Logical Methods in Computer Science
Vol. 14(1:3)2018, pp. 1–44
www.lmcs-online.org

Submitted Jun. 1, 2016
Published Jan. 10, 2018

COINDUCTIVE FOUNDATIONS OF INFINITARY REWRITING AND INFINITARY EQUATIONAL LOGIC *

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Abstract. We present a coinductive framework for defining and reasoning about the infinitary analogues of equational logic and term rewriting in a uniform way. We define $\overset{\infty}{=}$, the infinitary extension of a given equational theory $=_\mathcal{R}$, and $\to^\infty$, the standard notion of infinitary rewriting associated to a reduction relation $\to_\mathcal{R}$, as follows:

$$\begin{aligned} \overset{\infty}{=} \;&:=\; \nu R.\, (=_\mathcal{R} \cup \overline{R})^* \\ \to^\infty \;&:=\; \mu R.\, \nu S.\, (\to_\mathcal{R} \cup \overline{R})^* \,;\, \overline{S} \end{aligned}$$

Here μ and ν are the least and greatest fixed-point operators, respectively, and

$$\overline{R} \;:=\; \{\, \langle f(s_1,\ldots,s_n),\, f(t_1,\ldots,t_n)\rangle \mid f \in \Sigma,\, s_1\, R\, t_1, \ldots, s_n\, R\, t_n \,\} \cup \mathrm{Id}\,.$$

The setup captures rewrite sequences of arbitrary ordinal length, but it has neither the need for ordinals nor for metric convergence. This makes the framework especially suitable for formalizations in theorem provers.

1. Introduction

We present a coinductive framework for defining infinitary equational reasoning and infinitary rewriting in a uniform way. The framework is free of ordinals, metric convergence and partial orders on terms which have been essential in earlier definitions of the concept of infinitary rewriting [12, 28, 31, 27, 26, 3, 2, 4, 21].

Infinitary rewriting is a generalization of the ordinary finitary rewriting to infinite terms and infinite reductions (including reductions of ordinal length greater than ω). For the

Key words and phrases: infinitary rewriting, infinitary equational reasoning, coinduction.

* This is a modified and extended version of [16] which appeared in the proceedings of RTA 2015.

LOGICAL METHODS IN COMPUTER SCIENCE DOI:10.23638/LMCS-14(1:3)2018

definition of rewrite sequences of ordinal length, there is a design choice concerning the exclusion of jumps at limit ordinals, as illustrated in the ill-formed rewrite sequence

$$\underbrace{\mathsf{a} \to \mathsf{a} \to \mathsf{a} \to \cdots}_{\omega\text{-many steps}} \ \mathsf{b} \to \mathsf{b}$$

where the rewrite system is $\mathcal{R} = \{\, \mathsf{a} \to \mathsf{a},\, \mathsf{b} \to \mathsf{b} \,\}$. The rewrite sequence remains for ω steps at a and in the limit step 'jumps' to b. To ensure connectedness at limit ordinals, the usual choices are:

(i) *weak convergence* (also called 'Cauchy convergence'), where it suffices that the sequence of terms converges towards the limit term, and
(ii) *strong convergence*, which additionally requires that the 'rewriting activity', i.e., the depth of the rewrite steps, tends to infinity when approaching the limit.

The notion of strong convergence incorporates the flavor of 'progress', or 'productivity', in the sense that there is only a finite number of rewrite steps at every depth. Moreover, it leads to a more satisfactory metatheory where redex occurrences can be traced over limit steps.

While infinitary rewriting has been studied extensively, notions of infinitary equational reasoning have not received much attention. Some of the few works in this area are by Kahrs [26] and by Lombardi, Ríos and de Vrijer [32], see *Related Work* below. The reason is that the usual definition of infinitary rewriting is based on ordinals to index the rewrite steps, and hence the rewrite direction is incorporated from the start. This is different for the framework we propose here, which enables us to define several natural notions: infinitary equational reasoning, bi-infinite rewriting, and the standard concept of infinitary rewriting. All of these have strong convergence 'built-in'.

We define *infinitary equational reasoning* with respect to a system of equations $\mathcal{R}$, as a relation $\overset{\infty}{\cong}$ on potentially infinite terms by the following mutually coinductive rules:

$$\frac{s \ (=_{\mathcal{R}} \cup \overset{\infty}{\leftrightarrows})^{*} \ t}{s \overset{\infty}{\cong} t} \qquad\qquad \frac{s_1 \overset{\infty}{\cong} t_1 \quad \cdots \quad s_n \overset{\infty}{\cong} t_n}{f(s_1, s_2, \ldots, s_n) \overset{\infty}{\leftrightarrows} f(t_1, t_2, \ldots, t_n)} \tag{1.1}$$

The relation $\overset{\infty}{\leftrightarrows}$ stands for infinitary equational reasoning below the root. The coinductive nature of the rules means that the proof trees need not be well-founded. Reading the rules bottom-up, the first rule allows for an arbitrary, but finite, number of rewrite steps at any finite depth (of the term tree). The second rule enforces that we eventually proceed with the arguments, and hence the activity tends to infinity.

$$\frac{\dfrac{\dfrac{}{\mathsf{C}^{\omega} \overset{\infty}{\cong} \mathsf{a}}}{\mathsf{C}^{\omega} \overset{\infty}{\leftrightarrows} \mathsf{C}(\mathsf{a})} \quad \mathsf{C}(\mathsf{a}) =_{\mathcal{R}} \mathsf{a}}{\mathsf{C}^{\omega} \overset{\infty}{\cong} \mathsf{a}}$$

Figure 1: Derivation of $\mathsf{C}^{\omega} \overset{\infty}{\cong} \mathsf{a}$.

Example 1.1. Let $\mathcal{R}$ consist of the equation

$$\mathsf{C}(\mathsf{a}) = \mathsf{a}\ .$$

We write C^ω to denote the infinite term $\mathsf{C}(\mathsf{C}(\mathsf{C}(\ldots)))$, the solution of the equation $X = \mathsf{C}(X)$. Using the rules (1.1), we can derive $\mathsf{C}^\omega \overset{\infty}{=} \mathsf{a}$ as shown in Figure 1. This is an infinite proof tree as indicated by the loop $\dashrightarrow$ in which the sequence $\mathsf{C}^\omega \overset{\infty}{\leftrightarrows} \mathsf{C}(\mathsf{a}) =_{\mathcal{R}} \mathsf{a}$ is written by juxtaposing $\mathsf{C}^\omega \overset{\infty}{\leftrightarrows} \mathsf{C}(\mathsf{a})$ and $\mathsf{C}(\mathsf{a}) =_{\mathcal{R}} \mathsf{a}$.

Many of the proof trees we consider in this paper are regular trees, that is, trees having only a finite number of distinct subtrees. These infinite trees are convenient since they can be depicted by a 'finite tree' enriched with loops $\dashrightarrow$. However, we emphasise that our framework is not limited to regular trees.

Example 1.2. For an example involving non-regular proof trees, let $\mathcal{R}$ consist of the equation

$$\mathsf{b}(x) = \mathsf{C}(\mathsf{b}(\mathsf{S}(x)))\ .$$

Then we can derive $\mathsf{b}(x) \overset{\infty}{=} \mathsf{C}^\omega$ by the non-regular proof tree shown in Figure 2.

$$\dfrac{\mathsf{b}(x) =_{\mathcal{R}} \mathsf{C}(\mathsf{b}(\mathsf{S}(x))) \qquad \dfrac{\dfrac{\mathsf{b}(\mathsf{S}(x)) =_{\mathcal{R}} \mathsf{C}(\mathsf{b}(\mathsf{S}(\mathsf{S}(x)))) \qquad \dfrac{\dfrac{\vdots}{\mathsf{b}(\mathsf{S}(\mathsf{S}(x))) \overset{\infty}{=} \mathsf{C}^\omega}}{\mathsf{C}(\mathsf{b}(\mathsf{S}(\mathsf{S}(x)))) \overset{\infty}{\leftrightarrows} \mathsf{C}^\omega}}{\mathsf{b}(\mathsf{S}(x)) \overset{\infty}{=} \mathsf{C}^\omega}}{\mathsf{C}(\mathsf{b}(\mathsf{S}(x))) \overset{\infty}{\leftrightarrows} \mathsf{C}^\omega}}{\mathsf{b}(x) = \mathsf{C}^\omega}$$

Figure 2: Non-regular proof tree for $\mathsf{b}(x) = \mathsf{C}^\omega$.

Using the greatest fixed-point constructor ν, we can define $\overset{\infty}{=}$ equivalently as follows:

$$\overset{\infty}{=} \quad := \quad \nu R.\, (=_{\mathcal{R}} \cup \overline{R})^*\,, \tag{1.2}$$

where $\overline{R}$, corresponding to the second rule in (1.1), is defined by

$$\overline{R} \ := \ \{\, \langle f(s_1, \ldots, s_n),\, f(t_1, \ldots, t_n)\rangle \mid f \in \Sigma,\ s_1\, R\, t_1,\, \ldots, s_n\, R\, t_n \,\} \cup \mathrm{Id}\,. \tag{1.3}$$

This is a new and interesting notion of infinitary (strongly convergent) equational reasoning.

Now let $\mathcal{R}$ be a term rewriting system (TRS). If we use $\to_{\mathcal{R}}$ instead of $=_{\mathcal{R}}$ in the rules (1.1), we obtain what we call *bi-infinite rewriting* $\overset{\infty}{\rightrightarrows}$:

$$\dfrac{s\ (\to_{\mathcal{R}} \cup \overset{\infty}{\rightrightarrows})^*\ t}{s \overset{\infty}{\rightrightarrows} t} \qquad\qquad \dfrac{s_1 \overset{\infty}{\rightrightarrows} t_1 \quad \cdots \quad s_n \overset{\infty}{\rightrightarrows} t_n}{f(s_1, s_2, \ldots, s_n) \overset{\infty}{\rightrightarrows} f(t_1, t_2, \ldots, t_n)} \tag{1.4}$$

corresponding to the following fixed-point definition:

$$\overset{\infty}{\to} \;:=\; \nu R.\, (\to_{\mathcal{R}} \cup \overline{R})^* \,. \tag{1.5}$$

We write $\overset{\infty}{\to}$ to distinguish bi-infinite rewriting from the standard notion $\to^\infty$ of (strongly convergent) infinitary rewriting [35]. The symbol ∞ is centered above $\to$ in $\overset{\infty}{\to}$ to indicate that bi-infinite rewriting is 'balanced', in the sense that it allows rewrite sequences to be extended infinitely forwards, but also infinitely backwards. Here backwards does *not* refer to reversing the arrow $\leftarrow_{\mathcal{R}}$. For example, for $\mathcal{R} = \{\, \mathsf{C}(\mathsf{a}) \to \mathsf{a} \,\}$ we have the backward-infinite rewrite sequence $\cdots \to \mathsf{C}(\mathsf{C}(\mathsf{a})) \to \mathsf{C}(\mathsf{a}) \to \mathsf{a}$ and hence $\mathsf{C}^\omega \overset{\infty}{\to} \mathsf{a}$. The proof tree for $\mathsf{C}^\omega \overset{\infty}{\to} \mathsf{a}$ has the same shape as the proof tree displayed in Figure 1; the only difference is that $\overset{\infty}{=}$ is replaced by $\overset{\infty}{\to}$ and $\overset{\infty}{\leftrightarrow}$ by $\overset{\infty}{\rightharpoondown}$. In contrast, the standard notion $\to^\infty$ of infinitary rewriting only takes into account forward limits and we do *not* have $\mathsf{C}^\omega \to^\infty \mathsf{a}$.

We have the following strict inclusions:

$$\to^\infty \;\subsetneq\; \overset{\infty}{\to} \;\subsetneq\; \overset{\infty}{=} \tag{1.6}$$

In our framework, these inclusions follow directly from the fact that the proof trees for $\to^\infty$ (see below) are a restriction of the proof trees for $\overset{\infty}{\to}$ which in turn are a restriction of the proof trees for $\overset{\infty}{=}$. It is also easy to see that each inclusion is strict. For the first, see above. For the second, just note that $\overset{\infty}{\to}$ is not symmetric.

Finally, by a further restriction of the proof trees, we obtain the standard concept of (strongly convergent) infinitary rewriting $\to^\infty$. Using least and greatest fixed-point operators, we define:

$$\to^\infty \;:=\; \mu R.\, \nu S.\, (\to \cup \overline{R})^* \,;\, \overline{S} \,, \tag{1.7}$$

where $;$ denotes relational composition in diagrammatic order, that is:

$$x \;(R \,;\, S)\; y \iff \exists z.\; x \; R \; z \wedge z \; S \; y \,.$$

The greatest fixed point defined using the variable S is a coinductively defined relation. Thus only the last step in the sequence $(\to \cup \overline{R})^* \,;\, \overline{S}$ is coinductive. This corresponds to the following fact about reductions σ of ordinal length: every strict prefix of σ must be shorter than σ itself, while strict suffixes may have the same length as σ.

If we replace μ by ν in (1.7), we get a definition equivalent to $\overset{\infty}{\to}$ defined by (1.5). To see that it is at least as strong, note that $\mathrm{Id} \subseteq \overline{S}$.

Conversely, $\to^\infty$ can be obtained by a restriction of the proof trees obtained by the rules (1.4) for $\overset{\infty}{\to}$. Assume that in a proof tree using the rules (1.4), we mark those occurrences of $\overset{\infty}{\rightharpoondown}$ that are followed by another step in the premise of the rule (i.e., those that are not the last step in the premise). Thus we split $\overset{\infty}{\rightharpoondown}$ into $\rightharpoondown^\infty$ and $\overset{\le}{\rightharpoondown}{}^\infty$. Then the restriction to obtain the relation $\to^\infty$ is to forbid infinite nesting of marked symbols $\overset{\le}{\rightharpoondown}{}^\infty$. This marking is made precise in the following rules:

$$\frac{s \;(\to \cup \overset{\le}{\rightharpoondown}{}^\infty)^* \,;\, \rightharpoondown^\infty \; t}{s \to^\infty t} \qquad \frac{s_1 \to^\infty t_1 \quad \cdots \quad s_n \to^\infty t_n}{f(s_1, s_2, \ldots, s_n) \overset{(\le)}{\rightharpoondown}{}^\infty f(t_1, t_2, \ldots, t_n)} \qquad \frac{}{s \overset{(\le)}{\rightharpoondown}{}^\infty s} \tag{1.8}$$

Here $\rightharpoondown^\infty$ stands for infinitary rewriting below the root, and $\overset{\le}{\rightharpoondown}{}^\infty$ is its marked version. The symbol $\overset{(\le)}{\rightharpoondown}{}^\infty$ stands for both $\rightharpoondown^\infty$ and $\overset{\le}{\rightharpoondown}{}^\infty$. Correspondingly, the rule in the middle is an abbreviation for two rules. The axiom $s \rightharpoondown^\infty s$ serves to 'restore' reflexivity, that is, it models the identity steps in $\overline{S}$ in (1.7). Intuitively, $s \overset{\le}{\to}{}^{\infty} t$ can be thought of as an infinitary rewrite sequence below the root, shorter than the sequence we are defining.

We have an infinitary strongly convergent rewrite sequence from s to t if and only if $s \to^{\infty} t$ can be derived by the rules (1.8) in a (not necessarily well-founded) proof tree without infinite nesting of $\overset{\le}{\to}{}^{\infty}$, that is, proof trees in which all paths (ascending through the proof tree) contain only finitely many occurrences of $\overset{\le}{\to}{}^{\infty}$. The depth requirement in the definition of strong convergence arises naturally in the rules (1.8), in particular the middle rule pushes the activity to the arguments.

The fact that the rules (1.8) capture the infinitary rewriting relation $\to^{\infty}$ is a consequence of a result due to [28] which states that every strongly convergent rewrite sequence contains only a finite number of steps at any depth $d \in \mathbb{N}$, in particular only a finite number of root steps $\to_{\varepsilon}$. Hence every strongly convergent reduction is of the form $(\overset{\le}{\to}{}^{\infty} \,;\, \to_{\varepsilon})^{*} ; \overset{\le}{\to}{}^{\infty}$ as in the premise of the first rule, where the steps $\overset{\le}{\to}{}^{\infty}$ are reductions of shorter length.

We conclude with an example of a TRS that allows for a rewrite sequence of length beyond ω.

$$\dfrac{\mathsf{a} \to_{\varepsilon} \mathsf{C}(\mathsf{a}) \qquad \dfrac{\dfrac{}{\mathsf{a} \to^{\infty} \mathsf{C}^{\omega}}}{\mathsf{C}(\mathsf{a}) \overset{\le}{\to}{}^{\infty} \mathsf{C}^{\omega}}}{\mathsf{a} \to^{\infty} \mathsf{C}^{\omega}}$$

Figure 3: A reduction $\mathsf{a} \to^{\infty} \mathsf{C}^{\omega}$.

$$\dfrac{\dfrac{\dfrac{\text{like Figure 3}}{\mathsf{a} \to^{\infty} \mathsf{C}^{\omega}} \qquad \dfrac{\text{like Figure 3}}{\mathsf{b} \to^{\infty} \mathsf{C}^{\omega}}}{\mathsf{f}(\mathsf{a},\mathsf{b}) \overset{\le}{\to}{}^{\infty} \mathsf{f}(\mathsf{C}^{\omega},\mathsf{C}^{\omega})} \qquad \mathsf{f}(\mathsf{C}^{\omega},\mathsf{C}^{\omega}) \to_{\varepsilon} \mathsf{D}}{\mathsf{f}(\mathsf{a},\mathsf{b}) \to^{\infty} \mathsf{D}}$$

Figure 4: A reduction $\mathsf{f}(\mathsf{a},\mathsf{b}) \to^{\infty} \mathsf{D}$.

Example 1.3. We consider the term rewriting system from [12] with the following rules:

$$\mathsf{f}(x,x) \to \mathsf{D} \qquad\qquad \mathsf{a} \to \mathsf{C}(\mathsf{a}) \qquad\qquad \mathsf{b} \to \mathsf{C}(\mathsf{b})\,.$$

We then have $\mathsf{a} \to^{\infty} \mathsf{C}^{\omega}$, that is, an infinite reduction from a to C^{ω} in the limit:

$$\mathsf{a} \to \mathsf{C}(\mathsf{a}) \to \mathsf{C}(\mathsf{C}(\mathsf{a})) \to \mathsf{C}(\mathsf{C}(\mathsf{C}(\mathsf{a}))) \to \cdots \to^{\omega} \mathsf{C}^{\omega}\,.$$

Using the proof rules (1.8), we can derive $\mathsf{a} \to^{\infty} \mathsf{C}^{\omega}$ as shown in Figure 3. The proof tree in Figure 3 can be described as follows: We have an infinitary rewrite sequence from a to C^{ω} since we have a root step from a to $\mathsf{C}(\mathsf{a})$, and an infinitary reduction below the root from

$\mathsf{C}(\mathsf{a})$ to C^ω. The latter reduction $\mathsf{C}(\mathsf{a}) \to^\infty \mathsf{C}^\omega$ is in turn witnessed by the infinitary rewrite sequence $\mathsf{a} \to^\infty \mathsf{C}^\omega$ on the direct subterms.

We also have the following reduction, now of length $\omega + 1$:

$$\mathsf{f}(\mathsf{a},\mathsf{b}) \to \mathsf{f}(\mathsf{C}(\mathsf{a}),\mathsf{b}) \to \mathsf{f}(\mathsf{C}(\mathsf{a}),\mathsf{C}(\mathsf{b})) \to \cdots \to^\omega \mathsf{f}(\mathsf{C}^\omega,\mathsf{C}^\omega) \to \mathsf{D}\,.$$

That is, after an infinite rewrite sequence of length ω, we reach the limit term $\mathsf{f}(\mathsf{C}^\omega,\mathsf{C}^\omega)$, and we then continue with a rewrite step from $\mathsf{f}(\mathsf{C}^\omega,\mathsf{C}^\omega)$ to D. Figure 4 shows how this rewrite sequence $\mathsf{f}(\mathsf{a},\mathsf{b}) \to^\infty \mathsf{D}$ can be derived in our setup. We note that the rewrite sequence $\mathsf{f}(\mathsf{a},\mathsf{b}) \to^\infty \mathsf{D}$ cannot be 'compressed' to length ω. So there is no reduction $\mathsf{f}(\mathsf{a},\mathsf{b}) \to^{\leq\omega} \mathsf{D}$.

Related Work. While a coinductive treatment of infinitary rewriting is not new [8, 25, 22], the previous approaches only capture rewrite sequences of length at most ω. The coinductive framework that we present here captures all strongly convergent rewrite sequences of arbitrary ordinal length.

From the topological perspective, various notions of infinitary rewriting and infinitary equational reasoning have been studied in [26]. The closure operator S_E from [26] is closely related to our notion of infinitary equational reasoning $\stackrel{\infty}{=}$. The operator S_E is defined by $S_E(R) = (S \circ E)^\star(R)$ where

(i) $E(R)$ is the equivalence closure of R, and
(ii) $S(R)$ is the strongly convergent rewrite relation obtained from (single steps) R,
(iii) and $f^\star(R)$ is defined as $\mu x.\, R \cup f(x)$.

Although defined in very different ways, the relations $S_E(\to)$ and $\stackrel{\infty}{=}$ typically coincide.

In [32], Lombardi, Ríos and de Vrijer introduce infinitary equational reasoning based on limits to reason about permutation equivalence of infinitary reductions that are modelled by proof terms.

Martijn Vermaat has formalized infinitary rewriting using metric convergence (in place of strong convergence) in the Coq proof assistant [36], and proved that weakly orthogonal infinitary rewriting does not have the property UN of unique normal forms, see [20]. While his formalization could be extended to strong convergence, it remains to be investigated to what extent it can be used for the further development of the theory of infinitary rewriting.

Ketema and Simonsen [29] introduce the notion of 'computable infinite reductions' [29], where terms as well as reductions are computable, and provide a Haskell implementation of the Compression Lemma for this notion of reduction.

This current paper is an extended version of [16]. The most important changes are:

(i) We have introduced a novel notion of permutation equivalence on infinitary rewrite sequences, which we call *parallel permutation equivalence*. We show that two rewrite sequences are parallel permutation equivalent if and only if they are represented by the same proof tree in our framework, see Section 8.
(ii) We have rewritten and extended the description of the Coq formalisation of the Compression Lemma (Section 9).

Outline. In Section 2 we introduce infinitary rewriting in the usual way based on ordinals, and with convergence at every limit ordinal. Section 3 is a short explanation of (co)induction and fixed-point rules. The two new definitions of infinitary rewriting $\to^\infty$ based on mixing induction and coinduction, as well as their equivalence, are spelled out in Section 4. Then, in Section 5, we prove the equivalence of these new definitions of infinitary rewriting with the standard definition. In Section 6 we present the above introduced relations $\overset{\infty}{=}$ and $\overset{\infty}{\to}$ of infinitary equational reasoning and bi-infinite rewriting. In Section 7 we compare the three relations $\overset{\infty}{=}$, $\overset{\infty}{\to}$ and $\to^\infty$. In Section 8 we present our new work on parallel permutation equivalence and canonical proof trees. As an application, we show in Section 9 that our framework is suitable for formalizations in theorem provers. We conclude in Section 10.

2. Preliminaries on Term Rewriting

We give a brief introduction to infinitary rewriting. For further reading on infinitary rewriting we refer to [31, 35, 6, 21], for an introduction to finitary rewriting to [30, 35, 1, 5].

A *signature* Σ is a set of symbols f each having a fixed arity $ar(f) \in \mathbb{N}$. Let $\mathcal{X}$ be an infinite set of variables such that $\mathcal{X} \cap \Sigma = \varnothing$. The set $Ter^\infty(\Sigma, \mathcal{X})$ of (finite and) *infinite terms* over Σ and $\mathcal{X}$ is coinductively defined by the following grammar:

$$t ::=^{\text{co}} x \mid f(\underbrace{t, \ldots, t}_{ar(f) \text{ times}})\ (x \in \mathcal{X},\ f \in \Sigma)\,.$$

This means that $Ter^\infty(\Sigma, \mathcal{X})$ is defined as the largest set T such that for all $t \in T$, either $t \in \mathcal{X}$ or $t = f(t_1, t_2, \ldots, t_n)$ for some $f \in \Sigma$ with $ar(f) = n$ and $t_1, t_2, \ldots, t_n \in T$. So the grammar rules may be applied an infinite number of times, and equality on the terms is bisimilarity. See further Section 3 for a brief introduction to coinduction.

We write Id for the identity relation on terms, $\mathrm{Id} := \{\langle s, s\rangle \mid s \in Ter^\infty(\Sigma, \mathcal{X})\}$.

Remark 2.1. Alternatively, the set $Ter^\infty(\Sigma, \mathcal{X})$ arises from the set of finite terms, $Ter(\Sigma, \mathcal{X})$, by metric completion, using the well-known distance function d defined by $\mathsf{d}(t, s) = 2^{-n}$ if the n-th level of the terms $t, s \in Ter(\Sigma, \mathcal{X})$ (viewed as labeled trees) is the first level where a difference appears, in case t and s are not identical; furthermore, $\mathsf{d}(t, t) = 0$. It is standard that this construction yields $\langle Ter(\Sigma, \mathcal{X}), \mathsf{d}\rangle$ as a metric space. Now, infinite terms are obtained by taking the completion of this metric space, and they are represented by infinite trees. We will refer to the complete metric space arising in this way as $\langle Ter^\infty(\Sigma, \mathcal{X}), \mathsf{d}\rangle$, where $Ter^\infty(\Sigma, \mathcal{X})$ is the set of finite and infinite terms over Σ.

Let $t \in Ter^\infty(\Sigma, \mathcal{X})$ be a finite or infinite term. The set of *positions* $\mathcal{P}os(t) \subseteq \mathbb{N}^*$ *of* t is defined by: $\varepsilon \in \mathcal{P}os(t)$, and $ip \in \mathcal{P}os(t)$ whenever $t = f(t_1, \ldots, t_n)$ with $1 \le i \le n$ and $p \in \mathcal{P}os(t_i)$. For $p \in \mathcal{P}os(t)$, the *subterm* $t|_p$ of t at position p is defined by $t|_\varepsilon = t$ and $f(t_1, \ldots, t_n)|_{ip} = t_i|_p$. The set of *variables* $\mathcal{V}ar(t) \subseteq \mathcal{X}$ *of* t is $\mathcal{V}ar(t) = \{x \in \mathcal{X} \mid \exists p \in \mathcal{P}os(t).\, t|_p = x\}$.

A *substitution* σ is a map $\sigma : \mathcal{X} \to Ter^\infty(\Sigma, \mathcal{X})$; its domain is extended to $Ter^\infty(\Sigma, \mathcal{X})$ essentially by corecursion: $\sigma(f(t_1, \ldots, t_n)) = f(\sigma(t_1), \ldots, \sigma(t_n))$ (cf. [33, Example 2.5(iv) and Remark 3.2]). For a term t and a substitution σ, we write $t\sigma$ for $\sigma(t)$. We write $x \mapsto s$ for the substitution defined by $\sigma(x) = s$ and $\sigma(y) = y$ for all $y \neq x$. Let $\Box$ be a fresh variable. A *context* C is a term $Ter^\infty(\Sigma, \mathcal{X} \cup \{\Box\})$ containing precisely one occurrence of $\Box$. For contexts C and terms s we write $C[s]$ for $C(\Box \mapsto s)$.

A *rewrite rule* $\ell \to r$ over Σ and $\mathcal{X}$ is a pair (ℓ, r) of terms $\ell, r \in \mathit{Ter}^\infty(\Sigma, \mathcal{X})$ such that the left-hand side ℓ is not a variable ($\ell \notin \mathcal{X}$), and all variables in the right-hand side r occur in ℓ, $\mathcal{V}ar(r) \subseteq \mathcal{V}ar(\ell)$. Note that we require neither the left-hand side nor the right-hand side of a rule to be finite.

A *term rewriting system (TRS)* $\mathcal{R}$ over Σ and $\mathcal{X}$ is a set of rewrite rules over Σ and $\mathcal{X}$. A TRS induces a rewrite relation on the set of terms as follows. For $p \in \mathbb{N}^*$ we define $\to_{\mathcal{R},p} \subseteq \mathit{Ter}^\infty(\Sigma, \mathcal{X}) \times \mathit{Ter}^\infty(\Sigma, \mathcal{X})$, a *rewrite step at position* p, by $C[\ell\sigma] \to_{\mathcal{R},p} C[r\sigma]$ if C is a context with $C|_p = \Box$, $\ell \to r \in \mathcal{R}$, and $\sigma : \mathcal{X} \to \mathit{Ter}^\infty(\Sigma, \mathcal{X})$. We write $\to_\varepsilon$ for *root steps*, $\to_\varepsilon = \{\, (\ell\sigma, r\sigma) \mid \ell \to r \in \mathcal{R},\ \sigma \text{ a substitution} \,\}$. We write $s \to_{\mathcal{R}} t$ if $s \to_{\mathcal{R},p} t$ for some $p \in \mathbb{N}^*$. A *normal form* is a term without a redex occurrence, that is, a term that is not of the form $C[\ell\sigma]$ for some context C, rule $\ell \to r \in \mathcal{R}$ and substitution σ.

A natural consequence of this construction is the notion of *weak convergence*: we say that $t_0 \to t_1 \to t_2 \to \cdots$ is an infinite reduction sequence with limit t, if t is the limit of the sequence $t_0, t_1, t_2, \ldots$ in the usual sense of metric convergence. In contrast, the central notion of *strong convergence* requires, in addition to weak convergence, that the depth of the redexes contracted in successive steps tends to infinity when approaching a limit ordinal from below. This condition rules out the possibility that the action of redex contraction stays confined at the top, or stagnates at some finite level of depth.

Definition 2.2. A *transfinite rewrite sequence* (of ordinal length α) consists of an initial term t_0 and a sequence of rewrite steps $(t_\beta \to_{\mathcal{R},p_\beta} t_{\beta+1})_{\beta<\alpha}$ such that for every limit ordinal $\lambda < \alpha$ we have that if β approaches λ from below, then

(i) the distance $\mathsf{d}(t_\beta, t_\lambda)$ tends to 0 and, moreover,
(ii) the depth of the rewrite action, i.e., the length of the position p_β, tends to infinity.

The sequence is called *strongly convergent* if α is a successor ordinal, or there exists a term t_α such that the conditions (i) and (ii) are fulfilled for every limit ordinal $\lambda \leq \alpha$; we then write $t_0 \to^\infty_{\mathit{ord}} t_\alpha$. The subscript *ord* is used in order to distinguish $\to^\infty_{\mathit{ord}}$ from the equivalent relation $\to^\infty$ as defined in Definition 4.3. We sometimes write $t_0 \to^\alpha_{\mathit{ord}} t_\alpha$ to explicitly indicate the length α of the sequence. The sequence is called *divergent* if it is not strongly convergent.

There are several reasons why strong convergence is beneficial; the foremost being that in this way we can define the notion of *descendant* (also *residual*) over limit ordinals. Also the well-known Parallel Moves Lemma and the Compression Lemma fail for weak convergence, see [34] and [12] respectively.

3. (Co)induction, Fixed Points and Relations

We briefly introduce the relevant concepts from (co)algebra and (co)induction that will be used later throughout this paper. For a more thorough introduction, we refer to [24]. There will be two main points where coinduction will play a role, in the definition of terms and in the definition of term rewriting.

Terms are usually defined with respect to a type constructor F. For instance, consider the type of lists with elements in a given set A, given in a functional programming style:

```
type List a = Nil | Cons a (List a)
```

The above grammar corresponds to the type constructor $F(X) = 1 + A \times X$ where the 1 is used as a placeholder for the empty list `Nil` and the second component represents the `Cons` constructor. Such a grammar can be interpreted in two ways: The *inductive* interpretation yields as terms the set of finite lists, and corresponds to the *least fixed point* of F. The *coinductive* interpretation yields as terms the set of all finite or infinite lists, and corresponds to the *greatest fixed point* of F. More generally, the inductive interpretation of a type constructor yields closed finite terms (with well-founded syntax trees), and dually, the coinductive interpretation yields closed possibly infinite terms. For readers familiar with the categorical definitions of algebras and coalgebras, these two interpretations amount to defining closed finite terms as the *initial F-algebra*, and closed possibly infinite terms as the *final F-coalgebra*.

In order to formally define finite and infinite terms over a signature Σ and a set of variables $\mathcal{X}$, consider the associated type constructor $G_{\Sigma,\mathcal{X}}(Y) = X + F_\Sigma(Y)$ where $F_\Sigma(Y) = \{f(y_1,\ldots,y_n) \mid f \in \Sigma, y_1,\ldots,y_n \in Y, n = ar(f)\}$. Then $Ter(\Sigma,\mathcal{X})$ is the least fixed point of $G_{\Sigma,\mathcal{X}}$ and $Ter^\infty(\Sigma,\mathcal{X})$ is the greatest fixed point of $G_{\Sigma,\mathcal{X}}$.

Equality on finite terms is the expected syntactic/inductive definition. Equality of possibly infinite terms is *bisimilarity*. For instance, in the above example, two finite or infinite lists are equal if and only if they are related by a `List`-bisimulation, which is a relation $R \subseteq$ `List a` $\times$ `List a` such that for all pairs in R are of the form

(i) $(\texttt{Nil}, \texttt{Nil})$, or
(ii) $(\texttt{Cons}\, a\, \sigma, \texttt{Cons}\, b\, \tau)$ such that $a = b$ and $(\sigma,\tau) \in R$.

Throughout this paper, we define and reason about relations on the set $T := Ter^\infty(\Sigma,\mathcal{X})$ of terms. Such relations are elements of the powerset of $T \times T$, which we view as a complete lattice $L := \mathcal{P}(T \times T)$ in which joins and meets are given by unions and intersections of relations. Relations on terms can thus be defined as least and greatest fixed points of monotone operators on L, using the Knaster-Tarski fixed point theorem. In L, an *inductively defined relation* is a least fixed point $\mu X.\, F(X)$ of a monotone $F : L \to L$. Dually, a *coinductively defined relation* is a greatest fixed point $\nu X.\, F(X)$ of a monotone $F : L \to L$. We will make frequent use of the fact that $\nu Y.\, F(Y)$ is the greatest post-fixed point of F, that is,

$$\nu Y.\, F(Y) = \bigcup \{\, X \in L \mid X \subseteq F(X) \,\}, \tag{3.1}$$

and $\mu Y.\, F(Y)$ is the least pre-fixed point of F, that is,

$$\mu Y.\, F(Y) = \bigcap \{\, X \in L \mid F(X) \subseteq X \,\} \tag{3.2}$$

The above properties can be expressed as the following fixed point rules:

$$\frac{X \subseteq F(X)}{X \subseteq \nu Y.\, F(Y)}\,(\nu\text{-rule}) \qquad\qquad \frac{F(X) \subseteq X}{\mu Y.\, F(Y) \subseteq X}\,(\mu\text{-rule}) \tag{3.3}$$

These proof rules, in fact, show the connection to the more abstract categorical notions of induction and coinduction. This can be seen by viewing L as a partial order $(L,\subseteq)$. A partial order $(P,\leq)$ can, in turn, be seen as a category in which the objects are the elements of P and there is a unique arrow $X \to Y$ if $X \leq Y$. A functor on $(P,\leq)$ is then nothing but a monotone map F; an F-coalgebra $X \to F(X)$ is a post-fixed point of F; and a final F-coalgebra is a greatest fixed point of F. Dually, an F-algebra $F(X) \to X$ is a pre-fixed point of F, and an initial F-algebra is a least fixed point of F. The two proof rules express the universal properties of these final and initial objects.

We will use a number of basic operations on relations. These include union ($\cup$), reflexive, transitive closure (*), relation composition in diagrammatic order ($\mathbin{;}$), and relation lifting which we define now.

Definition 3.1. For a relation $R \subseteq T \times T$ we define its *lifting* $\overline{R}$ (with respect to Σ) by

$$\overline{R} \;:=\; \{\, \langle f(s_1,\ldots,s_n),\, f(t_1,\ldots,t_n)\rangle \mid f \in \Sigma,\ ar(f) = n\,, s_1 \mathrel{R} t_1, \ldots, s_n \mathrel{R} t_n \,\} \cup \mathrm{Id}\,.$$

It is straightforward to verify that all these operations are monotone (in all arguments). Hence any map $F\colon L \to L$ built from these operations will have a unique least and greatest fixed point.

4. New Definitions of Infinitary Term Rewriting

We present two new definitions of infinitary rewriting $s \to^\infty t$, based on mixing induction and coinduction, and prove their equivalence. In Section 5 we show they are equivalent to the standard definition based on ordinals. We summarize the definitions:

A. *Derivation Rules.* First, we define $s \to^\infty t$ via a syntactic restriction on the proof trees that arise from the coinductive rules (1.8). The restriction excludes all proof trees that contain ascending paths with an infinite number of marked symbols.

B. *Mixed Induction and Coinduction.* Second, we define $s \to^\infty t$ based on mutually mixing induction and coinduction, that is, least fixed points μ and greatest fixed points ν.

In contrast to previous coinductive definitions [8, 25, 22], the setup proposed here captures all strongly convergent rewrite sequences (of arbitrary ordinal length).

Throughout this section, we fix a signature Σ and a term rewriting system $\mathcal{R}$ over Σ. We also abbreviate $T := \mathit{Ter}^\infty(\Sigma, \mathcal{X})$.

Notation 4.1. Instead of introducing separate derivation rules for transitivity, we write a reduction of the form $s_0 \leadsto s_1 \leadsto \cdots \leadsto s_n$ as a sequence of single steps:

$$\frac{s_0 \leadsto s_1 \quad s_1 \leadsto s_2 \quad \cdots \quad s_{n-1} \leadsto s_n}{\text{conclusion}}$$

This allows us to write the subproof immediately above a single step.

4.1. Derivation Rules.

Definition 4.2. We define the relation $\to^\infty \subset T \times T$ as follows. We have $s \to^\infty t$ if there exists a (finite or infinite) proof tree δ deriving $s \to^\infty t$ using the following five rules:

$$\frac{s \;(\stackrel{\le}{\rightharpoondown}^{\infty} \cup \to_\varepsilon)^* \mathbin{;} \rightharpoondown^\infty t}{s \to^\infty t}\ \mathsf{split} \qquad \frac{s_1 \to^\infty t_1 \quad \cdots \quad s_n \to^\infty t_n}{f(s_1, s_2, \ldots, s_n) \stackrel{(\le)}{\rightharpoondown}^{\infty} f(t_1, t_2, \ldots, t_n)}\ \mathsf{lift} \qquad \frac{}{s \stackrel{(\le)}{\rightharpoondown}^{\infty} s}\ \mathsf{id}$$

such that δ does not contain an infinite nesting of $\stackrel{\le}{\rightharpoondown}^{\infty}$, that is, such that there exists no path ascending through the proof tree that meets an infinite number of symbols $\stackrel{\le}{\rightharpoondown}^{\infty}$. The symbol $\stackrel{(\le)}{\rightharpoondown}^{\infty}$ stands for $\rightharpoondown^\infty$ or $\stackrel{\le}{\rightharpoondown}^{\infty}$; so the second rule is an abbreviation for two rules; similarly for the third rule.

In the above definition, we tacitly assume that the root steps are derived by axioms of the form

$$\frac{}{\ell\sigma \to_{\varepsilon} r\sigma}\ \ell \to r \in \mathcal{R},\ \sigma \text{ a substitution} \tag{4.1}$$

For keeping the proof trees compact, we will just write $\ell\sigma \to_{\varepsilon} r\sigma$ in the proof trees not mentioning rule and substitution.

We give some intuition for the rules in Definition 4.2. The relations $\overset{\le}{\rightharpoondown}{}^{\infty}$ and $\rightharpoondown^{\infty}$ are infinitary reductions below the root. We use $\overset{\le}{\rightharpoondown}{}^{\infty}$ for constructing parts of the prefix (between root steps), and $\rightharpoondown^{\infty}$ for constructing a suffix of the reduction that we are defining. When thinking of ordinal indexed rewrite sequences σ, a suffix of σ can have length equal to σ, while the length of every prefix of σ must be strictly smaller than the length of σ. The five rules (split, and the two versions of lift and id) can be interpreted as follows:

(i) The split-rule: the term s rewrites infinitarily to t, $s \to^{\infty} t$, if s rewrites to t using a finite sequence of (a) root steps, and (b) infinitary reductions $\rightharpoondown^{\infty}$ below the root — where infinitary reductions preceding root steps must be shorter than the derived reduction.

(ii) The lift-rules: the term s rewrites infinitarily to t below the root, $s \overset{(\le)}{\rightharpoondown}{}^{\infty} t$, if the terms are of the shape $s = f(s_1, s_2, \dots, s_n)$ and $t = f(t_1, t_2, \dots, t_n)$ and there exist reductions between the arguments: $s_1 \to^{\infty} t_1, \dots, s_n \to^{\infty} t_n$.

(iii) The id-rules allow for the rewrite relations $\overset{(\le)}{\rightharpoondown}{}^{\infty}$ to be reflexive, and this in turn yields reflexivity of $\to^{\infty}$. For variable-free terms, reflexivity can already be derived using the other rules. For terms with variables, this rule is needed (unless we treat variables as constant symbols).

For an example of a proof tree, we refer to Example 1.3 in the introduction.

4.2. **Mixed Induction and Coinduction.** The next definition is based on mixing induction and coinduction. The inductive part is used to model the restriction to finite nesting of $\overset{\le}{\rightharpoondown}{}^{\infty}$ in the derivations of Definition 4.2. The induction corresponds to a least fixed point μ, while a coinductive rule to a greatest fixed point ν.

Definition 4.3. We define the relation $\to^{\infty} \subseteq T \times T$ by

$$\to^{\infty} \ := \ \mu R.\, \nu S.\, (\to_{\varepsilon} \cup \overline{R})^{*} \,;\, \overline{S}\,. \tag{4.2}$$

We argue why $\to^{\infty}$ is well-defined. Let $L := \mathcal{P}(T \times T)$ be the set of all relations on terms. Define functions $G : L \times L \to L$ and $F : L \to L$ by

$$G(R, S) := (\to_{\varepsilon} \cup \overline{R})^{*} \,;\, \overline{S} \quad \text{and} \quad F(R) := \nu S.\, G(R, S) = \nu S.\, (\to_{\varepsilon} \cup \overline{R})^{*} \,;\, \overline{S}\,. \tag{4.3}$$

It can easily be verified that F and G are monotone, in all their arguments, with respect to set-theoretic inclusion. Hence F and G have unique least and greatest fixed points.

In particular, the relation $\to^{\infty}$ given by (4.2) is well-defined.

4.3. **Equivalence.** We show equivalence of Definitions 4.2 and 4.3. Intuitively, the μR in the fixed point definition corresponds to the nesting restriction in the definition using derivation rules. If one thinks of Definition 4.3 as $\mu R.\, F(R)$ with $F(R) = \nu S.\, G(R, S)$ (see equation (4.3)), then $F^{n+1}(\varnothing)$ are all infinite rewrite sequences that can be derived using proof trees where the nesting depth of the marked symbol $\stackrel{<}{\rightharpoondown}{}^{\infty}$ is at most n.

To avoid confusion we write $\to^{\infty}_{der}$ for the relation $\to^{\infty}$ defined in Definition 4.2, and $\to^{\infty}_{fp}$ for the relation $\to^{\infty}$ defined in Definition 4.3. We show $\to^{\infty}_{der} = \to^{\infty}_{fp}$. Definition 4.2 requires that the nesting structure of $\stackrel{<}{\rightharpoondown}{}^{\infty}_{der}$ in proof trees is well-founded. As a consequence, we can associate to every proof tree a (countable) ordinal that allows to embed the nesting structure in an order-preserving way. We use ω_1 to denote the first uncountable ordinal, and we view ordinals as the set of all smaller ordinals (then the elements of ω_1 are all countable ordinals).

Definition 4.4. Let δ be a proof tree as in Definition 4.2, and let α be an ordinal. An *α-labeling of* δ is a labeling of all symbols $\stackrel{<}{\rightharpoondown}{}^{\infty}_{der}$ in δ with elements from α such that each label is strictly greater than all labels occurring in the subtrees (all labels above).

Lemma 4.5. *Every proof tree as in Definition 4.2 has an α-labeling for some $\alpha \in \omega_1$.*

Proof. Let δ be a proof tree and let $L(\delta)$ be the set of positions of the symbol $\stackrel{<}{\rightharpoondown}{}^{\infty}_{der}$ in δ. For positions $p, q \in L(\delta)$ we write $p < q$ if p is a strict prefix of q. Then we have that $<^{-1}$ is well-founded, that is, there is no infinite sequence $p_0 < p_1 < p_2 < \cdots$ with $p_i \in L(\delta)$ $(i \geq 0)$ as a consequence of the nesting restriction on $\stackrel{<}{\rightharpoondown}{}^{\infty}_{der}$.

By transfinite recursion, the well-founded order on $L(\delta)$ extends to a well-order, isomorphic to some ordinal α — and $\alpha < \omega_1$ since $L(\delta)$ is a countable set. ☐

Definition 4.6. Let δ be a proof tree as in Definition 4.2. We define the *nesting depth* of δ as the least ordinal $\alpha \in \omega_1$ such that δ admits an α-labeling. For every $\alpha \leq \omega_1$, we define a relation $\to^{\infty}_{\alpha,der} \subseteq \to^{\infty}_{der}$ as follows: $s \to^{\infty}_{\alpha,der} t$ whenever $s \to^{\infty}_{der} t$ can be derived using a proof with nesting depth $< \alpha$. Likewise we define relations $\rightharpoondown^{\infty}_{\alpha,der}$ and $\stackrel{<}{\rightharpoondown}{}^{\infty}_{\alpha,der}$.

As a direct consequence of Lemma 4.5 we have:

Corollary 4.7. *We have $\to^{\infty}_{\omega_1,der} = \to^{\infty}_{der}$.*

Theorem 4.8. *Definitions 4.2 and 4.3 define the same relation, $\to^{\infty}_{der} = \to^{\infty}_{fp}$.*

Proof. We begin with $\to^{\infty}_{fp} \subseteq \to^{\infty}_{der}$. Recall that $F(\to^{\infty}_{der})$ is the greatest fixed point of $G(\to^{\infty}_{der}, \_)$, see (4.3). Also, we have $\rightharpoondown^{\infty}_{der} = \stackrel{<}{\rightharpoondown}{}^{\infty}_{der} = \overline{\to^{\infty}_{der}}$, and hence

$$F(\to^{\infty}_{der}) = (\to_{\varepsilon} \cup \overline{\rightharpoondown^{\infty}_{der}})^* \,;\, \overline{F(\to^{\infty}_{der})} = (\to_{\varepsilon} \cup \stackrel{<}{\rightharpoondown}{}^{\infty}_{der})^* \,;\, \overline{F(\to^{\infty}_{der})} \tag{4.4}$$

$$\overline{F(\to^{\infty}_{der})} = \mathrm{Id} \cup \{\, \langle f(\vec{s}),\, f(\vec{t})\rangle \mid \vec{s}\ F(\to^{\infty}_{der})\ \vec{t}\,\} \tag{4.5}$$

where $\vec{s}$, $\vec{t}$ abbreviate $s_1, \ldots, s_n$ and $t_1, \ldots, t_n$, respectively, and we write $\vec{s}\ R\ \vec{t}$ if we have $s_1\ R\ t_1, \ldots, s_n\ R\ t_n$. Employing the μ-rule from (3.3), it suffices to show that $F(\to^{\infty}_{der}) \subseteq \to^{\infty}_{der}$. Assume $\langle s, t\rangle \in F(\to^{\infty}_{der})$. Then $\langle s, t\rangle \in (\to_{\varepsilon} \cup \stackrel{<}{\rightharpoondown}{}^{\infty}_{der})^* \,;\, \overline{F(\to^{\infty}_{der})}$. Then there exists s' such that $s\ (\to_{\varepsilon} \cup \stackrel{<}{\rightharpoondown}{}^{\infty}_{der})^*\ s'$ and $s'\ \overline{F(\to^{\infty}_{der})}\ t$. Now we distinguish cases according to (4.5):

$$\dfrac{s\ (\to_{\varepsilon} \cup \stackrel{<}{\rightharpoondown}{}^{\infty}_{der})^*\ t \qquad \dfrac{}{t \rightharpoondown^{\infty} t}\ \mathsf{id}}{s \to^{\infty} t}\ \mathsf{split} \qquad\qquad \dfrac{s\ (\to_{\varepsilon} \cup \stackrel{<}{\rightharpoondown}{}^{\infty}_{der})^*\ s' \qquad \dfrac{\delta_1 \quad \cdots \quad \delta_n}{s' \rightharpoondown^{\infty} t}\ \mathsf{lift}}{s \to^{\infty} t}\ \mathsf{split}$$

Here, for $i \in \{1, \dots, n\}$, δ_i is the proof tree for $s_i \to^\infty t_i$ obtained from $s_i\ F(\to^\infty_{der})\ t_i$ by corecursively applying the same procedure.

Next we show that $\to^\infty_{der} \subseteq \to^\infty_{fp}$. By Corollary 4.7 it suffices to show $\to^\infty_{\omega_1,der} \subseteq \to^\infty_{fp}$. We prove by well-founded induction on $\alpha \le \omega_1$ that $\to^\infty_{\alpha,der} \subseteq \to^\infty_{fp}$. Since $\to^\infty_{fp}$ is a fixed point of F, we obtain $\to^\infty_{fp} = F(\to^\infty_{fp})$, and since $F(\to^\infty_{fp})$ is the greatest fixed point of $G(\to^\infty_{fp}, \_)$, using the ν-rule from (3.3), it suffices to show the inclusion

$(*)$ $\to^\infty_{\alpha,der} \subseteq G(\to^\infty_{fp}, \to^\infty_{\alpha,der}) := (\to_\varepsilon \cup \overline{\to^\infty_{fp}})^* \mathbin{;} \overline{\to^\infty_{\alpha,der}}$.

Thus assume that $s \to^\infty_{\alpha,der} t$, and let δ be a proof tree of nesting depth $\le \alpha$ deriving $s \to^\infty_{\alpha,der} t$. The only possibility to derive $s \to^\infty_{der} t$ is an application of the split-rule with the premise $s\ (\to_\varepsilon \cup \stackrel{\le}{\rightharpoondown}{}^\infty_{der})^* \mathbin{;} \rightharpoondown^\infty_{der} t$. Since $s \to^\infty_{\alpha,der} t$, we have $s\ (\to_\varepsilon \cup \stackrel{\le}{\rightharpoondown}{}^\infty_{\alpha,der})^* \mathbin{;} \rightharpoondown^\infty_{\alpha,der} t$. Let τ be one of the steps $\stackrel{\le}{\rightharpoondown}{}^\infty_{\alpha,der}$ displayed in the premise. Let u be the source of τ and v the target, so $\tau : u \stackrel{\le}{\rightharpoondown}{}^\infty_{\alpha,der} v$. The step τ is derived either via the id-rule or the lift-rule. The case of the id-rule is not interesting since we then can drop τ from the premise. Thus let the step τ be derived using the lift-rule. Then the terms u, v are of form $u = f(u_1, \dots, u_n)$ and $v = f(v_1, \dots, v_n)$ and for every $1 \le i \le n$ we have $u_i \to^\infty_{\beta,der} v_i$ for some $\beta < \alpha$. Thus by induction hypothesis we obtain $u_i \to^\infty_{fp} v_i$ for every $1 \le i \le n$, and consequently $u\ \overline{\to^\infty_{fp}}\ v$. We then have $s\ (\to_\varepsilon \cup \overline{\to^\infty_{fp}})^* \mathbin{;} \rightharpoondown^\infty_{\alpha,der} t$, and hence $s\ G(\to^\infty_{fp}, \to^\infty_{\alpha,der})\ t$. This concludes the proof. □

5. Equivalence with the Standard Definition

In this section we prove the equivalence of the coinductively defined infinitary rewrite relations $\to^\infty$ from Definitions 4.2 (and 4.3) with the standard definition based on ordinal length rewrite sequences with metric and strong convergence at every limit ordinal (Definition 2.2). The crucial observation is the following theorem from [31]:

Theorem 5.1 (Theorem 2 of [31])**.** *A transfinite reduction is divergent if and only if for some $n \in \mathbb{N}$ there are infinitely many steps at depth n.*

We are now ready to prove the equivalence of both notions:

Theorem 5.2. *We have $\to^\infty = \to^\infty_{ord}$.*

Proof. We write $\rightharpoondown^\infty_{ord}$ to denote a reduction $\to^\infty_{ord}$ without root steps, and we write $\to^\alpha_{ord}$ and $\rightharpoondown^\alpha_{ord}$ to indicate the ordinal length α.

We begin with the direction $\to^\infty_{ord} \subseteq \to^\infty$. We define a function $\mathfrak{T}$ (and $\mathfrak{T}'_{(<)}$) by guarded corecursion [9], mapping rewrite sequences $s \to^\alpha_{ord} t$ (and $s \rightharpoondown^\alpha_{ord} t$) to infinite proof trees derived using the rules from Definition 4.2. This means that every recursive call produces a constructor, contributing to the construction of the infinite tree. Note that the arguments of $\mathfrak{T}$ (and $\mathfrak{T}'_{(<)}$) are not required to be structurally decreasing.

We do case distinction on the ordinal α. If $\alpha = 0$, then $t = s$ and we define

$$\mathfrak{T}(s \to^0_{ord} s) \quad = \quad \frac{\mathfrak{T}'(s \to^0_{ord} s)}{s \to^\infty s}\ \mathsf{split}$$

$$\mathfrak{T}'_{(<)}(x \to^0_{ord} x) \quad = \quad \frac{}{x \overset{(\le)}{\to}{}^{\infty} x}\ \mathsf{id}$$

$$\mathfrak{T}'_{(<)}(f(t_1,\ldots,t_n) \to^0_{ord} f(t_1,\ldots,t_n)) \quad = \quad \frac{\mathfrak{T}(t_1 \to^0_{ord} t_1) \quad \cdots \quad \mathfrak{T}(t_n \to^0_{ord} t_n)}{f(t_1,\ldots,t_n) \overset{(<)}{\to}{}^{\infty} f(t_1,\ldots,t_n)}\ \mathsf{lift}$$

If $\alpha > 0$, then, by Theorem 5.1 the rewrite sequence $s \to^\alpha_{ord} t$ contains only a finite number of root steps. As a consequence, it is of the form:

$$s = s_0 \rightsquigarrow s_1 \cdots \rightsquigarrow s_{2n} \rightsquigarrow s_{2n+1} = t$$

where for every $i \in \{0,\ldots,2n\}$:

(i) for even i, $s_i \rightsquigarrow s_{i+1}$ is an infinite reduction below the root $S_i : s_i \to^{\beta_i}_{ord} s_{i+1}$, and
(ii) for odd i, $s_i \rightsquigarrow s_{i+1}$ is a root step $s_i \to_\varepsilon s_{i+1}$,

where $\beta_i < \alpha$ if $i < 2n$ and $\beta_i \le \alpha$ if $i = 2n$. For $i < 2n$ we have $\beta_i < \alpha$ since every strict prefix must be shorter than the sequence itself. We define

$$\mathfrak{T}(s \to^\alpha_{ord} t) \quad = \quad \frac{\delta_0 \quad \delta_1 \quad \cdots \quad \delta_{2n}}{s \to^\infty t}\ \mathsf{split}$$

where, for $0 \le i < n$,

$$\delta_i = \begin{cases} s_i \to_\varepsilon s_{i+1} & \text{for odd } i, \\ \mathfrak{T}'_<(S_i : s_i \to^\beta_{ord} s_{i+1}) & \text{for even } i \text{ with } i < 2n, \\ \mathfrak{T}'(S_i : s_i \to^\beta_{ord} s_{i+1}) & \text{for even } i \text{ with } i = 2n. \end{cases}$$

For reductions $S : s \to^\alpha_{ord} t$ with $\alpha > 0$ we have $s = f(s_1,\ldots,s_n)$ and $t = f(t_1,\ldots,t_n)$ for some $f \in \Sigma$ of arity n and terms $s_1,\ldots,s_n,t_1,\ldots,t_n \in \mathit{Ter}^\infty(\Sigma,\mathcal{X})$. Moreover, for every i with $1 \le i \le n$, there are rewrite sequences $S_i : s_i \to^{\le\alpha}_{ord} t_i$ obtained by selecting from S the subsequence of steps on the i-th argument. These steps are not necessarily consecutive, but selecting them nonetheless gives rise to a well-defined reduction. We define:

$$\mathfrak{T}'_{(<)}(s \to^\alpha_{ord} t) \quad = \quad \frac{\mathfrak{T}(S_1 : s_1 \to^{\le\alpha}_{ord} t_1) \quad \cdots \quad \mathfrak{T}(S_n : s_n \to^{\le\alpha}_{ord} t_n)}{s \overset{(<)}{\to}{}^{\infty} t}\ \mathsf{lift}$$

The obtained proof tree $\mathfrak{T}(s \to^\alpha_{ord} t)$ derives $s \to^\infty t$. To see that the requirement that there is no ascending path through this tree containing an infinite number of symbols $\overset{<}{\to}{}^{\infty}$ is fulfilled, we note the following. The symbol $\overset{<}{\to}{}^{\infty}$ is produced by $\mathfrak{T}'_<(s \to^\beta_{ord} t)$ which is invoked in $\mathfrak{T}(s \to^\alpha_{ord} t)$ for a β that is strictly smaller than α. By well-foundedness of $<$ on ordinals, no such path exists.

We now show $\to^\infty \subseteq \to^\infty_{ord}$. We prove by well-founded induction on $\alpha \le \omega_1$ that $\to^\infty_\alpha \subseteq \to^\infty_{ord}$. This suffices since $\to^\infty = \to^\infty_{\omega_1}$. Let $\alpha \le \omega_1$ and assume that $s \to^\infty_\alpha t$. Let δ be a proof tree of nesting depth $< \alpha$ witnessing $s \to^\infty_\alpha t$. The only possibility to derive $s \to^\infty t$ is an application of the split-rule with the premise $s\ (\to_\varepsilon \cup \overset{<}{\to}{}^{\infty})^*\ ;\ \to^\infty t$. Since $s \to^\infty_\alpha t$, we have $s\ (\to_\varepsilon \cup \overset{<}{\to}{}^{\infty}_\alpha)^*\ ;\ \to^\infty_\alpha t$. By induction hypothesis we have $s\ (\to_\varepsilon \cup \overline{\to^\infty_{ord}})^*\ ;\ \to^\infty_\alpha t$, and thus $s \to^\infty_{ord}\ ;\ \to^\infty_\alpha t$. We have $\to^\infty_\alpha = \overline{\to^\infty_\alpha}$, and consequently $s \to^\infty_{ord} s_1 \overline{\to^\infty_\alpha}\ t$ for some term s_1. Repeating this argument on $s_1 \overline{\to_\alpha^\infty}\ t$, we get $s \to_{ord}^\infty s_1 \overline{\to_{ord}^\infty}\ s_2 \overline{\overline{\to_\alpha^\infty}}\ t$. After n iterations, we obtain

$$s \to_{ord}^\infty s_1 \overline{\to_{ord}^\infty}\ s_2 \overline{\overline{\to_{ord}^\infty}}\ s_3 \overline{\overline{\overline{\to_{ord}^\infty}}}\ s_4 \cdots (\to_{ord}^\infty)^{-(n-1)}\ s_n\ (\to_\alpha^\infty)^{-n}\ t$$

where $(\to_\alpha^\infty)^{-n}$ denotes the nth iteration of $x \mapsto \overline{x}$ on $\to_\alpha^\infty$.

Clearly, the limit of $\{s_n\}$ is t. Furthermore, each of the reductions $s_n \to_{ord}^\infty s_{n+1}$ are strongly convergent and take place at depth greater than or equal to n. Thus, the infinite concatenation of these reductions yields a strongly convergent reduction from s to t (there is only a finite number of rewrite steps at every depth n). □

6. Infinitary Equational Reasoning and Bi-Infinite Rewriting

6.1. **Infinitary Equational Reasoning.**

Definition 6.1. Let $\mathcal{R}$ be a TRS over Σ, and let $T = \mathit{Ter}^\infty(\Sigma, \mathcal{X})$. We define *infinitary equational reasoning* as the relation $\overset{\infty}{\cong} \subseteq T \times T$ by the mutually coinductive rules:

$$\frac{s\ (\leftarrow_\varepsilon \cup \to_\varepsilon \cup \overset{\infty}{\leftrightarrows})^*\ t}{s \overset{\infty}{\cong} t} \qquad\qquad \frac{s_1 \overset{\infty}{\cong} t_1 \quad \cdots \quad s_n \overset{\infty}{\cong} t_n}{f(s_1, s_2, \ldots, s_n) \overset{\infty}{\leftrightarrows} f(t_1, t_2, \ldots, t_n)}$$

where $\overset{\infty}{\leftrightarrows} \subseteq T \times T$ stands for infinitary equational reasoning below the root.

Note that, in comparison with the rules (1.1) for $\overset{\infty}{\cong}$ from the introduction, we now have used $\leftarrow_\varepsilon \cup \to_\varepsilon$ instead of $=_\mathcal{R}$. It is not difficult to see that this gives rise to the same relation. The reason is that we can 'push' non-root rewriting steps $=_\mathcal{R}$ into the arguments of $\overset{\infty}{\leftrightarrows}$.

Example 6.2. Let $\mathcal{R}$ be a TRS consisting of the following rules:

$$\mathsf{a} \to \mathsf{f}(\mathsf{a}) \qquad\qquad \mathsf{b} \to \mathsf{f}(\mathsf{b}) \qquad\qquad \mathsf{C}(\mathsf{b}) \to \mathsf{C}(\mathsf{C}(\mathsf{a}))\,.$$

Then we have $\mathsf{a} \overset{\infty}{\cong} \mathsf{b}$ as derived in Figure 5 (top), and $\mathsf{C}(\mathsf{a}) \overset{\infty}{\cong} \mathsf{C}^\omega$ as in Figure 5 (bottom).

Definition 6.1 of $\overset{\infty}{\cong}$ can also be defined using a greatest fixed point as follows:

$$\overset{\infty}{\cong} \;:=\; \nu R.\,(\leftarrow_\varepsilon \cup \to_\varepsilon \cup \overline{R})^*\,,$$

where $\overline{R}$ was defined in Definition 3.1. The equivalence of these definitions can be established in a similar way as in Theorem 4.8. As remarked at the end of section 3, the map $R \mapsto (\leftarrow_\varepsilon \cup \to_\varepsilon \cup \overline{R})^*$ is monotone, and consequently the greatest fixed point exists.

We note that, in the presence of collapsing rules (i.e., rules $\ell \to r$ where $r \in \mathcal{X}$), everything becomes equivalent: $s \overset{\infty}{\cong} t$ for all terms s, t. For example, having a rule $\mathsf{f}(x) \to x$ we obtain that $s \overset{\infty}{\cong} \mathsf{f}(s) \overset{\infty}{\cong} \mathsf{f}^2(s) \overset{\infty}{\cong} \cdots \overset{\infty}{\cong} \mathsf{f}^\omega$ for every term s. This can be overcome by forbidding certain infinite terms and certain infinite limits.

$$
\dfrac{
\mathsf{a} \to_\varepsilon \mathsf{f}(\mathsf{a})
\qquad
\dfrac{\dfrac{\mathsf{a} \to_\varepsilon \mathsf{f}(\mathsf{a}) \quad \dfrac{}{\mathsf{f}(\mathsf{a}) \overset{\infty}{\leftrightharpoons} \mathsf{f}^\omega}}{\mathsf{a} \overset{\infty}{=} \mathsf{f}^\omega}}{\mathsf{f}(\mathsf{a}) \overset{\infty}{\leftrightharpoons} \mathsf{f}^\omega}
\qquad
\dfrac{\dfrac{\dfrac{}{\mathsf{f}^\omega \overset{\infty}{\leftrightharpoons} \mathsf{f}(\mathsf{b})} \quad \mathsf{f}(\mathsf{b}) \leftarrow_\varepsilon \mathsf{b}}{\mathsf{f}^\omega \overset{\infty}{=} \mathsf{b}}}{\mathsf{f}^\omega \overset{\infty}{\leftrightharpoons} \mathsf{f}(\mathsf{b})}
\qquad
\mathsf{f}(\mathsf{b}) \leftarrow_\varepsilon \mathsf{b}
}{\mathsf{a} \overset{\infty}{=} \mathsf{b}}
$$

$$
\dfrac{
\dfrac{\dfrac{\text{(as above)}}{\mathsf{a} \overset{\infty}{=} \mathsf{b}}}{\mathsf{C}(\mathsf{a}) \overset{\infty}{\leftrightharpoons} \mathsf{C}(\mathsf{b})}
\qquad
\mathsf{C}(\mathsf{b}) \to_\varepsilon \mathsf{C}(\mathsf{C}(\mathsf{a}))
\qquad
\dfrac{\dfrac{}{\mathsf{C}(\mathsf{a}) \overset{\infty}{=} \mathsf{C}^\omega}}{\mathsf{C}(\mathsf{C}(\mathsf{a})) \overset{\infty}{\leftrightharpoons} \mathsf{C}^\omega}
}{\mathsf{C}(\mathsf{a}) \overset{\infty}{=} \mathsf{C}^\omega}
$$

Figure 5: An example of infinitary equational reasoning, deriving $\mathsf{C}(\mathsf{a}) \overset{\infty}{=} \mathsf{C}^\omega$ in the TRS $\mathcal{R}$ of Example 6.2. Recall Notation 4.1.

6.2. **Bi-Infinite Rewriting.** Another notion that arises naturally in our setup is that of bi-infinite rewriting, allowing rewrite sequences to extend infinitely forwards and backwards. We emphasize that each of the steps $\to_\varepsilon$ in such sequences is a forward step.

Definition 6.3. Let $\mathcal{R}$ be a term rewriting system over Σ, and let $T = \mathit{Ter}^\infty(\Sigma, \mathcal{X})$. We define the *bi-infinite rewrite relation* $\overset{\infty}{\rightrightarrows} \subseteq T \times T$ by the following coinductive rules

$$
\dfrac{s\ (\to_\varepsilon \cup \overset{\infty}{\rightharpoondown})^*\ t}{s \overset{\infty}{\rightrightarrows} t}
\qquad\qquad
\dfrac{s_1 \overset{\infty}{\rightrightarrows} t_1 \quad \cdots \quad s_n \overset{\infty}{\rightrightarrows} t_n}{f(s_1, s_2, \ldots, s_n) \overset{\infty}{\rightharpoondown} f(t_1, t_2, \ldots, t_n)}
$$

where $\overset{\infty}{\rightharpoondown} \subseteq T \times T$ stands for bi-infinite rewriting below the root.

If we replace $\overset{\infty}{=}$ and $\to^\infty$ by $\overset{\infty}{\rightrightarrows}$, and $\overset{\infty}{\leftrightharpoons}$ and $\rightharpoondown^\infty$ by $\overset{\infty}{\rightharpoondown}$, then Examples 1.1 and 1.3 are illustrations of this rewrite relation.

Again, like $\overset{\infty}{=}$, the relation $\overset{\infty}{\rightrightarrows}$ can also be defined using a greatest fixed point:

$$
\overset{\infty}{\rightrightarrows} \;:=\; \nu R.\, (\to_\varepsilon \cup \overline{R})^* \,.
$$

As remarked at the end of section 3, $R \mapsto (\to_\varepsilon \cup \overline{R})^*$ is monotone, and hence the greatest fixed point exists. Also, the equivalence of Definition 6.3 with this ν-definition can be established similarly.

7. Relating the Notions

Lemma 7.1. *Each of the relations $\to^{\infty}$, $\overset{\infty}{\to}$ and $\overset{\infty}{=}$ is reflexive and transitive. The relation $\overset{\infty}{=}$ is also symmetric.*

Proof. Follows immediately from the fact that the relations are defined using the reflexive-transitive closure in each of their first rules. □

Theorem 7.2. *For every TRS $\mathcal{R}$ we have the following inclusions:*

$$\begin{array}{ccccccc} \to^{\infty} & \subseteq & \overset{\infty}{\to} & \subseteq & (\overset{\infty}{\leftarrow} \cup \overset{\infty}{\to})^* & \subseteq & \overset{\infty}{=} \\ & \rotatebox{-45}{$\subsetneq$} & (^{\infty}\!\!\leftarrow \cup \to^{\infty})^* & \rotatebox{45}{$\subsetneq$} & & & \end{array}$$

Moreover, for each of these inclusions there exists a TRS for which the inclusion is strict.

Proof. The inclusions $\to^{\infty} \subsetneq \overset{\infty}{\to} \subsetneq \overset{\infty}{=}$ have already been established in the introduction, see equation (1.6). The inclusion $\to^{\infty} \subsetneq (^{\infty}\!\!\leftarrow \cup \to^{\infty})^*$ is well-known (and obvious). Also $\overset{\infty}{\to} \subsetneq (\overset{\infty}{\leftarrow} \cup \overset{\infty}{\to})^*$ is immediate since $\overset{\infty}{\to}$ is not symmetric.

The inclusion $(^{\infty}\!\!\leftarrow \cup \to^{\infty})^* \subseteq (\overset{\infty}{\leftarrow} \cup \overset{\infty}{\to})^*$ is immediate since $\to^{\infty} \subseteq \overset{\infty}{\to}$. Example 1.1 witnesses strictness of this inclusion. The reason is that, for this example, $\to^{\infty} = \to^*$ as the system does not admit any forward limits. Hence $(^{\infty}\!\!\leftarrow \cup \to^{\infty})^*$ is just finite conversion on potentially infinite terms. Thus $\mathsf{C}^{\omega} \overset{\infty}{\to} \mathsf{a}$, but not $\mathsf{C}^{\omega}\ (^{\infty}\!\!\leftarrow \cup \to^{\infty})^*\ \mathsf{a}$.

The inclusion $(\overset{\infty}{\leftarrow} \cup \overset{\infty}{\to})^* \subseteq \overset{\infty}{=}$ follows from the fact that $\overset{\infty}{=}$ includes $\overset{\infty}{\to}$ and is symmetric and transitive. Example 6.2 witnesses strictness: $\mathsf{C}(\mathsf{a}) = \mathsf{C}^{\omega}$ can only be derived by a rewrite sequence of the form:

$$\mathsf{C}(\mathsf{a}) \overset{\infty}{\to} \mathsf{C}(\mathsf{f}^{\omega}) \overset{\infty}{\leftarrow} \mathsf{C}(\mathsf{b}) \to \mathsf{C}(\mathsf{C}(\mathsf{a})) \overset{\infty}{\to} \mathsf{C}(\mathsf{C}(\mathsf{f}^{\omega})) \overset{\infty}{\leftarrow} \mathsf{C}(\mathsf{C}(\mathsf{b})) \to \mathsf{C}(\mathsf{C}(\mathsf{C}(\mathsf{a}))) \overset{\infty}{\to} \cdots$$

and hence we need to change rewriting directions infinitely often whereas $(\overset{\infty}{\leftarrow} \cup \overset{\infty}{\to})^*$ allows to change the direction only a finite number of times. □

Definition 7.3. For relations $S \subseteq \mathit{Ter}^{\infty}(\Sigma, \mathcal{X}) \times \mathit{Ter}^{\infty}(\Sigma, \mathcal{X})$ we define

$$\overset{\infty}{T}(S) \;\; := \;\; \nu R.\, (S^{-1} \cup S \cup \overline{R})^* \ .$$

Lemma 7.4. *We have $\overset{\infty}{T}(S) = \overset{\infty}{T}(\overset{\infty}{T}(S))$ for every $S \subseteq \mathit{Ter}^{\infty}(\Sigma, \mathcal{X}) \times \mathit{Ter}^{\infty}(\Sigma, \mathcal{X})$.*

Proof. For every relation S we have $S \subseteq (S^{-1} \cup S \cup \overline{R})^*$ and hence $S \subseteq \overset{\infty}{T}(S)$ by (3.3). Hence it follows that $\overset{\infty}{T}(S) \subseteq \overset{\infty}{T}(\overset{\infty}{T}(S))$. For $\overset{\infty}{T}(\overset{\infty}{T}(S)) \subseteq \overset{\infty}{T}(S)$ we note that

$$\begin{aligned} \overset{\infty}{T}(\overset{\infty}{T}(S)) &= (\ \overset{\infty}{T}(S)^{-1} \cup \overset{\infty}{T}(S) \cup \overline{\overset{\infty}{T}(\overset{\infty}{T}(S))}\ )^* && \text{by definition} \\ &= (\ \overset{\infty}{T}(S) \cup \overline{\overset{\infty}{T}(\overset{\infty}{T}(S))}\ )^* && \text{by symmetry of } \overset{\infty}{T}(S) \\ &= (\ (\ S^{-1} \cup S \cup \overline{\overset{\infty}{T}(S)}\ )^* \cup \overline{\overset{\infty}{T}(\overset{\infty}{T}(S))}\ )^* && \text{by definition} \\ &= (\ S^{-1} \cup S \cup \overline{\overset{\infty}{T}(S)} \cup \overline{\overset{\infty}{T}(\overset{\infty}{T}(S))}\ )^* && \\ &= (\ S^{-1} \cup S \cup \overline{\overset{\infty}{T}(\overset{\infty}{T}(S))}\ )^* && \text{since } \overset{\infty}{T}(S) \subseteq \overset{\infty}{T}(\overset{\infty}{T}(S)) \end{aligned}$$

Thus $\overset{\infty}{T}(\overset{\infty}{T}(S))$ is a fixed point of $R \mapsto (S^{-1} \cup S \cup \overline{R})^*$, and hence $\overset{\infty}{T}(\overset{\infty}{T}(S)) \subseteq \overset{\infty}{T}(S)$. □

It follows immediately that $\overset{\infty}{=}$ is closed under $\overset{\infty}{T}(\cdot)$.

Corollary 7.5. *We have* $\overset{\infty}{=} \;=\; \overset{\infty}{T}(\overset{\infty}{=})$ *for every TRS* $\mathcal{R}$.

Proof. We have $\overset{\infty}{=} \;=\; \overset{\infty}{T}(\to_\varepsilon) \;=\; \overset{\infty}{T}(\overset{\infty}{T}(\to_\varepsilon)) \;=\; \overset{\infty}{T}(\overset{\infty}{=})$. □

The work [26] introduces various notions of infinitary rewriting. We comment on the notions that are closest to the relations $\overset{\infty}{\to}$ and $\overset{\infty}{=}$ introduced in our paper. First, we note that it is not difficult to see that $\overset{\infty}{\to} \subsetneq \twoheadrightarrow_t$ where $\twoheadrightarrow_t$ is the topological graph closure of $\to$. The paper [26] also introduces a notion of infinitary equational reasoning with a strongly convergent flavour, namely:

$$S_E(R) = (S \circ E)^\star(R)$$

where

(i) $E(R)$ is the equivalence closure of R, and
(ii) $S(R)$ is the strongly convergent rewrite relation obtained from (single steps) R,
(iii) and $f^\star(R)$ is defined as $\mu x.\, R \cup f(x)$.

Lemma 7.6. *We have* $S_E(\to) \;\subseteq\; \overset{\infty}{=}$ *for every TRS* $\mathcal{R}$.

Proof. The following containments are immediate:

(i) $\to \;\subseteq\; \overset{\infty}{=}$,
(ii) $E(\overset{\infty}{=}) \;=\; \overset{\infty}{=}$, and
(iii) $S(\overset{\infty}{=}) \;\subseteq\; \overset{\infty}{T}(\overset{\infty}{=}) \;=\; \overset{\infty}{=}$ (Corollary 7.5).

From the definition of $S_E(\cdot)$ as a least fixed point, the claim follows. □

It could be reasonable to conjecture that $S_E(\to)$ and $\overset{\infty}{=}$ coincide. We now show that this is not the case.

Example 7.7. Consider the iTRS $\mathcal{R}$ consisting of the rules

$$c(b(x)) \to a(a(x))$$
$$c(a(x)) \to b(b(x))$$

Notice that $a^\omega \overset{\infty}{=} b^\omega$ in $\mathcal{R}$. One possible derivation δ of this fact is given below where $\overline{\delta}$ is the same as δ, but with all pairs mirrored and premises of the split rule are listed in reverse order. We also use that $b(b^\omega) = b^\omega$ and $a(a^\omega) = a^\omega$.

$$\dfrac{a(a(a^\omega)) \leftarrow_\varepsilon c(b(a^\omega)) \quad \dfrac{\delta}{c(b(a^\omega)) \overset{\infty}{\leftrightarrows} c(b(b^\omega))} \quad \dfrac{\overline{\delta}}{c(b^\omega) \overset{\infty}{\leftrightarrows} c(a^\omega)} \quad \dfrac{\delta}{c(a(a^\omega)) \overset{\infty}{\leftrightarrows} c(a(b^\omega))} \quad c(a(b^\omega)) \to_\varepsilon b(b(b^\omega))}{a^\omega \overset{\infty}{=} b^\omega}$$

One does not have $\langle a^\omega, b^\omega \rangle \in S_E(\mathcal{R})$, however. Let us sketch a proof of this. First, notice that, for any relation R, $S_E(R)$ can alternatively be described as

$$S_E(R) := \mu x.R \cup S(E(x)) = \mu x.R \cup E(x) \cup S(x) = \mu x.R \cup E(S(x)) \qquad (7.1)$$

This is because a prefixed point of the composition $S \circ E$ is a prefixed point of both monotone operators E and S, and vice versa. We are particularly interested in the operator appearing on the right-hand side of (7.1). After a single iteration, it yields the usual concept of infinitary conversion in the iTRS $\mathcal{R}$.

Observe that $\mathcal{R}$ is in fact a string rewrite system. Being orthogonal, and having no collapsing rules, we know that $\mathcal{R}$ satisfies both iCR and iSN. Therefore, infinitary conversion in $\mathcal{R}$ is characterized by canonical semantics $\mathcal{S}$, consisting of infinitary normal forms. It is easy to see that these are precisely

$$\begin{aligned}\mathcal{S} &= \{w \in \{a,b\}^m \mid m \leq \omega\} \sqcup \{wc^n \mid w \in \{a,b\}^*, n \leq \omega\} \\ &= \{wc^n \mid w \in \{a,b\}^m, m+n \leq \omega\}\end{aligned}$$

It is a curious fact that $E(S(\cdot))$ does not yet stabilize at $(=_\mathcal{S})$, the equality of infinitary normal forms. But it does stabilize after one more iteration — without relating a^ω and b^ω.

To see this, consider a sequence of $=_\mathcal{S}$-steps

$$s_0 \to^\infty \cdot {}^\infty\!\!\leftarrow s_1 \to^\infty \cdot {}^\infty\!\!\leftarrow s_2 \to^\infty \cdots \qquad \text{with } \lim_{n\to\infty} s_n = s_\infty \tag{7.2}$$

By infinitary rewriting theory, for each n, we can find standard, ω-compressed reductions

$$\rho_n : s_n \to^\infty s := \mathrm{NF}^\infty(s_0) \in \mathcal{S} \tag{7.3}$$

Let us say that a finite prefix $w \leq s$ is *stable* if we can find a number N, a prefix $v \leq s_\infty$, and a reduction $\nu : v \to^* w$ such that, for all $n \geq N$:

– $s_n = vs'_n$
– ρ_n factors as $\rho_n = \nu \circ \rho'_n$, where $\rho'_n : s'_n \to s'$ and $s = ws'$.

If every prefix of s is stable, then it is easy to see that $s_\infty \to^\infty s$; then the infinite conversion (7.2) yields no new pairs in $S_E(\mathcal{R})$.

Otherwise, there is a maximal stable prefix w — which may or may not be empty. Fixing this $w = w_{\max}$, with respective N, v, and ν, we find that

– $s = ws'$, $s_\infty = vs'_\infty$, $\nu : v \to w$;
– $s'_N \to^\infty \cdot {}^\infty\!\!\leftarrow s'_{N+1} \to^\infty \cdots$, with $\lim_{n\to\infty} s'_n = s'_\infty$;
– $s'_n \to^\infty s'$ for $n \geq N$;

We claim that $s'_\infty = c^\omega$. For suppose $s'_\infty \in \{c^k xu \mid k \geq 0, x \in \{a,b\}, u \in \{a,b,c\}^{\leq\omega}\}$. After k steps of outermost reduction, the outermost letter becomes $y \in \{a,b\}$, and there is no longer a redex present at the root. By continuity of the sequence $\{s'_n\}$, this even happens at each $n \geq M$, for some $M \geq N$. But now y becomes a stable prefix of s', and wy a stable prefix of s — contradicting maximality of w. So $s'_\infty = c^\omega$. Now, unless $s' = c^\omega$ as well, trivializing the whole thing, the reductions ρ'_n must be non-trivial, yielding subterms of form $aa(x)$ or $bb(x)$. Then s' has a prefix resulting from a reduction of a term of form $c^k au$ or $c^k bu$ to normal form — for arbitrarily large k. A cursory examination of the rules reveals that the only two possibilities for such prefixes are $a(ab)^k$ and $b(ba)^k$. Since the k indeed is unbounded as $s'_n \to s'_\infty$, we conclude that $s' \in \{a(ab)^\omega, b(ba)^\omega\}$.

Thus, the only pairs added to $(=_\mathcal{S})$ by the operator $S(\cdot)$ are those of the form $\langle w\alpha, vc^\omega\rangle$, where $v \to^* w$ and $\alpha \in \{a(ab)^\omega, b(ba)^\omega\}$. The equivalence generated by these relations corresponds to infinitary conversion in the augmented iTRS:

$$\mathcal{R}^+ := \mathcal{R} \cup \begin{cases} a(ab)^\omega \to c^\omega \\ b(ba)^\omega \to c^\omega \end{cases}$$

Let us denote this conversion by $=_{\mathcal{S}^+}$. It remains to show that $S_E(=_{\mathcal{S}^+})$ coincides with $=_{\mathcal{S}^+}$. We proceed as before, starting with a chain of $\mathcal{R}^+$-conversions

$$s_0 \to^\infty \cdot {}^\infty\!\!\leftarrow s_1 \to^\infty \cdot {}^\infty\!\!\leftarrow s_2 \to^\infty \cdots \qquad \text{with } \lim_{n\to\infty} s_n = s_\infty \tag{7.4}$$

We remark that $\mathcal{R}^+$ is still confluent, owing to lack of overlap.

Even though compression fails due to the presence of rules with infinite left sides, this failure happens to be completely innocuous: if any of these rules are ever used in a reduction sequence, they will replace an infinite part of the term by a normal form which cannot interact with anything — and the finite prefix which remains is strongly normalizing (since $\mathcal{R}^+$ is finitarily SN). In particular, for any $\mathcal{R}^+$-reduction $\rho : u \to^\infty u'$, the following are equivalent:

– ρ cannot be compressed to length ω;
– ρ factors as $u \to^\infty w\alpha \to wc^\omega$, where $\alpha \in \{a(ab)^\omega, b(ba)^\omega\}$ and $u \to^\infty w\alpha$ is is an infinite reduction.

We thus again obtain standard, $(\omega+1)$-compressed reductions

$$\rho_n : s_n \to^\infty s := \mathrm{NF}^\infty(s_0) \in \mathcal{S} \tag{7.5}$$

If it so happens that, no reduction ρ_n fires any of the new rules, then it is evident that (s, s_∞) is already included in $=_{\mathcal{S}^+}$. Otherwise, if one of the new rules is ever fired, then $s = wc^\omega$. If t is any term, and ρ is a reduction from t to a normal form of the shape wc^ω, then ρ factors as $\rho^f \circ \rho^i \circ \rho^!$, where

– $t = t^0 t^i$;
– $\rho^f : t_0 \to^* w$;
– $\rho^i : t_i \to^\infty \alpha$ are reductions in $\mathcal{R}$, where $\alpha \in \{a(ab)^\omega, b(ba)^\omega\}$;
– $\rho^! : \alpha \to c^\omega$.

Applying this observation for each ρ_n from (7.5), we conclude that the initial part ρ_n^f must eventually stabilize (due to stabilization of prefixes as $s_n \overset{n\to\infty}{\longrightarrow} s_\infty$). For the same reason, we have that $\rho_n^!$ must eventually settle on one of the two new rules, with the target of ρ^i converging to its left side. The remaining reductions ρ^i, being pure $\mathcal{R}$-reductions, are covered by our earlier analysis, and so we conclude that $s_\infty = s_\infty^0 s_\infty^i$, with $s_\infty^0 =_{\mathcal{S}^+} w$, and $s_\infty^i =_{\mathcal{S}^+} c^\omega$.

Finally, let us remark why it suffices to consider limits of length ω. This is settled by induction on the sequence length β.

As decisively settled in TeReSe, a strongly convergent sequence is necessarily of at-most-countable length β.

If β is a successor ordinal, then we conclude by finite induction from the greatest limit ordinal less than β.

Otherwise, β will be a limit ordinal, and we shall be able to produce a sequence $s_0, s_{\beta(1)}, s_{\beta(2)}, \ldots$ as before, with $s_{\beta(i)} \to^\infty \cdot {}^\infty\!\leftarrow s_0$.

The same analysis applies, and we deduct that $s_\beta \to^\infty_{\mathcal{R}^+} \mathsf{NF}(s_0)$. □

8. Correspondence of Proof Trees and Rewrite Sequences

In this section, we investigate the correspondence between ordinal-indexed rewrite sequences and coinductive proof trees. We define a *correspondence relation* that makes precise when a rewrite sequence is represented by a certain proof tree. In general, this correspondence is a many-to-many relation: a proof tree represents a class of rewrite sequences, and a rewrite sequence can be represented by different proof trees.

We then define *canonical proof trees* for $\to^\infty$ in such a way that every ordinal-indexed rewrite sequence has a unique representative among the canonical proof trees. More precisely, there is a many-to-one correspondence between rewrite sequences and canonical proof trees. To characterise the class of rewrite sequences represented by the same canonical proof tree, we introduce a notion of equivalence on infinitary rewrite sequences, called *parallel permutation equivalence*. Thereby two rewrite sequences are considered equivalent if they differ only in the order of steps in parallel subtrees.

Notation 8.1. For a rewrite sequence $S : s_0 \to_{\mathcal{R}}^{\alpha} s_\alpha$ consisting of steps $(s_\beta \to s_{\beta+1})_{\beta<\alpha}$ arising from the application of the rule $\ell_\beta \to r_\beta$ with substitution σ_β at position p_β, respectively, we introduce the following notation

$$\begin{aligned} \mathit{rul}(S,\beta) &= \ell_\beta \to r_\beta \\ \mathit{pos}(S,\beta) &= p_\beta \\ \mathit{sub}(S,\beta) &= \sigma_\beta \end{aligned}$$

for every $\beta < \alpha$.

8.1. **The Correspondence Relation.** Assume that we have a term $f(s_1,\ldots,s_n)$ and rewrite sequences on the direct subterms $S_1 : s_1 \to^{\alpha_1} t_1$, $\ldots$, $S_n : s_n \to^{\alpha_n} t_n$. As these rewrite sequences occur in parallel subterms, any interleaving of them gives rise to a rewrite sequence $f(s_1,\ldots,s_n) \to^{\beta} f(t_1,\ldots,t_n)$. The following definition introduces the notion of *interleaving* on the basis of a monotonic bijective embedding of the disjoint union $\alpha_1 \uplus \ldots \uplus \alpha_n$ into β.

Definition 8.2. Let $f \in \Sigma$ of arity n. Let $S_i : s_i \to^{\alpha_i} t_i$ be rewrite sequences of length α_i for every $i \in \{1,\ldots,n\}$. A rewrite sequence $T : f(s_1,\ldots,s_n) \to^{\beta} f(t_1,\ldots,t_n)$ of length β is called an *interleaving of* $S_1,\ldots,S_n$ *with root* f if there exists a bijection

$$\xi : (\{1\} \times \alpha_1 \cup \ldots \cup \{n\} \times \alpha_n) \to \beta$$

such that for every $i \in \{1,\ldots,n\}$ and every $\gamma < \alpha_i$ we have:

(i) $\mathit{pos}(T,\xi(i,\gamma)) = i \cdot \mathit{pos}(S_i,\gamma)$ (*corresponding position in the i-th argument*),
(ii) $\mathit{rul}(T,\xi(i,\gamma)) = \mathit{rul}(S_i,\gamma)$ (*same rule*),
(iii) $\mathit{sub}(T,\xi(i,\gamma)) = \mathit{sub}(S_i,\gamma)$ (*same substitution*), and
(iv) for every $\gamma' < \gamma$ it holds that $\xi(i,\gamma') < \xi(i,\gamma)$ (*monotonic embedding*).

The following definition introduces the correspondence between coinductive proof trees and ordinal-indexed rewrite sequences.

Definition 8.3. Let $\mathcal{R}$ be a term rewriting system. We define the *correspondence relation* between proof trees (with respect to Definition 4.2) and ordinal-indexed rewrite sequences as the largest relation such that the following conditions hold:

(i) A proof tree of the form

$$\frac{\delta_1 \quad \delta_2 \quad \cdots \quad \delta_n}{s \to^{\infty} t}\ \mathsf{split}$$

corresponds to a rewrite sequence $S : s \to^{\alpha} t$ if S is the concatenation of rewrite sequences $S_1,\ldots,S_n$ such that δ_i corresponds to S_i for every $i \in \{1,\ldots,n\}$.

(ii) A proof tree of the form $s \to_{\varepsilon} t$ only corresponds to the rewrite sequence $s \to_{\varepsilon} t$.

(iii) A proof tree of the form

$$\dfrac{\delta_1 \quad \delta_2 \quad \cdots \quad \delta_n}{f(s_1, s_2, \ldots, s_n) \overset{(<)}{\to}{}^{\infty} f(t_1, t_2, \ldots, t_n)}\ \mathsf{lift}$$

corresponds to a rewrite sequence $S : s \to^{\alpha} t$ if S is an interleaving of rewrite sequences $S_1, \ldots, S_n$ with root f such that the proof tree δ_i corresponds to the rewrite sequence S_i for every $i \in \{1, \ldots, n\}$.

(iv) A proof tree of the form

$$\dfrac{}{s \overset{(<)}{\to}{}^{\infty} s}\ \mathsf{id}$$

only corresponds to the empty rewrite sequence $s \to^0 s$.

Remark 8.4. Note that a proof tree corresponds to more than one rewrite sequence if and only if it contains an application of the lift-rule with (at least) two premises that do not correspond to empty rewrite sequences. The lift-rule introduces choice in the 'construction' of the rewrite sequence by allowing for an arbitrary interleaving of the rewrite sequences on the arguments.

The following example illustrates that a proof term can correspond to an infinite number of ordinal-indexed rewrite sequences.

Example 8.5. We consider the proof trees in Figures 3 and 4:

(i) The proof tree for $\mathsf{a} \to^{\infty} \mathsf{C}^{\omega}$ corresponds to the only rewrite sequence $\mathsf{a} \to^{\omega} \mathsf{C}^{\omega}$.
(ii) The proof tree for $\mathsf{b} \to^{\infty} \mathsf{C}^{\omega}$ corresponds to the only rewrite sequence $\mathsf{b} \to^{\omega} \mathsf{C}^{\omega}$.
(iii) The proof tree for $\mathsf{f}(\mathsf{a}, \mathsf{b}) \overset{\leq}{\to}{}^{\infty} \mathsf{f}(\mathsf{C}^{\omega}, \mathsf{C}^{\omega})$ corresponds to all possible interleavings of $\mathsf{a} \to^{\omega} \mathsf{C}^{\omega}$ and $\mathsf{b} \to^{\omega} \mathsf{C}^{\omega}$ applied to the respective subterms $\mathsf{f}(\mathsf{a}, \mathsf{b})$. Note that there are continuum many rewrite sequences that all have length ω or $\omega \cdot 2$.

The next example shows that some rewrite sequences can be represented by multiple proof trees.

Example 8.6. There are multiple proof trees for the rewrite sequence $\mathsf{a} \to^{\omega} \mathsf{C}^{\omega}$, for example the proof trees shown in Figures 3 and 6.

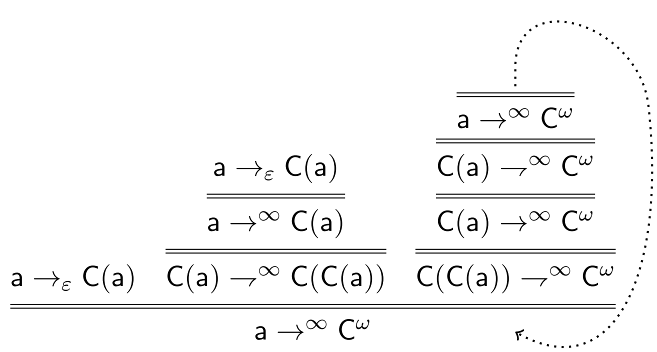

Figure 6: A second proof tree for $\mathsf{a} \to^{\infty} \mathsf{C}^{\omega}$.

Example 8.7. Let $\mathcal{R}$ consist of the rule $\mathsf{f}(x) \to \mathsf{g}(x)$. Figures 7 and 8 show proof trees corresponding to rewrite sequences $\mathsf{f}^\omega \to^\omega \mathsf{g}^\omega$. The proof tree in Figure 7 corresponds to the rewrite sequence

$$\mathsf{f}^\omega \to \mathsf{g}(\mathsf{f}^\omega) \to \mathsf{g}(\mathsf{g}(\mathsf{f}^\omega)) \to \cdots \to^\omega \mathsf{g}^\omega \ ,$$

and the proof tree in Figure 8 corresponds to

$$\mathsf{f}^\omega \to \mathsf{g}(\mathsf{f}^\omega) \to \mathsf{g}(\mathsf{f}(\mathsf{g}(\mathsf{f}^\omega))) \to \mathsf{g}^3(\mathsf{f}^\omega) \to \mathsf{g}^4(\mathsf{f}^\omega) \to \mathsf{g}^5(\mathsf{f}^\omega) \to \cdots \to^\omega \mathsf{g}^\omega \ .$$

Note that, by Remark 8.4, these rewrite sequences are unique. Both proof trees correspond to precisely one rewrite sequence since they do not contain applications of the lift-rule (rules with conclusion $\rightharpoondown^\infty$) with multiple premises.

$$\dfrac{\mathsf{f}^\omega \to_\varepsilon \mathsf{g}(\mathsf{f}^\omega) \qquad \dfrac{\dfrac{}{\mathsf{f}^\omega \to^\infty \mathsf{g}^\omega}}{\mathsf{g}(\mathsf{f}^\omega) \rightharpoondown^\infty \mathsf{g}^\omega}}{\mathsf{f}^\omega \to^\infty \mathsf{g}^\omega}$$

Figure 7: A proof tree for $\mathsf{f}^\omega \to \mathsf{g}(\mathsf{f}^\omega) \to \mathsf{g}(\mathsf{g}(\mathsf{f}^\omega)) \to \cdots \to^\omega \mathsf{g}^\omega$.

$$\dfrac{\mathsf{f}^\omega \to_\varepsilon \mathsf{g}(\mathsf{f}^\omega) \qquad \dfrac{\dfrac{\dfrac{\dfrac{\mathsf{f}^\omega \to_\varepsilon \mathsf{g}(\mathsf{f}^\omega)}{\mathsf{f}^\omega \to^\infty \mathsf{g}(\mathsf{f}^\omega)}}{\mathsf{f}^\omega \rightharpoondown^\infty \mathsf{f}(\mathsf{g}(\mathsf{f}^\omega))} \quad \mathsf{f}(\mathsf{g}(\mathsf{f}^\omega)) \to_\varepsilon \mathsf{g}(\mathsf{g}(\mathsf{f}^\omega)) \quad \dfrac{\dfrac{\dfrac{\dfrac{\text{Figure 7}}{\mathsf{f}^\omega \to^\infty \mathsf{g}^\omega}}{\mathsf{g}(\mathsf{f}^\omega) \rightharpoondown^\infty \mathsf{g}^\omega}}{\mathsf{g}(\mathsf{f}^\omega) \to^\infty \mathsf{g}^\omega}}{\mathsf{g}(\mathsf{g}(\mathsf{f}^\omega)) \rightharpoondown^\infty \mathsf{g}^\omega}}{\mathsf{f}^\omega \to^\infty \mathsf{g}^\omega}}{\mathsf{g}(\mathsf{f}^\omega) \rightharpoondown^\infty \mathsf{g}^\omega}}{\mathsf{f}^\omega \to^\infty \mathsf{g}^\omega}$$

Figure 8: A proof tree for $\mathsf{f}^\omega \to \mathsf{g}(\mathsf{f}^\omega) \to \mathsf{g}(\mathsf{f}(\mathsf{g}(\mathsf{f}^\omega))) \to \mathsf{g}^3(\mathsf{f}^\omega) \to \mathsf{g}^4(\mathsf{f}^\omega) \to \cdots \to^\omega \mathsf{g}^\omega$.

8.2. **Canonical Proof Trees and Parallel Permutation Equivalence.** In order to have a unique proof tree for every ordinal-indexed rewrite sequence, we introduce 'canonical' proof trees for $\to^\infty$. We show that the correspondence of canonical proof trees and rewrite sequences is a one-to-many relationship, and we characterise the class of rewrite sequences represented by a canonical proof tree in terms of 'parallel permutation equivalence'.

Definition 8.8. A proof tree for $\to^\infty$ is called *canonical* if

(i) every application of the split rule is an instance of the *canonical* split rule

$$\frac{s\ (\overset{\le}{\to}{}^{\infty}\ ;\ \to_\varepsilon)^*\ ;\ \to^{\infty}\ t}{s \to^{\infty} t}\ \textit{canonical}\ \mathsf{split} \quad \text{, and}$$

(ii) every application of the id rule is an instance of

$$\frac{}{x \overset{(\le)}{\to}{}^{\infty}\ x}\ \textit{canonical}\ \mathsf{id}$$

In contrast with the split rule from Definition 4.2, the *canonical* split rule replaces

$$(\overset{\le}{\to}{}^{\infty} \cup \to_\varepsilon)^* \quad \text{by} \quad (\overset{\le}{\to}{}^{\infty}\ ;\ \to_\varepsilon)^*\ .$$

Thereby the canonical form enforces that $\overset{\le}{\to}{}^{\infty}$ and $\to_\varepsilon$ alternate.

The canonical id rule replaces

$$s \overset{(\le)}{\to}{}^{\infty}\ s \quad \text{by} \quad x \overset{(\le)}{\to}{}^{\infty}\ x\ .$$

Thereby it enforces unique proof trees for empty rewrite sequences (using rules lift and canonical id).

Definition 8.9. Let $p, q \in \mathbb{N}^*$ be positions. We define

(i) $p \le q$ if $pr = q$ for some $r \in \mathbb{N}^*$,
(ii) $p \parallel q$ if $p \not\le q$ and $q \not\le p$.

If $p \parallel q$, then we say that p and q are *parallel* (to each other).

Recall that we consider ordinals α to be the set of all smaller ordinals: $\alpha = \{\beta \mid \beta < \alpha\}$. This allows us to speak about functions $f : \alpha \to \beta$.

Definition 8.10. Let $\mathcal{R}$ be a term rewriting system. Let $S : s \to_{\mathcal{R}}^{\infty} t_1$ and $T : s \to_{\mathcal{R}}^{\infty} t_2$ be strongly convergent rewrite sequences of length α and β, respectively.

The rewrite sequence S is called *parallel permutation equivalent* to T if there exists a bijection $f : \alpha \to \beta$ such that

(i) $\mathit{rul}(S, \gamma) = \mathit{rul}(T, f(\gamma))$ and $\mathit{pos}(S, \gamma) = \mathit{pos}(T, f(\gamma))$ for every $\gamma < \alpha$, and
(ii) $\mathit{pos}(S, \gamma_1) \parallel \mathit{pos}(S, \gamma_2)$ whenever $\gamma_1 < \gamma_2 < \alpha$ and $f(\gamma_1) > f(\gamma_2)$.

Observe that, the bijective mapping $f : \alpha \to \beta$ defines a permutation of the steps in the sequence S. The condition (i) guarantees that the step indexed by γ in S corresponds to the step indexed by $f(\gamma)$ in T as follows: both steps must arise from the same rule applied at the same position. The condition (ii) ensures that steps that have been permuted (changed their relative order in the sequence), arise from contractions at parallel positions.

In the following definition we select a subsequence of the steps of T that corresponds to (the permutation of) a prefix of S. For this purpose, we consider a step to be the application of a certain rule at a certain position. We do not take into account the source and the target of the steps as these may change due to preceding steps being dropped (not selected).

Definition 8.11. Let $S : s_0 \to_{\mathcal{R}}^{\alpha} s_\alpha$ and $T : t_0 \to_{\mathcal{R}}^{\beta} t_\beta$ be parallel permutation equivalent with respect to the bijection $f : \alpha \to \beta$. Let $\kappa \le \alpha$, and define S' as the prefix of S of length κ. We define the *permutation of S' with respect to f* as the rewrite sequence obtained from T by selecting the subsequence of steps at indexes $\gamma < \beta$ for which $f^{-1}(\gamma) < \kappa$.

As dropping (not selecting) a step changes all subsequent terms in the sequence, we need to show that the selected steps still form a rewrite sequence (the rules are applicable at the designated positions).

Proof of well-definedness of Definition 8.11. To prove that the selected subsequence of steps from T forms a rewrite sequence, we show that every non-selected (dropped) step is parallel to all subsequent selected steps. As a consequence, the dropped steps do not influence the applicability of the selected steps. Let us consider a step that is dropped, that is $\gamma < \beta$ with $f^{-1}(\gamma) \geq \kappa$, and a subsequent step that is selected, γ' with $\gamma < \gamma' < \beta$ and $f^{-1}(\gamma') < \kappa$. From parallel permutation equivalence of S and T it follows immediately that $\mathit{pos}(T, \gamma) \parallel \mathit{pos}(T, \gamma')$ since $f^{-1}(\gamma') < \kappa \leq f^{-1}(\gamma)$ and $\gamma < \gamma'$. □

Lemma 8.12. *Let $S : s_0 \to_{\mathcal{R}}^{\alpha} s_\alpha$ and $T : t_0 \to_{\mathcal{R}}^{\beta} t_\beta$ be parallel permutation equivalent with respect to the bijection $f : \alpha \to \beta$. Every prefix S' of S is parallel permutation equivalent to the permutation of S' with respect to f.*

Proof. Follows immediately from parallel permutation equivalence of the rewrite sequences S and T together with Definition 8.11 since the order of the selected steps is preserved (both from S to S' as well as from T to the subsequence of selected steps of T). □

The following lemma states that parallel permutation equivalent sequences converge to the same target term.

Lemma 8.13. *If the rewrite sequences $S : s_0 \to_{\mathcal{R}}^{\alpha} s_\alpha$ and $T : t_0 \to_{\mathcal{R}}^{\beta} t_\beta$ are parallel permutation equivalent, then we have $s_\alpha = t_\beta$.*

Proof. We prove the claim by induction on α. Let $S : s_0 \to_{\mathcal{R}}^{\alpha} s_\alpha$ and $T : t_0 \to_{\mathcal{R}}^{\beta} t_\beta$ be rewrite sequences that are parallel permutation equivalent. Let $f : \alpha \to \beta$ be such that the conditions of Definition 8.10 are fulfilled. We distinguish cases for α:

(i) If $\alpha = 0$, it follows that $\beta = 0$. Then $s_0 = t_0$ by definition of parallel permutation equivalence (the starting terms of the reductions must be equal).

(ii) If α is a successor ordinal $\alpha = \alpha' + 1$, we proceed as follows.

Let S' be the prefix $s \to^{\alpha'} s_{\alpha'}$ of S of length α'. In other words, S' is the rewrite sequence obtained from S by dropping the last step $s_{\alpha'} \to s_\alpha$.

Let T' be the permutation of S' with respect to f. So, T' is the rewrite sequence $s \to^{\beta'} u_{\beta'}$ with $\beta' \leq \beta$ obtained from T by dropping the step $t_{f(\alpha')} \to t_{f(\alpha')+1}$. Let

$$(\ell \to r) = \mathit{rul}(T, f(\alpha')) \qquad p = \mathit{pos}(T, f(\alpha')) \qquad \sigma = \mathit{sub}(T, f(\alpha'))$$

Recall that the step $t_{f(\alpha')} \to t_{f(\alpha')+1}$ can be dropped from T as its position is parallel to the positions of all subsequent steps in T: $\mathit{pos}(T, f(\alpha')) \parallel \mathit{pos}(T, \gamma)$ for every γ with $f(\alpha') < \gamma < \beta$. From this fact it also follows that

$$u_{\beta'}|_p = t_{f(\alpha')}|_p = \ell\sigma \qquad t_\beta|_p = t_{f(\alpha')+1}|_p \qquad t_\beta = u_{\beta'}[r\sigma]_p \tag{8.1}$$

Observe that S' is shorter than S as we have removed the last step. In contrast, the length of T' may be less or equal to the length of T. For example, dropping the 5th step from an ω-long sequence does not decrease its length.

From parallel permutation equivalence of S and T, it follows by Lemma 8.12 that S' and T' are parallel permutation equivalent. By induction hypothesis, we may conclude that $s_{\alpha'} = u_{\beta'}$. From (8.1) it follows that

$$s_{\alpha'}|_p = \ell\sigma \qquad\qquad s_\alpha = s_{\alpha'}[r\sigma]_p$$

since $rul(S, \alpha') = rul(T, f(\alpha')) = \ell \to r$ and $pos(S, \alpha') = pos(T, f(\alpha')) = p$. Then

$$s_\alpha = s_{\alpha'}[r\sigma]_p = u_{\beta'}[r\sigma]_p = t_\beta \; .$$

(iii) Assume α is a limit ordinal. We use $s =_n t$ to denote that the terms s and t coincide up to depth n. For $s_\alpha = t_\alpha$ it suffices to show that $s_\alpha =_n t_\alpha$ for every $n \in \mathbb{N}$.

Let $n \in \mathbb{N}$ be arbitrary. By strong convergence of S there exists a strict prefix S' of S that contains all steps of S at depth $\leq n$. So, if κ is the length of S', then all steps in S at index an $\gamma \geq \kappa$ have depth $> n$. Let T' be the permutation of S' with respect to f. By Lemma 8.12, S' and T' are permutation equivalent, and by induction hypothesis, they have the same final term, say final term u. As all steps in S after the last step of S' are at depth $> n$ we obtain $u =_n s_\alpha$. Likewise, we get $u =_n t_\beta$ since T' contains all steps of T that are at depth $\leq n$. Hence $s_\alpha =_n t_\beta$.

This concludes the proof. □

Proposition 8.14. *Parallel permutation equivalence is an equivalence relation.*

Proof. We prove reflexivity, symmetry and transitivity:

(a) *Reflexivity.* Parallel permutation equivalence of a rewrite sequence $S : s \to^\alpha t$ to itself is witnessed by choosing f as the identity function on α in Definition 8.10; then both conditions in the definition are trivially fulfilled.

(b) *Symmetry.* Let $S : s \to^\alpha t$ be parallel permutation equivalent to $T : s \to^\beta t$ with witnessing bijection $f : \alpha \to \beta$. Then parallel permutation equivalence of T to S is witnessed by f^{-1}; we check both conditions of Definition 8.10:

  (i) $rul(T, \gamma) = rul(T, f(f^{-1}(\gamma))) = rul(S, f^{-1}(\gamma))$ and
  $pos(T, \gamma) = pos(T, f(f^{-1}(\gamma))) = pos(S, f^{-1}(\gamma))$ for every $\gamma < \beta$
  (since $f^{-1}(\gamma) < \alpha$)

  (ii) We have

$$\begin{aligned} &\gamma_1 < \gamma_2 < \beta \text{ with } f^{-1}(\gamma_1) > f^{-1}(\gamma_2) \\ &\iff f^{-1}(\gamma_2) < f^{-1}(\gamma_1) < \alpha \text{ with } f(f^{-1}(\gamma_2)) = \gamma_2 > \gamma_1 = f(f^{-1}(\gamma_1)) \end{aligned}$$

  Hence $pos(T, \gamma_1) \parallel pos(T, \gamma_2)$ whenever $\gamma_1 < \gamma_2 < \beta$ and $f^{-1}(\gamma_1) > f^{-1}(\gamma_2)$.

(c) *Transitivity.* Let $S : s \to^\alpha t$ be parallel permutation equivalent to $T : s \to^\beta t$ and $T : s \to^\beta t$ parallel permutation equivalent to $U : s \to^\delta t$ witnessed by bijections $f : \alpha \to \beta$ and $g : \beta \to \delta$, respectively. Then parallel permutation equivalence of S to U is witnessed by $g \circ f$; we check both conditions of Definition 8.10:

  (i) $rul(S, \gamma) = rul(T, f(\gamma)) = rul(U, g(f(\gamma)))$ and
  $pos(S, \gamma) = pos(T, f(\gamma)) = pos(U, g(f(\gamma)))$ for every $\gamma < \alpha$, and

  (ii) Assume that $\gamma_1 < \gamma_2 < \alpha$ with $g(f(\gamma_1)) > g(f(\gamma_2))$. Then either $f(\gamma_1) > f(\gamma_2)$ or $f(\gamma_1) < f(\gamma_2) \;\wedge\; g(f(\gamma_1)) > g(f(\gamma_2))$. In the former case, $pos(S, \gamma_1) \parallel pos(S, \gamma_2)$ follows from parallel permutation equivalence of S to T. In the latter case, it follows from parallel permutation equivalence of T to U. □

The following lemma implies that the witnessing function f in the definition of parallel permutation equivalence is unique (for fixed rewrite sequences S and T).

Lemma 8.15. *Let $S : s \to^{\alpha} t$ and $T : s \to^{\beta} t$ be rewrite sequences. Let $\alpha' \leq \alpha$ and let $f : \alpha' \to \beta$ be an an injective function with the properties*

(i) *$rul(S,\gamma) = rul(T, f(\gamma))$ and $pos(S,\gamma) = pos(T, f(\gamma))$ for every $\gamma < \alpha'$, and*
(ii) *$pos(S,\gamma_1) \parallel pos(S,\gamma_2)$ whenever $\gamma_1 < \gamma_2 < \alpha'$ and $f(\gamma_1) > f(\gamma_2)$.*
(iii) *$pos(T,\gamma_1) \parallel pos(T,\gamma_2)$ whenever $\gamma_1 < \gamma_2 < \beta$, γ_1 is not in the image of f, but γ_2 is,*

Then f is unique with these properties (for fixed S, T, α').

Moreover, among the ordinals $\leq \alpha$, there exists a largest ordinal α' such that a function f with these properties exists.

Before we prove the lemma, let us give some intuition for the conditions (i)–(iii). Item (i) ensures that f correctly embeds S into T in the sense it respects redex position and the applied rule. Condition (ii) guarantees that f only swaps steps that are at parallel positions. Finally, condition (iii) requires that steps in T that are not in the image of f must be parallel to all subsequent steps in T that are in the image of f.

Proof of Lemma 8.15. We prove uniqueness of f by induction on α'

The base case $\alpha' = 0$ is trivial.

For α' a successor ordinal, $\alpha' = \alpha'' + 1$, we argue as follows. Let $f, g : \alpha' \to \beta$ be injective functions fulfilling the properties (i)–(iii). Then by induction hypothesis, the functions $f|_{\alpha''}$ and $g|_{\alpha''}$ coincide. Assume, for a contradiction, $f(\alpha'') \neq g(\alpha'')$. Without loss of generality we may assume $f(\alpha'') < g(\alpha'')$. Then $f(\alpha'') < g(\alpha'') < \beta$ and $f(\alpha'')$ is not in the image of g, but $g(\alpha'')$ is. However, $pos(T, f(\alpha'')) = pos(S, \alpha'') = pos(T, g(\alpha''))$, contradicting property (iii) for the function g. Hence f and g coincide.

Let α' be a limit ordinal and let $f, g : \alpha' \to \beta$ be injective functions fulfilling the properties (i)–(iii). Assume, for a contradiction, that $f(\alpha'') \neq g(\alpha'')$ for some $\alpha'' < \alpha'$. However, we have that $\alpha'' + 1 < \alpha'$ since α' is a limit ordinal. By induction hypothesis we obtain $f|_{\alpha''+1}$ coincides with $g|_{\alpha''+1}$. Thus $f(\alpha'') = g(\alpha'')$, contradicting our assumption. Hence, f and g coincide. This concludes the proof of uniqueness of f.

We write $P(\alpha')$ if there exists an injective function $f : \alpha' \to \beta$ with properties (i)–(iii). It remains to be shown that there exists a largest ordinal $\alpha' \leq \alpha$ for which $P(\alpha')$ holds. Let ξ be the supremum (union) of all ordinals $\xi' \leq \alpha$ for which $P(\xi')$; note that this set is non-empty since always $P(0)$ holds. If $P(\xi)$, then ξ is the largest of these ordinals. Thus assume $\neg P(\xi)$. Then ξ is a limit ordinal. Note that $\xi'' < \xi'$ and $P(\xi')$ imply $P(\xi'')$. As a consequence we have $P(\xi')$ for every $\xi' < \xi$. As the length of every rewrite sequence is countable [35], it follows that α and thus ξ are countable. Every countable limit ordinal has cofinality ω. Thus there exist ordinals $\xi_1 < \xi_2 < \xi_3 < \ldots$, each of which $< \xi$, such that $\xi = \bigcup \{ \xi_i \mid i \in \mathbb{N} \}$. Then $P(\xi_i)$ for every $i \in \mathbb{N}$. For every $i \in \mathbb{N}$, there exists an injective function $f_i : \xi_i \to \beta$ fulfilling properties (i)–(iii) for $\alpha' = \xi_i$. From the uniqueness (shown above), it follows that f_i coincides with $f_j|_{\xi_i}$ for every $i < j$. As a consequence, we can define $f : \xi \to \beta$ by

$$f(\xi') = f_i(\xi') \text{ whenever } i \in \mathbb{N} \text{ and } \xi' < \xi_i\,.$$

We claim that f is injective and has the properties (i)–(iii) and hence $P(\xi)$ holds; contradicting the above assumption. Injectivity and property (i) are immediate. Property (ii) follows from the fact that for $\gamma_1 < \gamma_2 < \xi$ there exists $i \in \mathbb{N}$ such that $\gamma_1, \gamma_2 < \xi_i$ (since ξ is the supremum of the ξ_i's). Then f_i fulfilling property (ii) for γ_1, γ_2 implies f fulfilling property (ii) for γ_1, γ_2. Analogously, property (iii) follows from the following observation: whenever $\gamma_1 < \gamma_2 < \beta$, if γ_2 is in the image of f and γ_1 is not, then there exists some i such that γ_2 is in the image of

f_i (because f is the union of all f_j), while γ_1 is not (since the image of f contains that of all f_j). □

Lemma 8.16. *Let $S : s \to^{\alpha} t$ be a rewrite sequence. Then S is of the form*

$$S = S_1 \,;\, S_2 \,;\, S_3 \,;\, \cdots \,;\, S_{2n+1}$$

for some $n \in \mathbb{N}$ such that for every $i \in \{1, \ldots, 2n\}$ we have:

(i) *for odd i, $S_i : s_i \to^{\alpha} s_{i+1}$ is a reduction below the root,*
(ii) *for even i, $S_i : s_i \to_{\varepsilon} s_{i+1}$ is a root step.*

If S is parallel permutation equivalent to $T : s \to^{\beta} t$, then T is of the form:

$$T = T_1 \,;\, T_2 \,;\, T_3 \,;\, \cdots \,;\, T_{2n+1}$$

such that S_i is parallel permutation equivalent to T_i for every $i \in \{1, \ldots, 2n+1\}$.

Proof. From the definition of parallel permutation equivalence, it follows that

($\star$) no steps can swap the order with a root step.

As a consequence, T contains the same root steps as S, in the same order. Hence T is of the form $T = T_1 \,;\, T_2 \,;\, T_3 \,;\, \cdots \,;\, T_{2n+1}$ such that for every $i \in \{1, \ldots, 2n\}$ we have:

(i) for odd i, $T_i : t_i \to^{\alpha} t_{i+1}$ is a reduction below the root,
(ii) for even i, $T_i : t_i \to_{\varepsilon} t_{i+1}$ is a root step, the same as S_i.

From ($\star$) it moreover follows that S_i is parallel permutation equivalent to T_i for every $i \in \{1, \ldots, 2n+1\}$. The reason is that there cannot be a step $s \in S_i$ with $f(s) \in T_j$ where i, j are odd and $j \neq i$. For otherwise, this would imply a swap of s with respect to the root step S_{i+1} (if $i < j$) or the root step S_{i-1} (if $i > j$). □

The next definition extracts the order in which certain rule applications occur in a rewrite sequence. A *rule application* is formally a pair $\langle \rho, p \rangle \in \mathcal{R} \times \mathbb{N}^*$ consisting of a rewrite rule and a position. A *step* in a rewrite sequence of length α is a triple $\langle \beta, \rho, p \rangle$ where $\beta < \alpha$ is an index and $\langle \rho, p \rangle$ is a rule application. Given a rewrite sequence S, consider the sequence of rule applications that take place at each step in S. We are interested in the subsequence of all those $\langle \rho, p \rangle$ that fall in a given set $P \subseteq \mathcal{R} \times \mathbb{N}^*$.

Definition 8.17. Let $S : s \to^{\alpha} t$ be a rewrite sequence. The *rule application sequence of S* is the sequence $\mathit{rulapp}(S) \colon \alpha \to \mathcal{R} \times \mathbb{N}^*$ given by $\mathit{rulapp}(S)(\beta) = \langle \mathit{rul}(S, \beta), \mathit{pos}(S, \beta) \rangle$ for all $\beta < \alpha$. Given a set $P \subseteq \mathcal{R} \times \mathbb{N}^*$ of *rule applications*, we define the *P-projection of S* as the subsequence $\mathit{proj}_P(S)$ of $\mathit{rulapp}(S)$ obtained by picking all rule applications in P.

In other words, $\mathit{proj}_P(S)$ is a function $\mathit{proj}_P(S) : \beta \to P$ for some ordinal $\beta \leq \alpha$ together with an embedding $f : \beta \to \alpha$ such that:

(i) *f is an embedding:* $\mathit{proj}_P(S)(\gamma) = \langle\, \mathit{rul}(S, f(\gamma)),\, \mathit{pos}(S, f(\gamma)) \,\rangle$ for every $\gamma < \beta$,
(ii) *f is increasing:* $f(\gamma_1) < f(\gamma_2)$ whenever $\gamma_1 < \gamma_2 < \beta$, and
(iii) *f selects all rule applications in P:* for all steps $\langle \gamma, \mathit{rul}(S, \gamma),\, \mathit{pos}(S, \gamma) \,\rangle$ in S, if $\langle\, \mathit{rul}(S, \gamma),\, \mathit{pos}(S, \gamma) \,\rangle \in P$ then γ is in the image of f.

We say that rewrite sequences S_1 and S_2 have the *same order of rule applications in* $P \subseteq \mathcal{R} \times \mathbb{N}^*$ if $\mathit{proj}_P(S_1) = \mathit{proj}_P(S_2)$ (here we mean point-wise equality).

It should be clear that for given S and P, $\mathit{proj}_P(S)$ is well-defined.

Example 8.18. Consider a rewrite system with rules

$$\begin{array}{llcl} (\rho_1) & \mathsf{a}(x) & \to & \mathsf{a}(\mathsf{a}(x)) \\ (\rho_2) & \mathsf{b}(x) & \to & \mathsf{b}(\mathsf{b}(x)) \\ (\rho_3) & \mathsf{f}(x,y) & \to & \mathsf{f}(\mathsf{a}(\mathsf{c}),\mathsf{b}(\mathsf{c})) \end{array}$$

Then we can rewrite

$$\begin{array}{rcll} S : \mathsf{f}(\mathsf{a}(\mathsf{c}),\mathsf{b}(\mathsf{c})) & \to & \mathsf{f}(\mathsf{a}(\mathsf{a}(\mathsf{c})),\mathsf{b}(\mathsf{c})) & \text{using } \langle \rho_1, 0\rangle \\ & \to & \mathsf{f}(\mathsf{a}(\mathsf{a}(\mathsf{a}(\mathsf{c}))),\mathsf{b}(\mathsf{c})) & \text{using } \langle \rho_1, 0\rangle \\ & \to & \mathsf{f}(\mathsf{a}(\mathsf{a}(\mathsf{a}(\mathsf{c}))),\mathsf{b}(\mathsf{b}(\mathsf{c}))) & \text{using } \langle \rho_2, 1\rangle \\ & \to & \mathsf{f}(\mathsf{a}(\mathsf{c}),\mathsf{b}(\mathsf{c})) & \text{using } \langle \rho_3, \epsilon\rangle \\ & \to & \mathsf{f}(\mathsf{a}(\mathsf{c}),\mathsf{b}(\mathsf{b}(\mathsf{c}))) & \text{using } \langle \rho_2, 1\rangle \\ & \to & \mathsf{f}(\mathsf{a}(\mathsf{a}(\mathsf{c})),\mathsf{b}(\mathsf{b}(\mathsf{c}))) & \text{using } \langle \rho_1, 0\rangle \end{array}$$

Taking $P = \{\langle \rho_1, 0\rangle, \langle \rho_2, 1\rangle\}$, we get that $\mathit{proj}_P(S) = \langle \rho_1, 0\rangle, \langle \rho_1, 0\rangle, \langle \rho_2, 1\rangle, \langle \rho_2, 1\rangle, \langle \rho_1, 0\rangle$, and, in particular, $\beta = 5$.

The following lemma states, for a given proof tree, the order of non-parallel rule applications is the same in every rewrite sequence corresponding to the proof tree.

Lemma 8.19. *Let δ be a proof tree (Definition 4.2) and let $P \subseteq \mathcal{R} \times \mathbb{N}^*$ be a set of rule applications such that for every $\langle \rho_1, p_1\rangle, \langle \rho_2, p_2\rangle \in P$ we have $p_1 \not\parallel p_2$. If rewrite sequences S and T both correspond to δ, then $\mathit{proj}_P(S) = \mathit{proj}_P(T)$.*

Proof. It suffices to consider the case that P consists of at most 2 elements since:

$$\mathit{proj}_P(S) = \mathit{proj}_P(T) \iff \forall Q \subseteq P.\, (|Q| \le 2 \implies \mathit{proj}_Q(S) = \mathit{proj}_Q(T)) \,.$$

We prove that for all proof trees δ and all $P \subseteq \mathcal{R} \times \mathbb{N}^*$ with $|P| \le 2$, if rewrite sequences S and T correspond to δ, then $\mathit{proj}_P(S) = \mathit{proj}_P(T)$. So let δ, S, T, P be as stated, and assume that $P = \{\langle \rho_1, p_1\rangle, \langle \rho_2, p_2\rangle\}$; we allow $\langle \rho_1, p_1\rangle = \langle \rho_2, p_2\rangle$. Without loss of generality, we assume that $p_1 \le p_2$. The proof is by well-founded induction on the length of p_2. First we note that if δ is a single root step or a single application of id, the result is immediate.

Base case: Here we have $p_2 = p_1 = \varepsilon$ which means that P contains only root step applications. We distinguish cases according to the root of δ. If the root of δ is obtained from a split-application, then the root steps in S and T must occur in the same order, hence $\mathit{proj}_P(S) = \mathit{proj}_P(T)$. In case the root of δ is obtained from a lift-application, then S and T cannot contain any root steps, hence we are also done.

Induction step: Assume that $p_2 = ip_2'$ and define

$$P' = \begin{cases} \{\langle \rho_2, p_2'\rangle\}, & \text{if } p_1 = \varepsilon \\ \{\langle \rho_1, p_1'\rangle, \langle \rho_2, p_2'\rangle\}, & \text{if } p_1 = ip_1' \end{cases}$$

By induction, we may assume that the property holds for the set of rule applications P'.

For $(\rho, p) \in (\mathcal{R} \times \mathbb{N}^*)$, we define $i \cdot (\rho, p) = (\rho, ip)$. For functions $h : \beta \to \mathcal{R} \times \mathbb{N}^*$, we define $(i \cdot h) : \beta \to \mathcal{R} \times \mathbb{N}^*$ by by $(i \cdot h)(\gamma) = i \cdot h(\gamma)$.

We distinguish cases for the shape of δ. If δ is of the form:

$$\frac{\delta_1 : s_1 \to^\infty t_1 \quad \delta_2 : s_2 \to^\infty t_2 \quad \cdots \quad \delta_n : s_n \to^\infty t_n}{f(s_1, \ldots, s_n) \overset{(<)}{\to}{}^\infty f(t_1, \ldots, t_n)}\ \mathsf{lift}$$

then S, respectively T, is an interleaving of rewrite sequences $S_1, \ldots, S_n$, respectively $T_1, \ldots, T_n$, corresponding to $\delta_1, \ldots, \delta_n$. In particular, $S_i \colon s_i \to^\infty t_i$ and $T_i \colon s_i \to^\infty t_i$ correspond to δ_i. Since S cannot contain root steps, we have that $\langle \rho_1, \varepsilon\rangle$ is not in the image of $\mathit{proj}_P(S)$, so if $p_1 = \varepsilon$ then $\mathit{proj}_P(S) = i \cdot \mathit{proj}_{P'}(S_i)$. If $p_1 = ip_i'$, then all

ρ_1- and ρ_2-applications in P are on the i'th subterm, and since interleaving preserves order of steps in this subterm, we again have that $\mathit{proj}_P(S) = i \cdot \mathit{proj}_{P'}(S_i)$. Similarly, $\mathit{proj}_P(T) = i \cdot \mathit{proj}_{P'}(T_i)$. Now, $\mathit{proj}_P(S) = \mathit{proj}_P(T)$ follows from the induction hypothesis.

If δ is of the form

$$\frac{\delta_1 \quad \delta_2 \quad \cdots \quad \delta_n}{s \to^\infty t}\ \mathsf{split}$$

then S, respectively T, is a concatenation of $S_1, \ldots, S_n$, respectively $T_1, \ldots, T_n$, corresponding to the trees $\delta_1, \ldots, \delta_n$. Hence it suffices to show that $\mathit{proj}_P(S_i) = \mathit{proj}_P(T_i)$ for all $i = 1, \ldots, n$. For all δ_i's that are a single node or an id-application, this holds. The remaining δ_i's must have a root that is obtained from a lift-application, and the result now follows from the lift-case. □

Next, we characterise the correspondence of canonical proof trees and rewrite sequences.

Theorem 8.20. *We have the following facts about canonical proof trees:*

(i) *Every rewrite sequence corresponds to precisely one canonical proof tree.*
(ii) *Rewrite sequences $S : s_0 \to_{\mathcal{R}}^{\alpha} s_\alpha$ and $T : t_0 \to_{\mathcal{R}}^{\beta} t_\beta$ are represented by the same canonical proof tree if and only if they are parallel permutation equivalent.*

Proof. First, note that every rewrite sequence corresponds to a canonical proof tree. This follows by an inspection of the proof of Theorem 5.2: for every rewrite sequence $S : s \to^\alpha t$, the proof tree $\mathfrak{T}(S)$ is canonical and corresponds to S.

Second, we show that canonical proof trees coincide if they correspond to rewrite sequences that are parallel permutation equivalent. As a direct consequence, we obtain that every rewrite sequence corresponds to precisely one canonical proof tree, establishing (i).

Notation: For a strongly convergent rewrite sequence $R : s \to_{ord}^{\alpha} t$, we write $\mathfrak{T}_R$ to denote *an arbitrary* canonical proof tree $\mathfrak{T}_R : s \to^\infty t$ corresponding to R. Note that, a priori, the tree $\mathfrak{T}_R$ can be different from $\mathfrak{T}(R)$. If R is a rewrite sequence below the root, then we moreover write $\mathfrak{T}_R'^{(<)}$ for a canonical proof tree $\mathfrak{T}_R'^{(<)} : s \xrightarrow{(<)\infty} t$ corresponding to R.

Let $S : s \to_{ord}^{\alpha} t$ and $T : s \to_{ord}^{\beta} t$ be rewrite sequences that are parallel permutation equivalent. We show that $\mathfrak{T}_S = \mathfrak{T}_T$ by coinduction, that is: if S and T are parallel permutation equivalent, then $\mathfrak{T}_S$ and $\mathfrak{T}_T$ have the same root and all subtrees arise again from parallel permutation equivalent rewrite sequences. As S and T are parallel permutation equivalent, from Lemma 8.16 it follows that S and T are of the forms:

$$\begin{aligned} S &= S_1 \mathbin{;} S_2 \mathbin{;} S_3 \mathbin{;} \cdots \mathbin{;} S_{2n+1} \\ T &= T_1 \mathbin{;} T_2 \mathbin{;} T_3 \mathbin{;} \cdots \mathbin{;} T_{2n+1} \end{aligned} \tag{8.2}$$

for some $n \in \mathbb{N}$ such that for every $i \in \{1, \ldots, 2n+1\}$ we have:

(a) for odd i, S_i and T_i are reductions below the root,
(b) for even i, S_i and T_i are a root steps, and
(c) S_i is parallel permutation equivalent to T_i for every $i \in \{1, \ldots, 2n+1\}$.

In particular, by Lemma 8.13, S_i and T_i have the same source and target term for every $i \in \{1, \ldots, 2n\}$, say source u_i and target u_{i+1}.

We consider the root of the proof trees $\mathfrak{T}_S : s \to^\infty t$ and $\mathfrak{T}_T : s \to^\infty t$. The only way to derive $\to^\infty$ is by an application of the split-rule. Due to (8.2), (a) and (b), and the form of the canonical split-rule, it follows that $\mathfrak{T}_S$ and $\mathfrak{T}_T$ must be of the form:

$$\mathfrak{T}_S = \dfrac{\mathfrak{T}'^{<}_{S_1} \quad u_2 \to_\varepsilon u_3 \quad \mathfrak{T}'^{<}_{S_3} \quad u_4 \to_\varepsilon u_5 \quad \cdots \quad u_{2n} \to_\varepsilon u_{2n+1} \quad \mathfrak{T}'_{S_{2n+1}}}{u_1 \to^\infty u_{2n+2}} \text{ split}$$

$$\mathfrak{T}_T = \dfrac{\mathfrak{T}'^{<}_{T_1} \quad u_2 \to_\varepsilon u_3 \quad \mathfrak{T}'^{<}_{T_3} \quad u_4 \to_\varepsilon u_5 \quad \cdots \quad u_{2n} \to_\varepsilon u_{2n+1} \quad \mathfrak{T}'_{T_{2n+1}}}{u_1 \to^\infty u_{2n+2}} \text{ split}$$

As a consequence $\mathfrak{T}_S$ and $\mathfrak{T}_T$ have the same root and, due to (c), the subtrees arise from rewrite sequences below the root that are again parallel permutation equivalent.

Let $S : s \to^\alpha_{\mathit{ord}} t$ and $T : s \to^\beta_{\mathit{ord}} t$ be rewrite sequences below the root that are parallel permutation equivalent. We consider the root of the proof trees $\mathfrak{T}'^{(<)}_S$ and $\mathfrak{T}'^{(<)}_T$ and observe that $\overset{(<)}{\to}{}^\infty$ can only be derived using the id-rule or the lift-rule. In case one of the trees is derived using the canonical id-rule, it follows that $s = t = x$ for some variable $x \in \mathcal{X}$, and there is only one possible proof tree deriving $x \overset{(<)}{\to}{}^\infty x$:

$$\mathfrak{T}'^{(<)}_S = \mathfrak{T}'^{(<)}_T = \dfrac{}{x \overset{(<)}{\to}{}^\infty x} \text{ id}$$

Thus, assume that both $\mathfrak{T}'^{(<)}_S$ and $\mathfrak{T}'^{(<)}_T$ are derived using the lift-rule. Then $s = f(s_1, \dots, s_n)$ and $t = f(t_1, \dots, t_n)$ for some $f \in \Sigma$ of arity n and terms $s_1, \dots, s_n, t_1, \dots, t_n$. The proof trees must be of the following forms:

$$\mathfrak{T}'^{(<)}_S = \dfrac{\mathfrak{T}_{S_1} \quad \dots \quad \mathfrak{T}_{S_n}}{f(s_1, \dots, s_n) \overset{(<)}{\to}{}^\infty f(t_1, \dots, t_n)} \text{ lift}$$

$$\mathfrak{T}'^{(<)}_T = \dfrac{\mathfrak{T}_{T_1} \quad \dots \quad \mathfrak{T}_{T_n}}{f(s_1, \dots, s_n) \overset{(<)}{\to}{}^\infty f(t_1, \dots, t_n)} \text{ lift}$$

Where, for $i \in \{1, \dots, n\}$, S_i is the subsequence of S on the i-th argument of f, and T_i is the subsequence of T on the i-th argument of f. Since S and T are parallel permutation equivalent, it follows that S_i and T_i are, for every $i \in \{1, \dots, n\}$. As a consequence, $\mathfrak{T}'^{(<)}_S$ and $\mathfrak{T}'^{(<)}_T$ have the same root and the subtrees arise from rewrite sequences that are parallel permutation equivalent. This concludes the coinduction, hence $\mathfrak{T}_S = \mathfrak{T}_T$.

It remains to be shown that rewrite sequences that are not parallel permutation equivalent have different canonical proof trees. Let $S : s \to^\alpha_{\mathit{ord}} t$ and $T : s \to^\beta_{\mathit{ord}} t$ be rewrite sequences that are not parallel permutation equivalent. Note that, due to strong convergence, S and T contain only a finite number of steps at every depth n. Moreover, we may assume that:

($\star$) For every rule $\rho \in \mathcal{R}$ and position $p \in \mathbb{N}^*$, S and T contain the same number of steps arising from an application of ρ at position p.

If ($\star$) was violated, then S and T cannot correspond to the same proof tree. The reason is that, from a given proof tree one can derive the steps at position p with respect to rule ρ.

By Lemma 8.15 there exists a largest ordinal $\alpha' \leq \alpha$ such that there exists an injective function $f : \alpha' \to \beta$ with the properties

(i) $\mathit{rul}(S, \gamma) = \mathit{rul}(T, f(\gamma))$ and $\mathit{pos}(S, \gamma) = \mathit{pos}(T, f(\gamma))$ for every $\gamma < \alpha'$, and
(ii) $\mathit{pos}(S, \gamma_1) \parallel \mathit{pos}(S, \gamma_2)$ whenever $\gamma_1 < \gamma_2 < \alpha'$ and $f(\gamma_1) > f(\gamma_2)$.
(iii) $\mathit{pos}(T, \gamma_1) \parallel \mathit{pos}(T, \gamma_2)$ whenever $\gamma_1 < \gamma_2 < \beta$, γ_1 is not in the image of f, but γ_2 is,

and this function f is unique. As S and T are not parallel permutation equivalent, it follows that either

(a) $\alpha' < \alpha$, or

(b) $\alpha' = \alpha$ and there exists $\beta' < \beta$ that is not in the image of f.

Note that (b) would contradict ($\star$) as all of the steps of S are in the domain of f but the image of f does not contain all of the steps of T. Then T would have to contain more steps than S at some position $p \in \mathbb{N}^*$.

Thus assume that $\alpha' < \alpha$. Then there exists some index $\gamma < \beta$ such that γ is not in the image of f, $\mathit{rul}(S, \alpha') = \mathit{rul}(T, \gamma)$ and $\mathit{pos}(S, \alpha') = \mathit{pos}(T, \gamma)$. For otherwise, if γ would not exist, we had a contradiction with ($\star$) as then S contained more steps than T at position $\mathit{pos}(S, \alpha')$ with respect to rule $\mathit{rul}(S, \alpha')$. We define $g : \alpha' + 1 \to \beta$ by $g(\xi) = f(\xi)$ for every $\xi < \alpha'$ and $g(\alpha') = \gamma$. Then g is injective and satisfies property (i). For property (ii) of g, we only need to consider the case that $\gamma_2 = \alpha'$, so $\gamma_1 < \gamma_2 = \alpha' < \alpha' + 1$ and $g(\gamma_2) = \gamma < g(\gamma_1)$. Then $\gamma < f(\gamma_1)$ and γ is not in the image of f, while $f(\gamma_1)$ is. Then it follows from property (iii) of f that $\mathit{pos}(T, \gamma) \parallel \mathit{pos}(T, f(\gamma_1))$. Hence $\mathit{pos}(T, g(\gamma_2)) \parallel \mathit{pos}(T, g(\gamma_1))$ and $\mathit{pos}(S, \gamma_2) \parallel \mathit{pos}(S, \gamma_1)$. Thus g satisfies property (ii). However, due to the choice of α', g cannot satisfy all properties (i)– (iii); hence property (iii) has to fail. For property (iii) of g, it suffices to consider the case that $\gamma_2 = \gamma$ since γ is the only element in the image of g that is not in the image of f. So, due to failure of property (iii) for g, we have:

(†) There exist $\gamma_1 < \gamma$ such that γ_1 is not in the image of g and $\mathit{pos}(T, \gamma_1) \nparallel \mathit{pos}(T, \gamma)$. Note that, in particular, $\gamma > 0$ since otherwise property (iii) would hold.

Define a set of rule applications $P = \{\, \chi_1,\, \chi_2 \,\}$, where

$$\chi_1 = \langle \mathit{rul}(T, \gamma_1), \mathit{pos}(T, \gamma_1) \rangle$$
$$\chi_2 = \langle \mathit{rul}(T, \gamma), \mathit{pos}(T, \gamma) \rangle$$

We compare the order of rule applications in P in the rewrite sequences S and T. From properties (ii) and (iii) it follows that f maps the i-th χ_1-step in S to the i-th χ_1-step in T, and likewise the i-th χ_2-step in S to the i-th χ_2-step in T. Then it follows from (†) that S and T do not contain the same order of rule applications in P. By Lemma 8.19 it follows that S and T cannot arise from the same proof tree. □

9. A Formalization in Coq

The standard definition of infinitary rewriting, using ordinal length rewrite sequences and strong convergence at limit ordinals, is difficult to formalize. The coinductive framework we propose, is easy to formalize and work with in theorem provers. We discuss the important steps of the formalisation of infinitary rewriting and the compression lemma.

9.1. **Formalisation of Relations and Vectors.** We have formalised binary relations and properties of relations, such as reflexivity, transitivity, inclusion and equality, as follows.

Formalisation 1 (Relations and properties of relations)**.**

```
Variables A : Type.

Definition relation := A -> A -> Prop.
```

```
Definition reflexive  (R : relation) := forall x, R x x.
Definition transitive (R : relation) := forall x y z, R x y -> R y z -> R x z.
Definition subrel  (R R' : relation) := forall x y, R x y -> R' x y.
Definition eqrel   (R R' : relation) := subrel R R' /\ subrel R' R.
```

Note that we have formalised relations as functions.

We have moreover formalised operations on relations such as composition, union and the reflexive transitive closure.

Formalisation 2 (Operations on relations)**.**

```
Definition compose (R S : relation) : relation :=
  fun a c =>  exists b, R a b /\ S b c.

Definition Runion  (R S : relation) : relation :=
  fun a b =>  R a b \/ S a b.

Inductive refl_trans_close (R : relation) : relation :=
  | refl_trans_step  : subrel R (refl_trans_close R)
  | refl_trans_refl  : reflexive (refl_trans_close R)
  | refl_trans_trans : transitive (refl_trans_close R).

Notation "R ;; S" := (compose R S) (right associativity).
Notation "R (+) S" := (Runion R S) (right associativity).
Notation "R *" := (refl_trans_close R) (left associativity).
```

Our formalisation of vectors is based on the formalisation of vectors by Pierre Boutillier (in the Coq standard library). Thereby a vector v of length n with elements from A is a function $v : \texttt{Fin n} \to A$. Here `Fin n` is an n-element set; for example `Fin 4` consists of the following elements

$$\begin{aligned}\texttt{Fin 4} = \{\, &\texttt{First 3},\\ &\texttt{Next (First 2)},\\ &\texttt{Next (Next (First 1))}\\ &\texttt{Next (Next (Next (First 0)))}\,\}\end{aligned}$$

Formalisation 3 (Vectors and a map operation)**.**

```
Inductive Fin : nat -> Type :=
  | First : forall n, Fin (S n)
  | Next  : forall n, Fin n -> Fin (S n).

Definition vector (n : nat) :=  Fin n -> A.

Definition vmap (n : nat) (f : A -> B) : vector A n -> vector B n :=
  fun v i =>  f (v i).
```

9.2. **Formalisation of Infinite Terms.** For the formalisation of terms, we begin with the set of variables and the signature.

Formalisation 4 (Variables and Signature)**.**

```
Record variables : Type := Variables {
  variable :> Type;
  beq_var : variable -> variable -> bool;
  beq_var_ok : forall x y, beq_var x y = true <-> x = y
}.

Record signature : Type := Signature {
  symbol :> Type;
  arity : symbol -> nat;
  beq_symb : symbol -> symbol -> bool;
  beq_symb_ok : forall x y, beq_symb x y = true <-> x = y
}.
```

Notably, next to the set of variables and functions symbols itself, our formalisation includes functions `beq_var` and `beq_symb` for a *decidable equality* on the variables and function symbols, respectively. The functions `beq_var_ok` and `beq_symb_ok` guarantee that the decidable equality coincides with the standard equality in Coq.

Recall that a term is either

(i) a variable $x \in \mathcal{X}$, or
(ii) a function symbol $f \in \Sigma$ together with a vector of terms of length $ar(f)$.

The inductive interpretation of this principle yields the finite terms (`finite_term` in Coq), the coinductive interpretation gives rise to the finite and infinite terms (`term` in Coq).

Formalisation 5 (Finite and Infinite Terms)**.**

```
Variable F : signature.
Variable X : variables.

Inductive finite_term : Type :=
  | FVar : X -> finite_term
  | FFun : forall f : F, vector finite_term (arity f) -> finite_term.

CoInductive term : Type :=
  | Var : X -> term
  | Fun : forall f : F, vector term (arity f) -> term.
```

In Coq, there is no extensional equality, that is, $\forall x.\ f(x) = g(x)$ does not imply $f = g$. Similarly, infinite terms s, t that coincide on every position, are not necessarily equal with respect to the standard equality $=$ in Coq. As a consequence, equality $=$ is not suitable to work with infinite terms in Coq. We therefore work with bisimilarity $\sim$, as is common practice in coalgebra. Terms are bisimilar if and only if they coincide on every position.

Formalisation 6 (Bisimilarity on terms)**.**

```
CoInductive term_bis : term -> term -> Prop :=
  | Var_bis : forall x, term_bis (Var x) (Var x)
  | Fun_bis : forall f v w, (forall i, term_bis (v i) (w i)) ->
                term_bis (Fun f v) (Fun f w).

Infix " [~] " := term_bis (no associativity, at level 70).
```

In the sequel, we write $\sim$ for bisimilarity of terms in Coq.

9.3. **Formalisation of Term Rewriting Systems.** Next, we formalise rewrite rules and term rewriting systems as lists of rules. In the following definition, we leave `is_var`, `vars`, `inc` and `list` implicit; the function

(i) `is_var : term` $\rightarrow$ `bool` returns `true` if and only if the given term is a variable,
(ii) `vars : finite_term` $\rightarrow$ `(X` $\rightarrow$ `Prop)` returns the set of variables in a term,
(iii) `inc` stands for set inclusion, and
(iv) `list` is the implementation of lists in the standard library of Coq.

Formalisation 7 (Rules and term rewriting systems)**.** These are formalized as

```
Record rule : Type := Rule {
  lhs     : finite_term;
  rhs     : term;
  rule_wf : is_var lhs = false /\ inc (vars rhs) (vars lhs)
}.

Definition trs := list rule.
```

A rule consists of a *finite* left-hand side (`lhs`) and *finite or infinite* right-hand side (`rhs`), and a proposition `rule_wf` stating that the `lhs` is not a variable, and the variables in `rhs` are a subset of the variables in `lhs`.

Remark 9.1. We note that the definition of `rules` could easily be generalised to infinite left-hand sides. This is not a restriction of our coinductive framework for infinitary rewriting. In the literature, *infinitary term rewriting systems* are typically defined to have finite left-hand sides to keep matching (with respect to left-linear rules) decidable. We have chosen to adopt this restriction as our goal was a formalisation of the Compression Lemma which only holds for term rewrite systems with finite (and linear) left-hand sides.

We introduce substitutions as maps from variables to finite or infinite terms.

Formalisation 8 (Substitution)**.**

```
Definition substitution := X -> term.

CoFixpoint substitute (sigma : substitution) (t : term) : term :=
  match t with
  | Var x      => sigma x
  | Fun f args => Fun f (vmap (substitute sigma) args)
  end.
```

The `substitute` function applies a substitution to a finite or infinite term.

9.4. **Formalisation of Infinitary Rewriting.** The closure of the rules under substitutions gives rise to root steps on finite and infinite terms.

Formalisation 9 (Root steps on finite and infinite terms)**.**

```
Variable system : trs.

Inductive root_step : relation term :=
  | Root_step :
      forall (s t : term) (r : rule) (u : substitution),
        In r system ->
```

```
        substitute u (lhs r) [~] s ->
        substitute u (rhs r) [~] t ->
          root_step s t.
```

As discussed above, we must include bisimilarity $\sim$ since the equality $=$ in Coq is too strict. Here '`In r system`' checks whether `r` (a rule) occurs in `system` (a list of rules).

We formalise the lifting operation $\overline{R}$ (Definition 3.1) as follows.

Formalisation 10 (Lifting)**.**

```
Inductive lift (R : relation term) : relation term :=
  | lift_id : forall s t, s [~] t -> lift R s t
  | lift_step :
      forall (f : F) (s t : vector term (arity f)),
             (forall i, R (s i) (t i)) ->
             forall fs ft, Fun f s [~] fs -> Fun f t [~] ft -> lift R fs ft.
```

Again, we include bisimilarity $\sim$ instead of the standard equality $=$ in Coq.

We use the root step rewrite relation and the lifting operation to introduce infinitary strongly convergent reductions. Our coinductive definition (Definition 4.3) is based on mixed induction and coinduction:

$$\to^{\infty} \;\;:=\;\; \mu R.\, \nu S.\, (\to_{\varepsilon} \cup \overline{R})^{*} \,\mathbf{;}\, \overline{S}\,.$$

Unfortunately, Coq has no support for mutual inductive and coinductive definitions.

To overcome this problem, we employ the fact that the greatest fixed point $\nu S.\, F(S)$ is the union of all S for which $S \subseteq F(S)$ (under the condition that $F : L \to L$ is monotone); see further Section 3. In other words, $\nu S.\, F(S)$ is the smallest relation T such that $S \subseteq T$ whenever $S \subseteq F(S)$. Hence, we have

$$\to^{\infty} \;\;:=\;\; \mu R. \left(\begin{array}{l} \text{the smallest } T \text{ such that:} \\ \text{for all relations } S, \\ \quad S \subseteq (\to_{\varepsilon} \cup \overline{R})^{*} \,\mathbf{;}\, \overline{S} \implies S \subseteq T \end{array} \right) \tag{9.1}$$

This definition lends itself to a formalisation in Coq.

Formalisation 11. Formalisation of (9.1):

```
Inductive ired : relation (term F X) :=
  | Ired :
      forall S : relation (term F X),
      subrel S ((root_step (+) lift ired)* ;; lift S) ->
      subrel S ired.
```

Here `;;` is relation composition, `(+)` is the union, and `*` the reflexive-transitive closure. The statement `Inductive ired` in the formalisation corresponds to μR in (9.1), and thus `ired` corresponds to R.

While Formalisation 11 is correct, it turns out that Coq is not able to generate a good induction principle from the definition. The generated induction principle is:

```
ired_ind :  forall P : term -> term -> Prop,
            ( forall S : relation term,
              subrel S ((root_step (+) lift ired)* ;; lift S) -> subrel S P )
            -> subrel ired P
```

In mathematical notation this reads as follows:

$$\forall P.\ \Big(\ \forall S.\ \Big(\ S \subseteq (\to_\varepsilon \cup\, \overline{\texttt{ired}})^*\,;\overline{S}\ \Big)\ \to\ S \subseteq P\ \Big)\ \to\ \texttt{ired} \subseteq P$$

Note that, in particular we have $\texttt{ired} = (\to_\varepsilon \cup \overline{\texttt{ired}})^*$. As a consequence, we already have to show $\texttt{ired} \subseteq P$ as part of the induction step. Hence, this induction principle is void.

To overcome this problem, we adapt (9.1) and Formalisation 11 as follows:

$$\to^\infty \ := \ \mu R. \left(\begin{array}{l} \text{the smallest } T \text{ such that:} \\ \text{for all relations } S \text{ and } I, \\ \quad I \subseteq R \implies S \subseteq (\to_\varepsilon \cup \overline{I})^*\,;\overline{S} \implies S \subseteq T \end{array} \right) \tag{9.2}$$

To help Coq generate a good induction principle, we introduce an auxiliary relation I, and we replace the occurrences of R in the body of the definition by I. To preserve the semantics of the definition, we require $\texttt{I} \subseteq \texttt{ired}$. (In other words, I is a lower-approximant of R.)

To see that (9.1) and (9.2) give rise to the same relation $\to^\infty$, we argue as follows. If we were to replace $I \subseteq R$ by $I = R$ in (9.2), then both definitions would obviously coincide. However, both definitions also coincide without the replacement since sets $I \subsetneq R$ do not contribute due to monotonicity.

We formalise (9.2) as follows.

Formalisation 12 (Strongly convergent rewrite relation)**.** [1]

```
Inductive ired : relation term :=
  | Ired :
      forall S I : relation term,
      subrel I ired ->
      subrel S ((root_step (+) lift I)* ;; lift S) ->
      subrel S ired.
```

Remark 9.2. For this definition, Coq generates the following good induction principle:

```
ired_ind : forall P : term -> term -> Prop,
           ( forall S I : relation (term F X),
             subrel I ired ->
             subrel I P ->
             subrel S ((root_step (+) lift I)* ;; lift S) ->
             subrel S P) ->
           subrel ired P
```

Thus in order to prove $\to^\infty \subseteq P$, it suffices to show

$$I \subset \to^\infty \implies I \subseteq P \implies S \subseteq (\to_\varepsilon \cup \overline{I})^*\,;\overline{S} \implies S \subseteq P$$

for every relation I and S. Below, we will see in several examples, that this is a useful induction principle that is easy to work with.

[1]Note that canonical proof trees could be formalised by a small adaptation of this formalization. On the one hand, the restricted (canonical) proof trees can simplify proofs whose source is (a proof tree for) an infinite reduction. On the other hand, the restriction might complicate proofs whose target is (a proof tree for) an infinite reduction. To combine the advantages of both choices, it would be interesting to formally prove that every proof tree can be transformed into an equivalent canonical proof tree.

9.5. **Formalisation of Omega Rewriting.** In order to formalise the Compression Lemma, we need rewrite sequences of length $\leq \omega$. To formalise $\to^{\leq\omega}$, we use

$$\to^{\leq\omega} \;:=\; \nu R.\, (\to^* \mathbin{;} \overline{R}) \tag{9.3}$$

That is, a rewrite sequence of length $\leq \omega$ is a finite rewrite sequence followed by rewrite sequences of length $\leq \omega$ on subterms (below the root). Using dovetailing, this gives rise to the usual concept of rewrite sequences indexed by an ordinal $\leq \omega$.

As above for $\to^\infty$, we have:

$$\to^{\leq\omega} \;:=\; \begin{array}{l}\text{the smallest } T \text{ such that:}\\ \text{for all relations } S,\ S \subseteq (\to^* \mathbin{;} \overline{S}) \implies S \subseteq T\end{array} \tag{9.4}$$

This definition can be formalised in Coq as follows.

Formalisation 13 (Rewrite sequences of length at most ω)**.**

```
Inductive ored : relation term :=
  | Ored :
      forall S : relation term,
      subrel S (mred ;; lift S) ->
      subrel S ored.
```

Here `mred` are finite rewrite sequences $\to^*$.

To keep the proof of compression in Coq as simple as possible, we have chosen to define the finite rewrite relation $\to^*$ in a non-standard way. We introduce `mred` as the smallest relation S that fulfils the following conditions:

(i) `mred_bis`: if $s \mathrel{S} t$, $s \sim s'$ and $t \sim t'$, then $s' \mathrel{S} t'$,
(ii) `mred_refl`: S is reflexive,
(iii) `mred_trans`: S is transitive,
(iv) `mred_root`: $\to_\varepsilon \subseteq S$, and
(v) `mred_fun`: if $u_1 \mathrel{S} v_1, \ldots, u_n \mathrel{S} v_n$, then $f(u_1, \ldots, u_n) \mathrel{S} f(v_1, \ldots, v_n)$.

Let us check that `mred` indeed is the finite rewrite relation $\to^*$:

(a) For `mred` $\subseteq \to^*$ note that the finite rewrite relation $\to^*$ fulfils all these criteria.
(b) For $\to^* \subseteq$ `mred` we argue as follows. We have $\to_\varepsilon \subseteq$ `mred` by `mred_root`. By `mred_fun` together with `mred_refl` it follows that `mred` is closed under contexts. Thus $\to \subseteq$ `mred`. By `mred_refl` and `mred_trans` we obtain that $\to^* \subseteq$ `mred`.

Hence `mred` $= \to^*$.

Formalisation 14 (Finite rewriting relation on infinite terms)**.**

```
Inductive mred : relation term :=
  | mred_bis :
      forall s s' t t', s [~] s' -> t [~] t' -> mred s t -> mred s' t'
  | mred_refl :
      forall s, mred s s
  | mred_trans :
      forall s t u, mred s t -> mred t u -> mred s u
  | mred_root :
      forall (s t : term F X) , root_step s t -> mred s t
  | mred_fun :
      forall (f : F) (u v : vector (term F X) (arity f)),
      (forall i : Fin (arity f), mred (u i) (v i)) ->
      mred (Fun f u) (Fun f v).
```

Notation 9.3. In the remainder of this section, we will write

$$\to_m \text{ for } \texttt{mred}\,, \quad \to_i \text{ for } \texttt{ired}\,, \quad \to_o \text{ for } \texttt{ored}\,, \text{ and } \quad \to_r \text{ for } \texttt{root\_step}.$$

We have formalised Lemma 9.6 to show that Formalisation 13 of `ored` indeed corresponds to Equation 9.3. We first prove two auxiliary facts.

Lemma 9.4 (`ored_subrel`)**.** *We have* $\to_o \subseteq \to_m ; \overline{\to_o}$.

Proof. Let $s \to_o t$. By definition of $\to_o$ there exists a relation $S \subseteq \to_o$ such that $S \subseteq \to_m ; \overline{S}$ and $s\ S\ t$. As $s \to_m ; \overline{S}\ t$ and $S \subseteq \to_o$, we have $s \to_m ; \overline{\to_o}\ t$. □

Lemma 9.5 (`subrel_lift_ored_ored`)**.** *We have* $\overline{\to_o} \subseteq \to_o$.

Proof. Let $s \overline{\to_o} t$. Define $S = \to_o \cup \langle s, t\rangle$. Then we have $s\ S\ t$. To prove $s \to_o t$, it suffices (by definition of $\to_o$) to show that $S \subseteq \to_m ; \overline{S}$. By Lemma 9.4, we have $\to_o \subseteq \to_m ; \overline{\to_o}$, and hence $\to_o \subseteq \to_m ; \overline{S}$. Moreover, we have $\langle s, t\rangle \in \to_m ; \overline{S}$ as a consequence of reflexively of $\to_m$, and since $s \overline{\to_o}\ t$ and $\to_o \subseteq S$ imply $s\ \overline{S}\ t$. This concludes the proof. □

Lemma 9.6 (`ored_ok`)**.** *We have* $\to_o = \to_m ; \overline{\to_o}$.

Proof. By Lemma 9.4 it suffices to show that $\to_m ; \overline{\to_o} \subseteq \to_o$. Let $s \to_m t \overline{\to_o} u$. Then, by Lemma 9.5, we have $s \to_m t \to_o u$. By definition of $\to_o$ there exists a relation $S \subseteq \to_o$ such that $S \subseteq \to_m ; \overline{S}$ and $t\ S\ u$. Thus $t \to_m t'\ \overline{S}\ u$ for some t'. Define $S' = S \cup \langle s, u\rangle$. Then $s\ S'\ u$ and $S' \subseteq \to_m ; \overline{S'}$ since $s \to_m t'\ \overline{S'}\ u$ by `mred_trans` and $S \subseteq S'$. Hence $s \to_o u$ by definition of $\to_o$ using S'. □

9.6. **Formalisation of the Compression Lemma.** Using the above definitions, we will now prove the Compression Lemma. The proof realises a transformation of $\to^{\infty}$ proof trees into $\to^{\leq\omega}$ proof trees.

Lemma 9.7 (Compression Lemma)**.** *Let $\mathcal{R}$ be a left-linear term rewriting system with finite left-hand sides. Whenever there is an infinite reduction from s to t ($s \to^{\infty} t$) then there exists a reduction of length at most ω from s to t ($s \to^{\leq\omega} t$).*

We have formalised the Compression Lemma as follows:

Lemma 9.8. *For left-linear* `trs`*'s we have* $\to_i \subseteq \to_o$.

The condition *finite left-hand sides* is part of the formalisation of `trs`'s, see above. In the remainder of this section we tacitly assume that the underlying `trs` is *left-linear*. This is a necessary condition for all lemmas 9.9–9.15.

We present the proof of Lemma 9.8 as close as possible to to our formalisation in Coq. We begin with a few auxiliary lemmas.

Lemma 9.9 (`ored_match`)**.** *If $s \to_o \ell\sigma$ for a finite, linear term ℓ, then $s \to_m \ell\tau$ for some substitution τ with $\forall x.\ \tau(x) \to_o \sigma(x)$.*

Proof. The proof proceeds by induction on ℓ. If $s \to_o \ell\sigma$, then there exists a term t such that $s \to_m t \overline{\to_o} \ell\sigma$. Assume that ℓ is a variable, say $\ell = x$. Then define $\tau(x) = t$ and $\tau(y) = \sigma(y)$ for every $y \neq x$. We have $\tau(x) = t \overline{\to_o} \ell\sigma = \sigma(x)$. Then $s \to_m \ell\tau$, and we have $\forall x.\ \tau(x) \to_o \sigma(x)$ since $\overline{\to_o} \subseteq \to_o$ by Lemma 9.5 and $\to_o$ is reflexive.

If ℓ is not a variable, let $\ell = f(\ell_1, \ldots, \ell_n)$. Then $t \overline{\to_o} \ell\sigma$ implies that $t = f(t_1, \ldots, t_n)$ for some terms $t_1, \ldots, t_n$, and we have $t_i \to_o \ell_i\sigma$ for every $i \in \{1, \ldots, n\}$. Then by induction hypothesis, we have $t_i \to_m \ell_i\tau_i$ and $\forall x.\ \tau_i(x) \to_o \sigma(x)$ for every $i \in \{1,\dots,n\}$. From linearity of ℓ it follows that $\ell_1,\dots,\ell_n$ do not share any variables. Consequently, we can define the substitution τ as follows: $\tau(x) = \tau_i(x)$ if $x \in \mathcal{V}ar(\ell_i)$ and $\tau(x) = \sigma(x)$ if $x \notin \mathcal{V}ar(\ell)$. Then $\ell_i\tau = \ell_i\tau_i$ for every $i \in \{1,\dots,n\}$ and $\ell\tau = f(\ell_1\tau_1,\dots,\ell_n\tau_n)$. Moreover we have $\forall x.\ \tau(x) \to_o \sigma(x)$ by definition of τ. It remains to show $s \to_m \ell\tau$. Using `mred_fun` we get $t = f(t_1,\dots,t_n) \to_m f(\ell_1\tau_1,\dots,\ell_n\tau_n) = \ell\tau$. By `mred_trans` we obtain $s \to_m \ell\tau$. $\square$

Lemma 9.10 (`subsitution_ored`). *Let t be a (finite or infinite) term and σ,τ substitutions such that $\sigma(x) \to_o \tau(x)$ for every $x \in \mathcal{V}ar(t)$. Then $t\sigma \to_o t\tau$.*

Proof. By definition of $\to_o$ it suffices to find a relation S such that $t\sigma\ S\ t\tau$ and $S \subseteq {\to_m} \mathbin{;} \overline{S}$. We define $S = {\to_o} \cup \{\ \langle u\sigma, u\tau\rangle \mid u \text{ a term }\}$. We show $S \subseteq {\to_m} \mathbin{;} \overline{S}$. We have

$$\to_o \;\subseteq\; \to_m \mathbin{;} \overline{\to_o} \;\subseteq\; \to_m \mathbin{;} \overline{S} \tag{9.5}$$

by Lemma 9.6.

So consider $u\sigma\ S\ u\tau$ for some term u. If u is a variable, then $u\sigma = \sigma(u) \to_o \tau(u) = u\tau$. Then $u\sigma \to_m \mathbin{;} \overline{S}\ u\tau$ follows from (9.5). Thus, let $u = f(u_1,\dots,u_n)$ for some symbol f and terms $u_1,\dots,u_n$. Then we have $u\sigma = f(u_1\sigma,\dots,u_n\sigma)\ \overline{S}\ f(u_1\tau,\dots,u_n\tau) = u\tau$, and clearly $\overline{S} \subseteq {\to_m} \mathbin{;} \overline{S}$. Hence $S \subseteq {\to_m} \mathbin{;} \overline{S}$. $\square$

Lemma 9.11 (`ored_rstep`). *We have ${\to_o} \mathbin{;} {\to_r} \subseteq {\to_o}$.*

Proof. Let $\ell \to r \in \mathcal{R}$ be a rule, σ a substitution and consider $s \to_o \ell\sigma \to_r r\sigma$. Then by Lemma 9.9 we have $s \to_m \ell\tau$ for some substitution τ with $\forall x.\ \tau(x) \to_o \sigma(x)$. We have $r\tau \to_o r\sigma$ by Lemma 9.10, and thus $r\tau \to_m t \overline{\to_o} r\sigma$ for some t by Lemma 9.6. Then also $s \to_m \ell\tau \to_r r\tau \to_m t \overline{\to_o} r\sigma$ and $s \to_m t \overline{\to_o} r\sigma$ by `mred_root` and `mred_trans`. Hence we have $s \to_o r\sigma$ by Lemma 9.6. $\square$

Lemma 9.12 (`ored_mred`). *We have ${\to_o} \mathbin{;} {\to_m} \subseteq {\to_o}$.*

Proof. The proof proceeds by induction on the definition of $\to_m$ in $s \to_o t \to_m u$. The induction step is trivial for `mred_bis`, `mred_refl` and `mred_trans`. For `mred_root` the claim follows from Lemma 9.11. Here we only consider the case of `mred_fun`. Then $t = f(t_1,\dots,t_n)$ and $u = f(u_1,\dots,u_n)$ for some symbol f and $t_i \to_m u_i$ for every $i \in \{1,\dots,n\}$. From $s \to_o t$ it follows that $s \to_m s' \overline{\to_o} t = f(t_1,\dots,t_n)$ for some s'. Hence $s' = f(s_1,\dots,s_n)$ with $s_i \to_o t_i$ for every $i \in \{1,\dots,n\}$. By the induction hypothesis, we obtain $s_i \to_o u_i$ for every $i \in \{1,\dots,n\}$, and, consequently, $s \to_m s' \overline{\to_o} u$. Finally, $s \to_o u$ by Lemma 9.6. $\square$

Lemma 9.13 (`ored_ored`). *We have ${\to_o} \mathbin{;} {\to_o} \subseteq {\to_o}$ and ${\to_o} \mathbin{;} \overline{\to_o} \subseteq {\to_o}$.*

Proof. Define $S = ({\to_o} \mathbin{;} {\to_o}) \cup ({\to_o} \mathbin{;} \overline{\to_o})$. We show $S \subseteq {\to_o}$. By definition of $\to_o$ it suffices that $S \subseteq {\to_m} \mathbin{;} \overline{S}$. We have:

$$\to_o \mathbin{;} \to_o \;\subseteq\; \to_o \mathbin{;} (\to_m \mathbin{;} \overline{\to_o}) \;\subseteq\; \to_o \mathbin{;} \overline{\to_o} \tag{9.6}$$

by Lemma 9.6 and Lemma 9.12, and

$$\to_o \mathbin{;} \overline{\to_o} \;\subseteq\; (\to_m \mathbin{;} \overline{\to_o}) \mathbin{;} \overline{\to_o} \;\subseteq\; \to_m \mathbin{;} (\overline{\to_o} \mathbin{;} \overline{\to_o}) \;\subseteq\; \to_m \mathbin{;} \overline{(\to_o \mathbin{;} \to_o)} \;\subseteq\; \to_m \mathbin{;} \overline{S}$$

by Lemma 9.6, associativity, definition of $\overline{\cdot}$. This proves $S \subseteq {\to_m} \mathbin{;} \overline{S}$. $\square$

Lemma 9.14 (`ored_mix`)**.** *We have* $(\to_r \cup \overline{\to_o})^* \subseteq \to_o$.

Proof. Define $S = (\to_r \cup \overline{\to_o})$. We prove $S^* \subseteq \to_o$ by induction on (the definition of) the reflexive transitive closure:

(i) `refl_trans_step`: For $s \; S \; t$ we have either $s \to_r t$ or $s \overline{\to_o} t$. Then $s \to_o t$ as a consequence of either Lemma 9.11 or Lemma 9.6, respectively.
(ii) `refl_trans_refl`: For $s \sim t$ we $s \to_m t$ by `mred_bis` and `mred_refl`. Hence $s \to_o t$ since $\to_m ; \overline{\to_o} \subseteq \to_o$ by Lemma 9.6 and $\overline{\to_o}$ is reflexive by `lift_id`.
(iii) `refl_trans_trans`: For $s \; S^* \; u \; S^* \; t$ we obtain $s \to_o u$ and $u \to_o t$ by induction hypothesis. Then $s \to_o t$ follows from Lemma 9.13. □

Lemma 9.15 (`ored_liftR`)**.** *We have* $R \subseteq \to_o ; \overline{R}$ *implies* $R \subseteq \to_o$.

Proof. Define $S = (\to_o ; R) \cup (\to_o ; \overline{R})$. We show $S \subseteq (\to_m ; \overline{S})$:

$$\to_o ; R \;\subseteq\; \to_o ; (\to_o ; \overline{R}) \;\subseteq\; \to_o ; \overline{R}$$

by $R \subseteq \to_o ; \overline{R}$ and Lemma 9.13, and

$$\to_o ; \overline{R} \;\subseteq\; (\to_m ; \overline{\to_o}) ; \overline{R} \;\subseteq\; \to_m ; \overline{(\to_o ; R)} \;\subseteq\; \to_m ; \overline{S}$$

by Lemma 9.6. Hence $S \subseteq (\to_m ; \overline{S})$ and consequently $S \subseteq \to_o$. Note that $R \subseteq \to_o ; \overline{R} \subseteq S$. Thus $R \subseteq S \subseteq \to_o$. □

We are ready to prove compression.

Proof of Lemma 9.8. The proof proceeds by induction on $\to_i$. By the induction principle discussed in Remark 9.2 we have to show

$$I \subset \to_i \implies I \subseteq \to_o \implies S \subseteq (\to_r \cup \overline{I})^* ; \overline{S} \implies S \subseteq \to_o$$

for every relation I and S. So let $I \subseteq (\to_i \cap \to_o)$ and $S \subseteq (\to_r \cup \overline{I})^* ; \overline{S}$. Since $I \subseteq \to_o$ we get $(\to_r \cup \overline{I})^* \subseteq (\to_r \cup \overline{\to_o})^* \subseteq \to_o$ by Lemma 9.14. Consequently, $S \subseteq \to_o ; \overline{S}$ and, by Lemma 9.15, we obtain $S \subseteq \to_o$. Hence $\to_i \subseteq \to_o$. □

To the best of our knowledge this is the first formal proof of this well-known lemma. The formalization is available at [17].

10. Conclusion

We have proposed a coinductive framework which gives rise to several natural variants of infinitary rewriting in a uniform way:

(a) infinitary equational reasoning $\overset{\infty}{=} \;:=\; \nu y.\, (\leftarrow_\varepsilon \cup \to_\varepsilon \cup \overline{y})^*$,
(b) bi-infinite rewriting $\overset{\infty}{\to} \;:=\; \nu y.\, (\to_\varepsilon \cup \overline{y})^*$, and
(c) infinitary rewriting $\to^\infty \;:=\; \mu x.\, \nu y.\, (\to_\varepsilon \cup \overline{x})^* ; \overline{y}$.

We believe that (a) and (b) are novel. As a consequence of the coinduction over the term structure, these notions have the strong convergence built-in, and thus can profit from the well-developed techniques (such as tracing) in infinitary rewriting.

We have given a mixed inductive/coinductive definition of infinitary rewriting and established a bridge between infinitary rewriting and coalgebra. Both fields are concerned with infinite objects and we would like to understand their relation better. In contrast to previous coinductive treatments, the framework presented here captures rewrite sequences of arbitrary ordinal length, and paves the way for formalizing infinitary rewriting in theorem provers (as illustrated by our proof of the Compression Lemma in Coq).

Concerning proof trees/terms for infinite reductions, let us mention that an alternative approach has been developed in parallel by Lombardi, Ríos and de Vrijer [32]. While we focus on proof terms for the reduction relation and abstract from the order of steps in parallel subterms, they use proof terms for modeling the fine-structure of the infinite reductions themselves. Another difference is that our framework allows for non-left-linear systems. We believe that both approaches are complementary. Theorems for which the fine-structure of rewrite sequences is crucial, must be handled using [32]. (But note that we can capture standard reductions by a restriction on proof trees and prove standardization using proof tree transformations, see [22]). If the fine-structure is not important, as for instance for proving confluence, then our system is more convenient to work with due to simpler proof terms.

Our work lays the foundation for several directions of future research:

(i) The coinductive treatment of infinitary λ-calculus [22] has led to elegant, significantly simpler proofs [10, 11] of some central properties of the infinitary λ-calculus. The coinductive framework that we propose enables similar developments for infinitary term rewriting with reductions of arbitrary ordinal length.

(ii) The concepts of bi-infinite rewriting is novel, and the theory of infinitary equational reasoning is still underdeveloped. It would be interesting to study these concepts. Is there an equivalent of ordinal-indexed rewrite sequences for bi-infinite rewriting (maybe using Conway's surreal numbers [7])? Is it possible to establish some sort of Compression Lemma for bi-infinite rewriting?

Moreover, it would be fruitful to compare the Church–Rosser properties

$$\stackrel{\infty}{=} \;\subseteq\; \to^{\infty} ; {}^{\infty}\!\leftarrow \qquad\qquad \text{and} \qquad\qquad ({}^{\infty}\!\leftarrow ; \to^{\infty})^{*} \;\subseteq\; \to^{\infty} ; {}^{\infty}\!\leftarrow \;.$$

(iii) The formalization of the proof of the Compression Lemma in Coq is just the first step towards the formalization of all major theorems in infinitary rewriting.

It would also be interesting to formalise infinitary equational reasoning $\stackrel{\infty}{=}$ and bi-infinite rewriting $\stackrel{\infty}{\rightrightarrows}$ in Coq. We expect that it is straightforward to adapt our Coq formalization of infinitary rewriting $\to^{\infty}$ to equational reasoning $\stackrel{\infty}{=}$ and bi-infinite rewriting $\stackrel{\infty}{\rightrightarrows}$. The latter two concepts have significantly simpler definitions in the fixed point calculus.

(iv) It is interesting to investigate how the coinductive framework should be extended to incorporate the infinitary analysis of meaningless terms. [3, 2, 4, 21] This would be the natural stepping-stone to the formalization of confluence theorems in infinitary rewriting extended with $\bot$-reduction.

(v) We believe that the coinductive definitions will ease the development of new techniques for automated reasoning about infinitary rewriting. For example, methods for proving (local) productivity [14, 18, 38], for (local) infinitary normalization [37, 15, 13], for (local) unique normal forms [20], and for analysis of infinitary reachability and infinitary confluence. Due to the coinductive definitions, the implementation and formalization of these techniques could make use of circular coinduction [23, 19].

Acknowledgments. We thank Patrick Bahr, Jeroen Ketema, and Vincent van Oostrom for fruitful discussions and comments on earlier versions of this paper. We are thankful to the reviewers for pointing out mistakes and suggesting several improvements.

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