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# MODELS OF TYPE THEORY BASED ON MOORE PATHS *

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Abstract. This paper introduces a new family of models of intensional Martin-Löf type theory. We use constructive ordered algebra in toposes. Identity types in the models are given by a notion of Moore path. By considering a particular gros topos, we show that there is such a model that is *non-truncated*, i.e. contains non-trivial structure at all dimensions. In other words, in this model a type in a nested sequence of identity types can contain more than one element, no matter how great the degree of nesting. Although inspired by existing non-truncated models of type theory based on simplicial and cubical sets, the notion of model presented here is notable for avoiding any form of Kan filling condition in the semantics of types.

## 1. Introduction

Homotopy Type Theory [Uni13] has re-invigorated the study of the intensional version of Martin-Löf type theory [Mar75]. On the one hand, the language of type theory helps to express synthetic constructions and arguments in homotopy theory and higher-dimensional category theory. On the other hand, the geometric and algebraic insights of those branches of mathematics shed new light on logical and type-theoretic notions. One might say that the familiar *propositions-as-types* analogy has been extended to *propositions-as-types-as-spaces*. In particular, under this analogy there is a *path-oriented* view of intensional (i.e. proof-relevant) equality: proofs of equality of two elements $x, y$ of a type $A$ correspond to elements of a Martin-Löf identity type $\mathtt{Id}_A\, x\, y$ and behave analogously to paths between two points $x, y$ in a space $A$. The complicated internal structure of intensional identity types relates to the homotopy classes of path spaces. To make the analogy precise and to exploit it, it helps to have a wide range of models of intensional type theory that embody this path-oriented view of equality in some way. This paper introduces a new family of such models, constructed from *Moore paths* [Moo55] in toposes.

Let $\mathbb{R}_+ = \{r \in \mathbb{R} \mid r \geq 0\}$ be the real half-line with the usual topology. Classically, a Moore path between points $x$ and $y$ in a topological space $X$ is a pair $p = (f, r)$ where $r \in \mathbb{R}_+$ and $f : \mathbb{R}_+ \longrightarrow X$ is a continuous function with $f\,0 = x$ and $f\,r' = y$ for all $r' \geq r$.

*Key words and phrases:* dependent type theory, homotopy theory, Moore path, topos.

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We will write $x \sim y$ for the set of Moore paths from $x$ to $y$, with $X$ understood from the context. Clearly there is a Moore path from $x$ to $y$ iff there is a conventional path, that is, a continuous function $f : [0,1] \longrightarrow X$ with $f\,0 = x$ and $f\,1 = y$. The advantage of Moore paths is that they admit degeneracy and composition operations that are unitary and associative up to equality; whereas for conventional paths these identities only hold up to homotopy. Specifically, one has the following Moore paths:

$$\mathtt{idp}\,x :\equiv (\lambda\, t \to x, 0) \in x \sim x$$

and for all $p = (f, r) \in x \sim y$ and $q = (g, s) \in y \sim z$

$$q \bullet p :\equiv (g \bullet^r f, r + s) \in x \sim z$$

$$\text{where } (g \bullet^r f)\,t :\equiv \begin{cases} f\,t & \text{if } t \leq r \\ g(t - r) & \text{if } r \leq t \end{cases} \qquad (t \in \mathbb{R}_+)$$

These definitions satisfy $p \bullet \mathtt{idp}\,x = p = \mathtt{idp}\,y \bullet p$ and $r \bullet (q \bullet p) = (r \bullet q) \bullet p$. In Section 3 we abstract from $\mathbb{R}$ some simple order-algebraic [Bou81, chapter VI] structure sufficient for the above definitions to work in a constructive algebraic setting, rather than a classical topological one. Initially the structure of an *ordered abelian group* in some topos [Joh02, MM92] suffices and then we extend that to an *ordered commutative ring* to ensure the models satisfy function extensionality.

In Sections 4–6 we use this structure in toposes to develop a family of models of intensional Martin-Löf type theory with: identity types given by Moore paths, $\Sigma$-types, $\Pi$-types satisfying function extensionality, inductive types (we just consider disjoint unions and $W$-types) and Tarski-style universes. By considering a particular *gros* topos [GV72] in Section 7.2 we get a *non-truncated* instance of our model construction, in other words one where iterated identity types $\mathtt{Id}_A, \mathtt{Id}_{\mathtt{Id}_A}, \mathtt{Id}_{\mathtt{Id}_{\mathtt{Id}_A}}, \ldots$ can be non-trivial to any depth of iteration.

The observation that the strictly associative and unitary nature of composition of Moore paths aids in the interpretation of Martin-Löf identity types is not new; see for example [vG12, Sections 5.1 and 5.5]. However, the fact that function extensionality can hold for identity based on Moore paths (Theorem 5.1) is new and quite surprising, given that such paths carry an intensional component, namely their "shape" (Definition 3.1). Another novelty of our approach concerns the fact that existing non-truncated models of type theory typically make use of some form of Kan filling condition [Kan58] to define a class of *fibrant* families of types with respect to which path types behave as identity types. One of the contributions of this paper is to show that one can avoid any form of Kan filling and still get a non-truncated model of intensional Martin-Löf type theory. Instead we use path composition and a simple notion of fibrant family phrased just in terms of the usual operation of transporting elements along equality proofs (Definition 4.1). As a consequence every type, regarded as a family over the terminal type, is fibrant in our setting. In particular, this means that intervals are first-class types in our models, something which is not true for existing path-oriented models, such as the classical simplicial [KL16] and constructive cubical sets [BCH14, CCHM18, ABC+17] models; and constructing universes in our setting does not need proofs of their fibrancy.

**Informal type theory.** The new models of type theory we present are given in terms of *categories with families* (CwF) [Dyb96, Hof97]. Specifically, we start with an arbitrary topos $\mathcal{E}$, to which can be associated a CwF, for example as in [LW15, Awo16]. We write $\mathcal{E}(\Gamma)$ for the set of families indexed by an object $\Gamma \in \mathcal{E}$ and $\mathcal{E}(\Gamma \vdash A)$ for the set of elements of a family $A \in \mathcal{E}(\Gamma)$. One can make $\mathcal{E}(\Gamma)$ into a category whose morphisms between two families $A, B \in \mathcal{E}(\Gamma)$ are elements in $\mathcal{E}(\Gamma.A \vdash B)$, where $\Gamma.A$ is the comprehension object associated with $A$. The category $\mathcal{E}(\Gamma)$ is equivalent to the slice category $\mathcal{E}/\Gamma$, the equivalence being given on objects by sending families $A \in \mathcal{E}(\Gamma)$ to corresponding projection morphisms $\Gamma.A \longrightarrow \Gamma$.

We then construct a new CwF by considering families in $\mathcal{E}$ equipped with certain extra structure (the *transport-along-paths* structure of Definition 4.1); the elements of a family in the new CwF are just those of the underlying family in $\mathcal{E}$. One could describe this construction using the language of category theory. Instead, as in [OP18, LOPS18] we find it clearer to express the construction using an internal language for the CwF associated with $\mathcal{E}$ – a combination of higher-order predicate logic and extensional type theory, with an impredicative universe of propositions given by the subobject classifier in $\mathcal{E}$; see [Mai05]. This use of internal language allows us to give an appealingly simple description of the type constructs in the new CwF. In the text we use this language informally (analogously to the way that [Uni13] develops Homotopy Type Theory). In particular the typing contexts of the judgements in the formal version, such as $[x_0 : A_0, x_1 : A_1(x_0), x_2 : A_2(x_0, x_1)]$, become part of the running text in phrases like "given $x_0 : A_0$, $x_1 : A_1(x_0)$ and $x_2 : A_2(x_0, x_1)$, then. . . "; and when we refer to a "function in the topos" (as opposed to one of its morphisms) we mean a term of function type in its internal language.

The arguments we give in Sections 3–6 are all constructively valid and in fact do not require the impredicative aspects of topos theory; indeed we have used Agda [Agd] (with uniqueness of identity proofs and postulates for quotient types) as a tool to develop and experiment with the material presented in those sections. The specific model presented in Section 7.2 uses topological spaces within classical mathematics.

## 2. Ordered Rings in a Topos

Let $\mathcal{E}$ be a topos with a natural number object [Joh02, MM92]. A *total order* on an object $\mathtt{R} \in \mathcal{E}$ is given by a subobject $\leqslant \rightarrowtail \mathtt{R} \times \mathtt{R}$ which is not only reflexive, transitive and anti-symmetric, but also satisfies that the join of the subobject $\leqslant$ and its opposite $\leqslant \circ \langle \pi_2 , \pi_1 \rangle$ is the whole of $\mathtt{R} \times \mathtt{R}$; in other words the following formula of the internal language of $\mathcal{E}$ is satisfied:

$$(\forall i, j : \mathtt{R})\ i \leqslant j \ \vee\ j \leqslant i \tag{2.1}$$

As mentioned in the Introduction, in this paper we use such formulas of the internal language to express properties of $\mathcal{E}$ instead of giving the category-theoretic version of the property. Note that since $\mathcal{E}$ may not be Boolean, we do not necessarily have the trichotomy property $(\forall i, j : \mathtt{R})\ i < j \ \vee\ i = j \ \vee\ j < i$ for the associated strict order relation $i < j \ :\equiv \neg(j \leqslant i)$. So when defining functions by cases using (2.1) we have to verify that the clauses for $i \leqslant j$ and for $j \leqslant i$ agree on the overlap, where $i = j$ holds by antisymmetry. For example, the *positive cone*

$$\mathtt{R}_{+} :\equiv \{i : \mathtt{R} \mid \mathtt{0} \leqslant i\} \tag{2.2}$$

*Total order*

$$(\forall i : \mathtt{R})\ i \leqslant i \tag{2.4}$$

$$(\forall i, j, k : \mathtt{R})\ i \leqslant j \ \wedge\ j \leqslant k \ \Rightarrow\ i \leqslant k \tag{2.5}$$

$$(\forall i, j : \mathtt{R})\ i \leqslant j \ \wedge\ j \leqslant i \ \Rightarrow\ i = j \tag{2.6}$$

$$(\forall i, j : \mathtt{R})\ i \leqslant j \ \vee\ j \leqslant i \tag{2.7}$$

*Abelian group*

$$(\forall i, j, k : \mathtt{R})\ i + (j + k) = (i + j) + k \tag{2.8}$$

$$(\forall i : \mathtt{R})\ \mathtt{0} + i = i \tag{2.9}$$

$$(\forall i, j : \mathtt{R})\ i + j = j + i \tag{2.10}$$

$$(\forall i : \mathtt{R})\ i + (-i) = \mathtt{0} \tag{2.11}$$

*Addition is order preserving*

$$(\forall i, j, k : \mathtt{R})\ i \leqslant j \ \Rightarrow\ k + i \leqslant k + j \tag{2.12}$$

*Multiplicative abelian monoid*

$$(\forall i, j, k : \mathtt{R})\ i \cdot (j \cdot k) = (i \cdot j) \cdot k \tag{2.13}$$

$$(\forall i : \mathtt{R})\ \mathtt{1} \cdot i = i \tag{2.14}$$

$$(\forall i, j : \mathtt{R})\ i \cdot j = j \cdot i \tag{2.15}$$

*Multiplication distributes over addition*

$$(\forall i, j, k : \mathtt{R})\ i \cdot (j + k) = (i \cdot j) + (i \cdot k) \tag{2.16}$$

*Positive elements are closed under multiplication*

$$(\forall i, j : \mathtt{R})\ \mathtt{0} \leqslant i \ \wedge\ \mathtt{0} \leqslant j \ \Rightarrow\ \mathtt{0} \leqslant i \cdot j \tag{2.17}$$

Figure 1: Ordered commutative ring axioms

associated with $\mathtt{R}$ has a binary operation of *minimum* $\mathtt{min} : \mathtt{R}_+ \times \mathtt{R}_+ \longrightarrow \mathtt{R}_+$ well-defined by the following properties

$$(\forall i, j : \mathtt{R}_+)\ \begin{array}{l} i \leqslant j \ \Rightarrow\ \mathtt{min}(i, j) = i \\ j \leqslant i \ \Rightarrow\ \mathtt{min}(i, j) = j \end{array} \tag{2.3}$$

In order to define Moore paths in $\mathcal{E}$ with respect to $\mathtt{R}$ we need it to have additive structure compatible with the total order; later, to get function extensionality we also need multiplicative structure and to construct universes we need a connectedness property for $\mathtt{R}$.

**Definition 2.1** (order-ringed topos)**.** An *order-ringed topos* $(\mathcal{E}, \mathtt{R})$ is a topos $\mathcal{E}$ together with an *ordered commutative ring object* [Bou81, chapter VI] $\mathtt{R}$ in $\mathcal{E}$. Thus $\mathtt{R}$ comes equipped with a subobject $\leqslant \rightarrowtail \mathtt{R} \times \mathtt{R}$ and morphisms

$$\mathtt{0} : 1 \longrightarrow \mathtt{R} \qquad \_ + \_ : \mathtt{R} \times \mathtt{R} \longrightarrow \mathtt{R} \qquad - : \mathtt{R} \longrightarrow \mathtt{R} \qquad \mathtt{1} : 1 \longrightarrow \mathtt{R} \qquad \_ \cdot \_ : \mathtt{R} \times \mathtt{R} \longrightarrow \mathtt{R}$$

satisfying the axioms in Fig. 1.

Specific examples of order-ringed toposes $(\mathcal{E}, \mathtt{R})$ will be considered in Section 7. In the next section we develop properties of Moore paths in $\mathcal{E}$ based on $\mathtt{R}$.

## 3. Moore Paths in a Topos

Fix an order-ringed topos $(\mathcal{E}, \mathtt{R})$. Recall from the Introduction that there is a CwF associated with $\mathcal{E}$ [LW15, Awo16] whose families at an object $\Gamma \in \mathcal{E}$ we denote by $\mathcal{E}(\Gamma)$. We continue to use an informal internal language based on extensional type theory to describe constructions and properties of the topos and its associated CwF. The development in this and the next section only makes use of the order and additive structure of the positive cone $\mathtt{R_+} = \{i : \mathtt{R} \mid \mathtt{0} \leqslant i\}$ of the ordered commutative ring object $\mathtt{R}$. (We will need to use its multiplicative structure in Section 5.)

**Definition 3.1** (Moore path objects)**.** For each object $\Gamma \in \mathcal{E}$, the family $(x \sim y \mid x, y : \Gamma) \in \mathcal{E}(\Gamma \times \Gamma)$ of *Moore path objects* is defined by:

$$x \sim y \;:\equiv\; \{(f, i) : (\mathtt{R_+} \to \Gamma) \times \mathtt{R_+} \mid f\,\mathtt{0} = x \;\wedge\; (\forall j \geqslant i)\; f\,j = y\} \tag{3.1}$$

We have the following functions associated with Moore paths:

$$|\_| : (x \sim y) \to \mathtt{R_+} \tag{3.2}$$
$$|(f, i)| = i$$
$$\_\,\mathtt{at}\,\_ : (x \sim y) \to \mathtt{R_+} \to \Gamma \tag{3.3}$$
$$(f, i)\;\mathtt{at}\;j = f\,j$$

Following [Bro09, Section 2], we call $|p|$ the *shape* of the path $p : x \sim y$. Thus if $p : x \sim y$, then $p\;\mathtt{at}\;\mathtt{0} = x$ and $p\;\mathtt{at}\;|p| = y$. *From now on we will just write $p\;\mathtt{at}\;i$ as $p\,i$.*

Note that morphisms in $\mathcal{E}$ respect Moore paths in the sense that for each $\gamma : \Gamma \longrightarrow \Delta$ in $\mathcal{E}$ there is a function mapping paths in $\Gamma$ to those in $\Delta$:

$$\gamma\;\text{'}\;\; : (x \sim y) \to (\gamma\,x \sim \gamma\,y) \tag{3.4}$$
$$\gamma\;\text{'}\;(f, i) :\equiv (\gamma \circ f, i)$$

**Definition 3.2** (Degenerate paths)**.** For each $\Gamma \in \mathcal{E}$, the *degenerate path* at $x : \Gamma$ is denoted $\mathtt{idp}\,x : x \sim x$ and is well-defined by the requirements:

$$|\mathtt{idp}\,x| = \mathtt{0} \tag{3.5}$$
$$(\forall i : \mathtt{R_+})\;(\mathtt{idp}\,x)\,i = x \tag{3.6}$$

**Definition 3.3** (Path composition)**.** Given $A \in \mathcal{E}$, if $i : \mathtt{R_+}$ and $f, g : \mathtt{R_+} \to A$ satisfy $f\,i = g\,\mathtt{0}$, then there is a function $g \bullet^i f : \mathtt{R_+} \to A$ satisfying

$$(g \bullet^i f)\,j = \begin{cases} f\,j & \text{if } j \leqslant i \\ g\,(j - i) & \text{if } i \leqslant j \end{cases}$$

(where, as usual, we write $j + (\text{-}i)$ as $j - i$). This is well-defined because when $i = j$, then $f\,j = f\,i = g\,\mathtt{0} = g(j - i)$. In particular, given paths $p = (f, i) : x \sim y$ and $q = (g, j) : y \sim z$, then $g \bullet^i f : \mathtt{R_+} \to A$ satisfies $(\forall k \geqslant i + j)\;(g \bullet^i f)\,k = z$, because $i + j \leqslant k$ implies $i = i + \mathtt{0} \leqslant i + j \leqslant k$ and $j = (i + j) - i \leqslant k - i$; and hence $(g \bullet^i f)\,k = g(k - i) = g\,j = z$. Therefore we get a well-defined path $q \bullet p : x \sim z$ satisfying

$$|q \bullet p| = |p| + |q| \tag{3.7}$$

$$(\forall i : \mathtt{R_+})\;\begin{array}{lll} i \leqslant |p| & \Rightarrow & (q \bullet p)\,i = p\,i \\ |p| \leqslant i & \Rightarrow & (q \bullet p)\,i = q\,(i - |p|) \end{array} \tag{3.8}$$

**Lemma 3.4.** *For each $\Gamma \in \mathcal{E}$, given $x, y, z, w : \Gamma$, $p : x \sim y$, $q : y \sim z$, $r : z \sim w$ and $\gamma : \Gamma \to \Gamma'$, one has:*

$$p \bullet (\mathtt{idp}\, x) = p = (\mathtt{idp}\, y) \bullet p \tag{3.9}$$

$$(r \bullet q) \bullet p = r \bullet (q \bullet p) \tag{3.10}$$

$$\gamma \mathbin{\text{'}} (\mathtt{idp}\, x) = \mathtt{idp}(\gamma\, x) \tag{3.11}$$

$$\gamma \mathbin{\text{'}} (q \bullet p) = (\gamma \mathbin{\text{'}} q) \bullet (\gamma \mathbin{\text{'}} p) \tag{3.12}$$

*Proof.* One just has to check that these properties follow constructively from the axioms (2.4)–(2.12). For example (3.10) holds because

$$|(r\bullet q)\bullet p| = |p| + |r\bullet q| = |p| + (|q| + |r|) = (|p| + |q|) + |r| = |q\bullet p| + |r| = |r\bullet(q\bullet p)|$$

and

$$((r \bullet q) \bullet p)\, i = \begin{cases} p\, i & \text{if } i \leqslant |p| \\ q\,(i - |p|) & \text{if } |p| \leqslant i \text{ and } i - |p| \leqslant |q| \\ r\,((i - |p|) - |q|) & \text{if } |p| \leqslant i \text{ and } |q| \leqslant i - |p| \end{cases}$$

$$(r \bullet (q \bullet p))\, i = \begin{cases} p\, i & \text{if } i \leqslant |p| + |q| \text{ and } i \leqslant |p| \\ q\,(i - |p|) & \text{if } i \leqslant |p| + |q| \text{ and } |p| \leqslant i \\ r\,(i - (|p| + |q|)) & \text{if } |p| + |q| \leqslant i \end{cases}$$

which are equal, because one can use axioms (2.4)–(2.12) to show that

$$\begin{aligned} i \leqslant |p| \;&\Leftrightarrow\; i \leqslant |p| + |q| \;\wedge\; i \leqslant |p| \\ |p| \leqslant i \;\wedge\; i - |p| \leqslant |q| \;&\Leftrightarrow\; i \leqslant |p| + |q| \;\wedge\; |p| \leqslant i \\ |p| \leqslant i \;\wedge\; |q| \leqslant i - |p| \;&\Leftrightarrow\; |p| + |q| \leqslant i \\ (i - |p|) - |q| \;&=\; i - (|p| + |q|). \end{aligned}$$

□

As well as composing Moore paths one can reverse them. To define this operation it is convenient to use the operation of *truncated subtraction* $\_ \dot{-} \_ : \mathtt{R}_{+} \times \mathtt{R}_{+} \longrightarrow \mathtt{R}_{+}$, which is well-defined by the following properties:

$$(\forall i, j : \mathtt{R}_{+}) \; \begin{array}{l} i \leqslant j \;\Rightarrow\; i \dot{-} j = 0 \\ j \leqslant i \;\Rightarrow\; i \dot{-} j = i - j \end{array} \tag{3.13}$$

**Lemma 3.5** (path reversal)**.** *For each $\Gamma \in \mathcal{E}$, given $x, y : \Gamma$ and $p : x \sim y$, there is a* reversed path $\mathtt{rev}\, p : y \sim x$ *well-defined by the requirements*

$$|\mathtt{rev}\, p| = |p|$$

$$(\forall i : \mathtt{R}_{+})\; (\mathtt{rev}\, p)\, i = p\,(|p| \dot{-} i)$$

*and satisfying*

$$\mathtt{rev}(\mathtt{idp}\, x) = \mathtt{idp}\, x \tag{3.14}$$

$$\mathtt{rev}(q \bullet p) = (\mathtt{rev}\, p) \bullet (\mathtt{rev}\, q) \tag{3.15}$$

$$\mathtt{rev}(\mathtt{rev}\, p) = p \tag{3.16}$$

$$\gamma \mathbin{\text{'}} (\mathtt{rev}\, p) = \mathtt{rev}(\gamma \mathbin{\text{'}} p) \tag{3.17}$$

*Proof.* As for the previous lemma, this follows straightforwardly from the axioms (2.4)–(2.12) within constructive logic. □

Although the definition of $\mathtt{rev}\,p$ is standard, the above equational properties are not often mentioned in the literature. However, they are crucial for the construction of identity types in Section 4 to work. What usually gets a mention is the fact that up to homotopy $\mathtt{rev}\,p$ is a two-sided inverse for $p$ with respect to the $\bullet$ operation. Paths $(\mathtt{rev}\,p) \bullet p \sim \mathtt{idp}\,x$ and $p \bullet (\mathtt{rev}\,p) \sim \mathtt{idp}\,y$ can be constructed using the path contraction operation given below; we do not bother to do that, because they are also a consequence of the *path induction* [Uni13, 1.12.1] property of identity types that follows from Theorem 4.10.

**Definition 3.6** (bounded abstractions)**.** The following binding syntax is very convenient for describing Moore paths. For each $\Gamma \in \mathcal{E}$, if $\lambda i \to \varphi(i)$ describes a function in $\mathtt{R_+} \to \Gamma$, then for each $j : \mathtt{R_+}$ using the $\mathtt{min}$ function (2.3) we get a path $\langle i \leqslant j\rangle\varphi(i) : \varphi(\mathtt{0}) \sim \varphi(j)$ in $\Gamma$ by defining:

$$\langle i \leqslant j\rangle\varphi(i) :\equiv (\lambda i \to \varphi(\mathtt{min}(i,j))\ ,\ j) \tag{3.18}$$

($i$ is bound in the above expression). It is easy to see that this form of bounded abstraction has the following properties:

$$\gamma\ \text{'}\ \langle i \leqslant j\rangle\varphi(i) = \langle i \leqslant j\rangle\gamma(\varphi(i)) \tag{3.19}$$

$$\langle i \leqslant \mathtt{0}\rangle\varphi(i) = \mathtt{idp}(\varphi(\mathtt{0})) \tag{3.20}$$

$$\langle i \leqslant |p|\rangle(p\,i) = p \tag{3.21}$$

**Lemma 3.7** (path contraction)**.** *Given $\Gamma \in \mathcal{E}$, for any path $p : x \sim y$ in $\Gamma$ and $i : \mathtt{R_+}$, there is a path $p_{\leqslant i} : x \sim p\,i$ satisfying*

$$p_{\leqslant \mathtt{0}} = \mathtt{idp}\,x \tag{3.22}$$

$$(\forall i \geqslant |p|)\ p_{\leqslant i} = p \tag{3.23}$$

*Proof.* Using the bounded abstraction notation and the $\mathtt{min}$ function (2.3), we define

$$p_{\leqslant i} :\equiv \langle j \leqslant \mathtt{min}(|p|, i)\rangle(p\,j) \tag{3.24}$$

Since $p\,(\mathtt{min}(|p|, i)) = p\,i$, this does give a path $x \sim p\,i$; and it has the required properties by (3.20) and (3.21). □

**Remark 3.8.** Using $p_{\leqslant i}$ we get for each $x : \Gamma$ that $\sum_{y:\Gamma}(x \sim y)$ is path-contractible [Uni13, Section 3.11] with centre $(x, \mathtt{idp}\,x)$, since for each $y : \Gamma$ and $p : x \sim y$ we have a path $\langle i \leqslant |p|\rangle(p\,i, p_{\leqslant i})$ in $(x, \mathtt{idp}\,x) \sim (y, p)$. This is part of the more general fact that Moore paths model identity types, which we show in the next section (see Theorem 4.10).

## 4. Transport along Paths

In this section we continue with the assumptions of the previous one: $\mathcal{E}$ is a topos (with associated CwF) containing an ordered commutative ring object $\mathtt{R}$. We want objects of Moore paths with respect to $\mathtt{R}$ (Definition 3.1) to give a model of identity types, as well as other type formers. Recall that in Martin-Löf Type Theory elements of identity types $\mathtt{Id}_\Gamma\,x\,y$ give rise to transport functions $A\,x \to A\,y$ between members of a family of types $(A\,x \mid x : \Gamma)$ over a type $\Gamma$; see for example [Uni13, Section 2.3]. Therefore, for each object $\Gamma \in \mathcal{E}$, we should restrict attention to families $A \in \mathcal{E}(\Gamma)$ that come equipped at least with some sort of transport operation taking a path $p : x \sim y$ in $\Gamma$ and an element $a : A\,x$ to an element $p_* a : A\,y$. This leads to the following definition.

**Definition 4.1** (tap fibrations)**.** Given an object $\Gamma \in \mathcal{E}$, a *transport-along-paths* (tap) structure for a family $A \in \mathcal{E}(\Gamma)$ is a $(\Gamma \times \Gamma)$-indexed family of morphisms $((\_)_* : x \sim y \to (A\,x \to A\,y) \mid x, y : \Gamma)$ satisfying for all $x : \Gamma$ and $a : A\,x$

$$(\mathtt{idp}\,x)_* a = a \tag{4.1}$$

We write $\mathbf{Fib}(\Gamma)$ for the families over $\Gamma$ equipped with a tap structure and call them *fibrations*. They are stable under re-indexing: given $\gamma : \Delta \longrightarrow \Gamma$ in $\mathcal{E}$ and $A \in \mathbf{Fib}(\Gamma)$, then $A\,\gamma :\equiv (A(\gamma\,x) \mid x : \Delta) \in \mathcal{E}(\Delta)$ has a tap structure via the congruence operation (3.4), taking the transport of $a : (A\,\gamma)\,x = A(\gamma\,x)$ along $p : x \sim y$ to be $(\gamma \mathbin{\text{'}} p)_* a : A(\gamma\,y)$. This re-indexing of tap structure respects composition and identities in $\mathcal{E}$. So $\mathbf{Fib}$ inherits the structure of a CwF from that of $\mathcal{E}$, with the set of elements of a fibration being the elements of the underlying family in $\mathcal{E}$, that is $\mathbf{Fib}(\Gamma \vdash A) = \mathcal{E}(\Gamma \vdash A)$.

We show that the CwF $\mathbf{Fib}$ inherits some type structure from $\mathcal{E}$. To do so involves definitions and calculations using the bounded abstraction formalism of Definition 3.6.

To describe $\Sigma$- and $\Pi$-types in $\mathbf{Fib}$ we have to lift paths in $\Gamma$ to paths in comprehension objects $\Gamma.A = \sum_{x:\Gamma} A\,x$ of fibrations $A \in \mathbf{Fib}(\Gamma)$:

**Lemma 4.2** (path lifting)**.** *Given $\Gamma \in \mathcal{E}$ and $A \in \mathbf{Fib}(\Gamma)$, for each path $p : x \sim y$ in $\Gamma$ and each $a : A\,x$, there is a path $\mathtt{lift}(p, a) : (x, a) \sim (y, p_* a)$ in $\Gamma.A$ satisfying*

$$\mathtt{lift}(\mathtt{idp}\,x, a) = \mathtt{idp}(x, a) \tag{4.2}$$

*and stable under re-indexing along morphisms $\Delta \longrightarrow \Gamma$ in $\mathcal{E}$.*

*Proof.* We can use path contraction (Lemma 3.7) to express path lifting using the bounded abstraction notation from Definition 3.6:

$$\mathtt{lift}(p, a) :\equiv \langle i \leqslant |p| \rangle (p\,i, (p_{\leqslant i})_* a) \tag{4.3}$$

Since the path contraction operation satisfies $p_{\leqslant \mathtt{0}} = \mathtt{idp}\,x$ and $p_{\leqslant |p|} = p$, this does indeed give a path in $(x, a) \sim (y, p_* a)$ . The desired properties of $\mathtt{lift}$ follow from corresponding properties of bounded abstraction and path contraction (3.19)–(3.23). Specifically, we have:

$$\begin{aligned}
\mathtt{lift}(\mathtt{idp}\,x, a) &= \langle i \leqslant |\mathtt{idp}\,x| \rangle (\mathtt{idp}\,x\,i, ((\mathtt{idp}\,x)_{\leqslant i})_* a) \\
&= \langle i \leqslant |\mathtt{0}| \rangle (x, ((\mathtt{idp}\,x)_{\leqslant i})_* a) \\
&= \mathtt{idp}(x, ((\mathtt{idp}\,x)_{\leqslant \mathtt{0}})_* a) \\
&= \mathtt{idp}(x, (\mathtt{idp}\,x)_* a) \\
&= \mathtt{idp}(x, a)
\end{aligned}$$

where $x : \Gamma$ and $a : A\,x$. We also have:

$$\begin{aligned}
\mathtt{lift}_A(\gamma \mathbin{\text{'}} p, a) &= \langle i \leqslant |\gamma \mathbin{\text{'}} p| \rangle ((\gamma \mathbin{\text{'}} p)\,i, ((\gamma \mathbin{\text{'}} p)_{\leqslant i})_* a) \\
&= \langle i \leqslant |p| \rangle (\gamma(p\,i), (\gamma \mathbin{\text{'}} p_{\leqslant i})_* a) \\
&= \langle i \leqslant |p| \rangle (\gamma \times \mathtt{id})(p\,i, (\gamma \mathbin{\text{'}} p_{\leqslant i})_* a) \\
&= (\gamma \times \mathtt{id}) \mathbin{\text{'}} (\langle i \leqslant |p| \rangle (p\,i, (\gamma \mathbin{\text{'}} p_{\leqslant i})_* a)) \\
&= (\gamma \times \mathtt{id}) \mathbin{\text{'}} (\mathtt{lift}_{A\gamma}(p, a))
\end{aligned}$$

where $\gamma : \Delta \longrightarrow \Gamma$, $p : x \sim y$ in $\Delta$, $a : A(\gamma\,x)$ and we write $\mathtt{lift}_A$ for the lifting operation on $A \in \mathbf{Fib}(\Gamma)$ and $\mathtt{lift}_{A\gamma}$ for the lifting operation on $A\gamma \in \mathbf{Fib}(\Delta)$. ∎

**Theorem 4.3** ($\Sigma$- and $\Pi$-types)**.** *Given $\Gamma \in \mathcal{E}$, $A \in \mathbf{Fib}(\Gamma)$ and $B \in \mathbf{Fib}(\Gamma.A)$, the families $\Sigma\, A\, B :\equiv (\sum_{a:A\,x} B(x,a) \mid x : \Gamma)$ and $\Pi\, A\, B :\equiv (\prod_{a:A\,x} B(x,a) \mid x : \Gamma)$ in $\mathcal{E}(\Gamma)$ have tap fibration structures that are stable under re-indexing along morphisms $\Delta \longrightarrow \Gamma$ in $\mathcal{E}$. Hence the CwF* **Fib** *supports $\Sigma$-and $\Pi$-types* [Hof97, Definitions 3.15 and 3.18].

*Proof.* Given a path $p : x \sim y$ in $\Gamma$, we get functions $\Sigma\, A\, B\, x \to \Sigma\, A\, B\, y$ and $\Pi\, A\, B\, x \to \Pi\, A\, B\, y$ using path lifting and (in the second case) path reversal:

$$p_*(a,b) :\equiv (p_* a, \mathtt{lift}(p,a)_* b) \in \Sigma\, A\, B\, y \qquad \text{where } (a,b) : \Sigma\, A\, B\, x \tag{4.4}$$

$$(p_* f)\, a :\equiv (\mathtt{rev}(\mathtt{lift}(\mathtt{rev}\, p, a)))_* f((\mathtt{rev}\, p)_* a) \in B(y,a) \qquad \text{where } f : \Pi\, A\, B\, x \text{ and } a : A\, y \tag{4.5}$$

To see why (4.5) has the correct type, consider an arbitrary $p : x \sim y$ in $\Gamma$, $f : \Pi\, A\, B\, x$ and $a : A\, y$. We have $(\mathtt{rev}\, p) : y \sim x$, and hence $(\mathtt{rev}\, p)_* a : A\, x$ and $f((\mathtt{rev}\, p)_* a) : B(x, (\mathtt{rev}\, p)_* a)$. Next, we have $\mathtt{lift}(\mathtt{rev}\, p, a) : (y, a) \sim (x, \mathtt{rev}\, p_* a)$ and therefore

$$\mathtt{rev}(\mathtt{lift}(\mathtt{rev}\, p, a)) : (x, \mathtt{rev}\, p_* a) \sim (y, a)$$

This allows us to transport along this path to get

$$(\mathtt{rev}(\mathtt{lift}(\mathtt{rev}\, p, a)))_* f((\mathtt{rev}\, p)_* a) \in B(y,a)$$

as required.

Note that these definitions satisfy the required property when transporting along the identity path. That is, given $x : \Gamma$ and $(a,b) : \Sigma\, A\, B\, x$, then using property (4.2) we have:

$$(\mathtt{idp}\, x)_*(a,b) = ((\mathtt{idp}\, x)_* a, \mathtt{lift}((\mathtt{idp}\, x), a)_* b) = ((\mathtt{idp}\, x)_* a, (\mathtt{idp}(x,a))_* b) = (a,b)$$

Similarly, given $x : \Gamma$, $f : \Pi\, A\, B\, x$ and $a : A\, x$ then using (4.2) and the properties of path reversal given in Lemma 3.5 we have:

$$\begin{aligned}
((\mathtt{idp}\, x)_* f)\, a &= (\mathtt{rev}(\mathtt{lift}(\mathtt{rev}(\mathtt{idp}\, x), a)))_* f((\mathtt{rev}(\mathtt{idp}\, x))_* a) \\
&= (\mathtt{rev}(\mathtt{lift}(\mathtt{idp}\, x, a)))_* f((\mathtt{idp}\, x)_* a) \\
&= (\mathtt{rev}(\mathtt{idp}(x,a)))_* f(a) \\
&= (\mathtt{idp}(x,a))_* f(a) \\
&= f(a)
\end{aligned}$$

Finally, the stability of these definitions under re-indexing comes from the fact that `rev` and `lift` are both stable under re-indexing. □

**Theorem 4.4** (Empty, unit, Boolean and natural number types)**.** *Given $\Gamma \in \mathcal{E}$, for each $A \in \mathcal{E}$, the constant family $(A \mid x : \Gamma)$ has a tap fibration structure given by $p_* a = a$ (for any $a : A$) and this is stable under re-indexing. Taking $A$ to be the initial object $\emptyset$, the terminal object $1$, the coproduct $1 + 1$ and the natural number object of the topos, we have that the CwF* **Fib** *supports empty, unit, Boolean and natural number types* [Hof97, Exercises E3.24–E3.26]. □

**Theorem 4.5** (Sum types)**.** *Given $\Gamma \in \mathcal{E}$ and $A, B \in \mathbf{Fib}(\Gamma)$, the family $A \oplus B :\equiv (A\, x + B\, x \mid x : \Gamma)$ of sum types has a tap fibration structure that is stable under re-indexing along morphisms $\Delta \longrightarrow \Gamma$ in $\mathcal{E}$. Hence the CwF* **Fib** *supports sum types.*

*Proof.* Given a path $p : x \sim y$ in $\Gamma$, we get a function $(A \oplus B)\,x \longrightarrow (A \oplus B)\,y$ by case analysis on the elements of $(A \oplus B)\,x = A\,x + B\,x$. Thus if $\mathtt{inl}$ and $\mathtt{inr}$ denote the constructors of the sum type $A\,x + B\,x$, we have

$$p_*(\mathtt{inl}\,a) = \mathtt{inl}(p_* a) \qquad \text{where } a : A\,x \tag{4.6}$$

$$p_*(\mathtt{inr}\,b) = \mathtt{inr}(p_* b) \qquad \text{where } b : B\,x \tag{4.7}$$

and clearly this definition inherits property (4.1) from the tap fibration structure of $A$ and $B$, and is stable under re-indexing. □

Since we only consider toposes with a natural number object, the CwF associated with $\mathcal{E}$ can interpret types of well-founded trees ($W$-types) [Uni13, Section 5.3]; see [MP00, Propositions 3.6 and 3.8]. We write $\mathtt{W}_{x:A}\,B\,x$ for the object of well-founded trees determined by a family $B \in \mathcal{E}(A)$, with constructor $\mathtt{sup} : \sum_{y:A}(B\,y \to \mathtt{W}_{x:A}\,B\,x) \to \mathtt{W}_{x:A}\,B\,x$.

**Theorem 4.6** ($W$-types)**.** *Given $A \in \mathbf{Fib}(\Gamma)$ and $B \in \mathbf{Fib}(\Gamma.A)$, the family*

$$W\,A\,B :\equiv (\mathtt{W}_{a:A\,x}\,B(x,a) \mid x : \Gamma) \in \mathcal{E}(\Gamma)$$

*has a tap fibration structure that is stable under re-indexing along morphisms $\Delta \longrightarrow \Gamma$ in $\mathcal{E}$. Therefore the CwF* **Fib** *supports $W$-types.*

*Proof.* Given a path $p : x \sim y$ in $\Gamma$, we get a function $W\,A\,B\,x \to W\,A\,B\,y$ via the following well-founded recursion equation:

$$p_*\,\mathtt{sup}(a\,,f) = \mathtt{sup}\,(p_*a\,,\lambda b \to p_*f((\mathtt{rev}(\mathtt{lift}(p,a)))_*b)) \qquad \begin{array}{l}\text{where } a : A\,x \text{ and}\\ f : B(x,a) \to W\,A\,B\,x\end{array} \tag{4.8}$$

Note that $p_*a : A\,y$. Therefore, assuming that the second argument to the $\mathtt{sup}$ constructor has type $B(y, p_*a) \to W\,A\,B\,y$, then the overall type of the constructor will be $W\,A\,B\,y$ as required. To see why the second component does have this type, consider an arbitrary $b : B(y, p_*a)$ and observe that $\mathtt{rev}(\mathtt{lift}(p,a)) : (y, p_*a) \sim (x,a)$. Therefore we have $(\mathtt{rev}(\mathtt{lift}(p,a)))_*b : B(x,a)$ and hence $f((\mathtt{rev}(\mathtt{lift}(p,a)))_*b) : W\,A\,B\,x$. Finally, recursively transporting along $p$ gives us $p_*f((\mathtt{rev}(\mathtt{lift}(p,a)))_*b) : W\,A\,B\,y$ as required.

Well-founded inductions using the properties of reversal and lifting given in Lemmas 3.5 and 4.2 suffice to show that this inherits the properties of a tap structure from those for $A$ and $B$:

$$\begin{aligned}(\mathtt{idp}\,x)_*\,\mathtt{sup}(a\,,f) &= \mathtt{sup}\,((\mathtt{idp}\,x)_*a\,,\lambda b \to (\mathtt{idp}\,x)_*f((\mathtt{rev}(\mathtt{lift}(\mathtt{idp}\,x,a)))_*b))\\ &= \mathtt{sup}\,((\mathtt{idp}\,x)_*a\,,\lambda b \to (\mathtt{idp}\,x)_*f((\mathtt{idp}(x,a))_*b))\\ &= \mathtt{sup}\,(a\,,\lambda b \to f(b))\\ &= \mathtt{sup}(a\,,f)\end{aligned}$$

The fact that this definition is stable under re-indexing follows immediately from the fact that $\mathtt{rev}$ and $\mathtt{lift}$ are both stable under re-indexing. □

So far we have considered type structure that lifts from the CwF associated with $\mathcal{E}$ to the CwF **Fib**. Now we consider identity types, where the structure of interest in **Fib** is not the one inherited from $\mathcal{E}$. Since $\mathcal{E}$ is a topos, it certainly has *extensional* identity types [Mar84], inhabitation of which coincides with judgemental equality, and those could be lifted to **Fib**. However, we wish to show that Moore path objects give the *intensional* version of identity types in **Fib**, the family of types $(\mathtt{Id}_A\,x\,y \mid x, y : A)$ inductively generated by a single constructor $\mathtt{Refl}_A : \prod_{x:A} \mathtt{Id}_A\,x\,x$. We will use Hofmann's version of the structure in a CwF needed to model such types [Hof97]. To do so, let us fix some notation for a CwF $\mathbf{C}$.

Re-indexing of a family $A \in \mathbf{C}(\Gamma)$ and an element $\alpha \in \mathbf{C}(\Gamma \vdash A)$ along a morphism $\gamma : \Delta \longrightarrow \Gamma$ will just be denoted by $A\,\gamma \in \mathbf{C}(\Delta)$ and $\alpha\,\gamma \in \mathbf{C}(\Delta \vdash A\,\gamma)$; and given an element $\beta \in \mathbf{C}(\Delta \vdash A\,\gamma)$, then $\langle \gamma\,,\beta\rangle$ denotes the unique morphism $\Delta \longrightarrow \Gamma.A$ whose composition with $\mathtt{fst} : \Gamma.A \longrightarrow \Gamma$ is $\gamma$ and whose re-indexing of the generic element $\mathtt{snd} \in \mathbf{C}(\Gamma.A \vdash A\,\mathtt{fst})$ is $\beta$:

$$\mathtt{fst} \circ \langle \gamma\,,\beta\rangle = \gamma \qquad \mathtt{snd}\langle \gamma\,,\beta\rangle = \beta.$$

**Definition 4.7.** Following Hofmann [Hof97, Definition 3.19], we say that a CwF $\mathbf{C}$ *supports the interpretation of intensional identity types* if for each object $\Gamma \in \mathbf{C}$ and each family $A \in \mathbf{C}(\Gamma)$ the following data is given and is stable under re-indexing along any $\gamma : \Delta \longrightarrow \Gamma$:

- a family $\mathtt{Id}_A \in \mathbf{C}(\Gamma.A.A\,\mathtt{fst})$,
- a morphism $\mathtt{Refl}_A : \Gamma.A \longrightarrow \Gamma.A.A\,\mathtt{fst}.\,\mathtt{Id}_A$ such that $\mathtt{fst} \circ \mathtt{Refl}_A$ equals the diagonal morphism $\langle \mathtt{id}\,,\mathtt{snd}\rangle : \Gamma.A \longrightarrow \Gamma.A.A\,\mathtt{fst}$,
- a function mapping each $B \in \mathbf{C}(\Gamma.A.A\,\mathtt{fst}.\,\mathtt{Id}_A)$ and $\beta \in \mathbf{C}(\Gamma.A \vdash B\ \mathtt{Refl}_A)$ to an element $\mathtt{J}_A\,B\,\beta \in \mathbf{C}(\Gamma.A.A\,\mathtt{fst}.\,\mathtt{Id}_A \vdash B)$ such that the re-indexing $(\mathtt{J}_A\,B\,\beta)\ \mathtt{Refl}_A$ equals $\beta$.

Given an object $\Gamma$ of the topos $\mathcal{E}$ and a family $A \in \mathcal{E}(\Gamma)$, we can use the family of Moore path objects $\_ \sim \_$ (Definition 3.1) to define a family $\mathtt{Id}_A \in \mathcal{E}(\Gamma.A.A\,\mathtt{fst})$ as follows:

$$\mathtt{Id}_A :\equiv (a_1 \sim a_2 \mid ((x, a_1), a_2) : \Gamma.A.A\,\mathtt{fst}) \tag{4.9}$$

We will show that this together with suitable $\mathtt{Refl}$ and $\mathtt{J}$ operations give an instance of Definition 4.7 for the CwF $\mathbf{Fib}$. In particular $\mathtt{Id}_A$ has a tap fibration structure when $A$ does. To see this we first need to analyse paths in $\Gamma.A$ in terms of paths in $\Gamma$ and in the fibres $A\,x$ (for $x : \Gamma$).

**Lemma 4.8.** *Given $\Gamma \in \mathcal{E}$ and $A \in \mathbf{Fib}(\Gamma)$, for each path $p : (x, a) \sim (y, b)$ in $\Gamma.A$, there is a path $\mathtt{snd}(p) : (\mathtt{fst}\ \text{'}\ p)_* a \sim b$ in $A\,y$ satisfying*

$$\mathtt{snd}(\mathtt{idp}(x, a)) = \mathtt{idp}\,a \tag{4.10}$$

*and stable under re-indexing along any $\gamma : \Delta \longrightarrow \Gamma$ in $\mathcal{E}$.*

*Proof.* We use a reversed version of the path-contraction operation from Lemma 3.7: for each path $q : x \sim y$ and each $i : \mathtt{R_+}$, define $q_{\geqslant i} : q\,i \sim y$ by

$$q_{\geqslant i} :\equiv \mathtt{rev}\left((\mathtt{rev}\,q)_{\leqslant(|q| \dot- i)}\right) \tag{4.11}$$

Given $p : (x, a) \sim (y, b)$, we get $\mathtt{fst}\ \text{'}\ p : x \sim y$ and hence for each $i : \mathtt{R_+}$ we have $(\mathtt{fst}\ \text{'}\ p)_{\geqslant i} : (\mathtt{fst}\ \text{'}\ p)\,i \sim y$; and since $\mathtt{snd}(p\,i) : A(\mathtt{fst}(p\,i)) = A((\mathtt{fst}\ \text{'}\ p)\,i)$, we get $((\mathtt{fst}\ \text{'}\ p)_{\geqslant i})_*\,\mathtt{snd}(p\,i) : A\,y$. So we can define

$$\mathtt{snd}(p) = \langle i \leqslant |p|\rangle\,(((\mathtt{fst}\ \text{'}\ p)_{\geqslant i})_*\,\mathtt{snd}(p\,i)) \tag{4.12}$$

to get a path $(\mathtt{fst}\ \text{'}\ p)_* a \sim b$ in $A\,y$. Property (4.10) holds since, using (3.20), we have

$$\begin{aligned}
\mathtt{snd}(\mathtt{idp}(x, a)) &= \langle i \leqslant |\mathtt{idp}(x, a)|\rangle\,(((\mathtt{fst}\ \text{'}\ \mathtt{idp}(x, a))_{\geqslant i})_*\,\mathtt{snd}(\mathtt{idp}(x, a)\,i)) \\
&= \langle i \leqslant 0\rangle\,(((\mathtt{idp}\,x)_{\geqslant i})_*\,\mathtt{snd}(x, a)) \\
&= \mathtt{idp}\,(((\mathtt{idp}\,x)_{\geqslant 0})_* a) \\
&= \mathtt{idp}\,((\mathtt{idp}\,x)_* a) \\
&= \mathtt{idp}\,a
\end{aligned}$$

Stability under re-indexing follows from (3.19) and the fact that `rev` is stable under re-indexing. □

**Lemma 4.9.** *Given* $\Gamma \in \mathcal{E}$ *and a fibration* $A \in \mathbf{Fib}(\Gamma)$*, the family* $\mathtt{Id}_A \in \mathcal{E}(\Gamma.A.A\,\mathtt{fst})$ *defined in* (4.9) *has a tap fibration structure that is stable under re-indexing along morphisms* $\Delta \longrightarrow \Gamma$ *in* $\mathcal{E}$*.*

*Proof.* First note that re-indexing the fibration $A \in \mathbf{Fib}(\Gamma)$ along $\mathtt{fst} : \Gamma.A \longrightarrow \Gamma$ we get $A\,\mathtt{fst} \in \mathbf{Fib}(\Gamma.A)$. Given a path $p : ((x,a_1),a_2) \sim ((y,b_1),b_2)$ in $\Gamma.A.A\,\mathtt{fst}$, define

$$p' :\equiv \mathtt{fst}\ \text{'}\ (\mathtt{fst}\ \text{'}\ p) : x \sim y$$

We can apply Lemma 4.8 to the paths $\mathtt{fst}\ \text{'}\ p : (x,a_1) \sim (y,b_1)$ and $\langle \mathtt{fst} \circ \mathtt{fst}\,, \mathtt{snd}\rangle\ \text{'}\ p : (x,a_2) \sim (y,b_2)$ in $\Gamma.A$ to get paths in $A\,y$

$$p_1 :\equiv \mathtt{snd}(\mathtt{fst}\ \text{'}\ p) : {p'}_* a_1 \sim b_1$$
$$p_2 :\equiv \mathtt{snd}(\langle \mathtt{fst} \circ \mathtt{fst}\,, \mathtt{snd}\rangle\ \text{'}\ p) : {p'}_* a_2 \sim b_2$$

(using the fact that $\mathtt{fst} \circ \langle \mathtt{fst} \circ \mathtt{fst}\,, \mathtt{snd}\rangle = \mathtt{fst} \circ \mathtt{fst}$). Thus for each path $q : a_1 \sim a_2$ in $A\,x$, we have

$$\begin{array}{ccc}
A\,x & \xrightarrow{{p'}_*} & A\,y \\
 & & \begin{array}{ccc} {p'}_* a_1 & \overset{({p'}_*)\text{'}q}{\rightsquigarrow} & {p'}_* a_2 \\ \mathtt{rev}(p_1) \uparrow & & \downarrow p_2 \\ b_1 & & b_2 \end{array} \\
a_1 \overset{q}{\rightsquigarrow} a_2 & &
\end{array}$$

and can compose together the paths in $A\,y$ to get $p_2 \bullet (({p'}_*)\ \text{'}\ q) \bullet \mathtt{rev}(p_1) : b_1 \sim b_2$. So altogether we get a function $p_* : \mathtt{Id}_A((x,a_1),a_2) \to \mathtt{Id}_A((y,b_1),b_2)$ defined by:

$$p_* q = \mathtt{snd}(\langle \mathtt{fst} \circ \mathtt{fst}\,, \mathtt{snd}\rangle\ \text{'}\ p) \bullet (((\mathtt{fst}\ \text{'}\ (\mathtt{fst}\ \text{'}\ p))_*)\ \text{'}\ q) \bullet \mathtt{rev}(\mathtt{snd}(\mathtt{fst}\ \text{'}\ p)) \tag{4.13}$$

The properties of Moore path reversal (Lemma 3.5) together with Lemma 4.8 suffice to show that this definition inherits the property of a tap fibration structure (4.1) from the one for $A$ and that it is stable under re-indexing. For example, given $x : \Gamma$, $a_1, a_2 : A\,x$ and $q : a_1 \sim a_2$ we have:

$$\begin{aligned}
(\mathtt{idp}((x,a_1),a_2))_* q &= \mathtt{snd}(\langle \mathtt{fst} \circ \mathtt{fst}\,, \mathtt{snd}\rangle\ \text{'}\ \mathtt{idp}((x,a_1),a_2)) \\
&\qquad \bullet (((\mathtt{fst}\ \text{'}\ (\mathtt{fst}\ \text{'}\ \mathtt{idp}((x,a_1),a_2)))_*)\ \text{'}\ q) \\
&\qquad \bullet \mathtt{rev}(\mathtt{snd}(\mathtt{fst}\ \text{'}\ \mathtt{idp}((x,a_1),a_2))) \\
&= \mathtt{snd}(\mathtt{idp}(x,a_2)) \bullet (((\mathtt{idp}\,x)_*)\ \text{'}\ q) \bullet \mathtt{rev}(\mathtt{snd}(\mathtt{idp}(x,a_1))) \\
&= (\mathtt{idp}\,a_2) \bullet q \bullet (\mathtt{idp}\,a_1) \\
&= q
\end{aligned}$$

as required. □

**Theorem 4.10** (Identity types)**.** *The CwF* **Fib** *of Definition 4.1 supports the interpretation of intensional identity types (Definition 4.7), given by Moore path objects as in* (4.9)*.*

*Proof.* In view of Lemma 4.9, it just remains to define the $\mathtt{Refl}$ and $\mathtt{J}$ operations as in Definition 4.7. Given $\Gamma \in \mathcal{E}$ and $A \in \mathbf{Fib}(\Gamma)$, we get $\mathtt{Refl}_A : \Gamma.A \longrightarrow \Gamma.A.A\,\mathtt{fst}.\,\mathtt{Id}_A$ by defining

$$\mathtt{Refl}_A(x, a) :\equiv (((x, a), a), \mathtt{idp}\, a) \tag{4.14}$$

Note that $(\mathtt{fst} \circ \mathtt{Refl}_A)(x, a) = ((x, a), a) = \langle \mathtt{id}\,, \mathtt{snd}\rangle(x, a)$, as required.

To define $\mathtt{J}$ we combine transport along paths with the fact that singleton types are contractible (Remark 3.8). More specifically, given $B \in \mathbf{Fib}(\Gamma.A.A\,\mathtt{fst}.\,\mathtt{Id}_A)$ and $\beta \in \mathcal{E}(\Gamma.A \vdash B\ \mathtt{Refl}_A)$, for each path $p : a_1 \sim a_2$ in $A\,x$ using Lemma 3.7 we have the following path in $\Gamma.A.A\,\mathtt{fst}.\,\mathtt{Id}_A$:

$$\langle i \leqslant |p|\rangle(((x, a_1), p\, i), p_{\leqslant i}) : (((x, a_1), a_1), \mathtt{idp}\, a_1) \sim (((x, a_1), a_2), p)$$

Since $B$ has a tap fibration structure we can transport $\beta(x, a_1) : B(((x, a_1), a_1), \mathtt{idp}\, a_1)$ along this path to get an element of $B(((x, a_1), a_2), p)$. So we can define $\mathtt{J}_A\, B\, \beta \in \mathcal{E}(\Gamma.A.A\,\mathtt{fst}.\,\mathtt{Id}_A \vdash B)$ by:

$$\mathtt{J}_A\, B\, \beta\, (((x, a_1), a_2), p) :\equiv (\langle i \leqslant |p|\rangle(((x, a_1), p\, i), p_{\leqslant i}))_*\, \beta(x, a_1) \tag{4.15}$$

$\mathtt{J}$ has the required computation property, because

$$\begin{aligned}
((\mathtt{J}_A\, B\, \beta)\, \mathtt{Refl}_A)(x, a) &= \mathtt{J}_A\, B\, \beta\, (((x, a), a), \mathtt{idp}\, a) && \text{by (4.14)}\\
&= (\langle i \leqslant 0\rangle(((x, a), a)\, \mathtt{idp}\, a))_*\, \beta(x, a) && \text{by (4.15) and (3.24)}\\
&= \mathtt{idp}(((x, a), a)\, \mathtt{idp}\, a)_* \beta(x, a) && \text{by (3.20)}\\
&= \beta(x, a) && \text{by (4.1)}
\end{aligned}$$

Stability of $\mathtt{Refl}$ under re-indexing follows from the fact that the congruence operations $\gamma$ ' preserve degenerate paths; and stability of $\mathtt{J}$ uses (3.19) and the fact that the path contraction operation $(\_)_{\leqslant i}$ is preserved by the congruence operations $\gamma$ ' . □

**Remark 4.11** (Associative tap fibrations)**.** Definition 4.1 does not require the transport action $((\_)_* : x \sim y \to (A\,x \to A\,y) \mid x, y : \Gamma)$ of a tap fibration $A \in \mathbf{Fib}(\Gamma)$ to be strictly associative. In other words, for all paths $p : x \sim y$ and $q : y \sim z$ and all $a : A\,x$ we do not necessarily have

$$(q \bullet p)_* a = q_*(p_* a) \tag{4.16}$$

As we have seen, that property is not needed to prove that fibrations give a model of type theory. Nevertheless, if one changes the definition by requiring (4.16), then the analogues of Theorems 4.3–4.6 and 4.10 do hold for this stronger notion of fibration, although we do not prove that here.

With Definition 4.1 as it stands, one has associativity up to homotopy, i.e. there is a path $(q \bullet p)_* a \sim q_*(p_* a)$. This is a consequence of Theorem 4.10, which allows one to use path induction [Uni13, Section 2.9]) to construct a path $(q \bullet p)_* a \sim q_*(p_* a)$ for any $p$ from the case when $p = \mathtt{idp}\, x$, where one has the degenerate path for $(q \bullet (\mathtt{idp}\, x))_* a = q_* a = q_*((\mathtt{idp}\, x)_* a)$.

**Remark 4.12** (Weak fibrations)**.** In the definition of tap fibration, if one replaces the equality (4.1) by a homotopy, $(\mathtt{idp}\, x)_* a \sim a$, the resulting weak notion of tap fibration satisfies versions of Theorems 4.3–4.6, albeit with more complicated proofs. The same is true for Theorem 4.10 except that one gets only *propositional* identity types, where there is a path between $(\mathtt{J}_A\, B\, \beta)\, \mathtt{Refl}_A$ and $\beta$, rather than an equality in $\mathcal{E}$ (cf. [CCHM18, Section 9.1] and [van18]).

## 5. Function Extensionality

Let $(\mathcal{E}, \mathtt{R})$ be an order-ringed topos (Definition 2.1). In this section we show that in the CwF **Fib** constructed from $(\mathcal{E}, \mathtt{R})$ as in the previous section, it is the case that functions behave extensionally with respect to the intensional identity types given by Moore paths.

So far we have only used the order and additive structure of $\mathtt{R}$ (axioms (2.4)–(2.12) in Fig. 1). For function extensionality to hold, we need more than that. To see why, consider the constant function $\mathtt{K_0} = \lambda i \to \mathtt{0} : \mathtt{R_+} \to \mathtt{R_+}$ and the identity function $\mathtt{id} : \mathtt{R_+} \to \mathtt{R_+}$.[1] If equality means existence of a Moore path, then these functions are extensionally equal, because we have $\lambda i \to \langle j \leqslant i \rangle j : \prod_{i:\mathtt{R_+}} (\mathtt{K_0}\, i \sim \mathtt{id}\, i)$. So if the principle of function extensionality is to hold with respect to Moore paths, then there will have to be a path $p : \mathtt{K_0} \sim \mathtt{id}$ in $\mathtt{R_+} \to \mathtt{R_+}$. What is its shape $|p|$? Since $p$ does not depend upon any assumptions, $|p|$ would have to be a global element $1 \longrightarrow \mathtt{R_+}$ in the topos $\mathcal{E}$. Axioms (2.4)–(2.12) only guarantee the existence of one such global element, namely $\mathtt{0}$. However, if $|p| = \mathtt{0}$ then $\mathtt{K_0} = \mathtt{id}$, so $(\forall i : \mathtt{R})\ \mathtt{0} = i$ and the model of type theory is degenerate. Therefore we need some extra assumptions about $\mathtt{R}$ if function extensionality is to hold without collapsing everything. This is where axioms (2.13)–(2.17) come into play. These may not be the minimal assumptions needed for function extensionality, but they are a well-known part of (constructive, ordered) Algebra [Bou81, chapter VI] that does the job, and this makes easier the task of finding specific models with good properties (which we address in Section 7.2).

Given an object $\Gamma \in \mathcal{E}$ and a family $A \in \mathcal{E}(\Gamma)$, if $p : f \sim g$ is a path between dependent functions $f, g : \prod_{x:\Gamma} A\, x$, then for each $x : \Gamma$ we can apply the path congruence operation (3.4) to $\lambda f \to f\, x : (\prod_{x:\Gamma} A\, x) \to A\, x$ and $p$ to obtain a path $(\lambda f \to f\, x)\ \text{'}\ p : f\, x \sim g\, x$ in $A\, x$. This gives us the following function (which coincides with the canonical function obtained by path induction as in [Uni13, Section 2.9]):

$$\begin{array}{l} \mathtt{happly} : (f \sim g) \to \prod_{x:\Gamma}(f\, x \sim g\, x) \\ \mathtt{happly}\, p\, x :\equiv (\lambda f \to f\, x)\ \text{'}\ p \end{array} \qquad \text{(where } f, g : \textstyle\prod_{x:\Gamma} A\, x\text{)} \tag{5.1}$$

**Theorem 5.1** (function extensionality modulo $\sim$)**.** *Given an order-ringed topos $(\mathcal{E}, \mathtt{R})$, an object $\Gamma \in \mathcal{E}$ and a family $A \in \mathcal{E}(\Gamma)$, for all $f, g : \prod_{x:\Gamma} A\, x$ there is a function $\mathtt{funext} : (\prod_{x:\Gamma}(f\, x \sim g\, x)) \to (f \sim g)$. Furthermore this function is quasi-inverse* [Uni13, Definition 2.4.6] *to $\mathtt{happly}$, that is, that for all $e : \prod_{x:\Gamma}(f\, x \sim g\, x)$ and $p : f \sim g$ there are paths $\varepsilon\, e : \mathtt{happly}(\mathtt{funext}\, e) \sim e$ in $\prod_{x:\Gamma}(f\, x \sim g\, x)$ and $\eta\, p : p \sim \mathtt{funext}(\mathtt{happly}\, p)$ in $f \sim g$.*

*Proof.* If $e : \prod_{x:\Gamma}(f\, x \sim g\, x)$, then for all $x : \Gamma$ we have $e\, x\, (\mathtt{0} \cdot |e\, x|) = e\, x\, \mathtt{0} = f\, x$ and $e\, x\, (\mathtt{1} \cdot |e\, x|) = e\, x\, |e\, x| = g\, x$. So there is a path $\mathtt{funext}\, e : f \sim g$ in $\prod_{x:\Gamma} A\, x$ given by:

$$\mathtt{funext}\, e = \langle i \leqslant \mathtt{1} \rangle \lambda x \to e\, x\, (i \cdot |e\, x|) \tag{5.2}$$

Note that by (3.19) we have for each $x : \Gamma$

$$\begin{aligned} \mathtt{happly}(\mathtt{funext}\, e)\, x &= (\lambda f \to f\, x)\ \text{'}\ \langle i \leqslant \mathtt{1} \rangle \lambda x \to e\, x\, (i \cdot |e\, x|) \\ &= \langle i \leqslant \mathtt{1} \rangle (e\, x\, (i \cdot |e\, x|)) \end{aligned}$$

whereas $e\, x = \langle i \leqslant |e\, x| \rangle (e\, x\, i)$ by (3.21). So to get a path from $\mathtt{happly}(\mathtt{funext}\, e)\, x$ to $e\, x$ we need to interpolate shapes from $\mathtt{1}$ to $|e\, x|$ while at the same time interpolating the

[1] Function extensionality for **Fib** only concerns functions between fibrant families; but as we noted in Section 4, in **Fib** all objects, and in particular $\mathtt{R_+}$, are fibrant as trivial families over the terminal object.

argument of $e\,x\,\_$ from $i \cdot |e\,x|$ to $i \cdot 1 = i$. So for each $j : \mathtt{R}$ with $0 \leqslant j \leqslant 1$, consider

$$u_{j,x} :\equiv (1 - j) + j \cdot |e\,x| \quad \text{and} \quad v_{j,x} :\equiv (1 - j) \cdot |e\,x| + j$$

Calculating with the ring axioms we find that $u_{j,x} \cdot v_{j,x} = |e\,x| + j \cdot (1 - j) \cdot (|e\,x| - 1)^2$. Since $0 \leqslant j$ and $0 \leqslant 1 - j$ by assumption and since squares are always positive in an ordered ring [Bou81, VI.19], we have that $|e\,x| \leqslant u_{j,x} \cdot v_{j,x}$; hence $e\,x\,(u_{j,x} \cdot v_{j,x}) = g\,x$. So for all $j : \mathtt{R}$ with $0 \leqslant j \leqslant 1$ we have a path $\langle i \leqslant u_{j,x}\rangle(e\,x\,(i \cdot v_{j,x})) : f\,x \sim g\,x$ in $A\,x$. When $j = 0$ this path is $\mathtt{happly}(\mathtt{funext}\,e)\,x$ (because $u_{0,x} = 1$ and $v_{0,x} = |e\,x|$); when $j = 1$ the path is $e\,x$ (because $u_{1,x} = |e\,x|$ and $v_{1,x} = 1$). Therefore we can define

$$\varepsilon\,e :\equiv \langle j \leqslant 1\rangle \lambda x \to \langle i \leqslant u_{j,x}\rangle(e\,x\,(i \cdot v_{j,x})) \tag{5.3}$$

to get the desired path $\mathtt{happly}(\mathtt{funext}\,e) \sim e$ in $\prod_{x:\Gamma}(f\,x \sim g\,x)$. Since for any $p : f \sim g$ it is the case that $p = \langle i \leqslant |p|\rangle(p\,i)$ and $\mathtt{funext}(\mathtt{happly}\,p) = \langle i \leqslant 1\rangle(p\,(i \cdot |p|))$, a similar argument to the one for $\varepsilon$ shows that

$$\eta\,p :\equiv \langle j \leqslant 1\rangle\langle i \leqslant (1 - j) \cdot |p| + j\rangle(p\,(i \cdot (1 - j + j \cdot |p|))) \tag{5.4}$$

gives a path $p \sim \mathtt{funext}(\mathtt{happly}\,p)$ in $f \sim g$. □

## 6. Universes

Let $(\mathcal{E}, \mathtt{R})$ be an order-ringed topos (Definition 2.1). In this section we show how to construct Tarski-style universes [Mar84, p. 88] in the CwF **Fib** constructed from $(\mathcal{E}, \mathtt{R})$ as in Section 4. To do so we assume that (the CwF associated with) $\mathcal{E}$ supports *inductive-recursive* definitions [Dyb00]. This will be the case if $\mathcal{E}$ is a Grothendieck topos, that is, a category of **Set**-valued sheaves on a site, if we assume **Set** is a model of a sufficiently strong set theory; see [DS99, section 6]. As well as the topos $\mathcal{E}$, we also need to assume something about the ordered ring $\mathtt{R}$, namely that it is a connected object in $\mathcal{E}$; see the definition below. The examples of order-ringed toposes that we give in Section 7 have these properties. For simplicity, we confine attention to universes containing a type of Booleans and closed under taking dependent function and identity types (given by Moore paths); other typing constructs can be dealt with in the same way.

Let the object $\mathtt{U} \in \mathcal{E}$ and the family $\mathtt{T} \in \mathcal{E}(\mathtt{U})$ be defined simultaneously by induction-recursion in $\mathcal{E}$ so that $\mathtt{U}$ has inductive constructors

$$\mathtt{bool} : \mathtt{U} \qquad \mathtt{pi} : \textstyle\prod_{u:\mathtt{U}}(\mathtt{T}\,u \to \mathtt{U}) \to \mathtt{U} \qquad \mathtt{eq} : \textstyle\prod_{u:\mathtt{U}} \mathtt{T}\,u \to \mathtt{T}\,u \to \mathtt{U} \tag{6.1}$$

and $\mathtt{T}$ satisfies the following recursion equations

$$\mathtt{T}\,\mathtt{bool} = 1 + 1 \tag{6.2}$$

$$(\forall u : \mathtt{U})(\forall f : \mathtt{T}\,u \to \mathtt{U})\ \mathtt{T}(\mathtt{pi}\,u\,f) = \textstyle\prod_{x:\mathtt{T}\,u} \mathtt{T}(f\,x) \tag{6.3}$$

$$(\forall u : \mathtt{U})(\forall x, y : \mathtt{T}\,u)\ \mathtt{T}(\mathtt{eq}\,u\,x\,y) = x \sim y \tag{6.4}$$

Here $x \sim y$ is the Moore path object (Definition 3.1) and $1 + 1$ is the coproduct of the terminal object $1 \in \mathcal{E}$ with itself—this gives an object of Booleans in $\mathcal{E}$ with elements

$$\mathtt{true} : 1 + 1 \qquad \mathtt{false} : 1 + 1 \tag{6.5}$$

Recall from the proof of Theorem 4.4 that any object of $\mathcal{E}$, and in particular $\mathtt{U}$, has a tap fibration structure when regarded as a family over 1. So the main task is to show that there is a tap fibration structure for the family $\mathtt{T} \in \mathcal{E}(\mathtt{U})$. The way that we will construct this structure agrees with the tap fibration structure for Boolean, Π- and identity types (as in Theorems 4.4, 4.3 and 4.10 respectively) when it is re-indexed along the coding functions $\mathtt{bool}$, $\mathtt{pi}$ and $\mathtt{id}$. In this way we get a Tarski-style universe $\mathtt{T} \in \mathbf{Fib}(\mathtt{U})$ in $\mathbf{Fib}$ containing a type of Booleans and closed under taking dependent function and identity types.

Recalling the definition of a tap structure (Definition 4.1), given a Moore path $p : u_0 \sim u_1$ in $\mathtt{U}$ we wish to construct a function $p_* : \mathtt{T}\,u_0 \to \mathtt{T}\,u_1$ which is the identity when $p$ is degenerate. To do so we recurse over the structure of $u_0 : \mathtt{U}$, which is of the form $\mathtt{bool}$, or $\mathtt{pi}\,v_0\,f_0$ (for some unique $v_0$ and $f_0$), or $\mathtt{eq}\,v_0\,x_0\,y_0$ (for some unique $v_0$, $x_0$ and $y_0$). In each case we would like the whole path $p$ to remain within whichever constructor form $u_0$ has, so that previously constructed transport functions can be combined appropriately using the recipes in the proofs of Theorems 4.4, 4.3 and 4.10. This property of paths in $\mathtt{U}$ holds if $\mathtt{R}$ is connected in the following sense.

**Definition 6.1** (Connected objects in a topos)**.** For each object $X$ of the topos $\mathcal{E}$, consider the morphism $\mathtt{K} : 1+1 \longrightarrow (X \to 1+1)$ from the Booleans $1+1$ into the exponential of $1+1$ by $X$ which is the transpose of the first projection $\pi_1 : (1+1) \times X \longrightarrow 1+1$; thus $\mathtt{K}$ sends a Boolean $b : 1+1$ to the constant function with value $b$. We will say that $X$ is *connected* if $\mathtt{K}$ is an isomorphism.

If $(\mathcal{E}, \mathtt{R})$ is an order-ringed topos for which $\mathtt{R}$ is connected, then given a predicate $\varphi : \mathtt{R}_+ \to \Omega$ that is decidable $((\forall i : \mathtt{R}_+)\ \varphi\, i \vee \neg(\varphi\, i))$, consider the function $f : \mathtt{R} \to 1+1$ well-defined by

$$f\,x = \begin{cases} \mathtt{true} & \text{if } (x \leqslant \mathtt{0} \wedge \varphi\,\mathtt{0}) \ \vee\ (\mathtt{0} \leqslant x \wedge \varphi\,x) \\ \mathtt{false} & \text{if } (x \leqslant \mathtt{0} \wedge \neg\varphi\,\mathtt{0}) \ \vee\ (\mathtt{0} \leqslant x \wedge \neg\varphi\,x) \end{cases} \qquad (x : \mathtt{R})$$

Since $\mathtt{R}$ is connected, either $f = \mathtt{K}\,\mathtt{true}$, or $f = \mathtt{K}\,\mathtt{false}$. If $\varphi\,\mathtt{0}$ holds, then $f\,\mathtt{0} = \mathtt{true}$ and therefore we cannot have $f = \mathtt{K}\,\mathtt{false}$; so in this case we must have $f = \mathtt{K}\,\mathtt{true}$. Thus $\mathtt{R}$ being connected implies that the half-line $\mathtt{R}_+$ satisfies

$$(\forall \varphi : \mathtt{R}_+ \to \Omega)\ ((\forall i : \mathtt{R}_+)\ \varphi\,i \vee \neg(\varphi\,i)) \ \Rightarrow\ \varphi\,\mathtt{0} \Rightarrow (\forall i : \mathtt{R}_+)\ \varphi\,i \tag{6.6}$$

We use this property to construct the required tap fibration structure for $\mathtt{T}$.

**Theorem 6.2** (Universes)**.** *Given an order-ringed topos* $(\mathcal{E}, \mathtt{R})$*, assuming the topos* $\mathcal{E}$ *supports the inductive-recursive definition* (6.1)–(6.4) *and that* $\mathtt{R}$ *is connected, then* $\mathtt{T} \in \mathcal{E}(\mathtt{U})$ *has a tap fibration structure that make it into a Tarski-style universe in* $\mathbf{Fib}$ *containing a Boolean type and closed under dependent function and identity types.*

*Proof.* We can construct a transport function

$$(\forall u_0, u_1 : \mathtt{U})(\forall p : u_0 \sim u_1)(\forall a : \mathtt{T}\,u_0)\ p_* a : \mathtt{T}\,u_1 \tag{6.7}$$

and prove

$$(\forall u : \mathtt{U})(\forall a : \mathtt{T}\,u)\ (\mathtt{idp}\,u)_* a = a \tag{6.8}$$

simultaneously by recursion and induction on the size (number of nested constructors) of elements of $\mathtt{U}$. Given $(f, i) : f\,\mathtt{0} \sim f\,i$ and $a : \mathtt{T}(f\,\mathtt{0})$ consider the structure of $f\,\mathtt{0}$:

**Case** $f\,\mathtt{0} = \mathtt{bool}$. Let $\varphi : \mathtt{R}_+ \to \Omega$ be $\varphi\,j :\equiv (f\,j = \mathtt{bool})$. Since $\mathtt{U}$ is the disjoint union of the images of the constructors $\mathtt{bool}$, $\mathtt{pi}$ and $\mathtt{eq}$, we have that $\varphi$ is decidable; and it holds when $j = \mathtt{0}$. Therefore by (6.6) we have $f\,i = \mathtt{bool}$. So $a : \mathtt{T}(f\,\mathtt{0}) = \mathtt{T}\,\mathtt{bool} = \mathtt{T}(f\,i)$ and in this case we can define $(f, i)_* a :\equiv a$.

**Case $f\,\mathtt{0} = \mathtt{pi}\,v_0\,g_0$, for some $v_0, g_0$.** Letting $\varphi : \mathtt{R_+} \to \Omega$ be

$$\varphi\,j :\equiv (\exists v : \mathtt{U}, g : \mathtt{T}\,v \to \mathtt{U})\; f\,j = \mathtt{pi}\,v\,g$$

once again this is a decidable predicate that holds at $\mathtt{0}$. Therefore by (6.6) and the fact that $\mathtt{pi}$ is injective, we have that there are functions $v : \mathtt{R_+} \to \mathtt{U}$ and $g : \prod_{j:\mathtt{R_+}} \mathtt{T}(v\,j) \to \mathtt{U}$ with

$$(\forall j : \mathtt{R_+})\; f\,j = \mathtt{pi}\,(v\,j)(g\,j) \;\wedge\; (v, i) : v_0 \sim v\,i \;\wedge\; g\,\mathtt{0} = g_0 \;\wedge\; (\forall j \geqslant i)\; g\,j = g\,i$$

Note that the type of $a$ is $\mathtt{T}(f\,\mathtt{0}) = \prod_{x:\mathtt{T}\,v_0} \mathtt{T}(g_0\,x) = \prod_{x:\mathtt{T}(v\,\mathtt{0})} \mathtt{T}(g\,\mathtt{0}\,x)$. To transport this to an element of $\mathtt{T}(f\,i) = \prod_{x:\mathtt{T}(v\,i)} \mathtt{T}(g\,i\,x)$ we can use the construction (4.5) from the proof of Theorem 4.3. More explicitly, for each $x : \mathtt{T}(v\,i)$ and $k : \mathtt{R_+}$ we have a path $\langle j \leqslant i \dot{-} k\rangle(v(i \dot{-} j)) : v\,i \sim v\,k$ and by recursion we can form

$$\overline{x}\,k :\equiv (\langle j \leqslant i \dot{-} k\rangle(v(i \dot{-} j)))_* x : \mathtt{T}(v\,k)$$

Hence we get $\overline{g} : \mathtt{R_+} \to \mathtt{U}$ and $a_0 : \mathtt{T}(\overline{g}\,\mathtt{0})$ by defining

$$\overline{g}\,k :\equiv g\,k\,(\overline{x}\,k)$$
$$a_0 :\equiv a(\overline{x}\,\mathtt{0})$$

By the induction hypothesis $\overline{x}$ satisfies $(\forall k \geqslant i)\; \overline{x}\,k = x$ and hence $(\overline{g}, i) : \overline{g}\,\mathtt{0} \sim g\,i\,x$. So by recursion once again, we get $(\overline{g}, i)_* a_0 : \mathtt{T}(g\,i\,x)$; and by induction hypothesis, this is equal to $a\,x$ in the case $i = \mathtt{0}$. Abstracting over $x$ gives the required element $(f, i)_* a$ of $\prod_{x:\mathtt{T}(v\,i)} \mathtt{T}(g\,i\,x)$.

**Case $f\,\mathtt{0} = \mathtt{eq}\,v_0\,x_0\,y_0$, for some $v_0, x_0, y_0$.** The argument is similar to the previous case, but using the decidable predicate $\varphi\,j :\equiv (\exists v : \mathtt{U}, x : \mathtt{T}\,u, y : \mathtt{T}\,u)\; f\,j = \mathtt{eq}\,v\,x\,y$ and the construction (4.13) from the proof of Lemma 4.9. □

**Remark 6.3** (Univalence)**.** An element $u$ of the universe $\mathtt{U}$ is a code for the corresponding type $\mathtt{T}\,u$. By the above theorem, every Moore path $p : u_0 \sim u_1$ in $\mathtt{U}$ induces a transport function $p_* : \mathtt{T}\,u_0 \to \mathtt{T}\,u_1$. Because Moore paths give identity types in **Fib**, these functions are *equivalences* [Uni13, Chapter 4], the inverse equivalence being given by transport along the reverse path $\mathtt{rev}\,p$. Were $\mathtt{T}$ to satisfy Voevodsky's *univalence axiom* [Uni13, Section 2.10], every equivalence between $\mathtt{T}\,u_0$ and $\mathtt{T}\,u_1$ would have to be of the form $p_*$ for some path $p : u_0 \sim u_1$. This cannot be the case, because we have seen that connectedness of $\mathtt{R}$ implies that given $p : u_0 \sim u_1$, the elements $u_0$ and $u_1$ must have equal outermost constructor form; whereas it is quite possible for $1 + 1$ to be equivalent (indeed, isomorphic) to, for example, a Π-type named by a code in $\mathtt{U}$. So the universes constructed in this section do not satisfy the interesting form of extensionality embodied by the univalence axiom. We return to this point in the Conclusion.

## 7. Models

In the previous sections we have seen how any order-ringed topos $(\mathcal{E}, \mathtt{R})$ (Definition 2.1) gives rise to a model of Martin-Löf type theory with intensional identity types given by Moore paths on $\mathtt{R}$. If $\mathtt{R}$ is trivial, that is, satisfies $\mathtt{0} = \mathtt{1}$, then existence of a Moore path just coincides with extensional equality in the topos $\mathcal{E}$. If the object $\mathtt{R}$ is non-trivial, but has decidable equality in $\mathcal{E}$ (for example, if it is the object of integers, or of rationals), then there is a path $\langle i \leqslant \mathtt{1}\rangle(\mathtt{if}\ i = \mathtt{0}\ \mathtt{then}\ \mathtt{true}\ \mathtt{else}\ \mathtt{false}) : \mathtt{true} \sim \mathtt{false}$ in the object $1 + 1$ of Booleans and so in this case the model of type theory we get is logically degenerate.

Therefore, when searching for examples of order-ringed toposes, we should at least look for ones with $\mathtt{R}$ non-trivial and not decidable. We first give a simple example of such an order-ringed topos, where the underlying topos is a presheaf category. Then we give a more sophisticated example, using a sheaf topos and for which the associated model of type theory has identity types that are not necessarily truncated at any level; in other words, ones in which iterated identity types $\mathtt{Id}_A, \mathtt{Id}_{\mathtt{Id}_A}, \mathtt{Id}_{\mathtt{Id}_{\mathtt{Id}_A}}, \ldots$ can be homotopically non-trivial for any level of iteration.

7.1. **A presheaf model.** Consider the category whose objects are ordered rings in the topos **Set** (sets and functions) and whose morphisms are ordered ring homomorphisms, that is, functions preserving the order $\leqslant$ and the ring operations $\mathtt{0}, \mathtt{1}, +, \cdot$. Let $\mathbf{C}$ be a small full subcategory of this category. For the purposes of this example it is not important which ordered rings $\mathbf{C}$ contains, so long as it contains the terminal ordered ring 1 (which has one element $\mathtt{0} = \mathtt{1}$) and a non-trivial ordered ring (one for which $\mathtt{0} \neq \mathtt{1}$); for definiteness let us assume that $\mathbf{C}$ contains the reals $\mathbb{R}$ with the usual order and ring operations.

7.1.1. *The topos.* We use the topos $\mathbf{Set}^{\mathbf{C}}$ of covariant presheaves on $\mathbf{C}$. Thus the objects of $\mathbf{Set}^{\mathbf{C}}$ are functors $\mathbf{C} \longrightarrow \mathbf{Set}$ and the morphisms are natural transformations between such functors. Let $\Delta : \mathbf{Set} \longrightarrow \mathbf{Set}^{\mathbf{C}}$ denoted the functor assigning to each set $S$ the constant presheaf $\Delta\, S : \mathbf{C} \longrightarrow \mathbf{Set}$, whose value at each $X \in \mathbf{C}$. is $\Delta\, S(X) = S$. $\Delta$ is the inverse image part of the unique geometric morphism from the topos $\mathbf{Set}^{\mathbf{C}}$ to **Set**; its direct image part, the right adjoint to $\Delta$, is the global sections functor $\mathbf{Set}^{\mathbf{C}}(1, \_) : \mathbf{Set}^{\mathbf{C}} \longrightarrow \mathbf{Set}$. Below we will also need to use the fact that $\Delta$ has a left adjoint

$$\begin{aligned} &\pi_0 : \mathbf{Set}^{\mathbf{C}} \longrightarrow \mathbf{Set} \\ &\pi_0(F) :\equiv F(1) \end{aligned} \tag{7.1}$$

(The adjunction $\pi_0 \dashv \Delta$ follows from the fact that 1 is a terminal object in $\mathbf{C}$.)

7.1.2. *The ordered commutative ring object* $\mathtt{R}$. Let $\mathtt{R} : \mathbf{C} \longrightarrow \mathbf{Set}$ denote the forgetful functor sending each ordered ring in $\mathbf{C}$ to its underlying set. As an object of $\mathbf{Set}^{\mathbf{C}}$, $\mathtt{R}$ has the structure of an ordered ring object:

- $\leqslant\ \rightarrowtail \mathtt{R} \times \mathtt{R}$ is the sub-presheaf of $\mathtt{R} \times \mathtt{R}$ whose value each object $X \in \mathbf{C}$ is the subset $\leqslant(X) \subseteq \mathtt{R}(X) \times \mathtt{R}(X) = X \times X$ given by the order on $X$:
$$\leqslant(X) :\equiv \{(x, y) \in X \times X \mid x \leqslant y\} \tag{7.2}$$
Note that these subsets do form a subpresheaf, because each morphism in $\mathbf{C}$ is in particular an order-preserving function. Furthermore, $\leqslant\ \rightarrowtail \mathtt{R} \times \mathtt{R}$ is a total order (2.4)–(2.7) in the topos $\mathbf{Set}^{\mathbf{C}}$, because disjunction in a presheaf topos is computed component-wise and each $\leqslant(X)$ is a total order on $X$.
- The ring structure on $\mathtt{R}$ is given component-wise by the ring structure on each $X \in \mathbf{C}$. For example, the addition morphism $+ : \mathtt{R} \times \mathtt{R} \longrightarrow \mathtt{R}$ has component at $X \in \mathbf{C}$ given by addition in $X$: $+_X(x, y) = x + y$; these are the components of a morphism in $\mathbf{Set}^{\mathbf{C}}$ because they are natural in $X$, since each morphism $\theta$ in $\mathbf{C}$ satisfies $\theta(x + y) = \theta\, x + \theta\, y$. Similarly, this structure on $\mathtt{R}$ satisfies axioms (2.8)–(2.17) because each $\mathtt{R}(X) = X$ is an ordered ring.

7.1.3. $\mathtt{R}$ *is connected.* To apply the results of Section 6, we need to verify that $\mathtt{R}$ is a connected object in $\mathbf{Set}^{\mathbf{C}}$ (Definition 6.1). Since the object of Booleans $1+1$ in the presheaf topos $\mathbf{Set}^{\mathbf{C}}$ is isomorphic to the constant functor $\Delta\, 2 : \mathbf{C} \longrightarrow \mathbf{Set}$, where $2 = \{0, 1\}$ is a two-element set, we have to show that $\mathtt{K} : \Delta\, 2 \longrightarrow (\mathtt{R} \to \Delta\, 2)$ is an isomorphism in $\mathbf{Set}^{\mathbf{C}}$. To see this, we just need to show that $\mathtt{K}_X : \Delta\, 2(X) \longrightarrow (\mathtt{R} \to \Delta\, 2)(X)$ is a bijection for each $X \in \mathbf{C}$. Letting $\mathtt{Y} : \mathbf{C}^{\mathrm{op}} \longrightarrow \mathbf{Set}^{\mathbf{C}}$ denote the Yoneda embedding and using the adjunction $\pi_0 \dashv \Delta$ mentioned in Sec. 7.1.1, there are bijections

$$
\begin{aligned}
\Delta\, 2(X) &= 2 && \text{(definition of } \Delta) \\
&\cong \mathbf{Set}(\mathbf{C}(X,1) \times 1, 2) && \text{(since 1 is terminal in } \mathbf{C}) \\
&= \mathbf{Set}((\mathtt{Y}X \times \mathtt{R})(1), 2) && \text{(definition of } \mathtt{Y}X \times \mathtt{R}) \\
&= \mathbf{Set}(\pi_0(\mathtt{Y}X \times \mathtt{R}), 2) && \text{(definition of } \pi_0) \\
&\cong \mathbf{Set}^{\mathbf{C}}(\mathtt{Y}X \times \mathtt{R}, \Delta\, 2) && (\pi_0 \text{ left adjoint to } \Delta) \\
&\cong \mathbf{Set}^{\mathbf{C}}(\mathtt{Y}X, \mathtt{R} \to \Delta\, 2) && \text{(universal property of exponential)} \\
&\cong (\mathtt{R} \to \Delta\, 2)(X) && \text{(Yoneda Lemma [Mac71, III.2])}
\end{aligned}
$$

and one can check that their composition is $\mathtt{K}_X$.

So $(\mathbf{Set}^{\mathbf{C}}, \mathtt{R})$ is an order-ringed topos from which we can construct a model $\mathbf{Fib}$ of intensional Martin-Löf Type Theory with universes as in the previous sections. Since we assumed that $\mathbf{C}$ contains the non-trivial ordered ring $\mathbb{R}$, it is not the case that $\mathtt{R}$ is trivial; that is, the sentence $\mathtt{0} = \mathtt{1}$ is not satisfied by $\mathtt{R}$ (because it is not satisfied at component $X = \mathbb{R}$).[2] More than this, the associated model of Type Theory built from $(\mathbf{Set}^{\mathbf{C}}, \mathtt{R})$ as in the previous sections is not logically trivial: there is no proof in it of $\mathtt{Id}_{\mathtt{Bool}}\,\mathtt{true}\,\mathtt{false}$ (so in particular, $\mathtt{R}$ cannot be a decidable object of $\mathbf{Set}^{\mathbf{C}}$). To see this, note that giving a proof of $\mathtt{Id}_{\mathtt{Bool}}\,\mathtt{true}\,\mathtt{false}$ in the CwF $\mathbf{Fib}$ associated with $(\mathbf{Set}^{\mathbf{C}}, \mathtt{R})$ is the same as giving an $\mathtt{R}$-based Moore path from $\mathtt{true} : 1 \longrightarrow \Delta\, 2$ to $\mathtt{false} : 1 \longrightarrow \Delta\, 2$. There is no such path because we saw above that $\mathtt{R} \to \Delta\, 2 \cong \Delta\, 2$ and hence every path in $\Delta\, 2$ is constant.

7.2. **A gros topos model.** Being logically non-trivial is rather a weak condition for models of type theory with intensional identity types. A more interesting one is that the model contains types $A$ whose iterated identity types $\mathtt{Id}_A, \mathtt{Id}_{\mathtt{Id}_A}, \mathtt{Id}_{\mathtt{Id}_{\mathtt{Id}_A}}, \ldots$ are all non-trivial (not isomorphic to the unit type). An interesting way of demonstrating that for the model associated with an order-ringed topos $(\mathcal{E}, \mathtt{R})$ is to show that the homotopy types of a rich collection of topological spaces (including all the $n$-dimensional spheres, say) are faithfully represented by the internal, $\mathtt{R}$-based notion of homotopy on a corresponding collection of objects of the topos $\mathcal{E}$. We give one such order-ringed topos in this section.

7.2.1. *The topos.* We use a topos of sheaves $\mathbf{Sh}(\mathbf{T}, J)$ [MM92, III] for the following site $(\mathbf{T}, J)$. Let $\mathbf{Haus}$ denote the category of Hausdorff topological spaces and continuous functions. We take the small category $\mathbf{T}$ to be the least full subcategory of $\mathbf{Haus}$ containing the reals $\mathbb{R}$ and closed under the following operations:

(a) If $X, Y \in \mathbf{T}$, then $\mathbf{T}$ contains the product space $X \times Y$.

[2] On the other hand, since we assume $\mathbf{C}$ also contains the trivial ring 1, neither is it the case that the sentence $\neg(\mathtt{0} = \mathtt{1})$ is satisfied by $\mathtt{R}$. Note that $\mathbf{Set}^{\mathbf{C}}$ is not a Boolean topos – it does not satisfy the Law of Excluded Middle.

(b) If $X \in \mathbf{T}$ and $C \subseteq X$ is a closed subset, then $C$ (with the subspace topology) is in $\mathbf{T}$.
(c) If $X, Y \in \mathbf{T}$ and $X$ is locally compact, then $\mathbf{T}$ contains the exponential space $Y^X$ (the set of continuous functions from $X$ to $Y$ endowed with the compact-open topology).

Since the spaces in $\mathbf{T}$ are Hausdorff, equalizers of continuous functions give closed subspaces and hence by (a) and (b) we have that $\mathbf{T}$ is closed under taking finite limits in **Haus**. Hence $\mathbf{T}$ contains $\mathbb{R}_+$ (as well as $[0,1]$ and all the spheres $S^n$ for any $n \in \mathbb{N}$). Then by (c) we have that $\mathbf{T}$ is closed under taking Moore path spaces (with their usual topology):

$$X \in \mathbf{T} \;\Rightarrow\; MX :\equiv \{(f,r) \in X^{\mathbb{R}_+} \times \mathbb{R}_+ \mid (\forall r' \geq r)\; f\,r' = f\,r\} \in \mathbf{T} \tag{7.3}$$

We define a coverage [Joh02, Definition A2.1.9] on $\mathbf{T}$ as follows. Following Dyer and Eilenberg [DE88] we say that a set $S$ of closed subsets of a topological space $X$ is a *local cover* if for each $x \in X$ there is a finite subset $\{C_1, \ldots, C_n\} \subseteq S$ with $x$ in the interior of $C_1 \cup \cdots \cup C_n$. Of course every finite cover of $X$ by closed subsets is trivially a local cover. Note that if $f : Y \longrightarrow X$ in $\mathbf{T}$ and $S$ is a local cover of $X$, then $f^{-1}S :\equiv \{f^{-1}C \mid C \in S\}$ is a local cover of $Y$. Hence we get a coverage on $\mathbf{T}$ with $J(X) :\equiv \{S \mid S \text{ is a local cover of } X\}$ for each $X \in \mathbf{T}$.

Given a functor $F : \mathbf{T}^{\mathrm{op}} \longrightarrow \mathbf{Set}$, if $C \subseteq X$ is a closed subspace of a space in $\mathbf{T}$, then for each $x \in F\,X$ we will just write $x|_C$ for the element $F\,i\,x \in F\,C$, where the $\mathbf{T}$-morphism $i : C \longrightarrow X$ is the inclusion function. Recall that $F$ is a sheaf with respect to $J$ iff for all $X \in \mathbf{T}$, $S \in J(X)$ and all $S$-indexed families $(x_C \in F\,C \mid C \in S)$ satisfying $(\forall C, D \in S)\; x_C|_{C\cap D} = x_D|_{C \cap D}$, there is a unique $x \in F\,X$ with $(\forall C \in S)\; x_C = x|_C$. The topos $\mathbf{Sh}(\mathbf{T}, J)$ is by definition the full subcategory of the functor category $\mathbf{Set}^{\mathbf{T}^{\mathrm{op}}}$ whose objects are sheaves.

7.2.2. *The ordered commutative ring object* R. Let $\mathtt{Y} : \mathbf{T} \longrightarrow \mathbf{Set}^{\mathbf{T}^{\mathrm{op}}}$ denote the Yoneda embedding for the small category $\mathbf{T}$. Because elements of $J(X)$ are local covers of $X \in \mathbf{T}$, for each $S \in J(X)$ if we have a family of continuous functions $f_C : C \longrightarrow X'$ for each $C \in S$ that agree where they overlap ($f_C|_{C\cap D} = f_D|_{C\cap D}$), then the unique function $f : X \longrightarrow X'$ that agrees with each of them ($f_C = f|_C$) is necessarily continuous. Hence each representable presheaf $\mathtt{Y}\,X' = \mathbf{T}(\_\,, X')$ is a sheaf (in such a case one says that the coverage $J$ is *subcanonical*) and the Yoneda embedding gives a functor $\mathtt{Y} : \mathbf{T} \longrightarrow \mathbf{Sh}(\mathbf{T}, J)$.

The axioms for *partially* ordered commutative rings are those in Fig. 1 except for (2.7). They make sense in any category with finite limits once we have an interpretation of the binary operation $\leqslant$, the constants $\mathtt{0}, \mathtt{1}$ and the operations $+, \mathtt{-}, \cdot$; and satisfaction of those axioms is preserved by functors that preserve finite limits. Since $\{(x,y) \in \mathbb{R} \times \mathbb{R} \mid x \leq y\}$ is a closed subset of $\mathbb{R} \times \mathbb{R}$ and the usual ring operations on $\mathbb{R}$ are continuous, it follows that $\mathbb{R}$ is a partially ordered commutative ring object in the finitely complete category $\mathbf{T}$. Then since $\mathtt{Y}$ preserves finite limits, the representable sheaf $\mathtt{R} :\equiv \mathtt{Y}\mathbb{R}$ is a partially ordered commutative ring object in the topos $\mathbf{Sh}(\mathbf{T}, J)$. In fact it is totally ordered, that is, satisfies (2.7). This is simply because we have a local cover $\{\{(x,y) \mid x \leq y\}, \{(x,y) \mid y \leq x\}\} \in J(\mathbb{R} \times \mathbb{R})$.[3]

[3]This fact is the motivation for considering a site based on covers by closed subsets rather than the more familiar case of open covers used for Giraud's gros topos [GV72, IV, 2.5]. However, as Spitters has pointed out [private communication], in view of [Joh79, Theorem 8.1] we could have used the reals in Johnstone's *topological topos* instead of $\mathbf{Sh}(\mathbf{T}, J)$ in this section.

7.2.3. $\mathtt{R}$ *is connected.* To apply the results of Section 6, we need to verify that $\mathtt{R}$ is a connected object in $\mathbf{Sh}(\mathbf{T}, J)$ (Definition 6.1). First note that $\mathbf{T}$ contains the discrete spaces $\emptyset$, 1 and $2 = \{0, 1\}$. Since $\emptyset$ is covered by the empty family of closed sets and 2 is covered by the two closed inclusions $\{0\} \hookrightarrow 2 \hookleftarrow \{1\}$, for any sheaf $F \in \mathbf{Sh}(\mathbf{T}, J)$ we have that $F(\emptyset) \cong 1$ and then $F(2) \cong F(1) \times F(1)$. So by the Yoneda Lemma

$$\mathbf{Sh}(\mathbf{T}, J)(\mathtt{Y}2, F) \cong F(2) \cong F(1) \times F(1) \cong \mathbf{Sh}(\mathbf{T}, J)(\mathtt{Y}1 + \mathtt{Y}1, F) \cong \mathbf{Sh}(\mathbf{T}, J)(1 + 1, F)$$

naturally in $F$. So the object of Booleans in $\mathbf{Sh}(\mathbf{T}, J)$ is representable by $2 \in \mathbf{T}$. Thus to see that $\mathtt{R}$ is connected, it suffices to show that $\mathtt{K} : \mathtt{Y}2 \longrightarrow (\mathtt{R} \to \mathtt{Y}2)$ is an isomorphism in $\mathbf{Sh}(\mathbf{T}, J)$. But $\mathtt{R} = \mathtt{Y}\mathbb{R}$ is also representable and $\mathtt{Y}$ preserves exponentials. It follows that $\mathtt{K} : \mathtt{Y}2 \longrightarrow (\mathtt{R} \to \mathtt{Y}2)$ is the image under $\mathtt{Y}$ of the continuous function $2 \longrightarrow 2^{\mathbb{R}}$ given by taking the exponential transpose of $\pi_1 : 2 \times \mathbb{R} \longrightarrow 2$ in $\mathbf{T}$. But $2 \longrightarrow 2^{\mathbb{R}}$ is an isomorphism in $\mathbf{T}$ because $\mathbb{R}$ is topologically connected; and hence so is $\mathtt{K} : \mathtt{Y}2 \longrightarrow (\mathtt{R} \to \mathtt{Y}2)$ in $\mathbf{Sh}(\mathbf{T}, J)$.

7.2.4. *Homotopy types in* $\mathbf{Sh}(\mathbf{T}, J)$*.* From the order-ringed topos $(\mathbf{Sh}(\mathbf{T}, J), \mathtt{Y}\mathbb{R})$ we get a CwF **Fib** modelling Martin-Löf Type Theory with universes and with intensional identity types $\mathtt{Id}_A$ given by Moore paths. Consider the congruence on the category $\mathbf{Sh}(\mathbf{T}, J)$ that identifies morphisms $\gamma, \delta : B \longrightarrow A$ when there is a global section of $\mathtt{Id}_{B \to A}\, \gamma\, \delta$. Let $\mathbf{Ho}(\mathbf{Sh}(\mathbf{T}, J))$ denote the associated quotient category.

By Theorem 5.1, two morphisms $\gamma, \delta : B \longrightarrow A$ are identified in $\mathbf{Ho}(\mathbf{Sh}(\mathbf{T}, J))$ iff $\prod_{x:B}(\gamma\, x \sim \delta\, x)$ has a global section. This is equivalent to requiring the existence of a morphism $H : B \longrightarrow \wp\, A$ in $\mathbf{Sh}(\mathbf{T}, J)$ satisfying $\partial^- \circ H = \gamma$ and $\partial^+ \circ H = \delta$, where

$$\wp\, A :\equiv \{(f, i) : (\mathtt{R}_+ \to A) \times \mathtt{R}_+ \mid (\forall j \geqslant i)\ f\, j = f\, i\} \tag{7.4}$$
$$\partial^-, \partial^+ : \wp\, A \longrightarrow A \qquad \partial^-(f, i) :\equiv f\, \mathtt{0} \qquad \partial^+(f, i) :\equiv f\, i$$

is the total object of the family of Moore path objects (3.1) with associated source and target morphisms. Note that since $\{(x, y) \in \mathbb{R} \times \mathbb{R} \mid x \leq y\}$ is a closed subset of $\mathbb{R} \times \mathbb{R}$, it is an object of $\mathbf{T}$ and the order relation on $\mathtt{R}$ in $\mathbf{Sh}(\mathbf{T}, J)$ is given by the representable $\mathtt{Y}\{(x, y) \mid x \leq y\} \rightarrowtail \mathtt{Y}(\mathbb{R} \times \mathbb{R}) \cong \mathtt{R} \times \mathtt{R}$. It follows that the positive cone of $\mathtt{R}$ is also a representable sheaf: $\mathtt{R}_+ \cong \mathtt{Y}(\mathbb{R}_+)$. Recall that, as well as preserving finite limits, the Yoneda embedding preserves any exponentials that happen to exist. It follows that for each representable sheaf $\mathtt{Y}X$ its total path object is representable by the Moore path space of $X$ (7.3): $\wp(\mathtt{Y}X) \cong \mathtt{Y}(MX)$; and under this isomorphism the source and target morphisms $\partial^-, \partial^+$ correspond to the representable morphisms induced by the usual source and target functions for $MX$. Since $\mathtt{Y}$ is full and faithful, it follows that for two continuous functions $f, g : X \longrightarrow X'$ in $\mathbf{T}$, the morphisms $\mathtt{Y}f$ and $\mathtt{Y}g$ get identified in $\mathbf{Ho}(\mathbf{Sh}(\mathbf{T}, J))$ iff $f$ and $g$ are homotopic in the classical sense. Thus $\mathtt{Y}$ induces a full and faithful embedding of the category $\mathbf{Ho}(\mathbf{T})$ of homotopy types of spaces in $\mathbf{T}$ into $\mathbf{Ho}(\mathbf{Sh}(\mathbf{T}, J))$. Since $\mathbf{T}$ contains all the spheres $S^n$, we deduce that identity types in this particular **Fib** are not necessarily truncated at any level of iteration.

## 8. Related Work

The classical topological notion of Moore path is a standard, if somewhat niche topic within homotopy theory. The Schedule Theorem of Dyer and Eilenberg [DE88] for globalising Hurewicz fibrations is a nice example of their usefulness. They have been used in connection with higher-dimensional category theory by Kapranov and Voevodsky [KV91] and by Brown [Bro09].

Although our use of constructive algebra within toposes to make models of intensional type theory appears to be new, we are not the only ones to consider using some form of path with strictly unitary and associative composition to model identity types with a judgemental computation rule. Van den Berg and Garner [vG12] use topological Moore paths and a simplicial version of them to get instances of their notion of *path object category* for modelling identity types. The results of Sections 3 and 4 show that any ordered abelian group in a topos induces a path object category structure on that topos; and since the notion of fibration we use (Definition 4.1) is closely related to the one used in [vG12] (see Proposition 6.1.5 of that paper), one can get alternative, more abstract categorical proofs of Theorems 4.3 and 4.10 from the work of Van den Berg and Garner. However, the concrete calculations in the internal language that we give are quite simple by comparison; and this approach proves its worth in Section 5, whose results on obtaining function extensionality from the ordered ring structure are new.

The PhD thesis of North [Nor17] uses a category-theoretic abstraction of the notion of Moore paths, called *Moore relation systems*, as part of a complete analysis of when a weak factorization system gives a model (in terms of display map categories, rather than CwFs) of identity-, Σ- and Π-types. A Moore relation system is a piece of category theory comparable to our use of ordered abelian groups in categorical logic in Section 3; it would be interesting to see if it can be extended in the way we extended from groups to rings in Section 5 in order to validate function extensionality.

Spitters [Spi16, Section 3] uses a somewhat different formulation of Moore path in the cubical topos [CCHM18]. His notion is the reflexive-transitive closure of the usual path types given by the bounded interval. For better properties and to get a closer relationship with our version, one would like to quotient these cubical "Spitters-Moore" path objects up to degenerate paths; but the undecidability of degeneracy seems to stop one being able to do that while retaining the (uniform) Kan-fibrancy of such path objects. Here we can side-step such issues, since notably our models manage to avoid using a notion of Kan fibrancy at all.

## 9. Conclusion

We have shown that any connected ordered ring in a topos gives rise to a model of Martin-Löf's intensional Type Theory with universes in which proofs of identity are given by Moore paths. We gave an example to show that such models of type theory can contain highly non-trivial identity types that faithfully represent the homotopy types of a wide class of topological spaces and in particular are not truncated at any level of iteration.

It is an open question whether there is an order-ringed topos that gives rise to such a model of type theory containing a *univalent* [Uni13, Section 2.10] universe. (We saw in Section 6 that the universes constructed there are not univalent.) The known examples of non-truncated univalent universes, such as the classical simplicial sets model [KL16] and the various constructive cubical sets models [BCH14, CCHM18, ABC$^{+}$17] make use of a modified form of the Hofmann-Streicher [HS99] construction in presheaf categories. Streicher [Str05] points out that the basic Hofmann-Streicher universe construction works for sheaf toposes through a suitable use of sheafification. So there are Hofmann-Streicher universes in both of the example toposes from Section 7, one of which is a presheaf topos and one a sheaf topos. However, the analysis of [LOPS18, Section 5] shows that in the examples of univalence mentioned above, one gets from the Hofmann-Streicher universe to a univalent universe classifying fibrations (of various kinds) by using the fact that in those models the path functor $\wp(\_)$ has a right adjoint. Unfortunately this is not the case for the models in Section 7, where it seems that the very property of the interval (half-line) that allows us to avoid all uses of Kan filling in favour of path composition when building our models of type theory, namely the total order (2.1), prevents the interval from being "tiny" and hence prevents $\wp(\_)$ from having a right adjoint.

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