Logical Methods in Computer Science
Vol. 14(2:1)2018, pp. 1–41
https://lmcs.episciences.org/

Submitted Jan. 03, 2017
Published Apr. 10, 2018

CLOSED SETS AND OPERATORS THEREON: REPRESENTATIONS, COMPUTABILITY AND COMPLEXITY

CARSTEN RÖSNICK-NEUGEBAUER

Technische Universität Darmstadt, Germany
e-mail address: research@carstenrn.com

Abstract. The TTE approach to Computable Analysis is the study of so-called representations (encodings for continuous objects such as reals, functions, and sets) with respect to the notions of computability they induce. A rich variety of such representations had been devised over the past decades, particularly regarding closed subsets of Euclidean space plus subclasses thereof (like compact subsets). In addition, they had been compared and classified with respect to both non-uniform computability of single sets and uniform computability of operators on sets. In this paper we refine these investigations from the point of view of computational complexity. Benefiting from the concept of second-order representations and complexity recently devised by Kawamura & Cook (2012), we determine parameterized complexity bounds for operators such as union, intersection, projection, and more generally function image and inversion. By indicating natural parameters in addition to the output precision, we get a uniform view on results by Ko (1991-2013), Braverman (2004/05) and Zhao & Müller (2008), relating these problems to the P/UP/NP question in discrete complexity theory.

1. Introduction

Closed subsets of Euclidean space, and in particular subclasses thereof like compact subsets, are important throughout many parts of pure theoretical mathematics, but also of no less relevance in disciplines like numerical analysis, convex optimization, or computational geometry. It is necessary to first define encodings for sets in order to describe computations on them which can be performed in a reasonably realistic computational model (which can even be implemented and used in practice [Mül00]). We choose the function oracle Turing machine model as in [KF82, Ko91, KC12] with encodings (functions of form $\phi\colon \Sigma^* \to \Sigma^*$) of continuous objects (reals, functions, sets) are given as oracles. Computational efficiency is gauged by second-order polynomial runtime bounds [KC12] – with the explicit addition of parameters which leads to a second-order equivalent of discrete parameterized complexity [KMRZ12, Ret13].

Key words and phrases: computational complexity, logic in computer science.

Supported by the *German Research Foundation* (DFG) with project `Zi 1009/4` and by the *Marie Curie International Research Staff Exchange Scheme Fellowship* `294962` within the 7th European Community Framework Program. A preliminary version (extended abstract) have been appeared in Proc. CCA'2013. Some parts have also been published in the author's PhD thesis [Rös14].

DOI:10.23638/LMCS-14(2:1)2018

The introduction of such encodings for sets, called *representations* in the TTE-branch of Computable Analysis, constitutes the first of two parts of this paper. One possible representation, $\boldsymbol{\delta}$, of a closed non-empty set $S \subseteq \mathbb{R}^d$ is by a function ϕ approximating its distance function d_S at any point up to arbitrary precision. Another representation, $\boldsymbol{\psi}$, asserts, given a point q and a precision parameter n, that either q is of distance less than 2^{-n} to S, or that it is of distance greater than $2 \cdot 2^{-n}$. Both representations allow for printing an arbitrarily precise picture of the respective set. So are these two representations computably equivalent, and if, how are they related complexity-wise? While computably equivalent in any dimension d, they are only polynomial-time equivalent in dimension $d = 1$. From dimension $d = 2$ onward, the question of whether a set S is polynomial-time computable with respect to $\boldsymbol{\delta}$ iff it is with respect to $\boldsymbol{\psi}$, has been linked to the P vs. NP question [Bra04]. Several more representations had been suggested [WK87, Her02, Ret08] and compared with respect to their computable equivalence [KS95, BW99, Wei00, Zie02, Her02, BP03]. The complexities of these relations, and in particular the *uniform* formulations (i. e., the complexity of an operator translating between two representations) of them, appear to be mostly unmentioned or unknown (except for a few examples [GLS88, Bra04]). We strengthen these previously known equivalence results from mere computable equivalence to *parameterized polynomial-time equivalence*, and prove *uniform exponential lower bounds* for the other relations. For dimension $d = 2$ these uniform (non-)polynomial-time equivalence results relate to complexity results for subsets of $\mathbb{R}^2$ with respect to various representations [CK95, CK05]; and they allow us to restate complexity results like for Julia sets [RW02, Bra05a] with respect to polynomial-time equivalent representations.

The second part of this work investigates the parameterized computational complexity of natural operators on sets, such as binary intersection and union, or projection to lower dimensional subspaces, but also forming the image or local inverse of a function with respect to a given set. The situation concerning their parameterized complexity is similar to that for representations of sets: Operators on closed, compact or regular subsets have been considered with respect to computability (e. g., [Zho96, Zie04, ZB04]) and non-uniform complexity bounds (e. g., [Ko91, Chap. 4], [KY08]), but it appears that less is known about the uniform complexity of operators (exceptions include [ZM08]). In addition, complexity bounds of e. g. projection and function inversion had been linked to classical problems from discrete complexity theory, namely P vs. NP and P vs. UP. Results like these are in the spirit of well-known ones for maximization and integration of functions [KF82, Fri84] (we refer the reader e. g. to [Ko98, BHW08] for an overview and more examples of this kind). In this paper we present uniform worst-case parameterized complexity bounds for all of the aforementioned operators. Providing operators through parameters with more information about their arguments turns out to be valuable and fruitful approach to achieve uniformity and also allows for a fine-grained perspective on their inherent complexity.[1] In addition to upper bounds we also present exponential-time lower bounds, thus extending upon the former non-uniform bounds that depended on believed-to-be-hard problems from discrete complexity theory.

[1] Take the projection of a subset of $\mathbb{R}^d$ to its first component as an example: The question whether a given point q is contained in the projection is uncomputable as long as no information about a bound of the set is given.

Results obtained in this paper. Primarily based on [GLS88, Chap. 4], [Wei00, §5], [Zie02] and [Ret08], we introduce five representations in Section 2.2: $\boldsymbol{\delta}$ and $\boldsymbol{\delta}_{\mathsf{rel}}$, $\boldsymbol{\psi}$, $\boldsymbol{\kappa}$, and $\boldsymbol{\omega}$. Each representation will depend on a fixed yet arbitrary norm—a dependence we will show in Section 3.2 to be of "polynomial-time irrelevance" for all but one representation ($\boldsymbol{\delta}$: Thm. 3.8). We furthermore compare relations between representations by means of (parameterized) polynomial-time translations; and observe that, although they are all (uniformly) computably equivalent over *convex regular* sets [Zie02, Cor. 4.13], mutual (parameterized) polynomial-time reductions exist only for intervals (Thm. 3.10), but in general not from dimension $d = 2$ onward. In fact, we identify a kind of hierarchy: $\boldsymbol{\omega}$ forms the *poorest*, $\boldsymbol{\delta}$ the *richest* representation, and $\boldsymbol{\delta}_{\mathsf{rel}}$, $\boldsymbol{\psi}$, $\boldsymbol{\kappa}$ are parameterized polynomial-time equivalent over compact sets (Thm. 3.11, 3.12 and 3.19). Parameterized polynomial-time equivalence of $\boldsymbol{\omega}$ with $\boldsymbol{\delta}$ (hence of all of the former representations) is finally achieved if restricted to compact convex regular sets (Thm. 3.14).

Section 4 then uses the formerly unveiled connections between representations by discussing the complexity of operators. Operators include Choice (finding *some* point in a set; the presumably most basic set operation) and Union, which are fully polynomial-time computable for all of the above representations but $\boldsymbol{\omega}$); and $\mathsf{Intersection}$, which (in contrast to the former result) polynomial-time computable *only* for $\boldsymbol{\omega}$. More involved operators, $\mathsf{Inversion}$ and Image, are discussed in Sections 5.2 and 5.3: We prove that $\mathsf{Inversion}$ is parameterized polynomial-time computable for Lipschitz-continuous functions whose inverse is also Lipschitz continuous (Thm. 5.9)—which fits right in the gap between naive exponential-time algorithms and results of Ko [Ko91, Thm. 4.23+4.26], the latter relating *non-uniform* $\mathsf{Inversion}$ for a more general class of functions to the (considered to be hard) question whether $\mathsf{P} \neq \mathsf{UP}$ holds true.

Preliminaries, nomenclature. We introduce some notations and concepts we frequently use throughout this paper. Let Σ be the binary alphabet $\{0, 1\}$, Σ^* denotes the set of finite 0/1-strings, Σ^ω the set of 0/1 *sequences* (isomorphic to $\Sigma^{(\Sigma^*)}$, the set of all total functions from Σ^* to Σ), and $\Sigma^{**} := (\Sigma^*)^{(\Sigma^*)}$ the set of all total functions from Σ^* to Σ^*. A finite string $s = s_1 \cdots s_k \in \Sigma^*$ with $s_i \in \Sigma$ is also called *word*. The *length* of s (as above) is defined by $\ell(s) := k$, and ε denotes the unique word of length 0 (the *empty word*). We further consider the sets Σ^ω and Σ^{**} as topological spaces equipped with the product topology (equip Σ, Σ^* with the discrete topology).

Let $\mathbb{N} := \{0\} \cup \mathbb{N}_+$ with $\mathbb{N}_+ := \{1, 2, 3, \dots\}$, and abbreviate the *binary logarithm of* x by $\operatorname{lb} x := \log_2 x$. By $\mathbb{D}_m := \{a/2^m \mid a \in \mathbb{Z}\}$ we denote the set of *dyadic rationals of precision* $m \in \mathbb{Z}$, and set $\mathbb{D} := \bigcup_{m \in \mathbb{Z}} \mathbb{D}_m$. Let further $\mathsf{bin}_{\mathbb{N}} \colon \mathbb{N} \to \Sigma^*$ denote the usual *binary coding* of naturals as words, $\mathsf{un}_{\mathbb{N}}$ denotes the *unary coding* which we usually abbreviate by $0^n := \mathsf{un}_{\mathbb{N}}(n)$. Even though $\mathsf{un}_{\mathbb{N}}$ is not surjective and thus does not admit an inverse, we like to understand by $\mathsf{un}_{\mathbb{N}}^{-1}$ the mapping $s \in \Sigma^* \mapsto \ell(s) \in \mathbb{N}$. Note that the notions of binary and unary encodings naturally extend to the set of integers by embedding $\mathbb{Z}$ into $\mathbb{N}$. By abuse of notations, we casually write 0^k for the unary coding of an *integer* k. Pairing functions (usually total, bijective, computable and invertible in polynomial time, although we do not need them to be surjective) are denoted by $\langle \cdot, \cdot \rangle_X \colon X \times X \to X$ whereby the "$_X$" will be usually clear from context (typically $\mathbb{N}$, Σ^* or Σ^{**}) and henceforth omitted. Explicitly, $\langle s, t \rangle_{\Sigma^*} := \mathsf{bin}_{\mathbb{N}}\big(\big\langle \mathsf{bin}_{\mathbb{N}}^{-1}(s), \mathsf{bin}_{\mathbb{N}}^{-1}(t) \big\rangle_{\mathbb{N}}\big)$ with $\langle \cdot, \cdot \rangle_{\mathbb{N}}$ being the *Cantor pairing function*. Further define the pairing function $\langle \phi, \psi \rangle$ on Baire space Σ^{**} through $\langle \phi, \psi \rangle(\varepsilon) := \varepsilon$, $\langle \phi, \psi \rangle(0\, s) := 0^{\ell(\psi(s))}\, 1\, \phi(s)$, and $\langle \phi, \psi \rangle(1\,s) \coloneqq 0^{\ell(\phi(s))}\,1\,\psi(s)$ for all $s \in \Sigma^*$. The binary encoding $\mathsf{bin}_{\mathbb{D}}^{(d)} \colon \mathbb{D}^d \to \Sigma^*$ of dyadic rationals is recursively defined: Let $\mathsf{bin}_{\mathbb{D}}^{(1)} \colon a/2^n \mapsto \big\langle \mathsf{bin}_{\mathbb{Z}}(a), 0^n \big\rangle_{\Sigma^*}$ and for $d \geq 2$ let $\mathsf{bin}_{\mathbb{D}}^{(d)} \colon (q_1, \ldots, q_d) \mapsto \big\langle \mathsf{bin}_{\mathbb{D}}^{(1)}(q_1), \mathsf{bin}_{\mathbb{D}}^{(d-1)}(q_2, \ldots, q_d) \big\rangle_{\Sigma^*}$.

A *normed (vector) space* is a pair $(X, \|\cdot\|)$ of a vector space X together with a norm $\|\cdot\|$ on X. A set $S \subseteq X$ is *open in* X if it is the set of its inner points, i. e., $S^\circ = S$, and closed (in X) if it is the closure of itself, i. e., $\overline{S} = S$. The boundary is defined as $\partial S \coloneqq S \setminus S^\circ$.

On $(\mathbb{R}^d, \|\cdot\|)$ we denote *closed* balls with center $x \in \mathbb{R}^d$ and radius $\delta > 0$ by $\overline{\mathrm{B}}_{\|\cdot\|}(x, \delta) \coloneqq \{y \in \mathbb{R}^d \mid \|x - y\| \leq \delta\}$—or simply by $\overline{\mathrm{B}}(x, \delta)$ if the norm used is understood. Similarly denote *open* balls as $\mathrm{B}_{\|\cdot\|}(x, \delta)$. Whenever useful, we use the abbreviation $\mathbb{D}_n^d(R) \coloneqq \mathbb{D}_n^d \cap \overline{\mathrm{B}}(0, R)$. A "ball" (actually a neighborhood) around a set $S \subseteq \mathbb{R}^d$ of radius $\delta > 0$ is defined through the union of balls around the points of S, i. e., $\overline{\mathrm{B}}(S, \delta) \coloneqq \bigcup_{x \in S} \overline{\mathrm{B}}(x, \delta)$, and similarly for open balls. The same works in the reverse direction, i. e., for negative radii: Denote by $\overline{\mathrm{B}}(S, -\delta) \coloneqq \big\{x \in \mathbb{R}^d \,\big|\, \overline{\mathrm{B}}(x, \delta) \subseteq S\big\}$ the (possibly empty) set of all points x lying *δ-deep in* S. Further define *hollow* closed balls centered at x with inner radius $\delta' > 0$ and outer radius $\delta \geq \delta'$ through $\overline{\mathrm{B}}_{\|\cdot\|}(x, \delta, \delta') \coloneqq \overline{\mathrm{B}}_{\|\cdot\|}(x, \delta) \setminus \mathrm{B}_{\|\cdot\|}(x, \delta')$.

The domain and co-domain (also: image) of a function f mapping from a set X into Y are denoted by $\mathrm{dom}(f)$ and $\mathrm{cod}(f)$, respectively. Besides total functions, $f \colon X \to Y$ with $\mathrm{dom}(f) = X$, we also consider partial functions $f \colon {\subseteq X} \to Y$ (f is defined only on a subset of X, thus the "$\subseteq X$"), and partial multi-valued functions as $f \colon X \rightrightarrows Y$, $f(x) \subseteq Y$. Phrased differently, a multi-valued partial function $f \colon X \rightrightarrows Y$ is a partial function from X into the powerset of Y. The multi-valued assignment of an element $x \in X$ to a subset $Y' \subseteq Y$ is abbreviated by $x \Mapsto Y'$.

2. Model, representations and complexity

All the concepts we discuss in this section are guided by the question how computations on subsets of real vector spaces could be carried out on a reasonably realistic machine model. Using a *Turing-like machine model* (Section 2.1), we define encodings of sets through *representations* (Section 2.2) and proceed by giving definitions for *computability* and *parameterized complexity* (Sections 2.3 and 2.4).

2.1. Computational models.

Two mainstream models in Computable Analysis are the *Type-2 Theory of Effectivity* [Wei00] and the *oracle Turing machine model* [KF82, Ko91]. The former even yields a topological interpretation of computability. Therefore, we start by introducing computability using the former model. Carrying these notation over to the latter model will then naturally give rise to a suitable notion of complexity. From a computational point, however, both models are equivalent. The notions discussed in this section are based on [Kaw11, §2.1+2.2] and [KMRZ12, §2].

2.1.1. *Type-2 machines: computations on finite and infinite strings.*

Type-2 machines extend upon classical Turing machines by operating on *infinite* strings instead of finite ones, i. e., on Σ^ω instead of Σ^*. Such a machine consists of finitely many left-to-right readable input and bidirectional working tapes, and one left-to-right writable output tape. A *computation* on finite strings is carried out as on classical Turing machines: Given a type-2 machine M with $k \in \mathbb{N}$ input tapes plus an input $(s_1, \ldots, s_k) \in \Sigma^* \times \cdots \times \Sigma^*$ to it, M either reads one symbol from one of its k input tapes, reads or writes one symbol on one of its working tapes, or writes a symbol on its output tape. The same applies if the input is not a tuple of finite strings, but of infinite ones from $\Sigma^\omega \times \cdots \times \Sigma^\omega$. Given such a machine M, we say it *computes a partial function* $f\colon \subseteq\Sigma^* \times \cdots \times \Sigma^*$ if it terminates on all inputs $(s_1, \ldots, s_k)$ from f's domain and writes $f(s_1, \ldots, s_k)$ symbol-by-symbol on the output tape. As for the infinite case, M computes a partial function $f\colon \subseteq\Sigma^\omega \times \cdots \times \Sigma^\omega \to \Sigma^\omega$ if M continues forever on input $(s_1, \ldots, s_k)$ whilst producing $s \coloneqq f(s_1, \ldots, s_k)$ on its output tape.

Machines which have to run infinitely long to produce their answer would certainly not deserve to be called "practicable". The key here, however, is that for every *finite prefix* of the input read a type-2 machine produces a *non-revisable finite prefix* of the still infinitely long correct output. We postpone discussing of the strong topological implications to computability until Thm. 2.3 and Section 2.5.

As for classical Turing machines, type-2 machines are also capable to compute functions $f\colon \subseteq X_1 \times \cdots \times X_k \to X'$ for sets X_i, X' different from Σ^ω by encoding their elements through items from Σ^ω (appropriate cardinalities assumed). Following [Wei00, Def. 2.3.1(2)], we call such encodings *representations*. Building upon them, we formulate the computability of functions $f\colon \subseteq X \to X'$ through *realizers*.

Definition 2.1 (representations, realizers)**.** Let X and X' be sets.

(1) A *representation of* X is a surjective partial function $\boldsymbol{\xi}\colon \subseteq\Sigma^\omega \to X$.
(2) An element $\phi \in \boldsymbol{\xi}^{-1}[\{x\}]$ is said to be a $\boldsymbol{\xi}$-*name* of x.

Further assume $\boldsymbol{\xi}$ and $\boldsymbol{\xi}'$ to be representations of X and X', respectively, and let $f\colon \subseteq X \to X'$ be some function.

(3) A function $g\colon \subseteq\Sigma^\omega \to \Sigma^\omega$ is called a $(\boldsymbol{\xi}, \boldsymbol{\xi}')$-*realizer* of function f if for all $\phi \in \boldsymbol{\xi}^{-1}\big[\mathrm{dom}(f)\big]$, $(\boldsymbol{\xi}' \circ g)(\phi) = (f \circ \boldsymbol{\xi})(\phi)$ holds true. A more visual way to think of realizers is by a commuting diagram:

$$\begin{tikzcd} \Sigma^\omega \arrow[r, "g"] \arrow[d, "\boldsymbol{\xi}"] & \Sigma^\omega \arrow[d, "\boldsymbol{\xi}'"] \\ X \arrow[r, "f"] & X' \end{tikzcd}$$

(4) Function f is called $(\boldsymbol{\xi}, \boldsymbol{\xi}')$-*computable* (-*continuous*) if it has a computable (continuous) $(\boldsymbol{\xi}, \boldsymbol{\xi}')$-realizer.

The above notions are reasonable in the sense that topological continuity of a function corresponds to $(\boldsymbol{\xi}, \boldsymbol{\xi}')$-continuity if $\boldsymbol{\xi}$ and $\boldsymbol{\xi}'$ are both *admissible* (cf. [Wei00, §3.2], [Sch02]). Note that all representations mentioned in this paper are admissible.

We present a few examples of representations in form of a definition.

Definition 2.2 (representations)**.** (1) Representations $\mathsf{\mathbf{un}}_{\mathbb{N}}$ and $\mathsf{\mathbf{bin}}_{\mathbb{N}}$ extend upon the unary and binary encodings, $\mathsf{un}_{\mathbb{N}}^{-1}$ and $\mathsf{bin}_{\mathbb{N}}^{-1}$ from Σ^* to Σ^ω.[2] We say, a natural number $k \in \mathbb{N}$ is represented by a $\mathsf{\mathbf{bin}}_{\mathbb{N}}$-name $\phi = 0\,b_0\,0\,b_1 \ldots 0\,b_\ell\,1^\omega \in \Sigma^\omega$, $\mathsf{\mathbf{bin}}_{\mathbb{N}} \colon \subseteq \Sigma^\omega \to \mathbb{N}$, if ϕ essentially is k's binary encoding: $\mathsf{\mathbf{bin}}_{\mathbb{N}}(\phi) = \sum_{i=0}^{\ell} b_i 2^i = k$. Its unary counterpart, denoted $\mathsf{\mathbf{un}}_{\mathbb{N}}$, can be obtained through representing a natural number $k \in \mathbb{N}$ by $\phi := (0\,1)^k\,1^\omega \in \Sigma^\omega$.

(2) We define a $\boldsymbol{\rho}$-name ϕ of a real number $x \in \mathbb{R}$ to be a suitably encoded sequence $(q_n)_n$ of dyadic rationals $q_n \in \mathbb{D}_n = \{a/2^n \mid a \in \mathbb{Z}\}$ (i. e., $\phi = \langle (q_n)_{n\in\mathbb{N}} \rangle \in \Sigma^\omega$; cf. [Wei00, Def. 4.1.5+4.1.17]) converging to x in the sense that $|q_n - x| \leq 2^{-n}$ holds true for all $n \in \mathbb{N}$.

(3) Based on $\boldsymbol{\rho}$, a representation $[\boldsymbol{\rho}\to\boldsymbol{\rho}] \colon \subseteq \Sigma^\omega \to \mathrm{C}(\mathbb{R})$ of continuous functions $f \colon \mathbb{R} \to \mathbb{R}$ may intuitively be understood as follows: A $[\boldsymbol{\rho}\to\boldsymbol{\rho}]$-name encodes how ($\boldsymbol{\rho}$-names of) $x \in \mathbb{R}$ are translated into $\boldsymbol{\rho}$-names of $f(x)$ (cf. [Wei00, Def. 3.3.13] and [Grz57]).

An important property of the TTE model and its representations is due to its concise topological roots, resulting in the *Main Theorem* in the TTE-branch of Computable Analysis.

Fact 2.3. Computability implies (topological) continuity.

Recall that $(\boldsymbol{\xi}, \boldsymbol{\xi}')$-computability by a Type-2 machine M means that M maps finite prefixes of a $\boldsymbol{\xi}$-name ϕ to finite prefixes of a $\boldsymbol{\xi}'$-name ϕ'. The reader is referred to [Wei00, Thm. 2.2.3+3.2.11] for detailed explanations and proofs.

2.1.2. *Oracle machines.* The type-2 model, and in particular the way we have introduced representations so far, does not yield a viable notion of complexity: Say ϕ is a $\boldsymbol{\rho}$-name of a real number $x \in \mathbb{R}$ as defined in Thm. 2.2(2), i. e., an encoded sequence $(q_n)_{n\in\mathbb{N}}$ of dyadic rationals. In order to access a specific element encoded through ϕ, say q_N, a type-2 machine has first to skip over a possibly large (compared to the coding length of q_N) prefix of ϕ. Such an initial motion has to reflect in some way in any complexity notion, although the search for q_N does not contribute anything to the *actual* computation on it. Granting a machine access to individual information encoded through ϕ (black-box approach) without charging too much for such access can be realized by *oracle Turing machines* (oracle machines, or OTMs, for short).

Recall that an oracle machine is a classical (possibly multi-tape) Turing machine with the addition of a special query tape and two new states: One to initiate the query to the oracle with the content of the question written on the query tape, and a second to mark that the oracle has written its respective answer on the query tape. The oracle attached to a machine can either be a subset of Σ^* (a possibly undecidable decision problem), or a string function. We choose the latter type, a *function-oracle* machine model (cf. [Ko91, Def. 2.11]).

Definition 2.4 (second-order representations)**.**

(1) A *second-order representation* $\boldsymbol{\xi}$ of a set X is a partial surjective function $\boldsymbol{\xi} \colon \subseteq \Sigma^{**} \to X$.

[2]To justify notational between $\mathsf{bin}_{\mathbb{N}}^{-1}$ and $\mathsf{\mathbf{bin}}_{\mathbb{N}}$: Computations are performed on the level of names (i. e., objects from Σ^ω). Objects like natural numbers or dyadic rationals, on the contrary, are usually used "as the are", i. e., not encoded as words or sequences. They are encoded back into words (via $\mathsf{bin}_{\mathbb{N}}$ or $\mathsf{un}_{\mathbb{N}}$) not before the end of the respective argument.

(2) Any ordinary representation $\boldsymbol{\xi}\colon \subseteq\Sigma^\omega \to X$ (i. e., in the sense of Thm. 2.1) induces a second-order representation $\tilde{\boldsymbol{\xi}}$: Any $\boldsymbol{\xi}$-name $\phi = (b_i)_i$, $b_i \in \Sigma$, yields a $\tilde{\boldsymbol{\xi}}$-name $\tilde{\phi}$ through $\tilde{\phi}(s) := b_{\ell(s)}$ for any $t \in \Sigma^*$.[3]

(3) Second-order representations $\tilde{\boldsymbol{\xi}}_1$ and $\tilde{\boldsymbol{\xi}}_2$ of X_1 and X_2, respectively, induce a second-order representation $\tilde{\boldsymbol{\xi}}_1 \times \tilde{\boldsymbol{\xi}}_2$ of $X_1 \times X_2$: If ϕ_i is a $\tilde{\boldsymbol{\xi}}_i$-name of $x_i \in X_i$, then $\langle \phi_1, \phi_2\rangle_{\Sigma^{**}}$ is a $\tilde{\boldsymbol{\xi}}_1 \times \tilde{\boldsymbol{\xi}}_2$-name of $X_1 \times X_2$.

Functions computable by oracle machines can be defined over realizers similar to Thm. 2.1.

Definition 2.5 (computable functions, realizers)**.** Assume $\boldsymbol{\xi}$ and $\boldsymbol{\xi}'$ to be second-order representations of X and X', respectively, and let $f\colon \subseteq X \to X'$ be some function.

(1) A function $g\colon \subseteq\Sigma^{**} \times \Sigma^* \to \Sigma^*$ is computable by an oracle machine $M^?$ if for all $(\phi, s) \in \Sigma^{**} \times \Sigma^*$, M^ϕ started with s halts and writes $g(\phi, s)$ on its output tape.

(2) A function $g\colon \subseteq\Sigma^{**} \times \Sigma^* \to \Sigma^*$ is called a $(\boldsymbol{\xi}, \boldsymbol{\xi}')$-realizer of function f if for all $\phi \in \boldsymbol{\xi}^{-1}\left[\mathrm{dom}(f)\right]$, $(f \circ \boldsymbol{\xi})(\phi) = \boldsymbol{\xi}'\big(g(\phi, \cdot)\big)$ holds true. (Note that $g(\phi, \cdot) \in \Sigma^{**}$.)

(3) Function f is called $(\boldsymbol{\xi}, \boldsymbol{\xi}')$-computable if it has a $(\boldsymbol{\xi}, \boldsymbol{\xi}')$-realizer computable by some oracle machine.

2.1.3. *Relation between both models.* Although the type-2 machines on one hand side and oracle machines on the other are seemingly different approaches to *Real Computability*, they are actually computably identical.

Fact 2.6. Let $\boldsymbol{\xi}$ and $\boldsymbol{\xi}'$ be ordinary representations of X and X', respectively. Every $(\boldsymbol{\xi}, \boldsymbol{\xi}')$-computable (-continuous) function (i. e., realized by some type-2 machine computable function) $f\colon \subseteq X \to Y$ is also $(\tilde{\boldsymbol{\xi}}, \tilde{\boldsymbol{\xi}}')$-computable (-continuous) (i. e., realized by some oracle machine computable function); and vice versa.

This follows by type conversion [Wei00, Lem. 2.1.6]: Since f is $(\boldsymbol{\xi}, \boldsymbol{\xi}')$-computable (-continuous), it has some computable (continuous) realizer $g\colon \subseteq\Sigma^\omega \to \Sigma^\omega$. By the aforementioned Lemma, a function $G\colon \subseteq\Sigma^\omega \times \Sigma^* \to \Sigma^*$, $(\boldsymbol{\xi}, \boldsymbol{\xi}')$-computable by some type-2 machine, exists such that the following hold true:

(1) Function G has a suitable domain: for all $\phi \in \Sigma^\omega$, $\phi \in \mathrm{dom}(g)$ if $\forall\, s \in \Sigma^* \,.\, (\phi, s) \in \mathrm{dom}(G)$;

(2) and G does behave like g (extensionally): $\forall\, \phi \in \mathrm{dom}(g) \,.\, \forall\, s \in \Sigma^* \,.\, G(\phi, s) = g(\phi)(s)$.

Since G is computable by some type-2 machine, it is computable by some oracle machine as well, thus $(\tilde{\boldsymbol{\xi}}, \tilde{\boldsymbol{\xi}}')$-realizing f. The reverse direction follows similarly when used that representations in TTE can equivalently written stated over Σ^{**} instead of Σ^ω [Wei00, Ex. 3.2(17)].

Convention. From now on we omit the “tilde” and simply write “$\boldsymbol{\xi}$” whenever we reason about second-order representations $\tilde{\boldsymbol{\xi}}$. Arguments $s \in \Sigma^*$ of names ϕ are usually of form $s = \langle q, 0^n\rangle$ for a dyadic point $q \in \mathbb{D}^d$ and a precision parameter $n \in \mathbb{N}$. We use $\phi(q, 0^n)$ as a shorthand for the correct but more verbose $\phi\big(\langle \mathsf{bin}_{\mathbb{D}}^{(d)}(q), \mathsf{un}_{\mathbb{N}}(n)\rangle\big)$.

[3] The reader is referred to [KMRZ12, Def. 1.16] for more (formal) definitions and extensions of second-order representations.

2.2. **Second-order representations of sets.** Throughout this paper, we solely concentrate on closed *non-empty* subsets of $\mathbb{R}^d$ (for various d) and subclasses thereof. More precisely: In any dimension $d \in \mathbb{N}$ we denote the class of *closed non-empty subsets of* $\mathbb{R}^d$ by $\mathcal{A}^{(d)}$, the class of *compact subsets* by $\mathcal{K}^{(d)} := \{S \in \mathcal{A}^{(d)} \mid \exists\, \delta > 0 \,.\, S \subseteq \overline{\mathrm{B}}(0,\delta)\}$, *convex subsets* are in $\mathcal{C}^{(d)} := \{S \in \mathcal{A}^{(d)} \cap (\mathbb{R}^d, \|\cdot\|_2) \mid \forall\, x, y \in S \,.\, \forall\, \lambda \in [0,1] \,.\, \lambda x + (1-\lambda) y \in S\}$, and *regular subsets* in $\mathcal{R}^{(d)} := \{S \subseteq \mathcal{A}^{(d)} \mid \overline{S^\circ} = S\}$.[4] All notions are depicted on Fig. 1.

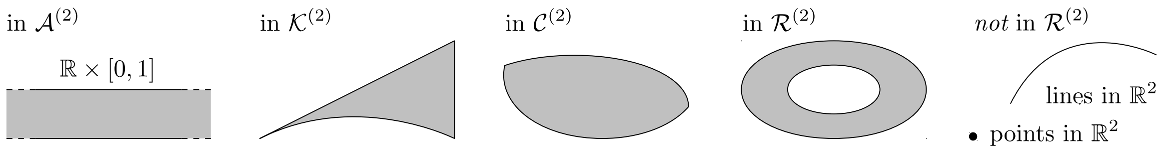


Figure 1: Classes of subsets of Euclidean space: Closed $\mathcal{A}$, compact $\mathcal{K}$, convex $\mathcal{C}$, and regular $\mathcal{R}$.

Intersections of the above subclasses will also be of interest, e. g., the class $\mathcal{KR}^{(d)} := \mathcal{K}^{(d)} \cap \mathcal{R}^{(d)}$ of *bounded bodies*; understand $\mathcal{CR}^{(d)}$ (*convex bodies*), $\mathcal{KC}^{(d)}$, and $\mathcal{KCR}^{(d)}$ (*bounded convex bodies*) similarly. Omitting the dimension on any class of subsets denotes the union over all d, i. e., $\mathcal{A} := \bigcup_{d \in \mathbb{N}} \mathcal{A}^{(d)}$ and so forth.

For a set $S \in \mathcal{A}^{(d)}$ and a norm $\|\cdot\|$ we define the distance function $d_{\|\cdot\|,S} \colon \mathbb{R}^d \to \mathbb{R}_{\geq 0}$, mapping any point $q \in \mathbb{R}^d$ to its minimal distance to set S, by $d_{\|\cdot\|,S}(q) := \min_{x \in S} \|q - x\|$.

Every representation for a class of sets provides approximate information to the specific set S it encodes in terms of answers to a specific type of questions: Given a point x, is $x \in S$? If not, is x far from S? How far? From Thm. 2.3 we can infer that only trivial sets S (i. e., $S = \emptyset$ or the whole space, $S = \mathbb{R}^d$) are representable by their characteristic functions since they are discontinuous in all other cases. We thus have to allow any name ϕ of a reasonable representation to be in some sense vague or "fuzzy" when queries are close to the boundary of the set S it encodes. To be more precise, we have to allow any name ϕ to make errors somewhere if ϕ represents $S \in \mathcal{A}^{(d)}$ with $\emptyset \subset S \subset \mathbb{R}^d$. This error, however, has to be controllable through a precision parameter n, just like for reals and functions.

Subsequently, we cite five different definitions (visualized in Fig. 2) for representations of sets.

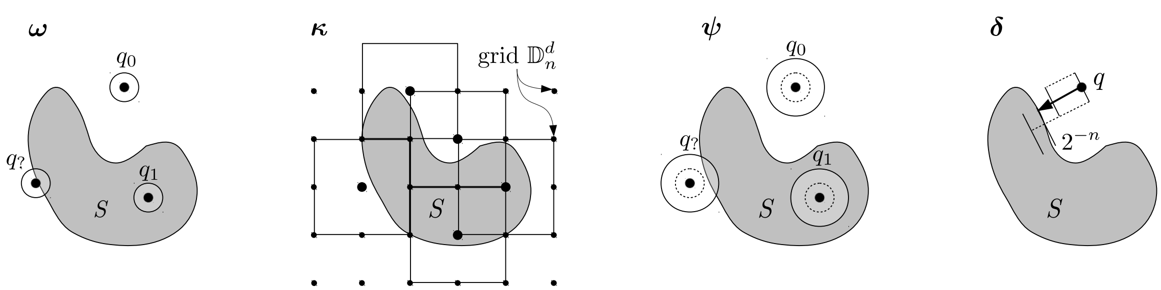


Figure 2: Representations of sets: Illustration of Thm. 2.7

[4]Normed vector spaces over equivalent norms are homeomorphic, thus imply the same topology. It therefore is not necessary to tie any of these subclasses (except for $\mathcal{C}$) to a concrete norm.

Definition 2.7. Fix a dimension $d \in \mathbb{N}$ and a norm $\|\cdot\|$ on $\mathbb{R}^d$. Points q are chosen from $\mathbb{D}^d$, and *precision parameters* are denoted by $n \in \mathbb{N}$.

(1) *Weak-membership representation:* A $\boldsymbol{\omega}^{(d)}_{\|\cdot\|}$-name ϕ of $S \in \mathcal{R}^{(d)}$ satisfies
 (a) $\phi(q, 0^n) = 1$ if $\overline{\mathrm{B}}_{\|\cdot\|}(q, 2^{-n}) \subseteq S$ (i. e., q lies 2^{-n}-deep within S), or
 (b) $\phi(q, 0^n) = 0$ if $\overline{\mathrm{B}}_{\|\cdot\|}(q, 2^{-n}) \cap S = \emptyset$ (i. e., q is off by more than 2^{-n}).

(2) A $\boldsymbol{\kappa}^{(d)}_{\|\cdot\|}$-name ϕ of $S \in \mathcal{K}^{(d)}$ satisfies
 (a) $b := \mathsf{un}_{\mathbb{Z}}^{-1}(\phi(\varepsilon))$ is an upper bound on the size of S, i. e., $S \subseteq \overline{\mathrm{B}}_{\|\cdot\|}(0, 2^b)$, and
 (b) ϕ encodes a sequence of sets $B_n \subset \mathbb{D}^d_n$ through $\phi(q, 0^n) = \chi_{B_n}(q)$ that is 2^{-n}-close to S in the Hausdorff-distance d_{H}, i. e., $d_{\mathsf{H}}(B_n, S) \le 2^{-n}$:
  (κ1) $\forall n \in \mathbb{N} \,.\, \forall x \in S \,.\, \exists q \in B_n \,.\, \|q - x\| \le 2^{-n}$, and
  (κ2) $\forall n \in \mathbb{N} \,.\, \forall q \in B_n \,.\, \exists x \in S \,.\, \|q - x\| \le 2^{-n}$.

(3) A $\boldsymbol{\psi}^{(d)}_{\|\cdot\|}$-name ϕ of S satisfies
 (a) $\phi(q, 0^n) = 1$ if $\mathrm{B}_{\|\cdot\|}(q, 2^{-n}) \cap S \neq \emptyset$ (q is 2^{-n}-close to S), or
 (b) $\phi(q, 0^n) = 0$ if $\overline{\mathrm{B}}_{\|\cdot\|}(q, 2^{-n+1}) \cap S = \emptyset$ (q is at least 2^{-n+1}-far off of S).

(4) A $\phi \in \Sigma^{**}$ is a $\boldsymbol{\delta}^{(d)}_{\|\cdot\|}$-name of $S \in \mathcal{A}^{(d)}$ whenever ϕ encodes S' distance function, i. e., $|\phi(q, 0^n) - d_{\|\cdot\|,S}(q)| \le 2^{-n}$.

(5) Relative version of $\boldsymbol{\delta}^{(d)}$ (specialization of [Ret08, Def. 1.27]): A $\phi \in \Sigma^{**}$ is a $\boldsymbol{\delta}\mathsf{rel}^{(d)}_{\|\cdot\|}$-name of $S \in \mathcal{A}^{(d)}$ if for all $q \in \mathbb{D}^d$ and $n \in \mathbb{N}$, ϕ satisfies

$$3/4 \cdot d_{\|\cdot\|,S}(q) - 2^{-n} \le \phi(q, 0^n) \le 5/4 \cdot d_{\|\cdot\|,S}(q) + 2^{-n}. \tag{2.1}$$

2.2.1. *A few historical remarks.* The concept underlying $\boldsymbol{\delta}$ (representation of the distance function; cf. [Wei00, $\boldsymbol{\psi}^{\text{dist}}$: Def. 5.1.6]) is the same as for *Turing located sets* ([GN94]; dating even back to Brouwer [Bro19, *katalogisierte Mengen*]), and the concept of *recognizable sets* [CK95, Def. 3.5] is underlying the *weak membership problem/representation* $\boldsymbol{\omega}$ ([GLS88, Def. 2.1.14]; also in [KS95, Def. 4.2]). A strengthening of $\boldsymbol{\omega}$, where the positive information is exact (in the sense that condition "x is 2^{-n}-close to S" is replaced with "$x \in S$"), was also considered under the notion of *strong recognizability* [CK95, Def. 4.1] and revisited later as *weak computability* by [Bra05b, Def. 3, Thm. 4]. Although Chou/Ko seemed to be the first to *formally* present this strengthening, this concept of one-sided error also has already been present implicitly as part of [GLS88, Lem. 4.3.3]. Representation $\boldsymbol{\kappa}$ was defined and used e. g. in [Wei00, Def. 5.2.1], [ZM08, $\boldsymbol{\kappa}_{\mathrm{G}}$: Def. 2.2]; and $\boldsymbol{\psi}$ in [Wei00, $\boldsymbol{\psi}$: Def. 5.1.1], [KC12, $\boldsymbol{\psi}_{\circledcirc}$: §2.2.3]. Questions regarding both the computability of sets with respect to different representations (which, however, are not part of this work) and the *computability relation* of representations (which are and will be discussed to some extent in Section 3) has been covered in many articles (see, e. g., [BW99, Wei00, Zie02, Her02, BP03]).

2.3. **Enrichments.** The representations we have seen in the previous section are rather generic. In practice, however, additional parameters are usually known, e. g., bounds on diameters of sets or rate of growth of functions. Such additional discrete information (*discrete advice parameters*, or just *advice parameters* for short [KMRZ12, p.18]) may be *uncomputable* from a given representation, but will turn out to be of great use (complexity-wise) in Sections 3 to 5.

Definition 2.8 (enrichments; cf. [KMRZ12, Def. 2.4(c+d)])**.** Let $\boldsymbol{\nu}_{\Sigma^*} : \subseteq\Sigma^{**} \to \Sigma^*$ denote a representation of Σ^*. Further, let $\boldsymbol{\xi} : \subseteq\Sigma^{**} \to X$ be a representation of a set X, and $\mathsf{E}: X \rightrightarrows \Sigma^*$ a multi-valued function (encodes information $\boldsymbol{\xi}$-names are *enriched* with). Then ϕ is a $\boldsymbol{\xi} \sqcap \mathsf{E}$-name of $x \in X$ if it is of form $\phi = \langle\phi_1, \phi_2\rangle$ with $\boldsymbol{\xi}(\phi_1) = x$ and $\boldsymbol{\nu}_{\Sigma^*}(\phi_2) \in \mathsf{E}(x)$.

More specific, we use the following four concrete enrichments in the remainder of this paper.

Definition 2.9 (concrete enrichments for sets)**.**

(1) Outer radii: Consider the enrichment

$$\mathsf{b}: \mathcal{K}^{(d)} \rightrightarrows \Sigma^* , \quad \mathsf{b}: S \mapsto \left\{\mathsf{un}_{\mathbb{Z}}(b) \;\middle|\; b \in \mathbb{Z} \text{ and } S \subseteq \overline{\mathrm{B}}_{\|\cdot\|}(0, 2^b)\right\}$$

By definition, $\boldsymbol{\psi}^{(d)}|^{\mathcal{K}} \sqcap \mathsf{b}$ then is a representation of $\mathcal{K}^{(d)}$ whereby each name contains an *outer radius parameter* b (encoded in unary according to b) on the encoded compact set. We refer to 2^b as an *outer radius* with respect to the outer radius parameter b.

(2) Inner radii and inner points: In a similar fashion to b define enrichments

$$\mathsf{r}: \mathcal{R}^{(d)} \rightrightarrows \Sigma^* , \quad \mathsf{r}: S \mapsto \left\{\mathsf{un}_{\mathbb{Z}}(r) \mid r \in \mathbb{Z},\, \exists\, x \in S^\circ \,.\, \overline{\mathrm{B}}_{\|\cdot\|}(x, 2^{-r}) \subseteq S\right\} ,$$
$$\mathsf{a}: \mathcal{R}^{(d)} \rightrightarrows \Sigma^* , \quad \mathsf{a}: S \mapsto \left\{\mathsf{bin}^{(d)}_{\mathbb{D}}(a) \mid a \in \mathbb{D}^d,\, \exists\, \delta > 0 \,.\, \overline{\mathrm{B}}_{\|\cdot\|}(a, \delta) \subseteq S\right\} .$$

We refer to decoded images under r as *inner radii parameter* (giving an *inner radius of* 2^{-r}), and to decoded images under a as *inner points.*

(3) Information encoded by a and r is independent of the other, i. e., the bound on an inner radius parameter according to r need not necessarily be centered at a. If we need both information, i. e., an inner *ball*, then we combine it to

$$\mathsf{ar}: S \in \mathcal{R}^{(d)} \mapsto \left\{\left\langle \mathsf{bin}^{(d)}_{\mathbb{D}}(a), \mathsf{un}_{\mathbb{Z}}(r)\right\rangle \mid \mathrm{B}_{\|\cdot\|}(a, 2^{-r}) \subseteq S\right\} .$$

These choices of encodings also meet both theory and practice: cf., for example, [GLS88, Def. 2.1.20] and [Hoo90, Def. 2.2+2.3]. Note that both the dimension and the norm will be always understood from the context and therefore is not considered enrichment.

Convention. The correct way to work with enriched representations would be like this: Let E be an enrichment, and $\langle\phi_1, \phi_2\rangle$ be a $\boldsymbol{\xi} \sqcap \mathsf{E}$-name. Then $E \coloneqq \boldsymbol{\nu}_{\Sigma^*}(\phi_2)$ is a concrete instance of enrichment E of object $x \coloneqq \boldsymbol{\xi}(\phi_1)$. As this intermediate step of "extracting" E from $\langle\phi_1, \phi_2\rangle$ is just a technical though necessary detail which does not add to any proof argument, we use the typographical convention to denote a concrete decoded instance of $\mathsf{E}(x)$ "variable style", that is, as E. In the above spirit, further abbreviate a $\boldsymbol{\xi}^{(d)}|^{\mathcal{KR}} \sqcap \mathsf{a} \sqcap \mathsf{r} \sqcap \mathsf{b}$-name $\langle\phi_1, \phi_2, \phi_3, \phi_4\rangle$ by $\left\langle\phi_1, a, 0^r, 0^b\right\rangle$. This notation has been purposefully chosen as a reminder that (a) inner points as advice parameters are encoded in *binary*, while (b) both inner and outer radii are encoded in *unary* according to enrichments r and b, respectively.

2.4. **Complexity of functions and operators: upper and lower bounds.** We briefly recap some facts from discrete complexity theory. Assume M to be a Turing machine that either accepts its input $s \in \Sigma^*$ or rejects it; i. e., M always terminates. The computation time of such a machine M is bounded by some non-decreasing function $t: \mathbb{N} \to \mathbb{N}$ (or: t-time bounded) if for all $s \in \Sigma^*$, M started on s holds within $t(\ell(s))$ steps.

Unless stated otherwise, we use "Turing machine" as a synonym for "*deterministic* Turing machine". Allowing a machine also to *guess* strings from Σ^* makes it *non-deterministic*. Through the course of this paper we need three complexity classes: P marks the class of all problems $A \subseteq \Sigma^*$ decidable by a deterministic polynomial-time bounded Turing machine, and NP the class of problems decidable by a *non*-deterministic polynomial-time Turing machine. Decision problems $A \in \mathsf{NP}$ can equivalently be stated as being *polynomial-time verifiable* by a deterministic Turing machine; i. e., there exists a decision problem $A' \subseteq \Sigma^*$ which is polynomial-time equivalent to A and satisfies $\exists\, B \in \mathsf{P} \,.\, A' = \{s \in \Sigma^* \mid \exists\, w \in \Sigma^{\ell(s)} \,.\, \langle w, s\rangle \in B\}$. Given an $s \in \Sigma^*$, a w which *verifies* $\langle w, s\rangle \in B$ is usually called a *witness for* $s \in A'$.

The class UP contains problems $A \subseteq \Sigma^*$ decidable by an *unambiguous* non-deterministic polynomial-time Turing machine; that is, a machine that for each $s \in A$ has *exactly one* accepting path. It is easy to see that $\mathsf{P} \subseteq \mathsf{UP} \subseteq \mathsf{NP}$, but whether any of these inclusions is proper is a wide-open problem. We defer the discussion of the hypothetical case $\mathsf{P} \neq \mathsf{UP}$ and its implications until Section 5.

As for example pointed out in [FG06], problems usually come with a variety of structural information (like the number of nodes in a graph, number of variables in a formula, number of faces of a polyhedron), which however is not reflected in the above one-ary notion of complexity. *Parameterized complexity* extends upon that: A parameterized decision problem (A, k) ($A \subseteq \Sigma^*$ with parameterization $k\colon \Sigma^* \to \mathbb{N}$, which typically is required to be at least computable) is (τ, t)-time computable, iff a deterministic Turing machine M exists whose computation time is bounded by $\tau(k(s)) \cdot t(\ell(s))$ for all $s \in \Sigma^*$. If t moreover is a polynomial, then (A, k) is said to be *parameterized polynomial-time* decidable (also: fixed-parameter tractable).

2.4.1. *Time complexity.* The complexity notion for oracle machines is similar to that for classical Turing machines, except for the extension that it takes the oracle in total one step to read the content written on the oracle tape and to produce its answer. The computation time of an oracle machine M^ϕ (and thus the complexity of the element of Σ^{**} it computes) with *set-oracle* (or equivalently a function-oracle $\phi \in \Sigma^\omega$) can solely be measured in the length of the discrete input given to $M^?$; which is the case for representations $\boldsymbol{\omega}$, $\boldsymbol{\kappa}$ and $\boldsymbol{\psi}$.

Definition 2.10. Let $t\colon \mathbb{N} \to \mathbb{N}$ be some non-decreasing function.

(1) A function $g\colon \subseteq\Sigma^\omega \times \Sigma^* \to \Sigma^*$ is t-time computable if an oracle machine $M^?$ exists which for all $\phi \in \Sigma^\omega$ and $s \in \Sigma^*$ computes $g(\phi, s)$ in time bounded by $t(\ell(s))$.
(2) Let $\boldsymbol{\xi}$ and $\boldsymbol{\xi}'$ be a second-order representations of sets X and X', respectively, and let $\mathrm{dom}(\boldsymbol{\xi}) \subseteq \Sigma^\omega$.[5] A $(\boldsymbol{\xi}, \boldsymbol{\xi}')$-computable function $f\colon \subseteq X \to X'$ is *t-time computable* if it is realized by a t-time $(\boldsymbol{\xi}, \boldsymbol{\xi}')$-computable function $g\colon \subseteq\Sigma^\omega \times \Sigma^* \to \Sigma^*$.

As hinted prior to the definition, it is not that obvious how to define complexity in case of names ϕ from Σ^{**} instead of Σ^ω. The problem with names from Σ^{**} is that oracle answers in general are *not* bounded in the length of its argument as it has been the case for $\phi \in \Sigma^\omega$. However, combining enrichments with Thm. 2.10 allows us to define time bounds whenever the oracle answers can be bounded in terms of the parameters a representation has been enriched with. For that purpose, force both $\boldsymbol{\delta}^{(d)}$- and $\boldsymbol{\delta}_{\mathsf{rel}}{}^{(d)}$-names ϕ to additionally satisfy $\phi(q, 0^n) \in \mathbb{D}^d_{n+1}$.[6] The subsequent definition is based on [KMRZ12, Def. 2.1+2.2].

[5] That is, $\boldsymbol{\xi}$-names are predicates.

[6] Note that the latter condition is only added to prevent unnecessarily long answers as they are not more accurate (with respect to the conditions on $\boldsymbol{\delta}$- and $\boldsymbol{\delta}_{\mathsf{rel}}$-names) when provided with a precision much higher than n. This restriction is not necessary in the general theory of *second-order polynomials* and *second-order polynomial-time*, but we defer this discussion until Section 5.

Definition 2.11 (parameterized complexity)**.** Let $\boldsymbol{\xi}$ and $\boldsymbol{\xi}'$ be second-order representations of sets X and X', respectively, and $\mathsf{E}\colon X \rightrightarrows \Sigma^*$ an enrichment of X. Moreover, let $\tau, t\colon \mathbb{N} \to \mathbb{N}$ be non-decreasing functions.

(1) A function $f\colon \subseteq X \to X'$ is (τ, t)-time $(\boldsymbol{\xi} \sqcap \mathsf{E}, \boldsymbol{\xi}')$-computable if it has a $(\boldsymbol{\xi} \sqcap \mathsf{E}, \boldsymbol{\xi}')$-realizer $g\colon \subseteq \Sigma^{**} \times \Sigma^* \to \Sigma^*$ such that the computation time on every input $(\phi, s) \in \mathrm{dom}(\boldsymbol{\xi}) \times \Sigma^* \mapsto g(\phi, s)$ is bounded by $\tau\big(\ell(\mathsf{E}(\boldsymbol{\xi}(\phi)))\big) \cdot t(\ell(s))$.[7]
(2) If t is a polynomial, then f is said to be *parameterized polynomial-time* $(\boldsymbol{\xi} \sqcap \mathsf{E}, \boldsymbol{\xi}')$-*computable.*
(3) If both t and τ are polynomials, then f is said to be *fully polynomial-time* $(\boldsymbol{\xi} \sqcap \mathsf{E}, \boldsymbol{\xi}')$-*computable.*

As advice parameters are part of a name anyway, we simply speak about "polynomial time" whenever "fully polynomial-time" is meant. This identification is justified as fully polynomial-time and unparameterized polynomial-time coincide for $\mathsf{E}\colon x \mapsto \{\varepsilon\}$.

On compact sets $K \in \mathcal{K}^{(d)}$, this definition allows to bound the answer lengths in terms of an outer radius parameter b as in Thm. 2.8(1). Take representation $\boldsymbol{\delta}$ as an example: Assume $\phi := \langle \phi', 0^b \rangle$ to be a $\boldsymbol{\delta}^{(d)} \sqcap \mathsf{b}$-name of K. Then $\ell(\phi'(q, 0^n))$ can be bounded linearly in $|b| + \ell(\langle q, 0^n \rangle)$ for all $q \in \mathbb{D}^d$ and $n \in \mathbb{N}$.

2.5. **Common proof arguments.** We review two common arguments that allows us to prove lower bounds or even the uncomputablitiy of operations.

2.5.1. *Adversary argument.* The *adversary method* is used to prove lower bounds on the *uniform* computational complexity of functions. Let $\boldsymbol{\xi}$, $\boldsymbol{\xi}'$, X, X' and f as in the previous subsections. For any discrete argument $s \in \Sigma^*$ pick an element $x \in X$ and construct a subset $Y \subset X$ of cardinality at least exponential in $\ell(s)$ such that every $y \in Y$ has a $\boldsymbol{\xi}$-name ϕ close to one of x, yet $f(x)$ differs by at least 2^{-n} from $f(y)$. Then any machine $M^?$ that $(\boldsymbol{\xi}, \boldsymbol{\xi}')$-realizes f necessarily has to ask exponentially many queries to ϕ.

This approach is similar to the adversary method from *Information-Based Complexity* [TWW88] where computations are exact, but only finite information is known about the input. As an example, take Riemann integration, done on 2^n many sampling points in order to achieve an approximation which is always guaranteed to be within error 2^{-n}.

2.5.2. *Topological discontinuity.* Given second-order representations $\boldsymbol{\xi}$ and $\boldsymbol{\xi}'$ of sets X and X', respectively, and a function $f\colon \subseteq X \to X'$. By Thm. 2.3 we already know that f is not $(\boldsymbol{\xi}, \boldsymbol{\xi}')$-computable whenever it is not $(\boldsymbol{\xi}, \boldsymbol{\xi}')$-continuous. Recall that Σ^{**} comes equipped with the product topology, providing a way to prove the latter: Construct an $x \in X$ and an appropriate $\boldsymbol{\xi}$-name ϕ. Any machine for a hypothetical $(\boldsymbol{\xi}, \boldsymbol{\xi}')$-realizer for f does only inspect finitely many values of ϕ. Now pick a slightly different $\boldsymbol{\xi}'$-name, say ϕ', for a different value, say x', which coincides with ϕ on values observed by $M^?$, but leads it to produce an answer exceeding the prescribed error bound.

[7]Notice the term "$\mathsf{E}(\boldsymbol{\xi}(\phi))$" in the time bound: Enrichments are by definition multi-valued but τ is not set-valued. Although technically incorrect, the meaning "this bound has to hold true for *every advice parameter in* $\mathsf{E}(...)$" clearly is supported by this notation while the correct statement would be "the computation time has to be bounded by $\tau(\ell(E)) \cdot t(\ell(s))$ for *all* $E \in \mathsf{E}(\boldsymbol{\xi}(\phi))$".

3. Comparing representations of sets

In this section, we compare the representations introduced in Thm. 2.7 with respect to their mutual polynomial-time *reducibility.* Two aspects will play a key role in these comparisons: Whether a representation $\boldsymbol{\xi}$ is *norm-invariant*, i. e., if $\boldsymbol{\xi}_{\|\cdot\|}$ and $\boldsymbol{\xi}_{\|\cdot\|'}$ are polynomial-time equivalent, and the influence of the dimension parameter. Both together will prove $\boldsymbol{\delta}^{(d)}$ to be *richer* (intuition: to carry *more* information) than all of the other representations from dimension $d = 2$ onward by combining that(a) $\boldsymbol{\delta}^{(d)}$ is not norm-invariant for $d \geq 2$, (b) all of the other representations we discuss *are* norm-invariant, and (c) $\boldsymbol{\delta}$ reduces to all of the other representations in polynomial time. Representation $\boldsymbol{\omega}$, on the other hand, will prove to be *poorer* than all of the other representations. However, this gap between $\boldsymbol{\delta}$ and $\boldsymbol{\omega}$ can be closed by restricting to $\mathcal{KCR}$, adding parameters to $\boldsymbol{\omega}$ and applying techniques from discrete optimization (Thm. 3.14), which proves *all* representations to be polynomial-time equivalent in this particular setting.

We now turn to the formalization of what has been described above.

Definition 3.1 (translations/reductions; cf. [Wei00, Def. 2.3.2])**.** Let $\boldsymbol{\xi}$ and $\boldsymbol{\xi}'$ be representations of the same set X. Then $\boldsymbol{\xi}$ uniformly *translates* (or: *reduces*) to $\boldsymbol{\xi}'$—$\boldsymbol{\xi} \preceq \boldsymbol{\xi}'$ for short—if id_X is $(\boldsymbol{\xi}, \boldsymbol{\xi}')$-computable. If id_X is parameterized polynomial-time $(\boldsymbol{\xi}, \boldsymbol{\xi}')$-computable, then we write $\boldsymbol{\xi} \preceq_{\mathrm{pp}} \boldsymbol{\xi}'$. If id_X is even fully polynomial-time $(\boldsymbol{\xi}, \boldsymbol{\xi}')$-computable, then we write $\boldsymbol{\xi} \preceq_{\mathrm{p}} \boldsymbol{\xi}'$.

Note that while the notation $\boldsymbol{\xi} \preceq \boldsymbol{\xi}'$ makes sense when read as "$\boldsymbol{\xi}$ translates to $\boldsymbol{\xi}'$", it is counter-intuitive when read as a reduction: $\boldsymbol{\xi}$ reduces to $\boldsymbol{\xi}'$ if $\boldsymbol{\xi}$-names carry *more* information than $\boldsymbol{\xi}'$-names; hence, $\boldsymbol{\xi}$-names are *harder* to compute than $\boldsymbol{\xi}'$-names. Reductions in classical complexity theory are usually thought the other way around, i. e., the *harder* problem being "greater or equal" to *easier* problems.

The intuition about representations encoding more or less information also explains the following fact which we will use in many places throughout this paper.

Fact 3.2. Let $\boldsymbol{\xi}_1, \boldsymbol{\xi}_2$ be representations of X, and $\boldsymbol{\xi}'_1, \boldsymbol{\xi}'_2$ be representations of X'. Then every $(\boldsymbol{\xi}_1, \boldsymbol{\xi}'_1)$-computable function is also $(\boldsymbol{\xi}_2, \boldsymbol{\xi}'_2)$-computable whenever $\boldsymbol{\xi}_2 \preceq \boldsymbol{\xi}_1$ (providing potentially *more* information about the input) and $\boldsymbol{\xi}'_1 \preceq \boldsymbol{\xi}'_2$ (requiring potentially *less* information about the output) hold. The same applies if $\preceq$ is replaced with $\preceq_{\mathrm{p}}$.

Convention. For $Y \subseteq X$ let $\boldsymbol{\xi} \preceq^Y \boldsymbol{\xi}'$ be an abbreviation for $\boldsymbol{\xi}|^Y \preceq \boldsymbol{\xi}'|^Y$. This new representation $\boldsymbol{\xi}|^Y : \subseteq\Sigma^{**} \to Y$ is the result of the restriction of $\boldsymbol{\xi}$'s image to Y. Apply the same to $\preceq_{\mathrm{p}}$, $\prec_{\mathrm{p}}$, $\equiv$ and $\equiv_{\mathrm{p}}$.

3.1. Technicalities.

Two subtle details need to be addressed before we can start comparing representations with respect to polynomial-time reducibility: (a) their dependence on the choice of norm underlying $\mathbb{R}^d$, and (b) considering also 'negative' values of the precision parameter n, that is, absolute error boundy larger than one. The former leads to the notion of *well-behaved norms*, while the latter introduces *scale-invariant representations.*

3.1.1. *Restriction to well-behaved norms.*

Representations $\boldsymbol{\psi}$, $\boldsymbol{\kappa}$ and $\boldsymbol{\omega}$ depend on the notion of "being close". Practically speaking, a point q gets printed on the screen whenever $\overline{\mathrm{B}}_{\|\cdot\|}(q, 2^{-n})$ meets the represented set, where points q were chosen from $\mathbb{D}_n^d$. The implicit assumption underlying all representations from Thm. 2.7 is *compatibility with the grid* $\mathbb{D}_n^d$: The whole space, say X, can be covered with $\|\cdot\|$-balls with radii $2^{-(n+c)}$ and centers from $X \cap \mathbb{D}_n^d$, where $c \in \mathbb{Z}$ is a constant depending on the pair $\|\cdot\|, \|\cdot\|_\infty$. As an example, take $\|\cdot\| := 4 \cdot \|\cdot\|_\infty$ with $c := -\operatorname{lb} 4$.

The implied necessity to incorporate a norm-dependent constant c into precision parameters is cumbersome and we avoid it by *imposing* the above mentioned "compatibility" on the respective norm. For the same reason we disallow *slanted* (or otherwise distorted) norms like $\|(x_1, x_2)\| := (|x_1/2|^2 + |x_2|^2)^{1/2}$ although this restriction can be avoided (as we will see in Thm. 3.7) and is only present to simplify things.

We denote such norms satisfying both of the above motivated properties as being *well-behaved*.

Definition 3.3 (well-behaved norms)**.** A norm $\|\cdot\|$ on $\mathbb{R}^d$ is said to be *well-behaved* if it has the following two properties:

(1) $\|\cdot\|$ is invariant under 90-degree rotations. More precisely: Let $\{e_1, \dots, e_d\}$ be the canonical basis of $\mathbb{R}^d$. Then $\|e_i\| = \|e_j\|$ for all $1 \le i, j \le d$.
(2) $\|\cdot\|$-balls are not too small, i. e., $\overline{\mathrm{B}}_{\|\cdot\|_\infty}(q, 2^{-(n+1)}) \subseteq \overline{\mathrm{B}}_{\|\cdot\|}(q, 2^{-n})$ for all $n \in \mathbb{N}$ and $q \in \mathbb{D}_n^d$.

It then follows by the second condition that $\mathbb{R}^d$ can be covered by $\|\cdot\|$-balls with centers from $\mathbb{D}_n^d$ and radii 2^{-n}. Examples of well-behaved norms include the p-norms $\|(x_1, \dots, x_d)\|_p := (|x_1| + \cdots + |x_d|)^{1/p}$ for $p \ge 1$, while $\|(x_1, x_2)\| := (|x_1/2|^2 + |x_2|^2)^{1/2}$ (violates the first condition) and $3/2\|\cdot\|_1$ (violates the second condition) are not.

Convention. For the rest of this paper, we only consider well-behaved norms unless stated otherwise.

3.1.2. *Scale-invariance.* Starting with [KF82], complexity results were stated for functions whose domains were a subset of the *unit* hypercube; the same was true for sets. This restriction rendered (at least for sets) the question about precision parameters *smaller than* 0 (i. e., absolute error bounds $> 2^{-0}$) pointless, which allowed for a complexity notion solely in the precision parameter. However, as we will see many times in Sections 4 and 5, algorithms for operators on sets often involve an unavoidable preprocessing step (given the representations we have seen so far): Given $b \in \mathbb{N}$, chop $\overline{\mathrm{B}}_{\|\cdot\|_\infty}\big(0, 2^b\big)$ into $2^{d(b+1)}$ unit hypercubes, pick a subset of them (usually one cube), then proceed by applying the given $\boldsymbol{\xi}$-name to this subset. It is this preprocessing step which seems to be artificial and superfluous as the *real* algorithm often starts only *after* this step. For this reason the author believes that a closed subset S of $[0, 1]$ (or any fixed compact set) should admit, up to a polynomial rather than exponential in k, the same complexity as S inflated by a factor of 2^k. Both sets are still *structurally* the same!

To this end, let $\widehat{\boldsymbol{\psi}}$ be the extension (or: relaxation) of representation $\boldsymbol{\psi}$ to *integer precisions*, i. e., a $\widehat{\boldsymbol{\psi}}^{(d)}$-name ϕ satisfies

$$\phi(s) = 1 \text{ if } S \cap \mathrm{B}\big(q, 2^{-n}\big) \neq \emptyset\,; \qquad \phi(s) = 0 \text{ if } S \cap \overline{\mathrm{B}}\big(q, 2^{-n+1}\big) = \emptyset\,.$$

with $s := \big\langle \mathsf{bin}_{\mathbb{D}}^{(d)}(q), \mathsf{un}_{\mathbb{Z}}(n) \big\rangle$. Recall that we agreed to equivalently write s as $\big\langle q, 0^n \big\rangle$ where 0^n abbreviates the "unary encoding" of integer n. With the following Lemma we attempt to provide a way around the above described dilemma.

Lemma 3.4 (properties of $\widehat{\boldsymbol{\psi}}$)**.**

(1) *Scaling a closed set by factor* 2^k *for* $k \in \mathbb{Z}$ *is a* parameterized *polynomial-time operation in the* absolute value *of* k *(cf.* [ZM08, Lem. 2.7(4)]*), that is, the binary length of* 2^k*. More precisely: Operator* $\mathsf{Scale}\colon (S,k) \mapsto \{2^k x \mid x \in S\}$ *is* $(\boldsymbol{\psi}^{(d)} \times \mathbf{un}_{\mathbb{Z}}, \boldsymbol{\psi}^{(d)})$*-computable in parameterized polynomial time.*

(2) *In contrast,* Scale *is* fully *polnomial-time* $(\widehat{\boldsymbol{\psi}}^{(d)} \times \mathbf{un}_{\mathbb{Z}}, \widehat{\boldsymbol{\psi}}^{(d)})$*-computable.*

(3) $\boldsymbol{\psi}^{(d)} \preceq_{\mathrm{pp}} \widehat{\boldsymbol{\psi}}^{(d)} \preceq_{\mathrm{p}} \boldsymbol{\psi}^{(d)}$*.*

sketch. The first statement follows by the argument hinted prior to this Lemma: Let $q \in \mathbb{D}^d$ and $n \in \mathbb{N}$. If $n-k \geq 0$, then simply query the $\boldsymbol{\psi}^{(d)}$-name with $\langle 2^k q, 0^{n-k}\rangle$. If $n-k < 0$, then first split $\overline{\mathrm{B}}(2^k q, 2^{k-n})$ into unit-balls and combine the queries on the center and precision 0 on each of these balls. For the second statement, use the argument from the above first case, namely, query the $\widehat{\boldsymbol{\psi}}^{(d)}$-name with precision $n-k$. The first reduction in statement three follows immediatiely from 1. and 2. For the second reduction, use the split of $\overline{\mathrm{B}}(2^k q, 2^{k-n})$ into unit-balls and argue as in the first case of statement one. □

It follows by the previous statement that *all* scaled versions of a set are polynomially equivalent with respect to $\widehat{\boldsymbol{\psi}}$.

Remark 3.5. As the concept of a scale-invariant representation avoids the above described deficiencies, we like to impose it on *every* representation $\boldsymbol{\xi}$ from Thm. 2.7. Therefore, we will denote $\widehat{\boldsymbol{\xi}}$ to be understood as the scale-invariant version of $\boldsymbol{\xi}$, and then associate $\boldsymbol{\xi}$ with $\widehat{\boldsymbol{\xi}}$ (i. e., drop the explicit hat). As a consequence, precision parameters shall now usually be *integers*.

3.2. **Topological versus computable equivalence of norms.** In this section we examine the question which representations $\boldsymbol{\xi}$ are *norm-invariant*, i. e., whether $\boldsymbol{\xi}_{\|\cdot\|} \equiv_{\mathrm{p}} \boldsymbol{\xi}_{\|\cdot\|'}$ holds true for *all* topological equivalent well-behaved norms $\|\cdot\|, \|\cdot\|'$. Notice that "norm-invariance" inherently asks about polynomial-time equivalence: norm-exchange is a computable operation for all representations from Thm. 2.7.

Our first result generalizes Braverman's remark [Bra05b, following Def. 2] on the interchangeability of $\boldsymbol{\psi}_{\|\cdot\|_2}$ and $\boldsymbol{\psi}_{\|\cdot\|_\infty}$.

Proposition 3.6. $\boldsymbol{\psi}^{(d)}_{\|\cdot\|} \equiv_{\mathrm{p}} \boldsymbol{\psi}^{(d)}_{\|\cdot\|'}$ *holds in any dimension* $d \in \mathbb{N}$ *and for any two norms* $\|\cdot\|, \|\cdot\|'$ *on* $\mathbb{R}^d$*.*

The key to prove this proposition is its *non-uniformity* with respect to any two well-behaved norms $\|\cdot\|, \|\cdot\|'$: The necessary information (here: the "coverage pattern" of the unit $\|\cdot\|'$-ball) for a machine to translate from $\boldsymbol{\psi}^{(d)}_{\|\cdot\|}$ to $\boldsymbol{\psi}^{(d)}_{\|\cdot\|'}$ can be directly encoded into it.

Remark 3.7. For every two norms $\|\cdot\|, \|\cdot\|'$ on $\mathbb{R}^d$ exists a constant $k \in \mathbb{N}$ and a finite set $D \subset \mathbb{D}^d_k$ ("coverage pattern") such that

$$\overline{\mathrm{B}}_{\|\cdot\|'}(0,1) \subseteq \bigcup\nolimits_{p \in D} \overline{\mathrm{B}}_{\|\cdot\|}(p, 2^{-k}) \subseteq \overline{\mathrm{B}}_{\|\cdot\|'}(0, 3/2)\ .$$

Note that $\|\cdot\|'$ can only be approximated by $\|\cdot\|$-balls up to a *constant factor* by the above coverage pattern D. Approximating the shape of a $\|\cdot\|'$-ball up to *arbitrary precision*, however, might still be uncomputable.

of Thm. 3.6. Let $k \in \mathbb{N}$ and $\mathbb{D}^d_k$ as in Thm. 3.7. Let further ϕ be a $\boldsymbol{\psi}^{(d)}_{\|\cdot\|}$-name of $S \in \mathcal{A}^{(d)}$, $q \in \mathbb{D}^d$, and $n \in \mathbb{Z}$. Claim: ϕ', defined as

$$\phi'(q, 0^n) := \max_{p \in D} \phi(p', 0^{n+k}) \ , \quad p' := q + 2^{-(n+k)} p \ ,$$

is a $\boldsymbol{\psi}^{(d)}_{\|\cdot\|'}$-name of S. Note that since D is finite, the maximum ranges only over finitely many values and is therefore computable in time linear in $n + k + \ell(\langle q \rangle)$.

If $\mathrm{B}_{\|\cdot\|'}(q, 2^{-n}) \cap S \neq \emptyset$, then by Thm. 3.7 there exists a point $p_0 \in D$ such that $\overline{\mathrm{B}}_{\|\cdot\|}(p', 2^{-(n+k)})$ meets S, justifying $\phi'(q, 2^{-n}) = 1$. If, on the other hand, $\overline{\mathrm{B}}_{\|\cdot\|'}(q, 2^{-n+1}) \cap S = \emptyset$, then in particular $\overline{\mathrm{B}}_{\|\cdot\|}(p', 2^{-(n+k)}) \cap S = \emptyset$ for all $p \in D$. Their union covers $\overline{\mathrm{B}}_{\|\cdot\|'}(q, 2^{-n})$ which renders $\phi'(q, 2^{-n}) = 0$ to be correct. $\square$

Noteworthy: Neither one of the norms has actually to be *computable*—a direct consequence of the note following Thm. 3.7.

The argument from Thm. 3.6 generalizes to $\boldsymbol{\omega}^{(d)}$ (over $\mathcal{R}^{(d)}$) and $\boldsymbol{\kappa}^{(d)}$ (over $\mathcal{K}^{(d)}$), rendering both representation to be norm-invariant, too. Representation $\boldsymbol{\delta}$, however, turns out to be *not* norm-invariant—not even non-uniformly (provided $\mathsf{P} \neq \mathsf{NP}$):

Theorem 3.8 ($\boldsymbol{\delta}$ is not polynomial-time invariant under a change of norms *unless* $\mathsf{P} \neq \mathsf{NP}$)**.** *In any dimension $d \geq 2$ there is a set $S \in \mathcal{K}^{(d)}$ that is polynomial-time $\boldsymbol{\delta}^{(d)}_{\|\cdot\|_1}$-computable but* not *polynomial-time $\boldsymbol{\delta}^{(d)}_{\|\cdot\|_\infty}$-computable if and only if $\mathsf{P} \neq \mathsf{NP}$.*

Proof. Only if ($\mathsf{P} = \mathsf{NP}$ implies that $\boldsymbol{\delta}$ is (non-uniformly) norm-invariant). Suppose $\mathsf{P} = \mathsf{NP}$ and let ϕ_1 be a polynomial-time $\boldsymbol{\delta}^{(d)}_{\|\cdot\|_1}$-computable name of S. Now consider the sets N and P,

$$N := \left\{ \langle p, \delta, 0^n, 0^m \rangle \;\middle|\; \exists\, p' \in \mathbb{D}^d_{n+2},\ \phi_1(p', 0^{n+2}) \leq 2^{-(n+2)} \,.\, \langle p, p', \delta, 0^n, 0^m \rangle \in P \right\} \ ,$$
$$P := \left\{ \langle p, p', \delta, 0^n, 0^m \rangle \;\middle|\; \left| \delta - \|p - p'\|_\infty \right| \leq 2^{-m} \right\} \ ,$$

which in turn are polynomial-time decidable by the above assumption.

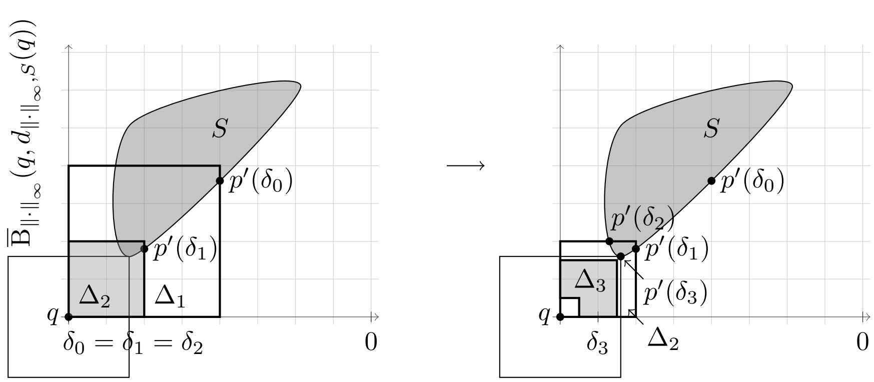


Figure 3: Search for a 2^{-n}-approximation δ_{n+1} to $d_{\|\cdot\|_\infty, S}(q)$ by iteratively determining distances δ_i and associated narrowed sets $\Delta_{i+1} := \left\{ x \in [0,1]^2 \;\middle|\; |\delta_i - \|x - q\|_\infty| \leq 2^{-(i+1)} \right\}$ such that $\delta_i \leq d_{\|\cdot\|_\infty, S}(q)$ and $\exists\, p'(\delta_i) \in \Delta_{i+1} \cap \mathbb{D}^d_{n+2} \,.\, \langle q, p'(\delta_i), \delta_i, 0^n, 0^{i+1} \rangle \in P$.

A $\boldsymbol{\delta}^{(d)}_{\|\cdot\|_\infty}$-name for S can be recovered from queries "$\langle q, \delta_i, 0^n, 0^i\rangle \in N$?" by the following iterative procedure (cf. Fig. 3): Let $\delta_0 := 0$. Then for each $1 \le i \le n+1$ set $\delta_i := \delta_{i-1}$ if $\langle q, \delta_{i-1}, 0^n, 0^i\rangle \in N$, and $\delta_i := \delta_{i-1} + 2^{-i}$ otherwise. This way,

$$\delta_i \le d_{\|\cdot\|_\infty, S}(q) \le \delta_i + 2^{-i} + 2 \cdot 2^{-(n+2)} \,, \tag{3.1}$$

and therefore $\left| d_{\|\cdot\|_\infty, S}(q) - \delta_{n+1} \right| \le 2^{-n}$. We prove the correctness of Eqn. (3.1) by induction. For $i = 0$ it surely is true, so consider the case $i > 0$. If $\langle q, \delta_{i-1}, 0^n, 0^i\rangle \in N$, then (3.1) holds true for $\delta_i := \delta_{i-1}$ by the construction of N. If, on the other hand, $\langle q, \delta_{i-1}, 0^n, 0^i\rangle \notin N$, then for all $p' \in \mathbb{D}^d_{n+2}$ with $\phi_1(p', 0^{n+2}) \le 2^{-(n+2)}$ we have $|\delta_{i-1} - \|q - p'\|_\infty| > 2^{-i}$. Since (3.1) holds for δ_{i-1}, it firstly implies $d_{\|\cdot\|_\infty, S}(q) > \delta_{i-1} + 2^{-i}$. But then (3.1) rewrites as

$$\delta_{i-1} + 2^{-i} \le d_{\|\cdot\|_\infty, S}(q) \le \delta_{i-1} + 2^{-i+1} + 2^{-n}$$

which is exactly (3.1) for $\delta_i := \delta_{i-1} + 2^{-i}$.

Consequently, $\phi(q, 0^n) := \delta_{n+1}$ gives a $\boldsymbol{\delta}^{(d)}_{\|\cdot\|_\infty}$-name of S.

If. We prove this direction only for $d = 2$, but the generalization to higher dimensions follows by similar constructions. Assuming $\mathsf{P} \ne \mathsf{NP}$, we construct an adversary set A through a proper encoding of an NP-complete problem $N \subset \Sigma^*$ of form $N = \{ s \in \Sigma^* \mid \exists\, w \in \Sigma^{\ell(s)} .\, \langle w, s\rangle \in P \}$, $P \in \mathsf{P}$, into A. To this end, for $n \in \mathbb{N}$ and $0 \le i < 2^n$ associate the i-th string $s \in \Sigma^n$ with the set $A_{n,i} \subset [s_{n,i}, s_{n,i+1}] \times [0, 2^{-(2n+1)}]$ where $s_{n,0} := 1 - 2^{-n}$, $s_{n,i} := s_{n,0} + i \cdot 2^{-(2n+1)}$ and (just to simplify the notation) $s_{n,2^n} := s_{n+1,0}$. For each word $s \in \Sigma^n$ we then split its associated set $A_{n,i}$ into 2^n slices $A_{n,i,j}$, $0 \le j < 2^n$, where $A_{n,i,j}$ is associated with the j-th string $w \in \Sigma^n$. To this end, let $s_{n,i,j} := s_{n,i} + j \cdot 2^{-(3n+1)}$ and $s_{n,i,j+1}$. Whenever $\langle w, x\rangle$ is in P we code a "bump" in $A_{n,i,j}$, and a simple line otherwise; i. e., for $w, s \in \Sigma^n$, $A_{n,i,j} := s_{n,i,j} + 2^{-(3n+1)} \cdot A_\wedge$ if $\langle w, s\rangle \in P$, and $A_{n,i,j} := s_{n,i,j} + 2^{-(3n+1)} \cdot A_-$ otherwise; $A_\wedge := \{(x,y) \in [0,1]^2 \mid x - y = 0 \text{ for } x \le 1/2, \text{ and } x + y = 1 \text{ for } x > 1/2\}$, $A_- := \{(x,y) \in [0,1]^2 \mid y = 0\}$. Thus $A := \bigcup_{n,i,j \in \mathbb{N},\, 0 \le i,j < 2^n} A_{n,i,j}$ encodes N.

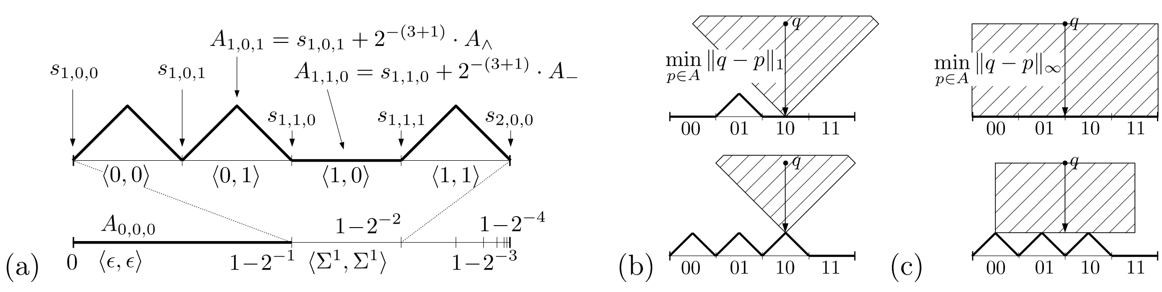


Figure 4: Encoding a certain NP-set N into a polynomial-time $\boldsymbol{\delta}^{(2)}_{\|\cdot\|_1}$-computable set A such that A being also polynomial-time $\boldsymbol{\delta}^{(2)}_{\|\cdot\|_\infty}$-computable would imply $\mathsf{P} = \mathsf{NP}$.

Without further notational overhead associate each point $q \in \mathbb{D}^1 \cap [0,1]$ with the (lexicographically) largest triple of indices (n,i,j) such that q belongs to $[s_{n,i,j}, s_{n,i,j+1}]$. As before, $\langle w, s\rangle$ is also uniquely identified by this triple. Now it is easy to construct a $\boldsymbol{\delta}^{(2)}_{\|\cdot\|_1}$-name for A, while it is hard (i. e., not computable in polynomial time) to construct one with respect to $\boldsymbol{\delta}^{(2)}_{\|\cdot\|_\infty}$ (both cases are also sketched in Fig. 4(b) and Fig. 4(c), respectively).

- A $\boldsymbol{\delta}^{(2)}_{\|\cdot\|_1}$-name ϕ of A can be constructed in polynomial time:

$$\phi\big((q_1,q_2),0^n\big) := \big|q_2 - \chi_P\langle w,s\rangle \cdot \big(2^{-(3n+2)} - |q_1 - (s_{n,i,j}+s_{n,i,j+1})/2|\big)\big|$$

- Now consider $\boldsymbol{\delta}^{(2)}_{\|\cdot\|_\infty}$. Assume there was a polynomial-time OTM $M^?$ which could compute a $\boldsymbol{\delta}^{(d)}_{\|\cdot\|_\infty}$-name ϕ' for A. Evaluating ϕ' at $(q_1',q_2') := \big((s_{n,i,0}+s_{n,i,0})/2,\, 2^{-(2n+1)}\big)$ with precision $n' := 3n+4$ then decides N because $\phi'\big((q_1',q_2'),2^{-n'}\big) \geq 2^{-(2n+2)} - 2^{-(3n+3)}$ if and only if a witness $w \in \Sigma^n$ exists with $\langle w,s\rangle \in P$. $\square$

3.3. **Polynomial-time relations.** Representations $\boldsymbol{\psi}^{(d)}$, $\boldsymbol{\kappa}^{(d)}$ and $\boldsymbol{\omega}^{(d)}$ are uniformly polynomial-time invariant under a change of norms; and so is $\boldsymbol{\delta}\mathsf{rel}^{(d)}$ according Thm. 3.13 below—however representation $\boldsymbol{\delta}^{(d)}$ in general is not, even non-uniformly subject to $\mathsf{P} \neq \mathsf{NP}$, although it is computably equivalent to $\boldsymbol{\psi}^{(d)}$ [BW99, Theorem 3.12]. In fact, restricted to the class $\mathcal{CR}$ of convex bodies, four of our five representations are known computably equivalent.

Fact 3.9 ([Zie02, Cor. 4.13])**.** $\boldsymbol{\delta}^{(d)} \equiv^{\mathcal{CR}} \boldsymbol{\psi}^{(d)} \equiv^{\mathcal{CR}} \boldsymbol{\omega}^{(d)}$ in any dimension $d \in \mathbb{N}$.

They are all equivalent because (intuitively speaking) points can be found due to regularity (regular sets are full-dimensional), and can be checked (locally) to be of the desired precision due to convexity (check if all points in a small neighborhood are also contained in the set).

In this section we now systematically compare all representations from Thm. 2.7 regarding their polynomial-time reducabilites in (a) dimension $d = 1$ and for $d \geq 2$, and (b) over various subclasses of $\mathcal{A}^{(d)}$. As a result, representations $\boldsymbol{\psi}^{(d)} \sqcap \mathsf{b}$, $\boldsymbol{\delta}\mathsf{rel}^{(d)} \sqcap \mathsf{b}$, and $\boldsymbol{\kappa}$ prove to be $\preceq_{\mathrm{p}}$-equivalent over $\mathcal{K}^{(d)}$ for every $d \in \mathbb{N}$. Taking $\boldsymbol{\psi}^{(d)}$ as a representative for this equivalence class, $\boldsymbol{\delta}^{(d)} \prec_{\mathrm{p}} \boldsymbol{\psi}^{(d)}$ holds true for $d \geq 2$, and $\boldsymbol{\psi}^{(d)} \prec_{\mathrm{p}} \boldsymbol{\omega}^{(d)}$ in any dimension (both even on $\mathcal{KCR}^{(d)}$!), which leaves us in a very different situation compared to Thm. 3.9. However: This distinction between $\boldsymbol{\delta}^{(d)}$ and $\boldsymbol{\omega}^{(d)}$ disappears when given the right set of additional parameters (Thm. 3.14), yielding one equivalence class of representations for sets as the result.

3.3.1. *Polynomial-time reducibilities in dimension $d = 1$.*

Proposition 3.10. $\boldsymbol{\delta}^{(1)}_{\|\cdot\|} \sqcap \mathsf{b} \equiv^{\mathcal{K}}_{\mathrm{p}} \boldsymbol{\delta}\mathsf{rel}^{(1)}_{\|\cdot\|} \sqcap \mathsf{b} \equiv^{\mathcal{K}}_{\mathrm{p}} \boldsymbol{\psi}^{(1)}_{\|\cdot\|} \sqcap \mathsf{b} \equiv^{\mathcal{K}}_{\mathrm{p}} \boldsymbol{\kappa}^{(1)}_{\|\cdot\|}$*,* $\boldsymbol{\psi}^{(1)} \preceq^{\mathcal{R}}_{\mathrm{p}} \boldsymbol{\omega}^{(1)}$*, and* $\boldsymbol{\omega}^{(1)}_{\|\cdot\|} \sqcap \mathsf{ar} \preceq^{\mathcal{CR}}_{\mathrm{p}} \boldsymbol{\psi}^{(1)}_{\|\cdot\|}$*.*

Proof. Without loss of generality, let $\|\cdot\| := \|\cdot\|_\infty$. Notice that the reductions $\boldsymbol{\delta}^{(1)} \sqcap \mathsf{b} \preceq^{\mathcal{K}}_{\mathrm{p}} \boldsymbol{\delta}\mathsf{rel}^{(d)} \sqcap \mathsf{b} \preceq^{\mathcal{K}}_{\mathrm{p}} \boldsymbol{\psi}^{(1)} \sqcap \mathsf{b} \preceq^{\mathcal{K}}_{\mathrm{p}} \boldsymbol{\kappa}^{(1)}$ already follow by definition of the respective representations.

- Reduction $\boldsymbol{\psi}^{(1)} \sqcap \mathsf{b} \preceq^{\mathcal{K}}_{\mathrm{p}} \boldsymbol{\delta}^{(1)}$: Let $\langle \phi, 0^b\rangle$ be a $\boldsymbol{\psi}^{(1)} \sqcap \mathsf{b}$-name of a closed $S \subseteq \overline{\mathrm{B}}(0,2^b)$. Further, set $b' := \max\{1,b\}$ and $c' := \operatorname{lb}\max\{2,\|q\|\}$. For any $q \in \mathbb{D}$ and $n \in \mathbb{Z}$, test if $\phi(q,0^{n+1}) = 1$. If it is, then 0 is a valid 2^{-n}-approximation of $d_S(q)$. If, on the other hand, $\phi(q,0^{n+1}) = 0$, then first find the smallest $i \in \mathbb{N}+$, $1 \leq i \leq n+b'+c'+1$, with $\phi(q,0^{n+1-i}) = 1$. Having found i, continue with two binary searches, one in $[q-2^{i-n},q]$ and the other in $[q,q+2^{i-n}]$, for points $p_-,p_+ \in \mathbb{D}_{n+1}$ eventually satisfying $\phi(p_\pm,0^{n+1}) = 1$ and minimizing $\|q-p_\pm\|$. Then $\min\{|q-p_-|,|q-p_+|\}$ consitutes a valid 2^{-n}-approximation of $d_S(q)$.

- Reduction $\boldsymbol{\delta}_{\mathsf{rel}}{}^{(1)} \sqcap \mathsf{b} \preceq_{\mathrm{p}}^{\mathcal{K}} \boldsymbol{\delta}^{(1)}$ follows by $\boldsymbol{\delta}_{\mathsf{rel}}{}^{(1)} \preceq_{\mathrm{p}}^{\mathcal{K}} \boldsymbol{\psi}^{(1)}$ and $\boldsymbol{\psi}^{(1)} \sqcap \mathsf{b} \preceq_{\mathrm{p}}^{\mathcal{K}} \boldsymbol{\delta}^{(1)}$ from above.
- Reduction $\boldsymbol{\kappa}^{(1)} \preceq_{\mathrm{p}}^{\mathcal{K}} \boldsymbol{\psi}^{(1)}$: Any $\boldsymbol{\kappa}^{(1)}$-name ϕ induces a $\boldsymbol{\psi}^{(1)}$-name ϕ' of the same set by $\phi'(q, 0^n) := \max_p\{\phi(p, 0^{n+1}) \mid p \in \mathbb{D}_{n+1} \cap [q - 2^{-n}, q + 2^{-n}]\}$. Since $|\mathbb{D}_{n+1} \cap [q - 2^{-n}, q + 2^{-n}]| \leq 5$, constantly many queries to ϕ suffice to devise ϕ'.
- Reduction $\boldsymbol{\psi}^{(1)} \preceq_{\mathrm{p}}^{\mathcal{R}} \boldsymbol{\omega}^{(1)}$: Every $\boldsymbol{\psi}^{(1)}$-name ϕ constitutes a $\boldsymbol{\omega}^{(1)}$-name ϕ' of the same set through $\phi'(q, 0^n) := \phi(q, 0^{n+1})$.
- Reduction $\boldsymbol{\omega}^{(1)} \sqcap \mathsf{ar} \preceq_{\mathrm{p}}^{\mathcal{R}} \boldsymbol{\psi}^{(1)}$: Given a $\boldsymbol{\omega}^{(1)} \sqcap \mathsf{ar}$-name $\langle \phi, a, 0^r \rangle$ of $S \in \mathcal{CR}^{(1)}$, do a binary search between a and q for a point $p \in \mathbb{D}_m$, $m := \max\{n, |r|\} + 1$, which minimizes $|q - p|$ over all such points satisfying $\phi(p, 0^m)$. Then ϕ' with $\phi'(q, 0^n) := 1$ if $|q - p| \leq 3 \cdot 2^{-(n+1)}$, and defined as 0 otherwise, consitutes a $\boldsymbol{\psi}^{(1)}$-name of S. Note that convexity is crucial in order to perform a binary search given only a $\boldsymbol{\omega}$-name. $\square$

3.3.2. *Arbitrary yet fixed dimension.* Some of the formerly explained relations change onward from dimension $d = 2$. As a first example we note a result due to Braverman.

Fact 3.11 ([Bra04, Thm. 3.2.1])**.** Let $d \geq 2$. $\mathsf{P} = \mathsf{NP}$ holds true iff every polynomial-time $\boldsymbol{\psi}_{\|\cdot\|_2}^{(d)}$-computable $S \in \mathcal{K}^{(d)}$ is also polynomial-time $\boldsymbol{\delta}_{\|\cdot\|_2}^{(d)}$-computable.

In short: Finding the distance from a point to a set only from local information (that is, a $\boldsymbol{\psi}$-name) about the latter is as hard as solving NP-problems in polynomial time. Thus, $\boldsymbol{\delta}^{(d)}$ is *richer* (i. e., it in a sense provides *more* information about closed non-empty sets) than any of the other representations (i. e., the others are *poorer*).

We note two implications, following immediately from the proof of Thm. 3.11.

- The statement also holds true over $\mathcal{KR}^{(d)}$. In fact, it uniformizes by an adversary argument as sketched in Section 2.5.1; i. e., $\boldsymbol{\psi}_{\|\cdot\|_2}^{(d)} \sqcap \mathsf{b} \not\preceq_{\mathrm{p}}^{\mathcal{KR}} \boldsymbol{\delta}_{\|\cdot\|_2}^{(d)}$ for $d \geq 2$.
- Theorem 3.11 is stated with respect to $\|\cdot\|_2$, but it easily generalizes to arbitrary well-behaved norms $\|\cdot\|$ by properly adapting the adversary set's shape; i. e., from $\|\cdot\|_2$-balls to $\|\cdot\|$-balls.

These two statements also apply to $\boldsymbol{\kappa}$ due to the following observation.

3.3.3. *Representation $\boldsymbol{\psi}$ with outer radii.* $\boldsymbol{\kappa}^{(d)}$ can be reformulated as $\boldsymbol{\psi}^{(d)} \sqcap \mathsf{b}$ with *necessary* outer radius parameter b as every $\boldsymbol{\psi}^{(d)} \sqcap \mathsf{b}$-name $\langle \phi, 0^b \rangle$ constitutes a $\boldsymbol{\kappa}^{(d)}$-name ϕ' through $\phi'(\varepsilon) := 0^b$ and $\phi'(q, 0^n) := \phi(q, 0^{n+1})$ for $q \in \mathbb{D}^d, n \in \mathbb{Z}$. The reverse direction requires a little bit more care: A point q which does not belong to B_n might still be arbitrarily close to the represented set, hence $\boldsymbol{\psi}^{(d)}$-name would have to give 1 when queried with $\langle q, 0^n \rangle$. However, any $\boldsymbol{\kappa}^{(d)}$-name *does* provide enough information if only queried on a finite set of points close to q.

Proposition 3.12. $\boldsymbol{\psi}_{\|\cdot\|}^{(d)} \sqcap \mathsf{b} \equiv_{\mathrm{p}}^{\mathcal{K}} \boldsymbol{\kappa}_{\|\cdot\|}^{(d)}$ *holds in any dimension* $d \in \mathbb{N}$.

Proof. By the above argumentation it just remains to prove the reduction $\boldsymbol{\kappa}^{(d)} \preceq_{\mathrm{p}}^{\mathcal{K}} \boldsymbol{\psi}^{(d)} \sqcap \mathsf{b}$.

Let $q \in \mathbb{D}^d$ and $n \in \mathbb{Z}$, and be ϕ a $\boldsymbol{\kappa}^{(d)}$-name of $S \in \mathcal{K}^{(d)}$. Firstly, an outer radius parameter according to b can be obtained through $\phi(\varepsilon)$. It thus remains to construct a $\boldsymbol{\psi}^{(d)}$-name ϕ' from queries to ϕ. We claim that $\phi'(q, 0^n) := \max_{p \in P} \phi(p, 0^{n+2})$ with $P := \overline{\mathrm{B}}(q, 3 \cdot 2^{-(n+1)}) \cap \mathbb{D}_{n+2}^d$ is such a name. The correctess follows by checking the two cases from definition of $\boldsymbol{\psi}$. If $\mathrm{B}(q, 2^{-n}) \cap S \neq \emptyset$, then by ($\kappa$1) there must exist a $p \in P$ with $\phi(p, 0^{n+2}) = 1$, which leads to $\phi'(q, 0^n) = 1$. Now consider the second case: $\overline{\mathrm{B}}(q, 2^{-n+1}) \cap S = \emptyset$. We prove it by contradition. To this end, assume $\phi'(q, 0^n) = 1$. Then there is a $p \in P$ which satisfies (κ2), i. e., there exists an $x \in S$ such that $x \in \overline{\mathrm{B}}(p, 2^{-(n+2)})$ which in turn produces a contradiction because of $\overline{\mathrm{B}}(p, 2^{-(n+2)}) \subset \overline{\mathrm{B}}(q, 2^{-n+1})$. □

3.3.4. *Local information and relative distance.* On compact sets and enriched with outer radius parameter $b \in \mathbb{Z}$, representation $\boldsymbol{\delta}_{\mathsf{rel}}{}^{(d)}$ is polynomial-time equivalent to $\boldsymbol{\psi}^{(d)}$.[8]

Proposition 3.13. $\boldsymbol{\psi}^{(d)}_{\|\cdot\|} \sqcap \mathsf{b} \equiv^{\mathcal{K}}_{\mathrm{p}} \boldsymbol{\delta}_{\mathsf{rel}}{}^{(d)}_{\|\cdot\|} \sqcap \mathsf{b}$ *holds in any dimension* $d \in \mathbb{N}$.

Proof. We prove the polynomial-time equivalence of $\boldsymbol{\delta}_{\mathsf{rel}}{}^{(d)} \sqcap \mathsf{b}$ and $\boldsymbol{\psi}^{(d)} \sqcap \mathsf{b}$ for $\|\cdot\| \coloneqq \|\cdot\|_\infty$. The full statement then is a direct application of Thm. 3.6.

Direction $\boldsymbol{\delta}_{\mathsf{rel}}{}^{(d)} \sqcap \mathsf{b} \preceq^{\mathcal{K}}_{\mathrm{p}} \boldsymbol{\psi}^{(d)} \sqcap \mathsf{b}$: Let $n \in \mathbb{Z}$ and $q \in \mathbb{D}^d$. If $\langle \phi, 0^b \rangle$ is a $\boldsymbol{\delta}_{\mathsf{rel}}{}^{(d)}|^{\mathcal{K}} \sqcap \mathsf{b}$-name of some $S \in \mathcal{K}^{(d)}$, then

$$\phi'\big(q, 0^n\big) \coloneqq \begin{cases} 1, & \text{if } \phi(q, 0^{n+4}) \leq 5/4 \cdot 2^{-n} + 2^{-(n+4)} \\ 0, & \text{if } \phi(q, 0^{n+4}) \geq 3/4 \cdot 2^{-n} + 2^{-(n+4)} \end{cases}$$

yields $\langle \phi', 0^b \rangle$ to be a $\boldsymbol{\psi}^{(d)} \sqcap \mathsf{b}$-name of S.

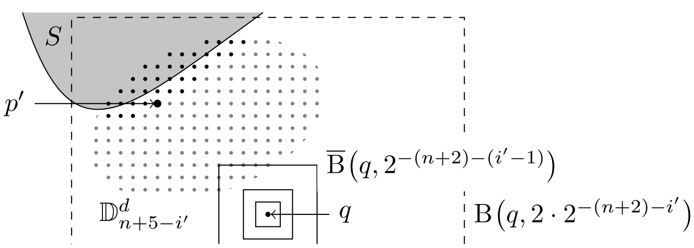


Figure 5: Reducing $\boldsymbol{\psi}^{(d)}|^{\mathcal{K}} \sqcap \mathsf{b}$ to $\boldsymbol{\delta}_{\mathsf{rel}}{}^{(d)}|^{\mathcal{K}} \sqcap \mathsf{b}$. Highlighted in black are points p with $\phi(p, 0^{n+5-i'}) = 1$.

Direction $\boldsymbol{\psi}^{(d)} \sqcap \mathsf{b} \preceq_{\mathrm{p}} \boldsymbol{\delta}_{\mathsf{rel}}{}^{(d)} \sqcap \mathsf{b}$: Let $b' \coloneqq \max\{1, b\}$ and $c' \coloneqq \operatorname{lb}\max\{2, \|q\|\}$. We start by determining an initial approximation to $d_S(q)$. To this end, start with $k \coloneqq 0$ and search for the smallest value $k \leq n + b' + c' + 1$ with $\phi(q, 0^{n+1-k}) = 1$. Denote this particular integer by k'. Note that such a k' does exist because of $S \subseteq \overline{\mathrm{B}}(0, 2^b + \|q\|) \subseteq \overline{\mathrm{B}}(0, 2^{b'+c'})$. This k' then yields the bound $d_S(q) \in [2^{-(n+2)+k'}, 2^{-n+k'}]$.

Now that we have a bound on $d_S(q)$ we can decompose $\overline{\mathrm{B}}(q, 2^{-n+k'}, 2^{-(n+2)+k'})$ into a constant number of regions to search in for a good approximation to $d_S(q)$. More precisely, let $p' \in \mathbb{D}^d_{n+5-k'} \cap \overline{\mathrm{B}}(q, 2^{-n+k'}, 2^{-(n+2)+k'})$ be a dyadic point with $\phi(p', 0^{n+5-k'}) = 1$ which minimizes $\|q - p'\|$ over all points from the above hollow set (this argument is also depicted in Fig. 5). This leads to $|d_S(q) - \|q - p'\|| \leq 2^{-(n+4)+k'}$. Moreover, $\phi'(q, 0^n) \coloneqq \|q - p'\|$ satisfies Eqn. (2.1). The first half, the lower bound on $\phi'(q, 0^n)$ in (2.1), follows by validating that the above bound on $\|q - p'\|$ implies $d_S(q) - 2^{-(n+4)-k'} \leq \|q - p'\|$. Comparing this

[8]Recall that by Thm. 3.5 we assume all representations to be scale-invariant. Without it, only $\widehat{\boldsymbol{\psi}}$ would have been fully polynomial-time equivalent to $\boldsymbol{\delta}_{\mathsf{rel}}$, while $\boldsymbol{\psi}$ would have been only parameterized polynomial-time equivalent.

bound with $3/4 \cdot d_S(q) - 2^{-n}$ from (2.1) shows that $d_S(q) \geq 2^{-(n+2)+k'} - 2^{-n+2}$ has to hold in order to prove the lower bound from (2.1) to be true—which it does (cf. the initial approximation we got on $d_S(q)$). The upper bound follows analogously. Hence, $\left\langle \phi', 0^b \right\rangle$ is a $\boldsymbol{\delta}_{\mathsf{rel}}{}^{(d)} \sqcap \mathsf{b}$-name of S. $\square$

3.3.5. *Comparing local information.* The situation regarding representation $\boldsymbol{\omega}$ is more diverse: Although $\boldsymbol{\omega}$ is *computably* equivalent to $\boldsymbol{\psi}$ over $\mathcal{CR}$-sets (Thm. 3.9), Thm. 3.10 already showed that additional local information (an inner point a and an inner radius 2^{-r}) is necessary to reduce a $\boldsymbol{\omega}^{(1)}$-name to a $\boldsymbol{\psi}^{(1)}$-name. The reduction itself was no more than a binary search, but the applicability was tied to dimension 1 and therefore does not extend to dimension $d = 2$ onward. Nonetheless, $\boldsymbol{\omega}^{(d)}$ *can* be shown to be polynomial-time reducible to $\boldsymbol{\psi}^{(d)}$—and even to $\boldsymbol{\delta}!$—in dimension $d \geq 2$ given enough additional information, although by a very different argument. We start by sketching the positive result about $\boldsymbol{\omega}$'s relation to $\boldsymbol{\delta}$ (extending upon [GLS88, Cor. 4.3.12]), and then show that none of the enrichments could have been directly computed (in polynomial-time) from a $\boldsymbol{\omega}$-name alone.

Theorem 3.14. $\boldsymbol{\omega}^{(d)} \sqcap \mathsf{ar} \sqcap \mathsf{b} \preceq_{\mathrm{p}}^{\mathcal{KCR}} \boldsymbol{\delta}^{(d)}$ *in any dimension d.*

This result follows by applying arguments from *Convex Optimization*: an adaption of the *Ellipsoid Method* plus a *polarity argument*. The Ellipsoid Method allows to first reduce $\boldsymbol{\omega}^{(d)} \sqcap \mathsf{ar} \sqcap \mathsf{b}$ to an intermediate representation $\boldsymbol{\varpi}^{(d)}$, called *weak optimization* representation [GLS88, WOPT: Def. 2.1.10]. A $\phi \in \Sigma^{**}$ is a $\boldsymbol{\varpi}^{(d)}$-name of $S \in \mathcal{KR}^{(d)}$, if for every directional (or: cost-) vector $c \in \mathbb{D}^d$ and precision $n \in \mathbb{Z}$, it satisfies

(1) $\phi(c, 0^n) = \varepsilon$ if $\overline{\mathrm{B}}(S, -2^{-n})$ is empty; and
(2) $\phi(c, 0^n) = p$ for some $p \in \overline{\mathrm{B}}(S, 2^{-n}) \cap \mathbb{D}^d$ such that $c^\mathsf{T} x \leq c^\mathsf{T} p + 2^{-n}$ holds true for all $x \in \overline{\mathrm{B}}(S, -2^{-n})$.

In the second case we also say that p is an *almost optimal point* with respect to the cost vector c (cf. Fig. 6). This case can moreover be reformulated by means of *halfspaces* and *hyperplanes*: Let c be some real-valued vector and $\alpha \in \mathbb{R}$. Then $H_c^{\leq\alpha} := \left\{ x \in \mathbb{R}^d \;\middle|\; c^\mathsf{T} x \leq \alpha \right\}$ and, analogously, $H_c^{\geq\alpha}$ are halfspaces, and their intersection constitutes the hyperplane $H_c^{=\alpha} := H_c^{\leq\alpha} \cap H^{\geq\alpha}$. The aforementioned second case now reads as $\bigcup_{x \in \overline{\mathrm{B}}(x, 2^{-n})} H_c^{\leq c^\mathsf{T} x} \subseteq H_c^{\leq c^\mathsf{T} p + 2^{-n}}$ for p as above.

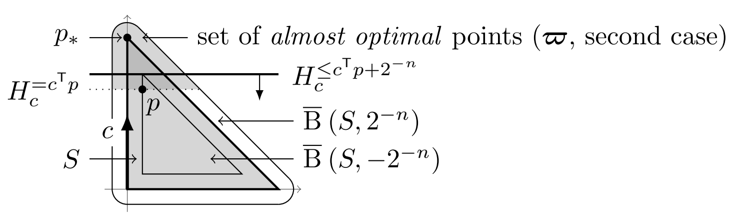


Figure 6: Weak optimization: Cost vector c, and a set $S \in \mathcal{KR}^{(2)}$ along with an optimal solution p_* and the set of almost optimal solutions.

Remark 3.15. Like for $\boldsymbol{\omega}$, representation $\boldsymbol{\varpi}$ only makes sense for regular sets since the first condition would otherwise always hold true, e. g., for singletons. The additional restriction to bounded sets moreover is necessary for the existence of a point p almost optimizing over S in direction of c. To get the "usual" notion of optimization in direction of c, first approximate the normalized vector to c (i. e., compute $c \cdot \|c\|_2^{-1}$ up to the desired precision) and then apply the $\boldsymbol{\varpi}^{(d)}$-name.

Further note that we tied representation $\boldsymbol{\varpi}$ to the Euclidean norm: The second condition in the definition of $\boldsymbol{\varpi}$ is stated by means of the the scalar product $\langle\cdot,\cdot\rangle$ induced by $\|\cdot\|_2$ (i. e., $x^\mathsf{T} y = \langle x, y\rangle = 1/4(\|x+y\|_2^2 - \|x-y\|_2^2)$). This does not lead to the most generic definition of $\boldsymbol{\varpi}$, however, it is a sensible choice because $\|\cdot\|_2$ is the *only* norm on $\mathbb{R}^d$ amongst the p-norms $\|(x_1,\dots,x_d)\|_p := \left(\sum_{i=1}^d |x_i|^p\right)^{1/p}$ (for $1 \le p \le \infty$) that induces a scalar product. We therefore only write $\boldsymbol{\varpi}^{(d)}$, but obviously mean $\boldsymbol{\varpi}^{(d)}_{\|\cdot\|_2}$

Using $\boldsymbol{\varpi}$ and the following Thm. 3.16, we translate an $\boldsymbol{\omega}$-name of a set S to a $\boldsymbol{\delta}$-name of its *polar set* $S^\bullet$ (a related but in most instances not the same set), and then use this as an intermediate step to prove the above Theorem.

Fact 3.16 ([GLS88, Cor. 4.3.12]). $\boldsymbol{\omega}^{(d)} \sqcap \mathsf{ar} \sqcap \mathsf{b} \preceq_{\mathrm{p}}^{\mathcal{KCR}} \boldsymbol{\varpi}^{(d)}$.

Lemma 3.17. *Define* $^\bullet\colon \mathcal{A}^{(d)} \to \mathcal{A}^{(d)}$ *through* $S \mapsto S^\bullet := \{y \in \mathbb{R}^d \mid \forall\, x \in S\,.\, y^\mathsf{T} x \le 1\}$*, and call* $S^\bullet$ *the* polar *of* S *(a well-known concept in convex geometry and optimization; cf.* [BL00, §4.1]*,* [dBCvKO08, §8.2]*). Further, call* S centered *if* $0 \in S^\circ$*.*

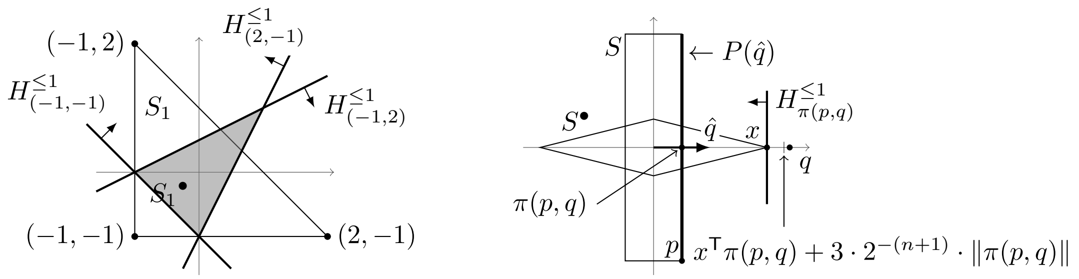
(1) *For all* $S \in \mathcal{A}^{(d)}$ *and* $r, b \in \mathbb{Z}$*,* $\overline{\mathrm{B}}(0, 2^{-r}) \subseteq S$ *implies* $S^\bullet \subseteq \overline{\mathrm{B}}(0, 2^r)$ *and* $S \subseteq \overline{\mathrm{B}}(0, 2^b)$ *implies* $\overline{\mathrm{B}}(0, 2^{-b}) \subseteq S^\bullet$*.*
(2) *Let* $S \in \mathcal{KCR}^{(d)}$ *be centered. Then* $S^\bullet$ *is also contained in* $\mathcal{KCR}^{(d)}$*, centered, and* $S = S^{\bullet\bullet}$*.*
(3) *Let* $\mathcal{Z} := \{S \in \mathcal{KCR}^{(d)} \mid S \text{ centered}\}$*. Further define an enrichment ("inner radius of centered sets")* $\mathsf{r}_0\colon S \rightrightarrows \{\mathsf{un}_\mathbb{Z}(r) \mid \overline{\mathrm{B}}(0, 2^{-r}) \subseteq S\}$*. Then* $^\bullet|_\mathcal{Z}$ *is* $\left(\boldsymbol{\omega}^{(d)} \sqcap \mathsf{r}_0 \sqcap \mathsf{b}, \boldsymbol{\delta}^{(d)} \sqcap \mathsf{r}_0 \sqcap \mathsf{b}\right)$*-computable in polynomial time.*

of Thm. 3.14. Let $\langle \phi, a, 0^r, 0^b\rangle$ be a $\boldsymbol{\omega}^{(d)} \sqcap \mathsf{ar} \sqcap \mathsf{b}$-name of $S \in \mathcal{Z}$. Then $\phi'(q, 0^n) := \phi(q-a, 0^n)$ gives a $\boldsymbol{\omega}^{(d)} \sqcap \mathsf{r}_0 \sqcap \mathsf{b}$-name $\langle \phi', 0^r, 0^b\rangle$ of $S' := \{x - a \mid x \in S\}$. Since S' is centered and therefore contains 0 as an inner point, Thm. 3.17(3) can be applied to get a $\boldsymbol{\delta}^{(d)} \sqcap \mathsf{r}_0 \sqcap \mathsf{b}$-name $\langle \phi'', 0^{r''}, 0^{b''}\rangle$ of $S'^\bullet$ out of ϕ' with $r'' := b$ and $b'' := r$. Use the reduction $\boldsymbol{\delta}^{(d)} \preceq_{\mathrm{p}} \boldsymbol{\omega}^{(d)}$ and apply Thm. 3.17(3) once again to get a $\boldsymbol{\delta}^{(d)}$-name ϕ''' of $S'^{\bullet\bullet} = S'$ (by Thm. 3.17(2)) out of ϕ''. The final translation $S' \mapsto \{x + a \mid x \in S'\}$ through $\psi(q, 0^n) := \phi'''(q + a, 0^n)$ yields a $\boldsymbol{\delta}^{(d)}$-name ψ of S. □

The key ingredient in the proof of Thm. 3.17 is to take the ratio $2^{-r}/2^b$ into account. If done correctly, this then ensures that we get a sufficiently good approximation of a bounding hyperplane $H_{p'}^{\le 1}$ from which the distance—and hence a $\boldsymbol{\delta}$-name—can be easily calculated. We wrap the necessary technical details in the following statement.

Proposition 3.18. *Let* $S \in \mathcal{KCR}^{(d)}$ *be centered. Further let* $r \in \mathbb{Z}$ *and* $b \in \mathbb{Z}$ *be inner and outer radius parameter, respectively.*

(1) *Let* ϕ *be a* $\boldsymbol{\varpi}^{(d)}$*-name of* S*. Then* $p := \phi(q, 0^m)$ *satisfies*
$$\exists\, p_* \in S\,.\left(p_* \in \overline{\mathrm{B}}(p, 2^{-n}) \text{ and } \forall\, x \in S\,.\, q^\mathsf{T} x \le q^\mathsf{T} p_*\right)$$
if $m \ge n + |b| + |r| + 1$*.*
(2) *Denote by* $\pi\colon (p, q) \mapsto p^\mathsf{T} q \cdot 1/(q^\mathsf{T} q) \cdot q$ *the projection of* $p \in \mathbb{R}^d$ *onto the line spanned by* $q \in \mathbb{R}^d$*. If* $\pi(p, q)$ *with* $p \in \partial S$ *is approximated by* $p' \in \mathbb{D}^d$ *with precision* $m \ge$



(a) Illustrating the definition of $S^\bullet$.

(b) Optimization in direction of $\hat q$: Pick an arbitrary point $p \in P(\hat q)$—the set of optimal points with respect to $\hat q$—and project it onto $\hat q$. The resulting point $\pi(p,q)$ then allows to recover the distance of q to $S^\bullet$ from $H^{=1}_{\pi(p,q)}$.

Figure 7: Construction of and argumentation using polar sets.

$n + |b| + |r| + 1$*, then*

$$d_\mathsf{H}\big(\overline{\mathrm{B}}(0, 2^b + 2^r) \cap H^{=1}_{p'},\, \overline{\mathrm{B}}(0, 2^b + 2^r) \cap H^{=1}_{\pi(p,q)}\big) \le 2^{-n} .$$

Stated differently, vectors $\pi(p,q)$ and p' describe approximately the same hyperplane (with respect to bounds b and r).

of Thm. 3.17. Note that the polar of a closed set is closed, too, as per definition it is the intersection of closed halfspaces, i. e., $S^\bullet = \bigcap_{x\in S} H^{\le 1}_x$ with halfspaces $H^{\le 1}_x := \{y \in \mathbb{R}^d \mid x^\mathsf{T} y \le 1\}$. Statement (2) now is a special case of the *Bipolar Theorem* (cf. [BL00, Thm. 4.1.5]), while statement (1) follows by examining the proof of the aforementioned theorem (cf. [BL00, Exercise 4.1(5)]).

Concerning (3): Let $\langle \phi, 0^r, 0^b\rangle$ be a $\boldsymbol{\omega}^{(d)}|^{\mathcal{Z}} \sqcap \mathsf{r_0} \sqcap \mathsf{b}$-name of $S \in \mathcal{Z}$, $q \in \mathbb{D}^d$ and $n \in \mathbb{N}$. Apply Thm. 3.16 to the above name to get a $\boldsymbol{\varpi}^{(d)}|^{\mathcal{Z}} \sqcap \mathsf{r_0} \sqcap \mathsf{b}$-name $\langle \phi', 0^r, 0^b\rangle$ of S.

Notice beforehand that Thm. 3.18 allows us to describe all of the following steps in terms of *exact* computations while they actually have to be carried out approximately. To get the approximative (and hence correct) version, use the aforementioned results and the closure of polynomial-time function computation under composition.

As already mentioned in Thm. 3.15, optimization in a certain direction in the usual sense is obtained from a $\boldsymbol{\varpi}$-name by first normalizing the respective cost vector; i. e., $q' := q/\|q\|$ in this setting. Now take the $\boldsymbol{\varpi}$-name ϕ' and apply it to q' to obtain an optimal point $p := \phi'(q', 0^m)$. The point p itself usually does not describe the distance between q and $S^\bullet$ appropriately as depicted in Fig. 7b, but its projection onto q' encodes precisely this information. To this end, let $p' := \pi(p, q')$ (use Thm. 3.18(2) to get a good approximations) and observe that the distance of q from $S^\bullet$ can be obtained from the distance of q to the hyperplane $\{y \in \mathbb{R}^d \mid y^\mathsf{T} p' = 1\} = H^{=1}_{p'}$ under the premise that $q^\mathsf{T} p' \ge 1$. More concretely, a valid $\boldsymbol{\delta}^{(d)}|^{\mathcal{Z}}$-name φ of $S^\bullet$ can be defined as follows:

$$\varphi(q, 0^n) := 0 \text{ if } q^\mathsf{T} p' \le 1 + 3\cdot 2^{-(n+1)} \cdot \|p'\| \,; \quad \varphi(q, 0^n) := (q^\mathsf{T} p' - 1)\cdot \|p'\|^{-1} \text{ otherwise} .$$

$\square$

of Thm. 3.18. Considering (1): First note that $d_{\mathsf{H}}\big(S, \overline{\mathrm{B}}(S, -2^{-m})\big) \leq 2^{-(n+1)}$ if $m \geq n + |b| + |r| + 1$ which follows by a geometric argument: Observe that S does contain a filled right-angled triangle T with adjacent side of length $\leq 2^b$ and opposite side of length $\geq 2^{-r}$. The ratio $2^{-r}/2^b$ bounds how "steep" this triangle can be. Stated differently, for all $x \in \partial T$ there exists a $y \in \overline{\mathrm{B}}(T, -2^{-m})$ with $\|x - y\| \leq 2^{-(n+1)}$ for m as above; which implies the above statement about the Hausdorff-distance of $\overline{\mathrm{B}}(S, -2^{-m})$ to S.

The definition of $\boldsymbol{\varpi}$ now implies that each almost optimal $p \in \overline{\mathrm{B}}(S, 2^{-m})$ fulfills $q^{\mathsf{T}} x \leq q^{\mathsf{T}} p + 2^{-m}$ for all $x \in \overline{\mathrm{B}}(S, -2^{-m})$. Combine this bound with the first argument over the Hausdorff distance to obtain the claimed result, namely that there exists an optimal point $p_* \in S \cap \overline{\mathrm{B}}(p, \delta)$ with $\delta := (2^{-(n+1)} + 2 \cdot 2^{-m})/2 < 2^{-(n+1)}$ with respect to optimization direction q.

Considering (2): First note that $p \in \partial S$ implies $\|p\| \geq 2^{-r}$, and also $\|p\| \leq 2^b$ due to $S \subseteq \overline{\mathrm{B}}(0, 2^b)$. Without loss of generality, let $\pi(p, q) =: (\lambda, 0, \dots, 0)$ and $p' := (\lambda \pm 2^{-m}, 0, \dots, 0)$ (the latter being a boundary case of $p' \in \overline{\mathrm{B}}(\pi(p, q), 2^{-m})$) with $2^{-r} \leq \lambda \leq 2^b$ (as noted before). In this particular case $\pi(p, q)$ and p' are codirectional which simplifies the following argument. Note that $\pi(p, q)^{\mathsf{T}} x = 1$ if $x_1 = 1/\lambda$, and ${p'}^{\mathsf{T}} x' = 1$ if $x'_1 = 1/(\lambda + 2^{-m})$. Then the codirectionality of $\pi(p, q)$ and p' imply that $H^{=1}_{\pi(p,q)}$ and $H^{=1}_{p'}$ are parallel, and they are of Hausdorff distance $|1/\lambda - 1/(\lambda + 2^{-m})|$. By $2^{-r} \leq \lambda \leq 2^b$. Therefore, $|1/\lambda - 1/(\lambda + 2^{-m})| \leq 2^{-n}$ by $m \geq n + |r| + |b| + 1$, and thus implies

$$d_{\mathsf{H}}\big(\overline{\mathrm{B}}(0, 2^b + 2^r) \cap H^{=1}_{p'}, \overline{\mathrm{B}}(0, 2^b + 2^r) \cap H^{=1}_{\pi(p,q)}\big) \leq 2^{-n} . \qquad \square$$

Both the enrichments (a, r and b) as well as the restriction to bounded convex bodies $\mathcal{KCR}$ were necessary to make Thm. 3.14 work, as we summarize in the following statement.

Proposition 3.19 (enrichments of $\boldsymbol{\omega}$)**.** *For all $d \in \mathbb{N}$ we get the following negative results:*

(1) *The multi-valued operation* $\mathsf{Bound}\colon \mathcal{KR}^{(d)} \rightrightarrows \mathbb{Z}$*,* $\mathcal{KR}^{(d)} \ni S \mapsto \big\{b \in \mathbb{Z} \;\big|\; S \subseteq \overline{\mathrm{B}}(0, 2^b)\big\}$ *is* $(\boldsymbol{\omega}^{(d)} \sqcap \mathsf{ar}, \mathbf{un}_{\mathbb{N}})$*-discontinuous. The analogous fact holds for $\boldsymbol{\psi}^{(d)}$ (cf.* [Wei00, Exercise 5.2.4]*).*

(2) *Convexity is crucial for Thm. 3.14 to hold:* $\boldsymbol{\omega}^{(d)} \sqcap \mathsf{ar} \sqcap \mathsf{b} \not\preceq^{\mathcal{KR}} \boldsymbol{\delta}^{(d)}$.

(3) *Addendum to the previous point: Convexity helps only in the presence of all of the above enrichments; i. e., there is* no *machine operating on $\mathcal{KCR}^{(d)}$ that provided with $\boldsymbol{\omega}^{(d)} \sqcap \chi_1 \sqcap \chi_2$ computes χ_3 in* polynomial time *for any permutation $\{\chi_1, \chi_2, \chi_3\}$ of* $\{\mathsf{a}, \mathsf{r}, \mathsf{b}\}$.

(4) *Over $\mathcal{KCR}^{(d)}$, advice parameters* a*,* r *and* b *are uniformly computable from $\boldsymbol{\psi}^{(d)}$. The same statement fails, however, for computability in* polynomial time.

Proof.

(1) Let $M^?$ be a hypothetical OTM to compute Bound. Further, let $\langle \phi, a, 0^r \rangle$ be a $\boldsymbol{\omega}^{(d)}|^{\mathcal{KR}} \sqcap \mathsf{ar}$-name for $S \in \mathcal{KR}^{(d)}$. Machine $M^?$ terminates (in finite time) and produces a potential bound b. During its computation it can only have made finitely many queries to ϕ and thus has checked only points in, say, $\overline{\mathrm{B}}_{\|\cdot\|}(0, 2^{b'})$ for some $b' \in \mathbb{Z}$. Therefore, $M^?$ would have produced the same potential bound $b \leq b'$ if S were replaced with the set $S' := S \cup \overline{\mathrm{B}}_{\|\cdot\|}(p, 1)$ for some point p satisfying $\overline{\mathrm{B}}_{\|\cdot\|}(0, 2^{b'}) \cap \overline{\mathrm{B}}_{\|\cdot\|}(p, 1) = \emptyset$.

(2) We prove the stronger statement $\boldsymbol{\omega}^{(d)} \sqcap \mathsf{ar} \sqcap \mathsf{b} \not\preceq^{\mathcal{KR}} \boldsymbol{\psi}^{(d)}$.

Let $S := \overline{\mathrm{B}}_{\|\cdot\|}(0, 2^b, 2^b - 2^{-3})$, and be $\phi := \big\langle \phi', a, 0^r, 0^b \big\rangle$ with $\phi'(q', n') := \chi_S(q')$ a concrete $\boldsymbol{\omega}^{(d)}|^{\mathcal{KR}} \sqcap \mathsf{ar} \sqcap \mathsf{b}$-name of S. Further let $M^?$ be a hypothetical OTM translating any ϕ into a $\boldsymbol{\psi}^{(d)}|^{\mathcal{KR}}$-name. The discrete inputs (tailor-made for the adversary argument) are $q := (0,\dots,0)$ and $n := 3$. On this input, $M^?$ does asks queries of precision at most $m \geq |r|$. Therefore, it states "0" as the correct answer a $\boldsymbol{\psi}^{(d)}$-name would have given on $\langle q, 0^n\rangle$ because of $\overline{\mathrm{B}}(0, 2^{-3+1}) \cap S = \emptyset$. $M^?$ surely produces the right answer for S, but it also does so on the slightly modified (adversary) set $S' := S \cup \overline{\mathrm{B}}(0, 2^{-(m+2)})$ for all $\boldsymbol{\omega}^{(d)}|^{\mathcal{KR}}$-names ϕ'' for S' with $\forall\, p \in \mathbb{D}^d \,.\, \forall\, k \in \mathbb{N},\ k \leq m \,.\, \phi'(p, 0^k) = \phi''(p, 0^k)$; thus misleading $M^?$ to produce the wrong answer (0 instead of 1).

(3) Parameter b can not be computed in polynomial time from r, a (and n) because the outer radius of a set S is simply not bounded in this (local) information about S. Finding an inner point a from b and r requires to query a $\boldsymbol{\omega}^{(d)}|^{\mathcal{KCR}}$-name ϕ in roughly $2^{d\cdot\max\{0,b+r\}}$ many points. To see why, consider the collection of adversary sets $\{S_p := \overline{\mathrm{B}}(p, 2^{-r}) \mid p \in \overline{\mathrm{B}}(0, 2^b) \cap \mathbb{D}^d_{r+1}\}$ and observe that S_p can only be distinguished from any other $S_{p'}$ if ϕ is evaluated in p *and* p' with precision $r+1$. An analogous argument shows why an inner radius parameter r can not be bounded in terms of a and b only.

(4) Computability of b, a and r: Let ϕ be a $\boldsymbol{\psi}^{(d)}_{\|\cdot\|}$-name of $S \in \mathcal{KCR}^{(d)}$. An outer radius parameter b exists since S is bounded, and it is computable from ϕ by exploiting convexity: Starting at 0, systematically ask queries $\phi(p, 0^1)$ with $p \in \mathbb{D}^d_1$ in order to find a value $b \geq 1$ such that (a) $\exists\, p \in \mathbb{D}^d_1 \cap \overline{\mathrm{B}}(0, 2^{b-1}) \,.\, \phi(p, 0^1) = 1$ and (b) $\forall\, p \in \mathbb{D}^d_1 \cap \overline{\mathrm{B}}(0, 2^b) \,.\, \phi(p, 0^1) = 0$. It then follows $S \cap \overline{\mathrm{B}}(0, 2^{b-1}) \neq \emptyset$, and $S \subset \overline{\mathrm{B}}(0, 2^b)$ is implied by using the convexity of S.

From b one can find a point a and also an inner radius parameter r by gradually increasing the precision: Starting with $r' := -b + 3$, increase r' until a point $p' \in \mathbb{D}^d_{r'}$ in $\overline{\mathrm{B}}(0, 2^b)$ is found such that all $p \in \mathbb{D}^d_{r'} \cap \overline{\mathrm{B}}(p', 2^{-r'+3})$ satisfy $\phi(p, 0^{r'}) = 1$. Then $\overline{\mathrm{B}}(p', 2^{-r'}) \subseteq S$ follows by convexity of S. Now choose $a := p'$ and $r := r'$.

Non-polynomial-time computability: Any (deterministic) computation of b and r must necessarily be unbounded in n (and, obviously, $|\phi|$), simply because the *values* of both b and r are usually unbounded in n. The same is true for an inner point a since it depends on (the unknown) inner radius parameter r; take $S = \overline{\mathrm{B}}(a, 2^{-r})$, $a \in \mathbb{D}^d_r \setminus \mathbb{D}^d_{r-1}$ as an example. □

Theorem 3.19(2) covers, in fact, *several* constellations of enrichments of $\boldsymbol{\omega}$ because it asserts that, informally, *"if we can not deduce χ_3 from $\boldsymbol{\omega}^{(d)}$ and two-thirds of other information (χ_1 and χ_2), then particularly neither none nor one-third of it would help, too".*

4. Geometric operations on sets

By definition, both the computability and complexity of an operator is inextricably linked to the choice of representations of elements it is based on; examples can be found in [Bra99], [Wei00, Thm. 5.1.13], [Zie02], [ZB04] (for computability), and [ZM08] (for complexity). While the computability is pretty well-studied, the complexity has been left behind as, again, a result of the missing generic framework to formulate explicit complexity bounds in. In this section, we do our small part to shine a light on the complexity of Choice (finding *some* point in a set), set operators Union, Intersection and Projection, and basic function operators Inversion (local inverse of a function) and Image.

4.1. Choice: **Finding a point in a set.** We analyze the complexity to compute *some* (multi-valued) member of a set S, given only a name of S; i. e., the complexity of the in general uncomputable (cf. [BG11, BdBP12]) operator $\mathsf{Choice}\colon \mathcal{A} \rightrightarrows \bigcup_{d\in\mathbb{N}} \mathbb{R}^d$, $\mathcal{A} \ni S \mapsto S$. It is an interesting operator because, intuitively, at least this operator should be (parameterized) polynomial-time computable for reasonable representations of sets; like the operator $\mathsf{Evaluation}\colon (f,x) \mapsto f(x)$ is in the realm of continuous functions [KC12].

The following statement indeed proves parameterized complexity results for Choice. In particular, $\boldsymbol{\psi}$ enriched with b suffices, while even more information is necessary for $\boldsymbol{\omega}$.

Theorem 4.1 (complexity of Choice)**.**

(1) *On compact sets,* $\mathsf{Choice}|_{\mathcal{K}}$ *is fully polynomial-time* $(\boldsymbol{\psi}^{(d)} \sqcap \mathsf{b}, \boldsymbol{\rho}^d)$*-computable.*
(2) $\mathsf{Choice}|_{\mathcal{KR}}$ *is* $(\boldsymbol{\omega}^{(d)} \sqcap \mathsf{r} \sqcap \mathsf{b}, \boldsymbol{\rho}^d)$*-computable in time polynomial exponential in* $|b| + |r|$*. This bound also is sharp (i. e., no fully polynomial-time bound holds).*

Proof. (1) Let $S \in \mathcal{K}^{(d)}$, $\langle \phi, 0^b \rangle$ be a $\boldsymbol{\psi}^{(d)} \sqcap \mathsf{b}$-name of S with $b \geq 0$, and $n \in \mathbb{Z}$. A point $q \in \mathbb{D}^d$ with $\phi(q, 0^n) = 1$ can then be found by the following iterative procedure. First, let $p_0 := 0$. Now assume that p_{i-1} for $1 \leq i \leq n+b+2$ is already given. Then deterministically pick one point p_i out of $\overline{\mathrm{B}}(p_{i-1}, 2^{b-(i-1)}) \cap \mathbb{D}^d_{i-b}$ with $\phi(p_i, 0^{i-b}) = 1$. Then p_{n+b+2} is guaranteed to be 2^{-n}-close to S.

(2) Let $\langle \phi, 0^r, 0^b \rangle$ be an $\boldsymbol{\omega}^{(d)}|^{\mathcal{KR}} \sqcap \mathsf{r} \sqcap \mathsf{b}$-name of $S \in \mathcal{KR}^{(d)}$.

Upper bound: Perform an exhaustive search on $\mathbb{D}^d_{r+1}(2^b)$. This way $\phi(p, 0^{r+1}) = 1$ is guaranteed by 2^{-r} being an inner radius of S for some point $p \in \mathbb{D}^d_{r+1+n}(2^b)$. Moreover, such a point will be found and is close-enough (in the sense of representation $\boldsymbol{\rho}^d$) to S.

Sharpness: Consider the class of sets $\mathcal{B} := \{\overline{\mathrm{B}}(a, 2^{-r}) \mid a \in \mathbb{D}^d_{r+n'}(2^b), n' \in \mathbb{Z}\} \subset \mathcal{KR}^{(d)}$. The sharpness then is a consequence of Thm. 3.19(2), $\boldsymbol{\omega}^{(d)} \sqcap \mathsf{r} \sqcap \mathsf{b} \not\preceq^{\mathcal{B}}_{\mathrm{p}} \boldsymbol{\psi}^{(d)}$: Exponentially many points $p \in \mathbb{D}^d$ have to be considered in order to tell any of the above sets apart. □

4.2. **Binary union.** [ZM08, Lem. 2.7] proved Union to be polynomial-time computable over $\boldsymbol{\kappa}$ with respect to an output-sensitive measure of complexity: Given two $\boldsymbol{\kappa}^{(d)} \equiv^{\mathcal{K}}_{\mathrm{p}} \boldsymbol{\psi}^{(d)} \sqcap \mathsf{b}$-names ϕ_1 and ϕ_2, taking the maximum over the outer radii parameter as well as the maximum over the answers at any point, $\phi(q, 0^n) = \max_i \phi_i(q, 0^n)$, constitutes a name of the union. Linear-time algorithms for $\boldsymbol{\psi}^{(d)}$ and $\boldsymbol{\delta}^{(d)}$ follow analogously.

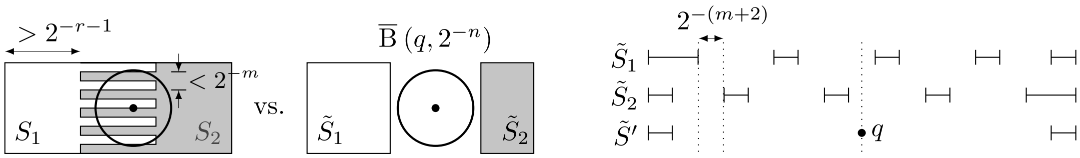
However, the same method applied to $\boldsymbol{\omega}^{(d)}$ over regular sets does *not* yield a valid $\boldsymbol{\omega}^{(d)}$-name of the union. As it turns out, Union is even *uncomputable* over $\boldsymbol{\omega}^{(d)}$. Convexity, again, proves to be the key to render Union computable, even in *polynomial time.*

Theorem 4.2.

(1) $\mathsf{Union}|_{\mathcal{KR}\times\mathcal{KR}}$ *is* $\big((\boldsymbol{\omega}^{(d)} \sqcap \mathsf{r} \sqcap \mathsf{b}) \times (\boldsymbol{\omega}^{(d)} \sqcap \mathsf{r} \sqcap \mathsf{b}), \boldsymbol{\omega}^{(d)}\big)$-discontinuous.
(2) *On* $\mathcal{CR}^{(r)}$*, however,* $\mathsf{Union}|_{\mathcal{CR}\times\mathcal{CR}}$ *becomes polynomial-time* $(\boldsymbol{\omega}^{(d)} \times \boldsymbol{\omega}^{(d)}, \boldsymbol{\omega}^{(d)})$*-computable.*[9]

Proof. (1) The basic adversary construction of sets S_i and $\tilde{S}_i$ is depicted in Fig. 8a. First choose S_i such that $\overline{\mathrm{B}}(q, 2^{-n+1})$ has an empty intersection with $S_1 \cup S_2$ (e. g., as a simple rectangle/cuboid as depicted). Further construct $\tilde{S}_i$ as follows: (a) r_i is an inner

[9]Keep in mind that the result may not be convex.



(a) $\mathsf{Union}|_{\mathcal{KR}\times\mathcal{KR}}$ is $\boldsymbol{\omega}^{(d)}$-discontinuous: The adversary set on the left is indistinguishable from the right one provided the "stripes" are just small enough to *not* contain a ball of radius 2^{-m}.

(b) Alternately cut small pieces of "size" $2^{-(m+2)}$ off of two copies of the unit interval (in general: unit hypercube) to obtain an empty intersection around point q.

Figure 8: Adversary arguments, proving the discontinuity of (a) Union over $\boldsymbol{\omega}$ and (b) $\mathsf{Intersection}$ over $\boldsymbol{\psi}$.

radius parameter of $\tilde{S}_i$; (b) the "teeth", being of length $> 2^{-n+2}$, are placed around q as depicted; (c) each rectangle/cuboid is of width $\le 2^{-(m+1)}$, where $m \in \mathbb{N}$ marks the maximal precision a hypothetical OTM $M^?$ for Union asks on input $\langle q, 0^n\rangle$. Now the only (and in this case correct) choice $M^{\langle \phi_1,\phi_2\rangle}$ started with $\langle q, 0^n\rangle$ has is to assert 0 since $\overline{\mathrm{B}}(q, 2^{-n+1}) \cap (S_1 \cup S_2) = \emptyset$. Now exchange the names for S_i by names $\tilde{\phi}_i$ for $\tilde{S}_i$ which coincide with the previous ones on all queries up to precision m. Then $M^{\langle \tilde{\phi}_1,\tilde{\phi}_2\rangle}$ sill asserts 0 in this case, although now $\mathrm{B}(q, 2^{-n}) \subset \tilde{S}_1 \cup \tilde{S}_2$ proves 1 to be the only correct answer.

(2) Let ϕ_i be $\boldsymbol{\omega}^{(d)}$-names of S_i, $i = 1, 2$. Claim: Then ϕ', defined through

$$\phi'(q, 0^n) := \max_{i=1,2} \left\{ \phi_i(p, 0^{n+2}) \;\middle|\; p \in B := \overline{\mathrm{B}}(q, 3 \cdot 2^{-(n+2)}) \cap \mathbb{D}^d_{n+2} \right\},$$

constitutes a $\boldsymbol{\omega}^{(d)}$-name of $S' := S_1 \cup S_2$.

Let $\overline{\mathrm{B}}(q, 2^{-n}) \cap S' = \emptyset$. Then $\bigcup_{p\in B} \overline{\mathrm{B}}(p, 2^{-(n+2)}) \subset S'$ must also have empty intersection with S', hence $\phi_i(p, 0^{n+2}) = 0$ for all $p \in B$ and $i \in \{1, 2\}$. Now let $\overline{\mathrm{B}}(q, 2^{-n}) \subseteq S'$. We prove the correctness of $\phi'(q, 0^n) := 1$ by contradiction. To this end, assume $\phi'(q, 0^n) = 0$, i. e., $\overline{\mathrm{B}}(p, 2^{-(n+2)}) \not\subseteq S_1, S_2$ would have to hold for all $p \in B$. Because of $\overline{\mathrm{B}}(p, 2^{-(n+2)}) \subset \overline{\mathrm{B}}(q, 2^{-n}) \subseteq S'$ and the convexity of S_i, there must be a $p' \in P$ such that $\overline{\mathrm{B}}(p', 2^{-(n+2)})$ is contained entirely either in $S_1 \cap S_2$, $S_1 \setminus S_2$ or $S_2 \setminus S_1$. If $\overline{\mathrm{B}}(p', 2^{-(n+2)})$ were contained in the first (convex) set, then we would get a contradiction because of $\overline{\mathrm{B}}(p', 2^{-(n+2)}) \not\subseteq S_1, S_2$. If it were contained in (one of the connected regions of) $S_1 \setminus S_2$, then we would also get a contradiction to the assumption that $\overline{\mathrm{B}}(p', 2^{-(n+2)}) \not\subseteq S_1$. The analogous argument also holds for the third set, thus proving $\boldsymbol{\omega}^{(d)}(\phi') = S'$. □

4.3. **Binary intersection.** $\mathsf{Intersection}$ proves to be discontinuous for $\boldsymbol{\delta}$ over the class $\mathcal{A}$ of closed sets [Wei00, Ex. 5.1(14)] by the usual adversary argument: Whenever a (hypothetical) algorithm decides upon a certain point x to be a member of the intersection, we can slightly modify the original sets by excluding points from a small neighborhood of x, rendering x to be far off the actual intersection and therefore leading any hypothetical OTM to produce a wrong answer.

Even when requiring the intersection of two *regular* sets to be regular again, this discontinuity remains [Zie02, §3]. We show how convexity helps to establish *computability*, and how the complexity is bounded in terms of an *inner radius of the intersection.*

Theorem 4.3. *Let*

$$\mathcal{D} := \left\{(S_1,S_2)\in\mathcal{R}^{(d)}\times\mathcal{R}^{(d)} \;\middle|\; S_1\cap S_2\in\mathcal{R}^{(d)}\right\};$$
$$\mathcal{E} := \left\{(S_1,S_2)\in\mathcal{KCR}^{(d)}\times\mathcal{KCR}^{(d)} \;\middle|\; S_1\cap S_2\in\mathcal{KCR}^{(d)}\right\},$$

and further define the enrichment r' *to encode an inner radius parameter of the intersection of two sets, i. e.,* $\mathsf{r}'\colon (S_1,S_2)\in\mathcal{D}\rightrightarrows\left\{\mathsf{un}_{\mathbb{Z}}(r')\;\middle|\;\exists\, a\in\mathbb{R}^d\,.\,\overline{\mathrm{B}}(a,2^{-r'})\subseteq S_1\cap S_2\right\}$.

(1) $\mathsf{Intersection}|_{\mathcal{D}}$ *is* $(\boldsymbol{\xi}^{(d)}\times\boldsymbol{\xi}^{(d)},\boldsymbol{\xi}^{(d)})$-dis*continuous for all representations* $\boldsymbol{\xi}$ *from Thm. 2.7.*
(2) $\mathsf{Intersection}|_{\mathcal{E}}$ *is* $(\boldsymbol{\xi}^{(d)}\times\boldsymbol{\xi}^{(d)},\boldsymbol{\xi}^{(d)})$*-computable for all representations* $\boldsymbol{\xi}$ *from Thm. 2.7.*
(3) $\mathsf{Intersection}|_{\mathcal{E}}$ *is parameterized polynomial-time* $\big((\boldsymbol{\xi}^{(d)}\sqcap\mathsf{b})\times(\boldsymbol{\xi}^{(d)}\sqcap\mathsf{b})\sqcap\mathsf{r}',\boldsymbol{\xi}^{(d)}\big)$*-computable for* $\boldsymbol{\xi}^{(d)}:=\boldsymbol{\psi}^{(d)}$*, and even fully polynomial-time computable for* $\boldsymbol{\xi}^{(d)}:=\boldsymbol{\omega}^{(d)}$.

Notice the duality in the $\boldsymbol{\omega}^{(d)}$-result for $\mathsf{Intersection}|_{\mathcal{E}}$ compared with Union: While for the first a correct answer was easy to produce when the point was not deep-enough in at least one of the two sets, it was easy to produce a correct answer for the latter if the point resided deep in both sets. Further note that this is the direct opposite of what holds for $\boldsymbol{\psi}$; an indication that $\boldsymbol{\psi}$ is dual to $\boldsymbol{\omega}$, just like the union of sets is the lattice-dual operation of intersection.

Proof. (1) We only show the $\boldsymbol{\psi}^{(d)}$-discontinuity of $\mathsf{Intersection}$ (the proof loosely follows [Wei00, Thm. 5.1.13]); the remaining statements follow by the same construction.

Let $S_1,S_2:=[-1,1]^d$ be the sets provided to $\mathsf{Intersection}$ through $\boldsymbol{\psi}^{(d)}$-names ϕ_1,ϕ_2. It follows that any $\boldsymbol{\psi}^{(d)}$-name ϕ' of $S':=S_1\cap S_2=[-1,1]^d$ has to satisfy $\phi'(q,0^n)=1$ for all $q\in\mathbb{D}^d\cap[-1,1]^d$ and $n\in\mathbb{N}$. Assume that a name for S' is computed by a hypothetical OTM $M^?$ for $\mathsf{Intersection}$, and let $m\in\mathbb{N}$ be the maximal precision of queries $M^{\langle\phi_1,\phi_2\rangle}$ asks when started on input $\langle q,0^n\rangle$ with $q:=(0,\dots,0)$. Now exchange ϕ_i by a $\boldsymbol{\psi}^{(d)}$-name $\tilde{\phi}_i$ for $\tilde{S}_i$ (depicted in Fig. 8b) which fulfills $\tilde{\phi}_i(p,0^k)=\phi_i(p,0^k)$ for all $p\in\mathbb{D}^d$ and $k\le m$. Then $M^{\langle\tilde{\phi}_1,\tilde{\phi}_2\rangle}\langle q,0^n\rangle=M^{\langle\phi_1,\phi_2\rangle}\langle q,0^n\rangle=1$ although $\tilde{S}':=\tilde{S}_1\cap\tilde{S}_2=\overline{\mathrm{B}}_{\|\cdot\|_\infty}(q,1,1-2^{-n+1})$ and therefore $\overline{\mathrm{B}}(q,2^{-n+1})\cap\tilde{S}'=\emptyset$ for $n\ge 3$.

(2) Apply $\boldsymbol{\xi}^{(d)}\preceq^{\mathcal{KCR}}\boldsymbol{\psi}^{(d)}\sqcap\mathsf{r}\sqcap\mathsf{b}$ (Thm. 3.9 + Thm. 3.19(3)) to the domain-side and $\boldsymbol{\psi}^{(d)}\preceq^{\mathcal{KCR}}\boldsymbol{\xi}^{(d)}$ (Thm. 3.9) to the codomain-side, then use statement 3.

(3) The second part, i. e. with $\boldsymbol{\xi}^{(d)}:=\boldsymbol{\omega}^{(d)}$, has been proved in [GLS88, p.129].

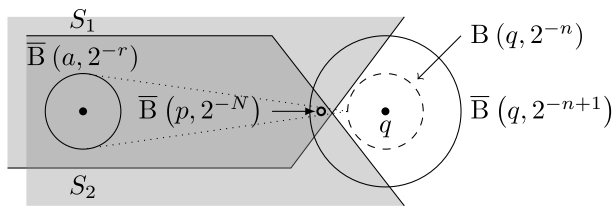


Figure 9: Intersecting two *convex* bodies when additional information about their intersection (an inner ball and an outer radius) is given.

Let $\langle\phi_1,\phi_2,0^{b_1},0^{b_2},0^{r'}\rangle$ be a $(\boldsymbol{\psi}^{(d)}\sqcap\mathsf{b}\times\boldsymbol{\psi}^{(d)}\sqcap\mathsf{b})\sqcap\mathsf{r}'$-name of $(S_1,S_2)\in\mathcal{E}$. Due to convexity, $S':=S_1\cap S_2$ only meets $\overline{\mathrm{B}}(q,2^{-n})$ if S' contains a polyhedron with precisely one vertex lying in $\mathrm{B}(q,2^{-n})$. Therefore, we derive a lower bound $N\in\mathcal{O}(n+b_1+b_2+r')$ on the inner radius parameter of S' close to q; i. e., a radius that guarantees the existence of a ball, say $\overline{\mathrm{B}}\big(p, 2^{-N}\big)$, which is contained in S' and also is sufficiently close to q; i. e., $\overline{\mathrm{B}}(p, 2^{-N}) \subseteq S' \cap \overline{\mathrm{B}}(q, 2^{-n+1})$. This argument is also depicted in Fig. 9; and it yields the following bound on N:

$$3/2 \cdot 2^{-n} \cdot \Big(2 \cdot 2^{-r'} \cdot (2^{\max\{b_1,b_2\}+1} - 2^{-r'} - 2^{-n})^{-1}\Big) \geq 2^{-(n+r'+\max\{b_1,b_2\})} \geq 2^{-N} .$$

Finally, construct ϕ' by a local search around q:

$$\phi'(q, 0^n) := \max\Big\{ \min_{i=1,2} \phi_i\big(p, 0^N\big) \;\Big|\; p \in \overline{\mathrm{B}}(q, 3/2 \cdot 2^{-(n+1)}) \cap \mathbb{D}_N^d \Big\} .$$

Due to the locality of this search, the number of points to be considered is exponential in r and $\max\{b_1, b_2\}$, but polynomial in n. □

4.4. **Projection operator.** For $d \in \mathbb{N}$ and $d \geq e$ let $\mathsf{Projection}$ be the operator

$$\mathsf{Projection}_{d,e} \colon \mathcal{K}^{(d)} \to \mathcal{K}^{(e)},$$
$$S \mapsto \mathsf{Projection}_{d,e}(S) := \Big\{x \in \mathbb{R}^d \;\Big|\; \exists\, y \in \mathbb{R}^{d-e} . (x, y) \in S\Big\} ,$$

pointwise projecting a subset S of d-dimensional Euclidean space down to dimension e.[10] Convexity again turns out to be the key to prove polynomial-time bounds.

Fact 4.4 ([ZM08, Thm. 3.2+Lem. 3.3])**.** (1) Let $d \geq 2$. The statement "if a set $S \in \mathcal{K}^{(d)}$ is polynomial-time $\boldsymbol{\kappa}^{(d)}$-computable, then operator $\mathsf{Projection}_{d,1}(S)$ is polynomial-time $\boldsymbol{\kappa}^{(1)}$-computable" is equivalent to $\mathsf{P} = \mathsf{NP}$.
(2) $\mathsf{Projection}_{2,1}|_{\mathcal{KC}}$ is polynomial-time $(\boldsymbol{\kappa}^{(2)}, \boldsymbol{\kappa}^{(1)})$-computable.

The proof of the second argument can be extended to $\mathsf{Projection}_{d,d-1}$—which by composition $\mathsf{Projection}_{e+1,e} \circ \cdots \circ \mathsf{Projection}_{d,d-1}$ implies the polynomial-time $(\boldsymbol{\kappa}^{(d)}, \boldsymbol{\kappa}^{(e)})$-computability of $\mathsf{Projection}_{d,e}$. Moreover, the second result carries over to the seemingly poorest representation $\boldsymbol{\omega}$, but only if restricted to bounded convex *bodies* $\mathcal{KCR}$.

Proposition 4.5 ($\boldsymbol{\omega}$-computability & -complexity of $\mathsf{Projection}$)**.** *Let* $d > e \in \mathbb{N}$.
(1) $\mathsf{Projection}|_{\mathcal{KR}}$ *is* $(\boldsymbol{\omega}^{(d)} \sqcap \mathsf{r} \sqcap \mathsf{b}, \boldsymbol{\omega}^{(e)})$-discontinuous *for* $d \geq 2$.
(2) *Nonetheless:* $\mathsf{Projection}|_{\mathcal{KCR}}$ *is* $(\boldsymbol{\omega}^{(d)} \sqcap \mathsf{b}, \boldsymbol{\omega}^{(e)})$*-computable, and even* $(\boldsymbol{\omega}^{(d)} \sqcap \mathsf{ar} \sqcap \mathsf{b}, \boldsymbol{\omega}^{(e)})$*-computable in parameterized polynomial time.*

Proof. (1) Apply the adversary argument we have seen several times before. The discontinuity then follows by cutting the unit hypercube $[0, 1]^d$ up via a "chess-board"-like pattern.
(2) Use $\boldsymbol{\omega}^{(d)} \sqcap \mathsf{ar} \sqcap \mathsf{b} \preceq_{\mathrm{p}}^{\mathcal{KCR}} \boldsymbol{\psi}^{(d)}$ (Thm. 3.14) and $\boldsymbol{\psi}^{(d)} \sqcap \mathsf{b} \equiv_{\mathrm{p}}^{\mathcal{K}} \boldsymbol{\kappa}^{(d)}$ Thm. 3.12) on the domain side, $\boldsymbol{\kappa}^{(e)} \preceq_{\mathrm{p}}^{\mathcal{K}} \boldsymbol{\psi}^{(e)} \preceq_{\mathrm{p}}^{\mathcal{KR}} \boldsymbol{\omega}^{(e)}$ on the co-domain side, and then apply the comment on the complexity of $\mathsf{Projection}_{d,e}|_{\mathcal{KC}}$ following Thm. 4.4. □

[10]Note that the projection of a compact/convex/regular set is again compact/convex/regular.

5. Function inversion and image computation

In this section we discuss the complexity of function inversion and image computation; i. e., of $(f, S) \mapsto (f|_S)^{-1}$ for the former, and $(f, S) \mapsto f[S]$ for the latter.

Recall that, for the representations in Thm. 2.7, a name would return either a bit ($\boldsymbol{\omega}$, $\boldsymbol{\psi}$) or a dyadic rational ($\boldsymbol{\delta}$, $\boldsymbol{\delta}_{\mathsf{rel}}$) and/or an integer ($\boldsymbol{\kappa}$), all bounded in binary length by that of the query and/or parameter. This becomes different when encoding (approximations to) arbitrary continuous real functions f. To this end, we refine the previous notion of complexity (Thm. 2.10 and 2.11) and measure the running time in both the discrete argument and the length of the name encoding f. This generalization is covered by Thm. 2.11(3) and permits to bound the complexity of the aforementioned operators.

For convenience and supported by the results from Section 3, we formulate the following definitions and results with respect to $\|\cdot\| := \|\cdot\|_\infty$.

5.1. Prerequisites.

Following Thm. 2.11 we already discussed the need to add advice parameters in order to state the complexity of operators solely in the coding length of their discrete arguments. As an example we saw $\boldsymbol{\delta}|^{\mathcal{K}}$ with advice parameter b. This approach works for sigma-compact metric spaces, but not for the space of continuous real functions: According to Arzela-Ascoli, its compact subsets are parameterized by a modulus of equicontinuity, that is, an integer sequence as opposed to a single integer. The following definition of *second-order polynomials* and *second-order polynomial time* (devised and investigated in a sequence of papers [Meh76, KC96, Lam06, KC12]) provides a solution by defining a notion of length in both the discrete argument *and* the oracle. A recent attempt to generalize from second-order to higher-order complexity can be found in [FH13].

Definition 5.1 (second-order polynomials and complexity; cf. [KC12, §3.2])**.**

(1) A total function $\phi\colon \Sigma^* \to \Sigma^*$ is *length-monotone* if $\ell(\phi(s)) \leq \ell(\phi(t))$ holds true whenever $\ell(s) \leq \ell(t)$ for $s, t \in \Sigma^*$. We denote the set of length-monotone functions by LM.

(2) On $\phi \in \mathsf{LM}$ define a notion of *length* through

$$\ell(\phi)(m) := \ell(\phi(0^m)) = \max_{s \in \Sigma^*.\, \ell(s) \leq m} \ell(\phi(s)).^{11}$$

(3) A *second-order polynomial* $P\colon (\mathbb{N} \to \mathbb{N}) \to (\mathbb{N} \to \mathbb{N})$ in arguments $L\colon \mathbb{N} \to \mathbb{N}$ and $n \in \mathbb{N}$ is defined inductively: Every constant $m \in \mathbb{N}$ is a second-order polynomial, as well as variable n; assuming Q and Q' are second-order polynomials, then $Q + Q'$, $Q \cdot Q'$ and $L(Q)$ are, too.

We make a few remarks why the above definitions are useful, and how they subsume Thm. 2.11.

Remark 5.2.

(1) As by construction, the class of second-order polynomials is closed under addition, multiplication and composition (just like its first-order counterpart, $\mathbb{N}[X]$).

(2) Thm. 2.4 already showed how to encode multiple length-monotone functions ϕ_1, ϕ_2 into one, $\phi := \langle \phi_1, \phi_2 \rangle$. This way, $\ell(\phi)(m+1) = \ell(\phi_1)(m) + \ell(\phi_2)(m) + 1$ for all $m \in \mathbb{N}$.

[11] Notice the overloading of the length-function $\ell(\cdot)$: Depending on the context, it denotes the length of either words or length-monotonic functions. But this overloading "behaves well" in the sense that every word $s \in \Sigma^*$ can be associated with the constant function $\phi_s\colon t \mapsto s$ so that the length of s coincides with the length of ϕ_s, i. e., $\ell(s) = \ell(\phi_s)$.

Let $\boldsymbol{\xi}$ and $\boldsymbol{\xi}'$ be second-order representations of sets X and X', respectively, and $\mathsf{E}\colon X \rightrightarrows \Sigma^*$ be a multi-valued function.

(3) As per Thm. 2.4(3), any $\boldsymbol{\xi} \times \boldsymbol{\xi}'$-name is of form $\langle \phi, \phi' \rangle$ for $\phi, \phi' \in \mathsf{LM}$, and thus itself in LM by the previous point.

(4) Similarly, any $\boldsymbol{\xi} \sqcap \mathsf{E}$-name $\phi = \langle \phi_1, \phi_2 \rangle$ is of length $\ell(\phi)(m{+}1) = \ell(\phi_1)(m){+}\ell(\mathsf{E}(\boldsymbol{\xi}(\phi_2))){+}1$.

As per the last two points, fully polynomial-time computability implies computability in second-order polynomial-time.

We are now ready to define representations for functions and lengths of names thereof.

Definition 5.3 (moduli, and representations for total functions)**.** Let $X \in \mathcal{K}^{(d)}$, and $f\colon X \to \mathbb{R}^e$ be a continuous function.

(1) A function $\overline{\mu}\colon \mathbb{N} \to \mathbb{N}$ is called *modulus of (uniform) continuity of* f if $\|x - y\| \le 2^{-\overline{\mu}(n)}$ implies $\|f(x) - f(y)\| \le 2^{-n}$ for all $x, y \in \operatorname{dom}(f)$ and precisions $n \in \mathbb{N}$.[12]

(2) A $\langle \phi, \varphi \rangle \in \mathsf{LM}$ is a $\boldsymbol{\lambda}^{d,e}_X$-name of f if

 (a) ϕ satisfies

$$\forall\, q \in \mathbb{D}^d \cap X \,.\, \forall\, n \in \mathbb{N} \,.\, \|\phi(q, 0^n) - f(q)\| \le 2^{-n} \tag{5.1}$$

 and

 (b) φ encodes a modulus of continuity $\overline{\mu}$ of f, i. e., $\varphi\colon s \in \Sigma^* \to 0^{\overline{\mu}(\ell(s))}$.

 In order to simplify the notation we associate φ with $\overline{\mu}$ and just write $\langle \phi, \overline{\mu} \rangle$ instead of $\langle \phi, \varphi \rangle$.

(3) A function $\underline{\mu}\colon \mathbb{N} \to \mathbb{N}$ is called *modulus of (uniform) unicity of* f (cf. [Ko91, §4.1]; introduced by Kohlenbach in [Koh90, Koh93], although in a more general way than we need it here) if $\|f(x) - f(y)\| \le 2^{-\underline{\mu}(n)}$ implies $\|x - y\| \le 2^{-n}$ for all $x, y \in \operatorname{dom}(f)$ and precisions $n \in \mathbb{N}$.

(4) A $\big\langle \phi, \overline{\mu}, \underline{\mu} \big\rangle \in \mathsf{LM}$ is a $\boldsymbol{\iota}^{d,e}_X$-name of f if $\overline{\mu}$ and $\underline{\mu}$ are, respectively, moduli of continuity and unicity of f, and $\boldsymbol{\lambda}^{d,e}_X\big(\langle \phi, \overline{\mu} \rangle\big) = f$.

Representations $\boldsymbol{\lambda}$ and $\boldsymbol{\iota}$ only cover subclasses of total functions with *a priori known domains*; thus asking about the $(\boldsymbol{\lambda}^{d,d}_X, \boldsymbol{\lambda}^{d,d}_Y)$-computability and -complexity of function inversion would only make sense if they were restricted to total injective and *surjective* functions of signature $X \to Y$ only. Phrased differently, formulating function inversion over a class $\mathcal{F}$ of functions and with respect to $\boldsymbol{\lambda}$ only makes sense in case that for all functions $f \in \mathcal{F}$ the (a priori known) codomain matches $\operatorname{img}(f)$; thus, the inverses of functions in $\mathcal{F}$ had to be total and (more importantly) had to share the same domain. This is too restrictive a requirement In general, the inverse g of an injective function $f\colon X \to Y$ is a *partial* function from Y to X; but Eqn. (5.1) does not work in case of partial functions: Any $\phi \in \mathsf{LM}$ satisfying Eqn. (5.1) and associated to a partial function $g\colon {\subseteq} X \to Y$ is only defined for *dyadic points in* $\operatorname{dom}(g)$, but $\operatorname{dom}(g)$ does not necessarily contain *any* dyadic point.

By relaxing on the first universal quantification in Eqn. (5.1) we obtain new representations $\boldsymbol{\lambda}^{d,e}_{\subseteq}$ and $\boldsymbol{\iota}^{d,e}_{\subseteq}$ (i. e., *multi-representation*; cf. [GWX08]) which extend $\boldsymbol{\lambda}^{d,e}$ and $\boldsymbol{\iota}^{d,e}$, respectively, and are tailor-made for *partial functions.* They render any name to be defined on *all* dyadic inputs (not only those from $X \cap \mathbb{D}^d$), but only give good approximations (in the usual sense) if the input is close to the domain of the respective function (specializing [KMRZ12, Ex. 1.19(h)]).

[12] The concepts and arguments in this section generalize to integer parameters.

Definition 5.4 (representing partial functions)**.** Let $f\colon \subseteq\mathbb{R}^d \to \mathbb{R}^e$ be a (possibly partial) function with compact domain. A $\langle \phi, \overline{\mu}, \underline{\mu} \rangle$ is a $\boldsymbol{\iota}^{d,e}_{\subseteq}$-name of f if $\overline{\mu}$ and $\underline{\mu}$ are moduli of f, respectively, and ϕ satisfies

$$\forall\, q \in \mathbb{D}^d \,.\, \forall\, n \in \mathbb{N} \,.\, \Big(\mathrm{dom}(f) \cap \overline{\mathrm{B}}\big(q, 2^{-\overline{\mu}(n+1)}\big) \neq \emptyset \\ \implies \exists\, x \in \mathrm{dom}(f) \cap \overline{\mathrm{B}}\big(q, 2^{-\overline{\mu}(n+1)}\big) \,.\, \big\|\phi\big(q, 0^n\big) - f(x)\big\| \leq 2^{-(n+1)} \Big). \tag{5.2}$$

Similarly define $\boldsymbol{\lambda}^{d,e}_{\subseteq}$ as the generalization of $\boldsymbol{\lambda}^{d,e}$ to continuous partial functions.

Note that by the above construction, every $\boldsymbol{\iota}^{d,e}_{\subseteq}$-name ϕ of some *total* function f in particular is a $\boldsymbol{\iota}^{d,e}$-name of f, too: For each $q \in \mathrm{dom}(f)$ there is an $x \in \mathrm{dom}(f) \cap \overline{\mathrm{B}}(q, 2^{-\overline{\mu}(n+1)})$ such that $\|f(q) - f(x)\| \leq 2^{-(n+1)}$. Applying (5.2) then yields

$$\|\phi(q, 0^n) - f(q)\| \leq \|\phi(q, 0^n) - f(x)\| + \|f(x) - f(q)\| \leq 2^{-(n+1)} + 2^{-(n+1)} = 2^{-n} \,.$$

5.2. **Function inversion: some upper and lower bounds.** The $\mathsf{Inversion}$ operator takes a function f and a subset $\mathcal{A} \ni S \subseteq \mathrm{dom}(f)$, and (under the assumption on f having a local inverse on S) maps (f, S) to the inverse of $f|_S$. In this section we focus on the *parameterized complexity* of this operator.

While $\mathsf{Inversion}$ is polynomial-time computable for injective functions *from* $[0,1]$ *to* $\mathbb{R}$ [Ko91, Thm. 4.6], its complexity is linked to the existence of one-way functions from dimension two onwards [Ko91, Thm. 4.23+4.26]. If f is *bi-Hölder continuous* (i. e., both f and its inverse are Hölder continuous), then $\mathsf{Inversion}$ still is only *computable in exponential time*, but becomes *parameterized polynomial-time* computable for *bi-Lipschitz functions* (Thm. 5.9). It turns out that this bound is actually the best we can achieve: There is *no* parameterized polynomial-time algorithm for $\mathsf{Inversion}$ over bi-Hölder functions that are not bi-Lipschitz assuming that *one-way permutations* exist (Thm. 5.11; an assumption stronger than the existence of one-way string functions underlying contemporary cryptography).

We start to formally prove the above claims by reviewing a few non-uniform bounds on function inversion. The first fact is a uniform reformulation of the above mentioned inversion result, [Ko91, Thm. 4.6], for one-dimensional functions.

Fact 5.5. $\mathsf{Inversion}$ is polynomial-time $\big(\boldsymbol{\iota}^{1,1}_{[0,1]}, \boldsymbol{\iota}^{1,1}_{\subseteq}\big)$-computable.

Notice the *necessity* of adding an inverse modulus $\underline{\mu}$ to make this result work. The algorithm behind the proof is based on *trisection on* $[0,1]$ [Wei00, Ex. 6.3.6]: For a given point q in the range of f, start with $p = 1/2$ as a candidate for a 2^{-n}-approximation to $f^{-1}(q)$ and use that injectivity implies strict monotonicity for injective functions $f\colon [0,1] \to \mathbb{R}$ to determine whether to continue this binary search in $[0,p]$ or $[p,1]$. This algorithm stops and returns p when it is of precision roughly $\overline{\mu}(\underline{\mu}(n))$. By unrolling the definitions of both $\overline{\mu}$ and $\underline{\mu}$ one verifies that this indeed gives a 2^{-n}-approximation to $f^{-1}(q)$. This approach, however, fails from dimension two on due to lack of total order.

The following two results recall known lower and upper bounds on the complexity of non-uniform function inversion.

Fact 5.6 (non-uniform bounds for function inversion; [Ko91, Thm. 4.23+4.26])**.**

(1) If $\mathsf{P} = \mathsf{NP}$, then f^{-1} is polynomial-time $(\boldsymbol{\rho}^2, \boldsymbol{\rho}^2)$-computable on $\mathrm{img}(f)$ whenever $f\colon [0,1]^2 \to \mathbb{R}^2$ is injective, $(\boldsymbol{\rho}^2|^{[0,1]^2}, \boldsymbol{\rho}^2)$-computable in polynomial time and $\underline{\mu}$ is polynomially bounded.[13]

(2) If $\mathsf{P} \neq \mathsf{UP}$, then there exists an injective, polynomial-time $(\boldsymbol{\rho}^2, \boldsymbol{\rho}^2)$-computable function $f\colon [0,1]^2 \to [0,1]^2$ with polynomial modulus of unicity $\underline{\mu}$ for which f^{-1} is *not* $(\boldsymbol{\rho}^2, \boldsymbol{\rho}^2)$-computable in polynomial time on $\mathrm{dom}(f^{-1}) = \mathrm{img}(f)$.

The second statement has been proved using the following result that connects the P vs. UP question with the existence of *one-way functions* (which we discuss thereafter).

Fact 5.7 ([Ko85, GS88])**.** Total one-way functions exist if and only if $\mathsf{P} \neq \mathsf{UP}$.

Notice the emphasis on totality (and implicitly on injectivity) since there are other types of one-way functions whose existence, in contrast, are not always connected to just P vs. UP [HT03, Thm. 3.2]. An injective polynomial-time computable function $\phi\colon \subseteq\Sigma^* \to \Sigma^*$ is said to be a *(worst-case) one-way function* if (a) some polynomial p exists such that $\ell(\phi^{-1}(s)) \leq p(\ell(s))$ whenever $s \in \mathrm{img}(\phi)$ (*polynomial honesty*); and (b) if no polynomial-time computable function ψ satisfies $\psi(\phi(s')) = s'$ for all $s' \in \mathrm{dom}(\phi)$ (not *polynomial-time invertible*).

Now we are equipped to talk about the proof of Thm. 5.6(2): Assume $\mathsf{P} \neq \mathsf{UP}$, and let $\phi\colon \Sigma^* \to \Sigma^*$ be a total one-way function. Based on ϕ, construct a piecewise-linear function f with the properties described in Thm. 5.6(2) which is hard to invert if ϕ is. This is achieved by encoding the image of ϕ into the domain of f in a way which only allows to recover the inverse $s = \phi^{-1}(t)$ from t and f if ϕ is polynomial-time computable. The moduli (of continuity/unicity) of f are, moreover, polynomials. More precisely, $\overline{\mu}(n) = \underline{\mu}(n) = cn + p(n) + \mathrm{const}$, where $p(\ell(s)) = \ell(\phi^{-1}(s))$ for any $s \in \Sigma^*$. Since $p(n)$ is super-logarithmic[14], the moduli are bounded linear (from below) in n. This suggests that $\mathsf{Inversion}$ could be polynomial-time computable for the class of Lipschitz- or even Hölder-continuous functions; which we prove to be almost correct in Thm. 5.9; and it can not be generalized to *arbitrary* polynomially bounded moduli (Thm. 5.6(2)).

The inversion algorithm we devise in Thm. 5.9 will involve partial injective functions, encoded using $\boldsymbol{\iota}^{d,e}$ together with $\boldsymbol{\psi}^{(d)}$ for their domain:

Definition 5.8 (representation $\boldsymbol{\theta}$)**.** Let $f\colon \subseteq\mathbb{R}^d \to \mathbb{R}^e$ be a (possibly partial) function with compact domain, and let $S \in \mathcal{K}^{(d)}$. A $\langle \phi, \overline{\mu}, \underline{\mu}, \psi \rangle \in \mathsf{LM}$ is a $\boldsymbol{\theta}^{d,e}$-name of (f, S) if (a) $S \subseteq \mathrm{dom}(f)$; (b) ψ is a $\boldsymbol{\kappa}^{(d)}$-name of S; (c) $\langle \phi, \overline{\mu}, \underline{\mu} \rangle$ is a $\boldsymbol{\iota}^{d,e}_{\subseteq}$-name of f.

Recall that a function $f\colon X \to Y$ on normed spaces $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ is (α, C)-*Hölder (continuous)* with *Hölder exponent* $0 < \alpha \leq 1$ and *Hölder constant* $C > 0$ if it satisfies

$$\|f(x) - f(y)\|_Y \leq C \cdot \|x - y\|_X^\alpha$$

for any two $x, y \in X$.

In particular, an (α, C)-Hölder function has modulus of continuity $\overline{\mu}(n) \coloneqq (n + \mathrm{lb}(C)) \cdot \alpha^{-1}$, and any L-Lipschitz (continuous) function is in fact $(1, L)$-Hölder. Take $[0,1] \ni x \mapsto \sqrt{x}$

[13] Note that f being polynomial-time computable already implies $\overline{\mu}$ to be polynomially bounded.

[14] If not, one could just try out all of the $2^{\mathrm{lb}(\ell(s))}$ many possible preimages for s under ϕ, thus computing ϕ in polynomial time, contradicting the existence of one-way functions (since ϕ is an arbitrary one), thus implying $\mathsf{P} = \mathsf{UP}$.

as an example: It is $(\alpha, 1)$-Hölder for $\alpha \leq 1/2$, and its inverse $[0,1] \ni y \mapsto y^2$ is even $(1,1)$-Hölder (hence 1-Lipschitz).

If the inverse of a Hölder function f exists and if it moreover is a Hölder function, than we call f *bi-Hölder*. If f is bi-Hölder, then there exist bounds $0 < \alpha, \alpha' \leq 1$ and $C, C' > 0$ such that

$$1/C' \cdot \|x-y\|_X^{1/\alpha'} \leq \|f(x)-f(y)\|_Y \leq C \cdot \|x-y\|_X^{\alpha} .$$

If $\alpha = \alpha' = 1$, then we call f *bi-Lipschitz.* [15]

For convenience, denote by $\mathcal{H}$ the class of partial functions $f\colon \subseteq\mathbb{R}^d \to \mathbb{R}^e$ that are also bi-Hölder, and by $\mathcal{L}$ the class of those f being bi-Lipschitz. Now we are equipped to state our result about inversion.

Theorem 5.9 (complexity of $\mathsf{Inversion}$)**.** *Operator* $\mathsf{Inversion}$ *is* $\big(\boldsymbol{\theta}^{d,e}, \boldsymbol{\iota}_{\subseteq}^{e,d}\big)$*-computable and its time complexity is bounded* exponentially *in* $\overline{\mu} \circ \underline{\mu} \circ \overline{\mu}(n)$*. This exponential dependence still holds true when restricted to* $\mathcal{H}$*, but leads to a* parameterized polynomial-time *bound when further restricted to* $\mathcal{L}$*.*

The exponential dependence on Hölder parameters in the above theorem is actually optimal unless $\mathsf{P} = \mathsf{UP} \cap \mathsf{coUP}$. To see why, we consider the notion of *one-way permutations*, that is, bijective one-way functions. It is known by [HT03, Thm. 3.1] that total one-way permutations exist if and only if $\mathsf{P} \neq \mathsf{UP} \cap \mathsf{coUP}$. Recall that the moduli of the function specifically constructed to prove Thm. 5.6(2) were polynomially bounded in n. Assuming the existence of total one-way permutations, they even become *linear* in n. Noting that f is a Hölder function if and only if it has a linearly bounded modulus then implies the claimed optimality of Thm. 5.9.

Lemma 5.10. *Let* φ *be a total one-way permutation. Then a* partial *one-way permutation* $\psi\colon \subseteq\Sigma^* \to \Sigma^*$ *with the following properties can be constructed from* φ*:*
(1) ψ *is length-preserving, i. e.,* $\psi[\Sigma^m] \subseteq \Sigma^m$ *for all* $m \in \mathbb{N}$;
(2) $\psi \in \mathsf{FP}$;
(3) $\psi^{-1} \in \mathsf{FP} \implies \varphi^{-1} \in \mathsf{FP}$.

Corollary 5.11. *Assume* $\mathsf{P} \neq \mathsf{UP} \cap \mathsf{coUP}$*. Then there exists an injective, polynomial-time* $(\boldsymbol{\rho}^2, \boldsymbol{\rho}^2)$*-computable function with moduli of continuity and unicity* $\overline{\mu}, \underline{\mu}$ *both of the form* $n \mapsto an + b$ *with* $a, b \in \mathbb{N}$ *for which* f^{-1} *is* not $(\boldsymbol{\rho}^2, \boldsymbol{\rho}^2)$*-computable in polynomial time on* $\mathrm{dom}(f^{-1}) = \mathrm{img}(f)$*.*

Theorem 5.11 leads to the conclusion that the exponential-time bound in Thm. 5.9 for $\mathsf{Inversion}$ restricted to Hölder functions is optimal (assuming $\mathsf{P} \neq \mathsf{UP} \cap \mathsf{coUP}$) as a continuous function is Hölder continuous if and only if it admits a linear modulus of continuity.

We now sketch how to prove Thm. 5.9 (also illustrated in Fig. 10) and postpone the respective proofs of the last two statements until the end of this subsection.

Let q be the point to compute $(f|_S)^{-1}(q)$ for. Testing for all points p on a fine grid, say $\mathbb{D}^d_{k(n)}$, whether their image approximate image is close to q would be a pure brute-force approach, and as such having an exponential running time. Instead we search iteratively: Start with a coarse grid $\mathbb{D}^d_{k(0)}$ and keep all these points from this (coarse) grid whose images are not too far from q. The key idea in this step, which will lead to a low(er) complexity, is

[15] Exponents $\alpha \in \{0\} \cup (1, \infty)$ excluded by purpose: $\alpha = 0$ if the respective function is bounded, and $\alpha > 1$ if it is constant.

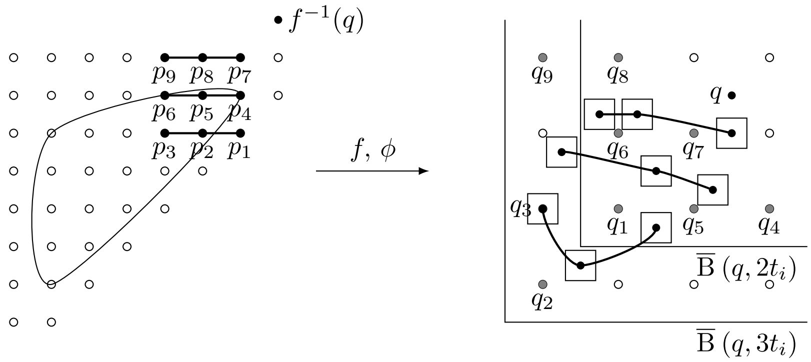


Figure 10: Points p_j along with their correct and approximate images $f(p_j)$ and $q_j :=$ $\phi\big(p_j, 0^{m_i+1}\big)$, respectively. All approximate images q_j, except for q_2, q_3 and q_9, are close enough to q (all that lies within the blue-highlighted ball), thus being candidates of being an approximate inverse image of q in round $i+1$.

that the number of points that have to be kept in this step can be bounded in terms of both moduli ($\overline{\mu}$ and $\underline{\mu}$):

- Any two distinct points $p, p' \in \mathbb{D}^d_{\overline{\mu}(n)}$ are $\|p - p'\| > 2^{-(\overline{\mu}(n)+1)}$ apart,
- thus (by definition of the inverse modulus) their images satisfy
$$\overline{\mathrm{B}}\big(f(p), 2^{-\underline{\mu}(\overline{\mu}(n)+1)-1}\big) \cap \overline{\mathrm{B}}\big(f(p'), 2^{-\underline{\mu}(\overline{\mu}(n)+1)-1}\big) = \emptyset$$
- which implies that only finitely many points from a fixed grid can be close to q—and we can bound their number in terms of $\overline{\mu}$, $\underline{\mu}$ and n.

For the next iteration, the grid will be refined to $\mathbb{D}^d_{k(1)}$. But instead of checking all these points, we will consider only those being close to a point p from the former grid $\mathbb{D}^d_{k(0)}$ whose image has turned out to be not too far from q. The complexity of this algorithm for finding a good approximation to $f^{-1}(q)$ will then be of form $\mathcal{O}(n \cdot \#\text{of points that have to be kept in each iteration})$.

of Thm. 5.9. Let $\langle \phi, \overline{\mu}, \underline{\mu}, \phi', 0^b \rangle$ be a $\boldsymbol{\iota}^{d,e}_{\subseteq}$-name of (f, S). Further, let $n \in \mathbb{N}$ and $q \in \mathbb{D}^e \cap S$; we postpone the discussion about the general case where $d_{f[S]}(q) \leq 2^{-\underline{\mu}(n+1)}$ to a later stage in this proof. Without loss of generality, we assume $\overline{\mu}(n+1) - \overline{\mu}(n) \geq 1$ and $\underline{\mu}(n+1) - \underline{\mu}(n) \geq 1$.[16] Moreover, we prove the theorem only for $b := 0$ (just for convenience) although the arguments extend to arbitrary outer radii parameter b.

To shorten the frequently used terms, we define precisions $k_i := \overline{\mu}(\underline{\mu}(i) + 1) + 1$ and $m_i := \underline{\mu}(i)$, radii $r_i := 2^{-k_i+1}$ and $t_i := 2^{-m_i}$, as well as approximations $q_{p,i} := \phi(p, 0^{k_i})$. The

[16] Hölder functions with Hölder exponent $\alpha \in (0, 1]$ have this property since $\mu(n+1) - \mu(n) = 1/\alpha \in [1, \infty)$ for $\mu(n) = 1/\alpha \cdot (n + \mathrm{lb}\, H)$.

proof is centered around the following sets:

$$\begin{aligned} S_0 &\coloneqq \left\{ p \in \mathbb{D}^d_{k_0} \;\middle|\; \phi'(p, 0^{m_0}) = 1 \right\} , \\ C_i &\coloneqq \left\{ p \in S_i \;\middle|\; p \in S_i \text{ and } \|q - q_{p,i}\| \le 2t_i \right\} , \\ S_{i+1} &\coloneqq \bigcup_{p \in C_i} S_{p,i+1} , \quad S_{p,i+1} \coloneqq \left\{ p' \in \overline{\mathrm{B}}(p, r_i) \cap \mathbb{D}^d_{k_{i+1}} \;\middle|\; \phi'(p', 0^{m_{i+1}}) = 1 \right\} . \end{aligned}$$

All we now have to do is to iteratively compute the *candidate sets* C_i and finally deterministically pick a point $p \in C_{n+2}$. We claim that such a p exists and that it is a 2^{-n}-approximation to $f^{-1}(q)$.

An important note before we continue. Since f is a partial function the term "$f(p)$" might be undefined for some $p \in S_i$. We nonetheless want to talk about objects like "$\overline{\mathrm{B}}(f(p), \cdot)$". The definition of $\boldsymbol{\iota}^{d,e}$ solves this problem: For $i \in \mathbb{N}$ and $p \in \mathbb{D}^d_{k_i}$ let $x_{p,i}$ be *any* point from $S \cap \overline{\mathrm{B}}(p, r_i)$ as in (5.2). Then $f\left[\overline{\mathrm{B}}(p, r_i) \cap S\right] \subseteq \overline{\mathrm{B}}(f(x_{p,i}), t_i)$, and we will therefore always reason about $\overline{\mathrm{B}}(f(x_{p,i}), \delta)$ instead of the maybe undefined $\overline{\mathrm{B}}(f(p), \delta/2)$.

Correctness: We have to show that $C_i \neq \emptyset$ for all $0 \le i \le n+2$, and that $f^{-1}(q) \in \overline{\mathrm{B}}(p, 2^{-n})$ for any $p \in C_{n+2}$. Instead of the statment "$C_i \neq \emptyset$" we prove the stronger proposition "$\exists\, p_i \in C_i \,.\, q \in \overline{\mathrm{B}}(f(x_{p_i,i}), t_i)$".

For $i = 0$ we first note that $\bigcup_{p \in S_0} \overline{\mathrm{B}}(p, r_0)$ is a superset of S. This plus the definition of $\overline{\mu}$ imply $f[S] \subseteq \bigcup_{p \in S_0} \overline{\mathrm{B}}(f(x_{p,0}), t_0)$. Therefore, there must exist a point $p_0 \in S_0$ whose image is close to q in the sense that $q \in \overline{\mathrm{B}}(f(x_{p_0,0}), t_0)$. Hence, $\|q - q_{p_0,0}\| \le 2t_0$ which gives $C_0 \neq \emptyset$.

Now let $i \ge 1$. By construction of C_{i-1} and S_i it holds that

$$q \in \bigcup_{p \in C_{i-1}} f\left[\overline{\mathrm{B}}(p, r_{i-1}) \cap S\right] \subseteq \bigcup_{p \in C_{i-1}} \bigcup_{p' \in P_{p,i}} f\left[\overline{\mathrm{B}}(p', r_i) \cap S\right] .$$

Therefore the exists a $p' \in P_i$ with $\left\|q - f(x_{p',i})\right\| \le t_i$, implying $\left\|q - q_{p',i}\right\| \le 2t_i$. Thus, $C_i \neq \emptyset$.

In the end (i. e., for $i = n+2$), the definition of $\underline{\mu}$ implies that for any $p \in C_{n+2}$ holds $q \in \overline{\mathrm{B}}(q_{p,n+2}, 2t_{n+2})$, which first leads to $q \in \overline{\mathrm{B}}(f(x_{p,n+2}), 3t_{n+2})$. Using that $\underline{\mu}(n+2) - \underline{\mu}(n) \ge 2$ implies $3t_{n+2} < 4t_{n+2} \le t_n$ finally allows to conclude $f^{-1}(q) \in \overline{\mathrm{B}}(p, 2^{-n})$.

A note on the general case of $d_{f[S]}(q) \le t_{n+1}$: By assumption, there exists an $x \in S$ such that $\|f(x) - q\| \le t_{n+1}$. Therefore, $\|f(x) - f(x_{p,n+2})\| \le t_{n+2}$ holds true for all $p \in \mathbb{D}^d_{k_{n+2}} \cap \overline{\mathrm{B}}(x, r_{n+2})$, implying $\|f(x) - q_{p,n+2}\| \le 2t_{n+2}$. Combining both bounds then gives $\|q_{p,n+2} - q\| \le 4t_{n+2} \le t_n$.

Complexity: We have to bound the number of points in S_0, C_i and S_{i+1} for $i \in \mathbb{N}$. The set S_0 contains at most $2^{d(b+k_0)}$ many points[17], and $|S_{i+1}|$ is bounded by

$$|S_{i+1}| \le \sum_{p \in C_i} \left|\overline{\mathrm{B}}(p, r_i) \cap \mathbb{D}^d_{k_{i+1}}\right| \le |C_i| \cdot (2r_i/r_{i+1})^d \le |C_i| \cdot 2^{d(k_{i+1} - k_i + 2)} .$$

[17]This is true modulo details: The exponential dependence on $k_0 = \overline{\mu}(\underline{\mu}(0) + 1) + 1$ only leads to an exponential dependence on the respective Hölder exponents. This exponential bound, however, can be reduced to linear: Extend the definition of $\overline{\mu}$ and $\underline{\mu}$ to integers and, instead of k_0, start with k_{-j} for $j \in \mathbb{N}_+$ being maximal with property $k_{-j} \ge 0$. Such a j can be found in time logarithmic in the absolute value of both Hölder constants.

The bound on $|C_i|$ requires a bit more care (as hinted prior to this proof): Any two distinct points $p, p' \in C_i$ have the property that $\overline{\mathrm{B}}(p, r_i/4)$ and $\overline{\mathrm{B}}(p', r_i/4)$ are disjoint. It then follows by definition of $\underline{\mu}$ that $\overline{\mathrm{B}}(x_{p,i}, 2^{-\underline{\mu}(k_i+2)})$ and $\overline{\mathrm{B}}(x_{p',i}, 2^{-\underline{\mu}(k_i+2)})$ are also disjoint. This fact now allows to bound $|C_i|$ by counting how many disjoint balls of radius $2^{-\underline{\mu}(k_i+2)}$ fit into $\overline{\mathrm{B}}(q, 2t_i + t_i)$:

$$|C_i| \le (2 \cdot 3t_i / 2^{-\underline{\mu}(k_i+2)})^d < (4 \cdot 2^{\underline{\mu}(k_i+2)-m_i})^d \ .$$

The above describe procedure for computing $\mathsf{Inversion}$ therefore checks at most

$$\mathcal{O}\Big(\sum\nolimits_{i=0}^{n+2} \big|C_i\big| + \big|S_i\big| \Big)$$

many points. Their number is bounded by (and thus further simplifies to)

$$\mathcal{O}\big(n \cdot 2^{\underline{\mu}(k_{n+2}+2)-m_{n+2}+k_{n+2}-k_{n+1}}\big) \ . \tag{5.3}$$

If $f|_S$ is bi-Hölder continuous, then its moduli are of form $\overline{\mu}(n) = \overline{\alpha}^{-1}(n + \overline{c})$ and $\underline{\mu}(n) = \underline{\alpha}^{-1}(n + \underline{c})$ with $\overline{c} \coloneqq \mathrm{lb}\, \overline{C}$, $\underline{c} \coloneqq \mathrm{lb}\, \underline{C}$. Moreover,

$$\underline{\mu}(k_{n+2} + 2) - m_{n+2} = n \cdot ((\overline{\alpha}\underline{\alpha}^2)^{-1} - \underline{\alpha}^{-1}) + 2 \cdot (\overline{\alpha}\underline{\alpha}^2)^{-1} + k_0/\underline{\alpha}$$

and $k_{i+1} - k_i = (\overline{\alpha}\underline{\alpha})^{-1}$. Assuming $\overline{\alpha}\underline{\alpha} = 1$ (which holds exactly for bi-Lipschitz functions) allows to rewrite Eqn. (5.3) to $\mathcal{O}\big(n \cdot 2^{k_0}\big)$ by applying the identities we just obtained.

Note that the encoding length of each p and $q_{p,i}$ is bounded linearly in $b + k_{n+2} + \ell(\langle q \rangle)$. Finally, this bound combined with the former bound on the number of points to check gives the claimed parameterized polynomial-time bound for $\mathsf{Inversion}$ over $\mathcal{L}$. □

of Thm. 5.11. Follows directly from the proof of Thm. 5.6(2) by replacing the one-way function with a partial one-way permutation as in Thm. 5.10. Since ψ is length-preserving it satisfies $p(\ell(s)) = \ell(\psi^{-1}(s))$ with $p \coloneqq \mathrm{id}$. By the remarks following Thm. 5.6(2), the moduli of the function constructed to prove this direction are of form $\mu(n) = cn + p(n) + \mathrm{const}$—a bound linear in n. □

of Thm. 5.10. Let φ be a total one-way permutation and $p \in \mathbb{N}[X]$ such that $\ell(s) \le p(\ell(\varphi(s)))$ for all $s \in \Sigma^*$. Set

$$\Gamma_n \coloneqq \sum\nolimits_{i=0}^{n} \big(p(i) + 2\big) \ , \quad \gamma_n \coloneqq \Gamma_n - \big(p(n) + 2\big) \ , \quad \delta_{s,n} \coloneqq p(n) - \ell(s) \ ,$$

and construct a partial function $\psi \colon \subseteq\Sigma^* \to \Sigma^*$ by

$$\psi \colon 0^{\gamma_n}\, 1\, 0^{\delta_{s,n}}\, 1\, s \longmapsto 0^{\Gamma_n-(n+1)}\, 1\, \varphi(s) \quad \text{for } \varphi(s) \in \Sigma^n \ .$$

The idea behind the construction of ψ is to first pad the all arguments to φ with length-n images to be of length Γ_n, and then to pad the image of each $t \in \Sigma^{\gamma_n}$ also to length Γ_n. This way, ψ will be length-preserving.

Concerning (2): Given a $t \in \Sigma^*$, use $\ell(t)$ to determine whether t is contained in Σ^{Γ_n} for some n. To this end, check if t is of form $0^{\gamma_n}\, 1\, 0^{\delta_{s,n}}\, 1\, s$ for some $s \in \Sigma^{\le p(n)}$ and also if $\varphi(s) \in \Sigma^n$. Note that the respective n is bounded from above by $\ell(t)$. If t is not of this particular form, then $t \notin \mathrm{dom}(\psi)$ follows immediately. If, on the contrary, t is of this form, but $\varphi(s) \notin \Sigma^n$, then $t \notin \mathrm{dom}(\psi)$ follows, too. If, however, $\varphi(s) \in \Sigma^n$, then the (easy to compute) string $0^{\Gamma_n-(n+1)}\, 1\, \varphi(s)$ is the image of t under ψ.

Concerning (3): Let $\psi^{-1} \in \mathsf{FP}$. Given $t \in \Sigma^*$, construct $t' \coloneqq 0^{\Gamma_n}\, 1\, t$. Note that by surjectivity of φ we know that elements of $\mathrm{dom}(\psi)$ can only be of the above form. It thus suffices to compute $s' := \psi^{-1}(t') = 0^{\gamma_n}\,1\,0^{\delta_{s,n}}\,1\,s$ and extract s from it which by construction of ψ satisfies $\varphi^{-1}(t) = s$. $\square$

5.3. Image. The operator $\mathsf{Image}\colon \subseteq\mathrm{C}(\mathbb{R}^d, \mathbb{R}^e) \times \mathcal{K}^{(d)} \ni (f, S) \mapsto f[S] \in \mathcal{K}^{(e)}$ has been proven to be $(\boldsymbol{\lambda}_{\subseteq}^{d,e} \times \boldsymbol{\kappa}^{(d)}, \boldsymbol{\psi}^{(e)})$-computable [Wei00, Thm. 6.2.4(4)] which, however, fails if we relax the restriction on S from compact to closed [Wei00, Thm. 6.2.4(3)]. The respective proof unfortunately does not yield any complexity bounds. However: Restricting Image to Hölder functions does give parameterized bounds. To this end, define a representation $\boldsymbol{\Lambda}_{\subseteq}^{d,e}$ as follows: A ϕ' is a $\boldsymbol{\Lambda}_{\subseteq}^{d,e}$-name of $(f, S) \in \mathcal{H} \times \mathcal{K}^{(d)}$ if $\phi' = \langle \phi, \varphi \rangle$ with $\boldsymbol{\lambda}_{\subseteq}^{d,e}(\phi) = f$, $\boldsymbol{\kappa}^{(d)}(\varphi) = S$, and $S \subseteq \mathrm{dom}(f)$. Further denote by $\alpha\mathsf{C}$ the enrichment by Hölder parameters, i. e.,

$$\alpha\mathsf{C}\colon \mathcal{H} \rightrightarrows \Sigma^* , \quad \alpha\mathsf{C}\colon f \Mapsto \{\langle \mathsf{un}_{\mathbb{N}}(1/\alpha), \mathsf{bin}_{\mathbb{N}}(C)\rangle \mid f \text{ is } (\alpha, C)\text{-Hölder continuous}\} .$$

Then the complexity of Image restricted to Hölder functions follows immediately from Thm. 5.9.

Corollary 5.12. $\mathsf{Image}|_{\mathcal{H}\times\mathcal{K}}$ *is* parameterized *polynomial-time* $(\boldsymbol{\Lambda}_{\subseteq}^{d,e} \sqcap \alpha\mathsf{C}, \boldsymbol{\kappa}^{(e)})$*-computable, and* $\mathsf{Image}|_{\mathcal{L}\times\mathcal{K}}$ *is* fully *polynomial-time* $(\boldsymbol{\Lambda}_{\subseteq}^{d,e} \sqcap \alpha\mathsf{C}, \boldsymbol{\kappa}^{(e)})$*-computable.*

For the proof it essentially suffices to modify the proof of Thm. 5.9 as follows: Replace all k_i with $\overline{\mu}(i+1)+1$, m_i with i, and instead of deterministically picking a point $p \in S_{n+2}$ we check whether S_{n+2} is empty. If S_{n+2} is empty, then it is a witness for $d_S(q) \geq 2^{-n}$, while a non-empty S_{n+2} witnesses $d_S(q) \leq 2^{-n+1}$.

6. Future research

Sections 4 and 5 can be understood as the base for further interesting questions about operators and parameters that render them to be polynomial-time computable; like the complexity of preimage $\mathsf{PreImage}\colon (f, S') \mapsto f^{-1}[S']$ [ZB04, Lem. 24], or generalizations of the solution operator for Poisson equations to arbitrary compact domains [KSZ13]. We also left open questions raised about the complexity of $\mathsf{Inversion}$ for more restricted classes of functions (continuous, smooth, Gevrey [LLM01, KMRZ12]) and about improvements of Thm. 5.6. For example: Can Ko's construction be modified to produce a smooth function instead of only a continuous one? And do Ziegler and McNicholl's computability results on the implicit and inverse function theorem [Zie06, McN08] (parameterized) polynomial-time if restricted to a subset of C^2 or Gevrey functions?

Acknowledgements

I am grateful to Martin Ziegler for continuous advice and many helpful suggestions; and to Akitoshi Kawamura, Ulrich Kohlenbach, Robert Rettinger, and Florian Steinberg for seminal discussions, hints and lots of advice. I also like to express my gratitude to the anonymous referees who have provided invaluable suggestions on how to improve both the structure of this paper and the presentation of results plus their respective proofs.

References

[BB85] E. Bishop and D.S. Bridges. *Constructive Analysis*, volume 2. Springer Verlag, 1985.

[BBH01] J. Blanck, V. Brattka, and P. Hertling, editors. *4th International Workshop on Computability and Complexity in Analysis (CCA 2000), Swansea, UK, Selected Papers*, volume 2064 of *Lecture Notes in Computer Science*. Springer, September 17–19 2001.

[BdBP12] V. Brattka, M. de Brecht, and A. Pauly. Closed Choice and a Uniform Low Basis Theorem. *Annals of Pure and Applied Logic*, 163(8):986–1008, 2012.

[BG11] V. Brattka and G. Gherardi. Effective Choice and Boundedness Principles in Computable Analysis. *The Bulletin of Symbolic Logic*, pages 73–117, 2011.

[BHW08] V. Brattka, P. Hertling, and K. Weihrauch. A Tutorial on Computable Analysis. In *New Computational Paradigms*, pages 425–491. Springer, 2008.

[Bis67] M. Bishop. Foundations of Constructive Analysis. 1967.

[BL00] J.M. Borwein and A.S. Lewis. *Convex Analysis and Nonlinear Optimization: Theory and Examples*. Springer, 2000.

[BP03] V. Brattka and G. Presser. Computability on Subsets of Metric Spaces. *Theoretical Computer Science*, 305:43–76, 2003.

[Bra99] V. Brattka. Computable Invariance. *Theoretical Computer Science*, 210(1):3–20, 1999.

[Bra03] V. Brattka. The Emperor's New Recursiveness: The Epigraph of the Exponential Function in Two Models of Computability. *Words, Languages & Combinatorics*, 3:63–72, 2003.

[Bra04] M. Braverman. Computational Complexity of Euclidean Sets: Hyperbolic Julia-Sets are Poly-Time Computable. Master's thesis, University of Toronto, 2004.

[Bra05a] M. Braverman. Hyperbolic Julia Sets are Poly-Time Computable. *Electronic Notes in Theoretical Computer Science*, 120:17–30, 2005.

[Bra05b] M. Braverman. On the Complexity of Real Functions. In *Foundations of Computer Science, 2005. FOCS 2005. 46th Annual IEEE Symposium on* [con05], pages 155–164.

[Bra05c] M Braverman. On the Complexity of Real Functions. *arXiv preprint cs.CC/0502066*, 2005.

[Bro19] L.E.J. Brouwer. Begründung der Mengenlehre unabhängig vom logischen Satz vom ausgeschlossenen Dritten. Zweiter Teil. Verhandelingen Akad. *Wet. Amsterdam Natuurk. Sect. I*, 12(7), 1919.

[BW99] V. Brattka and K. Weihrauch. Computability on Subsets of Euclidean Space I: Closed and Compact Subsets. *Theoretical Computer Science*, 219(1):65–93, 1999.

[CCA13] *10th International Workshop on Computability and Complexity in Analysis (CCA 2013), Nancy, France, Proceedings*, July 8–10 2013.

[CK95] A.W. Chou and Ker-I Ko. Computational Complexity of Two-Dimensional Regions. *SIAM Journal on Computing*, 24(5):923–947, 1995.

[CK05] A.W. Chou and Ker-I Ko. The Computational Complexity of Distance Functions of Two-Dimensional Domains. *Theoretical Computer Science*, 337(1):360–369, 2005.

[con95] *12th Annual Symposium on Theoretical Aspects of Computer Science (STACS 1995), Munich, Germany, Proceedings*, March 2–4 1995.

[con05] *46th Annual IEEE Symposium on Foundations of Computer Science (FOCS 2005), Pittsburgh, PA, USA, Proceedings*. IEEE Computer Society, October 23–25 2005.

[dBCvKO08] M. de Berg, O. Cheong, M. van Kreveld, and M. Overmars. *Computational Geometry: Algorithms and Applications*. Springer, 3 edition, 2008.

[FG06] J. Flum and M. Grohe. *Parameterized Complexity Theory*, volume 3. Springer Heidelberg, 2006.

[FH13] H. Férée and M. Hoyrup. Higher-Order Complexity in Analysis. In CCA13 [CCA13], pages 22–35. Extended abstract.

[FHHP] H. Férée, E. Hainry, M. Hoyrup, and R. Péchoux. Polynomial-Time Computable Real Functions.

[Fri84] H. Friedman. The Computational Complexity of Maximization and Integration. *Advances in Mathematics*, 53(1):80–98, 1984.

[GLS88] M. Grötschel, L. Lovász, and A. Schrijver. *Geometric Algorithms and Combinatorial Optimization*. Springer, Berlin, 1988.

[GN94] X. Ge and A. Nerode. On Extreme Points of Convex Compact Turing Located Sets. *Logical Foundations of Computer Science*, pages 114–128, 1994.

[Grz57] A. Grzegorczyk. On the Definition of Computable Real Continuous Functions. *Fundamenta Mathematicae*, 42:61–71, 1957.

[GS88] S. Grollmann and A.L. Selman. Complexity Measures for Public-Key Cryptosystems. *SIAM Journal on Computing*, 17(2):309–335, 1988.

[GWX08] T. Grubba, K. Weihrauch, and Y. Xu. Effectivity on Continuous Functions in Topological Spaces. *Electronic Notes in Theoretical Computer Science*, 202:237–254, 2008.

[Her02] P. Hertling. A Comparison of certain Representations of Regularly Closed Sets. *Electronic Notes in Theoretical Computer Science*, 66(1):65–78, 2002.

[Hoo90] H.J. Hoover. Feasible Real Functions and Arithmetic Circuits. *SIAM Journal on Computing*, 19(1):182–204, 1990.

[Hoy12] M. Hoyrup. On the Inversion of Computable Functions. 2012.

[HT03] C.M. Homan and M. Thakur. One-Way Permutations and Self-Witnessing Languages. *Journal of Computer and System Sciences*, 67:608–622, 2003.

[Kaw11] A. Kawamura. *Computational Complexity in Analysis and Geometry*. PhD thesis, University of Toronto, 2011.

[KC96] B.M. Kapron and S.A. Cook. A new Characterization of Type-2 Feasibility. *SIAM Journal on Computing*, 25(1):117–132, 1996.

[KC12] A. Kawamura and S.A. Cook. Complexity Theory for Operators in Analysis. *ACM Transactions on Computation Theory (TOCT)*, 4(2):5, 2012.

[KF82] Ker-I Ko and H. Friedman. Computational Complexity of Real Functions. *Theoretical Computer Science*, 20(3):323–352, 1982.

[KMRZ12] A. Kawamura, N.Th. Müller, C. Rösnick, and M. Ziegler. Parameterized Uniform Complexity in Numerics: from Smooth to Analytic, from NP-Hard to Polytime. *CoRR*, abs/1211.4974, 2012.

[Ko85] Ker-I Ko. On Some Natural Completeness Operators. *SIAM Journal on Computing*, 37(1):1–30, 1985.

[Ko91] Ker-I Ko. *Complexity Theory of Real Functions*. Birkhäuser Boston Inc., 1991.

[Ko98] K. Ko. Polynomial-Time Computability in Analysis. *Studies in Logic and the Foundations of Mathematics*, 139:1271–1317, 1998.

[KO01] U. Kohlenbach and P.B. Oliva. Effective Bounds on Strong Unicity in L_1-Approximation. *Basic Research in Computer Science (BRICS)*, 14, 2001.

[Koh90] U. Kohlenbach. *Theorie der majorisierbaren und stetigen Funktionale und ihre Anwendungen bei der Extraktion von Schranken aus inkonstruktiven Beweisen: Effektive Eindeutigkeitsmodule bei besten Approximationen aus ineffektiven Beweisen*. PhD thesis, Goethe-Universität Frankfurt (Main), 1990.

[Koh93] U. Kohlenbach. Effective Moduli from ineffective uniqueness proofs. An Unwinding of de La Vallée Poussin's proof for Chebycheff approximation. *Num. Funct. Anal. Optim.*, 14:581–606, 1993.

[KS95] M. Kummer and M. Schäfer. Computability of Convex Sets. In *STACS* [con95], pages 550–561.

[KSZ13] A. Kawamura, F. Steinberg, and M. Ziegler. On the Computational Complexity of Poisson's and Laplace's Equation. In CCA13 [CCA13], page 138. Abstract.

[KY08] K. Ko and F. Yu. On the Complexity of Convex Hulls of Subsets of the Two-Dimensional Plane. *Electronic Notes in Theoretical Computer Science*, 202:121–135, 2008.

[Lam06] B. Lambov. The Basic Feasible Functionals in Computable Analysis. *Journal of Complexity*, 22(6):909–917, 2006. Computability and Complexity in Analysis.

[LLM01] S. Labhalla, H. Lombardi, and E. Moutai. Espaces métriques rationnellement présentés et complexité, le cas de l'espace des fonctions réelles uniformément continues sur un intervalle compact. *Theoretical Computer Science*, 250(1):265–332, 2001.

[McN08] T.H. McNicholl. A Uniformly Computable Implicit Function Theorem. *Mathematical Logic Quarterly*, 54(3):272–279, 2008.

[Meh76] K. Mehlhorn. Polynomial and Abstract Subrecursive Classes. *Journal of Computer and System Sciences*, 12(2):147–178, 1976.

[Mül87] N.Th. Müller. Uniform Computational Complexity of Taylor Series. In Ottmann [Ott87], pages 435–444.

[Mül00] N.Th. Müller. The iRRAM: Exact Arithmetic in C++. In Blanck et al. [BBH01], pages 222–252.

[Ott87] Thomas Ottmann, editor. *14th International Colloquium on Automata, Languages and Programming (ICALP 1987), Karlsruhe, Germany, Proceedings*, volume 267 of *Lecture Notes in Computer Science*. Springer, July 13–17 1987.

[Ret08] R. Rettinger. Computability and Complexity Aspects of Univariate Complex Analysis, 2008. Habilitation thesis.

[Ret13] R. Rettinger. Computational Complexity in Analysis. In CCA13 [CCA13], pages 100–109. Extended abstract.

[Rös14] Carsten Rösnick. *Parametrisierte uniforme Berechnungskomplexität in Geometrie und Numerik.* PhD thesis, Technische Universität Darmstadt, Wiesbaden, 2014.

[RW02] R. Rettinger and K. Weihrauch. The Computational Complexity of Some Julia Sets. *Electronic Notes in Theoretical Computer Science*, 66(1):154–164, 2002.

[Sch02] M. Schröder. Extended Admissibility. *Theoretical Computer Science*, 284:519–538, 2002.

[TWW88] J.F. Traub, G.W. Wasilkowski, and H. Woźniakowski. *Information-Based Complexity.* Academic Press, 1988.

[Wei00] K. Weihrauch. *Computable Analysis: An Introduction.* Springer, 2000.

[WK87] K. Weihrauch and C. Kreitz. Representations of the Real Numbers and of the Open Subsets of the Set of Real Numbers. *Annals of Pure and Applied Logic*, 35:247–260, 1987.

[ZB04] M. Ziegler and V. Brattka. Computability in Linear Algebra. *Theoretical Computer Science*, 326(1):187–211, 2004.

[Zho96] Q. Zhou. Computable Real-Valued Functions on Recursive Open and Closed Subsets of Euclidean Space. *Mathematical Logic Quarterly*, 42(1):379–409, 1996.

[Zie02] M. Ziegler. Computability on Regular Subsets of Euclidean Space. *Mathematical Logic Quarterly*, 48(1):157–181, 2002.

[Zie04] M. Ziegler. Computable Operators on Regular Sets. *Mathematical Logic Quarterly*, 50(4-5):392–404, 2004.

[Zie06] M. Ziegler. Effectively Open Real Functions. *Journal of Complexity*, 22(6):827–849, 2006.

[ZM08] X. Zhao and N.Th. Müller. Complexity of Operators on Compact Sets. *Electronic Notes in Theoretical Computer Science*, 202:101–119, 2008.