Logical Methods in Computer Science
Vol. 14(1:16)2018, pp. 1–23
https://lmcs.episciences.org/

Submitted Aug. 01, 2017
Published Feb. 27, 2018

AN ENRICHED VIEW ON THE EXTENDED FINITARY MONAD–LAWVERE THEORY CORRESPONDENCE

RICHARD GARNER AND JOHN POWER

Department of Mathematics, Macquarie University, NSW 2109, Australia
e-mail address: richard.garner@mq.edu.au

Department of Computer Science, University of Bath, Claverton Down, Bath BA2 7AY, United Kingdom
e-mail address: A.J.Power@bath.ac.uk

Dedicated to Jiří Adámek on the occasion of his retirement

ABSTRACT. We give a new account of the correspondence, first established by Nishizawa–Power, between finitary monads and Lawvere theories over an arbitrary locally finitely presentable base. Our account explains this correspondence in terms of enriched category theory: the passage from a finitary monad to the corresponding Lawvere theory is exhibited as an instance of *free completion* of an enriched category under a class of absolute colimits. This extends work of the first author, who established the result in the special case of finitary monads and Lawvere theories over the category of sets; a novel aspect of the generalisation is its use of enrichment over a *bicategory*, rather than a monoidal category, in order to capture the monad–theory correspondence over all locally finitely presentable bases simultaneously.

1. INTRODUCTION

A key theme of Jiří Adámek's superlative research career has been the study of the subtle interaction between monads and theories, especially within computer science. We hope he might see this paper as a development of the abstract mathematics underlying this aspect of his body of work. Jiří has been an inspiration to both of us, on both a scientific and a personal level, as he has been to many in category theory and beyond, and we are therefore pleased to dedicate this paper to him.

The starting point of our development is the well-known fact that categorical universal algebra provides two distinct ways to approach the notion of (single-sorted, finitary) equational algebraic theory. On the one hand, any such theory $\mathbb{T}$ gives rise to a Lawvere theory whose models coincide (to within coherent isomorphism) with the $\mathbb{T}$-models. Recall

Key words and phrases: Lawvere theory, finitary monad, locally finitely presentable category, enrichment in a bicategory, absolute colimits.

The authors gratefully acknowledge the support of Australian Research Council grants DP160101519 and FT160100393 and of the Royal Society International Exchanges scheme, grant number IE151369. No data was generated in the research for this paper.

that a *Lawvere theory* is a small category $\mathcal{L}$ equipped with an identity-on-objects, strict finite-power-preserving functor $\mathbb{F}^{\mathrm{op}} \to \mathcal{L}$ and that a *model* of a Lawvere theory is a finite-power-preserving functor $\mathcal{L} \to \mathbf{Set}$, where here $\mathbb{F}$ denotes the category of finite cardinals and mappings.

On the other hand, an algebraic theory $\mathbb{T}$ gives rise to a finitary (i.e., filtered-colimit-preserving) monad on the category of sets, whose Eilenberg–Moore algebras also coincide with the $\mathbb{T}$-models. We can pass back and forth between the presentations using finitary monads and Lawvere theories in a manner compatible with semantics; this is encapsulated by an equivalence of categories fitting into a triangle

$$\begin{tikzcd} \mathbf{Mnd}_f(\mathbf{Set})^{\mathrm{op}} \arrow[rr, "\simeq"] \arrow[dr, "\mathbf{Alg}(-)"'] & \simeq & \mathbf{Law}^{\mathrm{op}} \arrow[dl, "\mathbf{Mod}(-)"] \\ & \mathbf{CAT} & \end{tikzcd} \tag{1.1}$$

which commutes up to pseudonatural equivalence. Both these categorical formulations of equational algebraic structure are *invariant* with respect to the models, meaning that whenever two algebraic theories have isomorphic categories of models, the Lawvere theories and the monads they induce are also isomorphic. However, the two approaches emphasise different aspects of an equational theory $\mathbb{T}$. On the one hand, the Lawvere theory $\mathcal{L}$ encapsulates the *operations* of $\mathbb{T}$: the hom-set $\mathcal{L}(m,n)$ comprises all the operations $X^m \to X^n$ which are definable in any $\mathbb{T}$-model X. On the other hand, the action of the monad T encapsulates the construction of the *free* $\mathbb{T}$*-model* on any set; though since there are infinitary equational theories which also admit free models, the restriction to finitary monads is necessary to recover the equivalence with Lawvere theories.

While the equivalence in (1.1) is not hard to construct, there remains the question of how it should be understood. Clarifying this point was the main objective of [Gar14b]: it describes a setting within which both finitary monads on **Set** and Lawvere theories can be considered on an equal footing, and in which the passage from a finitary monad to the associated Lawvere theory can be understood as an instance of the same process by which one associates:

(a) To a locally small category $\mathcal{C}$, its Karoubian envelope;
(b) To a ring R, its category of finite-dimensional matrices;
(c) To a metric space, its Cauchy-completion.

The setting is that of $\mathcal{V}$-enriched category theory [Kel82]; while the process is that of free completion under a class of *absolute colimits* [Str83a]—colimits that are preserved by any $\mathcal{V}$-functor. The above examples are instances of such a completion, since:

(a) Each locally small category is a **Set**-category, and splittings of idempotents are **Set**-absolute colimits;
(b) Each ring can be seen as a one-object **Ab**-category and the corresponding category of finite-dimensional matrices can be obtained by freely adjoining finite biproducts—which are **Ab**-absolute colimits;
(c) Each metric space can be seen as an $\mathbb{R}_+^\infty$-category—where $\mathbb{R}_+^\infty$ is the monoidal poset of non-negative reals extended by infinity, as defined in [Law73]—and its Cauchy-completion can be obtained by adding limits for Cauchy sequences which, again as in [Law73], are $\mathbb{R}_+^\infty$-absolute colimits.

In order to fit (1.1) into this same setting, one takes as enrichment base the category $\mathcal{F}$ of finitary endofunctors of **Set**, endowed with its composition monoidal structure. On the one hand, finitary monads on **Set** are the same as monoids in $\mathcal{F}$, which are the same as one-object $\mathcal{F}$-categories. On the other hand, Lawvere theories may also be identified with certain $\mathcal{F}$-categories; the argument here is slightly more involved, and may be summarised as follows.

A key result of [Gar14b], recalled as Proposition 2.4 below, identifies $\mathcal{F}$-categories admitting all absolute *tensors* (a kind of enriched colimit) with ordinary categories admitting all finite powers. In one direction, we obtain a category with finite powers from an absolute-tensored $\mathcal{F}$-category by taking the underlying ordinary category; in the other, we use a construction which generalises the *endomorphism monad* (or in logical terms the *complete theory*) of an object in a category with finite powers.

Using this key result, we may identify Lawvere theories with identity-on-objects strict absolute-tensor-preserving $\mathcal{F}$-functors $\mathbb{F}^{\mathrm{op}} \to \mathcal{L}$, where, overloading notation, we use $\mathbb{F}^{\mathrm{op}}$ and $\mathcal{L}$ to denote not just the relevant categories with finite powers, but also the corresponding $\mathcal{F}$-categories. In [Gar14b], the $\mathcal{F}$-categories equipped with an $\mathcal{F}$-functor of the above form were termed *Lawvere $\mathcal{F}$-categories*.

By way of the above identifications, the equivalence (1.1) between finitary monads and Lawvere theories can now be re-expressed as an equivalence between one-object $\mathcal{F}$-categories and Lawvere $\mathcal{F}$-categories: which can be obtained via the standard enriched-categorical process of *free completion* under all absolute tensors. The universal property of this completion also explains the compatibility with the semantics in (1.1); we recall the details of this in Section 2 below.

We thus have three categorical perspectives on equational algebraic theories: as Lawvere theories, as finitary monads on **Set**, and (encompassing the other two) as $\mathcal{F}$-categories. It is natural to ask if these perspectives extend so as to account for algebraic structure borne not by *sets* but by objects of an arbitrary locally finitely presentable category $\mathcal{A}$. This is of practical interest, since such structure arises throughout mathematics and computer science, as in, for example, sheaves of rings, or monoidal categories, or the second-order algebraic structure of [FPT99].

The approach using monads extends easily: we simply replace finitary monads on **Set** by finitary monads on $\mathcal{A}$. The approach using Lawvere theories also extends, albeit more delicately, by way of the *Lawvere $\mathcal{A}$-theories* of [NP09, LP09]. If we write $\mathcal{A}_f$ for a skeleton of the full subcategory of finitely presentable objects in $\mathcal{A}$, then a Lawvere $\mathcal{A}$-theory is a small category $\mathcal{L}$ together with an identity-on-objects finite-limit-preserving[1] functor $J\colon {\mathcal{A}_f}^{\mathrm{op}} \to \mathcal{L}$; while a model of this theory is a functor $\mathcal{L} \to \mathbf{Set}$ whose restriction along J preserves finite limits. These definitions are precisely what is needed to extend the equivalence (1.1) to one of the form:

$$\begin{tikzcd} \mathbf{Mnd}_f(\mathcal{A})^{\mathrm{op}} \arrow[rr, "\simeq"] \arrow[dr, "\mathbf{Alg}(-)"'] & \arrow[d, phantom, "\simeq"] & \mathbf{Law}(\mathcal{A})^{\mathrm{op}} \arrow[dl, "\mathbf{Mod}(-)"] \\ & \mathbf{CAT} & \end{tikzcd} \tag{1.2}$$

What does not yet exist in this situation is an extension of the third, enriched-categorical perspective; the objective of this paper is to provide one. Like in [Gar14b], this will provide a common setting in which the approaches using monads and using Lawvere theories can

[1]Note the absence of the qualifier "strict"; for a discussion of this, see Remark 3.2 below.

coexist; and, like before, it will provide an explanation as to *why* an equivalence (1.2) should exist, by exhibiting it as another example of a completion under a class of absolute colimits.

There is a subtlety worth remarking on in how we go about this. One might expect that, for each locally finitely presentable $\mathcal{A}$, one simply replaces the monoidal category $\mathcal{F}$ of finitary endofunctors of **Set** by the monoidal category $\mathcal{F}_{\mathcal{A}}$ of finitary endofunctors of $\mathcal{A}$, and then proceeds as before. This turns out not to work in general: there is a paucity of $\mathcal{F}_{\mathcal{A}}$-enriched absolute colimits, such that freely adjoining them to a one-object $\mathcal{F}_{\mathcal{A}}$-category does not necessary yield something resembling the notion of Lawvere $\mathcal{A}$-theory.

In overcoming this apparent obstacle, we are led to a global analysis which is arguably more elegant: it involves a single enriched-categorical setting in which finitary monads and Lawvere theories over all locally finitely presentable bases coexist simultaneously, and in which the monad–theory correspondences for each $\mathcal{A}$ arise as instances of the same free completion process.

This setting involves enrichment not in the monoidal category of finitary endofunctors of a particular $\mathcal{A}$, but in the *bicategory* **LFP** of finitary functors between locally presentable categories[2]. The theory of categories enriched in a bicategory was developed in [Wal81, Str83b] and will be recalled in Section 3 below; for now, note that a one-object **LFP**-enriched category is a monad in **LFP**, thus, a finitary monad on a locally finitely presentable category. This explains one side of the correspondence (1.2); for the other, we extend the key technical result of [Gar14b] by showing that absolute-tensored **LFP**-categories can be identified with what we call *partially finitely complete* categories. These will be defined in Section 5 below; they are categories $\mathcal{C}$, not necessarily finitely complete, that come equipped with a sieve of finite-limit-preserving functors expressing which finite limits do in fact exist in $\mathcal{C}$.

The relevance of this result is as follows. If $J \colon {\mathcal{A}_f}^{\mathrm{op}} \to \mathcal{L}$ is a Lawvere $\mathcal{A}$-theory, then we may view both ${\mathcal{A}_f}^{\mathrm{op}}$ and $\mathcal{L}$ as partially finitely complete, and so as absolute-tensored **LFP**-categories, on equipping the former with the sieve of *all* finite-limit-preserving functors into it, and the latter with the sieve of all finite-limit-preserving functors which factor through J. In this way, we can view a Lawvere $\mathcal{A}$-theory as a particular kind of **LFP**-functor ${\mathcal{A}_f}^{\mathrm{op}} \to \mathcal{L}$, where, as before, we overload notation by using the same names for the **LFP**-enriched categories as for the ordinary categories from which they are derived.

If we term the **LFP**-functors arising in this way *Lawvere* **LFP***-categories*, then our reconstruction of the equivalence (1.2) will follow, exactly as in [Gar14b], upon showing the equivalence of one-object **LFP**-categories, and Lawvere **LFP**-categories; and, exactly as before, we will obtain this equivalence via the enriched-categorical process of *free completion* under absolute tensors. Moreover, the universal property of this free completion once again explains the compatibility of this equivalence with the taking of semantics; thereby giving an enriched-categorical explanation of the entire triangle (1.2).

2. The one-object case

In this section, we summarise and discuss the manner in which [Gar14b] reconstructs the equivalence of finitary monads on **Set** and Lawvere theories from an enriched-categorical viewpoint. Most of what we will say is simply revision, but note that the points clarified by Proposition 2.3 and Example 2.8 are new.

[2] In fact, when it comes down to it, we will work not with **LFP** itself, but with a biequivalent bicategory **LexProf** of *lex profunctors*; see Section 4.

We start from the observation that finitary monads on **Set** are equally monoids in the monoidal category $\mathcal{F}$ of finitary endofunctors of **Set**, so equally one-object *$\mathcal{F}$-enriched categories* in the sense of [Kel82]. More precisely:

Proposition 2.1. *The category of finitary monads on* **Set** *is equivalent to the category of one-object $\mathcal{F}$-enriched categories.*

To understand Lawvere theories from the $\mathcal{F}$-enriched perspective is a little more involved. As a first step, note that if $J\colon \mathbb{F}^{\mathrm{op}} \to \mathcal{L}$ is a Lawvere theory, then both $\mathbb{F}^{\mathrm{op}}$ and $\mathcal{L}$ are categories with finite powers, and J is a finite-power-preserving functor between them. Thus the desired understanding flows from one of the key results of [Gar14b], which shows that categories admittings finite powers are equivalent to $\mathcal{F}$-enriched categories admitting all *absolute tensors* in the following sense.

Definition 2.2. If $\mathcal{V}$ is a monoidal category and $\mathcal{C}$ is a $\mathcal{V}$-category, then a *tensor* of $X \in \mathcal{C}$ by $V \in \mathcal{V}$ comprises an object $V \cdot X \in \mathcal{C}$ and morphism $u\colon V \to \mathcal{C}(X, V\cdot X)$ in $\mathcal{V}$, such that, for any $U \in \mathcal{V}$ and $Y \in \mathcal{C}$, the assignation

$$U \xrightarrow{f} \mathcal{C}(V\cdot X, Y) \quad \mapsto \quad U \otimes V \xrightarrow{f\otimes u} \mathcal{C}(V\cdot X, Y) \otimes \mathcal{C}(X, V\cdot X) \xrightarrow{\circ} \mathcal{C}(X,Y) \qquad (2.1)$$

gives a bijection between maps $U \to \mathcal{C}(V\cdot X, Y)$ and $U \otimes V \to \mathcal{C}(X,Y)$ in $\mathcal{V}$. Such a tensor is said to be *preserved* by a $\mathcal{V}$-functor $F\colon \mathcal{C} \to \mathcal{D}$ if the composite morphism $F_{X,V\cdot X} \circ u\colon V \to \mathcal{C}(X, V\cdot X) \to \mathcal{D}(FX, F(V\cdot X))$ in $\mathcal{V}$ exhibits $F(V\cdot X)$ as $V \cdot FX$. Tensors by $V \in \mathcal{V}$ are said to be *absolute* if they are preserved by any $\mathcal{V}$-functor.

There is a delicate point here. The theory of [Kel82] considers enrichment only over a symmetric monoidal closed base; by contrast, our $\mathcal{F}$ is non-symmetric and *right-closed*—meaning that there exists a right adjoint $[V, –]$ to the functor $(–) \otimes V$ tensoring on the *right* by an object V. The value $[V, W]$ of this right adjoint can be computed by first forming the right Kan extension $\mathrm{Ran}_V W \in [\mathbf{Set}, \mathbf{Set}]$, and then the finitary coreflection of that. On the other hand, $\mathcal{F}$ is not *left-closed*—meaning that there is not always a right adjoint to the functor $V \otimes (–)$ tensoring on the *left* by V—because $V \otimes (–)$ will not be cocontinuous if V itself is not cocontinuous.

While the non-symmetric, right-closed setting is too weak to allow constructions such as opposite $\mathcal{F}$-categories, functor $\mathcal{F}$-categories, or tensor product of $\mathcal{F}$-categories, it is strong enough to allow for a good theory of $\mathcal{F}$-enriched colimits—of which absolute tensors are an example. In particular, we may give the following tractable characterisation of the absolute tensors over a right-closed base. In the statement, recall that a *left dual* for $V \in \mathcal{V}$ is a left adjoint for V seen as a 1-cell in the one-object bicategory corresponding to $\mathcal{V}$.

Proposition 2.3. *Let $\mathcal{V}$ be a right-closed monoidal category. Tensors by $V \in \mathcal{V}$ are absolute if and only if V admits a left dual in $\mathcal{V}$.*

This result originates in [Str83a], though some adaptations to the proof are required in the non-left-closed setting; we defer giving these to Section 6.1 below, where we will give a proof which works in the more general bicategory-enriched context.

For now, applying this result when $\mathcal{V} = \mathcal{F}$, we see that tensors by an object $F \in \mathcal{F}$—a finitary endofunctor of **Set**—are absolute just when F has a (necessarily finitary) left adjoint G. In this case, by adjointness we must have $F \cong (–)^{G1}$; moreover, in order for F to be a finitary endofunctor, $G1$ must be finitely presentable in **Set**, thus, a finite set. So an $\mathcal{F}$-category is absolute-tensored just when it admits tensors by $(–)^n \in \mathcal{F}$ for all $n \in \mathbb{N}$. In [Gar14b], such $\mathcal{F}$-categories were called *representable*.

The following further characterisation of the absolute-tensored $\mathcal{F}$-categories is Proposition 3.8 of *ibid.*; in the statement, we write $\mathcal{F}\text{-}\mathbf{CAT}_{\mathrm{abs}}$ for the 2-category of absolute-tensored $\mathcal{F}$-categories and all (necessarily absolute-tensor-preserving) $\mathcal{F}$-functors and $\mathcal{F}$-transformations, and write **FPOW** for the 2-category of categories with finite powers and finite-power-preserving functors.

Proposition 2.4. *The underlying ordinary category of any absolute-tensored $\mathcal{F}$-category admits finite powers, while the underlying ordinary functor of any $\mathcal{F}$-functor between representable $\mathcal{F}$-categories preserves finite powers. The induced 2-functor $\mathcal{F}\text{-}\mathbf{CAT}_{\mathrm{abs}} \to \mathbf{FPOW}$ is an equivalence of 2-categories.*

This result is the technical heart of [Gar14b]; as there, the task of giving its proof will be eased if we replace the category of finitary endofunctors of **Set** by the equivalent functor category $[\mathbb{F}, \mathbf{Set}]$. The equivalence in question arises via left Kan extension and restriction along the inclusion $\mathbb{F} \to \mathbf{Set}$; and transporting the composition monoidal structure on finitary endofunctors across it yields the so-called *substitution* monoidal structure on $[\mathbb{F}, \mathbf{Set}]$, with tensor and unit given by $(A \otimes B)(n) = \int^k Ak \times (Bn)^k$ and $I(n) = n$. Henceforth, we re-define $\mathcal{F}$ to be this monoidal category. Having done so, we see that a general $\mathcal{F}$-category $\mathcal{C}$ involves objects $X, Y, \ldots$, hom-objects $\mathcal{C}(X,Y) \in [\mathbb{F}, \mathbf{Set}]$, and composition and identities notated as follows:

$$\begin{aligned} \textstyle\int^k \mathcal{C}(Y,Z)(k) \times \mathcal{C}(X,Y)(n)^k &\to \mathcal{C}(X,Z)(n) & I(n) &\to \mathcal{C}(X,X)(n) \\ [g, f_1, \ldots, f_k] &\mapsto g \circ (f_1, \ldots, f_k) & i &\mapsto \pi_i \ . \end{aligned} \tag{2.2}$$

Note moreover that, since the unit $I \in [\mathbb{F}, \mathbf{Set}]$ is represented by 1, the arrows $X \to Y$ in the underlying ordinary category $\mathcal{C}_0$ of $\mathcal{C}$ are the elements of $\mathcal{C}(X,Y)(1)$.

Proof of Proposition 2.4. Suppose $\mathcal{C}$ is an absolute-tensored $\mathcal{F}$-category. Then for each $X \in \mathcal{C}$ and $n \in \mathbb{N}$, we have $y_n \cdot X \in \mathcal{C}$ and a unit map $y_n \to \mathcal{C}(X, y_n \cdot X)$, or equally, an element $u \in \mathcal{C}(X, y_n \cdot X)(n)$, rendering each (2.1) invertible. When $U = y_1$, the function (2.1) is given, to within isomorphism, by

$$\begin{aligned} \mathcal{C}(y_n \cdot X, Y)(1) &\to \mathcal{C}(X,Y)(n) \\ f &\mapsto f \circ (u) \ ; \end{aligned} \tag{2.3}$$

thus, when $Y = X$ we obtain elements $p_1, \ldots, p_n \in \mathcal{C}(y_n \cdot X, X)(1)$ with $p_i \circ (u) = \pi_i$ in $\mathcal{C}(X,X)(n)$. It follows by the $\mathcal{F}$-category axioms that $(u \circ (p_1, \ldots, p_n)) \circ (u) = u$, and so, by (2.3) with $Y = y_n \cdot X$, that $u \circ (p_1, \ldots, p_n) = \mathrm{id}_{y_n \cdot X}$ in $\mathcal{C}(y_n \cdot X, y_n \cdot X)(1)$.

We claim that the maps $p_i \colon y_n \cdot X \to X$ in $\mathcal{C}_0$ exhibit $y_n \cdot X$ as the n-fold power X^n. Indeed, given $g_1, \ldots, g_n \colon Z \to X$ in $\mathcal{C}_0$, we define $g \colon Z \to y_n \cdot X$ by $g = u \circ (g_1, \ldots, g_n)$, and now $p_i \circ (u) = \pi_i$ implies $p_i \circ (g) = g_i$. Moreover, if $h \colon Z \to y_n \cdot X$ satisfies $p_i \circ (h) = g_i$ then $g = u \circ (p_1 \circ (h), \ldots, p_n \circ (h)) = (u \circ (p_1, \ldots, p_n)) \circ h = h$.

This proves the first claim. The second follows easily from the fact that any $\mathcal{F}$-functor preserves absolute tensors, and so we have a 2-functor $\mathcal{F}\text{-}\mathbf{CAT}_{\mathrm{abs}} \to \mathbf{FPOW}$. Finally, to show this is a 2-equivalence we construct an explicit pseudoinverse. To each category $\mathcal{D}$ with finite powers, we associate the $\mathcal{F}$-category $\underline{\mathcal{D}}$ with objects those of $\mathcal{D}$, with hom-objects $\underline{\mathcal{D}}(X,Y) = \mathcal{D}(X^{(-)}, Y)$, with composition operations (2.2) obtained using the universal property of power, and with identity elements π_i given by power projections. This $\underline{\mathcal{D}}$ is absolute-tensored on taking the tensor of $X \in \underline{\mathcal{D}}$ by y_n to be X^n, with unit element $1_{X^n} \in \underline{\mathcal{D}}(X, X^n)(n) = \mathcal{D}(X^n, X^n)$. It is now straightforward to extend the assignation

$\mathcal{D} \mapsto \underline{\mathcal{D}}$ to a 2-functor $\mathbf{FPOW} \to \mathcal{F}\text{-}\mathbf{CAT}_{\mathrm{abs}}$, and to show that this is pseudoinverse to the underlying ordinary category 2-functor. □

In particular, if $J\colon \mathbb{F}^{\mathrm{op}} \to \mathcal{L}$ is a Lawvere theory, then we can view both $\mathbb{F}^{\mathrm{op}}$ and $\mathcal{L}$ as $\mathcal{F}$-categories, and J as an $\mathcal{F}$-functor between them. Defining a *Lawvere $\mathcal{F}$-category* to be a representable $\mathcal{F}$-category $\mathcal{L}$ equipped with an identity-on-objects, strict absolute-tensor-preserving $\mathcal{F}$-functor $\mathbb{F}^{\mathrm{op}} \to \mathcal{L}$, it follows easily that:

Proposition 2.5. *The category of Lawvere $\mathcal{F}$-categories is equivalent to the category of Lawvere theories (where maps in each case are commuting triangles under $\mathbb{F}^{\mathrm{op}}$).*

Using Propositions 2.1 and 2.5, we may now re-express the the monad–theory correspondence (1.1) in $\mathcal{F}$-categorical terms as an equivalence between one-object $\mathcal{F}$-categories and Lawvere $\mathcal{F}$-categories. We obtain this using the process of *free completion* under absolute tensors—a description of which can be derived from [BC82].

Proposition 2.6. *Let $\mathcal{V}$ be a monoidal category and let $\mathcal{V}_d \subset \mathcal{V}$ be a subcategory equivalent to the full subcategory of objects with left duals in $\mathcal{V}$. The free completion under absolute tensors of a $\mathcal{V}$-category $\mathcal{C}$ is given by the $\mathcal{V}$-category $\bar{\mathcal{C}}$ with:*

- *Objects $V \cdot X$, where $X \in \mathcal{C}$ and $V \in \mathcal{V}_d$;*
- *Hom-objects $\bar{\mathcal{C}}(V \cdot X, W \cdot Y) = W \otimes \mathcal{C}(X, Y) \otimes V^*$ (for V^* a left dual for V);*
- *Composition built from composition in $\mathcal{C}$ and the counit maps $\varepsilon\colon W^* \otimes W \to I$;*
- *Identities built from the unit maps $\eta\colon I \to V \otimes V^*$ and identities in $\mathcal{C}$.*

Taking $\mathcal{V} = \mathcal{F}$ and $\mathcal{V}_d$ to be the full subcategory on the y_n's, we thus arrive at:

Proposition 2.7. *The category of one-object $\mathcal{F}$-categories is equivalent to the category of Lawvere $\mathcal{F}$-categories.*

Proof. The free completion under absolute tensors of the unit $\mathcal{F}$-category $\mathcal{I}$ is $\mathbb{F}^{\mathrm{op}}$; whence each one-object $\mathcal{F}$-category $\mathcal{C}$ yields a Lawvere $\mathcal{F}$-category on applying completion under absolute tensors to the unique $\mathcal{F}$-functor $\mathcal{I} \to \mathcal{C}$. In the other direction, we send a Lawvere $\mathcal{F}$-category $J\colon \mathbb{F}^{\mathrm{op}} \to \mathcal{L}$ to the one-object sub-$\mathcal{F}$-category of $\mathcal{L}$ on $J1$. □

The conjunction of Propositions 2.1, 2.5 and 2.7 now yields the equivalence on the top row of (1.1). More pedantically, it yields *an* equivalence, which we should check is in fact the usual one:

Example 2.8. Let T be a finitary monad on **Set**, and let $\mathcal{T}$ be the corresponding one-object $\mathcal{F}$-category; thus, $\mathcal{T}$ has a single object X with $\mathcal{T}(X, X)(n) = Tn$, and composition and identities coming from the monad structure of T. The free completion $\bar{\mathcal{T}}$ of $\mathcal{T}$ under absolute tensors has objects $y_n \cdot X$ for $n \in \mathbb{N}$—or equally, just natural numbers—and hom-objects given by

$$\bar{\mathcal{T}}(n, m) = y_m \otimes \mathcal{T}(X, X) \otimes (y_n)^* \cong (T(n \times -))^m \ . \tag{2.4}$$

The underlying ordinary category of $\bar{\mathcal{T}}$ is thus the category $\mathcal{L}$ with natural numbers as objects, and $\mathcal{L}(n, m) = (Tn)^m$. Similarly, the underlying strict finite-power-preserving functor of $\mathbb{F}^{\mathrm{op}} \to \bar{\mathcal{T}}$ is given by postcomposition with the unit of T, and so is precisely the Lawvere theory corresponding to the finitary monad T.

To reconstruct the whole pseudocommutative triangle in (1.1), we need the following result, which combines Propositions 2.5 and 4.4 of [Gar14b]; we omit the proof for now, though note that the corresponding generalisations over a general locally finitely presentable base will be proven as Propositions 4.3 and 7.3 below.

Proposition 2.9. *Let $\mathcal{S}$ denote the $\mathcal{F}$-category corresponding to the category-with-finite-powers* **Set**. *The embeddings of finitary monads and Lawvere theories into $\mathcal{F}$-categories fit into pseudocommutative triangles:*

$$\begin{tikzcd}[column sep=small] \mathbf{Mnd}_f(\mathbf{Set})^{\mathrm{op}} \arrow[rr] \arrow[dr, "\mathbf{Alg}(-)"'] & \simeq & \mathcal{F}\text{-}\mathbf{CAT}^{\mathrm{op}} \arrow[dl, "\mathcal{F}\text{-}\mathbf{CAT}(-{,}\mathcal{S})"] \\ & \mathbf{CAT} & \end{tikzcd} \quad \text{and} \quad \begin{tikzcd}[column sep=small] \mathbf{Law}^{\mathrm{op}} \arrow[rr] \arrow[dr, "\mathbf{Mod}(-)"'] & \simeq & \mathcal{F}\text{-}\mathbf{CAT}^{\mathrm{op}} \arrow[dl, "\mathcal{F}\text{-}\mathbf{CAT}(-{,}\mathcal{S})"] \\ & \mathbf{CAT} & \end{tikzcd} .$$

Given this, to obtain the compability with semantics in (1.1), it suffices to show that, for any one-object $\mathcal{F}$-category $\mathcal{C}$ with completion under absolute tensors $\bar{\mathcal{C}}$, there is an equivalence between the category of $\mathcal{F}$-functors $\mathcal{C} \to \mathcal{S}$ and the category of $\mathcal{F}$-functors $\bar{\mathcal{C}} \to \mathcal{S}$. But since by construction $\mathcal{S}$ is absolute-tensored, this follows directly from the universal property of free completion under absolute tensors.

3. Ingredients for generalisation

In the rest of the paper, we extend the analysis of the previous section to deal with the finitary monad–Lawvere correspondence over an arbitrary locally finitely presentable (lfp) base. In this section, we set up the necessary background for this: first recalling from [NP09] the details of the generalised finitary monad–Lawvere theory correspondence, and then recalling from [Wal81, Str83b] some necessary aspects of bicategory-enriched category theory. We will assume familiarity with the basic theory of lfp categories as found, for example, in [AR94, GU71].

3.1. The monad–theory correspondence over a general lfp base.

In extending the monad–theory correspondence (1.1) from **Set** to a given lfp category $\mathcal{A}$, one side of the generalisation is apparent: we simply replace finitary monads on **Set** by finitary monads on $\mathcal{A}$. On the other side, the appropriate generalisation of Lawvere theories is given by the *Lawvere $\mathcal{A}$-theories* of [NP09]:

Definition 3.1. A *Lawvere $\mathcal{A}$-theory* is a small category $\mathcal{L}$ together with an identity-on-objects, finite-limit-preserving functor $J\colon {\mathcal{A}_f}^{\mathrm{op}} \to \mathcal{L}$. A *morphism* of Lawvere $\mathcal{A}$-theories is a functor $\mathcal{L} \to \mathcal{L}'$ commuting with the maps from ${\mathcal{A}_f}^{\mathrm{op}}$. A *model* for a Lawvere $\mathcal{A}$-theory is a functor $F\colon \mathcal{L} \to \mathbf{Set}$ for which FJ preserves finite limits.

Here, and in what follows, we write $\mathcal{A}_f$ for a small subcategory equivalent to the full subcategory of finitely presentable objects in $\mathcal{A}$. Note that, while ${\mathcal{A}_f}^{\mathrm{op}}$ has all finite limits, we do not assume the same of $\mathcal{L}$; though, of course, it will admit all finite limits of diagrams in the image of J, and so in particular all finite products.

Remark 3.2. Our definition of Lawvere $\mathcal{A}$-theory alters that of [LP09, NP09] by dropping the requirement of *strict* finite limit preservation. However, this apparent relaxation does not in fact change the notion of theory. To see why, we must consider carefully what this strictness amounts to, which is delicate, since $\mathcal{L}$ need not be finitely complete. The correct interpretion is as follows: we fix some choice of finite limits in ${\mathcal{A}_f}^{\mathrm{op}}$, and also assume that, for each finite diagram $D\colon \mathcal{I} \to {\mathcal{A}_f}^{\mathrm{op}}$, the category $\mathcal{L}$ is endowed with a choice of limit for

JD (in particular, because J is the identity on objects, this equips $\mathcal{L}$ with a choice of finite products). We now require that $J\colon {\mathcal{A}_f}^{\mathrm{op}} \to \mathcal{L}$ send the chosen limits in ${\mathcal{A}_f}^{\mathrm{op}}$ to the chosen limits in $\mathcal{L}$. However, in this situation, the choice of limits in $\mathcal{L}$ is uniquely determined by that in ${\mathcal{A}_f}^{\mathrm{op}}$ *so long as* J preserves finite limits in the non-algebraic sense of sending limit cones to limit cones. Thus, if we interpret the preservation of finite limits in Definition 3.1 in this non-algebraic sense, then our notion of Lawvere $\mathcal{A}$-theory agrees with [LP09, NP09].

Note also that our definition of model for a Lawvere $\mathcal{A}$-theory is that of [LP09], rather than that of [NP09]: the latter paper defines a model to comprise $A \in \mathcal{A}$ together with a functor $F\colon \mathcal{L} \to \mathbf{Set}$ such that $FJ = \mathcal{A}(–, A)\colon {\mathcal{A}_f}^{\mathrm{op}} \to \mathbf{Set}$. The equivalence of these definitions follows since $\mathcal{A}$ is equivalent to the category $\mathbf{FL}({\mathcal{A}_f}^{\mathrm{op}}, \mathbf{Set})$ of finite-limit-preserving functors ${\mathcal{A}_f}^{\mathrm{op}} \to \mathbf{Set}$ via the assignation $A \mapsto \mathcal{A}(–, A)$.

Even bearing the above remark in mind, it is not immediate that Lawvere **Set**-theories and their models coincide with Lawvere theories and their models in the previous sense; however, this was shown to be so in [NP09, Theorem 2.4]. The correctness of these notions over a general base is confirmed by the main result of [NP09], which we restate here as:

Theorem 3.3. [NP09, Corollary 5.2] *The category of finitary monads on $\mathcal{A}$ is equivalent to the category of Lawvere $\mathcal{A}$-theories; moreover this equivalence is compatible with the semantics in the sense displayed in* (1.2)*.*

Proof (sketch). For a finitary monad T on $\mathcal{A}$, let $\mathcal{L}_T$ be the category with objects those of $\mathcal{A}_f$, hom-sets $\mathcal{L}_T(A, B) = \mathcal{A}(B, TA)$, and the usual Kleisli composition; now the identity-on-objects $J_T\colon {\mathcal{A}_f}^{\mathrm{op}} \to \mathcal{L}_T$ sending $f \in \mathcal{A}_f(B, A)$ to $\eta_A \circ f \in \mathcal{L}_T(A, B)$ is a Lawvere $\mathcal{A}$-theory. Conversely, if $J\colon {\mathcal{A}_f}^{\mathrm{op}} \to \mathcal{L}$ is a Lawvere $\mathcal{A}$-theory, then the composite of the evident forgetful functor $\mathbf{Mod}(\mathcal{L}) \to \mathbf{FL}({\mathcal{A}_f}^{\mathrm{op}}, \mathbf{Set})$ with the equivalence $\mathbf{FL}({\mathcal{A}_f}^{\mathrm{op}}, \mathbf{Set}) \to \mathcal{A}$ is finitarily monadic, and so gives a finitary monad $T_{\mathcal{L}}$ on $\mathcal{A}$. With some care one may now show that these processes are pseudoinverse in a manner which is compatible with the semantics. $\square$

3.2. **Bicategory-enriched category theory.**

We now recall some basic definitions from the theory of categories enriched over a bicategory as developed in [Wal81, Str83b].

Definition 3.4. Let $\mathcal{W}$ be a bicategory whose 1-cell composition and identities we write as $\otimes$ and I_x respectively. A *$\mathcal{W}$-category* $\mathcal{C}$ comprises:

- A set $\mathrm{ob}\,\mathcal{C}$ of objects;
- For each $X \in \mathrm{ob}\,\mathcal{C}$, an *extent* $\epsilon X \in \mathrm{ob}\,\mathcal{W}$;
- For each $X, Y \in \mathrm{ob}\,\mathcal{C}$, a hom-object[3] $\mathcal{C}(X, Y) \in \mathcal{W}(\epsilon X, \epsilon Y)$;
- For each $X, Y, Z \in \mathrm{ob}\,\mathcal{C}$, composition maps in $\mathcal{W}(\epsilon X, \epsilon Z)$ of the form
$$\mu_{xyz}\colon \mathcal{C}(Y, Z) \otimes \mathcal{C}(X, Y) \to \mathcal{C}(X, Z) \ ;$$
- For each $X \in \mathrm{ob}\,\mathcal{C}$ an identities map in $\mathcal{W}(\epsilon X, \epsilon X)$ of the form
$$\iota_X\colon I_{\epsilon X} \to \mathcal{C}(X, X) \ ;$$

[3]There are two conventions in the literature: we either take $\mathcal{C}(X, Y) \in \mathcal{W}(\epsilon X, \epsilon Y)$, as in [GP97] for example, or we take $\mathcal{C}(X, Y) \in \mathcal{W}(\epsilon Y, \epsilon X)$ as in [Str83b]. We have chosen the former convention here, and have adjusted results from the literature where necessary to conform with this.

subject to associativity and unitality axioms. A $\mathcal{W}$*-functor* $F\colon \mathcal{C} \to \mathcal{D}$ between $\mathcal{W}$-categories comprises an extent-preserving assignation on objects, together with maps $F_{XY}\colon \mathcal{C}(X,Y) \to \mathcal{D}(FX,FY)$ in $\mathcal{W}(\epsilon X, \epsilon Y)$ for each $X, Y \in \mathrm{ob}\,\mathcal{C}$, subject to the two usual functoriality axioms. Finally, a $\mathcal{W}$*-transformation* $\alpha\colon F \Rightarrow G\colon \mathcal{C} \to \mathcal{D}$ between $\mathcal{W}$-functors comprises maps $\alpha_x\colon I_{\epsilon X} \to \mathcal{D}(FX,GX)$ in $\mathcal{W}(\epsilon X, \epsilon X)$ for each $X \in \mathrm{ob}\,\mathcal{C}$ obeying a naturality axiom. We write $\mathcal{W}$-$\mathbf{CAT}$ for the 2-category of $\mathcal{W}$-categories, $\mathcal{W}$-functors and $\mathcal{W}$-transformations.

Note that, if $\mathcal{W}$ is the one-object bicategory corresponding to a monoidal category $\mathcal{V}$, then we re-find the usual definitions of $\mathcal{V}$-category, $\mathcal{V}$-functor and $\mathcal{V}$-transformation. A key difference in the general bicategorical situation is that a $\mathcal{W}$-category does not have a single "underlying ordinary category", but a whole family of them:

Definition 3.5. For any $x \in \mathcal{W}$, we write $\mathcal{I}_x$ for the $\mathcal{W}$-category with a single object $*$ of extent x and with $\mathcal{I}_A(*,*) = I_x$, and write $(–)_x$ for the representable 2-functor $\mathcal{W}\text{-}\mathbf{CAT}(\mathcal{I}_x, –)\colon \mathcal{W}\text{-}\mathbf{CAT} \to \mathbf{CAT}$. On objects, this 2-functor sends a $\mathcal{W}$-category $\mathcal{C}$ to the ordinary category $\mathcal{C}_x$ whose objects are the objects of $\mathcal{C}$ with extent x, and whose morphisms $X \to Y$ are morphisms $I_x \to \mathcal{C}(X,Y)$ in $\mathcal{W}(x,x)$.

3.3. **Enrichment through variation.** It was shown in [GP97] that there is a close link between $\mathcal{W}$-categories and $\mathcal{W}$*-representations*. A $\mathcal{W}$-representation is simply a homomorphism of bicategories $F\colon \mathcal{W} \to \mathbf{CAT}$, but thought of as a "left action"; thus, we notate the functors $F_{xy}\colon \mathcal{W}(x,y) \to \mathbf{CAT}(Fx,Fy)$ as $W \mapsto W *_F (–)$, and write the components of the coherence isomorphisms for F as maps $\lambda\colon I_x *_F X \to X$ and $\alpha\colon (V \otimes W) *_F X \to V *_F (W *_F X)$.

Theorem 3.7 of [GP97] establishes an equivalence between *closed* $\mathcal{W}$-representations and *tensored* $\mathcal{W}$-categories. Here, a $\mathcal{W}$-representation is called *closed* if each functor $(–) *_F W\colon \mathcal{W}(x,y) \to Fy$ has a right adjoint $\langle W, –\rangle_F\colon Fy \to \mathcal{W}(x,y)$; while a $\mathcal{W}$-category is *tensored* if it admits all tensors in the following sense:

Definition 3.6. If $\mathcal{W}$ is a bicategory and $\mathcal{C}$ is a $\mathcal{W}$-category, then a *tensor* of $X \in \mathcal{C}_x$ by $W \in \mathcal{W}(x,y)$ is an object $W \cdot X \in \mathcal{C}_y$ together with a map $u\colon W \to \mathcal{C}(X, W \cdot X)$ in $\mathcal{W}(x,y)$ such that, for any $U \in \mathcal{W}(y,z)$ and $Z \in \mathcal{C}_z$, the assignation

$$U \xrightarrow{f} \mathcal{C}(W \cdot X, Z) \quad \mapsto \quad U \otimes W \xrightarrow{f \otimes u} \mathcal{C}(V \cdot X, Z) \otimes \mathcal{C}(X, W \cdot X) \xrightarrow{\circ} \mathcal{C}(X,Z) \tag{3.1}$$

establishes a bijection between morphisms $U \to \mathcal{C}(W \cdot X, Z)$ in $\mathcal{W}(y,z)$ and morphisms $U \otimes W \to \mathcal{C}(X,Z)$ in $\mathcal{W}(x,z)$.

We note here for future use that a tensor $W \cdot X$ is said to be *preserved* by a $\mathcal{W}$-functor $F\colon \mathcal{C} \to \mathcal{D}$ if the composite $W \to \mathcal{C}(X, W \cdot X) \to \mathcal{D}(FX, F(W \cdot X))$ exhibits $F(W \cdot X)$ as $W \cdot FX$; and that tensors by a 1-cell W are called *absolute* if they are preserved by any $\mathcal{W}$-functor.

Proposition 3.7. [GP97, Theorem 3.7] *There is an equivalence between the* 2*-category* $\mathcal{W}\text{-}\mathbf{CAT}_{\mathrm{tens}}$ *of tensored* $\mathcal{W}$*-categories, tensor-preserving* $\mathcal{W}$*-functors and* $\mathcal{W}$*-natural transformations, and the* 2*-category* $\mathrm{Hom}(\mathcal{W}, \mathbf{CAT})_{\mathrm{cl}}$ *of closed* $\mathcal{W}$*-representations, pseudonatural transformations and modifications.*

Proof (sketch). In one direction, the closed $\mathcal{W}$-representation C associated to a tensored $\mathcal{W}$-category $\mathcal{C}$ is defined on objects by $C(x) = \mathcal{C}_x$, and with action by 1-cells given by tensors: $W *_C X = W \cdot X$. We do not need the further details here, and so omit them. In the other direction, the tensored $\mathcal{W}$-category $\mathcal{F}$ associated to a closed representation $F\colon \mathcal{W} \to \mathbf{CAT}$ has objects of extent a being objects of Fa; hom-objects given by $\mathcal{F}(X,Y) = \langle X,Y\rangle_F$; and composition and identities given by transposing the maps

$$(\langle Y,Z\rangle_F \otimes \langle X,Y\rangle_F) *_F X \xrightarrow{\alpha} \langle Y,Z\rangle_F *_F (\langle X,Y\rangle_F *_F X) \xrightarrow{1*\varepsilon} \langle Y,Z\rangle_F *_F Y \xrightarrow{\varepsilon} Z$$

and $\lambda\colon I_a *_F X \to X$ under the closure adjunctions. The $\mathcal{W}$-category $\mathcal{F}$ so obtained admits all tensors on taking $W \cdot X = W *_F X$ with unit $W \to \langle X, W *_F X\rangle_F$ obtained from the closure adjunctions. $\square$

We will make use of this equivalence in Section 4.2 below, and will require the following easy consequence of the definitions:

Proposition 3.8. *Let $F\colon \mathcal{W} \to \mathbf{CAT}$ be a closed representation, corresponding to the tensored $\mathcal{W}$-category $\mathcal{F}$, and let $T\colon a \to a$ be a monad in $\mathcal{W}$, corresponding to the one-object $\mathcal{W}$-category $\mathcal{T}$. There is an isomorphism of categories $\mathcal{W}\text{-}\mathbf{CAT}(\mathcal{T},\mathcal{F}) \cong F(T)\text{-}\mathbf{Alg}$, natural in maps of monads on a in $\mathcal{W}$.*

Proof. The action on objects of a $\mathcal{W}$-functor $\mathcal{T} \to \mathcal{F}$ picks out an object of extent a in $\mathcal{F}$, thus, an object $X \in Fa$. The action on homs is given by a map $x\colon T \to \langle X,X\rangle_F$ in $\mathcal{W}(a,a)$, while functoriality requires the commutativity of:

$$\begin{tikzcd} & I_a \arrow[dl, "\eta"'] \arrow[dr, "\iota"] & \\ T \arrow[rr, "x"] & & \langle X,X\rangle_F \end{tikzcd} \qquad \text{and} \qquad \begin{tikzcd} T \otimes T \arrow[r, "x\otimes x"] \arrow[d, "\mu"'] & \langle X,X\rangle_F \otimes \langle X,X\rangle_F \arrow[d, "\mu"] \\ T \arrow[r, "x"] & \langle X,X\rangle_F \end{tikzcd}.$$

Transposing under adjunction, this is equally to give $X \in Fa$ and a map $T *_F X \to X$ satisfying the two axioms to be an algebra for $F(T) = T *_F (-)$. Further, to give a $\mathcal{W}$-transformation $F \Rightarrow G\colon \mathcal{T} \to \mathcal{F}$ is equally to give $\varphi\colon I_x \to \langle X,Y\rangle_F$ such that

$$\begin{tikzcd} T \arrow[r, "y"] \arrow[d, "x"'] & \langle Y,Y\rangle_F \arrow[r, "\varphi\otimes 1"] & \langle X,Y\rangle_F \otimes \langle Y,Y\rangle_F \arrow[d, "\mu"] \\ \langle X,X\rangle_F \arrow[r, "1\otimes\varphi"] & \langle X,X\rangle_F \otimes \langle X,Y\rangle_F \arrow[r, "\mu"] & \langle X,Y\rangle_F \end{tikzcd}$$

commutes; which, transposing under adjunction and using the coherence constraint $I_a * X \cong X$, is equally to give a map $X \to Y$ commuting with the $F(T)$-actions. The naturality of the correspondence just described in T is easily checked. $\square$

4. Finitary monads and their algebras

We now begin our enriched-categorical analysis of the monad–theory correspondence over an lfp base. We first describe a bicategory **LexProf** of *lex profunctors* which is biequivalent to the bicategory **LFP** of finitary functors between lfp categories, but more convenient to work with; we then exhibit each finitary monad on an lfp category as a **LexProf**-category, and the associated category of algebras as a category of **LexProf**-enriched functors.

4.1. **Finitary monads as enriched categories.** The basic theory of lfp categories tells us that for any small finitely-complete $\mathbb{A}$, the category $\mathbf{FL}(\mathbb{A}, \mathbf{Set})$ of finite-limit-preserving functors $\mathbb{A} \to \mathbf{Set}$ is lfp, and moreover that every lfp category is equivalent to one of this form. So **LFP** is biequivalent to the bicategory whose objects are small finitely-complete categories, whose hom-category from $\mathbb{A}$ to $\mathbb{B}$ is $\mathbf{LFP}(\mathbf{FL}(\mathbb{A}, \mathbf{Set}), \mathbf{FL}(\mathbb{B}, \mathbf{Set}))$, and whose composition is inherited from **LFP**.

Now, since for any small finitely-complete $\mathbb{A}$, the inclusion $\mathbb{A}^{\mathrm{op}} \to \mathbf{FL}(\mathbb{A}, \mathbf{Set})$ exhibits its codomain as the free filtered-cocomplete category on its domain, there are equivalences $\mathbf{LFP}(\mathbf{FL}(\mathbb{A}, \mathbf{Set}), \mathbf{FL}(\mathbb{B}, \mathbf{Set})) \simeq [\mathbb{A}^{\mathrm{op}}, \mathbf{FL}(\mathbb{B}, \mathbf{Set})]$; thus transporting the compositional structure of **LFP** across these equivalences, we obtain:

Definition 4.1. The right-closed bicategory **LexProf** of *lex profunctors* has:

- As objects, small categories with finite limits.
- $\mathbf{LexProf}(\mathbb{A}, \mathbb{B}) = [\mathbb{A}^{\mathrm{op}}, \mathbf{FL}(\mathbb{B}, \mathbf{Set})]$; we typically identify objects therein with functors $\mathbb{A}^{\mathrm{op}} \times \mathbb{B} \to \mathbf{Set}$ that preserve finite limits in their second variable.
- The identity 1-cell $I_{\mathbb{A}} \in \mathbf{LexProf}(\mathbb{A}, \mathbb{A})$ is given by $I_{\mathbb{A}}(a', a) = \mathbb{A}(a', a)$, while the composition of $M \in \mathbf{LexProf}(\mathbb{A}, \mathbb{B})$ and $N \in \mathbf{LexProf}(\mathbb{B}, \mathbb{C})$ is given by:
$$(N \otimes M)(a, c) = \int^{b \in \mathbb{B}} N(b, c) \times M(a, b) \ . \tag{4.1}$$
- For $M \in \mathbf{LexProf}(\mathbb{A}, \mathbb{B})$ and $P \in \mathbf{LexProf}(\mathbb{A}, \mathbb{C})$, the right closure $[M, P] \in \mathbf{LexProf}(\mathbb{B}, \mathbb{C})$ is defined by
$$[M, P](b, c) = \int_a [M(a, b), P(a, c)] \ . \tag{4.2}$$

By the above discussion, **LFP** is biequivalent to **LexProf**, and this induces an equivalence between the category of monads on $\mathcal{A}$ in **LFP**—thus, the category of finitary monads on $\mathcal{A}$—and the category of monads on ${\mathcal{A}_f}^{\mathrm{op}}$ in **LexProf**. Such monads correspond with one-object **LexProf**-categories of extent ${\mathcal{A}_f}^{\mathrm{op}}$ and so:

Proposition 4.2. *For any locally finitely presentable category $\mathcal{A}$, the category $\mathbf{Mnd}_f(\mathcal{A})$ of finitary monads on $\mathcal{A}$ is equivalent to the category of* **LexProf***-categories with a single object of extent ${\mathcal{A}_f}^{\mathrm{op}}$.*

4.2. **Algebras for finitary monads as enriched functors.** We now explain algebras for finitary monads in the **LexProf**-enriched context. Composing the biequivalence $\mathbf{LexProf} \to \mathbf{LFP}$ with the inclusion 2-functor $\mathbf{LFP} \to \mathbf{CAT}$ yields a homomorphism $S \colon \mathbf{LexProf} \to \mathbf{CAT}$ which on objects sends $\mathbb{A}$ to $\mathbf{FL}(\mathbb{A}, \mathbf{Set})$, and for which the action of a 1-cell $M \in \mathbf{LexProf}(\mathbb{A}, \mathbb{B})$ on an object $X \in \mathbf{FL}(\mathbb{A}, \mathbf{Set})$ is given as on the left in
$$(M *_S X)(b) = \int^{a \in \mathbb{A}} M(a, b) \times Xa \qquad\qquad \langle X, Y \rangle_S(a, b) = \mathbf{Set}(Xa, Yb) \ .$$
This S is a closed representation, where for $X \in \mathbf{FL}(\mathbb{A}, \mathbf{Set})$ and $Y \in \mathbf{FL}(\mathbb{B}, \mathbf{Set})$ we define $\langle X, Y \rangle_S$ as to the right above; and so applying Proposition 3.7 gives a tensored **LexProf**-category $\mathcal{S}$ with objects of extent $\mathbb{A}$ being finite-limit-preserving functors $\mathbb{A} \to \mathbf{Set}$, and with hom-objects $\mathcal{S}(X, Y)(a, b) = \mathbf{Set}(Xa, Yb)$.

Proposition 4.3. *For any locally finitely presentable category $\mathcal{A}$, the embedding of finitary monads on $\mathcal{A}$ as one-object* **LexProf***-categories obtained in Proposition 4.2 fits into a triangle,* *commuting up to pseudonatural equivalence:*

$$\begin{tikzcd} \mathbf{Mnd}_f(\mathcal{A})^{\mathrm{op}} \arrow[rr] \arrow[dr, "(-)\text{-}\mathbf{Alg}"'] & \simeq & (\mathbf{LexProf}\text{-}\mathbf{CAT})^{\mathrm{op}} \arrow[dl, "\mathbf{LexProf}\text{-}\mathbf{CAT}(-,\mathcal{S})"] \\ & \mathbf{CAT} & \end{tikzcd} \quad . \tag{4.3}$$

Proof. Given $T \in \mathbf{Mnd}_f(\mathcal{A})$, which is equally a monad on $\mathcal{A}$ in $\mathbf{LFP}$, we can successively apply the biequivalences $\mathbf{LFP} \to \mathbf{LexProf}$ and $\mathbf{LexProf} \to \mathbf{LFP}$ to obtain in turn a monad T' on $\mathcal{A}_f^{\mathrm{op}} \in \mathbf{LexProf}$ and a monad T'' on $\mathbf{FL}(\mathcal{A}_f^{\mathrm{op}}, \mathbf{Set})$. It follows easily from the fact of a biequivalence that $T\text{-}\mathbf{Alg} \simeq T''\text{-}\mathbf{Alg}$.

Now, starting from $T \in \mathbf{Mnd}_f(\mathcal{A})$, the functor across the top of (4.3) sends it to the one-object $\mathbf{LFP}$-category $\mathcal{T}'$ corresponding to T'; whereupon by Proposition 3.8, we have pseudonatural equivalences

$$\mathbf{LexProf}\text{-}\mathbf{CAT}(\mathcal{T}', \mathcal{S}) \cong S(T')\text{-}\mathbf{Alg} = T''\text{-}\mathbf{Alg} \simeq T\text{-}\mathbf{Alg}\ . \qquad \square$$

4.3. **General LexProf-categories.** Before turning to the relationship of **LexProf**-categories and Lawvere $\mathcal{A}$-theories, we take a moment to unpack the data for a general **LexProf**-category $\mathcal{C}$. We have objects $X, Y, \ldots$ with associated extents $\mathbb{A}, \mathbb{B}, \ldots$ in **LexProf**; while for objects $X \in \mathcal{C}_{\mathbb{A}}$ and $Y \in \mathcal{C}_{\mathbb{B}}$, we have the hom-object $\mathcal{C}(X,Y) \colon \mathbb{A}^{\mathrm{op}} \times \mathbb{B} \to \mathbf{Set}$, which is a functor preserving finite limits in its second variable. By the coend formula (4.1) for 1-cell composition in **LexProf**, composition in $\mathcal{C}$ is equally given by functions

$$\begin{aligned} \mathcal{C}(Y,Z)(j,k) \times \mathcal{C}(X,Y)(i,j) &\to \mathcal{C}(X,Z)(i,k) \\ (g,f) &\mapsto g \circ f \end{aligned} \tag{4.4}$$

which are natural in $i \in \mathbb{A}$ and $k \in \mathbb{C}$ and dinatural in $j \in \mathbb{B}$. On the other hand, identities in $\mathcal{C}$ are given by functions $\iota_X \colon \mathbb{A}(i,j) \to \mathcal{C}(X,X)(i,j)$, natural in $i,j \in \mathbb{A}$; if we define $1_{X,i} := \iota_X(1_i)$, then the **LexProf**-category axioms for $\mathcal{C}$ say that $f \circ 1_{X,i} = f = 1_{Y,j} \circ f$ for all $f \in \mathcal{C}(X,Y)(i,j)$ and that the operation (4.4) is associative. Note that the naturality of each ι_X together with the unit axioms imply that the action on morphisms of the hom-object $\mathcal{C}(X,Y)$ is given by

$$\begin{aligned} \mathcal{C}(X,Y)(\varphi,\psi) \colon \mathcal{C}(X,Y)(i,j) &\to \mathcal{C}(X,Y)(i',j') \\ f &\mapsto \iota_Y(\psi) \circ f \circ \iota_X(\varphi)\ . \end{aligned} \tag{4.5}$$

Applying naturality of ι_X again to this formula yields the following functoriality equation for any pair of composable maps in $\mathbb{A}$:

$$\iota_X(\varphi' \circ \varphi) = \iota_X(\varphi') \circ \iota_X(\varphi)\ . \tag{4.6}$$

5. Partial finite completeness

In the following two sections, we will identify the absolute-tensored **LexProf**-categories with what we call *partially finitely complete* ordinary categories; this identification will take the form of a biequivalence between suitably-defined 2-categories. We will exploit this biequivalence in Section 7 in order to identify Lawvere $\mathcal{A}$-theories with certain functors between absolute-tensored **LexProf**-categories.

5.1. **Partially finitely complete categories.** We begin by introducing the 2-category of partially finitely complete categories and partially finite-limit-preserving functors.

Definition 5.1. By a *left-exact sieve* on a category $\mathcal{C}$, we mean a collection $\mathcal{S}$ of finite-limit-preserving functors $\mathbb{A} \to \mathcal{C}$, each with small, finitely-complete domain, and satisfying the following conditions, wherein we write $\mathcal{S}[\mathbb{A}]$ for those elements of $\mathcal{S}$ with domain $\mathbb{A}$:

(i) If $X \in \mathcal{S}[\mathbb{B}]$ and $G \in \mathbf{FL}(\mathbb{A}, \mathbb{B})$, then $XG \in \mathcal{S}[\mathbb{A}]$;
(ii) If $X \in \mathcal{S}[\mathbb{A}]$ and $X \cong Y \colon \mathbb{A} \to \mathcal{C}$, then $Y \in \mathcal{S}[\mathbb{A}]$;
(iii) Each object of $\mathcal{C}$ is in the image of some functor in $\mathcal{S}$.

A *partially finitely complete category* $(\mathcal{C}, \mathcal{S}_\mathcal{C})$ is a category $\mathcal{C}$ together with a left-exact sieve $\mathcal{S}_\mathcal{C}$ on it. Where confusion is unlikely, we may write $(\mathcal{C}, \mathcal{S}_\mathcal{C})$ simply as $\mathcal{C}$. A *partially finite-limit-preserving functor* $(\mathcal{C}, \mathcal{S}_\mathcal{C}) \to (\mathcal{D}, \mathcal{S}_\mathcal{D})$ is a functor $F \colon \mathcal{C} \to \mathcal{D}$ such that $FX \in \mathcal{S}_\mathcal{D}$ for all $X \in \mathcal{S}_\mathcal{C}$; we call such an F *sieve-reflecting* if, for all $Y \in \mathcal{S}_\mathcal{D}$, there exists $X \in \mathcal{S}_\mathcal{C}$ such that $FX \cong Y$. We write **PARFL** for the 2-category of partially finitely complete categories, partially finite-limit-preserving functors, and arbitrary natural transformations.

The following examples should serve to clarify the relevance of these notions to Lawvere theories over a general lfp base.

Example 5.2. Any finitely complete $\mathcal{C}$ can be seen as partially finitely complete when endowed with the sieve $\mathcal{S}_\mathcal{C}$ of all finite-limit-preserving functors into $\mathcal{C}$ with small domain. If $\mathcal{D}$ is also finitely complete, then any finite-limit-preserving $F \colon \mathcal{C} \to \mathcal{D}$ is clearly also partially finite-limit-preserving; conversely, if $F \colon \mathcal{C} \to \mathcal{D}$ is partially finite-limit-preserving, then for any finite diagram $D \colon \mathbb{I} \to \mathcal{C}$, closing its image in $\mathcal{C}$ under finite limits yields a small subcategory $\mathbb{A}$ for which the full inclusion $J \colon \mathbb{A} \to \mathcal{C}$ preserves finite limits. As F is partially finite-limit-preserving, the composite $FJ \colon \mathbb{A} \to \mathcal{D}$ also preserves finite limits; in particular, the chosen limit cone over D in $\mathcal{C}$—which lies in the subcategory $\mathbb{A}$—is sent to a limit cone in $\mathcal{D}$. It follows there is a full and locally full inclusion of 2-categories $\mathbf{FL} \to \mathbf{PARFL}$.

Example 5.3. If $J \colon \mathcal{A}_f^{\mathrm{op}} \to \mathcal{L}$ is a Lawvere $\mathcal{A}$-theory, then $\mathcal{L}$ becomes a partially finitely complete category when endowed with the sieve generated by J:

$$\mathcal{S}_\mathcal{L} = \{F \colon \mathbb{A} \to \mathcal{L} : F \cong JG \text{ for some finite-limit-preserving } G \colon \mathbb{A} \to \mathcal{A}_f^{\mathrm{op}}\} \ . \tag{5.1}$$

Clearly $\mathcal{S}_\mathcal{L}$ satisfies conditions (i) and (ii) above, and satisfies (iii) by virtue of J being bijective on objects. Moreover, a partially finite-limit-preserving $\mathcal{L} \to \mathcal{C}$ is precisely a functor $F \colon \mathcal{L} \to \mathcal{C}$ such that $FJ \in \mathcal{S}_\mathcal{C}$; so in particular, a partially finite-limit-preserving $\mathcal{L} \to \mathbf{Set}$ is precisely a model for the Lawvere $\mathcal{A}$-theory $\mathcal{L}$.

5.2. **Partial finite completeness and LexProf-enrichment.** Towards our identification of absolute-tensored **LexProf**-categories with partially finitely complete categories, we now construct a 2-adjunction

$$\begin{tikzcd} \mathbf{PARFL} \arrow[r, shift right, "\Gamma"'] \arrow[r, phantom, "\perp"] & \mathbf{LexProf\text{-}CAT} \arrow[l, shift right, "\int"'] \end{tikzcd} \ . \tag{5.2}$$

Definition 5.4. Let $\mathcal{C}$ be a partially finitely complete category. The **LexProf**-category $\Gamma(\mathcal{C})$ has objects of extent $\mathbb{A}$ given by elements $X \in \mathcal{S}_\mathcal{C}[\mathbb{A}]$, and remaining data defined as follows:

- For $X \in \mathcal{S}_{\mathcal{C}}[\mathbb{A}]$ and $Y \in \mathcal{S}_{\mathcal{C}}[\mathbb{B}]$, the hom-object $\Gamma(\mathcal{C})(X,Y) \in \mathbf{LexProf}(\mathbb{A},\mathbb{B})$ is given by $\Gamma(\mathcal{C})(X,Y)(i,j) = \mathcal{C}(Xi,Yj)$. Note that this preserves finite limits in its second variable since Y and each $\mathcal{C}(Xi,\text{–})$ do so.
- Composition in $\Gamma(\mathcal{C})$ may be specified, as in (4.4), by natural families of functions $\Gamma(\mathcal{C})(Y,Z)(j,k) \times \Gamma(\mathcal{C})(X,Y)(i,j) \to \Gamma(\mathcal{C})(X,Z)(i,k)$, which we obtain from composition in $\mathcal{C}$.
- Identities $\iota_X \colon \mathbb{A}(i,j) \to \Gamma(\mathcal{C})(X,X)(i,j) = \mathcal{C}(Xi,Xj)$ are given by the action of X on morphisms.

The **LexProf**-category axioms for $\Gamma(\mathcal{C})$ follow from the category axioms of $\mathcal{C}$ and functoriality of each X.

If $F \colon \mathcal{C} \to \mathcal{D}$ is a partially finite-limit-preserving functor, then we define the **LexProf**-functor $\Gamma(F) \colon \Gamma(\mathcal{C}) \to \Gamma(\mathcal{D})$ to have action on objects $X \mapsto FX$ (using the fact that $FX \in \mathcal{S}_{\mathcal{D}}$ whenever $X \in \mathcal{S}_{\mathcal{C}}$). The components of the action of $\Gamma(F)$ on hom-objects $\Gamma(\mathcal{C})(X,Y) \to \Gamma(\mathcal{D})(FX,FY)$ are functions $\mathcal{C}(Xi,Yj) \to \mathcal{D}(FXi,FYj)$, which are given simply by the action of F on morphisms. The **LexProf**-functor axioms are immediate from functoriality of F.

Finally, for a 2-cell $\alpha \colon F \Rightarrow G \colon \mathcal{C} \to \mathcal{D}$ in **PARFL**, we define a **LexProf**-transformation $\Gamma(\alpha) \colon \Gamma(F) \Rightarrow \Gamma(G)$ whose component $I_{\mathbb{A}} \to \Gamma(\mathcal{D})(FX,GX)$ is given by the dinatural family of elements $\alpha_{Xi} \in \mathcal{D}(FXi,GXi)$. The **LexProf**-naturality of $\Gamma(\alpha)$ amounts to the condition that $Gf \circ \alpha_{Xi} = \alpha_{Yj} \circ Ff \colon FXi \to GYj$ for all $f \colon Xi \to Yj$ in $\mathcal{C}$; which is so by naturality of α.

Proposition 5.5. *The data of Definition 5.4 comprise the action on* 0-, 1-, *and* 2-*cells of a* 2-*functor* $\Gamma \colon \mathbf{PARFL} \to \mathbf{LexProf}\text{-}\mathbf{CAT}$*, Moreover the* 2-*functor* Γ *admits a left* 2-*adjoint* $\int \colon \mathbf{LexProf}\text{-}\mathbf{CAT} \to \mathbf{PARFL}$.

Proof. The 2-functoriality of Γ is easy to check, and so it remains to construct its left 2-adjoint $\int$. Given a **LexProf**-category $\mathcal{C}$, we write $\int\mathcal{C}$ for the category with:
- **Objects** of the form (X,i) where $X \in \mathcal{C}_{\mathbb{A}}$ and $i \in \mathbb{A}$;
- **Morphisms** $f \colon (X,i) \to (Y,j)$ being elements $f \in \mathcal{C}(X,Y)(i,j)$;
- **Identities** given by the elements $1_{X,i} \in \mathcal{C}(X,X)(i,i)$;
- **Composition** mediated by the functions (4.4).

Given $X \in \mathcal{C}_{\mathbb{A}}$, we write $\iota_X \colon \mathbb{A} \to \int\mathcal{C}$ for the functor given by $i \mapsto (X,i)$ on objects and by the identities map $\iota_X \colon \mathbb{A}(i,i') \to \mathcal{C}(X,X)(i,i')$ of $\mathcal{C}$ on morphisms; note this is functorial by (4.6). By the definition of $\int\mathcal{C}$ and (4.5), we have that

$$\mathcal{C}(X,Y) = (\textstyle\int\mathcal{C})(\iota_X(\text{–}),\iota_Y(\text{–})) \colon \mathbb{A}^{\mathrm{op}} \times \mathbb{B} \to \mathbf{Set} \ ; \tag{5.3}$$

in particular, as each $\mathcal{C}(X,Y)$ preserves finite limits in its second variable, each functor $\int\mathcal{C}((X,i),\iota_Y(\text{–})) \colon \mathbb{B} \to \mathbf{Set}$ preserve finite limits, whence each $\iota_Y \colon \mathbb{B} \to \int\mathcal{C}$ preserves finite limits. It follows that $\int\mathcal{C}$ is partially finitely complete when endowed with the left-exact sieve

$$\mathcal{S}_{\int\mathcal{C}} = \{\, G \colon \mathbb{A} \to \Gamma(\mathcal{C}) : G \cong \iota_Y F \text{ for some } Y \in \mathcal{C}_{\mathbb{B}} \text{ and } F \in \mathbf{FL}(\mathbb{A},\mathbb{B}) \,\} \ . \tag{5.4}$$

We now show that $\int\mathcal{C}$ provides the value at $\mathcal{C}$ of a left 2-adjoint to Γ; thus, we must exhibit isomorphisms of categories, 2-natural in $\mathcal{D} \in \mathbf{PARFL}$, of the form:

$$\mathbf{PARFL}(\textstyle\int\mathcal{C},\mathcal{D}) \cong \mathbf{LexProf}\text{-}\mathbf{CAT}(\mathcal{C},\Gamma(\mathcal{D})) \ . \tag{5.5}$$

Now, to give a partially finite-limit-preserving functor $F \colon \int\mathcal{C} \to \mathcal{D}$ is to give:

- For all $X \in \mathcal{C}_{\mathbb{A}}$ and $i \in \mathbb{A}$ an object $F(X,i) \in \mathcal{D}$; and
- For all $f \in \mathcal{C}(X,Y)(i,j)$, a map $Ff \colon F(X,i) \to F(Y,j)$ in $\mathcal{D}$,

functorially with respect to the composition (4.4) and composition in $\mathcal{D}$, and subject to the requirement that $F\iota_X \in \mathcal{S}_{\mathcal{D}}[\mathbb{A}]$ for all $X \in \mathcal{C}_{\mathbb{A}}$. On the other hand, to give a **LexProf**-functor $G \colon \mathcal{C} \to \Gamma(\mathcal{D})$ is to give:

- For all $X \in \mathcal{C}_{\mathbb{A}}$, a functor $GX \in \mathcal{S}_{\mathcal{D}}[\mathbb{A}]$; and
- For all $f \in \mathcal{C}(X,Y)(i,j)$, an element of $\Gamma(\mathcal{D})(GX,GY) = \mathcal{D}((GX)i,(GY)j)$, i.e., a map $Gf \colon (GX)i \to (GY)j$ in $\mathcal{D}$,

subject to the same functoriality condition. Thus, given $F \colon \int\mathcal{C} \to \mathcal{D}$, we may define $\bar{F} \colon \mathcal{C} \to \Gamma(\mathcal{D})$ by taking $\bar{F}X = F\iota_X$ (which is in $\mathcal{S}_{\mathcal{D}}[\mathbb{A}]$ by assumption) and $\bar{F}f = Ff$; the functoriality is clear. On the other hand, given $G \colon \mathcal{C} \to \Gamma(\mathcal{D})$, we may define $\bar{G} \colon \int\mathcal{C} \to \mathcal{D}$ by taking $\bar{G}(X,i) = (GX)i$ and $\bar{G}f = Gf$. Functoriality is again clear, but we need to check that $\bar{G}\iota_X \in \mathcal{S}_{\mathcal{D}}[\mathbb{A}]$ for all $X \in \mathcal{C}_{\mathbb{A}}$. In fact we show that $\bar{G}\iota_X = GX$, which is in $\mathcal{S}_{\mathcal{D}}[\mathbb{A}]$ by assumption. On objects, $\bar{G}\iota_X(i) = \bar{G}(X,i) = (GX)i$ as required. On morphisms, the compatibility of G with identities in $\mathcal{C}$ and $\Gamma(\mathcal{D})$ gives a commuting triangle of sets and functions:

$$\begin{tikzcd} & \mathbb{A}(i,j) \arrow[dl, "\iota_X"'] \arrow[dr, "\iota_{GX}"] & \\ \mathcal{C}(X,X)(i,j) \arrow[rr, "G"] & & \Gamma(\mathcal{D})(GX,GX)(i,j) = \mathcal{D}(GXi,GXj)\ . \end{tikzcd}$$

The left-hand path maps $\varphi \in \mathbb{A}(i,j)$ to $G\iota_X(\varphi) = \bar{G}\iota_X(\varphi)$; while by definition of $\Gamma(\mathcal{D})$ the right-hand path maps φ to $GX(\varphi)$; whence $\bar{G}\iota_X = GX$ as required. It is clear from the above calculations that the assignations $F \mapsto \bar{F}$ and $G \mapsto \bar{G}$ are mutually inverse, which establishes the bijection (5.5) on objects.

To establish (5.5) on maps, let $F_1, F_2 \colon \int\mathcal{C} \rightrightarrows \mathcal{D}$. The components of a **LexProf**-transformation $\bar{\alpha} \colon \bar{F}_1 \Rightarrow \bar{F}_2 \colon \mathcal{C} \to \Gamma(\mathcal{D})$ comprise natural families of functions $\bar{\alpha}_{Xij} \colon \mathbb{A}(i,j) \to \Gamma(\mathcal{D})(\bar{F}_1X, \bar{F}_2X)(i,j) = \mathcal{D}(F_1\iota_X(i), F_2\iota_X(j))$ satisfying **LexProf**-naturality. By Yoneda, each $\bar{\alpha}_{Xij}$ is uniquely determined by elements $\bar{\alpha}_{X,i} = \bar{\alpha}_{Xii}(\mathrm{id}_i) \in \mathcal{D}(F_1(X,i), F_2(X,i))$ satisfying $F_2\iota_X(\varphi) \circ \bar{\alpha}_{X,i} = \bar{\alpha}_{X,j} \circ F_1\iota_X(\varphi)$ for all $\varphi \in \mathbb{A}(i,j)$; their **LexProf**-naturality is now the requirement that the square

$$\begin{tikzcd} \mathcal{C}(X,Y)(i,j) \arrow[r, "F_1"] \arrow[d, "F_2"'] & \mathcal{D}(F_1(X,i), F_1(Y,j)) \arrow[d, "\bar{\alpha}_{Y,j} \circ (-)"] \\ \mathcal{D}(F_2(X,i), F_2(Y,j)) \arrow[r, "(-)\circ\bar{\alpha}_{X,i}"] & \mathcal{D}(F_1(X,i), F_2(Y,j)) \end{tikzcd}$$

commute for each X, Y, i, j. Note that this *implies* the earlier condition that $F_2\iota_X(\varphi) \circ \bar{\alpha}_{X,i} = \bar{\alpha}_{X,j} \circ F_1\iota_X(\varphi)$ on taking $X = Y$ and evaluating at $\iota_X(\varphi)$; now evaluating at a general element, we get the condition that $F_2f \circ \bar{\alpha}_{X,i} = \bar{\alpha}_{Y,j} \circ F_1f$ for all $f \colon (X,i) \to (Y,j)$ in $\int\mathcal{C}$—which says precisely that we have a natural transformation $\bar{\alpha} \colon F_1 \Rightarrow F_2 \colon \int\mathcal{C} \to \mathcal{D}$. This establishes the bijection (5.5) on morphisms; the 2-naturality in $\mathcal{D}$ is left as an easy exercise for the reader. $\square$

6. Absolute-tensored **LexProf**-categories

In this section, we prove the key technical result of this paper, Theorem 6.4, which shows that the 2-adjunction (5.2) exhibits **PARFL** as biequivalent to the full sub-2-category of **LexProf**-**CAT** on the absolute-tensored **LexProf**-categories.

6.1. **Absolute tensors in $\mathcal{W}$-categories.** We begin by characterising absolute tensors in $\mathcal{W}$-categories for an arbitrary right-closed bicategory $\mathcal{W}$. Here, *right-closedness* is the condition that, for every 1-cell $W \in \mathcal{W}(x, y)$ and every $z \in \mathcal{W}$, the functor $(–) \otimes W : \mathcal{W}(y, z) \to \mathcal{W}(x, z)$ admits a right adjoint $[W, –] : \mathcal{W}(x, z) \to \mathcal{W}(y, z)$. In this setting, we will show that tensors by a 1-cell W are absolute if and only if W is a right adjoint in $\mathcal{W}$. If $\mathcal{W}$ were both left- and right- closed, this would follow from the characterisation of enriched absolute colimits given in [Str83a], but in the absence of left-closedness, we need a different proof. The first step is the following, which is a special case of [Gar14a, Theorem 1.2]:

Proposition 6.1. *Let $\mathcal{W}$ be a bicategory, let $\mathcal{C}$ be a $\mathcal{W}$-category, let $X \in \mathcal{C}_x$ and let $W \in \mathcal{W}(x, y)$. If W admits the left adjoint $W^* \in \mathcal{W}(y, x)$, then there is a bijective correspondence between data of the following forms:*

(a) *A map $u : W \to \mathcal{C}(X, Y)$ in $\mathcal{W}(x, y)$ exhibiting Y as $W \cdot X$;*
(b) *Maps $u : W \to \mathcal{C}(X, Y)$ in $\mathcal{W}(x, y)$ and $u^* : W^* \to \mathcal{C}(Y, X)$ in $\mathcal{W}(y, x)$ rendering commutative the squares:*

$$
\begin{tikzcd}
I_y \arrow[r, "\eta"] \arrow[d, "\iota"'] & W \otimes W^* \arrow[d, "u \otimes u^*"] \\
\mathcal{C}(Y,Y) & \mathcal{C}(X,Y) \otimes \mathcal{C}(Y,X) \arrow[l, "\mu"']
\end{tikzcd}
\qquad
\begin{tikzcd}
W^* \otimes W \arrow[r, "\varepsilon"] \arrow[d, "u^* \otimes u"'] & I_x \arrow[d, "\iota"] \\
\mathcal{C}(Y,X) \otimes \mathcal{C}(X,Y) \arrow[r, "\mu"] & \mathcal{C}(X,X)
\end{tikzcd} \; . \tag{6.1}
$$

Proof. Given (a), applying surjectivity in (3.1) to $\iota_X \circ \varepsilon : W^* \otimes W \to I_x \to \mathcal{C}(X, X)$ yields a unique map $u^* : W^* \to \mathcal{C}(Y, X)$ making the square right above commute. To see that the left square also commutes, it suffices by injectivity in (3.1) to check that the sides become equal after tensoring on the right with u and postcomposing with $\mu : \mathcal{C}(Y, Y) \otimes \mathcal{C}(X, Y) \to \mathcal{C}(X, Y)$. This follows by a short calculation using commutativity in the right square and the triangle identities.

To complete the proof, it remains to show that if u and u^* are given as in (b), then u exhibits Y as $W \cdot X$. Thus, given $g : U \otimes W \to \mathcal{C}(X, Z)$, we must show that $g = \mu \circ (f \otimes u)$ as in (3.1) for a unique $f : U \to \mathcal{C}(Y, Z)$. We may take f to be

$$U \xrightarrow{1 \otimes \eta} U \otimes W \otimes W^* \xrightarrow{g \otimes u^*} \mathcal{C}(X, Z) \otimes \mathcal{C}(Y, X) \xrightarrow{\mu} \mathcal{C}(Y, Z) \; ; \tag{6.2}$$

now that $g = \mu \circ (f \otimes u)$ follows on rewriting with the right-hand square of (6.1), the triangle identities and the $\mathcal{W}$-category axioms for $\mathcal{C}$. Moreover, if $f' : U \to \mathcal{C}(Y, Z)$ also satisfies $g = \mu \circ (f' \otimes u)$, then substituting into (6.2) gives

$$f = U \xrightarrow{1 \otimes \eta} U \otimes W \otimes W^* \xrightarrow{f' \otimes u \otimes u^*} \mathcal{C}(Y, Z) \otimes \mathcal{C}(X, Y) \otimes \mathcal{C}(Y, X) \xrightarrow{\mu \circ (1 \otimes \mu)} \mathcal{C}(Y, Z)$$

which is equal to f' via the category axioms for $\mathcal{C}$ and the left square of (6.1). ∎

Using this result, we may now prove:

Proposition 6.2. *Let $\mathcal{W}$ be a right-closed bicategory. Tensors by $W \in \mathcal{W}(x,y)$ are absolute if and only if the 1-cell W admits a left adjoint in $\mathcal{W}$.*

Proof. If W admits a left adjoint, then the data for a tensor by W can be expressed as in Proposition 6.1(b); since these data are clearly preserved by any $\mathcal{W}$-functor, tensors by W are absolute. Conversely, suppose that tensors by W are absolute; we will show that W admits the left dual $[W, I_x] \in \mathcal{W}(y,x)$. The counit ε is the evaluation map $\mathrm{ev}\colon [W,I_x] \otimes W \to I_x$, and it remains only to define the unit.

For each $a \in \mathcal{W}$, we have the $\mathcal{W}$-representation $\mathcal{W}(a,\text{–})\colon \mathcal{W} \to \mathbf{CAT}$ which is closed since $\mathcal{W}$ is right-closed. Thus, by the construction of Proposition 3.7, there is a tensored $\mathcal{W}$-category $\overline{\mathcal{W}(a,\text{–})} = a/\mathcal{W}$ whose objects of extent b are 1-cells $a \to b$, whose hom-objects are $(a/\mathcal{W})(X,Y) = [X,Y]$, and whose tensors are given by $Y \cdot X = Y \otimes X$.

Now, for any 1-cell $Z \in \mathcal{W}(a,b)$, there is a $\mathcal{W}$-functor $[Z,\text{–}]\colon a/\mathcal{W} \to b/\mathcal{W}$ given on objects by $X \mapsto [Z,X]$ and with action $[X,Y] \to [[Z,X],[Z,Y]]$ on hom-objects obtained by transposing the composition map in $a/\mathcal{W}$. Since tensors by W are absolute, they are preserved by $[Z,\text{–}]\colon a/\mathcal{W} \to b/\mathcal{W}$; it follows that the map

$$\theta_{ZX}\colon W \otimes [Z,X] \to [Z, W \otimes X]$$

in $\mathcal{W}(b,z)$ given by transposing $W \otimes \mathrm{ev}\colon W \otimes [Z,X] \otimes Z \to W \otimes X$ is invertible for all $Z \in \mathcal{W}(a,b)$ and $X \in \mathcal{W}(a,x)$. In particular, we have $\theta_{W,I_x}\colon W \otimes [W,I_x] \cong [W, W \otimes I_x]$ and so a unique $\eta\colon I_y \to W \otimes [W,I_x]$ such that $\theta_{VI} \circ \eta$ is the transpose of the morphism $\rho_W \lambda_W\colon I_y \otimes W \to W \to W \otimes I_x$. This condition immediately implies the triangle identity $(W \otimes \varepsilon) \circ (\eta \otimes W) = 1$, and implies the other triangle identity $(\varepsilon \otimes [W,I_x]) \circ ([W,I_x] \otimes \eta) = 1$ after transposing under adjunction and using bifunctoriality of $\otimes$. $\square$

6.2. **Absolute LexProf-tensors.** Using the above result, we may now characterise the absolute-tensored **LexProf**-categories via the construction $\int$ of Proposition 5.5.

Proposition 6.3. *A* **LexProf***-category $\mathcal{C}$ is absolute-tensored if and only if, for all $X \in \mathcal{C}_{\mathbb{B}}$ and all $F\colon \mathbb{A} \to \mathbb{B}$ in* **FL***, there exists $Y \in \mathcal{C}_{\mathbb{A}}$ and a natural isomorphism*

$$\begin{tikzcd} \mathbb{A} \arrow[rr, "F"] \arrow[dr, "\iota_Y"'] & \arrow[d, phantom, "\overset{\upsilon}{\Leftarrow}"] & \mathbb{B} \arrow[dl, "\iota_X"] \\ & \int\mathcal{C} & \end{tikzcd} \tag{6.3}$$

Proof. We write $(\text{–})_*\colon \mathbf{FL}^{\mathrm{op}} \to \mathbf{LexProf}$ for the identity-on-objects homomorphism sending $F\colon \mathbb{A} \to \mathbb{B}$ to the lex profunctor $F_*\colon \mathbb{B} \nrightarrow \mathbb{A}$ with $F_*(b,a) = \mathbb{B}(b,Fa)$. Each F_* has a left adjoint F^* in **LexProf** with $F^*(a,b) = \mathbb{B}(Fa,b)$, and—as all idempotents split in a finitely complete category—the usual analysis of adjunctions of profunctors adapts to show that, within isomorphism, *every* right adjoint 1-cell in **LexProf** arises thus. So by Proposition 6.2, a **LexProf**-category $\mathcal{C}$ is absolute-tensored just when it admits all tensors by 1-cells F_*.

By Proposition 6.1, this is equally to say that, for all $X \in \mathcal{C}_{\mathbb{B}}$ and $F \in \mathbf{FL}(\mathbb{A},\mathbb{B})$, we can find $Y \in \mathcal{C}_{\mathbb{A}}$ and maps $u\colon F_* \to \mathcal{C}(X,Y)$ and $u^*\colon F^* \to \mathcal{C}(Y,X)$ rendering commutative both squares in (6.1). To complete the proof, it suffices to show that the data of u and u^* are equivalent to those of an invertible transformation υ as in (6.3). Now, u comprises a natural family of maps $\mathbb{B}(j, Fi) \to \mathcal{C}(X, Y)(j, i)$; equally, by Yoneda, elements $\upsilon_i \in \mathcal{C}(X, Y)(Fi, i) = \int\mathcal{C}(\iota_X(Fi), \iota_Y(i))$ dinatural in $i \in \mathbb{A}$; or equally, the components of a natural transformation υ as in (6.3). Similar arguments show that giving $u^*\colon F^* \to \mathcal{C}(Y, X)$ is equivalent to giving a natural transformation $\upsilon^*\colon \iota_Y \Rightarrow \iota_X F$, and that commutativity in the two squares of (6.1) is equivalent to the condition that υ and υ^* are mutually inverse. □

We now have all the necessary ingredients to prove:

Theorem 6.4. *The* 2*-functor* $\Gamma\colon$ **PARFL** $\to$ **LexProf-CAT** *of* (5.2) *is an equivalence on hom-categories, and its biessential image comprises the absolute-tensored* **LexProf***-categories. Thus* Γ *exhibits* **PARFL** *as biequivalent to the full and locally full sub-*2*-category of* **LexProf-CAT** *on the absolute-tensored* **LexProf***-categories.*

Proof. By a standard argument, to say that Γ is an equivalence on homs is equally to say that each counit component $\varepsilon_{\mathcal{C}}\colon \int\Gamma\mathcal{C} \to \mathcal{C}$ of (5.2) is an equivalence in **PARFL**. Now, from the definitions, the category $\int\Gamma\mathcal{C}$ has:

- **Objects** being pairs $(X \in \mathcal{S}[\mathbb{A}], i \in \mathbb{A})$;
- **Morphisms** $(X, i) \to (Y, j)$ being maps $Xi \to Yj$ in $\mathcal{C}$;
- **Composition and identities** inherited from $\mathcal{C}$,

while $\varepsilon_{\mathcal{C}}\colon \int\Gamma\mathcal{C} \to \mathcal{C}$ sends (X, i) to Xi and is the identity on homsets. So clearly $\varepsilon_{\mathcal{C}}$ is fully faithful; while condition (iii) for a left-exact sieve ensures that it is essentially surjective, and so an equivalence of categories. However, for $\varepsilon_{\mathcal{C}}$ to be an equivalence in **PARFL**, its pseudoinverse must also be partially finite-limit-preserving. This is easily seen to be equivalent to $\varepsilon_{\mathcal{C}}$ being sieve-reflecting; but for each $X \in \mathcal{S}[\mathbb{A}]$, the functor $\iota_X\colon \mathbb{A} \to \mathcal{D}$ sending i to (X, i) and φ to $X\varphi$ is by definition in the sieve $\mathcal{S}_{\int\Gamma\mathcal{C}}$, and clearly $\varepsilon_{\mathcal{C}} \circ \iota_X = X$.

This shows that Γ is locally an equivalence; as for its biessential image, this comprises just those $\mathcal{C} \in$ **LexProf-CAT** at which the unit $\eta_{\mathcal{C}}\colon \mathcal{C} \to \Gamma(\int\mathcal{C})$ is an equivalence of **LexProf**-categories. Now, from the definitions, $\Gamma(\int\mathcal{C})$ has:

- **Objects** of extent $\mathbb{A}$ being functors $\mathbb{A} \to \int\mathcal{C}$ in the left-exact sieve $\mathcal{S}_{\int\mathcal{C}}$;
- **Hom-objects** given by $\Gamma(\int\mathcal{C})(X, Y) = \int\mathcal{C}(X-, Y-)$;
- **Composition and identities** inherited from $\int\mathcal{C}$,

while the **LexProf**-functor $\eta_{\mathcal{C}}\colon \mathcal{C} \to \Gamma(\int\mathcal{C})$ is given on objects by $X \mapsto \iota_X$, and on homs by the equality $\mathcal{C}(X, Y) = (\int\mathcal{C})(\iota_X-, \iota_Y-)$ of (5.3); in particular, it is always fully faithful. To characterise when it is essentially surjective, note first that isomorphisms in $\Gamma(\int\mathcal{C})_{\mathbb{A}}$ are equally natural isomorphisms in $[\mathbb{A}, \int\mathcal{C}]$. Now as objects of $\Gamma(\int\mathcal{C})_{\mathbb{A}}$ are elements of $\mathcal{S}_{\int\mathcal{C}}[\mathbb{A}]$, and since by (5.4) every such is isomorphic to $\iota_X F$ for some $X \in \mathcal{C}_{\mathbb{B}}$ and $F \in \mathbf{FL}(\mathbb{A}, \mathbb{B})$, we see that $\eta_{\mathcal{C}}$ is essentially surjective precisely when for all $X \in \mathcal{C}_{\mathbb{B}}$ and $F \in \mathbf{FL}(\mathbb{A}, \mathbb{B})$ there exists $Y \in \mathcal{C}_{\mathbb{A}}$ and a natural isomorphism $\upsilon\colon \iota_X F \cong \iota_Y$: which by Proposition 6.3, happens precisely when $\mathcal{C}$ admits all absolute tensors. □

We will use this result in the sequel to freely identify absolute-tensored **LexProf**-categories with partially finitely complete categories; note that, on doing so, the left 2-adjoint $\int\colon$ **LexProf-CAT** $\to$ **PARFL** of (5.2) provides us with a description of the *free completion* of an **LexProf**-category under absolute tensors.

7. Lawvere $\mathcal{A}$-theories and their models

We are now ready to give our **LexProf**-categorical account of Lawvere $\mathcal{A}$-theories and their models. We will identify each Lawvere $\mathcal{A}$-theory with what we term a *Lawvere* **LexProf***-category on* $\mathcal{A}$, and will identify the category of models with a suitable category of **LexProf**-enriched functors.

7.1. **Lawvere $\mathcal{A}$-theories as enriched categories.** Lawvere **LexProf**-categories will involve certain absolute-tensored **LexProf**-categories, which in light of Theorem 6.4, we may work with in the equivalent guise of partially finitely complete categories. In giving the following definition, and throughout the rest of this section, we view the finitely complete ${\mathcal{A}_f}^{\mathrm{op}}$ as being partially finitely complete as in Example 5.2.

Definition 7.1. Let $\mathcal{A}$ be an lfp category. A *Lawvere* **LexProf***-category over* $\mathcal{A}$ comprises a partially finitely complete category $\mathcal{L}$ together with a map $J\colon {\mathcal{A}_f}^{\mathrm{op}} \to \mathcal{L}$ in **PARFL** which is identity-on-objects and sieve-reflecting. A morphism of Lawvere **LexProf**-categories is a commuting triangle in **PARFL**.

Proposition 7.2. *For any locally finitely presentable $\mathcal{A}$, the category of Lawvere $\mathcal{A}$-theories is isomorphic to the category of Lawvere* **LexProf***-categories over $\mathcal{A}$.*

Proof. Any Lawvere $\mathcal{A}$-theory $J\colon {\mathcal{A}_f}^{\mathrm{op}} \to \mathcal{L}$ can be viewed as a Lawvere **LexProf**-category over $\mathcal{A}$ as in Example 5.3; it is moreover clear that under this assignation, maps of Lawvere $\mathcal{A}$-theories correspond bijectively with maps of Lawvere **LexProf**-categories. It remains to show that each Lawvere **LexProf**-category $J\colon {\mathcal{A}_f}^{\mathrm{op}} \to \mathcal{L}$ over $\mathcal{A}$ arises from a Lawvere $\mathcal{A}$-theory. Because in this context, ${\mathcal{A}_f}^{\mathrm{op}}$ is equipped with the maximal left-exact sieve, the fact that J is a morphism in **PARFL** is equivalent to the condition that $J \in \mathcal{S}_{\mathcal{L}}$—so that, in particular, J is finite-limit-preserving. Moreover, the fact of J being sieve-reflecting implies that $\mathcal{S}_{\mathcal{L}}$ must be exactly the left-exact sieve (5.1) generated by J. □

7.2. **Models for Lawvere $\mathcal{A}$-theories as enriched functors.** We now describe how models for a Lawvere $\mathcal{A}$-theory can be understood in **LexProf**-categorical terms. Recall from Section 4.2 that we defined $\mathcal{S}$ to be the **LexProf**-category whose objects of extent $\mathbb{A}$ are finite-limit-preserving functors $\mathbb{A} \to \mathbf{Set}$, and whose hom-objects are given by $\mathcal{S}(X,Y)(i,j) = \mathbf{Set}(Xi, Yj)$. By inspection of Definition 5.4, this is equally the **LexProf**-category $\Gamma(\mathbf{Set})$ when **Set** is seen as partially finitely complete as in Example 5.2.

Proposition 7.3. *For any locally finitely presentable category $\mathcal{A}$, the identification of Lawvere $\mathcal{A}$-theories with Lawvere* **LexProf***-categories on $\mathcal{A}$ fits into a triangle, commuting up to pseudonatural equivalence:*

$$\begin{tikzcd} \mathbf{Law}(\mathcal{A})^{\mathrm{op}} \arrow[rr] \arrow[dr, "(-)\text{-}\mathbf{Mod}"'] & \simeq & (\mathbf{LexProf}\text{-}\mathbf{CAT})^{\mathrm{op}} \arrow[dl, "\mathbf{LexProf}\text{-}\mathbf{CAT}(-{,}\mathcal{S})"] \\ & \mathbf{CAT} & \end{tikzcd}$$

wherein the horizontal functor is that sending a Lawvere $\mathcal{A}$-theory $J\colon {\mathcal{A}_f}^{\mathrm{op}} \to \mathcal{L}$ to the **LexProf***-category $\Gamma(\mathcal{L})$.*

Proof. We observed in Example 5.3 above that, if $J\colon {\mathcal{A}_f}^{\mathrm{op}} \to \mathcal{L}$ is a Lawvere $\mathcal{A}$-theory, then the category $\mathcal{L}$-**Mod** of $\mathcal{L}$-models is isomorphic to the hom-category $\mathbf{PARFL}(\mathcal{L}, \mathbf{Set})$. Since $\Gamma\colon \mathbf{PARFL} \to \mathbf{LexProf\text{-}CAT}$ is an equivalence on homs, and since $\Gamma(\mathbf{Set}) = \mathcal{S}$, we thus obtain the components of the required pseudonatural equivalence as the composites

$$\mathcal{L}\text{-}\mathbf{Mod} \xrightarrow{\cong} \mathbf{PARFL}(\mathcal{L}, \mathbf{Set}) \xrightarrow{\Gamma} \mathbf{LexProf\text{-}CAT}(\Gamma(\mathcal{L}), \mathcal{S}) \ . \qquad \square$$

8. Reconstructing the monad–theory correspondence

We have now done all the hard work necessary to prove our main result.

Theorem 8.1. *The process of freely completing a one-object* **LexProf***-category under absolute tensors induces, by way of the identifications of Propositions 4.2 and 7.2, an equivalence betwen the categories of finitary monads on $\mathcal{A}$ and of Lawvere $\mathcal{A}$-theories. This equivalence fits into a pseudocommuting triangle:*

$$\begin{tikzcd}[column sep=small] \mathbf{Mnd}_f(\mathcal{A})^{\mathrm{op}} \arrow[rr] \arrow[dr, "(-)\text{-}\mathbf{Alg}"'] & \simeq & \mathbf{Law}(\mathcal{A})^{\mathrm{op}} \ . \arrow[dl, "(-)\text{-}\mathbf{Mod}"] \\ & \mathbf{CAT} & \end{tikzcd} \tag{8.1}$$

Proof. To obtain the desired equivalence, it suffices by Propositions 4.2 and 7.2 to construct an equivalence between the category of **LexProf**-categories with a single object of extent $\mathcal{A}_f^{\mathrm{op}}$, and the category of Lawvere **LexProf**-categories on $\mathcal{A}$.

On the one hand, given the one-object **LexProf**-category $\mathcal{T}$, applying the free completion under absolute tensors $\int\colon \mathbf{LexProf\text{-}CAT} \to \mathbf{PARFL}$ to the unique **LexProf**-functor $!\colon \mathcal{I}_{{\mathcal{A}_f}^{\mathrm{op}}} \to \mathcal{T}$ yields a Lawvere **LexProf**-category:

$$J_{\mathcal{T}} = \textstyle\int !\colon {\mathcal{A}_f}^{\mathrm{op}} \to \int \mathcal{T} \ , \tag{8.2}$$

where direct inspection of the definition of $\int$ tells us that $\int \mathcal{I}_{{\mathcal{A}_f}^{\mathrm{op}}} = {\mathcal{A}_f}^{\mathrm{op}}$ and that $\int !$ is identity-on-objects and sieve-reflecting.

On the other hand, if $J\colon {\mathcal{A}_f}^{\mathrm{op}} \to \mathcal{L}$ is a Lawvere **LexProf**-category on $\mathcal{A}$, then we may form the composite around the top and right of the following square, wherein η is a unit component of the 2-adjunction (5.2):

$$\begin{tikzcd} \mathcal{I}_{{\mathcal{A}_f}^{\mathrm{op}}} \arrow[r, "\eta"] \arrow[d, "F"'] & \Gamma({\mathcal{A}_f}^{\mathrm{op}}) \arrow[d, "\Gamma(J)"] \\ \mathcal{T}_J \arrow[r, "G"] & \Gamma(\mathcal{L}) \ . \end{tikzcd} \tag{8.3}$$

We now factorise this composite as (identity-on-objects, fully faithful), as around the left and bottom, to obtain the required one-object **LexProf**-category $\mathcal{T}_J$.

The functoriality of the above assignations is direct; it remains to check that they are inverse to within isomorphism. First, if $\mathcal{T}$ is a one-object **LexProf**-category with associated

Lawvere **LexProf**-category (8.2), then in the naturality square

$$\begin{tikzcd} \mathcal{I}_{\mathcal{A}_f^{\mathrm{op}}} \arrow[r, "\eta"] \arrow[d, "!"'] & \Gamma(\mathcal{A}_f{}^{\mathrm{op}}) \arrow[d, "\Gamma(\int !)"] \\ \mathcal{T} \arrow[r, "\eta"] & \Gamma(\int \mathcal{T}) \end{tikzcd}$$

for η, the left-hand arrow is identity-on-objects, and the bottom fully faithful (by Theorem 6.4). Comparing with (8.3), we conclude by the essential uniqueness of (identity-on-objects, fully faithful) factorisations that $\mathcal{T} \cong \mathcal{T}_{J_\mathcal{T}}$ as required.

Conversely, if $J\colon \mathcal{A}_f{}^{\mathrm{op}} \to \mathcal{L}$ is a Lawvere **LexProf**-category on $\mathcal{A}$ with associated one-object **LexProf**-category $\mathcal{T}_J$ as in (8.3), then we may form the following diagram:

$$\begin{tikzcd} & \mathcal{A}_f{}^{\mathrm{op}} \arrow[dl, "J_{\mathcal{T}_J} = \int F"'] \arrow[dr, "J"] & \\ \int \mathcal{T}_J \arrow[r, "\int G"'] & \int \Gamma(\mathcal{L}) \arrow[r, "\varepsilon"'] & \mathcal{L} \end{tikzcd}$$

where ε is a counit component of (5.2). The composite around the left and bottom is the adjoint transpose of GF under (5.2); but by (8.3), $GF = \Gamma(J) \circ \eta$ which is in turn the adjoint transpose of J. It thus follows that the above triangle commutes. Since both $\int F$ and J are identity-on-objects, so is the horizontal composite; moreover, ε is an equivalence by Theorem 6.4 while $\int G$ is fully faithful since G is, by inspection of the definition of $\int$. So the lower composite is fully faithful and identity-on-objects, whence invertible, so that $J \cong J_{\mathcal{T}_J}$ as required.

We thus have an equivalence as across the top of (8.1), and it remains to show that this renders the triangle below commutative to within pseudonatural equivalence. To this end, consider the diagram

$$\begin{tikzcd} \mathbf{Mnd}_f(\mathcal{A})^{\mathrm{op}} \arrow[rr] \arrow[d] & & \mathbf{Law}(\mathcal{A})^{\mathrm{op}} \arrow[d] \\ \mathbf{LexProf\text{-}CAT}^{\mathrm{op}} \arrow[rr, "\Gamma\int"] \arrow[urr, phantom, "\cong"] \arrow[dr, "\mathbf{LexProf\text{-}CAT}(-,\mathcal{S})"'] & \simeq & \mathbf{LexProf\text{-}CAT}^{\mathrm{op}} \arrow[dl, "\mathbf{LexProf\text{-}CAT}(-,\mathcal{S})"] \\ & \mathbf{CAT}^{\mathrm{op}} & \end{tikzcd}$$

The top square commutes to within isomorphism by our construction of the equivalence $\mathbf{Mnd}_f(\mathbf{Set}) \simeq \mathbf{Law}$; whilst the lower triangle commutes to within pseudonatural equivalence because $\Gamma\int$ is a bireflector of **LexProf**-categories into absolute-tensored **LexProf**-categories, and $\mathcal{S}$ is by definition absolute-tensored. Finally, by Propositions 4.3 and 7.3, the composites down the left and the right are pseudonaturally equivalent to $(-)$-**Alg** and $(-)$-**Mod** respectively. □

The only thing that remains to check is:

Proposition 8.2. *The equivalence constructed in Theorem 8.1 agrees with the equivalence constructed by Nishizawa–Power in* [NP09].

Proof. We prove this by tracing through the steps by which we constructed the equivalence of Theorem 8.1, starting from a finitary monad $S\colon \mathcal{A} \to \mathcal{A}$ in **LFP**.

- We first transport S across the biequivalence $\mathbf{LFP} \simeq \mathbf{LexProf}$ to get a monad of the form $T\colon {\mathcal{A}_f}^{\mathrm{op}} \twoheadrightarrow {\mathcal{A}_f}^{\mathrm{op}}$ in $\mathbf{LexProf}$. From the description of this biequivalence in Section 4.1, the underlying lex profunctor $T\colon \mathcal{A}_f \times {\mathcal{A}_f}^{\mathrm{op}} \to \mathbf{Set}$ is given by $T(i,j) = \mathcal{A}(j, Si)$, while the unit and multiplication of T are induced by postcomposition with those of S.
- We next form the one-object $\mathbf{LexProf}$-category $\mathcal{T}$ which corresponds to T.
- We next construct the Lawvere $\mathbf{LexProf}$-category $J\colon {\mathcal{A}_f}^{\mathrm{op}} \to \int\mathcal{T}$ corresponding to $\mathcal{T}$ by applying $\int$ to the unique $\mathbf{LexProf}$-functor $\mathcal{I}_{{\mathcal{A}_f}^{\mathrm{op}}} \to \mathcal{T}$. From the explicit description of $\int$ in Proposition 5.5, we see that $\int\mathcal{T}$ has the same objects as ${\mathcal{A}_f}^{\mathrm{op}}$, hom-sets $\int\mathcal{T}(i,j) = \mathcal{A}(j, Si)$, and composition as in the Kleisli category of S. Moreover, the functor J is the identity on objects, and given on hom-sets by postcomposition with the unit of S.
- Finally, the Lawvere $\mathcal{A}$-theory associated to this Lawvere $\mathbf{LexProf}$-category is obtained by applying the forgetful functor $\mathbf{PARFL} \to \mathbf{CAT}$; which, comparing the preceding description with the proof of Theorem 3.3, is exactly the Lawvere $\mathcal{A}$-theory associated to the finitary monad $S\colon \mathcal{A} \to \mathcal{A}$. $\square$

References

[AR94] Jiří Adámek and Jiří Rosický. *Locally presentable and accessible categories*, volume 189 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, 1994.

[BC82] R. Betti and A. Carboni. Cauchy-completion and the associated sheaf. *Cahiers de Topologie et Géométrie Différentielle*, 23(3):243–256, 1982.

[FPT99] Marcelo Fiore, Gordon Plotkin, and Daniele Turi. Abstract syntax and variable binding. In *Logic in Computer Science 14 (Trento, 1999)*, pages 193–202. IEEE Computer Society Press, 1999.

[Gar14a] Richard Garner. Diagrammatic characterisation of enriched absolute colimits. *Theory and Applications of Categories*, 29:No. 26, 775–780, 2014.

[Gar14b] Richard Garner. Lawvere theories, finitary monads and Cauchy-completion. *Journal of Pure and Applied Algebra*, 218(11):1973–1988, 2014.

[GP97] R. Gordon and A. J. Power. Enrichment through variation. *Journal of Pure and Applied Algebra*, 120(2):167–185, 1997.

[GU71] Peter Gabriel and Friedrich Ulmer. *Lokal präsentierbare Kategorien*, volume 221 of *Lecture Notes in Mathematics*. Springer-Verlag, 1971.

[Kel82] G. Maxwell Kelly. *Basic concepts of enriched category theory*, volume 64 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, 1982. Republished as: *Reprints in Theory and Applications of Categories 10* (2005).

[Law73] F. William Lawvere. Metric spaces, generalized logic, and closed categories. *Rendiconti del Seminario Matematico e Fisico di Milano*, (43):135–166, 1973. Republished as: *Reprints in Theory and Applications of Categories 1* (2002).

[LP09] Stephen Lack and John Power. Gabriel-Ulmer duality and Lawvere theories enriched over a general base. *Journal of Functional Programming*, 19(3-4):265–286, 2009.

[NP09] Koki Nishizawa and John Power. Lawvere theories enriched over a general base. *Journal of Pure and Applied Algebra*, 213(3):377–386, 2009.

[Str83a] Ross Street. Absolute colimits in enriched categories. *Cahiers de Topologie et Geométrie Différentielle Catégoriques*, 24(4):377–379, 1983.

[Str83b] Ross Street. Enriched categories and cohomology. *Quaestiones Mathematicae*, 6:265–283, 1983. Republished as: *Reprints in Theory and Applications of Categories 14* (2005).

[Wal81] R. F. C. Walters. Sheaves and Cauchy-complete categories. *Cahiers de Topologie et Geométrie Différentielle Catégoriques*, 22(3):283–286, 1981.