}}_+, \mathsf{d}_\mathbb{R}$.*

(1) *If f is β-Lipschitz Yoneda-continuous, then αf is $\alpha\beta$-Lipschitz Yoneda-continuous;*
(2) *If f is α-Lipschitz Yoneda-continuous and g is β-Lipschitz Yoneda-continuous then $f+g$ is $(\alpha+\beta)$-Lipschitz Yoneda-continuous;*
(3) *If f, g are α-Lipschitz Yoneda-continuous, then so are* $\min(f,g)$ *and* $\max(f,g)$*;*
(4) *If $(f_i)_{i\in I}$ is any family of α-Lipschitz Yoneda-continuous maps, then the pointwise supremum* $\sup_{i\in I} f_i$ *is also α-Lipschitz Yoneda-continuous.*
(5) *If $\alpha \leq \beta$ and f is α-Lipschitz Yoneda-continuous then f is β-Lipschitz Yoneda-continuous.*

Proof. We use Lemma 6.4. (We let the reader check, as an exercise, that (1) and (5) hold even for non-standard quasi-metric spaces. But one cannot use Lemma 6.4, then.)

(1) If $(x,r) \mapsto f(x) - \beta r$ is Scott-continuous, then $(x,r) \mapsto \alpha f(x) - \alpha\beta r$ is, too, since multiplication by $\alpha \in \mathbb{R}_+$ is Scott-continuous. Note that it does not matter that we multiply α by a positive or a negative number, but the fact that α is non-negative does matter.

(2) If $(x,r) \mapsto f(x) - \alpha r$ and $(x,r) \mapsto g(x) - \beta r$ are Scott-continuous, so is their sum $(x,r) \mapsto (f+g)(x) - (\alpha+\beta)r$, because addition is Scott-continuous.

(3) Similarly, since min and max are Scott-continuous.

(4) The map $(x,r) \mapsto (\sup_{i\in I} f_i(x)) - \alpha r = \sup_{i\in I}(f_i(x) - \alpha r)$ is Scott-continuous, because any supremum of Scott-continuous maps with values in $\mathbb{R}\cup\{+\infty\}$ is Scott-continuous: this is a standard exercise, and reduces to showing that suprema commute.

(5) If $(x,r) \mapsto f(x) - \alpha r$ is Scott-continuous, then so is $(x,r) \mapsto f(x) - \beta r$, since it arises as the sum of the former plus the map $(x,r) \mapsto -(\beta-\alpha)r$, which is Scott-continuous by Proposition 2.4 (3). ☐

In a metric space X, d, there is a notion of distance to a closed set C: $d(x,C) = \inf\{d(x,y) \mid y \in C\}$, and $d(x,C) = 0$ if and only if $x \in C$. One can also define the *thinning* of an open subset U by $r \geq 0$ as the set of points x whose distance to the complement of U is strictly larger than r.

We generalize the notion to all standard quasi-metric spaces as follows.

Definition 6.8. Given any subset A of a quasi-metric space X, d, the largest open subset V of $\mathbf{B}(X,d)$ such that $V \cap X \subseteq A$ is written $\widehat{A}$.

$\widehat{A}$ is obtained as the union of all the open subsets V of $\mathbf{B}(X,d)$ such that $V \cap X \subseteq A$. By the definition of the d-Scott topology, $\widehat{A} \cap X$ is the interior of A in X. In particular, for every d-Scott open subset U of X, $\widehat{U} \cap X = U$.

By Proposition 2.4 (4), for every $r \in \mathbb{R}_+$, the map $\_+r$ is Scott-continuous, so $(\_+r)^{-1}(\widehat{U})$ is also open. Hence the following defines a d-Scott open subset of X.

Definition 6.9 (Thinning)**.** In a standard quasi-metric space X, d, the *thinning* U^{-r} of the d-Scott open subset U by $r \in \mathbb{R}_+$ is $(\_+r)^{-1}(\widehat{U}) \cap X = \{x \in X \mid (x,r) \in \widehat{U}\}$.

This allows us to (re)define the distance $d(x, \overline{U})$ of x to the complement $\overline{U}$ of an open subset U.

Definition 6.10. In a standard quasi-metric space X, d, define $d(x,\overline{U})$ for $x \in X$ and U open in X as $\sup\{r \in \mathbb{R}_+ \mid x \in U^{-r}\} = \sup\{r \in \mathbb{R}_+ \mid (x,r) \in \widehat{U}\}$.

This has the expected properties:

Lemma 6.11. *In a standard quasi-metric space X, d, and for all $x, y \in X$ and every d-Scott open subset U of X, the following hold:*

(1) $d(x, \overline{U}) = 0$ *if and only if* $x \not\in U$*;*
(2) $d(x, \overline{U}) \leq d(x, y) + d(y, \overline{U})$*;*
(3) *the map* $d(\_, \overline{U})$ *is* 1*-Lipschitz Yoneda-continuous from* X, d *to* $\overline{\mathbb{R}}_+, \mathsf{d}_{\mathbb{R}}$*.*

Proof. (1) If $x \in U$, then $(x, 0) \in \widehat{U}$, so $(x, 1/2^n)$ is in $\widehat{U}$ for some $n \in \mathbb{N}$, by appealing to Lemma 3.4. It follows that $d(x, \overline{U}) \geq 1/2^n > 0$. Conversely, if $d(x, \overline{U}) > 0$, then $(x, r) \in \widehat{U}$ for some $r > 0$. Since $(x, r) \leq^{d^+} (x, 0)$, $(x, 0)$ is also in $\widehat{U}$, so $x \in U$.

(2) Assume that $d(x, \overline{U}) > d(x, y) + d(y, \overline{U})$. Then there is an $r \in \mathbb{R}_+$ such that $(x, r) \in \widehat{U}$ and $r > d(x, y) + d(y, \overline{U})$. In particular, $d(x, y) < r$, and therefore $(y, r - d(x, y))$ is a well-defined formal ball. Moreover, $(x, r) \leq^{d^+} (y, r - d(x, y))$, by definition. Since $\widehat{U}$ is upwards-closed, $(y, r - d(x, y))$ is in $\widehat{U}$, so $d(y, \overline{U}) \geq r - d(x, y)$: contradiction.

(3) By (2), $f = d(\_, \overline{U})$ is 1-Lipschitz. Hence $f' \colon (x, r) \mapsto d(x, \overline{U}) - r$ is monotonic. Let us show that f' is Scott-continuous. We shall conclude by using Lemma 6.4. Let $(x_i, r_i)_{i \in I}$ be a directed family of formal balls with a supremum (x, r). We must show that $d(x, \overline{U}) - r = \sup_{i \in I} d(x_i, \overline{U}) - r_i$. By monotonicity, the left-hand side is larger than or equal to the right-hand side. Let us suppose, for the sake of contradiction, that it is strictly larger: for some $a \in \mathbb{R}_+$, $d(x, \overline{U}) > a > \sup_{i \in I} d(x_i, \overline{U}) + r - r_i$. By definition of $d(x, \overline{U})$, there is a radius $r' \in \mathbb{R}_+$ such that $(x, r') \in \widehat{U}$, and $r' > a$. By Proposition 2.4 (2), (x, r') is the supremum of the directed family $(x_i, r_i + r' - r)_{i \in I}$. Since $\widehat{U}$ is Scott-open, there is an $i \in I$ such that $(x_i, r_i + r' - r)$ is in $\widehat{U}$. In particular, $d(x_i, \overline{U}) \geq r_i + r' - r$, hence $a > \sup_{i \in I} d(x_i, \overline{U}) + r - r_i \geq r'$. This is impossible since $r' > a$. □

The following compares $d(x, \overline{U})$ with the more familiar formula $\inf_{y \in \overline{U}} d(x, y)$. We write $\downarrow$ for downward closure with respect to $\leq^d$. For a finite set E, $\downarrow E$ is closed, hence its complement is always open.

Proposition 6.12. *Let* X, d *be a standard quasi-metric space,* $x \in X$*, and* U *be a* d*-Scott open subset of* X*. Then* $d(x, \overline{U}) \leq \inf_{y \in \overline{U}} d(x, y)$*, with equality if* $\overline{U} = \downarrow E$ *for some finite set* E*, or if* x *is a center point.*

In particular, $d(x, \overline{U}) = \inf_{y \in \overline{U}} d(x, y)$ in all metric spaces, and in all Smyth-complete quasi-metric spaces, since all points are center points in those situations.

Proof. For every $r \in \mathbb{R}_+$, for every $y \in X$, if $d(x, y) \leq r$ then $(x, r) \leq^{d^+} (y, 0)$. Hence if $d(x, y) \leq r$ and $(x, r) \in \widehat{U}$ then $(y, 0) \in \widehat{U}$, hence $y \in U$. By contraposition, if $(x, r) \in \widehat{U}$ and $y \in \overline{U}$, then $d(x, y) > r$. Taking infima over y and suprema over r, we obtain $d(x, \overline{U}) \leq \inf_{y \in \overline{U}} d(x, y)$.

If $\overline{U} = \downarrow E$ for some finite set $E = \{y_1, y_2, \cdots, y_n\}$, then the downward closure of E in $\mathbf{B}(X, d)$, which we shall write as $\downarrow_{\mathbf{B}} E$ to avoid any confusion, is the closure of E in $\mathbf{B}(X, d)$, and its intersection with X is $\downarrow E$. It follows that $\widehat{U}$ is the complement of $\downarrow_{\mathbf{B}} E$. For every $r < \min_{1 \leq i \leq n} d(x, y_i)$, for every i, $1 \leq i \leq n$, (x, r) is not below $(y_i, 0)$, since that would imply $d(x, y_i) \leq r$. This means that (x, r) is not in $\downarrow_{\mathbf{B}} E$, hence is in its complement, $\widehat{U}$. By definition, it follows that $d(x, \overline{U}) \geq r$. As r is arbitrary, $d(x, \overline{U}) \geq \min_{1 \leq i \leq n} d(x, y_i) \geq \inf_{y \in \overline{U}} d(x, y)$.

For the second part, we no longer assume $\overline{U} = \downarrow E$, but we assume that x is a center point. We know that $d(x, \overline{U}) \leq \inf_{y \in \overline{U}} d(x, y)$, and we assume that the inequality is strict: there are two real numbers $s, t \in \mathbb{R}_+$ such that $d(x, \overline{U}) < s < t \leq \inf_{y \in \overline{U}} d(x, y)$. The rightmost

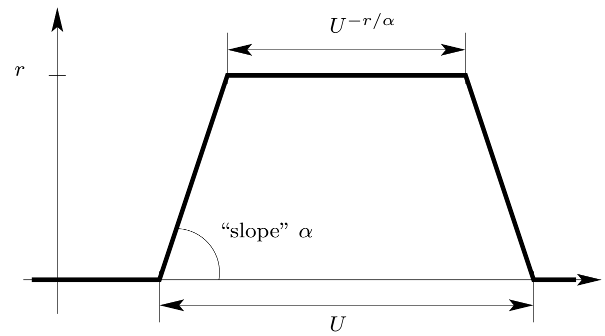

Figure 2: An α-Lipschitz Yoneda-continuous map approximating $r\chi_U$

inequality states that every $y \in \overline{U}$ is such that $d(x, y) \geq t$, hence, by contraposition, that $B^d_{x,<t} \subseteq U$. Since x is a center point, $B^d_{(x,s),<t-s}$ is open, by Lemma 5.7 (2). The intersection of $B^d_{(x,s),<t-s}$ with X is $B^d_{x,<t}$, which is included in U, so $B^d_{(x,s),<t-s} \subseteq \widehat{U}$. In particular, $(x, s) \in \widehat{U}$, so $d(x, \overline{U}) \geq s$, contradiction. □

Write χ_U for the characteristic function of the open subset U. We compare functions, and take suprema of functions, pointwise. The map $\min(r, \alpha d(\_, \overline{U}))$ studied below is probably best understood through a picture: see Figure 2.

Proposition 6.13. *Let X, d be a standard quasi-metric space. For all $\alpha, r \in \mathbb{R}_+$, $\min(r, \alpha d(\_, \overline{U}))$ is an α-Lipschitz Yoneda-continuous map from X, d to $\overline{\mathbb{R}}_+, \mathsf{d}_{\mathbb{R}}$, and is less than or equal to $r\chi_U$. Moreover the family $(\min(r, \alpha d(\_, \overline{U})))_{\alpha>0}$ is a chain, and its supremum is $r\chi_U$.*

Proof. The function $\min(r, \alpha d(\_, \overline{U}))$ is α-Lipschitz Yoneda-continuous by Lemma 6.11 (3) and Proposition 6.7.

For every $x \in U$, $\min(r, \alpha d(x, \overline{U})) \leq r = r\chi_U(x)$. For every $x \in \overline{U}$, we use Lemma 6.11 (1) to conclude that $\min(r, \alpha d(x, \overline{U})) = \min(r, 0) = 0 \leq \chi_U(x)$.

If $\alpha \leq \alpha'$, then clearly $\min(r, \alpha d(x, \overline{U})) \leq \min(r, \alpha' d(x, \overline{U}))$, so the family is a chain.

To show the final claim, take any $x \in U$. By Lemma 6.11 (1), $d(x, \overline{U})$ is non-zero, so $\alpha d(x, \overline{U}) \geq r$ for α large enough. Then $\min d(r, \alpha d(x, \overline{U})) = r = r\chi_U(x)$. □

Proposition 6.14. *Let X, d be a standard quasi-metric space. For all $\alpha, r \in \mathbb{R}_+$, $\min(r, \alpha d(\_, \overline{U}))$ is the largest α-Lipschitz Yoneda-continuous map from X, d to $\overline{\mathbb{R}}_+, \mathsf{d}_{\mathbb{R}}$ that is less than or equal to $r\chi_U$.*

Proof. The claim is clear if $r = 0$, so let us assume $r > 0$. Let f be α-Lipschitz Yoneda-continuous from X, d to $\overline{\mathbb{R}}_+, \mathsf{d}_{\mathbb{R}}$, and assume $f \leq r\chi_U$. Recall that the map $f' \colon \mathbf{B}(X, d) \to \mathbb{R} \cup \{+\infty\}$ defined by $f'(x, s) = f(x) - \alpha s$ is Scott-continuous (Lemma 6.4).

Consider the open subset $V = {f'}^{-1}(0, +\infty]$. For every open ball $(x, 0)$ with radius 0 in V, $f'(x, 0) = f(x) > 0$. Since $f \leq r\chi_U$, x is then in U. This shows that $V \cap X \subseteq U$, hence $V \subseteq \widehat{U}$. Said in another way, for every open ball (x, s) such that $f(x) - \alpha s > 0$, (x, s) is in $\widehat{U}$. Therefore, for every $x \in X$, every $s \in \mathbb{R}_+$ such that $f(x) > \alpha s$ is less than or equal to $d(x, \overline{U})$. Taking suprema over s, $f(x) \leq \alpha d(x, \overline{U})$. Hence $f \leq \alpha d(\_, \overline{U})$, and we conclude since $f \leq r\chi_U \leq r$. □

A map $f\colon X \to \overline{\mathbb{R}}_+$ that is continuous when $\overline{\mathbb{R}}_+$ is equipped with its Scott topology is classically known as a *lower semicontinuous* function from X to $\overline{\mathbb{R}}_+$.

We finally obtain the following result, which shows that we can approximate any $\overline{\mathbb{R}}_+$-valued lower semicontinuous map, as closely as we wish, by α-Lipschitz Yoneda-continuous maps, as α tends to $+\infty$.

Definition 6.15. For every lower semicontinuous map f from a standard quasi-metric space X, d to $\overline{\mathbb{R}}_+$, for every $\alpha \in \mathbb{R}_+$, let $f^{(\alpha)}$ be the largest α-Lipschitz Yoneda-continuous map from X, d to $\overline{\mathbb{R}}_+, \mathsf{d}_{\mathbb{R}}$ below f.

This exists, as the pointwise supremum of all α-Lipschitz Yoneda-continuous maps below f (Proposition 6.7). Proposition 6.14 can be recast as follows.

Fact 6.16. For every $r \in \mathbb{R}_+$, for every d-Scott open subset of X, $(r\chi_U)^{(\alpha)} = \min(r, \alpha d(\_, \overline{U}))$.

Theorem 6.17. *Let X, d be a standard quasi-metric space. For every lower semicontinuous map $f\colon X \to \overline{\mathbb{R}}_+$, the family $(f^{(\alpha)})_{\alpha\in\mathbb{R}_+}$ is a chain, and $\sup_{\alpha\in\mathbb{R}_+} f^{(\alpha)} = f$.*

Proof. The family is non-empty, since for example the constant 0 map is in it. If $\alpha \leq \beta$, then $f^{(\alpha)}$ is β-Lipschitz Yoneda-continuous by Proposition 6.7 (5), and since $f^{(\beta)}$ is largest, $f^{(\alpha)} \leq f^{(\beta)}$. Hence the family is a chain, and is in particular directed.

Clearly, $\sup_{\alpha\in\mathbb{R}_+} f^{(\alpha)} \leq f$. If the inequality were strict, there would be a point $x \in X$ and two real numbers $r, s \in \mathbb{R}_+$ such that, for every $\alpha \in \mathbb{R}_+$, $f^{(\alpha)}(x) \leq r < s < f(x)$. Let U be the open set $f^{-1}(s, +\infty]$. Then $s\chi_U \leq f$, so $(s\chi_U)^{(\alpha)} \leq f^{(\alpha)}$ for every $\alpha \in \mathbb{R}_+$. Using Fact 6.16 and Proposition 6.13, $\sup_\alpha (s\chi_U)^{(\alpha)} = s\chi_U$, so $s\chi_U(x) \leq \sup_\alpha f^{(\alpha)}(x) \leq r$. This is impossible, since $x \in U$. ☐

Our intended application of this result is the following. Given a topological space X, let $\mathcal{L}X$ be the set of lower semicontinuous maps from X to $\overline{\mathbb{R}}_+$. When X, d is a standard quasi-metric space, let $\mathcal{L}_1 X$ be the set of 1-Lipschitz Yoneda-continuous maps from X, d to $\overline{\mathbb{R}}_+, \mathsf{d}_{\mathbb{R}}$. A *prevision* on a topological space X is a Scott-continuous map $F\colon \mathcal{L}X \to \overline{\mathbb{R}}_+$ such that $F(\alpha h) = \alpha F(h)$ for all $\alpha \in \mathbb{R}_+$, $h \in \mathcal{L}X$. Various refinements of the notion yield semantic models for mixed probabilistic and non-deterministic choice (see [Gou07]).

Define the following variant of the Hutchinson-Kantorovitch metric, itself inspired from [Gou08]. The only difference is that h is not restricted to be 1-Lipschitz, but 1-Lipschitz and Yoneda-continuous:

$$d_{\mathrm{H}}(F, F') = \sup\{\mathsf{d}_{\mathbb{R}}(F(h), F'(h)) \mid h \in \mathcal{L}_1 X\}.$$

A complete study of that quasi-metric is out of scope of this paper, but showing that it is a quasi-metric at all requires Theorem 6.17. It satisfies the triangular inequality since $\mathsf{d}_{\mathbb{R}}$ does, and the challenge is to show that $d_{\mathrm{H}}(F, F') = d_{\mathrm{H}}(F', F) = 0$ if and only if $F = F'$. We show the more general claim that $d_{\mathrm{H}}(F, F') = 0$ if and only if $F \leq F'$, i.e., if and only if $F(h) \leq F'(h)$ for every $h \in \mathcal{L}X$. The if direction is obvious, while in the only if direction, we have $\mathsf{d}_{\mathbb{R}}(F(1/\alpha\, h^{(\alpha)}), F'(1/\alpha\, h^{(\alpha)})) = 0$ hence $F(h^{(\alpha)}) \leq F'(h^{(\alpha)})$ for every $\alpha > 0$; using Theorem 6.17 and the Scott-continuity of F and F', $F(h) \leq F'(h)$.

7. Continuous and Algebraic Quasi-Metric Spaces

Let us return for a moment to continuous and algebraic quasi-metric spaces. It is well-known that the continuous dcpos are exactly the retracts of algebraic dcpos. We shall prove a similar result for Yoneda-complete quasi-metric spaces.

Every quasi-metric space X, d has a so-called *Yoneda-completion* $\mathbf{Y}(X, d)$ [BvBR98]. It is built as a certain subspace of the space of all 1-Lipschitz maps from X, d to $\overline{\mathbb{R}}_+, \mathsf{d}_{\mathbb{R}}$, and comes with an isometric embedding $\eta^{\mathbf{Y}}_X \colon X \to \mathbf{Y}(X, d)$, which maps x to $d(\_, x)$. That allows us to see X, d as a sub-quasi-metric space of $\mathbf{Y}(X, d)$. $\mathbf{Y}(X, d)$ is algebraic, with X as set of d-finite elements. It has the following universal property: for every Yoneda-complete space Y, ∂, every α-Lipschitz map $f \colon X, d \to Y, \partial$ has a unique α-Lipschitz Yoneda-continuous extension to $\mathbf{Y}(X, d)$. This holds, more generally, if one replaces 'α-Lipschitz' by 'uniformly continuous' [BvBR98].

There is another notion of completion, the *formal ball completion* $\mathbf{S}(X, d)$ of X, d: that one is due to Vickers [Vic05]; we rely on the presentation of [Gou13, Section 7.5]. This rests on a familiar domain-theoretic construction known as the *rounded ideal completion* $\mathbf{RI}(B, \prec)$ of an abstract basis $B, \prec$. The elements of $\mathbf{RI}(B, \prec)$ are the *rounded ideals* of B, namely subsets D of B that are $\prec$-directed (for every finite subset $\{x_1, \cdots, x_n\}$ of D, there is an $x \in D$ such that $x_i \prec x$ for every i) and $\prec$-downwards-closed (if $y \in D$ and $x \prec y$ then $x \in D$). $\mathbf{RI}(B, \prec)$ is always a continuous dcpo under inclusion. B embeds into $\mathbf{RI}(B)$ through the map $b \mapsto \Downarrow b = \{b' \in B \mid b' \prec b\}$. Modulo that embedding, B is a basis of $\mathbf{RI}(B, \prec)$, and the way-below relation of $\mathbf{RI}(B, \prec)$ restricted to B is exactly $\prec$. Moreover, when B is a C-space, hence an abstract basis with $b \prec b'$ if and only if b' is in the interior of $\uparrow b$, then $\mathbf{RI}(B, \prec)$ is the sobrification of B, a result due to J. Lawson [Law97].

This applies notably to $\mathbf{B}(X, d)$, which is a C-space in its open ball topology, hence an abstract basis with $\prec$ defined by $(x, r) \prec (y, s)$ if and only if $d(x, y) < r - s$. Given a rounded ideal D of $\mathbf{B}(X, d)$, let its *aperture* $\alpha(D)$ be $\inf\{r \mid (x, r) \in D\}$. Then X embeds into $\mathbf{RI}(\mathbf{B}(X, d), \prec)$ through the map $x \mapsto \Downarrow(x, 0)$. Clearly, $\Downarrow(x, 0)$ has aperture 0. We now define $\mathbf{S}(X, d)$ as the subset of $\mathbf{RI}(\mathbf{B}(X, d), \prec)$ consisting of the rounded ideals with aperture zero. This comes with a quasi-metric $d^+_{\mathcal{H}}$, defined by $d^+_{\mathcal{H}}(D, D') = \sup_{b \in D} \inf_{b' \in D'} d^+(b, b')$, which makes the embedding $x \mapsto \Downarrow(x, 0)$ an isometric embedding—namely, $d^+_{\mathcal{H}}(\Downarrow(x, 0), \Downarrow(x', 0)) = d(x, x')$.

The key property of that construction is that $\mathbf{B}(\mathbf{S}(X, d), d^+_{\mathcal{H}})$ is isomorphic, as a poset, to the rounded ideal completion $\mathbf{RI}(\mathbf{B}(X, d), \prec)$, through the map $\sigma_X \colon (D, s) \mapsto D + s = \{(x, r{+}s) \mid (x, r) \in D\}$ [Gou13, Proposition 7.5.4]. It follows immediately that $\mathbf{B}(\mathbf{S}(X, d), d^+_{\mathcal{H}})$ is a continuous dcpo, so $\mathbf{S}(X, d)$ is continuous Yoneda-complete (Theorem 3.7). In fact, it is algebraic Yoneda-complete, and the elements $\Downarrow(x, 0)$ are its d-finite elements [Gou13, Exercise 7.5.11].

$\mathbf{S}(X, d)$ has the same universal property as $\mathbf{Y}(X, d)$: for every Yoneda-complete space Y, ∂, every α-Lipschitz map $f \colon X, d \to Y, \partial$ has a unique α-Lipschitz Yoneda-continuous extension to $\mathbf{S}(X, d)$. This holds, more generally, if one replaces 'α-Lipschitz' by 'uniformly continuous' [Gou13, Proposition 7.5.22, Fact 7.5.23].

By categorical generalities, it follows that $\mathbf{S}(X, d)$ and $\mathbf{Y}(X, d)$ are naturally isomorphic in $\mathbf{YQMet}$ [Gou13, Exercise 7.5.30]. If we restrict ourselves to the full subcategories of $\mathbf{QMet}$ and $\mathbf{YQMet}$ consisting of metric spaces, the same argument shows that $\mathbf{S}(X, d)$ and $\mathbf{Y}(X, d)$ are naturally isomorphic to the usual Cauchy completion of X, d. This requires one to check that $d^+_{\mathcal{H}}$ is a metric whenever d is, and that is indeed the case [Gou13, Lemma 7.5.17].

If instead we restrict ourselves to the full subcategory **Ord** of **QMet** consisting of posets and monotonic maps, and to the full subcategory **Dcpo** of **YQMet** consisting of dcpos and Scott-continuous maps, yet the same argument, using the fact that the ideal completion $\mathbf{I}(B)$ of a poset B is the free dcpo over B, leads to the conclusion that $\mathbf{S}(X, d_{\leq})$ and $\mathbf{Y}(X, d_{\leq})$ are naturally isomorphic to the ideal completion of the poset X [Gou13, Exercise 7.5.25].

All that is known. We would like to show that there is little more to do to obtain a few interesting new results.

In any category $\mathbf{C}$, there is a notion of ($\mathbf{C}$-)*retraction* of an object Y onto an object X: a pair of morphisms $\mathfrak{r}\colon Y \to X$ and $\mathfrak{s}\colon X \to Y$ such that $\mathfrak{r} \circ \mathfrak{s} = \mathrm{id}_X$. X is a *retract* of Y, $\mathfrak{s}$ is the *section* map, and $\mathfrak{r}$ itself is sometimes called a *retraction* map. Colloquially, we shall call a **Dcpo**-retract a Scott-continuous retract, and a **YQMet**-retract a 1-Lipschitz Yoneda-continuous retract.

Recall the construction $\mathbf{B}^1(f)\colon (x, r) \in \mathbf{B}(X, d) \mapsto (f(x), r) \in \mathbf{B}(Y, \partial)$, for each 1-Lipschitz map $f\colon X, d \to Y, \partial$. This defines a functor $\mathbf{B}^1$ from **QMet** to **Ord**. Recall that, for a 1-Lipschitz map f between Yoneda-complete spaces, f is Yoneda-continuous if and only if $\mathbf{B}^1(f)$ is Scott-continuous [Gou13, Proposition 7.4.38]. It follows that $\mathbf{B}^1$ also defines a functor from **YQMet** to **Dcpo**.

In particular, given a retraction $Y, \partial \underset{\mathfrak{s}}{\overset{\mathfrak{r}}{\rightleftarrows}} X, d$ in **YQMet**, we obtain a retraction $\mathbf{B}(Y, \partial) \underset{\mathbf{B}^1(\mathfrak{s})}{\overset{\mathbf{B}^1(\mathfrak{r})}{\rightleftarrows}} \mathbf{B}(X, d)$ in **Dcpo**. The Scott-continuous retracts of continuous dcpos are continuous dcpos, hence the following is obvious, in the light of Theorem 3.7.

Proposition 7.1. *Any 1-Lipschitz Yoneda-continuous retract (i.e., any retract in* **YQMet***) of a continuous Yoneda-complete quasi-metric space is itself continuous Yoneda-complete.*

Proposition 7.1 is due to P. Waszkiewicz, in the more general setting of domains over a Girard quantale [Was09, Theorem 3.3].

Proposition 7.1, together with Proposition 5.18, implies that every 1-Lipschitz Yoneda-continuous retract of a (standard) algebraic quasi-metric space is continuous.

We now proceed to show that every continuous Yoneda-complete space X, d is a retract of an algebraic Yoneda-complete space, and that will be $\mathbf{S}(X, d), d^+_{\mathcal{H}}$. That seems to be new.

Lemma 7.2. *Let X, d be a quasi-metric space. If $(x, r) \ll (y, s)$ then $(x, r) \prec (y, s)$.*

Proof. By Lemma 3.4, $(x, r) \leq^{d^+} (y, s+1/2^n)$ for some $n \in \mathbb{N}$. Hence $d(x, y) \leq r - s - 1/2^n < r - s$. ☐

Lemma 7.3. *Let X, d be a quasi-metric space with a continuous poset of formal balls. For every formal ball (x, r), $\mathrel{\rotatebox[origin=c]{-90}{$\twoheadrightarrow$}}(x, r)$ is an element of $\mathbf{RI}(\mathbf{B}(X, d), \prec)$. If $r = 0$, and X, d is standard, then it is an element of $\mathbf{S}(X, d)$.*

The notation $\mathrel{\rotatebox[origin=c]{-90}{$\twoheadrightarrow$}}(x, r)$ stands for $\{(y, s) \in \mathbf{B}(X, d) \mid (y, s) \ll (x, r)\}$. This should not be confused with $\Downarrow(x, r) = \{(y, s) \in \mathbf{B}(X, d) \mid (y, s) \prec (x, r)\}$.

Proof. Since $\mathbf{B}(X, d)$ is a continuous poset, $\mathrel{\rotatebox[origin=c]{-90}{$\twoheadrightarrow$}}(x, r)$ is $\ll$-directed, hence $\prec$-directed by Lemma 7.2. For every $(y, s) \in \mathrel{\rotatebox[origin=c]{-90}{$\twoheadrightarrow$}}(x, r)$, for every $(z, t) \prec (y, s)$, we have $(z, t) \leq^{d^+} (y, s) \ll (x, r)$, so $(z, t) \in \mathrel{\rotatebox[origin=c]{-90}{$\twoheadrightarrow$}}(x, r)$. Therefore $\mathrel{\rotatebox[origin=c]{-90}{$\twoheadrightarrow$}}(x, r)$ is $\prec$-downwards-closed, hence in $\mathbf{RI}(\mathbf{B}(X, d), \prec)$.

If $r = 0$ and X, d is standard, since $(x, 0)$ is the supremum of $\Downarrow(x, 0)$, the infimum of the radii of formal balls in $\Downarrow(x, 0)$ is equal to 0, by Proposition 2.4 (1). Hence the aperture of $\Downarrow(x, 0)$ is 0, whence $\Downarrow(x, 0) \in \mathbf{S}(X, d)$. □

Lemma 7.4. *Let X, d be a quasi-metric space, and assume that the way-below relation $\ll$ on $\mathbf{B}(X, d)$ is standard. For every formal ball (x, r), $\Downarrow(x, r) = \Downarrow(x, 0) + r$.*

Proof. If $(y, s) \ll (x, r)$, then in particular $(y, s) \leq^{d^+} (x, r)$, hence $s \geq r$. Using Definition 3.1 (or Lemma 3.2) with $a = r$, we obtain $(y, s-r) \ll (x, 0)$, and that exhibits (y, s) as an element of $\Downarrow(x, 0) + r$. Conversely, any element $(y, s + r)$ of $\Downarrow(x, 0) + r$, that is with $(y, s) \ll (x, 0)$, satisfies $(y, s + r) \ll (x, r)$ since $\ll$ is standard. □

Instead of embedding X into $\mathbf{S}(X, d)$ through $\eta_{\mathbf{S}}(x) = \Downarrow(x, 0)$, we consider $\eta'_{\mathbf{S}}(x) = \Downarrow(x, 0)$.

Lemma 7.5. *Let X, d be a continuous quasi-metric space. The map $\eta'_{\mathbf{S}} \colon X, d \to \mathbf{S}(X, d), d^+_{\mathcal{H}}$ is 1-Lipschitz Yoneda-continuous, and $\mathbf{B}^1(\eta'_{\mathbf{S}})$ maps (x, r) to $\sigma_X^{-1}(\Downarrow(x, r))$.*

Proof. Lemma 7.3 enables us to claim that $\eta'_{\mathbf{S}}$ takes its values in $\mathbf{S}(X, d)$. Now consider the map $f \colon \mathbf{B}(X, d) \to \mathbf{RI}(\mathbf{B}(X, d), \prec)$ defined by $f((x, r)) = \Downarrow(x, r)$. Since $\mathbf{B}(X, d)$ is a continuous poset, this is a Scott-continuous map. Using Lemma 7.4, we obtain that $f((x, r)) = \Downarrow(x, 0) + r = \sigma_X(\Downarrow(x, 0), r) = \sigma_X(\mathbf{B}^1(\eta'_{\mathbf{S}})(x, r))$. Since σ_X is an isomorphism, this suffices to show that $\mathbf{B}^1(\eta'_{\mathbf{S}})$ is Scott-continuous, and we have seen that this is equivalent to the fact that $\eta'_{\mathbf{S}}$ is 1-Lipschitz Yoneda-continuous, in the case of standard quasi-metric spaces. Recall from Definition 3.10 that all continuous quasi-metric spaces are standard. □

In the converse direction, given a Yoneda-complete quasi-metric space X, d, let d-lim map every $D = (x_i, r_i)_{i \in I, \sqsubseteq} \in \mathbf{S}(X, d)$ to the d-limit of the Cauchy-weightable net $(x_i)_{i \in I, \sqsubseteq}$. Let also sup denote the supremum map from $\mathbf{RI}(\mathbf{B}(X, d), \prec)$ to $\mathbf{B}(X, d)$.

Lemma 7.6. *Let X, d be a continuous Yoneda-complete quasi-metric space. The map d-lim: $\mathbf{S}(X, d), d^+_{\mathcal{H}} \to X, d$ is 1-Lipschitz Yoneda-continuous, and $\mathbf{B}^1(d\text{-lim}) = \sup \circ \sigma_X$.*

Proof. Every element of $\mathbf{RI}(\mathbf{B}(X, d), \prec)$ can be written uniquely as $D+r$, where $D \in \mathbf{S}(X, d)$ and $r = \alpha(D)$. We know that $\sup(D + r) = (d\text{-lim}(D), r)$, and this shows that $\sup \circ \sigma_X = \mathbf{B}^1(d\text{-lim})$. The claim follows immediately. □

Lemma 7.7. *Let X, d be a Yoneda-complete quasi-metric space, and assume that the way-below relation $\ll$ on $\mathbf{B}(X, d)$ is standard. Then d-lim $\circ\, \eta'_{\mathbf{S}}$ is the identity on X, and $\eta'_{\mathbf{S}} \circ d$-lim $\leq \mathrm{id}_{\mathbf{S}(X,d)}$.*

Proof. We note that $\mathbf{B}^1(d\text{-lim} \circ \eta'_{\mathbf{S}}) = \mathbf{B}^1(d\text{-lim}) \circ \mathbf{B}^1(\eta'_{\mathbf{S}}) = (\sup \circ \sigma_X) \circ (\sigma_X^{-1} \circ f)$, where $f \colon (x, r) \mapsto \Downarrow(x, r)$. Clearly, $\sup \circ f = \mathrm{id}_{\mathbf{B}(X,d)}$. Hence $\mathbf{B}^1(d\text{-lim} \circ \eta'_{\mathbf{S}}) = \mathrm{id}_{\mathbf{B}(X,d)}$. By applying each side of the equation to $(x, 0)$, we obtain that $d\text{-lim}(\eta'_{\mathbf{S}}(x)) = x$.

For the second claim, we consider any rounded ideal D of $\mathbf{B}(X, d), \prec$. Let $x = d\text{-lim}(D)$, so that $(x, 0) = \sup D$. For every $(y, s) \in \eta'_{\mathbf{S}}(x) = \Downarrow(x, 0)$, use interpolation to find (z, t) such that $(y, s) \ll (z, t) \ll (x, 0) = \sup D$, so (y, s) is way-below some element (y', s') of D. By Lemma 7.2, $(y, s) \prec (y', s')$, so $(y, s) \in D$ since D is $\prec$-downwards-closed. Therefore $\eta'_{\mathbf{S}}(d\text{-lim}(D)) \subseteq D$. □

The second part of Lemma 7.7 shows that d-lim and $\eta'_{\mathbf{S}}$ not only define a retraction, but an *embedding-projection pair* (a concept that has meaning in any order-enriched category; here, **YQMet**). A retract defined this way is called a *projection*.

Putting all this together, we obtain:

Proposition 7.8. *Every continuous Yoneda-complete quasi-metric space X, d is a projection of the algebraic Yoneda-complete quasi-metric space $\mathbf{S}(X,d), d^+_{\mathcal{H}}$ through the pair $\eta'_{\mathbf{S}}$, d-lim.*

Theorem 7.9. *The continuous Yoneda-complete quasi-metric spaces are exactly the* 1*-Lipschitz Yoneda-continuous retracts (resp., projections) of algebraic Yoneda-complete quasi-metric spaces.*

Remark 7.10. Theorem 7.9 is very similar to the well-known result that the continuous dcpos are exactly the Scott-continuous retracts (resp., projections) of algebraic dcpos, and our proof is also very similar. The main difference is our use of a rounded ideal completion instead of an ideal completion. In fact, Theorem 7.9 includes that domain-theoretic result as a special case. Notably, if X is a dcpo, we can consider it as a Yoneda-complete quasi-metric space $X, d_{\leq}$. Then $\mathbf{S}(X, d_{\leq})$ is easily seen to be exactly the ideal completion of X, and $d_{\leq}{}^+_{\mathcal{H}}$ is exactly $d_{\subseteq}$.

8. Quasi-Ideal Models

Keye Martin introduced the notion of an *ideal domain* [Mar03], namely dcpos where each non-finite element is maximal. All such domains are automatically algebraic, and first-countable. If we agree that a *model* of a space X is a dcpo in which X embeds as its space of maximal elements, Martin also showed that every space X that has an ω-continuous model has an ideal model; and that the metrizable spaces that have an ideal model are exactly the completely metrizable spaces.

By definition, a space that has a model must be T_1. It is tempting to try and generalize the notion of model to T_0 spaces, say as a dcpo in which X embeds as an upwards-closed subspace. To generalize ideal models, we shall require X to embed as the set of non-finite elements in a quasi-ideal domain, defined as follows.

Definition 8.1 (Quasi-Ideal Domain)**.** A *quasi-ideal domain* is an algebraic domain in which every element below a finite element is itself finite. An *ω-quasi-ideal domain* is a quasi-ideal domain that has only countably many finite elements.

Ideal domains are clearly quasi-ideal. In a quasi-ideal domain, we shall call *limit elements* those points that are not finite. A quasi-ideal domain is organized as two non-mixing layers: a layer of finite elements, all below a second layer of limit elements.

Example 8.2. For a quasi-ideal domain that is not ideal, consider $\mathbb{P}(A)$ under inclusion, for any infinite set A. This is an ω-quasi-ideal domain if and only if A is countable. □

Example 8.3. Any quasi-ideal domain is isomorphic to the ideal completion of its poset of finite elements, because that is the case for all algebraic domains. Conversely, given a poset B, its ideal completion $\mathbf{I}(B)$ is algebraic, but almost never quasi-ideal. For example, $\mathbf{I}(\mathbb{R}_+)$ consists of finite elements of the form $[0, a]$, and limit elements $[0, b)$, $b > 0$. The order is inclusion, and they are deeply interleaved. In general, we can show that $\mathbf{I}(B)$ is a quasi-ideal domain if and only if, for every directed family D in B with no largest element, D has no upper bound in B. Indeed, if D has no largest element, then $\downarrow D$ is a limit element in $\mathbf{I}(B)$, and if D has an upper bound x, this limit element is below the finite element $\downarrow x$, showing that $\mathbf{I}(B)$ is not quasi-ideal. Conversely, if the directed families in B with no largest element have no upper bound, then for every ideal D in $\mathbf{I}(B)$ that is a limit element, there is no finite element $\downarrow x$, $x \in B$, such that $D \subseteq \downarrow x$, so that $\mathbf{I}(B)$ is a quasi-ideal domain. That observation simplifies to the following when B is a dcpo: for a dcpo B, $\mathbf{I}(B)$ is a quasi-ideal domain if and only if B has the ascending chain condition. $\square$

An (ω-)*quasi-ideal model* of a topological space X is an (ω-)quasi-ideal domain, seen as a topological space with the Scott topology, whose subspace of limit elements is homeomorphic to X — in short, an (ω-)quasi-ideal domain in which X embeds as its subspace of limit elements.

De Brecht showed [dB13, Theorem 53] that the quasi-Polish spaces are exactly the spaces that embed as the non-finite elements of some ω-algebraic (equivalently here, ω-continuous) domain. One consequence of our results below will be a strengthening of one direction of that theorem, namely that all quasi-Polish spaces have an ω-quasi-ideal model. We also believe that the proof is simpler.

For now, our goal will be slightly different: to show that every continuous Yoneda-complete quasi-metric space has a quasi-ideal model; but the technique will be the same. The basic construction is inspired by what Martin did [Mar03].

Given a continuous Yoneda-complete quasi-metric space X, d, $\mathbf{B}(X, d)$ has a basis. We need slightly less than that.

Definition 8.4. Given a quasi-metric space X, d with a continuous poset of formal balls, a *local basis* of $\mathbf{B}(X, d)$ is a subset $\mathcal{B}$ of formal balls such that, for every $x \in X$, the set of formal balls $(y, s) \in \mathcal{B}$ such that $(y, s) \ll (x, 0)$ is directed, and has $(x, 0)$ as supremum.

In the sequel, we fix a Yoneda-complete quasi-metric space X, d with a continuous poset of formal balls, and a local basis $\mathcal{B}$ of $\mathbf{B}(X, d)$.

Definition 8.5. The poset $\mathbf{B}'(X, d; \mathcal{B})$ is defined as follows. Its elements are all the formal balls of the form $(x, 0)$, $x \in X$, and those in the local basis $\mathcal{B}$. Its ordering is defined by $(x, r) \sqsubseteq (y, s)$ if and only if $(x, r) \sqsubset (y, s)$ or $(x, r) = (y, s)$, where $(x, r) \sqsubset (y, s)$ if and only if:

(1) either $(x, r) \ll (y, s)$ and $r \geq 2s$, where $\ll$ is the way-below relation in $\mathbf{B}(X, d)$,
(2) or $r = s = 0$ and $x \leq^d y$.

The second clause ensures that, equating $x \in X$ with $(x, 0) \in \mathbf{B}'(X, d; \mathcal{B})$, the ordering on X is the restriction of $\sqsubseteq$. The first clause can be interpreted as saying that to move up (strictly) among the elements of $\mathcal{B}$, we must not only jump high—take an (y, s) that is way-above (x, r)—but also reduce radii by a constant factor. We take 2 for this factor, but this is arbitrary: any constant strictly larger than 1 would work equally well.

The first clause also allows one to compare an element $(x, r) \in \mathcal{B}$ with an element of the form $(y, 0)$, not just to compare two elements of $\mathcal{B}$. However, one must note that $(x, r) \ll (y, s)$ forces $r \neq 0$, hence $(x, r) \in \mathcal{B}$. This is a direct consequence of Lemma 3.4:

Remark 8.6. If $(x, r) \ll (y, s)$ then $r > s$; in particular, $r \neq 0$.

We shall say "$\sqsubseteq$-directed" or "$\leq^{d^+}$-directed" to make clear with respect to which ordering directedness is assumed, and similarly for other epithets.

Fact 8.7. Plainly, $(x, r) \sqsubseteq (y, s)$ implies $(x, r) \leq^{d^+} (y, s)$, and that implies that every $\sqsubseteq$-directed family is also $\leq^{d^+}$-directed.

The following technical lemma will be useful.

Lemma 8.8. *Let $(x_i, r_i)_{i \in I}$ be a $\sqsubseteq$-directed family in $\mathbf{B}'(X, d; \mathcal{B})$, with $r_i \neq 0$ for every $i \in I$, and assume that it has no $\sqsubseteq$-largest element. For every $i \in I$, there is a $j \in I$ such that $(x_i, r_i) \ll (x_j, r_j)$ and $r_i \geq 2r_j$.*

Proof. Since (x_i, r_i) is not $\sqsubseteq$-largest, there is a (x_k, r_k) such that $(x_k, r_k) \not\sqsubseteq (x_i, r_i)$. By directedness, find (x_j, r_j) such that $(x_i, r_i), (x_k, r_k) \sqsubseteq (x_j, r_j)$. It cannot be that $(x_i, r_i) = (x_j, r_j)$, it cannot be either that $r_i = r_j = 0$ and $x_i \leq^d x_j$ since all radii are assumed non-zero, so $(x_i, r_i) \ll (x_j, r_j)$ and $r_i \geq 2r_j$. □

Equally useful is the following consequence, which should be interpreted in the light of Example 8.3. There is a poset B consisting of those elements of $\mathbf{B}'(X, d; \mathcal{B})$ whose radius is non-zero, ordered by $\sqsubseteq$. The lemma below states that the directed families D in B that have no largest element have no upper bound in B. Hence $\mathbf{I}(B)$ will be a quasi-ideal domain. There is some remaining work to do to show that $\mathbf{I}(B)$ is in fact isomorphic to $\mathbf{B}'(X, d; \mathcal{B})$, but this is a good start.

Lemma 8.9. *Let $(x_i, r_i)_{i \in I}$ be a $\sqsubseteq$-directed family in $\mathbf{B}'(X, d; \mathcal{B})$, with $r_i \neq 0$ for every $i \in I$, and assume that it has no $\sqsubseteq$-largest element. Then $\inf_{i \in I} r_i = 0$.*

Proof. Iterating Lemma 8.8 from some arbitrary index $i_0 \in I$, we obtain a sequence $(x_{i_0}, r_{i_0}) \ll (x_{i_1}, r_{i_1}) \ll \cdots \ll (x_{i_k}, r_{i_k}) \ll \cdots$ with $r_{i_0} \geq 2r_{i_1} \geq \cdots \geq 2^k r_{i_k} \geq \cdots$. The infimum of those values is 0. □

Lemma 8.10. *$\mathbf{B}'(X, d; \mathcal{B})$ is a dcpo, and directed suprema are computed exactly as in the larger dcpo $\mathbf{B}(X, d)$.*

Proof. Let $(x_i, r_i)_{i \in I}$ be a $\sqsubseteq$-directed family in $\mathbf{B}'(X, d; \mathcal{B})$, and let (x, r) be its $\leq^{d^+}$-supremum, in $\mathbf{B}(X, d)$. If that supremum is reached, namely if $(x, r) = (x_i, r_i)$ for some $i \in I$, then we claim that (x, r) is also the $\sqsubseteq$-supremum. To show that it is an $\sqsubseteq$-upper bound, we must show that $(x_j, r_j) \sqsubseteq (x, r)$, for any $j \in I$. By directedness, find $k \in I$ so that $(x_i, r_i), (x_j, r_j) \sqsubseteq (x_k, r_k)$. Then $(x_i, r_i) \leq^{d^+} (x_k, r_k) \leq^{d^+} (x, r) = (x_i, r_i)$, so $(x_i, r_i) = (x_k, r_k)$. That implies $(x_j, r_j) \sqsubseteq (x_i, r_i) = (x, r)$. It is the $\sqsubseteq$-supremum because it is attained.

If $r_j = 0$ for some $j \in I$, then the subfamily of those formal balls (x_i, r_i) such that $(x_j, r_j) \sqsubseteq (x_i, r_i)$ is again $\sqsubseteq$-directed, and has the same $\sqsubseteq$-upper bounds and the same $\leq^{d^+}$-upper bounds as the original family. If $r_j = 0$, $(x_j, r_j) \sqsubseteq (x_i, r_i)$ implies $r_i = 0$, so that subfamily consists of formal balls of radius 0. On formal balls of radius 0, $\sqsubseteq$ and $\leq^{d^+}$ coincide, because of Remark 8.6. Therefore (x, r) is the $\sqsubseteq$-supremum of our original family again. Note that, since $r = 0$, it is, in particular, in $\mathbf{B}'(X, d; \mathcal{B})$.

Henceforth assume that the family has no $\sqsubseteq$-largest element, and that $r_j \neq 0$ for every $j \in I$. In particular, the whole family is inside $\mathcal{B}$. For every $i \in I$, use Lemma 8.8 and find $j \in I$ such that $(x_i, r_i) \ll (x_j, r_j)$ and $r_i \geq 2r_j$. Since $(x_j, r_j) \leq^{d^+} (x, r)$, it follows that $(x_i, r_i) \ll (x, r)$ and $r_i \geq 2r$, whence $(x_i, r_i) \sqsubset (x, r)$, at least if we can show that (x, r) is in $\mathcal{B}'(X, d; \mathcal{B})$, which we shall do below.

Let (y, s) be an arbitrary $\leq^{d^+}$-upper bound of the family. Note that $s \leq \inf_{i \in I} r_i$, and that $\inf_{i \in I} r_i = 0$ by Lemma 8.9. This applies to $(y, s) = (x, r)$, so $r = 0$ as well. That shows that (x, r) is in $\mathcal{B}'(X, d; \mathcal{B})$, as promised, so (x, r) is a $\sqsubseteq$-upper bound of the family.

Now consider any $\sqsubseteq$-upper bound (y, s) of the family. This is, in particular, a $\leq^{d^+}$-upper bound, so $s = 0$, as we have just seen. Since (x, r) is the least $\leq^{d^+}$-upper bound, $(x, r) \leq^{d^+} (y, s)$. We now use $r = s = 0$ to conclude that $(x, r) \sqsubseteq (y, s)$. ☐

Lemma 8.11. *Every element $(x, 0)$, $x \in X$, is the supremum in $\mathbf{B}'(X, d; \mathcal{B})$ of a $\sqsubseteq$-directed family of elements $(y, s) \in \mathcal{B}$ with $(y, s) \ll (x, 0)$.*

Proof. Since $\mathcal{B}$ is a local basis of $\mathbf{B}(X, d)$, $(x, 0)$ is the $\leq^{d^+}$-supremum of a $\leq^{d^+}$-directed family $D = (x_i, r_i)_{i\in I}$ of formal balls in $\mathcal{B}$, with $(x_i, r_i) \ll (x, 0)$ for every $i \in I$. In particular, $(x_i, r_i) \sqsubseteq (x, 0)$. There is no reason why D should be $\sqsubseteq$-directed, but we repair this by a variant of a standard trick, allowing us to obtain a subfamily of D that is $\sqsubseteq$-directed and will have the same supremum.

Note that $r_i \neq 0$ for each $i \in I$, by Remark 8.6. Also, since $(x, 0)$ is the $\leq^{d^+}$-supremum of D, $0 = \inf_{i\in I} r_i$. (Recall that X, d is Yoneda-complete, hence standard.)

Our new family consists of formal balls which we write (x_J, r_J), indexed by finite subsets J of I, and is constructed by induction on the cardinality of J. We will ensure that each (x_J, r_J) is in D, and that $(x_J, r_J) \ll (x, 0)$, for every J. We shall also make sure that $(x_K, r_K) \sqsubseteq (x_J, r_J)$ for every $K \subseteq J$, and that will ensure that our new family is $\sqsubseteq$-directed.

When $J = \emptyset$, we pick (x_J, r_J) arbitrarily from D. Otherwise, consider U, the intersection of the subsets $\Uparrow(y, s)$, $(y, s) \in J$, and of the subsets of the form $\Uparrow(x_K, r_K)$ for $K \subsetneq J$. All those subsets are Scott-open, because $\mathbf{B}(X, d)$ is a continuous poset. As a finite intersection of open subsets, U is open. Since $(x_i, r_i) \ll (x, 0)$ for every $i \in I$, and by induction hypothesis, U contains $(x, 0)$. Hence some (x_i, r_i) is in U. Since $\inf_{j\in I} r_j = 0$ and $r_i \neq 0$, we can also find (x_j, r_j) in D so that $r_i \geq 2r_j$, and by $\leq^{d^+}$-directedness, we can even require that $(x_i, r_i) \leq^{d^+} (x_j, r_j)$. Now decide to set $(x_J, r_J) = (x_j, r_j)$. (All that, naturally, uses the Axiom of Choice rather heavily.) That is enough to ensure that $(x_K, r_K) \sqsubset (x_J, r_J)$ for every $K \subsetneq J$, since $(x_K, r_K) \ll (x_i, r_i) \leq^{d^+} (x_j, r_j)$ and $r_K \geq r_i \geq 2r_j$.

By construction, $(x_i, r_i) \sqsubseteq (x_{\{i\}}, r_{\{i\}})$, too, so our new family of formal balls (x_J, r_J) is not only $\sqsubseteq$-directed, but also $\sqsubseteq$-cofinal in D, hence it has $(x, 0)$ as $\sqsubseteq$-supremum. ☐

Lemma 8.12. *The finite elements of $\mathbf{B}'(X, d; \mathcal{B})$ are its elements with non-zero radius. (In particular, they are all in $\mathcal{B}$.)*

Proof. Let $(x, r) \in \mathcal{B}$, with $r \neq 0$. We show that (x, r) is $\sqsubseteq$-finite. Consider a $\sqsubseteq$-directed family $(x_i, r_i)_{i\in I}$ with supremum (y, s) such that $(x, r) \sqsubseteq (y, s)$. If the supremum is attained, say at $i \in I$, then $(x, r) \sqsubseteq (x_i, r_i)$, and we are done.

Otherwise, we note that since $r \neq 0$ and $(x, r) \sqsubseteq (y, s)$, either $(x, r) = (y, s)$ or $(x, r) \ll (y, s)$ and $r \geq 2s$. In the latter case, for some $i \in I$, and using interpolation, $(x, r) \ll (x_i, r_i)$.

We now, again, enquire whether $r_i = 0$ for some i. If that is the case, then we can without loss of generality assume that $r_j = 0$ for every $j \in I$, taking a cofinal subfamily if necessary. Note that $s = \inf_j r_j = 0$. Since $(x, r) \sqsubseteq (y, s)$ and $r \neq 0$, a case analysis on the definition of $\sqsubseteq$ shows that $(x, r) \ll (y, s)$. Therefore, and using interpolation, $(x, r) \ll (x_i, r_i)$ for some $i \in I$. Since $r_i = 0$, $(x, r) \sqsubseteq (x_i, r_i)$ trivially.

If the supremum is not attained and $r_i \neq 0$ for every $i \in I$, then $s \leq \inf_{i\in I} r_i = 0$, using Lemma 8.9. Since $r \neq 0$ and $s = 0$, as above we find an index $i \in I$ such that $(x, r) \ll (x_i, r_i)$. Now use Lemma 8.8: there is an index $j \in I$ such that $(x_i, r_i) \ll (x_j, r_j)$ and $r_i \geq 2r_j$. It follows that $(x, r) \ll (x_j, r_j)$ and $r \geq 2r_j$, namely, $(x, r) \sqsubset (x_j, r_j)$.

Next, we show that no element of the form $(x, 0)$ is $\sqsubseteq$-finite. Use Lemma 8.11 to find a $\sqsubseteq$-directed family $(x_i, r_i)_{i \in I}$ whose supremum is $(x, 0)$, and with $(x_i, r_i) \ll (x, 0)$. By Remark 8.6, $r_i \neq 0$. If $(x, 0)$ were $\sqsubseteq$-finite, there would be an $i \in I$ such that $(x, 0) \sqsubseteq (x_i, r_i)$. In particular, $(x, 0) \leq^{d^+} (x_i, r_i)$, which implies $r_i = 0$: that is impossible. $\square$

Proposition 8.13. $\mathbf{B}'(X, d; \mathcal{B})$ *is a quasi-ideal domain.*

Proof. By Lemma 8.11 and Lemma 8.12, it is algebraic. Also, if (x, r) is below a finite element (y, s), namely one with $s \neq 0$, then $r \geq s$, which implies that $r \neq 0$, hence that (x, r) is finite, too. $\square$

Proposition 8.14. *Let X, d be a continuous Yoneda-complete quasi-metric space, and let $\mathcal{B}$ be a local basis of $\mathbf{B}(X, d)$. Then X with the d-Scott topology is homeomorphic to the subspace of limit elements of the quasi-ideal domain $\mathbf{B}'(X, d; \mathcal{B})$.*

Proof. Temporarily call *d-Scott'* topology the topology induced by the Scott topology on $\mathbf{B}'(X, d; \mathcal{B})$ on its subspace X. In the light of Proposition 8.13, it only remains to show that the d-Scott and the d-Scott' topologies are the same.

Let U be a d-Scott open in X, and write it as $V \cap X$, where V is Scott-open in $\mathbf{B}(X, d)$. Let $V' = V \cap \mathbf{B}'(X, d; \mathcal{B})$. Clearly, $V' \cap X = U$. We claim that V' is Scott-open in $\mathbf{B}'(X, d; \mathcal{B})$. It is $\sqsubseteq$-upwards-closed because V is $\leq^{d^+}$-upwards-closed, and since $\sqsubseteq$ implies $\leq^{d^+}$ (Fact 8.7). For a $\sqsubseteq$-directed family D in $\mathbf{B}'(X, d; \mathcal{B})$ with $\sqsubseteq$-supremum (x, r) in V', D is $\leq^{d^+}$-directed and has (x, r) as $\leq^{d^+}$-supremum, by Lemma 8.10. Since (x, r) is in V, some element of D is in V, and therefore necessarily in V' as well. Since V' is Scott-open in $\mathbf{B}'(X, d; \mathcal{B})$, $U = V' \cap X$ is then d-Scott' open.

Conversely, let U be d-Scott' open, and let V' be a Scott-open subset of $\mathbf{B}'(X, d; \mathcal{B})$ such that $U = V' \cap X$. For every $x \in U$, $(x, 0)$ is in V', therefore $(x, 0)$ has an open neighborhood $\uparrow(y, r)$ included in V', where (y, r) is a finite element of $\mathbf{B}'(X, d; \mathcal{B})$, and $\uparrow$ denotes upwards-closure in $\mathbf{B}'(X, d; \mathcal{B})$. Since (y, r) is finite, r is non-zero (Lemma 8.12). In particular, since $(y, r) \sqsubseteq (x, 0)$, we must have $(y, r) \ll (x, 0)$ in $\mathbf{B}(X, d)$. Therefore $\Uparrow(y, r) \cap X$ is a d-Scott open neighborhood of x.

For every $z \in \Uparrow(y, r) \cap X$, we have $(y, r) \ll (z, 0)$, so $(y, r) \sqsubseteq (z, 0)$. Therefore, $(z, 0) \subseteq \uparrow(y, r) \subseteq V'$, and that entails $z \in U$. Hence $\Uparrow(y, r) \cap X$ is included in U. Since U is a d-Scott open neighborhood of each of its points, U is d-Scott open. $\square$

Taking $\mathcal{B} = \mathbf{B}(X, d)$ itself, for example, we obtain:

Theorem 8.15. *Every continuous Yoneda-complete quasi-metric space, in its d-Scott topology, has a quasi-ideal model.*

Looking back at Example 1.8, we obtain the following.

Corollary 8.16. *Every continuous dcpo has a quasi-ideal model.*

Making some countability assumptions allows us to refine those results. Recall that a dcpo is ω-continuous if and only if it is continuous and has a countable basis. Equivalently, a continuous dcpo is ω-continuous if and only if its Scott topology is countably based [Nor89, Proposition 3.1].

Applying Proposition 8.14 to a countable basis $\mathcal{B}$ of $\mathbf{B}(X, d)$, $\mathbf{B}'(X, d; \mathcal{B})$ will be a quasi-ideal domain with countably many finite elements, that is, an ω-quasi-ideal domain. Hence:

Proposition 8.17. *Every Yoneda-complete quasi-metric space with an ω-continuous poset of formal balls, equipped with its d-Scott topology, has an ω-quasi-ideal model.*

In particular, every ω-continuous dcpo has an ω-quasi-ideal model.

Matthew de Brecht introduced the quasi-Polish spaces as the topological spaces underlying the countably based Smyth-complete spaces in their open ball topology [dB13] (equivalently, in their d-Scott topology, since the two topologies coincide on Smyth-complete spaces). Among the many equivalent characterizations of quasi-Polish spaces, Theorem 53 of op. cit. states that the quasi-Polish spaces are also the topological spaces that embed into an ω-algebraic dcpo as its subset of non-finite elements. The equivalence between (1) and (4) below strengthens this result.

Theorem 8.18. *The following are equivalent for a topological space X:*

(1) *X is quasi-Polish;*
(2) *X is countably based and there is a quasi-metric d that makes X, d a Smyth-complete quasi-metric space;*
(3) *there is a quasi-metric d that makes X, d a Yoneda-complete quasi-metric space with an ω-continuous poset of formal balls;*
(4) *X has an ω-quasi-ideal model.*

Proof. The previous discussion shows the implication (4) $\Rightarrow$ (1), in particular. (1) $\Rightarrow$ (2) is by definition. (3) $\Rightarrow$ (4) is by Proposition 8.17. It remains to show (2) $\Rightarrow$ (3). Assume X, d Smyth-complete. In particular, it is Yoneda-complete and continuous. We need to show that $\mathbf{B}(X, d)$ is ω-continuous, and we exhibit a countable basis to that end.

Let d^{sym} be the symmetrization of d: $d^{sym}(x, y) = \max(d(x, y), d(y, x))$. Since X is countably based in the open ball topology of d (which coincides with the d-Scott topology), it has a countable subset D that is dense with respect to the open ball topology of d^{sym}. In fact, this is an equivalence, due to H.-P. A. Künzi [Kün09], and mentioned as Proposition 13 in [dB13] (see also [Gou13, Lemma 6.3.48]): a quasi-metric space has a countably-based open ball topology if and only if its symmetrization has a countable dense subset. Let $\mathcal{B}$ be the countable set of formal balls (x, r) with $x \in D$ and r rational.

Fix a formal ball (y, s), and consider $A = \{(x, r) \in \mathcal{B} \mid d^{sym}(x, y) < r - s\}$. Given finitely many elements (x_1, r_1), ..., (x_n, r_n) of A, let ϵ be any positive number less than or equal to $\min_{i=1}^n (r_i - s - d^{sym}(x_i, y))$ such that $r = s + \epsilon/2$ is rational. By density, $B^{d^{sym}}_{y,<\epsilon/2}$ contains some $x \in D$. By symmetry, $d^{sym}(x, y) < \epsilon/2 = r - s$, so (x, r) is in A. Moreover, for each i, $d^{sym}(x_i, x) \leq d^{sym}(x_i, y) + d^{sym}(y, x) < d^{sym}(x_i, y) + \epsilon/2 \leq (r_i - s - \epsilon) + \epsilon/2 = r_i - s - \epsilon/2 = r_i - s - (r - s) = r_i - r$. In particular, $d(x_i, x) \leq r_i - r$, which entails $(x_i, x) \leq^{d^+} (x, r)$. It follows that A is a directed family of elements.

Every element (x, r) of A is such that $d(x, y) < r - s$, namely $(x, r) \prec (y, s)$. By the Romaguera-Valero Theorem, $\ll = \prec$, so every element of A is way-below (y, s). The net $(x)_{(x,r) \in A, \leq^{d^+}}$ is Cauchy-weightable by definition, and y is its limit in the open ball topology of X, d^{sym}: indeed every open neighborhood of y in that topology contains an open ball $B^{d^{sym}}_{y,<\eta}$ for some $\eta > 0$, which itself contains some $x \in D$ by density; A must contain (x, r) for some rational positive r, namely with r such that $0 < r - s \leq \eta$. Every limit in the open ball topology of d^{sym} is a d^{sym}-limit [Gou13, Proposition 7.1.19]. It follows that the supremum of A is the formal ball whose center is the d^{sym}-limit of $(x)_{(x,r) \in A, \leq^{d^+}}$, which is y as we have just seen, and whose radius is $\inf\{r \mid (x, r) \in A\} = s$. Hence $\sup A = (y, s)$.

Since every formal ball (y, s) is the supremum of a directed family A of elements from the countable set $\mathcal{B}$ and way-below (y, s), $\mathcal{B}$ is a countable basis of $\mathbf{B}(X, d)$. $\square$

9. Conclusion and Open Problems

We have shown a variety of results on quasi-metric spaces, and all share one feature: they are all proved domain-theoretically, by reasoning on the poset of formal balls. This proves to be a useful complement to the view of quasi-metric spaces as enriched categories, and works by relatively simple reductions to notions and techniques from ordinary domain theory.

Some questions remain open, as usual. Is there any form of converse to Theorem 8.15? In general, what are the spaces that have a quasi-ideal model? Theorem 8.18 answers the question completely for countably based spaces, but what about non-countably based spaces? Keye Martin showed that the metric spaces that have an ideal model are exactly the complete metric spaces [Mar03]. However, ideal models are not only algebraic, but also first-countable, and that is crucial. There is no reason to believe that quasi-ideal models are first-countable, and continuous Yoneda-complete quasi-metric spaces are not in general first-countable in their d-Scott topology either (as they contain all continuous dcpos already, see Example 1.6 and Example 1.8).

References

[AAHPR09] Mahdi Ali-Akbari, Bijan Honari, Massoud Pourmahdian, and Mir Mohammad Rezaii. The space of formal balls and models of quasi-metric spaces. *Mathematical Structures in Computer Science*, 19:337–355, 2009.

[AJ94] Samson Abramsky and Achim Jung. Domain theory. In S. Abramsky, D. M. Gabbay, and T. S. E. Maibaum, editors, *Handbook of Logic in Computer Science*, volume 3, pages 1–168. Oxford University Press, 1994.

[BvBR98] Marcello M. Bonsangue, Frank van Breugel, and J. J. M. M. Rutten. Generalized metric spaces: Completion, topology, and powerdomains via the Yoneda embedding. *Theoretical Computer Science*, 193:1–51, 1998.

[dB13] Matthew de Brecht. Quasi-Polish spaces. *Annals of Pure and Applied Logic*, 164(3):356–381, 2013.

[DM10] François G. Dorais and Carl Mummert. Stationary and convergent strategies in Choquet games. *Fundamenta Mathematicae*, 209:59–79, 2010.

[EH98] Abbas Edalat and Reinhold Heckmann. A computational model for metric spaces. *Theoretical Computer Science*, 193:53–73, 1998.

[Ern91] Marcel Erné. The ABC of order and topology. In *Category Theory at Work, Proc. Workshop*, volume 18 of *Research and Exposition in Mathematics*, pages 57–83, Bremen, Germany, 1991.

[Ern05] Marcel Erné. Minimal bases, ideal extensions, and basic dualities. *Topology Proceedings*, 29(2):445–489, 2005.

[GHK$^+$03] Gerhard Gierz, Karl Heinrich Hofmann, Klaus Keimel, Jimmie D. Lawson, Michael Mislove, and Dana S. Scott. Continuous lattices and domains. In *Encyclopedia of Mathematics and its Applications*, volume 93. Cambridge University Press, 2003.

[Gou07] Jean Goubault-Larrecq. Continuous previsions. In Jacques Duparc and Thomas A. Henzinger, editors, *Proceedings of the 16th Annual EACSL Conference on Computer Science Logic (CSL'07)*, pages 542–557, Lausanne, Switzerland, September 2007. Springer-Verlag LNCS 4646.

[Gou08] Jean Goubault-Larrecq. Simulation hemi-metrics between infinite-state stochastic games. In Roberto Amadio, editor, *Proceedings of the 11th International Conference on Foundations of Software Science and Computation Structures (FoSSaCS'08)*, pages 50–65, Budapest, Hungary, March-April 2008. Springer-Verlag LNCS 4962.

[Gou13] Jean Goubault-Larrecq. *Non-Hausdorff Topology and Domain Theory—Selected Topics in Point-Set Topology*, volume 22 of *New Mathematical Monographs*. Cambridge University Press, 2013. ISBN 978-1-107-03413-6.

[Hal70] Paul R. Halmos. How to write mathematics. *L'enseignement mathématique*, 16(2):123–152, 1970.

[HKL90] Leo A. Harrington, Alexander S. Kechris, and Alain Louveau. A Glimm-Effros dichotomy for Borel equivalence relations. *Journal of the American Mathematical Society*, 3(4):903–928, 1990.

[Kün09] Hans-Peter A. Künzi. An introduction to quasi-uniform spaces. In F. Mynard and E. Pearl, editors, *Beyond Topology*, volume 486 of *Contemporary Mathematics*, pages 239–304. American Mathematical Society, 2009.

[KW10] Mateusz Kostanek and Paweł Waszkiewicz. The formal ball model for $\mathcal{Q}$-categories. *Mathematical Structures in Computer Science*, 21(1):1–24, 2010.

[Law97] Jimmie Lawson. The round ideal completion via sobrification. In *Topology Proceedings*, volume 22, pages 261–274, 1997.

[Law02] F. William Lawvere. Metric spaces, generalized logic, and closed categories. *Reprints in Theory and Applications of Categories*, 1:1–27, 2002. Originally published in *Rendiconti del seminario matematico e fisico di Milano*, XLIII (1973), 135–166.

[Mar99] Keye Martin. Nonclassical techniques for models of computation. In *Topology Proceedings*, pages 375–405, 1999.

[Mar03] Keye Martin. Ideal models of spaces. *Theoretical Computer Science*, 305(1–3):277–297, August 2003.

[Nor89] Tommy Norberg. Existence theorems for measures on continuous posets, with applications to random set theory. *Mathematica Scandinavica*, 64:15–51, 1989.

[RV10] Salvador Romaguera and Oscar Valero. Domain theoretic characterisations of quasi-metric completeness in terms of formal balls. *Mathematical Structures in Computer Science*, 20:453–472, 2010.

[Vic05] Steven Vickers. Localic completion of generalized metric spaces I. *Theory and Application of Categories*, 14(15):328–356, 2005.

[Vic09] Steven Vickers. Localic completion of generalized metric spaces II: Powerlocales. *Journal of Logic and Analysis*, 1(11):1–48, 2009.

[Was09] Paweł Waszkiewicz. On domain theory over Girard quantales. *Fundamenta Informaticae*, 92(1):169–192, 2009.

[WS81] Klaus Weihrauch and Ulrich Schreiber. Embedding metric spaces into cpo's. *Theoretical Computer Science*, 16(1):5–24, 1981.