

# CAPTURING POLYNOMIAL TIME USING MODULAR DECOMPOSITION

BERIT GRUSSIEN

Humboldt-Universität zu Berlin, Unter den Linden 6, 10099 Berlin, Germany  
*e-mail address:* [grussien@informatik.hu-berlin.de](mailto:grussien@informatik.hu-berlin.de)

**ABSTRACT.** The question of whether there is a logic that captures polynomial time is one of the main open problems in descriptive complexity theory and database theory. In 2010 Grohe showed that fixed-point logic with counting captures polynomial time on all classes of graphs with excluded minors. We now consider classes of graphs with excluded induced subgraphs. For such graph classes, an effective graph decomposition, called modular decomposition, was introduced by Gallai in 1976. The graphs that are non-decomposable with respect to modular decomposition are called *prime*. We present a tool, the Modular Decomposition Theorem, that reduces (definable) canonization of a graph class  $\mathcal{C}$  to (definable) canonization of the class of prime graphs of  $\mathcal{C}$  that are colored with binary relations on a linearly ordered set. By an application of the Modular Decomposition Theorem, we show that fixed-point logic with counting captures polynomial time on the class of permutation graphs. Within the proof of the Modular Decomposition Theorem, we show that the modular decomposition of a graph is definable in symmetric transitive closure logic with counting. We obtain that the modular decomposition tree is computable in logarithmic space. It follows that cograph recognition and cograph canonization is computable in logarithmic space.

## 1. INTRODUCTION

The aim of descriptive complexity theory is to find logics that characterize, or *capture*, complexity classes. The first result in this field was made by Fagin in 1974 [Fag74]. He showed that existential second-order logic captures the complexity class NP. One of the most interesting open problems in descriptive complexity theory is the question of whether there exists a logic that captures PTIME.<sup>1</sup>

Independently of each other, Immerman [Imm86] and Vardi [Var82] obtained an early result towards a logical characterization for PTIME. They proved that fixed-point logic (FP) captures PTIME on ordered structures,<sup>2</sup> that is, on structures where a linear order is present. On structures that are not necessarily ordered, it is easy to prove that FP

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\* This article is an extended version of [Gru17b].

<sup>1</sup> Originally, this question was asked by Chandra and Harel in a study of database query languages [CH82].

<sup>2</sup> More precisely, Immerman and Vardi's theorem holds for least fixed-point logic (LFP) and the equally expressive inflationary fixed-point logic (IFP). Our indeterminate FP refers to either of these two logics.

does not capture PTIME. In order to obtain a candidate for a logic capturing PTIME on all structures, Immerman proposed in 1987 to add to FP the ability to count [Imm87a]. Although the resulting logic, fixed-point logic with counting (FP+C), is not strong enough to capture PTIME on all finite structures [CFI92], it does so on many interesting classes of structures: FP+C captures PTIME, for example, on planar graphs [Gro98], all classes of graphs of bounded treewidth [GM99] and on  $K_5$ -minor free graphs [Gro08]. Note that all these classes can be defined by a list of forbidden minors. In fact, Grohe showed in 2010 that FP+C captures PTIME on all graph classes with excluded minors [Gro10b]. This leads to the question whether a similar result can be obtained for graph classes that are characterized by excluded induced subgraphs, i.e., graph classes that are closed under taking induced subgraphs. For FP+C such a general result is not possible: Capturing PTIME on the class of chordal graphs, comparability graphs or co-comparability graphs is as hard as capturing PTIME on the class of all graphs for any “reasonable” logic [Gro10a, Lau11]. Yet, this gives us reason to consider the three mentioned graph classes and their subclasses more closely. So far, there are results showing that FP+C captures PTIME on the class of chordal line graphs [Gro10a] and on the class of interval graphs (chordal co-comparability graphs) [Lau10].

We add to these results the following results: FP+C captures PTIME on the class of permutation graphs (comparability co-comparability graphs) (see Section 5) and on the class of chordal comparability graphs (see [Gru17a]). Both results are based on modular decomposition (also called substitution decomposition), a graph decomposition which was introduced by Gallai in 1976 [Gal67]. Similar to treelike decomposition for classes with forbidden minors, modular decomposition is a suitable efficient graph decomposition for classes with forbidden induced subgraphs.

The modular decomposition of a graph partitions the vertex set of the graph into so called modules, that is, into subsets that share the same neighbors. A graph is *prime* if only the vertex set itself and all vertex sets of size 1 are modules of the graph. For every class  $\mathcal{C}$  of graphs that is closed under induced subgraphs, we let  $\mathcal{C}_{\text{prim}}^*$  be the class of all prime graphs from  $\mathcal{C}$  that are colored with binary relations on a linearly ordered set. Our Modular Decomposition Theorem states that there is an FP+C-canonization of  $\mathcal{C}$  if there is an FP+C-canonization of the class  $\mathcal{C}_{\text{prim}}^*$ . Note that the Modular Decomposition Theorem also holds for reasonable extensions of FP+C that are closed under FP+C-transductions.

The Modular Decomposition Theorem can be used for multiple purposes. One reason for this is that the existence of an FP+C-canonization of a graph class  $\mathcal{C}$  has various consequences. It implies that FP+C captures PTIME on class  $\mathcal{C}$ , the existence of a polynomial-time canonization algorithm for graph class  $\mathcal{C}$ , and that there is an easy-to-understand algorithm, the Weisfeiler-Leman Method, that solves the graph isomorphism problem on  $\mathcal{C}$ . Further, the Modular Decomposition Theorem itself can be transferred to polynomial time: There is a polynomial-time canonization algorithm for  $\mathcal{C}$  if there is a polynomial-time canonization algorithm for the class  $\mathcal{C}_{\text{prim}}^*$ . We will present even more variations of the Modular Decomposition Theorem that might be helpful for future applications.

By means of the Modular Decomposition Theorem, we can not only show that the canonization of the class of permutation graphs and chordal comparability graphs is definable in FP+C, but simplify the proof of Laubner in [Lau10] that there is an FP+C-canonization of the class of interval graphs.

Within the proof of the Modular Decomposition Theorem, we show that the modular decomposition of a graph is definable in symmetric transitive closure logic with counting.

As a consequence, the modular decomposition tree can be computed in logarithmic space. Previously, it was only known that the modular decomposition tree is computable in linear time [CH94, MS94], or in polylogarithmic time with a linear number of processors [Dah95].<sup>3</sup> It follows directly that cograph recognition is in LOGSPACE. As there is a logarithmic-space algorithm for tree canonization [Lin92], it also follows that there exists a logarithmic-space algorithm for cograph canonization.

**Structure.** After setting out the necessary preliminaries in Section 2, we introduce modular decomposition and show that it is **STC+C**-definable, and therefore LOGSPACE-computable, in Section 3. In Section 4, we introduce the Modular Decomposition Theorem, we prove it, and we present variations of it. Finally, we apply (a variation of) the Modular Decomposition Theorem in Section 5 and show that **FP+C** captures PTIME on the class of permutation graphs. We close with a few concluding remarks.

## 2. BASIC DEFINITIONS AND NOTATION

We write  $\mathbb{N}$  for the set of all non-negative integers. For all  $n, n' \in \mathbb{N}$ , we define  $[n, n'] := \{m \in \mathbb{N} \mid n \leq m \leq n'\}$  and  $[n] := [1, n]$ . We often denote tuples  $(a_1, \dots, a_k)$  by  $\bar{a}$ . Given a tuple  $\bar{a} = (a_1, \dots, a_k)$ , let  $\tilde{a} := \{a_1, \dots, a_k\}$ . Let  $n \geq 1$ , and  $\bar{a}^i = (a_1^i, \dots, a_{k_i}^i)$  be a tuple of length  $k_i$  for each  $i \in [n]$ . We denote the tuple  $(a_1^1, \dots, a_{k_1}^1, \dots, a_1^l, \dots, a_{k_l}^l)$  by  $(\bar{a}^1, \dots, \bar{a}^l)$ . Mappings  $f: A \rightarrow B$  are extended to tuples  $\bar{a} = (a_1, \dots, a_k)$  over  $A$  via  $f(\bar{a}) := (f(a_1), \dots, f(a_k))$ .

For a set  $S$ , we let  $\mathcal{P}(S)$  be the set of all subsets of  $S$  and  $\binom{S}{2}$  be the set of all 2-element subsets of  $S$ . If  $\mathcal{D}$  is a set of sets, then we let  $\bigcup \mathcal{D}$  be the union of all sets in  $\mathcal{D}$ . The *disjoint union* of two sets  $S$  and  $S'$  is denoted by  $S \dot{\cup} S'$ . A *partition* of a set  $S$  is a set  $\mathcal{D}$  of disjoint non-empty subsets of  $S$  such that  $S = \bigcup \mathcal{D}$ .

**2.1. Relations and Orders.** The reflexive, symmetric, transitive closure of a binary relation  $R$  on  $U$  is called the equivalence relation *generated* by  $R$  on  $U$ . Let  $\approx$  be an equivalence relation on  $U$ . For each  $a \in U$ , we denote the equivalence class of  $a$  by  $a/\approx$ . (We also use another notation, which we specify later.) We let  $U/\approx$  be the set of equivalence classes. For a  $k$ -ary relation  $R \subseteq U^k$  we let  $R/\approx$  be the set  $\{(a_1/\approx, \dots, a_k/\approx) \mid (a_1, \dots, a_k) \in R\}$ .

A binary relation  $\prec$  on a set  $U$  is a *strict partial order* if it is irreflexive and transitive. We say  $a$  and  $b$  are *comparable* with respect to a strict partial order  $\prec$  if  $a \prec b$  or  $b \prec a$ ; otherwise we call them *incomparable*. A strict partial order where no two elements  $a, b$  with  $a \neq b$  are incomparable is called a *strict linear order*. For each strict linear order  $\prec$  there exists an associated reflexive relation  $\preceq$ , called a *linear order*, which is defined by  $a \preceq b$  if and only if  $a \prec b$  or  $a = b$ . A binary relation  $\preceq$  is a linear order if and only if it is transitive, antisymmetric and total.

A *strict weak order* is a strict partial order where incomparability is transitive. Moreover, in a strict weak order incomparability is an equivalence relation. If  $a$  and  $a'$  are incomparable with respect to a strict weak order  $\prec$ , then  $a \prec b$  implies  $a' \prec b$ , and  $b \prec a$  implies  $b \prec a'$ . As a consequence, if  $\prec$  is a strict weak order on  $U$  and  $\sim$  is the equivalence relation defined by incomparability, then  $\prec$  induces a strict linear order on the set  $U/\sim$  of equivalence classes.

<sup>3</sup> For a survey of the algorithmic aspects of modular decomposition see [HP10].

**2.2. Graphs and LO-Colored Graphs.** A *graph* is a pair  $(V, E)$  consisting of a non-empty finite set  $V$  of *vertices* and a set  $E \subseteq \binom{V}{2}$  of *edges*.

Let  $G = (V, E)$  be a graph. For a subset  $W \subseteq V$  of vertices,  $G[W]$  denotes the *induced subgraph* of  $G$  with vertex set  $W$ . The *complement graph* of  $G$  is the graph  $\overline{G} := (V, \overline{E})$  where  $\overline{E} = \binom{V}{2} \setminus E$ . *Connectivity* and *connected components* are defined the usual way.

Let  $G = (V, E)$  be a graph and  $f: V \rightarrow C$  be a mapping from the vertices of  $G$  to a finite set  $C$ . Then  $f$  is a *coloring* of  $G$ , and the elements of  $C$  are called *colors*. Throughout this paper we color the vertices of a graph with binary relations on a linearly ordered set.<sup>4</sup> We call graphs with such a coloring *LO-colored graphs*. More precisely, an LO-colored graph is a tuple  $G^* = (V, E, M, \preceq, L)$  with the following properties:

- (1) The pair  $(V, E)$  is a graph. We call  $(V, E)$  the *underlying graph* of  $G^*$ .
- (2) The set of *basic color elements*  $M$  is a non-empty finite set with  $M \cap V = \emptyset$ .
- (3) The binary relation  $\preceq \subseteq M^2$  is a linear order on  $M$ .
- (4) The *color relation*  $L \subseteq V \times M^2$  is a ternary relation that assigns to each vertex  $v \in V$  a color  $L_v := \{(d, d') \mid (v, d, d') \in L\}$ .

Let  $d_0, \dots, d_{|M|-1}$  be the enumeration of the basic color elements in  $M$  according to their linear order  $\preceq$ . We call  $L_v^{\mathbb{N}} := \{(i, j) \in \mathbb{N}^2 \mid (d_i, d_j) \in L_v\}$  the *natural color* of  $v \in V$ .

We can use the linear order  $\preceq$  on  $M$  to obtain a linear order on the colors  $\{L_v \mid v \in V\}$  of  $G^*$ . Thus, an LO-colored graph is a special kind of colored graph with a linear order on its colors.

**2.3. Structures.** A *vocabulary* is a finite set  $\tau$  of relation symbols. Each relation symbol  $R \in \tau$  has a fixed arity  $\text{ar}(R) \in \mathbb{N}$ . A  $\tau$ -*structure* consists of a non-empty finite set  $U(A)$ , its *universe*, and for each relation symbol  $R \in \tau$  of a relation  $R(A) \subseteq U(A)^{\text{ar}(R)}$ .

An *isomorphism* between  $\tau$ -structures  $A$  and  $B$  is a bijection  $f: U(A) \rightarrow U(B)$  such that for all  $R \in \tau$  and all  $\bar{a} \in U(A)^{\text{ar}(R)}$  we have  $\bar{a} \in R(A)$  if and only if  $f(\bar{a}) \in R(B)$ . We write  $A \cong B$  to indicate that  $A$  and  $B$  are *isomorphic*.

Let  $E$  be a binary relation symbol. Each graph corresponds to an  $\{E\}$ -structure  $G = (V, E)$  where the universe  $V$  is the vertex set and  $E$  is an irreflexive and symmetric binary relation, the edge relation. To represent an LO-colored graph  $G^* = (V, E, M, \preceq, L)$  as a logical structure we extend the 5-tuple by a set  $U$  to a 6-tuple  $(U, V, E, M, \preceq, L)$ , and we require that  $U = V \dot{\cup} M$  additionally to properties 1-4. The set  $U$  serves as the universe of the structure, and  $V, E, M, \preceq, L$  are relations on  $U$ . We usually do not distinguish between (LO-colored) graphs and their representation as logical structures. It will be clear from the context which form we are referring to.

**2.4. Logics.** In this section we introduce first-order logic with counting, symmetric transitive closure logic (with counting) and fixed-point logic (with counting). Detailed introductions of these logics can be found, e.g., in [EF99, Gro13, Imm87b]. We assume basic knowledge in logic, in particular of *first-order logic* (FO).

*First-order logic with counting* (FO+C) extends FO by a counting operator that allows for counting the cardinality of FO+C-definable relations. It lives in a two-sorted context, where structures  $A$  are equipped with a *number sort*  $N(A) := [0, |U(A)|]$ . FO+C has two types

<sup>4</sup> In particular, we color graphs with representations of ordered copies of graphs on the number sort (defined in Section 4.2).

of *individual variables*: **FO+C**-variables are either *structure variables* that range over the universe  $U(A)$  of a structure  $A$ , or *number variables* that range over the number sort  $N(A)$ . For each individual variable  $u$ , let  $A^u := U(A)$  if  $u$  is a structure variable, and  $A^u := N(A)$  if  $u$  is a number variable. Let  $A^{(u_1, \dots, u_k)} := A^{u_1} \times \dots \times A^{u_k}$ . Tuples  $(u_1, \dots, u_k)$  and  $(v_1, \dots, v_\ell)$  of variables are *compatible* if  $k = \ell$ , and for every  $i \in [k]$  the variables  $u_i$  and  $v_i$  have the same type. An *assignment in  $A$*  is a mapping  $\alpha$  from the set of variables to  $U(A) \cup N(A)$ , where for each variable  $u$  we have  $\alpha(u) \in A^u$ . For tuples  $\bar{u} = (u_1, \dots, u_k)$  of variables and  $\bar{a} = (a_1, \dots, a_k) \in A^{\bar{u}}$ , the assignment  $\alpha[\bar{a}/\bar{u}]$  maps  $u_i$  to  $a_i$  for each  $i \in [k]$ , and each variable  $v \notin \bar{u}$  to  $\alpha(v)$ . By  $\varphi(u_1, \dots, u_k)$  we denote a formula  $\varphi$  with  $\text{free}(\varphi) \subseteq \{u_1, \dots, u_k\}$ , where  $\text{free}(\varphi)$  is the set of free variables in  $\varphi$ . Given a formula  $\varphi(u_1, \dots, u_k)$ , a structure  $A$  and  $(a_1, \dots, a_k) \in A^{(u_1, \dots, u_k)}$ , we write  $A \models \varphi[a_1, \dots, a_k]$  if  $\varphi$  holds in  $A$  with  $u_i$  assigned to  $a_i$  for each  $i \in [k]$ . We write  $\varphi[A, \alpha; \bar{u}]$  for the set of all tuples  $\bar{a} \in A^{\bar{u}}$  with  $(A, \alpha[\bar{a}/\bar{u}]) \models \varphi$ . For a formula  $\varphi(\bar{u})$  (with  $\text{free}(\varphi) \subseteq \bar{u}$ ) we also denote  $\varphi[A, \alpha; \bar{u}]$  by  $\varphi[A; \bar{u}]$ , and for a formula  $\varphi(\bar{v}, \bar{u})$  and  $\bar{a} \in A^{\bar{v}}$ , we denote  $\varphi[A, \alpha[\bar{a}/\bar{v}]; \bar{u}]$  also by  $\varphi[A, \bar{a}; \bar{u}]$ .

**FO+C** is obtained by extending **FO** with the following formula formation rules:

- $\phi := p \leq q$  is a formula if  $p, q$  are number variables. We let  $\text{free}(\phi) := \{p, q\}$ .
- $\phi' := \#\bar{u} \psi = \bar{p}$  is a formula if  $\psi$  is a formula,  $\bar{u}$  is a tuple of individual variables and  $\bar{p}$  a tuple of number variables. We let  $\text{free}(\phi') := (\text{free}(\psi) \setminus \bar{u}) \cup \bar{p}$ .

To define the semantics, let  $A$  be a structure and  $\alpha$  be an assignment. We let

- $(A, \alpha) \models p \leq q$  iff  $\alpha(p) \leq \alpha(q)$ ,
- $(A, \alpha) \models \#\bar{u} \psi = \bar{p}$  iff  $|\psi[A, \alpha; \bar{u}]| = \langle \alpha(\bar{p}) \rangle_A$ ,

where for tuples  $\bar{n} = (n_1, \dots, n_k) \in N(A)^k$  we let  $\langle \bar{n} \rangle_A$  be the number

$$\langle \bar{n} \rangle_A := \sum_{i=1}^k n_i \cdot (|U(A)| + 1)^{i-1}.$$

*Symmetric transitive closure logic (with counting)* **STC(+C)** is an extension of **FO(+C)** with stc-operators. The set of all **STC(+C)**-formulas is obtained by extending the formula formation rules of **FO(+C)** by the following rule:

- $\phi := [\text{stc}_{\bar{u}, \bar{v}} \psi](\bar{u}', \bar{v}')$  is a formula if  $\psi$  is a formula and  $\bar{u}, \bar{v}, \bar{u}', \bar{v}'$  are compatible tuples of structure (or number) variables. We let  $\text{free}(\phi) := \bar{u}' \cup \bar{v}' \cup (\text{free}(\psi) \setminus (\bar{u} \cup \bar{v}))$ .

Let  $A$  be a structure and  $\alpha$  be an assignment. We let

- $(A, \alpha) \models [\text{stc}_{\bar{u}, \bar{v}} \psi](\bar{u}', \bar{v}')$  iff  $(\alpha(\bar{u}'), \alpha(\bar{v}'))$  is contained in the symmetric transitive closure of  $\psi[A, \alpha; \bar{u}, \bar{v}]$ .

*(Inflationary) fixed-point logic (with counting)* **FP(+C)** is an extension of **FO(+C)** with atomic second order formulas and ifp-operators. **FP(+C)** has a further type of variables: *relational variables*. A relational variable  $X$  of arity  $k$  ranges over relations  $R \subseteq W_1 \times \dots \times W_k$  where  $W_i = U(A)$  (or  $W_i = N(A)$ ) for all  $i \in [k]$ . We let  $A^X := \mathcal{P}(W_1 \times \dots \times W_k)$ . We say a relational variable  $X$  and a tuple  $\bar{u}$  of individual variables are *compatible* if  $A^{\bar{u}} \in A^X$ . We extend the formula formation rules of **FO(+C)** by the following two rules:

- $\phi := X\bar{u}$  is a formula if  $X$  is a relational variable and  $\bar{u}$  is a tuple of structure (or number) variables such that  $X$  and  $\bar{u}$  are compatible. We let  $\text{free}(\phi) := \bar{u} \cup \{X\}$ .
- $\phi' := [\text{ifp}_{X, \bar{u}} \psi]\bar{u}'$  is a formula if  $\psi$  is a formula, and  $X$  is a relational variable,  $\bar{u}, \bar{u}'$  are tuples of structure (or number) variables such that  $X, \bar{u}, \bar{u}'$  are compatible. We let  $\text{free}(\phi') := \bar{u}' \cup (\text{free}(\psi) \setminus (\bar{u} \cup \{X\}))$ .

Let  $A$  be a structure and  $\alpha$  be an assignment. We let

- $(A, \alpha) \models X\bar{u}$  iff  $\alpha(\bar{u}) \in \alpha(X)$ ,
- $(A, \alpha) \models [\text{ifp}_{X, \bar{u}} \psi]\bar{u}'$  iff  $\alpha(\bar{u}') \in F_\infty$ ,

where  $F_\infty$  is defined as follows: Let  $F: A^X \rightarrow A^X$  be the mapping defined by  $F(R) := R \cup \psi[A, \alpha[R/X]; \bar{u}]$  for all  $R \in A^X$ . We let  $F_0 := \emptyset$  and  $F_{i+1} := F(F_i)$  for all  $i \geq 0$ . Let  $m \geq 0$  be such that  $F_m = F_{m+1}$ . Then  $F_\infty := F_m$ .

We also use the property that *simultaneous inflationary fixed-point logic* has the same expressive power as FP. For the syntax and semantics of this logic we refer the reader to [Gro13] or [EF99].

For logics  $L, L'$  we write  $L \leq L'$  if  $L$  is semantically contained in  $L'$ . We have  $\text{STC} \leq \text{FP}$  and  $\text{STC+C} \leq \text{FP+C}$ . Note that simple arithmetics like addition and multiplication are definable in  $\text{STC+C}$ .

**2.5. Transductions.** Transductions (also known as *syntactical interpretations*) define certain structures within other structures. More on transductions can be found in [Gro13, Gru17a]. In this section, we introduce transductions, consider compositions of transductions, and present the new notion of counting transductions.

In the following we introduce parameterized transductions for  $\text{FP+C}$ . As parameter variables of these transductions, we allow individual variables as well as relational variables. The domain variables are individual variables.

**Definition 2.1** (Parameterized  $\text{FP+C}$ -Transduction). Let  $\tau_1, \tau_2$  be vocabularies.

- (1) A *parameterized  $\text{FP+C}[\tau_1, \tau_2]$ -transduction* is a tuple

$$\Theta(\bar{X}) = \left( \theta_{\text{dom}}(\bar{X}), \theta_U(\bar{X}, \bar{u}), \theta_{\approx}(\bar{X}, \bar{u}, \bar{u}'), (\theta_R(\bar{X}, \bar{u}_{R,1}, \dots, \bar{u}_{R, \text{ar}(R)}))_{R \in \tau_2} \right)$$

of  $\text{FP+C}[\tau_1]$ -formulas, where  $\bar{X}$  is a tuple of individual or relational variables, and  $\bar{u}, \bar{u}'$  and  $\bar{u}_{R,i}$  for every  $R \in \tau_2$  and  $i \in [\text{ar}(R)]$  are compatible tuples of individual variables.

- (2) The *domain* of  $\Theta(\bar{X})$  is the class  $\text{Dom}(\Theta(\bar{X}))$  of all pairs  $(A, \bar{P})$  such that  $A \models \theta_{\text{dom}}[\bar{P}]$  and  $\theta_U[A, \bar{P}; \bar{u}]$  is not empty, where  $A$  is a  $\tau_1$ -structure and  $\bar{P} \in A^{\bar{X}}$ . The variables occurring in tuple  $\bar{X}$  are called *parameter variables*, and the ones occurring in  $\bar{u}$  are referred to as *domain variables*. The elements in  $\bar{P}$  are called *parameters*.
- (3) Let  $(A, \bar{P})$  be in the domain of  $\Theta(\bar{X})$ . We define a  $\tau_2$ -structure  $\Theta[A, \bar{P}]$  as follows. Let  $\approx$  be the equivalence relation generated by  $\theta_{\approx}[A, \bar{P}; \bar{u}, \bar{u}']$  on  $A^{\bar{u}}$ . We let

$$U(\Theta[A, \bar{P}]) := \theta_U[A, \bar{P}; \bar{u}] / \approx$$

be the universe of  $\Theta[A, \bar{P}]$ . Further, for each  $R \in \tau_2$ , we let

$$R(\Theta[A, \bar{P}]) := \left( \theta_R[A, \bar{P}; \bar{u}_{R,1}, \dots, \bar{u}_{R, \text{ar}(R)}] \cap \theta_U[A, \bar{P}; \bar{u}]^{\text{ar}(R)} \right) / \approx.$$

A parameterized  $\text{FP+C}[\tau_1, \tau_2]$ -transduction defines a parameterized mapping from  $\tau_1$ -structures into  $\tau_2$ -structures via  $\text{FP+C}[\tau_1]$ -formulas. If  $\theta_{\text{dom}} := \top$  or  $\theta_{\approx} := \perp$ , we omit the respective formula in the presentation of the transduction.

A parameterized  $\text{FP+C}[\tau_1, \tau_2]$ -transduction  $\Theta(\bar{X})$  is an  $\text{FP+C}[\tau_1, \tau_2]$ -transduction if  $\bar{X}$  is the empty tuple. Let  $\bar{X}$  be the empty tuple. For simplicity, we denote a transduction  $\Theta(\bar{X})$  by  $\Theta$ , and we write  $A \in \text{Dom}(\Theta)$  if  $(A, \bar{X})$  is contained in the domain of  $\Theta$ .

Let  $\mathcal{C}_1$  be a class of  $\tau_1$ -structures and  $\mathcal{C}_2$  be a class of  $\tau_2$ -structures. We call a mapping  $f$  from  $\mathcal{C}_1$  to  $\mathcal{C}_2$  *FP+C-definable*, if there exists an  $\text{FP+C}[\tau_1, \tau_2]$ -transduction  $\Theta$  such that  $\mathcal{C}_1 \subseteq \text{Dom}(\Theta)$  and for all  $\tau_1$ -structures  $A \in \mathcal{C}_1$  we have  $f(A) = \Theta[A]$ .

An important property of  $\text{FP+C}[\tau_1, \tau_2]$ -transductions is that, they allow to *pull back*  $\tau_2$ -formulas, which means that for each  $\tau_2$ -formula there exists an  $\tau_1$ -formula that expresses essentially the same. This property is the core of the Transduction Lemma. A proof of the Transduction Lemma can be found in [Gru17a].

**Proposition 2.2** (Transduction Lemma). *Let  $\tau_1, \tau_2$  be vocabularies. Let  $\Theta(\bar{X})$  be a parameterized  $\text{FP+C}[\tau_1, \tau_2]$ -transduction, where  $\ell$ -tuple  $\bar{u}$  is the tuple of domain variables. Further, let  $\psi(x_1, \dots, x_\kappa, p_1, \dots, p_\lambda)$  be an  $\text{FP+C}[\tau_2]$ -formula where  $x_1, \dots, x_\kappa$  are structure variables and  $p_1, \dots, p_\lambda$  are number variables. Then there exists an  $\text{FP+C}[\tau_1]$ -formula  $\psi^{-\Theta}(\bar{X}, \bar{u}_1, \dots, \bar{u}_\kappa, \bar{q}_1, \dots, \bar{q}_\lambda)$ , where  $\bar{u}_1, \dots, \bar{u}_\kappa$  are compatible with  $\bar{u}$  and  $\bar{q}_1, \dots, \bar{q}_\lambda$  are  $\ell$ -tuples of number variables, such that for all  $(A, \bar{P}) \in \text{Dom}(\Theta(\bar{X}))$ , all  $\bar{a}_1, \dots, \bar{a}_\kappa \in A^{\bar{u}}$  and all  $\bar{n}_1, \dots, \bar{n}_\lambda \in N(A)^\ell$ ,*

$$\begin{aligned} A \models \psi^{-\Theta}[\bar{P}, \bar{a}_1, \dots, \bar{a}_\kappa, \bar{n}_1, \dots, \bar{n}_\lambda] &\iff \bar{a}_1/\approx, \dots, \bar{a}_\kappa/\approx \in U(\Theta[A, \bar{P}]), \\ &\langle \bar{n}_1 \rangle_A, \dots, \langle \bar{n}_\lambda \rangle_A \in N(\Theta[A, \bar{P}]) \text{ and} \\ &\Theta[A, \bar{P}] \models \psi[\bar{a}_1/\approx, \dots, \bar{a}_\kappa/\approx, \langle \bar{n}_1 \rangle_A, \dots, \langle \bar{n}_\lambda \rangle_A], \end{aligned}$$

where  $\approx$  is the equivalence relation generated by  $\theta_{\approx}[A, \bar{P}; \bar{u}, \bar{u}']$  on  $A^{\bar{u}}$ .

The following proposition shows that the composition of a parameterized transduction and a transduction is again a parameterized transduction.

**Proposition 2.3** [Gru17a]. *Let  $\tau_1, \tau_2$  and  $\tau_3$  be vocabularies. Let  $\Theta_1(\bar{X})$  be a parameterized  $\text{FP+C}[\tau_1, \tau_2]$ -transduction and  $\Theta_2$  be an  $\text{FP+C}[\tau_2, \tau_3]$ -transduction. Then there exists a parameterized  $\text{FP+C}[\tau_1, \tau_3]$ -transduction  $\Theta(\bar{X})$  such that for all  $\tau_1$ -structures  $A$  and all  $\bar{P} \in A^{\bar{X}}$ ,*

$$(A, \bar{P}) \in \text{Dom}(\Theta(\bar{X})) \iff (A, \bar{P}) \in \text{Dom}(\Theta_1(\bar{X})) \text{ and } \Theta_1[A, \bar{P}] \in \text{Dom}(\Theta_2),$$

and for all  $(A, \bar{P}) \in \text{Dom}(\Theta(\bar{X}))$ ,

$$\Theta[A, \bar{P}] \cong \Theta_2[\Theta_1[A, \bar{P}]].$$

In the following we introduce the new notion of parameterized counting transductions. The universe of the structure  $\Theta^\#[A, \bar{P}]$  defined by a parameterized counting transduction  $\Theta^\#(\bar{X})$  automatically includes the number sort  $N(A)$  of  $A$ , for all structures  $A$  and tuples  $\bar{P}$  of parameters from the domain of  $\Theta^\#(\bar{X})$ . Parameterized counting transductions are as powerful as parameterized transductions. Presenting a parameterized counting transduction instead of a parameterized transduction will contribute to a clearer presentation.

**Definition 2.4** (Parameterized FP+C-Counting Transduction). Let  $\tau_1, \tau_2$  be vocabularies.

(1) A *parameterized  $\text{FP+C}[\tau_1, \tau_2]$ -counting transduction* is a tuple

$$\Theta^\#(\bar{X}) = \left( \theta_{\text{dom}}^\#(\bar{X}), \theta_U^\#(\bar{X}, \bar{u}), \theta_{\approx}^\#(\bar{X}, \bar{u}, \bar{u}'), (\theta_R^\#(\bar{X}, \bar{u}_{R,1}, \dots, \bar{u}_{R, \text{ar}(R)}))_{R \in \tau_2} \right)$$

of  $\text{FP+C}[\tau_1]$ -formulas, where  $\bar{X}$  is a tuple of individual or relational variables,  $\bar{u}, \bar{u}'$  are compatible tuples of individual variables but not tuples of number variables of length 1, and for every  $R \in \tau_2$  and  $i \in [\text{ar}(R)]$ ,  $\bar{u}_{R,i}$  is a tuple of variables that is compatible to  $\bar{u}$  or a tuple of number variables of length 1.

- (2) The *domain* of  $\Theta^\#(\bar{X})$  is the class  $\text{Dom}(\Theta^\#(\bar{X}))$  of all pairs  $(A, \bar{P})$  where  $A$  is a  $\tau_1$ -structure,  $\bar{P} \in A^{\bar{X}}$  and  $A \models \theta_{\text{dom}}^\#[\bar{P}]$ .
- (3) Let  $(A, \bar{P})$  be in the domain of  $\Theta^\#(\bar{X})$ . We define a  $\tau_2$ -structure  $\Theta^\#[A, \bar{P}]$  as follows. Let  $\approx$  be the equivalence relation generated by  $\theta_{\approx}^\#[A, \bar{P}; \bar{u}, \bar{u}']$  on the set  $A^{\bar{u}} \dot{\cup} N(A)$ . We let

$$U(\Theta^\#[A, \bar{P}]) := (\theta_U^\#[A, \bar{P}; \bar{u}] \dot{\cup} N(A)) /_{\approx}$$

be the universe of  $\Theta^\#[A, \bar{P}]$ . Further, for each  $R \in \tau_2$ , we let

$$R(\Theta^\#[A, \bar{P}]) := \left( \theta_R^\#[A, \bar{P}; \bar{u}_{R,1}, \dots, \bar{u}_{R,\text{ar}(R)}] \cap (\theta_U^\#[A, \bar{P}; \bar{u}] \dot{\cup} N(A))^{\text{ar}(R)} \right) /_{\approx}.$$

**Proposition 2.5** [Gru17a]. *Let  $\tau_1, \tau_2$  be vocabularies. Let  $\Theta^\#(\bar{X})$  be a parameterized  $\text{FP+C}[\tau_1, \tau_2]$ -counting transduction. Then there exists a parameterized  $\text{FP+C}[\tau_1, \tau_2]$ -transduction  $\Theta(\bar{X})$  such that*

- $\text{Dom}(\Theta(\bar{X})) = \text{Dom}(\Theta^\#(\bar{X}))$  and
- $\Theta[A, \bar{P}] \cong \Theta^\#[A, \bar{P}]$  for all  $(A, \bar{P}) \in \text{Dom}(\Theta(\bar{X}))$ .

**2.6. Canonization.** In this section we introduce ordered structures, (definable) canonization and the capturing of PTIME. A more detailed introduction can be found in [Gro13] and [EF99].

Let  $\tau$  be a vocabulary with  $\leq \notin \tau$ . A  $\tau \cup \{\leq\}$ -structure  $A'$  is *ordered* if the relation symbol  $\leq$  is interpreted as a linear order on the universe of  $A'$ . Let  $A$  be a  $\tau$ -structure. An ordered  $\tau \cup \{\leq\}$ -structure  $(A', \leq_{A'})$  is an *ordered copy* of  $A$  if  $A' \cong A$ . Let  $\mathcal{C}$  be a class of  $\tau$ -structures. A mapping  $f$  is a *canonization mapping* of  $\mathcal{C}$  if it assigns every structure  $A \in \mathcal{C}$  to an ordered copy  $f(A) = (A_f, \leq_{A_f})$  of  $A$  such that for all structures  $A, B \in \mathcal{C}$  we have  $f(A) \cong f(B)$  if  $A \cong B$ . We call the ordered structure  $f(A)$  the *canon* of  $A$ .

Let  $\Theta(\bar{x})$  be a parameterized  $\text{FP+C}[\tau, \tau \cup \{\leq\}]$ -transduction, where  $\bar{x}$  is a tuple of individual variables. We say  $\Theta(\bar{x})$  *canonizes* a  $\tau$ -structure  $A$  if there exists a tuple  $\bar{p} \in A^{\bar{x}}$  such that  $(A, \bar{p}) \in \text{Dom}(\Theta(\bar{x}))$ , and for all tuples  $\bar{p} \in A^{\bar{x}}$  with  $(A, \bar{p}) \in \text{Dom}(\Theta(\bar{x}))$ , the  $\tau \cup \{\leq\}$ -structure  $\Theta[A, \bar{p}]$  is an ordered copy of  $A$ .<sup>5</sup> A (parameterized)  $\text{FP+C}$ -*canonization* of a class  $\mathcal{C}$  of  $\tau$ -structures is a (parameterized)  $\text{FP+C}[\tau, \tau \cup \{\leq\}]$ -transduction that canonizes all  $A \in \mathcal{C}$ . A class  $\mathcal{C}$  of  $\tau$ -structures *admits  $\text{FP+C}$ -definable (parameterized) canonization* if  $\mathcal{C}$  has a (parameterized)  $\text{FP+C}$ -canonization.

The following lemma shows that parameters can be eliminated from  $\text{FP+C}$ -canonizations.

**Lemma 2.6** [Gro13, Lemma 3.3.18]<sup>6</sup>. *Let  $\mathcal{C}$  be a class of  $\tau$ -structures. If  $\mathcal{C}$  admits  $\text{FP+C}$ -definable parameterized canonization, then there exists an  $\text{FP+C}$ -canonization of  $\mathcal{C}$  without parameters.*

We can use definable canonization of a graph class to prove that PTIME is captured on this graph class. Let  $L$  be a logic and  $\mathcal{C}$  be a graph class.  $L$  *captures PTIME on  $\mathcal{C}$*  if for each class  $\mathcal{A} \subseteq \mathcal{C}$ , there exists an  $L$ -sentence defining  $\mathcal{A}$  if and only if  $\mathcal{A}$  is PTIME-decidable.<sup>7</sup> If  $L$

<sup>5</sup> Note that if the tuple  $\bar{x}$  of parameter variables is the empty tuple,  $\text{FP+C}[\tau, \tau \cup \{\leq\}]$ -transduction  $\Theta$  canonizes a  $\tau$ -structure  $A$  if  $A \in \text{Dom}(\Theta)$  and the  $\tau \cup \{\leq\}$ -structure  $\Theta[A]$  is an ordered copy of  $A$ .

<sup>6</sup> [Gro13, Lemma 3.3.18] is shown for  $\text{IFP+C}$ . Note that Lemma 3.3.18 states that there exists an  $\text{IFP+C}$ -canonization of  $\mathcal{C}$  without parameters that is also *normal*.

<sup>7</sup> A precise definition of what it means that a logic (*effectively strongly*) captures a complexity class can be found in [EF99, Chapter 11].



captures PTIME on the class of all graphs, then  $L$  captures PTIME [EF99, Theorem 11.2.6]. A fundamental result was shown by Immerman and Vardi:<sup>8</sup>

**Theorem 2.7** [Imm86, Var82]. *FP captures PTIME on the class of all ordered graphs.*

Let us suppose there exists a parameterized FP+C-canonization of a graph class  $\mathcal{C}$ . Since FP captures PTIME on ordered graphs and we can pull back each FP-sentence that defines a polynomial-time property on ordered graphs under this canonization, the capturing result of Immerman and Vardi transfers from ordered structures to the class  $\mathcal{C}$ .

**Proposition 2.8.** *Let  $\mathcal{C}$  be a class of graphs. If  $\mathcal{C}$  admits FP+C-definable parameterized canonization, then FP+C captures PTIME on  $\mathcal{C}$ .*

### 3. DEFINING THE MODULAR DECOMPOSITION IN STC+C

In this section we show that the modular decomposition of a graph is definable in STC+C.

First, we introduce modules and modular decomposition in this section. In order to show that the modular decomposition is definable in STC+C, we consider modules that are spanned by two vertices, that is, modules that contain the two vertices and are minimal with this property. We use the concept of edge classes introduced by Gallai in [Gal67] to show that these spanned modules are definable in STC+C. Afterwards, we show how the spanned modules are related to the modules occurring in the modular decomposition. We obtain that the modular decomposition is definable in STC+C. Consequently, it is computable in logarithmic space [Rei05]. Thus, the modular decomposition tree is computable in logarithmic space. We conclude that cograph recognition and cograph canonization is in logarithmic space.

We use that the modular decomposition is STC+C-definable (actually we only require FP+C-definable) in order to prove the Modular Decomposition Theorem in Section 4.

**3.1. Modules and their Basic Properties.** Let  $G = (V, E)$  be a graph. A non-empty subset  $M \subseteq V$  is a *module* of a graph  $G$  if for all vertices  $v \in V \setminus M$  and all  $w, w' \in M$  we have

$$\{v, w\} \in E \iff \{v, w'\} \in E.$$

All vertex sets of size 1 are modules. We call them *singleton modules*. Further, the vertex set  $V$  is a module. We also refer to the module  $V$  and the singleton modules as *trivial modules*. The connected components of  $G$  or of the complement graph  $\overline{G}$  are modules as well (see Figure 1A). The same holds for unions of connected components. Figure 1B shows a further example of modules in a graph. A module  $M$  is a *proper module* if  $M \subset V$ . We call a graph *prime* if all of its modules are trivial modules. The path  $P_i$  with  $i \geq 4$  vertices, e.g., is a prime graph. Notice that if  $M$  is a module of a graph  $G$ , then  $M$  is also a module of  $\overline{G}$ . Therefore, a graph  $G$  is prime if and only if  $\overline{G}$  is prime.

The following observations contain fundamental but easily provable properties of modules.

<sup>8</sup> Immerman and Vardi proved this capturing result not only for the class of ordered graphs but for the class of ordered structures.

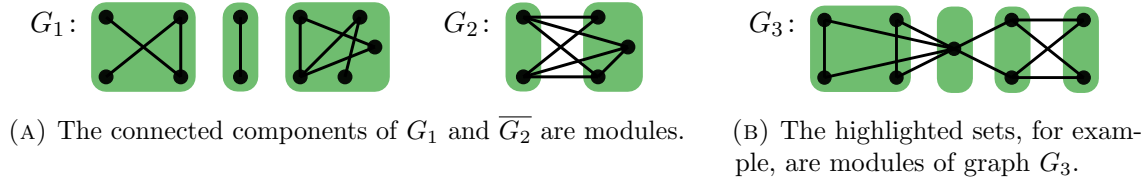


FIGURE 1. Modules of Graphs

**Observation 3.1.** If  $M_1$  and  $M_2$  are modules of a graph  $G$  with  $M_1 \cap M_2 = \emptyset$ , then either there exist no edges between vertices in  $M_1$  and vertices in  $M_2$ , or every vertex in  $M_1$  is adjacent to each vertex in  $M_2$ .

**Observation 3.2.** If  $M_1$  and  $M_2$  are modules of a graph  $G$  with  $M_1 \cap M_2 \neq \emptyset$ , then  $M_1 \cap M_2$  and  $M_1 \cup M_2$  are modules as well.

**Observation 3.3.** Let  $M'$  be a module of  $G$ , and let  $M \subseteq M'$ . Then  $M$  is a module of  $G$  if and only if it is a module of  $G[M']$ .

**3.2. Modular Decomposition.** In the following we present the modular decomposition of a graph, which was introduced by Gallai in 1967 [Gal67]. The modular decomposition decomposes a graph with at least two vertices into proper modules. It can be applied recursively.

Let  $G = (V, E)$  be an arbitrary graph with  $|V| > 1$ . We let  $n$  be the number of vertices in  $G$ . If  $G$  (or  $\overline{G}$ ) is not connected, then every connected component of  $G$  (or  $\overline{G}$ ) is a module, and we can partition the vertex set of  $G$  (or  $\overline{G}$ ) into its connected components. If  $G$  and  $\overline{G}$  are connected, then there also exists a unique partition of  $V$  into proper modules. Gallai showed that in this case the maximal proper modules of  $G$  form a partition of  $V$  (Satz 2.9 and 2.11 in [Gal67]). Figure 1B depicts the maximal proper modules of a graph  $G$  where  $G$  and  $\overline{G}$  are connected.

Thus, we can canonically partition each graph  $G$  with  $n > 1$  into proper modules. For a vertex  $v$  of graph  $G$  we let  $D_G(v)$  be the respective proper module containing  $v$ . Hence, for a vertex  $v$  of a graph  $G = (V, E)$  with  $|V| > 1$ , the set  $D_G(v)$  is<sup>9</sup>

- the connected component of  $G$  that contains  $v$  if  $G$  is not connected,
- the connected component of  $\overline{G}$  that contains  $v$  if  $\overline{G}$  is not connected, or
- the maximal proper module of  $G$  that contains  $v$  if  $G$  and  $\overline{G}$  are connected.

If the graph  $G$  has only one vertex  $v$ , we let  $D_G(v) := \{v\}$ .

We define the *(recursive) modular decomposition* of  $G$  as the following family of subsets  $D_{i,v} \subseteq V$  with  $i \in [0, n]$ ,  $v \in V$ . We let  $D_{0,v} := V$  for all  $v \in V$ , and for  $i \in [0, n - 1]$  we define  $D_{i+1,v}$  for all  $v \in V$  recursively:

$$D_{i+1,v} := D_{G[D_{i,v}]}(v).$$

As an example, a graph and its modular decomposition is illustrated in Figure 2.

<sup>9</sup> We can also say  $D_G(v)$  is the maximal strong proper module of  $G$  that contains  $v$ . (A module  $M$  is *strong*, if  $M \cap M' = \emptyset$ ,  $M \subseteq M'$  or  $M' \subseteq M$  for all other modules  $M'$ .) Gallai proved that the maximal strong proper modules partition  $V(G)$  (Satz 2.11 in [Gal67]), and that for each graph  $G$  they coincide with the sets  $D_G(v)$  as they are defined here [Gal67, Satz 2.9 and 2.10].

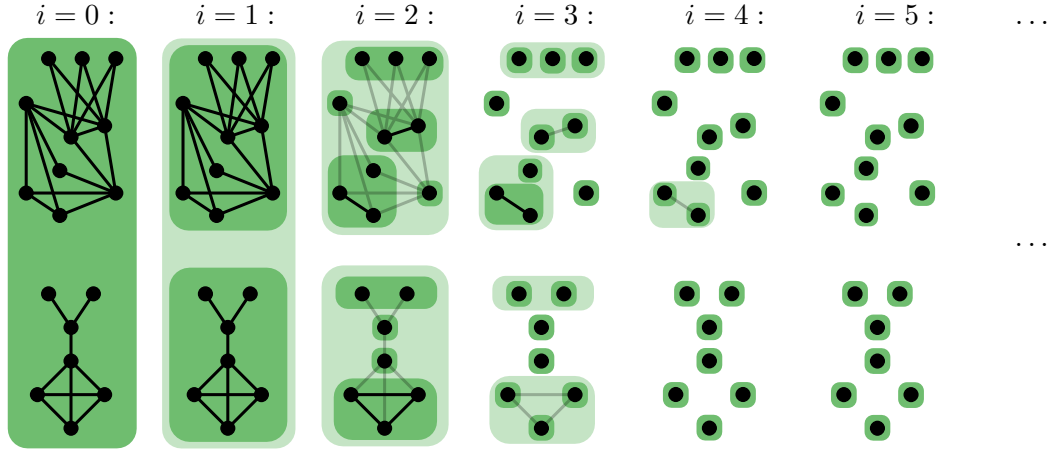


FIGURE 2. Modular decomposition of a graph

It is easy to see that there exists a  $k \in [0, n]$  such that  $V = D_{0,v} \supset D_{1,v} \supset \dots \supset D_{k,v} = \{v\}$  and that  $D_{i,v} = \{v\}$  for all  $i \geq k$ . Thus,  $D_{n,v} = \{v\}$  for all  $v \in V$ . For all  $i \in [0, n]$  and all  $v \in V$  the set  $D_{i,v}$  is a module of  $G$  as we can apply Observation 3.3 inductively. Further, an easy induction shows that the set  $\{D_{i,v} \mid v \in V\}$  is a partition of the vertex set  $V$  for all  $i \in [0, n]$ . Hence, we can conclude the following:

**Observation 3.4.** For all  $v, w \in V$  and all  $i \in [0, n]$ , the modules  $D_{i,v}$  and  $D_{i,w}$  are equal if and only if  $w \in D_{i,v}$ .

**3.3. Spanned Modules and (W)edge Classes.** Let  $v, w \in V$  be vertices of  $G$ . We let  $M_{v,w}$  be the intersection of all modules of  $G$  that contain  $v$  and  $w$ . Since  $V$  is a module, Observation 3.2 implies that  $M_{v,w}$  is a module. Consequently,  $M_{v,w}$  is the smallest module containing  $v$  and  $w$ . We say the vertices  $v, w \in V$  *span* a module  $M$  if  $M = M_{v,w}$ , and call  $M$  a *spanned module* if there exists  $v, w \in V$  that span  $M$ . Trivially,  $M_{v,v} = \{v\}$ .

Let  $e, e' \in E$  be two edges of  $G$ . The edges  $e$  and  $e'$  *form a wedge* in  $G$  (we also write  $e \wedge e'$ ) if there exist three distinct vertices  $u, v, w \in V$  such that  $e = \{u, v\}$ ,  $e' = \{u, w\}$  and there is no edge between  $v$  and  $w$ . Clearly,  $e \wedge e'$  implies  $e' \wedge e$ . We call  $\wedge$  the *wedge relation* on  $E$ . The edges  $e$  and  $e'$  are *wedge connected* if there exists a  $k \geq 1$  and a sequence of edges  $e_1, \dots, e_k$ , such that  $e = e_1$ ,  $e' = e_k$  and  $e_i \wedge e_{i+1}$  for all  $1 \leq i < k$ . It is not hard to see that wedge connectivity is an equivalence relation on the set of edges of the graph. The equivalence classes are the *edge classes* of  $G$ .<sup>10</sup> Thus, the edge classes partition the set of edges of a graph. Note that the edge classes of the complement graph  $\overline{G}$  of  $G$  partition the set of edges of  $\overline{G}$ , and therefore, they partition the set  $\binom{V}{2} \setminus E$  of non-edges of  $G$ . We define the *wedge class* of  $\{v, w\} \in \binom{V}{2}$  as the edge class of  $G$  that contains  $\{v, w\}$  if  $\{v, w\}$  is an edge of  $G$ , or as the edge class of  $\overline{G}$  that contains  $\{v, w\}$  otherwise. For distinct vertices  $v$  and  $w$  we let  $W_{v,w}$  be the union of all elements in the wedge class of  $\{v, w\}$ . Clearly, we have  $v, w \in W_{v,w}$ .

<sup>10</sup> Edge classes (or Kantenklassen) are defined in [Gal67]. We extend this definition to wedge classes.

**Example 3.5.** Consider the graph  $H$  that is depicted in Figure 3A. The edge classes of  $H$  are illustrated in Figure 3B and the edge classes of  $\overline{H}$  in Figure 3C. Further, we have  $W_{e,f} = \{d, e, f\}$  and  $W_{b,f} = V(H) \setminus \{c\}$ .

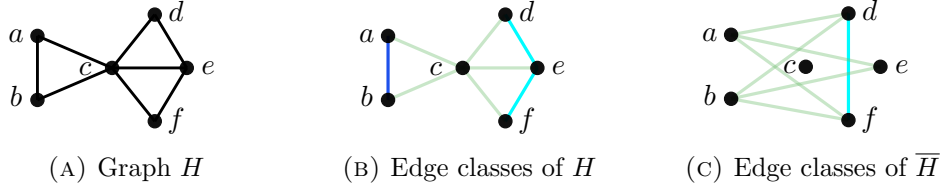


FIGURE 3

The following lemma follows directly from Satz 1.5 in [Gal67] where the lemma is shown for all  $v, w \in V$  with  $\{v, w\} \in E$ .

**Lemma 3.6.** *Let  $v, w \in V$  with  $v \neq w$ . Then  $W_{v,w} \subseteq M_{v,w}$  and  $W_{v,w}$  is a module.*

**Corollary 3.7.**  *$M_{v,w} = W_{v,w}$  for all  $v, w \in V$  with  $v \neq w$ .*

In the following lemma we show that spanned modules are definable in symmetric transitive closure logic.

**Lemma 3.8.** *There exists an STC-formula  $\varphi_M(x_1, x_2, y)$  such that for all pairs  $(v_1, v_2) \in V^2$  of vertices of  $G$ , the set  $\varphi_M[G, v_1, v_2; y]$  is the module spanned by  $v_1$  and  $v_2$ .*

*Proof.* Clearly, there exists an STC-formula that defines the module spanned by two vertices if the vertices are equal. In order to define the module  $M_{v_1, v_2}$  spanned by two distinct vertices  $v_1$  and  $v_2$ , we apply Corollary 3.7 and use the definition of  $W_{v_1, v_2}$ .

First of all, we need a formula for the wedge relation, that is, a formula which is satisfied for vertices  $w_1, w_2, w'_1, w'_2 \in V$  if, and only if,  $\{w_1, w_2\} \wedge \{w'_1, w'_2\}$  in  $G$ . Clearly, this is precisely the case if there exist  $i, j \in [2]$  such that

$$w_i = w'_j, w_{3-i} \neq w'_{3-j}, \quad \text{and} \\ \{w_1, w_2\} \in E, \{w'_1, w'_2\} \in E, \{w_{3-i}, w'_{3-j}\} \notin E.$$

Thus, we obtain an FO-formula for the wedge relation by taking the disjunction of the above statement over all  $i, j \in [2]$ . Since the wedge relation is symmetric, we can use the STC-operator to express wedge connectivity. Hence, there exists an STC-formula that expresses wedge connectivity in  $G$ , and similarly we obtain one for wedge connectivity in  $\overline{G}$ , as well. Using these formulas we are able to define the wedge classes of a graph. Consequently, we can also define the set  $W_{v_1, v_2}$  for distinct vertices  $v_1, v_2 \in V$  in STC.

Now, we can define  $\varphi_M$  such that it distinguishes between the cases of whether the spanning vertices are equal or not and defines the spanned module accordingly.  $\square$

**3.4. Defining the Modular Decomposition in STC+C.** Let us fix a vertex  $v \in V$ . In this section, our goal is to define the sets  $D_{i,v}$  for  $i \in [0, n]$  in STC+C. In order to do this, we show that each set  $D_{i,v}$  can be constructed out of certain modules  $M_{v,w}$  of  $G$  with  $w \in V$ .

First, we take a look at two results (Lemma 3.9 and 3.11) of Gallai. They will help us to gain a better understanding of the connection between  $D_{i,v}$  and the sets  $M_{v,w}$ .

**Lemma 3.9** [Gal67, Satz 2.9 and 2.11 in connection with Satz 1.2 (3b)<sup>11</sup>]. *Suppose  $G$  and  $\overline{G}$  are connected and let  $M', M''$  be maximal proper modules of  $G$  with  $M' \neq M''$ . Further let  $v \in M'$  and  $w \in M''$ . Then  $M_{v,w} = V$ .*

**Corollary 3.10.** *Let  $i \in [0, n-1]$  and  $v \in V$ . If  $G[D_{i,v}]$  and its complement are connected and  $|D_{i,v}| > 1$ , then for all vertices  $w \in D_{i,v} \setminus D_{i+1,v}$  we have  $D_{i,v} = M_{v,w}$ .<sup>12</sup>*

**Lemma 3.11** [Gal67, Satz 1.2 (2)<sup>11</sup>]. *Suppose  $G$  is not connected and let  $v$  and  $w$  be in different connected components  $C_v$  and  $C_w$  of  $G$ . Then  $M_{v,w} = C_v \dot{\cup} C_w$ .*

**Corollary 3.12.** *Let  $i \in [0, n-1]$  and  $v \in V$ . If  $G[D_{i,v}]$  or its complement is not connected, then for all  $w \in D_{i,v} \setminus D_{i+1,v}$  we have  $M_{v,w} = D_{i+1,w} \dot{\cup} D_{i+1,v}$ .<sup>12</sup>*

From Corollary 3.10 and 3.12 we can conclude that there exists a vertex  $w \in V$  such that  $D_{i,v} = M_{v,w}$  if  $G[D_{i,v}]$  and its complement are connected, or if  $G[D_{i,v}]$  or its complement consists of two connected components. If  $G[D_{i,v}]$  or its complement consists of more than two connected components, then for each  $w \in D_{i,v}$  we have  $M_{v,w} \neq D_{i,v}$ . However,  $D_{i,v}$  is the union of all connected components  $D_{i+1,w}$  with  $w \in D_{i,v}$ . Thus, Corollary 3.12 shows that  $D_{i,v}$  is the union of all  $M_{v,w}$  where  $w \in D_{i,v}$  is in a connected component different from the one containing  $v$ .

Let  $v \in V$  be fixed. So far, we have seen that we obtain each set  $D_{i,v}$  by taking the union of certain submodules  $M_{v,w}$  of  $D_{i,v}$ . We show in the following that we can partition the vertex set  $V$  into  $A_0^v, \dots, A_k^v$  such that

$$D_{i,v} = \bigcup \{M_{v,w} \mid w \in A_i^v\},$$

where  $k$  is minimal with  $D_{k,v} = \{v\}$ . In order to obtain this partition, we order the modules  $M_{v,w}$  with  $w \in V$  with respect to proper inclusion. This order is a strict weak order (Lemma 3.13). Hence, incomparability is an equivalence relation. We define the relation  $\prec_v$  on  $V$  by letting

$$w_1 \prec_v w_2 : \iff M_{v,w_2} \subset M_{v,w_1}.$$

Then incomparability regarding  $\prec_v$  is an equivalence relation on the vertex set  $V$ . The resulting equivalence classes form the partition  $\{A_0^v, \dots, A_k^v\}$ . Consequently, we obtain the sets  $D_{i,v}$  by taking the union of all sets  $M_{v,w}$  that are incomparable with respect to proper inclusion. An example showing the connection between  $D_{i,v}$ ,  $M_{v,w}$  for  $w \in V$ ,  $\prec_v$  and the sets  $A_0^v, \dots, A_k^v$  for a specific vertex  $v \in V$  is given in Figure 4.

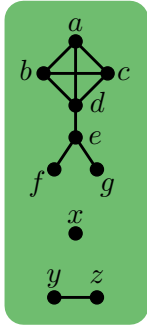
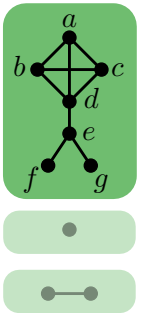
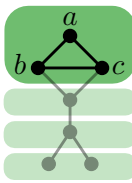
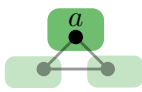
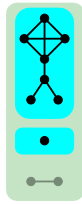


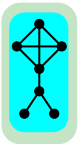
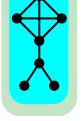





**Lemma 3.13.** *For every  $v \in V$  the relation  $\prec_v$  is a strict weak order.*

*Proof.* Let  $v \in V$ . It is easy to see that  $\prec_v$  is transitive and irreflexive. Let us show that incomparability is transitive. Thus, let  $w_1$  and  $w_2$ , and  $w_2$  and  $w_3$  be incomparable with respect to  $\prec_v$ , and let us assume that  $w_1$  and  $w_3$  are comparable, that is, without loss of generality we have  $w_1 \prec_v w_3$ , which means  $M_{v,w_1} \supset M_{v,w_3}$ . Let  $i \in \{0, \dots, n\}$  be maximal such that  $D_{i,v}$  contains  $M_{v,w_1}$ ,  $M_{v,w_2}$  and  $M_{v,w_3}$ .

First of all, we show that  $M_{v,w_j} \neq D_{i,v}$  for all  $j \in \{1, 2, 3\}$ .  $M_{v,w_3}$  cannot be equal to  $D_{i,v}$  as  $M_{v,w_3}$  is a proper subset of  $M_{v,w_1}$ . If the module  $M_{v,w_2}$  was equal to  $D_{i,v}$ , then  $M_{v,w_3} \subset M_{v,w_2}$ , and  $w_3$  and  $w_2$  would be comparable with respect to  $\prec_v$ . Thus,  $M_{v,w_2} \neq D_{i,v}$ . Finally,  $M_{v,w_1}$  cannot be equal to  $D_{i,v}$  either, since  $M_{v,w_1} = D_{i,v}$  implies that  $M_{v,w_2} \subset M_{v,w_1}$

<sup>11</sup> In [Gal67] Gallai showed this lemma for the set  $W_{v,w}$  instead of  $M_{v,w}$ .

<sup>12</sup> The module  $M_{v,w}$  always refers to the graph  $G$ .

	$i = 0:$	$i = 1:$	$i = 2:$	$i = 3:$			
$D_{i,a}$ :							
Sets $M_{a,w}$ for $w \in V$ : (ordered according to set inclusion)	$M_{a,x}$  $M_{a,y}$  $M_{a,z}$ 	$M_{a,d}$  $M_{a,e}$  $M_{a,f}$  $M_{a,g}$ 	$M_{a,b}$  $M_{a,c}$ 	$M_{a,a}$ 			
Relation $\prec_a$ :	$x, y, z$	$\prec_a$	$d, e, f, g$	$\prec_a$	$b, c$	$\prec_a$	$a$
Partition $A_0^a, \dots, A_3^a$ :	$A_0^a = \{x, y, z\}$	$A_1^a = \{d, e, f, g\}$	$A_2^a = \{b, c\}$	$A_3^a = \{a\}$			

(A)  $D_{i,a}$ ,  $M_{a,w}$ ,  $\prec_a$  and  $A_0^a, \dots, A_3^a$

$G[D_{i,a}]$ is	not connected	connected	connected	connected
$\overline{G}[D_{i,a}]$ is	connected	connected	not connected	connected
Result:	$D_{0,a} = M_{a,x} \cup M_{a,w}$ for $w \in \{y, z\}$	$D_{1,a} = M_{a,w}$ for $w \in \{d, e, f, g\}$	$D_{2,a} = M_{a,b} \cup M_{a,c}$	$D_{3,a} = M_{a,a}$

(B) Connection between the sets  $D_{i,a}$  and  $M_{a,w}$

FIGURE 4

and then  $w_2$  and  $w_1$  would be comparable. Consequently, neither of  $M_{v,w_1}$ ,  $M_{v,w_2}$  and  $M_{v,w_3}$  is equal to  $D_{i,v}$ , and  $|D_{i,v}| > 1$ .

Now, if  $G[D_{i,v}]$  and its complement are connected, we can partition  $D_{i,v}$  into maximal proper modules, and for all  $j \in [3]$  we obtain that  $M_{v,w_j}$  is a subset of module  $D_{i+1,v}$  if  $w_j \in D_{i+1,v}$  or equal to  $D_{i,v}$  if  $w_j \in D_{i,v} \setminus D_{i+1,v}$  (Corollary 3.10). As we have shown above that  $M_{v,w_j} \neq D_{i,v}$  for all  $j \in [3]$ , we have  $M_{v,w_1}, M_{v,w_2}, M_{v,w_3} \subseteq D_{i+1,v}$ , which is a contradiction to the choice of  $i$ .

If  $G[D_{i,v}]$  is not connected, we can partition  $D_{i,v}$  into its connected components. The case of  $\overline{G}[D_{i,v}]$  being not connected can be treated analogously. For every  $u \in D_{i,v}$ , the set  $D_{i+1,u}$  is the connected component of  $G[D_{i,v}]$  containing  $u$ . Let us denote  $D_{i+1,u}$  by  $C_u$ . Since  $i$  has been chosen maximal, there has to be a  $j \in \{1, 2, 3\}$  such that  $M_{v,w_j}$  is not contained in  $C_v$ . For this  $j$ , vertex  $w_j$  must be a vertex in  $M_{v,w_j} \setminus C_v$ , and by Corollary 3.12 we obtain that  $M_{v,w_j} = C_v \dot{\cup} C_{w_j}$ . As  $w_1$  and  $w_2$  are incomparable and  $w_2$  and  $w_3$  are incomparable, independent from our choice of  $j$ , there exists an index  $k \in \{1, 2, 3\} \setminus \{j\}$  such that  $w_j$  and  $w_k$  are incomparable. Thus,  $M_{v,w_k}$  cannot be a proper subset of  $C_v \dot{\cup} C_{w_j}$ , and consequently,  $M_{v,w_k} \setminus C_v \neq \emptyset$ . As above, we obtain that the module  $M_{v,w_k}$  is equal to  $C_v \dot{\cup} C_{w_k}$ . Let us assume  $j = 3$  or  $k = 3$ . The the module  $M_{v,w_3} = C_v \dot{\cup} C_{w_3}$  is a proper subset of the module  $M_{v,w_1}$ . Thus,  $M_{v,w_1} \setminus C_v \neq \emptyset$  and we can deduce  $M_{v,w_1} = C_v \dot{\cup} C_{w_1}$  as we did before. Since both  $M_{v,w_1}$  and  $M_{v,w_3}$  are the union of two connected components,  $M_{v,w_3}$  cannot be a proper subset of  $M_{v,w_1}$ . Therefore,  $j = 1$  and  $k = 2$ , or  $j = 2$  and  $k = 1$ . As a consequence, we have  $M_{v,w_1} = C_v \dot{\cup} C_{w_1}$  and  $M_{v,w_2} = C_v \dot{\cup} C_{w_2}$ . Now, if  $M_{v,w_3} \setminus C_v \neq \emptyset$ , then  $M_{v,w_3}$  is the disjoint union of the connected components  $C_v$  and  $C_{w_3}$ , a contradiction to  $M_{v,w_3} \subset M_{v,w_1}$ . If  $M_{v,w_3}$  is a subset of  $C_v$ , then  $M_{v,w_3}$  is a proper subset of  $M_{v,w_2}$ , which yields that  $w_2$  and  $w_3$  are comparable, a contradiction. Hence, incomparability is transitive.  $\square$

There exists an STC-formula  $\varphi_{\prec}(x, y_1, y_2)$  such that for all vertices  $v, w_1, w_2 \in V$  we have  $G \models \varphi_{\prec}[v, w_1, w_2]$  if, and only if,  $w_1 \prec_v w_2$ , that is, the module spanned by  $v, w_2$  is a proper subset of the module spanned by  $v, w_1$ . Let  $\varphi_M$  be the formula from Lemma 3.8. Then

$$\begin{aligned} \varphi_{\prec}(x, y_1, y_2) := & \forall z (\varphi_M(x, y_2, z) \rightarrow \varphi_M(x, y_1, z)) \wedge \\ & \exists z (\varphi_M(x, y_1, z) \wedge \neg \varphi_M(x, y_2, z)). \end{aligned} \quad (3.1)$$

According to Lemma 3.13 incomparability with respect to  $\prec_v$  is transitive. Hence, incomparability is an equivalence relation. We write  $w \sim_v w'$  if the vertices  $w$  and  $w'$  are incomparable. We let  $[w]_v$  be the equivalence class of  $w$ , and  $V/\sim_v$  be the set of all equivalence classes. Then  $V/\sim_v = \{A_0^v, \dots, A_k^v\}$ . We let  $[z]_v \prec_v [w]_v$  if there exist vertices  $z' \in [z]_v$  and  $w' \in [w]_v$  such that  $z' \prec_v w'$ . If  $w$  and  $w'$ , and  $z$  and  $z'$  are incomparable with respect to the strict weak order  $\prec_v$ , then  $z \prec_v w$  implies  $z' \prec_v w'$ , and  $\prec_v$  induces a strict linear order on  $V/\sim_v$ .

We use the strict linear order on the equivalence classes of the incomparability relation induced by  $\prec_v$  to assign numbers to the equivalence classes, which match their position within the strict linear order. We assign 0 to the smallest equivalence class regarding  $\prec_v$ . The largest equivalence class regarding  $\prec_v$  is  $[v]_v = \{v\}$ . Let  $p_v: V/\sim_v \rightarrow \mathbb{N}$  be this assignment. Then  $p_v(A_i^v) = i$  for all  $i \in [0, k]$ .

**Lemma 3.14.** *For all  $i \in [0, n - 1]$ ,  $v \in V$  and  $w \in D_{i,v} \setminus D_{i+1,v}$ , we have  $p_v([w]_v) = i$ .*

*Proof.* In order to show Lemma 3.14, we first prove the following three claims.

**Claim 3.15.** For all  $i \in [0, n-1]$ , and all vertices  $v \in V$  and  $w, w' \in D_{i,v} \setminus D_{i+1,v}$ , it holds that  $p_v([w]_v) = p_v([w']_v)$ .

*Proof.* Let  $i \in [0, n-1]$ ,  $v \in V$  and  $w, w' \in D_{i,v} \setminus D_{i+1,v}$ . If the graph  $G[D_{i,v}]$  and its complement are connected, then  $M_{v,w} = D_{i,v}$  and  $M_{v,w'} = D_{i,v}$  according to Corollary 3.10. If  $G[D_{i,v}]$  or its complement are not connected, then  $M_{v,w} = D_{i+1,v} \dot{\cup} D_{i+1,w}$  and  $M_{v,w'} = D_{i+1,v} \dot{\cup} D_{i+1,w'}$  by Corollary 3.12. In both cases,  $M_{v,w}$  and  $M_{v,w'}$  are incomparable with respect to proper set inclusion, and therefore  $w$  and  $w'$  are incomparable with respect to  $\prec_v$ . Consequently,  $p_v([w]_v) = p_v([w']_v)$ .  $\lrcorner$

**Claim 3.16.** For all  $i \in [0, n-1]$ ,  $v \in V$  and  $w \in D_{i,v} \setminus D_{i+1,v}$ , we have  $D_{i+1,v} \subset M_{v,w} \subseteq D_{i,v}$ .

*Proof.* Let  $i \in [0, n-1]$ ,  $v \in V$  and  $w \in D_{i,v} \setminus D_{i+1,v}$ . Then  $D_{i+1,v} \subset D_{i,v}$ . If the graph  $G[D_{i,v}]$  and its complement are connected, then  $M_{v,w} = D_{i,v}$  according to Corollary 3.10. If  $G[D_{i,v}]$  or its complement are not connected, then  $M_{v,w} = D_{i+1,v} \dot{\cup} D_{i+1,w}$  by Corollary 3.12. Clearly, we have  $D_{i+1,v} \subset M_{v,w} \subseteq D_{i,v}$  in both cases.  $\lrcorner$

**Claim 3.17.** For all  $i \in [0, n-1]$ , and all vertices  $v \in V$ ,  $w \in D_{i,v} \setminus D_{i+1,v}$  and  $u \in D_{i+1,v}$ , we have  $p_v([u]_v) > p_v([w]_v)$ .

*Proof.* Let  $i \in [0, n-1]$ ,  $v \in V$ ,  $w \in D_{i,v} \setminus D_{i+1,v}$  and  $u \in D_{i+1,v}$ . Since  $u \in D_{i+1,v}$ , we have  $M_{v,u} \subseteq D_{i+1,v}$ . According to Claim 3.16,  $D_{i+1,v} \subset M_{v,w}$ . Hence,  $M_{v,u} \subset M_{v,w}$ . It follows that  $p_v([u]_v) > p_v([w]_v)$ .  $\lrcorner$

We prove Lemma 3.14 by induction on  $i \in [0, n-1]$ , that is, we show that  $p_v([w]_v) = i$  for  $v \in V$  and  $w \in D_{i,v} \setminus D_{i+1,v}$ . Let  $v \in V$ .

First of all, let us consider the base case. Suppose  $i = 0$ . Claim 3.15 and Claim 3.17 imply that the equivalence classes  $[w]_v$  are minimal with respect to  $\prec_v$  for all  $w \in D_{0,v} \setminus D_{1,v}$ . Hence,  $p_v([w]_v) = 0$  for all  $w \in D_{0,v} \setminus D_{1,v}$ .

Next, let us consider the inductive case. Suppose  $i > 0$ . By inductive assumption we have  $p_v([z]_v) = i-1$  for all  $z \in D_{i-1,v} \setminus D_{i,v}$ , and  $p_v([z']_v) < i-1$  for all  $z' \in V \setminus D_{i-1,v}$ . For arbitrary vertices  $w \in D_{i,v} \setminus D_{i+1,v}$  and  $z \in D_{i-1,v} \setminus D_{i,v}$ , we show that  $M_{v,w} \subset M_{v,z}$ . Then it follows from Claim 3.15 and Claim 3.17 that  $p_v([z]_v) = i$  for all  $w \in D_{i,v} \setminus D_{i+1,v}$ .

Let  $w \in D_{i,v} \setminus D_{i+1,v}$  and  $z \in D_{i-1,v} \setminus D_{i,v}$ . By Claim 3.16, we have  $M_{v,w} \subseteq D_{i,v}$ . If the graph  $G[D_{i-1,v}]$  or its complement are not connected, then  $M_{v,z} = D_{i,v} \dot{\cup} D_{i,z}$  by Corollary 3.12, and  $M_{v,w} \subset M_{v,z}$ . Now suppose the graph  $G[D_{i-1,v}]$  and its complement are connected. Then  $M_{v,z} = D_{i-1,v}$  according to Corollary 3.10. Since  $z \in D_{i-1,v} \setminus D_{i,v}$ , we have  $D_{i,v} \subset D_{i+1,v}$ . Consequently,  $M_{v,w} \subset M_{v,z}$  holds also in this case.  $\square$

We define

$$S_{i,v} := \{v\} \cup \bigcup \{M_{v,w} \mid p_v([w]_v) = i, w \in V\}$$

for all  $i \in [0, n]$ . Thus,  $S_{i,v}$  is the union of  $\{v\}$  and all modules  $M_{v,w}$  where  $w$  belongs to the equivalence class at position  $i$  regarding  $\prec_v$ . If  $k+1$  is the number of equivalence classes of  $\sim_v$ , then

$$S_{i,v} = \begin{cases} \bigcup \{M_{v,w} \mid p_v([w]_v) = i, w \in V\} & \text{if } i \leq k \\ \{v\} & \text{if } i \geq k. \end{cases}$$



**Lemma 3.18.** *For all  $i \in \{0, \dots, n\}$  and  $v \in V$ , we have  $D_{i,v} = S_{i,v}$ .*

*Proof.* Let  $i \in [0, n]$  and  $v \in V$ . Suppose  $|D_{i,v}| = 1$ . Then  $D_{i,v} = \{v\}$ . If  $i = n$ , then clearly  $S_{i,v} = \{v\}$  and  $D_{i,v} = S_{i,v}$ . Now assume that  $i < n$ . Lemma 3.14 implies that  $w \in D_{i,v} \setminus D_{i+1,v}$  if and only if  $p_v([w]_v) = i$  for all  $w \in V \setminus \{v\}$ . We have  $D_{i,n} = \{v\}$  precisely if  $D_{i,v} \setminus D_{i+1,v} = \emptyset$ . Hence,  $D_{i,n} = \{v\}$  if and only if there does not exist a vertex  $w \in V \setminus \{v\}$  with  $p_v([w]_v) = i$ . It follows that  $D_{i,v} = S_{i,v}$ .

Suppose  $|D_{i,v}| > 1$ . Then  $i < n$  and  $D_{i,v} \setminus D_{i+1,v}$  is not empty. If  $G[D_{i,v}]$  and its complement are connected, then for all  $w \in D_{i,v} \setminus D_{i+1,v}$  we have  $D_{i,v} = M_{v,w}$  (Corollary 3.10). If  $G[D_{i,v}]$  or its complement are not connected, then for all vertices  $w \in D_{i,v} \setminus D_{i+1,v}$  we have  $M_{v,w} = D_{i+1,v} \dot{\cup} D_{i+1,w}$  (Corollary 3.12). Therefore in both cases, we have  $D_{i,v} = \bigcup \{M_{v,w} \mid w \in D_{i,v} \setminus D_{i+1,v}\}$ . Since  $w \in D_{i,v} \setminus D_{i+1,v}$  if and only if  $p_v([w]_v) = i$  for all  $w \in V \setminus \{v\}$  by Lemma 3.14, we obtain  $D_{i,v} = \bigcup \{M_{v,w} \mid p_v([w]_v) = i, w \in V \setminus \{v\}\}$ . As the vertex  $\{v\}$  is contained in  $D_{i,v}$ , it follows that  $D_{i,v} = \{v\} \cup \bigcup \{M_{v,w} \mid p_v([w]_v) = i, w \in V\}$ . Hence,  $D_{i,v} = S_{i,v}$ .  $\square$

**Theorem 3.19.** *There is an STC+C-formula  $\varphi_D(p, x, z)$  such that for all graphs  $G$ , all  $i \in N(G)$  and all vertices  $v \in V(G)$  the set  $\varphi_D[G, i, v; z]$  is the set  $D_{i,v}$  of the modular decomposition of  $G$ .*

*Proof.* First we need a formula  $\varphi_{\text{ord}}$  that assigns to vertices  $v, w \in W$  the position  $p_v([w]_v)$  of  $[w]_v$  within the strict linear order of the equivalence classes of the incomparability relation induced by  $\prec_v$ . More precisely,  $G \models \varphi_{\text{ord}}[v, w, i]$  if and only if  $p_v([w]_v) = i$ , for all  $v, w \in V(G)$  and  $i \in N(G)$ . Clearly,  $\varphi_{\text{ord}}$  is satisfied for  $v, w \in V$  and  $i \in N(V)$  exactly if  $i$  is the number of equivalence classes that are smaller than  $[w]_v$  regarding  $\prec_v$ . Thus, we need an STC+C-formula which counts the number of equivalence classes smaller than  $[w]_v$ .

Let  $\varphi_{\prec}$  be the formula defined in (3.1). Then the formula

$$\varphi_{\sim_v}(x, y, y') := \neg\varphi_{\prec}(x, y, y') \wedge \neg\varphi_{\prec}(x, y, y')$$

defines the equivalence relation  $\sim_v$  for every  $v \in V$ , that is,  $G \models \varphi_{\sim_v}[v, w, w']$  if and only if  $w \sim_v w'$ , for all vertices  $v, w, w' \in V(G)$ . The construction from the proof of Lemma 2.4.3 in [Lau11] shows how to count definable equivalence classes in deterministic transitive closure logic with counting DTC+C, which is a logic that is contained in STC+C. With the help of the formula  $\varphi_{\prec}$ , we can use this construction to count the equivalence classes smaller than  $[w]_v$  regarding  $\prec_v$  for all  $v, w \in V(G)$ . Thus, the formula  $\varphi_{\text{ord}}$  can be defined in STC+C.

Now, we let  $\varphi_M$  be the formula from Lemma 3.8, and we apply Lemma 3.18, that is, we use that  $D_{i,v} = S_{i,v}$ . Then it is easy to see that the following formula is as desired:

$$\varphi_D(p, x, z) := \exists y (\varphi_{\text{ord}}(x, y, p) \wedge \varphi_M(x, y, z)) \vee x = z. \quad \square$$

As STC+C-formulas can be evaluated in logarithmic space [Rei05], we obtain the following corollary.

**Corollary 3.20.** *There exists a logarithmic-space deterministic Turing machine that, given a graph  $G = (V, E)$ , a number  $i \leq |V|$  and vertices  $v, w \in V$ , decides whether  $w \in D_{i,v}$ .*

Let the family of subsets  $D_{i,v}$  with  $i \in [0, n]$ ,  $v \in V$  be the modular decomposition of graph  $G = (V, E)$ . The modular decomposition tree of  $G$  is the directed tree  $T = (V_T, E_T)$  with

$$\begin{aligned} V_T &:= \{D_{i,v} \mid i \in [0, n], v \in V\}, \\ E_T &:= \{(D_{i,v}, D_{i+1,v'}) \in V_T^2 \mid D_{i+1,v'} \subset D_{i,v}\}. \end{aligned}$$

**Corollary 3.21.** *There exists a logarithmic-space deterministic Turing machine that, given a graph  $G = (V, E)$ , outputs the modular decomposition tree of  $G$ .*

A graph is a *cograph* if it can be constructed from isolated vertices by disjoint union and join operations. We obtain the *join* of two graphs  $G$  and  $H$  by taking the disjoint union of  $G$  and  $H$  and adding all edges  $\{v, w\}$  where  $v$  is a vertex of  $G$  and  $w$  is a vertex of  $H$ .

The modular decomposition trees of cographs have a special property: Each inner node is the disjoint union or the join of its children. Moreover, only modular decomposition trees of cographs have this property.

**Corollary 3.22.** *Cograph recognition is in LOGSPACE.*

We obtain the *cotree* of a cograph  $G$  by coloring each inner node  $v$  of the modular decomposition tree of  $G$  with 0 if  $v$  is the disjoint union of its children, and with 1 if  $v$  is the join of its children. It is well known that for each cograph the cotree is a canonical tree representation. In [Lin92], Lindell presented a logarithmic-space algorithm for tree canonization, which can easily be extended to cotrees. Thus, Corollary 3.21 also implies the following:

**Corollary 3.23.** *There exists a logarithmic-space algorithm for cograph canonization.*

#### 4. MODULAR DECOMPOSITION THEOREM

For suitable graph classes  $\mathcal{C}$  that are closed under induced subgraphs, the Modular Decomposition Theorem is a tool which can be used to show that  $\mathcal{C}$  admits **FP+C**-definable canonization. More precisely, for graph classes  $\mathcal{C}$  that are closed under induced subgraphs, the Modular Decomposition Theorem states that  $\mathcal{C}$  admits **FP+C**-definable canonization if the class of LO-colored graphs with prime underlying graphs from  $\mathcal{C}$  admits **FP+C**-definable (parameterized) canonization. Note that the Modular Decomposition Theorem also holds for reasonable extensions of **FP+C** that are closed under parameterized **FP+C**-transductions.

In this section, we first introduce modular contractions and representations of ordered graphs. Then we present the Modular Decomposition Theorem and a proof of it. Finally, we show variations of the Modular Decomposition Theorem: We show that  $\mathcal{C}$  admits **FP+C**-definable canonization if the class of prime graphs of  $\mathcal{C}$  admits **FP+C**-definable orders, and we present an analog of the Modular Decomposition Theorem for polynomial-time computable canonization.

**4.1. Modular Contraction.** The modular contraction is the graph that we obtain by contracting the maximal proper modules of a graph to vertices.

For a graph  $G = (V, E)$ , let  $\sim_G$  be the equivalence relation on  $V$  defined by the partition  $\{D_G(v) \mid v \in V\}$  (see Section 3.2). Then the equivalence class  $v/\sim_G$  of a vertex  $v \in V$  is the module  $D_G(v)$  of  $G$ . We let  $G_\sim$  be the graph consisting of the vertex set  $V/\sim_G = \{v/\sim_G \mid v \in V\}$ , where two distinct vertices  $w/\sim_G$  and  $w'/\sim_G$  are adjacent if and only if  $w$  and  $w'$  are adjacent in  $G$ . According to Observation 3.1, the edges of  $G_\sim$  are well-defined. We call  $G_\sim$  the *modular contraction* of  $G$ . Thus, the modular contraction of a graph  $G$  is

- an edgeless graph with as many vertices as there are connected components in  $G$  if  $G$  is not connected,
- a complete graph with as many vertices as there are connected components in  $\bar{G}$  if  $\bar{G}$  is not connected, or

- if  $G$  and  $\overline{G}$  are connected and  $|V(G)| > 1$ , a set of vertices, one vertex for each maximal proper module, where there is an edge between two vertices exactly if the corresponding modules are (completely) connected with edges; or a single vertex if  $|V(G)| = 1$ .

Figure 5 depicts the graphs from Figure 1 and their modular contractions.

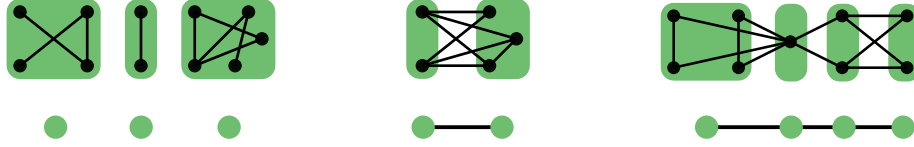


FIGURE 5. Graphs and their respective modular contractions

**Observation 4.1** [Gal67, Satz 1.8]. If  $G$  and  $\overline{G}$  are connected, then the modular contraction  $G_{\sim}$  of  $G$  is prime.

**Observation 4.2.** For every graph  $G$ , the modular contraction of  $G$  is isomorphic to an induced subgraph of  $G$ .

For all modules  $D_{i,v}$  of  $G$ , we denote the modular contraction  $G[D_{i,v}]_{\sim}$  of  $G[D_{i,v}]$  for all  $i \leq n$  and  $v \in V$  by  $G_{i,v}$ . Notice that  $G_{0,v}$  is the modular contraction  $G_{\sim}$  of  $G$ .

**4.2. Representation of an Ordered Graph.** In the following we introduce the representation of an ordered graph  $G$ .

Each ordered graph  $G = (V, E, \leq)$  is isomorphic to an ordered graph where the vertex set is  $[[V]]$  and the linear order is the natural order  $\leq_{[[V]]}$  on  $[[V]]$ . Thus, we suppose the vertex set of our ordered graph  $G$  is  $[[V]]$  and the linear order of  $G$  is  $\leq_{[[V]]}$ . We use the representation to encode the ordered graph in a binary relation. Later, when we want to color the vertices of graphs with ordered graphs, we use these representations as colors instead. As a result we obtain an LO-colored graph.

Let  $G$  be an ordered graph with vertex set  $[n]$  and linear order  $\leq_{[n]}$ . We encode the ordered graph  $G$  in a symmetric binary relation  $g_{\text{rep}}(G) \subseteq [n]^2$ :

$$g_{\text{rep}}(G) := \{(l, l') \mid \{l, l'\} \in E(G)\} \cup \{(n, n)\}.$$

We call  $g_{\text{rep}}(G)$  the *representation* of  $G$ . We can reinterpret every representation  $R \subseteq N(G)^2$  of an ordered graph as an ordered graph  $g_{\text{graph}}(R)$ . Let  $n' \in N(G)$  be the only number with  $(n', n') \in R$ . We let

$$\begin{aligned} V(g_{\text{graph}}(R)) &:= [n'], \\ E(g_{\text{graph}}(R)) &:= \{(l_1, l_2) \mid (l_1, l_2) \in R \setminus \{(n', n')\}\} \quad \text{and} \\ \leq(g_{\text{graph}}(R)) &:= \leq_{[n']}. \end{aligned}$$

We call  $g_{\text{graph}}(R)$  the *ordered graph* of relation  $R$ . It is easy to see that  $g_{\text{graph}}(g_{\text{rep}}(G)) = G$ .

**4.3. The Modular Decomposition Theorem.** In the following we present the Modular Decomposition Theorem and a proof of it.

For a class  $\mathcal{C}$  of graphs that is closed under induced subgraphs, we let  $\mathcal{C}_{\text{prim}}^*$  be the class of all LO-colored graphs  $H^* = (U, V, E, M, \trianglelefteq, L)$  where the underlying graph  $(V, E)$  is a prime graph in  $\mathcal{C}$  and  $|V| \geq 4$ .

**Theorem 4.3** (Modular Decomposition Theorem). *Let  $\mathcal{C}$  be a class of graphs that is closed under induced subgraphs. If  $\mathcal{C}_{\text{prim}}^*$  admits FP+C-definable (parameterized) canonization, then  $\mathcal{C}$  admits FP+C-definable canonization.*

Let  $\mathcal{G}_{\mathcal{KI}}^*$  be the class of all LO-colored graphs where the underlying graph is complete or edgeless. For a class  $\mathcal{C}$  of graphs that is closed under induced subgraphs, we let  $\mathcal{C}_{\mathcal{KI}}^* := \mathcal{C}_{\text{prim}}^* \cup \mathcal{G}_{\mathcal{KI}}^*$ . Notice that  $\mathcal{C}_{\mathcal{KI}}^*$  contains all LO-colored graphs where the underlying graph is a prime graph in  $\mathcal{C}$ , because every prime graph with less than 4 vertices is complete or edgeless. The following two observations show that it suffices to prove the Modular Decomposition Theorem under the assumption that the class  $\mathcal{C}_{\mathcal{KI}}^*$  admits FP+C-definable canonization.

Observation 4.4 is a direct consequence of Lemma 2.6.

**Observation 4.4.** If there exists a parameterized FP+C-canonization of  $\mathcal{C}_{\text{prim}}^*$ , then there exists an FP+C-canonization of  $\mathcal{C}_{\text{prim}}^*$ .

**Observation 4.5.** Let  $\mathcal{C}$  be a class of graphs that is closed under induced subgraphs. If  $\mathcal{C}_{\text{prim}}^*$  admits FP+C-definable canonization, then  $\mathcal{C}_{\mathcal{KI}}^*$  admits FP+C-definable canonization.

*Proof.* Let  $\mathcal{C}$  be closed under induced subgraphs, and let  $\Theta^c$  be an FP+C-canonization of  $\mathcal{C}_{\text{prim}}^*$ . We extend  $\Theta^c$  to an FP+C-canonization of the class  $\mathcal{C}_{\mathcal{KI}}^*$ .

It is easy to describe in FP+C whether the underlying graph  $H$  of an LO-colored graph  $H^*$  is complete or edgeless. Also, it is not hard to define the canon of an LO-colored graph  $H^* \in \mathcal{G}_{\mathcal{KI}}^*$  in FP+C. We can use the lexicographical order of the vertices' natural colors and the linear order of the basic color elements to define an ordered copy of  $H^*$  (see [Gru17a, Example 17]). Thus, we can extend  $\Theta^c$  in such a way that it first detects whether LO-colored graph  $H^*$  is in  $\mathcal{G}_{\mathcal{KI}}^*$  or not. If  $H^* \in \mathcal{G}_{\mathcal{KI}}^*$ , then  $\Theta^c$  defines the canon as explained above. If  $H^* \notin \mathcal{G}_{\mathcal{KI}}^*$ , then  $\Theta^c$  behaves as originally intended.  $\square$

For the remainder of this section, let  $\mathcal{C}$  be a graph class that is closed under induced subgraphs. Further, let  $\Theta^c$  be an FP+C-canonization of the class  $\mathcal{C}_{\mathcal{KI}}^*$ , and let  $f^*$  be the canonization mapping defined by  $\Theta^c$ . We show that there exists an FP+C-canonization of  $\mathcal{C}$ .

**Sketch of the Proof.** In order to show the Modular Decomposition Theorem the idea is to construct the canon of each  $G \in \mathcal{C}$  recursively using the modular decomposition. Let  $n$  be the number of vertices of  $G$ . Then for all  $i \in \{n, \dots, 0\}$ , starting with  $i = n$ , we inductively define the canons of the induced subgraphs  $G[D_{i,v}]$  for all  $v \in V$ . We can trivially define the canon for each module that is a singleton. For the inductive step we consider the modular contraction  $G_{i,v}$  of  $G[D_{i,v}]$ . For all  $i < n$  and  $v \in V$ , the graph  $G_{i,v}$  is prime if  $G[D_{i,v}]$  and  $\overline{G}[D_{i,v}]$  are connected, complete if  $\overline{G}[D_{i,v}]$  is not connected or edgeless if  $G[D_{i,v}]$  is not connected. We transform  $G_{i,v}$  into an LO-colored graph  $G_{i,v}^*$  by coloring every vertex  $w/\sim_{G[D_{i,v}]}$  of  $G_{i,v}$  with the representation of the canon of the graph  $G[D_{i+1,w}]$ . The canon of  $G[D_{i+1,w}]$  is definable by inductive assumption. Then  $G_{i,v}^* \in \mathcal{C}_{\mathcal{KI}}^*$ . Thus, we can apply  $f^*$  to get  $G_{i,v}^*$ 's canon  $K_{i,v}^*$ . Now each vertex of  $K_{i,v}^*$  stands for a module, and the color of every

vertex is the representation of the canon of the graph induced by this module. Therefore, we can use the coloring to replace each vertex of  $K_{i,v}^*$  by the graph induced by the module that the vertex represents. From the coloring, we also obtain a linear order on each module. We use the order on the vertices of  $K_{i,v}^*$  to extend the linear orders on the modules to a linear order on the vertex set of the resulting graph.

In the following we present a detailed proof of the Modular Decomposition Theorem.

First, we shortly introduce notation that simplifies the construction of an FP+C-canonization. Then, we start by recursively defining the canonization mapping  $f$  which maps each graph  $G \in \mathcal{C}$  to its canon  $f(G)$ . Afterwards we show that this canonization mapping is FP+C-definable.

**Notation.** Throughout this section, we use  $x, y, z$  and variants like  $x_1, y', z^*$  of these letters for structure variables, and  $o, p, q, r, s$  and variants for number variables. There exist FO+C-formulas  $\text{zero}(p)$ ,  $\text{one}(p)$  and  $\text{largest}(p)$  that define the numbers 0, 1 and  $|U(A)|$  for all structures  $A$ , and an FP+C-formula  $\text{plus}(p, q, r)$  that defines the addition function [Gro13, Example 2.3.5]. We write  $p = 0$ ,  $p = 1$  and  $p + q = r$  instead of  $\text{zero}(p)$ ,  $\text{one}(p)$  and  $\text{plus}(p, q, r)$ , respectively, and use similar abbreviations. We denote  $\neg u = v$  by  $u \neq v$  and  $p \leq q \wedge \neg p = q$  by  $p < q$ , and abbreviate  $\exists u_1 \dots \exists u_k$  by  $\exists u_1, \dots, u_k$  and  $\neg \#o \varphi = 0$  by  $\exists o \varphi$ .

**Canonization Mapping.** In the following we define the canonization mapping  $f$ , which maps each graph  $G \in \mathcal{C}$  to the canon  $f(G)$ . We let the vertex set of canon  $f(G)$  be  $[[V(G)]]$ . The linear order on the vertex set is the natural order  $\leq_{[[V(G)]]}$  on  $[[V(G)]]$ .

If  $|V(G)| = 1$ , then the canon of  $G$  is  $f(G) := (\{1\}, \emptyset, \leq_{\{1\}})$ . Now in order to define the canonization mapping  $f$  on graphs  $G$  with  $|V(G)| > 1$ , we use their decomposition into modules to recursively construct the canon of a graph from the canons of the induced subgraphs on its decomposition modules. In a first step we define  $G_{\sim}^*$ , the *LO-colored graph* of  $G$ , which has  $G_{\sim}$ , the modular contraction of  $G$ , as underlying graph. To obtain  $G_{\sim}^*$  we color every vertex  $w/\sim_G$  of  $G_{\sim}$  with the representation of the canon  $f(G[D_G(w)])$  of  $G[D_G(w)]$ . More precisely, we let

$$G_{\sim}^* := (U_{G_{\sim}^*}, V_{G_{\sim}^*}, E_{G_{\sim}^*}, M_{G_{\sim}^*}, \preceq_{G_{\sim}^*}, L_{G_{\sim}^*})$$

where

$$\begin{aligned} U_{G_{\sim}^*} &:= V_{G_{\sim}^*} \dot{\cup} M_{G_{\sim}^*}, \\ (V_{G_{\sim}^*}, E_{G_{\sim}^*}) &:= G_{\sim}, \\ M_{G_{\sim}^*} &:= [0, |V(G)|], \\ \preceq_{G_{\sim}^*} &:= \preceq_{[0, |V(G)|]}, \quad \text{and} \\ L_{G_{\sim}^*} &:= \{(v, i, j) \in V_{G_{\sim}^*} \times M_{G_{\sim}^*}^2 \mid (i, j) \in g_{\text{rep}}(f(G[D_G(v)]))\}. \end{aligned}$$

The construction of  $G_{\sim}^*$  is illustrated in Figure 6.

As  $G_{\sim}$ , the underlying graph of  $G_{\sim}^*$ , is a modular contraction,  $G_{\sim}$  is prime, complete or edgeless. Therefore, we can use the given canonization mapping  $f^*$  to obtain the canon of  $G_{\sim}^*$ :

$$K_{\sim}^* = (U_{K_{\sim}^*}, V_{K_{\sim}^*}, E_{K_{\sim}^*}, M_{K_{\sim}^*}, \preceq_{K_{\sim}^*}, L_{K_{\sim}^*}, \leq_{K_{\sim}^*}).$$

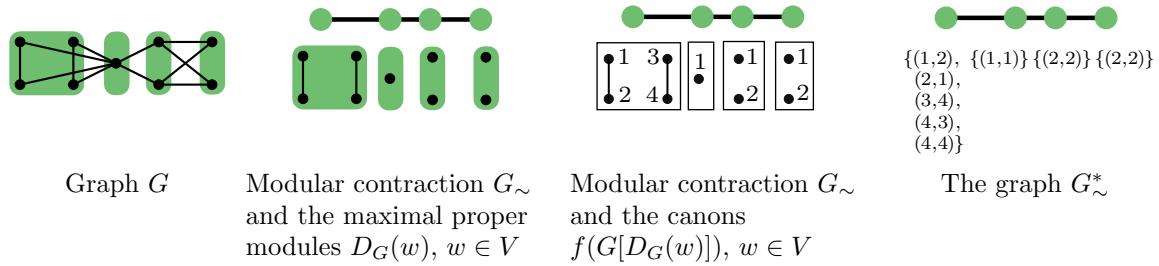


FIGURE 6. Construction of  $G_{\sim}^*$

To get the canon of  $G$ , we replace each vertex  $w \in V_{K_{\sim}^*}$  of the ordered LO-colored graph  $K_{\sim}^*$  by the graph represented by  $w$ 's natural color  $L_w^{\mathbb{N}}$ . Since each LO-colored graph consists of a linear order on the basic color elements, the natural colors of isomorphic LO-colored graphs are equal. Hence, the natural colors of  $K_{\sim}^*$  match the (natural) colors of  $G_{\sim}^*$ , which again encode the canons of the subgraphs induced by the modules the vertices represent. Thus, we replace the vertices of  $K_{\sim}^*$  by the corresponding canons. We use the linear order on the vertices (given by the linear order  $\leq_{K_{\sim}^*}$  restricted to the vertex set  $V_{K_{\sim}^*}$ ) to replace one vertex after the other. We name the new vertices consecutively according to the time of their installment (and their order in the respective canon). Figure 7 shows the construction of  $f(G)$ .

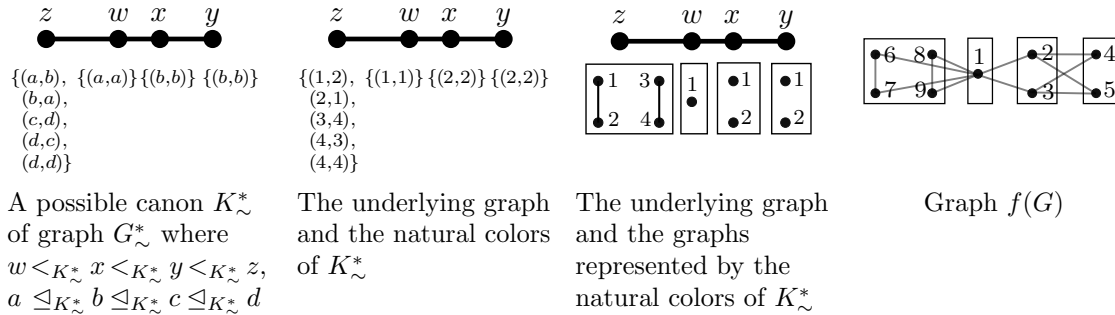


FIGURE 7. Construction of  $f(G)$

In the following we describe the construction of the canon  $f(G)$  more precisely. For all vertices  $w \in V_{K_{\sim}^*}$ , let  $L_w^{\mathbb{N}}$  be the natural color of  $w$ , and let  $n_w$  be the only element with  $(n_w, n_w) \in L_w^{\mathbb{N}}$ . Since the module that  $w$  stands for consists of at least one vertex, such an  $n_w$  exists and  $0 < n_w \leq |M_{K_{\sim}^*}|$ . To construct the canon we assign each vertex  $n$  of the ordered graph  $g_{\text{graph}}(L_w^{\mathbb{N}})$  of representation  $L_w^{\mathbb{N}}$  to the number

$$\text{nb}(w, n) := n + \sum_{\substack{w' <_{K_{\sim}^*} w, \\ w' \in V_{K_{\sim}^*}}} n_{w'}, \tag{4.1}$$

where  $w' <_{K_{\sim}^*} w$  if and only if  $w' \leq_{K_{\sim}^*} w$  and  $w' \neq w$ . Clearly, the mapping  $\text{nb}$  is a bijection, that maps  $(n, v)$ , where  $n$  is a vertex in the graph represented by vertex  $v$ 's natural color, to  $m \in [|V(G)|]$ .

We add a pair of numbers to the edges of  $f(G)$  if they represent vertices from different modules, and the modules are completely connected; or they represent vertices from the same module that are connected by an edge. Thus, we add  $\{m_1, m_2\}$  to the edges of  $f(G)$  if

- (1) there exist an edge  $\{w_1, w_2\} \in E_{K_{\sim}^*}$  and numbers  $n_1, n_2 \in [|M_{K_{\sim}^*}|]$  such that  $n_1 \leq n_{w_1}$ ,  $n_2 \leq n_{w_2}$  and  $(m_1, m_2) = (\text{nb}(w_1, n_1), \text{nb}(w_2, n_2))$ , or
- (2) there exist a vertex  $w \in V_{K_{\sim}^*}$  and a pair  $(n_1, n_2) \in L_w^{\mathbb{N}}$  such that  $n_1 \neq n_2$  and  $(m_1, m_2) = (\text{nb}(w, n_1), \text{nb}(w, n_2))$ .

Clearly, the ordered graph  $f(G)$  is an ordered copy of  $G$  on the number sort. Observation 4.6 shows that  $f$  maps isomorphic graphs from  $\mathcal{C}$  to the same ordered graph. Hence,  $f$  is a canonization mapping. Note that Observation 4.6 follows also directly from  $f$  being FP+C-definable, which is shown in the remainder of Section 4.3.

**Observation 4.6.** For all  $G, G' \in \mathcal{C}$ , we have  $f(G) = f(G')$  if  $G \cong G'$ .

*Proof.* Let  $h$  be an isomorphism between  $G$  and  $G'$ . We show that  $f(G) = f(G')$  by induction. Clearly, this is the case if  $G$  and  $G'$  consist of only one vertex. Therefore, suppose  $|V(G)| = |V(G')| > 1$ . As the modular decomposition of a graph is unique, the isomorphism  $h$  maps every decomposition module of  $G$  to a decomposition module of  $G'$ . Hence, the respective graphs induced by the decomposition modules of  $G$  and  $G'$  are isomorphic, and by inductive assumption  $f$  maps them to the same canon. Further,  $h$  induces an isomorphism  $h_{\sim}$  between  $G_{\sim}$  and  $G'_{\sim}$ . Consequently, the graphs  $G_{\sim}^*$  and  $G'_{\sim}^*$  are isomorphic. They are mapped to isomorphic copies  $K_{\sim}^*$  and  $K'_{\sim}^*$  by  $f^*$ . Let  $g$  be an isomorphism between them. Clearly, for each vertex  $v \in V_{K_{\sim}^*}$ , the vertices  $v$  and  $g(v)$  have the same natural color. Further, we have  $v_1 \leq_{K_{\sim}^*} v_2$  if and only if  $g(v_1) \leq_{K'_{\sim}^*} g(v_2)$ . As a consequence,  $f(G) = f(G')$ .  $\square$

**Defining the Canonization Mapping in FP+C.** We show that the canonization mapping  $f$  is FP+C-definable in the following five steps.

**Step 1: Counting Transduction from Graphs to LO-Colored Graphs.** For all modules  $D_{i,v}$  of  $G$  with  $i \leq V(G)$  and  $v \in V(G)$ , we denote the LO-colored graph  $(G[D_{i,v}])_{\sim}^*$  of  $G[D_{i,v}]$  by  $G_{i,v}^*$ . Notice that the underlying graph of  $G_{i,v}^*$  is  $G_{i,v}$ .

The first step in constructing an FP+C-transduction that defines  $f$  is to define the LO-colored graph  $G_{i,v}^*$  for all  $G \in \mathcal{C}$ ,  $i \in N(G)$  and  $v \in V(G)$ . For this purpose, we define a parameterized counting transduction  $\Theta^{\#}(o, z, X)$ , where  $o$  is a number variable,  $z$  is a structure variable, and  $X$  is a relational variable of arity 4 that ranges over relations  $R \subseteq N(G) \times V(G) \times N(G)^2$ . It is a parameterized FP+C $\{\{E\}, \{V, E, M, \sqsubseteq, L\}\}$ -counting transduction, which maps every graph  $G$  to an LO-colored graph  $G_{i,v}^R := \Theta^{\#}[G, i, v, R]$  for  $(G, i, v, R) \in \text{Dom}(\Theta^{\#}(o, z, X))$ . For some triples  $(i, v, R) \in G^{(o, z, X)}$  where  $R$  is a specific relation depending on  $i$  and  $v$ , the LO-colored graph  $G_{i,v}^R$  is isomorphic to  $G_{i,v}^*$ . We let

$$\begin{aligned} \Theta^{\#}(o, z, X) = & (\theta_{\text{dom}}(o, z, X), \theta_U(o, z, X, y), \theta_{\sim}(o, z, X, y, y'), \\ & \theta_V(o, z, X, y), \theta_E(o, z, X, y, y'), \\ & \theta_M(o, z, X, p), \theta_{\sqsubseteq}(o, z, X, p, p'), \theta_L(o, z, X, y, p, p')), \end{aligned}$$

where

$$\begin{aligned}
\theta_{\text{dom}}(o, z, X) &:= \neg \text{largest}(o), \\
\theta_U(o, z, X, y) &:= \varphi_D(o, z, y), \\
\theta_{\approx}(o, z, X, y, y') &:= \exists o'(o + 1 = o' \wedge \varphi_D(o', y, y')), \\
\theta_V(o, z, X, y) &:= \top, \\
\theta_E(o, z, X, y, y') &:= E(y, y'), \\
\theta_M(o, z, X, p) &:= \top, \\
\theta_{\leq}(o, z, X, p, p') &:= p \leq p' \quad \text{and} \\
\theta_L(o, z, X, y, p, p') &:= \exists o'(o + 1 = o' \wedge X(o', y, p, p')).
\end{aligned}$$

As a reminder, the formula  $\varphi_D(o, z, y)$ , which was introduced in Theorem 3.19, defines the set  $D_{i,v}$  of the modular decomposition, i.e., for all  $i \in N(G)$  and all vertices  $v \in V(G)$  we have  $\varphi_D[G, i, v; y] = D_{i,v}$ .

Let  $G \in \mathcal{C}$ . We say a triple  $(i, v, R) \in G^{(o,z,X)}$  is *suitable* for  $G$  if it satisfies  $i < |V(G)|$  and the following property: For all  $w \in D_{i,v}$  the relation

$$R_{i+1,w} := \{(n_1, n_2) \mid (i + 1, w, n_1, n_2) \in R\}$$

is the representation of the canon of  $G[D_{i+1,w}]$  defined by  $f$ . We let  $\text{Suit}(G)$  be the set of all suitable triples for  $G$ .

**Lemma 4.7.** *Let  $G \in \mathcal{C}$  and let  $(i, v, R) \in G^{(o,z,X)}$  be a suitable triple for graph  $G$ . Then  $(G, i, v, R) \in \text{Dom}(\Theta^\#(o, z, X))$  and  $G_{i,v}^R = G_{i,v}^*$ .*

*Proof.* Let  $G \in \mathcal{C}$ . Further, let  $i \in N(G)$ ,  $v \in V(G)$  and  $R \subseteq N(G) \times V(G) \times N(G)^2$  be such that  $(i, v, R) \in \text{Suit}(G)$ . Then,  $i < |V(G)|$ . Therefore,  $(G, i, v, R) \in \text{Dom}(\Theta^\#(o, z, X))$ . Clearly,  $\theta_U[G, i, v, R; y]$  is the set  $D_{i,v}$ . Further,  $\theta_{\approx}[G, i, v, R; y, y']$  is the equivalence relation  $\{D_{i+1,w} \mid w \in V(G)\}$ . Let  $\approx$  denote this equivalence relation. Then the universe of  $G_{i,v}^R$  is the set  $D_{i,v}/\approx \dot{\cup} [0, |V(G)|]$ . The vertex set  $V(G_{i,v}^R)$  is  $D_{i,v}/\approx$ , and it is not hard to see that the formulas  $\theta_V$ ,  $\theta_{\approx}$  and  $\theta_E$  of transduction  $\Theta^\#(o, z, X)$  define the graph  $G_{i,v}$ . Further,  $M(G_{i,v}^R) = [0, |V(G)|]$  and  $\leq(G_{i,v}^R)$  is the natural order on  $[0, |V(G)|]$ . Finally, the formula  $\theta_L$  defines the color relation. As  $(i, v, R) \in \text{Suit}(G)$ , the relation  $\{(m_1, m_2) \mid (i + 1, w, m_1, m_2) \in R\}$  is the representation of the canon of  $G[D_{i+1,w}]$  for all  $w \in V(G)$ , and we obtain that  $G_{i,v}^R$ , that is,  $\Theta^\#[G, i, v, R]$ , is equal to  $G_{i,v}^*$  for all  $(i, v, R) \in \text{Suit}(G)$ .  $\square$

Later, we will make sure that the triple of parameters  $(o, z, X)$  is always interpreted by a suitable triple.

**Step 2: Transduction from Graphs to ordered LO-Colored Graphs.**  $\Theta^\#(o, z, X)$  is a parameterized FP+C-counting transduction. Thus, there exists a parameterized FP+C-transduction  $\Theta^*(o, z, X)$  with the same domain, such that  $\Theta^\#[G, i, v, R]$  and  $\Theta^*[G, i, v, R]$  are isomorphic for all  $(G, i, v, R)$  in the domain (Proposition 2.5). As a consequence, Lemma 4.7 holds for FP+C-transduction  $\Theta^*(o, z, X)$  in a similar way: For a graph  $G \in \mathcal{C}$  and a suitable triple  $(i, v, R) \in G^{(o,z,X)}$  the tuple  $(G, i, v, R)$  is in the domain of  $\Theta^*(o, z, X)$ , and the LO-colored graph  $\Theta^*[G, i, v, R]$  is isomorphic to  $G_{i,v}^*$ .



Let  $G \in \mathcal{C}$  and let  $(i, v, R)$  be a suitable triple for  $G$ . Then  $\Theta^*[G, i, v, R]$ , as it is isomorphic to  $G_{i,v}^*$ , is an LO-colored graph in  $\mathcal{C}_{\mathcal{K}\mathcal{I}}^*$ . Transduction  $\Theta^c$  is an FP+C-canonicalization for the class  $\mathcal{C}_{\mathcal{K}\mathcal{I}}^*$  of LO-colored graphs. According to Proposition 2.3 we can compose  $\Theta^*(o, z, X)$  and  $\Theta^c$ . We obtain a parameterized FP+C[ $\{E\}, \{V, E, M, \trianglelefteq, L, \leq\}$ ]-transduction  $\Theta^{*c}(o, z, X)$  where  $(G, i, v, R) \in \text{Dom}(\Theta^{*c}(o, z, X))$  for all  $G \in \mathcal{C}$  and  $(i, v, R) \in \text{Suit}(G)$ .

As  $\Theta^*[G, i, v, R]$  and  $G_{i,v}^*$  are isomorphic for  $G \in \mathcal{C}$  and suitable triples  $(i, v, R)$  for  $G$ , and  $\Theta^c$  is a canonicalization, the ordered LO-colored graph  $\Theta^c[\Theta^*[G, i, v, R]]$  is an ordered copy of  $G_{i,v}^*$ . Further, for all  $(G, i, v, R) \in \text{Dom}(\Theta^{*c}(o, z, X))$  the ordered LO-colored graphs  $\Theta^{*c}[G, i, v, R]$  and  $\Theta^c[\Theta^*[G, i, v, R]]$  are isomorphic. Thus,  $\Theta^{*c}[G, i, v, R]$  also is an ordered copy of  $G_{i,v}^*$  for  $G \in \mathcal{C}$  and suitable triples  $(i, v, R)$  for  $G$ . We denote the ordered copy  $\Theta^{*c}[G, i, v, R]$  of  $G_{i,v}^*$  by  $K_{i,v}^*$ .

The relations between the different parameterized transductions used in Step 2 are illustrated in Figure 8.

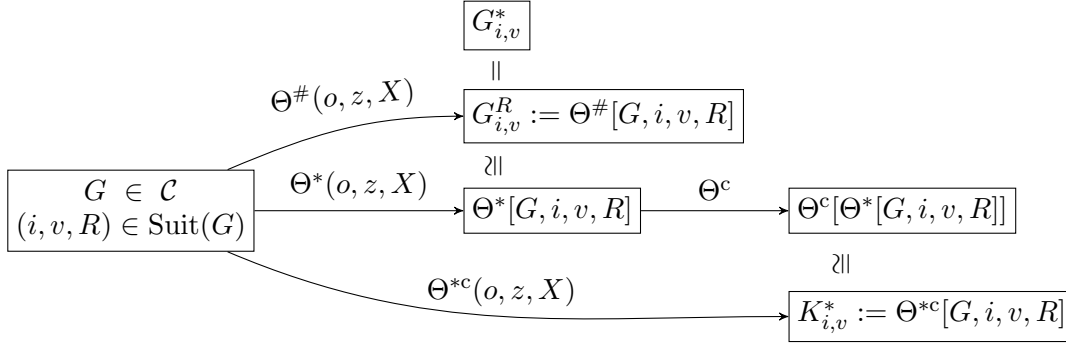


FIGURE 8. Overview of the parameterized transductions in Step 2

**Step 3: Defining the Edge Relation of the Canon  $f(G[D_{i,v}])$ .** In the following we construct an FP+C[ $\{V, E, M, \trianglelefteq, L, \leq\}$ ]-formula that given an ordered LO-colored graph  $K_{i,v}^*$  defines the edge relation of  $f(G[D_{i,v}])$ .

In order to do this, we have to define the function  $\text{nb}(w, n)$  from (4.1) in FP+C. For every vertex  $w$  of  $K_{i,v}^*$  and each vertex  $n$  occurring in the ordered graph of the natural color of the vertex  $w$ ,  $\text{nb}(w, n)$  is the number that the vertex  $n$  is assigned to in the canon of  $G[D_{i,v}]$ . The function  $\text{nb}(w, n)$  depends on the values  $n_{w'}$  for certain vertices  $w'$ . For a vertex  $w$ , the value  $n_w$  is the number of vertices in the graph represented by the natural color of the vertex  $w$ . We can determine this number by finding the only vertex  $u$  for which  $(u, u)$  belongs to the color of  $w$ . Then  $n_w$  is the number of vertices that are smaller than  $u$  with respect to the linear order  $\trianglelefteq(K_{i,v}^*)$  of the basic color elements. We define  $n_w$  via formula  $\varphi_{n_w}$ :

$$\varphi_{n_w}(x, p) := \exists y \left( L(x, y, y) \wedge \#y' (\trianglelefteq(y', y) \wedge y' \neq y) = p \right).$$

Then we have  $K_{i,v}^* \models \varphi_{n_w}[w, n_w]$  if, and only if,  $g_{\text{graph}}(L_w^{\mathbb{N}})$  has  $n_w$  vertices, where  $L_w^{\mathbb{N}}$  is the natural color of  $w$  in  $K_{i,v}^*$ . Notice that the formula  $\varphi_{n_w}$  cannot be satisfied if  $w$  is a basic color element.

In order to define function  $\text{nb}(w, n)$ , we first check whether  $n \in [n_w]$ . Then we count the vertices  $n'$  in the graph of the natural color of  $w$  with  $0 < n' \leq n$ , and the vertices occurring

in the graphs of the natural colors of all vertices  $w'$  that are smaller than  $w$  with respect to the linear order  $\leq(K_{i,v}^*)$ . Thus, we let<sup>13</sup>

$$\begin{aligned} \varphi_{\text{nb}}(x, r, s) := & \exists p \left( \varphi_{n_w}(x, p) \wedge "0 < r \leq p" \right) \wedge \\ & \#(x', r') \left( (x' = x \wedge "0 < r' \leq r") \vee \right. \\ & \left. \exists p' \left( \varphi_{n_w}(x', p') \wedge \leq(x', x) \wedge x' \neq x \wedge "0 < r' \leq p'" \right) \right) = s. \end{aligned}$$

Then  $K_{i,v}^* \models \varphi_{\text{nb}}[w, n, m]$  if and only if  $w$  is a vertex,  $n \in [n_w]$  and  $\text{nb}(w, n) = m$  in  $K_{i,v}^*$ .

With the formula  $\varphi_{\text{nb}}$  we are able to define the edge relation of the canon of  $G[D_{i,v}]$ . We let

$$\varphi_E(s_1, s_2) := \varphi_{E,1}(s_1, s_2) \vee \varphi_{E,2}(s_1, s_2)$$

where

$$\begin{aligned} \varphi_{E,1}(s_1, s_2) := & \exists x_1, x_2, r_1, r_2 \left( E(x_1, x_2) \wedge \bigwedge_{j \in \{1,2\}} \varphi_{\text{nb}}(x_j, r_j, s_j) \right), \\ \varphi_{E,2}(s_1, s_2) := & \exists x, y_1, y_2, r_1, r_2 \left( L(x, y_1, y_2) \wedge r_1 \neq r_2 \wedge \right. \\ & \left. \bigwedge_{j \in \{1,2\}} \left( r_j = \#y(\leq(y, y_j) \wedge y \neq y_j) \wedge \varphi_{\text{nb}}(x, r_j, s_j) \right) \right) \end{aligned}$$

It is not hard to see that  $\varphi_{E,1}[K_{i,v}^*; s_1, s_2]$  and  $\varphi_{E,2}[K_{i,v}^*; s_1, s_2]$  are exactly the edges of the canon of  $G[D_{i,v}]$  obtained by rule 1 and rule 2 from page 23.

**Step 4: Pulling Back the Formula for the Edge Relation.** The formula  $\varphi_E(s_1, s_2)$  is an  $\text{FP+C}[\{V, E, M, \leq, L, \leq\}]$ -formula that defines the edge relation of the canon  $f(G[D_{i,v}])$  for a given ordered LO-colored graph  $K_{i,v}$ . To construct an equivalent  $\text{FP+C}[\{E\}]$ -formula for the  $\text{FP+C}$ -canonization of the class  $\mathcal{C}$ , we pull back  $\varphi_E$  under the parameterized  $\text{FP+C}[\{E\}, \{V, E, M, \leq, L, \leq\}]$ -transduction  $\Theta^{*c}(o, z, X)$ . Hence, we apply the Transduction Lemma (Proposition 2.2) to the formula  $\varphi_E(s_1, s_2)$ . We obtain the  $\text{FP+C}[\{E\}]$ -formula  $\varphi_E^{-\Theta^{*c}}(o, z, X, \bar{q}_1, \bar{q}_2)$ . Let  $G \in \mathcal{C}$  and let  $(i, v, R)$  be a suitable triple for  $G$ . Then  $(G, i, v, R) \in \text{Dom}(\Theta^{*c}(o, z, X))$ . Thus, for all tuples  $\bar{m}_1 \in G^{\bar{q}_1}$  and  $\bar{m}_2 \in G^{\bar{q}_2}$ , we have

$$\begin{aligned} G \models \varphi_E^{-\Theta^{*c}}[i, v, R, \bar{m}_1, \bar{m}_2] & \iff \langle \bar{m}_1 \rangle_G, \langle \bar{m}_2 \rangle_G \in N(K_{i,v}^*) \text{ and} & (4.2) \\ & K_{i,v}^* \models \varphi_E[\langle \bar{m}_1 \rangle_G, \langle \bar{m}_2 \rangle_G]. \end{aligned}$$

The length of tuples  $\bar{q}_1, \bar{q}_2$ , and therefore also of  $\bar{m}_1, \bar{m}_2$ , is the same and depends on the length of the tuple of domain variables of the canonization  $\Theta^c$ . Let  $\ell$  be the length of the listed tuples. Let  $\bar{m}_1 = (m_1^1, \dots, m_1^\ell)$  and let the other tuples be defined analogously. In the following we show that in each tuple of variables we only need the first number variable, as the others are always assigned to 0.

<sup>13</sup> Note that we have  $|M(K_{i,v}^*)| > V(G)$ , and thus,  $|N(K_{i,v}^*)| > V(G)$ . Consequently,  $\text{nb}(w, n) \in N(K_{i,v}^*)$  and a single number variable can represent  $\text{nb}(w, n)$ .

Again, let  $G$  be a graph in  $\mathcal{C}$  and let  $(i, v, R)$  be a suitable triple for  $G$ . Since the vertex set of  $f(G)$  is  $[|V(G)|]$ , we have  $\langle \bar{m}_1 \rangle_G, \langle \bar{m}_2 \rangle_G \in [V(G)]$  for all  $\bar{m}_1, \bar{m}_2 \in N(G)^\ell$  with  $K_{i,v}^* \models \varphi_E[\langle \bar{m}_1 \rangle_G, \langle \bar{m}_2 \rangle_G]$ . Now remember that for a tuple  $\bar{n} = (n_1, \dots, n_\ell) \in N(G)^\ell$ ,

$$\langle \bar{n} \rangle_G = \sum_{i=1}^{\ell} n_i \cdot (|V(G)| + 1)^{i-1}.$$

Consequently, we have  $m_1^j = 0$  and  $m_2^j = 0$  for all  $j > 1$ , which means that  $m_1^1 = \langle \bar{m}_1 \rangle_G$  and  $m_2^1 = \langle \bar{m}_2 \rangle_G$ .

We define  $\phi_E$  as follows:

$$\phi_E(o, z, X, q_1, q_2) := \varphi_E^{-\Theta^{*c}}(o, z, X, (q_1, 0, \dots, 0), (q_2, 0, \dots, 0)).$$

Then, for  $G \in \mathcal{C}$ ,  $(i, v, R) \in \text{Suit}(G)$  and  $m_1, m_2 \in N(G)$  we have

$$G \models \phi_E[i, v, R, m_1, m_2] \iff \text{Vertices } m_1 \text{ and } m_2 \text{ are adjacent in } f(G[D_{i,v}]).$$

**Step 5: Inductive Definition of the Canon  $f(G)$ .** We are now able to inductively define the edge relation of the canon  $f(G)$  of  $G \in \mathcal{C}$ . We let

$$\phi_K(s_1, s_2) := \exists o', z' \left( o' = 0 \wedge s_1 \neq s_2 \wedge [\text{ifp}_{X, (o, z, q_1, q_2)} \phi](o', z', s_1, s_2) \right)$$

where

$$\phi := \phi_1 \vee (\phi_2 \wedge (\phi_E \vee \phi_{n_w}))$$

and

$$\begin{aligned} \phi_1(o, z, q_1, q_2) &:= \text{largest}(o) \wedge q_1 = 1 \wedge q_2 = 1, \\ \phi_2(o, z, X, q_1, q_2) &:= \neg \text{largest}(o) \wedge \exists o', z', q'_1, q'_2 (o + 1 = o' \wedge X(o', z', q'_1, q'_2)), \\ \phi_{n_w}(o, z, q_1, q_2) &:= q_1 = q_2 \wedge q_1 = \#y \varphi_D(o, z, y). \end{aligned}$$

The relational variable  $X$  within the inflationary fixed-point operator of the  $\text{FP+C}$ -formula  $\phi_K$  is of arity 4 and ranges over relations  $R \subseteq N(G) \times V(G) \times N(G)^2$ . Let  $X^\infty$  be the relation assigned to the variable  $X$  after the fixed-point is reached. We show in Lemma 4.9 that for each  $i \in N(G)$  and  $v \in V(G)$  the set of pairs  $\{(n_1, n_2) \mid (i, v, n_1, n_2) \in X^\infty\}$  is the representation of the canon  $f(G[D_{i,v}])$ . For  $i = 0$  and any vertex  $v \in V(G)$  we have  $D_{i,v} = V(G)$ . Therefore, Lemma 4.9 implies the following corollary.

**Corollary 4.8.** *For all  $G \in \mathcal{C}$  and all  $n_1, n_2 \in N(G)$ ,*

$$G \models \phi_K[n_1, n_2] \iff \{n_1, n_2\} \text{ is an edge of the canon } f(G) \text{ of } G.$$

The formula  $\phi$  of the inflationary fixed-point operator is constructed such that  $\phi_1$  defines the basis of the inductive definition. For  $i = |V(G)|$  and all vertices  $v \in V(G)$ , it ensures that the tuples describing the representation of the canon of  $G[D_{i,v}]$  are added to the fixed-point relation in the first step. Thus, all tuples in  $\{(|V(G)|, v, 1, 1) \mid v \in V(G)\}$  are added in the first step. The formulas  $\phi_2$  and  $\phi_E \vee \phi_{n_w}$  take effect in the inductive step. In step  $k$  we add all tuples  $(i, v, n_1, n_2) \in X^\infty$  to the fixed-point relation with  $i = |V(G)| - k + 1$ . The formula  $\phi_2$  ensures that we add only tuples  $(i, v, n_1, n_2)$  if the tuples for  $i + 1$  have already been included to the fixed-point relation. This way,  $i, v$  and the fixed-point relation form a suitable triple. Then,  $\phi_E \vee \phi_{n_w}$  defines the representation of the canon of  $G[D_{i,v}]$ .

In the following lemma we show inductively that the formula  $\phi_K$  uses an inflationary fixed-point operator which in stage  $k$  of its iteration defines the representations of the canons of all  $G[D_{i,v}]$  with  $v \in V(G)$  and  $i \geq |V(G)| - k + 1$ .

**Lemma 4.9.** *Let  $X^k$  be the fixed-point relation that we get at stage  $k$  of the iteration of the inflationary fixed-point operator in the formula  $\phi_K$ . Further, let  $S^k$  be the set of all tuples  $(i, v, n_1, n_2) \in N(G) \times V(G) \times N(G)^2$  where  $i \geq |V(G)| - k + 1$  and  $(n_1, n_2)$  is in  $g_{\text{rep}}(f(G[D_{i,v}]))$ , the representation of the canon of  $G[D_{i,v}]$ . Then  $X^k = S^k$ .*

*Proof.* Of course, for  $k = 0$  we have  $X^k = \emptyset$  and  $S^k = \emptyset$ . For  $k = 1$ , it is easy to see that there exists no tuple that satisfies the formula  $\phi_2$  since  $X^0 = \emptyset$ . Consequently,  $X^1$  is the set  $\phi_1[G; o, z, q_1, q_2] = \{(|V(G)|, v, 1, 1) \mid v \in V(G)\}$ . Further, for all  $v \in V(G)$  the representation of the canon of  $G[D_{|V(G)|,v}]$  is  $\{(1, 1)\}$ , and therefore,  $X^1 = S^1$ . Now assume  $k \geq 1$ , and let  $X^k = S^k$ . In the following we prove that  $X^{k+1} = S^{k+1}$  by showing that  $X_j^{k+1} = S_j^{k+1}$  for all  $j \in N(G)$ , where  $S_j^{k+1}$  is the set of all tuples  $(j, v, n_1, n_2) \in S^{k+1}$  and  $X_j^{k+1}$  is the set of tuples  $(j, v, n_1, n_2) \in X^{k+1}$ .

It is easy to see that  $X_j^{k+1} = S_j^{k+1}$  for  $j = |V(G)|$ : We have already shown that  $\phi_1[G; o, z, q_1, q_2] = X^1$  and that  $X_1 = S_1$ . Further, the relation  $\phi_2[G, \alpha[X^k/X]; o, z, q_1, q_2]$  cannot contain any tuples  $(i, v, n_1, n_2)$  with  $i = |V(G)|$ . Therefore,  $X_j^{k+1} = S_1$ . Since  $S_1 = S_j^{k+1}$  for  $j = |V(G)|$ , we have  $X_j^{k+1} = S_j^{k+1}$ .

Next, let us consider  $j < |V(G)| - k$ . Then  $j < |V(G)|$ , and there does not exist a tuple  $(i, v, n_1, n_2) \in \phi_1[G; o, z, q_1, q_2]$  with  $i = j$ . Further, by inductive assumption we have  $X^k = S^k$ , and by definition we know that the set  $S^k$  does not contain any tuples  $(j', v, n_1, n_2)$  with  $j' < |V(G)| - k + 1$ . Consequently, there cannot be a tuple  $(i, v, n_1, n_2)$  in  $\phi_2[G, \alpha[X^k/X]; o, z, q_1, q_2]$  with  $i = j$ . Thus, for  $j < |V(G)| - k$  we have  $X_j^{k+1} = \emptyset$ , and since  $S_j^{k+1}$  is also empty, we obtain  $X_j^{k+1} = S_j^{k+1}$ .

Now, suppose  $|V(G)| - k \leq j < |V(G)|$ . Then the relation  $\phi_1[G; o, z, q_1, q_2]$  does not contain any tuples  $(i, v, m_1, m_2)$  with  $i = j$ . However, there exist a vertex  $v \in V(G)$  and numbers  $n_1, n_2 \in N(G)$  such that  $(j, v, n_1, n_2) \in \phi_2[G, \alpha[X^k/X]; o, z, q_1, q_2]$  because  $X^k = S^k$ , by inductive assumption, and  $S_{j'}^k$  is non-empty for all  $j' \geq |V(G)| - k + 1$ , by definition. Since we have  $X^k = S^k$ , and  $j + 1 \geq |V(G)| - k + 1$  and  $j < |V(G)|$ , the relation  $\{(n_1, n_2) \mid (j + 1, w, n_1, n_2) \in X^k\}$  is the representation of the canon of  $G[D_{j+1,w}]$  for all  $w \in V(G)$ . Therefore,  $(j, v, X^k)$  is a suitable triple for all  $v \in V(G)$ . As shown in Step 4, the relation  $\phi_E[G, j, v, X^k; q_1, q_2]$  is the edge relation of the canon  $f(G[D_{j,v}])$  of  $G[D_{j,v}]$  for suitable triples  $(j, v, X^k)$ . Further,  $\phi_{n_w}[G, j, v; q_1, q_2] = \{(|D_{j,v}|, |D_{j,v}|)\}$ . Thus, the relation  $(\phi_E \vee \phi_{n_w})[G, j, v, X^k; q_1, q_2]$ , is the representation of the canon of  $G[D_{j,v}]$  for all vertices  $v \in V(G)$ , and it follows that  $X_j^{k+1} = S_j^{k+1}$ .  $\square$

*Proof of Theorem 4.3.* Let  $\mathcal{C}$  be a class of graphs that is closed under induced subgraphs. Further, let  $\mathcal{C}_{\text{prim}}^*$  admit FP+C-definable parameterized canonization. Then there exists an FP+C-canonization  $\Theta^c$  of the class  $\mathcal{C}_{\mathcal{I}}^*$  (Observations 4.4 and 4.5). Now according to Corollary 4.8, the FP+C-formula  $\phi_K$  defines the edge relation of the canon  $f(G)$  for all  $G \in \mathcal{C}$ .

Therefore,  $\Theta' = (\theta'_U, \theta'_E, \theta'_\leq)$  with

$$\begin{aligned}\theta'_U(s_1) &:= 0 \leq s_1, \\ \theta'_E(s_1, s_2) &:= \phi_K(s_1, s_2), \\ \theta'_\leq(s_1, s_2) &:= s_1 \leq s_2\end{aligned}$$

is an FP+C-canonization of the graph class  $\mathcal{C}$ .  $\square$

**4.4. Variations of the Modular Decomposition Theorem.** In this section we show variations of the Modular Decomposition Theorem, which might be helpful in future applications. Let  $\mathcal{C}$  be a graph class that is closed under taking induced subgraphs. We prove that  $\mathcal{C}$  admits FP+C-definable canonization if the class of prime graphs of  $\mathcal{C}$  admits FP+C-definable orders, and we present an analog of the Modular Decomposition Theorem for polynomial-time computable canonization. The Modular Decomposition Theorem and the just mentioned analog of it require a canonization of the class  $\mathcal{C}_{\text{prim}}^*$ . We also show that we can relax this requirement, and prove that a canonization of the class of all prime graphs from  $\mathcal{C}$  that are colored with elements from a linearly ordered set is sufficient.

An FP+C-formula  $\varphi(\bar{x}, y, y')$  defines orders on a class  $\mathcal{C}$  of graphs if for all graphs  $G \in \mathcal{C}$  there is a tuple  $\bar{v} \in G^{\bar{x}}$  such that the binary relation  $\varphi[G, \bar{v}; y, y']$  is a linear order on  $V(G)$ . We say a graph class  $\mathcal{C}$  admits FP+C-definable orders, if there exists an FP+C-formula that defines orders on  $\mathcal{C}$ .

For a graph class  $\mathcal{C}$ , let  $\mathcal{C}_{\text{prim}}$  be the class of all prime graphs from  $\mathcal{C}$ .

**Corollary 4.10.** *Let  $\mathcal{C}$  be a graph class that is closed under induced subgraphs. If  $\mathcal{C}_{\text{prim}}$  admits FP+C-definable orders, then  $\mathcal{C}$  admits FP+C-definable canonization.*

*Proof.* Let graph class  $\mathcal{C}$  be closed under induced subgraphs, and let  $\varphi(\bar{x}, y_1, y_2)$  be an FP+C-formula that defines orders on  $\mathcal{C}_{\text{prim}}$ . We use the formula  $\varphi$  to define a parameterized FP+C-canonization of the class  $\mathcal{C}_{\text{prim}}^*$ . Then Corollary 4.10 follows directly from the Modular Decomposition Theorem.

First of all, we use  $\varphi(\bar{x}, y_1, y_2)$  to define an FP+C-formula  $\varphi_{\text{lin}}(\bar{x})$  where for all  $G \in \mathcal{C}_{\text{prim}}$  and  $\bar{v} \in G^{\bar{x}}$  we have

$$G \models \varphi_{\text{lin}}(\bar{v}) \iff \varphi[G, \bar{v}; y, y'] \text{ is a linear order on } V(G).$$

As it can be tested in first-order logic whether a binary relation is a linear order, i.e., a transitive, antisymmetric and total relation, the formula  $\varphi_{\text{lin}}$  is FP+C-definable.

Since we can define orders on  $\mathcal{C}_{\text{prim}}$ , we can also define orders on the underlying graphs of the LO-colored graphs from  $\mathcal{C}_{\text{prim}}^*$ . We simply pull back the formula  $\varphi$  under FP+C[ $(\{V, E, M, \triangleleft, L\}, \{E\})$ ]-transduction  $\Theta = (V(x), E(x, x'))$ , which maps every LO-colored graph to (an isomorphic copy of) its underlying graph. We do the same for the formula  $\varphi_{\text{lin}}$ .

Let  $\leq_V$  be a linear order on the vertex set  $V(G^*)$  of the underlying graph of  $G^* \in \mathcal{C}_{\text{prim}}^*$ . We can use  $\leq_V$  and the linear order  $\triangleleft$  on the set  $M(G^*)$  of basic color elements to construct a linear order  $\leq^*$  on the universe  $U(G^*)$  of  $G^*$ . We let

$$\leq^* := \leq_V \cup \triangleleft \cup \{(v, m) \mid v \in V(G^*), m \in M(G^*)\}. \quad (4.3)$$

We now define a parameterized **FP+C**-canonization  $\Theta_{\leq}(x)$ , which maps each LO-colored graph  $G^* \in \mathcal{C}_{\text{prim}}^*$  to an ordered copy  $(G^*, \leq^*)$ . Valid parameters of this transduction are all tuples  $\bar{v} \in G^{\bar{x}}$  where  $\varphi[G, \bar{v}; y, y']$  is a linear order on the vertex set  $V(G)$ . We let

$$\Theta_{\leq}(\bar{x}) = (\theta_{\text{dom}}, \theta_U, \theta_V, \theta_E, \theta_M, \theta_{\triangleleft}, \theta_L, \theta_{\leq}),$$

where

$$\begin{aligned} \theta_{\text{dom}}(\bar{x}) &:= \varphi_{\text{lin}}^{-\Theta}(\bar{x}), & \theta_M(\bar{x}, y) &:= M(y), \\ \theta_U(\bar{x}, y) &:= \top, & \theta_{\triangleleft}(\bar{x}, y, y') &:= \triangleleft(y, y'), \\ \theta_V(\bar{x}, y) &:= V(y), & \theta_L(\bar{x}, y, y', y'') &:= L(y, y', y''), \\ \theta_E(\bar{x}, y, y') &:= E(y, y'), \end{aligned}$$

and

$$\theta_{\leq}(\bar{x}, y, y') := \varphi^{-\Theta}(\bar{x}, y, y') \vee \triangleleft(y, y') \vee (V(y) \wedge M(y')).$$

The formula  $\theta_{\text{dom}}$ , that is, the pull-back of  $\varphi_{\text{lin}}$ , defines the valid parameters, and the formula  $\theta_{\leq}$  defines the linear order  $\leq^*$  from (4.3) by using the pull-back of the **FP+C**-formula  $\varphi$ .  $\square$

In Section 4.3 we recursively defined a canonization mapping for the graph class  $\mathcal{C}$ . It is not hard to see that this canonization mapping can be computed in polynomial time if there exists a canonization mapping for  $\mathcal{C}_{\text{prim}}^*$  that is computable in polynomial time. Thus, the Modular Decomposition Theorem can be transferred to polynomial time:

**Corollary 4.11.** *Let  $\mathcal{C}$  be a class of graphs that is closed under induced subgraphs. If  $\mathcal{C}_{\text{prim}}^*$  admits polynomial-time canonization, then  $\mathcal{C}$  admits polynomial-time canonization.*

Within the Modular Decomposition Theorem and Corollary 4.11 it is possible to relax the requirement of a canonization of the class  $\mathcal{C}_{\text{prim}}^*$  to a canonization of the class  $\mathcal{C}'_{\text{prim}}$ , that is, the class of all prime graphs from  $\mathcal{C}$  that are colored with elements from a linearly ordered set. More precisely, for a class  $\mathcal{C}$  of graphs that is closed under induced subgraphs, we let  $\mathcal{C}'_{\text{prim}}$  be the class of all tuples  $G' = (V, E, C, \triangleleft, f)$  with the following properties:

- (1) The pair  $(V, E)$  is a prime graph from the class  $\mathcal{C}$ , and  $|V| \geq 4$ .
- (2) The set of *colors*  $C$  is a non-empty finite set with  $C \cap V = \emptyset$ .
- (3) The binary relation  $\triangleleft \subseteq C^2$  is a linear order on  $C$ .
- (4) The *coloring*  $f \subseteq V \times C$  is a binary relation where for each vertex  $v \in V$  there exists exactly one color  $c \in C$  with  $(v, c) \in f$ . We also denote this color  $c$  by  $f(v)$ . We say a color  $c \in C$  is *used* if there exists a vertex  $v \in V$  with  $c = f(v)$ .

To represent a colored graph  $G' = (V, E, C, \triangleleft, f) \in \mathcal{C}'_{\text{prim}}$  as a logical structure we extend the 5-tuple by a set  $U$  to a 6-tuple  $(U, V, E, M, \triangleleft, L)$ , and we require that  $U = V \dot{\cup} C$  additionally to the properties above.

In the following, we call the colors of an LO-colored graph *LO-colors*, and we say an LO-color  $D$  is *used* if there exists a vertex that is colored with  $D$ .

**Lemma 4.12.** *Let  $\mathcal{C}$  be a class of graphs. If there exists a (parameterized) **FP+C**-canonization of  $\mathcal{C}'_{\text{prim}}$ , then there also exists one of  $\mathcal{C}_{\text{prim}}^*$ .*

*Proof.* Let  $\mathcal{C}$  be a graph class, and let us suppose that the class  $\mathcal{C}'_{\text{prim}}$  admits **FP+C**-definable (parameterized) canonization. According to Lemma 2.6, we can assume that there exists an **FP+C**-canonization  $\Theta'^c$  of  $\mathcal{C}'_{\text{prim}}$ . We show that there exists an **FP+C**-canonization  $\Theta$  of  $\mathcal{C}_{\text{prim}}^*$ .

In order to do this, we first map each LO-colored graph  $G^* \in \mathcal{C}_{\text{prim}}^*$  to a colored graph  $G' \in \mathcal{C}'_{\text{prim}}$ . Within this mapping we replace the set of used LO-colors with the initial segment of the natural numbers. We define the mapping by an **FP+C**-counting transduction  $\Theta^\#$ , and apply Proposition 2.5, to show that there exists an **FP+C**-transduction  $\Theta^{*'}$  that defines essentially the same mapping. Then we compose  $\Theta^{*'}$  and the canonization  $\Theta^{/c}$  to a transduction  $\Theta^{*c}$  (Proposition 2.3). An overview of the different transductions can be found in Figure 9. Finally, we construct  $\Theta$  with the help of  $\Theta^{*c}$ , and substitute the colors used in the ordered colored graphs again by their corresponding LO-colors.

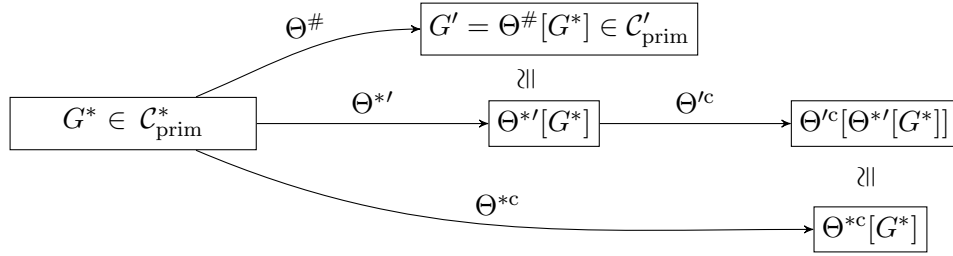


FIGURE 9. Overview of the transductions

First, we map each LO-colored graph  $G^* = (U, V, E, M, \leq, L) \in \mathcal{C}_{\text{prim}}^*$  to a colored graph  $G' = (U', V', E', C', \leq', f') \in \mathcal{C}'_{\text{prim}}$ . The linear order  $\leq$  on  $M$  induces a linear order  $\leq_{\mathcal{L}}$  on the set of LO-colors  $\mathcal{L} := \{L_v \mid v \in V\}$ . We construct  $G'$  by substituting for each vertex  $v \in V$  the LO-color  $L_v$  by the number corresponding to the position of  $L_v$  within the linear order  $\leq_{\mathcal{L}}$  on  $\mathcal{L}$ . It is not hard to see that there exists an **FP+C**-formula  $\varphi_{\text{pos}}^*(x, p)$  such that for all  $G^* \in \mathcal{C}_{\text{prim}}^*$ ,  $v \in V(G^*)$  and  $i \in N(G^*)$  we have  $G^* \models \varphi_{\text{pos}}^*[v, i]$  if, and only if, the LO-color  $L_v$  is at position  $i$  regarding  $\leq_{\mathcal{L}}$ . We let  $C'$  be the set of numbers  $N(G^*)$ , and we let  $\leq'$  be the natural linear order on  $N(G^*)$ .

The following **FP+C**-counting transduction  $\Theta^\#$  maps each LO-colored graph  $G^* \in \mathcal{C}_{\text{prim}}^*$  to a colored graph  $G' \in \mathcal{C}'_{\text{prim}}$  as described above. We let

$$\Theta^\# = (\theta_U^\#(x), \theta_V^\#(x), \theta_E^\#(x, x'), \theta_C^\#(p), \theta_{\leq}^\#(p, p'), \theta_f^\#(x, p)),$$

where

$$\begin{aligned} \theta_U^\#(x) &:= V(x), & \theta_C^\#(p) &:= \top, \\ \theta_V^\#(x) &:= V(x), & \theta_{\leq}^\#(p, p') &:= p \leq p', \\ \theta_E^\#(x, x') &:= E(x, x'), & \theta_f^\#(x, p) &:= \varphi_{\text{pos}}^*(x, p). \end{aligned}$$

According to Proposition 2.5, there exists an **FP+C**-transduction  $\Theta^{*'}$ , such that  $\Theta^\#[G^*]$  and  $\Theta^{*'}[G^*]$  are isomorphic for all LO-colored graphs  $G^* \in \mathcal{C}_{\text{prim}}^*$ . We compose the two transductions  $\Theta^{*'}$  and  $\Theta^{/c}$  (Proposition 2.3) and obtain an **FP+C**  $\{\{V, E, M, \leq, L\}, \{V, E, C, \leq, f, \leq\}\}$ -transduction  $\Theta^{*c} = (\theta_U^{*c}, \theta_{\approx}^{*c}, \theta_V^{*c}, \theta_E^{*c}, \theta_C^{*c}, \theta_{\leq}^{*c}, \theta_f^{*c}, \theta_{\leq}^{*c})$ . The transduction  $\Theta^{*c}$  maps each LO-colored graph  $G^* \in \mathcal{C}_{\text{prim}}^*$  to the canon  $\Theta^{*c}[G^*]$  of the colored graph  $\Theta^{*'}[G^*] \in \mathcal{C}'_{\text{prim}}$ . We use the linear order on the set of colors of  $\Theta^{*c}[G^*]$  to replace the color of each vertex again with the corresponding LO-color.

Let  $\bar{u}$  be the tuple of domain variables of  $\Theta^{*c}$ . Let  $\approx$  be the equivalence relation generated by  $\theta_{\approx}^{*c}[G^*; \bar{u}, \bar{u}']$  on  $(G^*)^{\bar{u}}$ . Similarly to the formula  $\varphi_{\text{pos}}^*(x, p)$ , we can define an **FP+C**-formula  $\varphi'_{\text{pos}}(\bar{u}, p)$  where for all  $G^* \in \mathcal{C}_{\text{prim}}^*$ ,  $\bar{a} \in (G^*)^{\bar{u}}$  and  $i \in N(G^*)$  we have  $G^* \models \varphi'_{\text{pos}}[\bar{a}, i]$  if, and

only if,  $\bar{a}/\approx \in V(\Theta^{*c}[G^*])$  and the position of the color of  $\bar{a}/\approx$  is  $i$  regarding the linear order induced by  $\preceq(\Theta^{*c}[G^*])$  on the set  $\{f(\Theta^{*c}[G^*])(v) \mid v \in V(\Theta^{*c}[G^*])\}$  of colors.

We use  $\Theta^{*c}$  and the formulas  $\varphi_{\text{pos}}^*$  and  $\varphi'_{\text{pos}}$  to define an **FP+C**-canonization  $\Theta$  for the class  $\mathcal{C}_{\text{prim}}^*$ . We let

$$\Theta = (\theta_U, \theta_{\approx}, \theta_V, \theta_E, \theta_M, \theta_{\preceq}, \theta_L, \theta_{\leq}),$$

where

$$\begin{aligned} \theta_U(\bar{u}, z) &:= \theta_V^{*c}(\bar{u}), \\ \theta_{\approx}(\bar{u}, z, \bar{u}', z') &:= (\theta_{\approx}^{*c}(\bar{u}, \bar{u}') \wedge \neg M(z) \wedge \neg M(z')) \vee (z = z' \wedge M(z) \wedge M(z')), \\ \theta_V(\bar{u}, z) &:= \theta_V^{*c}(\bar{u}) \wedge \neg M(z), \\ \theta_E(\bar{u}, z, \bar{u}', z') &:= \theta_E^{*c}(\bar{u}, \bar{u}') \wedge \neg M(z) \wedge \neg M(z'), \\ \theta_M(\bar{u}, z) &:= M(z), \\ \theta_{\preceq}(\bar{u}, z, \bar{u}', z') &:= \preceq(z, z') \wedge M(z) \wedge M(z'), \\ \theta_L(\bar{u}, z, \bar{u}', z', \bar{u}'', z'') &:= \theta_V(\bar{u}, z) \wedge \theta_M(\bar{u}', z') \wedge \theta_M(\bar{u}'', z'') \\ &\quad \wedge \exists p \exists x (\varphi'_{\text{pos}}(\bar{u}, p) \wedge \varphi_{\text{pos}}^*(x, p) \wedge L(x, z', z'')), \\ \theta_{\leq}(\bar{u}, z, \bar{u}', z') &:= (\theta_{\leq}^{*c}(\bar{u}, \bar{u}') \wedge \neg M(z) \wedge \neg M(z')) \vee \theta_{\preceq}(\bar{u}, z, \bar{u}', z') \vee (\neg M(z) \wedge M(z')). \end{aligned}$$

Let  $G^* \in \mathcal{C}_{\text{prim}}^*$ . Then the vertices of  $\Theta[G^*]$  correspond to the vertices of  $\Theta^{*c}[G^*]$ , and the underlying graph of  $\Theta[G^*]$  is isomorphic to the underlying graph of  $\Theta^{*c}[G^*]$ . The basic color elements of  $\Theta[G^*]$  and their linear order correspond to the basic color elements of  $G^*$  and their linear order. The LO-color of each vertex of  $\Theta[G^*]$  is defined as follows: First, the position  $i$  of the color of the corresponding vertex of  $\Theta^{*c}[G^*]$  is determined. Then we pick an (arbitrary) vertex  $v$  of  $G^*$  whose LO-color is at position  $i$  with respect to  $\preceq_{\mathcal{L}}$ . We assign the LO-color of  $v$  to the vertex of  $\Theta[G^*]$ . To obtain the linear order on the elements of  $\Theta[G^*]$ , we combine the linear order on the vertices of  $\Theta[G^*]$ , which is obtained from the linear on the vertices of  $\Theta^{*c}[G^*]$ , and the linear order  $\preceq(\Theta[G^*])$  on the basic color elements of  $\Theta[G^*]$ . Consequently,  $\Theta[G^*]$  is an ordered copy of  $G^*$  for all  $G^* \in \mathcal{C}_{\text{prim}}^*$ .  $\square$

The following corollary is a direct consequence of Lemma 4.12 and the Modular Decomposition Theorem.

**Corollary 4.13.** *Let  $\mathcal{C}$  be a class of graphs that is closed under induced subgraphs. If  $\mathcal{C}'_{\text{prim}}$  admits **FP+C**-definable (parameterized) canonization, then  $\mathcal{C}$  admits **FP+C**-definable canonization.*

Within the proof of Lemma 4.12, it is described how to obtain a polynomial-time canonization mapping for  $\mathcal{C}_{\text{prim}}^*$  if there exists one for  $\mathcal{C}'_{\text{prim}}$ . Hence, Corollary 4.11 implies the following:

**Corollary 4.14.** *Let  $\mathcal{C}$  be a class of graphs that is closed under induced subgraphs. If  $\mathcal{C}'_{\text{prim}}$  admits polynomial-time canonization, then  $\mathcal{C}$  admits polynomial-time canonization.*

**Remark 4.15.** Note that if  $\mathcal{C}_{\text{prim}}$  admits polynomial-time canonization, then it does not necessarily follow that  $\mathcal{C}$  admits polynomial-time canonization. Let us suppose there is a deterministic Turing machine  $M$  that computes in polynomial time for all graphs  $H \in \mathcal{C}_{\text{prim}}$  a linear order  $\leq_H$  on the vertex set of  $H$ , such that  $H \cong H'$  implies  $(H, \leq_H) \cong (H', \leq_{H'})$  for all graphs  $H, H' \in \mathcal{C}_{\text{prim}}$ . Then  $\mathcal{C}_{\text{prim}}$  admits polynomial-time canonization. Let us consider two graphs  $G$  and  $G'$  that are isomorphic. Assume there exists an isomorphism  $h$  between their modular contractions  $G_{\sim}$  and  $G'_{\sim}$  such that  $h(v/\sim) = v'/\sim$  but  $v/\sim$  and  $v'/\sim$



represent modules that induce non-isomorphic graphs. For example, suppose that  $G$  and  $G'$  are isomorphic to the graph in Figure 1B, and that  $v/\sim$  and  $v'/\sim$  correspond to the two ends of the modular contraction (a path of length 4). Depending on the input strings that represent  $G_\sim$  and  $G'_\sim$ , it is possible that  $M$  computes linear orders  $\leq_{G_\sim}$  and  $\leq_{G'_\sim}$  such that  $v/\sim$  and  $v'/\sim$  occur at the same position in  $\leq_{G_\sim}$  and  $\leq_{G'_\sim}$ , respectively. If we use  $\leq_{G_\sim}$  and the linear orders on the maximal proper modules of  $G$  to construct a linear order for an isomorphic copy of  $G$ , and use  $\leq_{G'_\sim}$  and the linear orders on the maximal proper modules of  $G'$  to construct a linear order for an isomorphic copy of  $G'$ , we obtain ordered copies of  $G$  and  $G'$  that are not isomorphic.

## 5. CAPTURING POLYNOMIAL TIME ON PERMUTATION GRAPHS

In this section we use the Modular Decomposition Theorem to prove that there exists an **FP+C**-canonization of the class of permutation graphs. More precisely, for prime permutation graphs  $G$  we show that there exist parameterized **FP**-formulas that define the strict linear orders of a realizer of  $G$ . This directly implies that the class of prime permutation graphs admits **FP**-definable orders. As the class of permutation graphs is closed under induced subgraphs, we can apply Corollary 4.10, and obtain that canonization of the class of permutation graphs is definable in **FP+C**. As a consequence, **FP+C** captures **PTIME** on the class of permutation graphs.

**5.1. Preliminaries.** Let  $G = (V, E)$  be a graph, and let  $<_1$  and  $<_2$  be two strict linear orders on the vertex set  $V$ . We call  $(<_1, <_2)$  a *realizer* of  $G$  if  $u, v \in V$  are adjacent in  $G$  if and only if they occur in different order in  $<_1$  and  $<_2$ , that is,  $u <_1 v$  and  $v <_2 u$ , or  $v <_1 u$  and  $u <_2 v$ . A graph  $G$  is a *permutation graph* if there exists a realizer of  $G$ . Figure 10 shows an example of a permutation graph and a realizer of it. A detailed introduction to permutation graphs can be found in [Gol91].

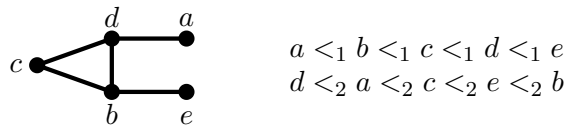


FIGURE 10. A permutation graph and a realizer

Let  $\triangleleft_1$  and  $\triangleleft_2$  be two binary relations. We call the pair  $(\triangleleft_1, \triangleleft_2)$  *transitive* if each of the binary relations  $\triangleleft_1$  and  $\triangleleft_2$  is transitive. Further, we let the transitive closure  $(\triangleleft_1, \triangleleft_2)^T$  of  $(\triangleleft_1, \triangleleft_2)$  be the pair  $(\triangleleft_1^T, \triangleleft_2^T)$  where  $\triangleleft_1^T$  and  $\triangleleft_2^T$  is the transitive closure of  $\triangleleft_1$  and  $\triangleleft_2$ , respectively. Let  $G = (V, E)$  be a graph and  $(\triangleleft_1, \triangleleft_2)$  be a pair of binary relations on  $V$ . The pair  $(\triangleleft_1, \triangleleft_2)$  is *closed under  $E$*  if for all vertices  $u, v \in V$  and all  $i \in [2]$  the following holds:

- If  $u \triangleleft_i v$  and  $\{u, v\} \in E$ , then  $v \triangleleft_{3-i} u$ .
- If  $u \triangleleft_i v$  and  $\{u, v\} \notin E$ , then  $u \triangleleft_{3-i} v$ .

Notice that each realizer of a graph  $G = (V, E)$  is closed under the edge relation  $E$ . Moreover, we observe the following.

**Observation 5.1.** Let  $G = (V, E)$  be a graph. Then a pair of binary relations  $(\triangleleft_1, \triangleleft_2)$  is a realizer of  $G$  if, and only if,  $\triangleleft_1$  and  $\triangleleft_2$  are strict linear orders and  $(\triangleleft_1, \triangleleft_2)$  is closed under the edge relation  $E$ .

Now for all  $i \in [2]$  we let

$$D_{3-i}^E := \{(v, u) \mid u \triangleleft_i v \text{ and } \{u, v\} \in E\} \text{ and}$$

$$D_{3-i}^F := \{(u, v) \mid u \triangleleft_i v \text{ and } \{u, v\} \notin E\},$$

and we let  $(\triangleleft_1, \triangleleft_2)^E$  be the pair  $(\triangleleft_1^E, \triangleleft_2^E)$  of relations where for all  $i \in [2]$  we have

$$\triangleleft_i^E := \triangleleft_i \cup D_i^E \cup D_i^F.$$

**Observation 5.2.** Let  $G = (V, E)$  be a graph and  $\triangleleft_1, \triangleleft_2$  be binary relations on  $V$ . Then  $(\triangleleft_1, \triangleleft_2)^E$  is closed under  $E$ .

Let  $\triangleleft_1, \triangleleft_2$  be binary relations on  $V$ . It is not hard to see that the relations  $\triangleleft_1^E$  and  $\triangleleft_2^E$  are minimal with the property that  $\triangleleft_1 \subseteq \triangleleft_1^E$ ,  $\triangleleft_2 \subseteq \triangleleft_2^E$  and  $(\triangleleft_1^E, \triangleleft_2^E)$  is closed under  $E$ . Thus, we call  $(\triangleleft_1, \triangleleft_2)^E$  the *closure of  $(\triangleleft_1, \triangleleft_2)$  under  $E$* .

**5.2. Defining Orders on Prime Permutation Graphs.** We now show that the class of prime permutation graphs admits FP-definable orders. It is known that the realizer of a prime permutation graph is unique up to reversal and exchange [Möh85]. Thus, a prime permutation graph has at most 4 different realizers. We prove that the strict linear orders of these realizers are definable in FP.

Let  $G = (V, E)$  be a prime permutation graph. For each  $w \in V$  we define two binary relations  $\triangleleft_1^w$  and  $\triangleleft_2^w$  on the vertex set  $V$ . If there exists a realizer  $(\triangleleft_1, \triangleleft_2)$  of  $G$  where  $w$  is the first vertex of the first strict linear order  $\triangleleft_1$ , then it will turn out that  $(\triangleleft_1^w, \triangleleft_2^w) = (\triangleleft_1, \triangleleft_2)$ .

Let  $w \in V$ . In order to construct the binary relations  $\triangleleft_1^w$  and  $\triangleleft_2^w$ , we recursively define relations  $\triangleleft_{1,k}^w$  and  $\triangleleft_{2,k}^w$  on the vertex set  $V$  for all  $k \geq 0$ . We begin with defining the relations for  $k = 0$ . As  $w$  is the first element of the first strict linear order of the realizer that we want to reconstruct, we let

$$\triangleleft_{1,0}^w := \{(w, v) \mid v \in V, v \neq w\} \quad \text{and} \quad \triangleleft_{2,0}^w := \emptyset.$$

Now, we recursively define  $\triangleleft_{1,k+1}^w$  and  $\triangleleft_{2,k+1}^w$  for all  $k > 0$  as follows:

$$(\triangleleft_{1,k+1}^w, \triangleleft_{2,k+1}^w) := ((\triangleleft_{1,k}^w, \triangleleft_{2,k}^w)^E)^T.$$

Clearly, for all vertices  $w \in V$  and all  $k \geq 0$  the relations satisfy the property that

$$\triangleleft_{1,k}^w \subseteq \triangleleft_{1,k+1}^w \quad \text{and} \quad \triangleleft_{2,k}^w \subseteq \triangleleft_{2,k+1}^w.$$

Since the vertex set is finite, there exists an  $m \geq 0$  such that  $\triangleleft_{i,m}^w = \triangleleft_{i,m+1}^w$  for all  $i \in [2]$ . We define  $\triangleleft_i^w := \triangleleft_{i,m}^w$  for  $i \in [2]$ .

In the following, let  $(\triangleleft_1, \triangleleft_2)$  be a realizer of the permutation graph  $G$ , and let  $w$  be the first element of  $\triangleleft_1$ . We show that  $(\triangleleft_1^w, \triangleleft_2^w) = (\triangleleft_1, \triangleleft_2)$ . By definition of  $(\triangleleft_{1,0}^w, \triangleleft_{2,0}^w)$  we have  $\triangleleft_{1,0}^w \subseteq \triangleleft_1$  and  $\triangleleft_{2,0}^w \subseteq \triangleleft_2$ . Further, we obtain  $(\triangleleft_1^w, \triangleleft_2^w)$  from  $(\triangleleft_{1,0}^w, \triangleleft_{2,0}^w)$  by recursively taking the closure under the edge relation  $E$  and the transitive closure. Since the realizer  $(\triangleleft_1, \triangleleft_2)$  is closed under both, the following observation holds.

**Observation 5.3.** For all  $k \geq 0$ , it holds that  $\triangleleft_{1,k}^w \subseteq \triangleleft_1$  and  $\triangleleft_{2,k}^w \subseteq \triangleleft_2$ .

For all  $k \geq 0$ , relations  $\triangleleft_{1,k}^w$  and  $\triangleleft_{2,k}^w$  are strict partial orders. By induction on  $k$ , it can be shown that incomparability with respect to  $\triangleleft_{i,k}^w$  for  $i \in [2]$  is transitive. It follows that  $\triangleleft_{1,k}^w$  and  $\triangleleft_{2,k}^w$  are strict weak orders.

**Lemma 5.4.** *Relations  $\triangleleft_{1,k}^w$  and  $\triangleleft_{2,k}^w$  are strict weak orders for all  $k \geq 0$ .*

*Proof.* In order to show that a relation is a strict weak order, we have to prove that it is a strict partial order and that incomparability is transitive. Let  $k \geq 0$ . As  $\triangleleft_1$  and  $\triangleleft_2$  are irreflexive, it follows from  $\triangleleft_{1,k}^w \subseteq \triangleleft_1$  and  $\triangleleft_{2,k}^w \subseteq \triangleleft_2$  (Observation 5.3) that  $\triangleleft_{1,k}^w$  and  $\triangleleft_{2,k}^w$  are irreflexive as well. Further, the relations  $\triangleleft_{1,k}^w$  and  $\triangleleft_{2,k}^w$  are transitive. Hence,  $\triangleleft_{1,k}^w$  and  $\triangleleft_{2,k}^w$  are strict partial orders. It remains to show that incomparability is transitive. Two vertices  $x$  and  $y$  that are incomparable with respect to  $\triangleleft_{i,k}^w$ , are denoted by  $x \sim_{i,k}^w y$ . Let us consider  $k = 0$ . With respect to  $\triangleleft_{1,0}^w$ , all elements in  $V \setminus \{w\}$  are pairwise incomparable and  $w$  is incomparable to itself. Further, all elements in  $V$  are pairwise incomparable with respect to  $\triangleleft_{2,0}^w$ . Thus, for  $\triangleleft_{1,0}^w$  and  $\triangleleft_{2,0}^w$  incomparability is transitive. To show that incomparability is transitive for  $k > 0$  we need the following claims.

**Claim 5.5.** Let  $\kappa \geq 0$ ,  $i \in [2]$  and  $x, y \in V$ . If  $x \sim_{i,\kappa+1}^w y$ , then  $x \sim_{1,\kappa}^w y$  and  $x \sim_{2,\kappa}^w y$ .

*Proof.* Let  $\kappa \geq 0$ ,  $i \in [2]$  and  $x, y \in V$ . Without loss of generality, suppose that  $i = 1$  and that  $x \sim_{1,\kappa+1}^w y$ . For a contradiction let us assume that  $x$  and  $y$  are comparable with respect to  $\triangleleft_{1,\kappa}^w$  or  $\triangleleft_{2,\kappa}^w$ . If  $x$  and  $y$  are comparable with respect to  $\triangleleft_{1,\kappa}^w$ , then it follows directly that  $x$  and  $y$  are comparable with respect to  $\triangleleft_{1,\kappa+1}^w$ , since  $\triangleleft_{1,\kappa} \subseteq \triangleleft_{1,\kappa+1}^w$ , and we have a contradiction. Thus, suppose  $x$  and  $y$  are comparable with respect to  $\triangleleft_{2,\kappa}^w$ . Then  $x$  and  $y$  are also comparable with respect to  $(\triangleleft_{1,\kappa}^w)^E$ , and therefore also with respect to  $((\triangleleft_{1,\kappa}^w)^E)^T = \triangleleft_{1,\kappa+1}^w$ , a contradiction.  $\square$

**Claim 5.6.** Let  $\kappa \geq 0$ ,  $i \in [2]$  and  $y, z \in V$ . Further, let  $\triangleleft_{1,\kappa}^w$  and  $\triangleleft_{2,\kappa}^w$  be strict weak orders, and suppose  $y \sim_{i,\kappa+1}^w z$ . Then for all vertices  $v \in V$  the following holds: If  $v(\triangleleft_{i,\kappa}^w)^E z$ , then  $v(\triangleleft_{i,\kappa}^w)^E y$ .

*Proof.* Let  $\kappa \geq 0$ ,  $i \in [2]$  and  $v, y, z \in V$ . Let  $\triangleleft_{1,\kappa}^w$  and  $\triangleleft_{2,\kappa}^w$  be strict weak orders, and suppose that  $y \sim_{i,\kappa+1}^w z$  and  $v(\triangleleft_{i,\kappa}^w)^E z$ . Without loss of generality, assume  $i = 1$ . Relation  $(\triangleleft_{1,\kappa}^w)^E$  contains only pairs that are in  $\triangleleft_{1,\kappa}^w$ , in  $D_{1,\kappa}^E$  or in  $D_{1,\kappa}^E$ . Therefore,  $v(\triangleleft_{1,\kappa}^w)^E z$  implies that either  $v \triangleleft_{1,\kappa}^w z$ ,  $z \triangleleft_{2,\kappa}^w v$  or  $v \triangleleft_{2,\kappa}^w z$ . If we have  $v \triangleleft_{1,\kappa}^w z$ , then we also have  $v \triangleleft_{1,\kappa}^w y$ , as  $y$  and  $z$  are incomparable with respect to  $\triangleleft_{1,\kappa}^w$  by Claim 5.5 and  $\triangleleft_{1,\kappa}^w$  is a strict weak order. Analogously,  $z \triangleleft_{2,\kappa}^w v$  and  $v \triangleleft_{2,\kappa}^w z$  imply  $y \triangleleft_{2,\kappa}^w v$  and  $v \triangleleft_{2,\kappa}^w y$ , respectively. Hence, in each of the cases we obtain  $v(\triangleleft_{1,\kappa}^w)^E y$ .  $\square$

Now, let us assume there exists a  $k > 0$  such that incomparability is not transitive for  $\triangleleft_{1,k}^w$  or  $\triangleleft_{2,k}^w$ , and suppose  $k$  is minimal. Without loss of generality, assume that incomparability is not transitive for  $\triangleleft_{1,k}^w$ . Consequently, there exist vertices  $x, y, z \in V$  such that  $x \sim_{1,k}^w y$ ,  $y \sim_{1,k}^w z$  and  $x \not\sim_{1,k}^w z$ . Hence,  $x$  and  $z$  are comparable, which means  $x \triangleleft_{1,k}^w z$  or  $z \triangleleft_{1,k}^w x$ . Without loss of generality, suppose  $x \triangleleft_{1,k}^w z$ . Since  $\triangleleft_{1,k}^w$  is the transitive closure of  $(\triangleleft_{1,k-1}^w)^E$ , there exists an  $l \geq 0$  and  $v_0, v_1, \dots, v_{l+1}$  such that

$$x = v_0(\triangleleft_{1,k-1}^w)^E \dots (\triangleleft_{1,k-1}^w)^E v_l (\triangleleft_{1,k-1}^w)^E v_{l+1} = z.$$

As we know that the relations  $\triangleleft_{1,k-1}^w$  and  $\triangleleft_{2,k-1}^w$  are strict weak orders, that  $y \sim_{1,k}^w z$ , and that  $v_l(\triangleleft_{1,k-1}^w)^E z$ , we can apply Claim 5.6. We obtain that  $v_l \triangleleft_{1,k-1}^E y$ . Thus, we have

$$x = v_0(\triangleleft_{1,k-1}^w)^E \dots (\triangleleft_{1,k-1}^w)^E v_l (\triangleleft_{1,k-1}^w)^E y,$$

and therefore,  $x \triangleleft_{1,k}^w y$ , a contradiction.  $\square$

**Corollary 5.7.** *Relations  $\triangleleft_1^w$  and  $\triangleleft_2^w$  are strict weak orders.*

According to Corollary 5.7, incomparability with respect to  $\triangleleft_i^w$  is an equivalence relation. Each equivalence class of this equivalence relation is of size 1, as every equivalence class of size at least 2 would be a module. As a consequence, we obtain Theorem 5.8.

**Theorem 5.8.** *Relations  $\triangleleft_1^w$  and  $\triangleleft_2^w$  are strict linear orders.*

*Proof.* Let us assume that  $\triangleleft_1^w$  is not a strict linear order. Since  $\triangleleft_1^w$  is a strict weak order by Corollary 5.7, there exist two distinct vertices  $u, v$  such that  $u \sim_1 v$ , i.e.,  $u$  and  $v$  are incomparable regarding  $\triangleleft_1^w$ . Hence, the equivalence class  $u/\sim_1$  contains at least two elements. In the following we prove that  $u/\sim_1$  is a module. Let us assume  $u/\sim_1$  is not a module. Then there exists a vertex  $z \notin u/\sim_1$  and vertices  $x, y \in u/\sim_1$  such that  $z$  and  $x$  are adjacent and  $z$  and  $y$  are not adjacent. As  $\triangleleft_1^w$  is a strict weak order, we either have  $z \triangleleft_1^w x$  and  $z \triangleleft_1^w y$ , or  $x \triangleleft_1^w z$  and  $y \triangleleft_1^w z$ . Let us assume  $z \triangleleft_1^w x$  and  $z \triangleleft_1^w y$ . The other case can be shown analogously. Since there is an edge between  $z$  and  $x$  and no edge between  $z$  and  $y$ , and  $(\triangleleft_1^w, \triangleleft_2^w)$  is closed under edge relation  $E$ , we have  $x \triangleleft_2^w z$  and  $z \triangleleft_2^w y$ . Therefore, we must also have  $x \triangleleft_2^w y$ , by transitivity of  $\triangleleft_2^w$ . As  $(\triangleleft_1^w, \triangleleft_2^w)$  is closed under  $E$ , we obtain that  $x \triangleleft_1^w y$  or  $y \triangleleft_1^w x$ . Hence,  $x$  and  $y$  are comparable with respect to  $\triangleleft_1^w$ , a contradiction. Consequently,  $u/\sim_1$  is a module with  $|u/\sim_1| \geq 2$ . Clearly,  $u/\sim_1$  cannot be the vertex set  $V$  since we know  $w \triangleleft_1^w v$  for all  $v \neq w$ , where  $w$  is the initial vertex. Thus,  $u/\sim_1$  is a non-trivial module, a contradiction to  $G$  being prime.

Similarly we can prove that  $\triangleleft_2^w$  is a strict linear order. To show that a module  $u/\sim_2$  with  $|u/\sim_2| \geq 2$  for  $u \in V$  cannot be the vertex set  $V$ , we argue as follows: Since  $w \triangleleft_1^w v$  for all  $v \in V$  with  $v \neq w$  and  $(\triangleleft_1^w, \triangleleft_2^w)$  is closed under  $E$ , vertex  $w$  is comparable to all  $v \neq w$  with respect to  $\triangleleft_2^w$ . Hence, the equivalence relation  $\sim_2$  has at least two equivalence classes.  $\square$

**Corollary 5.9.** *We have  $\triangleleft_1^w = \triangleleft_1$  and  $\triangleleft_2^w = \triangleleft_2$ .*

The relations  $\triangleleft_1^w$  and  $\triangleleft_2^w$  are definable in fixed-point logic, i.e., there are FP-formulas  $\varphi_{\triangleleft_1}(x, y, y')$  and  $\varphi_{\triangleleft_2}(x, y, y')$  such that for all prime permutation graphs  $G = (V, E)$  and all  $w, v, v' \in V$  we have

$$G \models \varphi_{\triangleleft_i}[w, v, v'] \iff v \triangleleft_i^w v'.$$

In order to define  $\varphi_{\triangleleft_i}$  we use a simultaneous inflationary fixed-point operator. Within this fixed-point operator, two binary relational variables  $X_1$  and  $X_2$  are used to create the strict linear orders  $\triangleleft_1^w$  and  $\triangleleft_2^w$ . Let  $X_1^k$  and  $X_2^k$  be the relations that we obtain after the  $k$ th iteration within the simultaneous fixed-point operator. We can design the operator such that  $X_1^k$  and  $X_2^k$  are precisely  $\triangleleft_{1,k}^w$  and  $\triangleleft_{2,k}^w$ . Since the transitive closure and the closure under the edge relation are definable in FP, this operator is definable in FP.

As a consequence of Corollary 5.9, we obtain the following:

**Corollary 5.10.** *Let  $\varphi(x, y, y') := \varphi_{\triangleleft_1}(x, y, y') \vee y = y'$ . Then the FP-formula  $\varphi$  defines orders on the class of prime permutation graphs.*

Thus, the class of prime permutation graphs admits FP-definable orders. Since the class of permutation graphs is closed under taking induced subgraphs, we can apply Corollary 4.10. As a result we obtain the following theorem:

**Theorem 5.11.** *The class of permutation graphs admits FP+C-definable canonization.*

**Corollary 5.12.** *FP+C captures PTIME on the class of permutation graphs.*

## 6. CONCLUSION

So far, little is known about logics capturing PTIME on classes of graphs that are closed under induced subgraphs. This paper makes a contribution in this direction. We provide a tool, the Modular Decomposition Theorem, which simplifies proving that canonization is definable on such graph classes. Therefore, it also simplifies proving that PTIME can be captured on them. By means of the Modular Decomposition Theorem, we have shown in this paper that there exists an FP+C-canonization of the class of permutation graphs. Thus, FP+C captures PTIME on this class of graphs. The Modular Decomposition Theorem can also be applied to show that the class of chordal comparability graphs admits FP+C-definable canonization (see [Gru17a]). It follows that FP+C captures PTIME on the class of chordal comparability graphs and that there exists a polynomial-time algorithm for chordal comparability graph canonization. The author is optimistic that the Modular Decomposition Theorem can be used to obtain new results on further classes of graphs.

It would be interesting to find out whether a tool similar to the Modular Decomposition Theorem can also be obtained for split (or join) decomposition. Such a “Split Decomposition Theorem” could be used to prove that FP+C captures PTIME on the class of circle graphs, which are a generalization of permutation graphs and well-structured with respect to split decompositions.

Within this paper, we have also shown that there exists a logarithmic-space algorithm that computes the modular decomposition tree of a graph, and presented a variation of the Modular Decomposition Theorem for polynomial time. In the context of algorithmic graph theory, where modular decomposition has been established as a fundamental tool, these should find various applications. As a first application, we directly obtained that cograph recognition and cograph canonization is computable in logarithmic space.

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