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UNIQUENESS OF DIRECTED COMPLETE POSETS BASED ON SCOTT CLOSED SET LATTICES

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Abstract. In analogy to a result due to Drake and Thron about topological spaces, this paper studies the dcpos (directed complete posets) which are fully determined, among all dcpos, by their lattices of all Scott-closed subsets (such dcpos will be called C_σ-unique). We introduce the notions of down-linear element and quasicontinuous element in dcpos, and use them to prove that dcpos of certain classes, including all quasicontinuous dcpos as well as Johnstone's and Kou's examples, are C_σ-unique. As a consequence, C_σ-unique dcpos with their Scott topologies need not be bounded sober.

1. Introduction

From a result by Drake and Thron in [1], one deduces the following result (see Fact 3 in the Appendix): a topological space X has the property that $C(X)$ isomorphic to $C(Y)$ implies X is homeomorphic to Y iff X is sober and T_D (every derived set $d(\{x\}) = cl(\{x\}) - \{x\}$ of point $x \in X$ is closed), where $C(X)$ and $C(Y)$ denote the lattices of closed sets of X and T_0 space Y, respectively (see also [13], line 11-13, page 504).

For any dcpo P, let $C_\sigma(P)$ denote the lattice of all Scott closed subsets of P (with the inclusion order). A directed complete poset (or dcpo, for short) P will be called a C_σ-unique dcpo (or C_σ-unique, for short) if for any dcpo Q, P is isomorphic to Q whenever the lattices $C_\sigma(P)$ and $C_\sigma(Q)$ are isomorphic. From a counterexample constructed in [6] recently, we know that not every dcpo is C_σ-unique. It is therefore natural to ask which dcpos are C_σ-unique. One of the classic results in domain theory is that a dcpo P is continuous iff the lattice $C_\sigma(P)$ is a completely distributive lattice (Theorem II-1.14 of [2]). From this it follows that every continuous dcpo is sober and C_σ-unique. In a similar way, one can deduce that every quasicontinuous dcpo is sober and C_σ-unique. Compared with Drake's

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and Thron's result, one naturally asks whether every C_σ-unique dcpo is sober in their Scott topology.

In [9], Johnstone constructed the first dcpo whose Scott topology is not sober. Later Isbell [8] constructed a complete lattice whose Scott topology is not sober. Kou [10] further gave a dcpo whose Scott topology is well-filtered but not sober. In this paper, we will introduce the concepts of quasicontinuous element and down-linear element in dcpos. With these concepts we identify some classes of C_σ-unique dcpos, that include all quasicontinuous dcpos as well as Johnstone's and Kou's examples. The full characterization of C_σ-unique dcpos is still open.

2. Preliminaries

For any subset A of a poset P, let $\uparrow A = \{x \in P : y \leq x \text{ for some } y \in A\}$ and $\downarrow A = \{x \in P : x \leq y \text{ for some } y \in A\}$. A subset A is called an upper set if $A = \uparrow A$, and a lower set if $A = \downarrow A$. A subset U of a poset P is Scott open if (i) $U = \uparrow U$ and (ii) for any directed subset D, $\bigvee D \in U$ implies $D \cap U \neq \emptyset$, whenever $\bigvee D$ exists. All Scott open sets of a poset P form a topology on P, denoted by $\sigma(P)$ and called the Scott topology on P. The complements of Scott open sets are called Scott closed sets. Clearly, a subset A is Scott closed iff (i) $A = \downarrow A$ and (ii) for any directed subset $D \subseteq A$, $\bigvee D \in A$ whenever $\bigvee D$ exists. The set of all Scott closed sets of P will be denoted by $C_\sigma(P)$. The space $(P, \sigma(P))$ is denoted by ΣP.

A poset P is directed complete if its every directed subset has a supremum. A directed complete poset is briefly called a dcpo.

A subset A of a topological space is irreducible if whenever $A \subseteq F_1 \cup F_2$ with F_1 and F_2 closed, then $A \subseteq F_1$ or $A \subseteq F_2$ holds. The set of all nonempty irreducible closed subsets of space X will be denoted by $Irr(X)$.

For any T_0 topological space (X, τ), the specialization order $\leq_\tau$ on X is defined by $x \leq_\tau y$ iff $x \in cl(\{y\})$ where "$cl(\cdot)$" means taking closure.

Remark 2.1.

(1) For any topological space X, $(Irr(X), \subseteq)$ is a dcpo. If $\mathcal{D}$ is a directed subset of $Irr(X)$, the supremum of $\mathcal{D}$ in $(Irr(X), \subseteq)$ equals $cl(\bigcup \mathcal{D})$ (the closure of $\bigcup \mathcal{D}$), which is the same as the supremum of $\mathcal{D}$ in the complete lattice of all closed sets of X.

(2) For any $x \in X$, $cl(\{x\}) \in Irr(X)$. A T_0 space X is called sober if $Irr(X) = \{cl(\{x\}) : x \in X\}$, that is, every nonempty irreducible closed set is the closure of a point.

(3) Assume that (X, τ) and (Y, η) are topological spaces such that the open set lattices $(\tau, \subseteq)$ and $(\eta, \subseteq)$ of X and Y are isomorphic, then the closed set lattice $(C(X), \subseteq)$ of X and the closed set lattice $(C(Y), \subseteq)$ of Y are also isomorphic (they are dual to the corresponding open set lattices). Since irreducibility is a lattice-intrinsic property of elements, it follows that the posets $(Irr(X), \subseteq)$ and $(Irr(Y), \subseteq)$ are isomorphic.

For a T_0 space X, a sobrification of X is a sober space Y together with a continuous mapping $\eta_X : X \longrightarrow Y$, such that for any continuous mapping $f : X \longrightarrow Z$ with Z sober, there is a unique continuous mapping $\hat{f} : Y \longrightarrow Z$ satisfying $f = \hat{f} \circ \eta_X$. The sobrification of a T_0 space is unique up to homeomorphism. Clearly if a space X is sober, then X is homeomorphic to any sobrification of X.

Remark 2.2. The following facts about sober spaces and sobrifications are well-known.

(1) The set $Irr(X)$ of all nonempty closed irreducible sets of a T_0 space X equipped with the hull-kernel topology is a sobrification of X, where the mapping $\eta_X : X \longrightarrow Irr(X)$ is defined by $\eta_X(x) = cl(\{x\})$ for all $x \in X$. The closed sets of the hull-kernel topology consists of all sets of the form $h(A) = \{F \in Irr(X) : F \subseteq A\}$ (A is a closed set of X). So the sobrification of a space X is totally determined by the lattice $C(X)$. (See Exercise V-4.9 of [2] for details, where the topology was given by means of open sets).

(2) If X and Y are both sober spaces and the closed set lattices $C(X)$ and $C(Y)$ are homomorphic, then the sobrification of X and that of Y are homeomorphic. Hence X and Y are homeomorphic.

(3) From (1) and (2), we easily deduce that if Y is a sober space, then Y is a sobrification of a T_0 space X iff the closed set lattices $C(X)$ and $C(Y)$ are isomorphic (equivalently, the open set lattice of Y is isomorphic to that of X).

A T_0 space X will be called Scott sobrifiable if there is a dcpo P such that the Scott space ΣP is a sobrification of X.

For any T_0 space (X, τ), let $\leq_\tau$ be the specialization order on X ($x \leq_\tau y$ iff $x \in cl(\{y\})$). It is well-known that the specialization order on the Scott space ΣP of a poset P coincides with the original order on P. Thus a T_0 space (X, τ) is homeomorphic to $\Sigma\, P$ for some poset P iff (X, τ) is homeomorphic to the Scott space $\Sigma(X, \leq_\tau)$ of the poset $(X, \leq_\tau)$. The specialization order on the space $Irr(X)$ (with the hull-kernel topology) equals the inclusion order of sets. From the above, we can easily deduce the following fact.

Remark 2.3. A T_0 space (X, τ) is Scott sobrifiable iff for any Scott closed set $\mathcal{F}$ of the dcpo $Irr(X)$, there is a closed set A of X such that $\mathcal{F} = h(A)$, where $h(A) = \{F \in Irr(X) : F \subseteq A\}$.

A topological space (X, τ) is called a d-space (or monotone convergence space) if (i) X is T_0, (ii) the poset $(X, \leq_\tau)$ is a dcpo, and (iii) for any directed subset $D \subseteq (X, \leq_\tau)$, D converges (as a net) to $\bigvee D$.

Remark 2.4.

(1) Every sober space is a d-space.

(2) Every Scott space ΣP of a dcpo P is a d-space.

(3) If (X, τ) is a d-space, then every closed set F of X is a Scott closed set of the dcpo $(X, \leq_\tau)$.

Lemma 2.5. *Let (X, τ) be a d-space. If $\{x_i : i \in I\}$ is a directed subset of $(X, \leq_\tau)$, then the supremum $\sup\{cl(\{x_i\}) : i \in I\}$ of $\{cl(\{x_i\}) : i \in I\}$ in $Irr(X)$ equals $cl(\{x\})$, where $x = \bigvee\{x_i : i \in I\}$.*

For more about dcpos, Scott topology and related topics we refer the reader to [2] and [3].

3. Main results

In this section, we identify some classes of C_σ-unique dcpos, using irreducible sets, down-linear elements, quasicontinuous elements and the M property, respectively.

A T_0 space is called bounded-sober if every nonempty upper bounded (with respect to the specialization order on X) closed irreducible subset of the space is the closure of a point [14]. Every sober space is bounded-sober, the converse implication is not true. If X is a T_0 space such that every irreducible closed *proper* subset is the closure of an element, then X is bounded-sober. In the following, a dcpo whose Scott topology is sober (bounded-sober) will be simply called a sober (bounded-sober) dcpo.

Lemma 3.1. *For a bounded-sober dcpo P, ΣP is Scott sobrifiable if and only if P is sober.*

Proof. We only need to check that if ΣP is not sober, then it is not Scott sobrifiable. Since ΣP is not sober, there is a nonempty irreducible closed set F such that F is not the closure of any point. From the assumption that ΣP is bounded-sober, one can verify that the set $\mathcal{F} = \downarrow_{Irr(\Sigma P)} \{cl(\{x\}) : x \in F\}$ consists precisely of the elements $cl(\{x\})$ $(x \in F)$, and is a Scott closed set of $Irr(\Sigma P)$. But any closed set B of ΣP containing all $cl(\{x\}) (x \in F)$ must contain F, thus $h(B) \neq \mathcal{F}$. By Remark 2.3, ΣP is not Scott sobrifiable. □

In the following, we shall write $P \cong Q$ if the two posets P and Q are isomorphic.

Theorem 3.2. *Let P be a sober dcpo. For any bounded-sober dcpo Q, if $C_\sigma(P) \cong C_\sigma(Q)$ then $P \cong Q$.*

Proof. Let Q be a bounded-sober dcpo such that $C_\sigma(P) \cong C_\sigma(Q)$. Then, by Remark 2.2 (3), ΣP is a sobrification of ΣQ. Thus ΣQ is Scott sobrifiable. By Lemma 3.1, ΣQ is also sober. Therefore, by Remark 2.2 (2), ΣP and ΣQ are homeomorphic, which then implies $P \cong Q$. □

Definition 3.3. An element a of a poset P is called down-linear if the subposet $\downarrow a = \{x \in P : x \leq a\}$ is a chain (for any $x_1, x_2 \in \downarrow a$, it holds that either $x_1 \leq x_2$ or $x_2 \leq x_1$).

Lemma 3.4. *Let (X, τ) be a d-space.*

1. *If $F \in Irr(X)$ is a down-linear element of the poset $Irr(X)$, then there exists an $x \in X$ such that $F = cl(\{x\})$.*
2. *If $F \in Irr(X)$ equals the supremum of a directed set of down-linear elements of $Irr(X)$, then $F = cl(\{x\})$ for some $x \in X$.*

Proof.

(1) First, the set $\{cl(\{x\}) : x \in F\}$ is a subset of $\downarrow F$ in $Irr(X)$, so it is a chain. Thus $\{x : x \in F\}$ is a chain of $(X, \leq_\tau)$. Since X is a d-space, $\widehat{x} = \sup\{x : x \in F\}$ exists. Then, noticing that F is closed, we have $\widehat{x} \in F$ by Remark 2.4 (3). Then $F \subseteq cl(\{\widehat{x}\}) \subseteq F$, implying $cl(\{\widehat{x}\}) = F$.

(2) Let F be the supremum of a directed set of down-linear irreducible closed sets in $Irr(X)$. Then by (1), $F = \sup\{cl(\{x_i\}) : i \in I\}$ in $Irr(X)$, where $\{cl(\{x_i\}) : i \in I\}$ is a directed family. Thus, $\{x_i : i \in I\}$ is a directed set of $(X, \leq_\tau)$. Again, as X is a d-space, $x = \sup\{x_i : i \in I\}$ exists. By Lemma 2.5, $cl(\{x\}) = \sup\{cl(\{x_i\}) : i \in I\} = F$. □

In the following, for a dcpo P, we shall use $Irr_\sigma(P)$ to denote the dcpo of all nonempty irreducible Scott closed subsets of P. Without specification, irreducible sets of a poset mean the irreducible sets with respect to the Scott topology.

Theorem 3.5. *Let P be a dcpo satisfying the following condition*

(DL-sup): *for any proper irreducible Scott closed set F, F is either a down-linear element of $Irr_\sigma(P)$ or it is the supremum of a directed set of down-linear elements of $Irr_\sigma(P)$.*

Then P is C_σ-unique.

Proof. Let dcpo P satisfy the above condition (DL-sup) and Q be a dcpo such that $C_\sigma(P) \cong C_\sigma(Q)$.

(1) By Lemma 3.4, if $F \in Irr_\sigma(P)$ and $F \neq P$, then $F = cl(\{x\})$ for some point.

(2) Since $C_\sigma(P) \cong C_\sigma(Q)$, Q also satisfies condition (DL-sup). So every nonempty closed irreducible proper subset of ΣQ is the closure of a point.

(3) Let F be a nonempty irreducible closed subset of P with an upper bound a. If $F \neq P$, then F is the closure of some point by (1). Otherwise $F = P$, thus $a \in P$ is the largest element in P, hence $F = P = \downarrow a = cl(\{a\})$. Therefore ΣP is bounded-sober. Similarly ΣQ is bounded-sober. If either ΣP or ΣQ is sober, then by Theorem 3.2, $P \cong Q$. Assume now that neither ΣP nor ΣQ is sober. Then there is a nonempty irreducible closed set F of P, which is not the closure of a singleton set. But by (1) and (2), F cannot be a proper subset, so $F = P$. Thus P is an irreducible closed set which does not equal to the closure of any singleton set. Similarly, Q is an irreducible closed set which is not the closure of any singleton set. Note that in this case, P and Q are the top elements of $Irr_\sigma(P)$ and $Irr_\sigma(Q)$, respectively. Thus $Q \cong \{cl(\{y\}) : y \in Q\} \cong Irr_\sigma(Q) - \{Q\} \cong Irr_\sigma(P) - \{P\} \cong \{cl(\{x\}) : x \in P\} \cong P$, as desired. □

Example 3.6. In [9], Johnstone constructed the first non-sober dcpo as $X = \mathbb{N} \times (\mathbb{N} \cup \{\infty\})$ with partial order defined by

$$(m, n) \leq (m', n') \Leftrightarrow \text{either } m = m' \text{ and } n \leq n' \text{or } n' = \infty \text{ and } n \leq m'.$$

Then

(a) $(X, \leq)$ is a dcpo, X is irreducible and $X \neq cl(\{x\})$ for any $x \in X$.
(b) If F is a proper irreducible Scott closed set of X, then $F = \downarrow(m, n)$ for some $(m, n) \in X$.
(c) If $n \neq \infty$, $\downarrow(m, n)$ is a down-linear element of $Irr_\sigma(X)$. If $n = \infty$, then $\downarrow(m, n)$ is the supremum of the chain $\{\downarrow(m, k) : k \neq \infty\}$ whose members are down-linear.

Hence by Theorem 3.5, we deduce that dcpo $X = \mathbb{N} \times (\mathbb{N} \cup \{\infty\})$ is C_σ-unique. Thus an C_σ-unique dcpo need not be sober.

Next, we provide a class of C_σ-unique dcpos via quasicontinuous elements.

Remark 3.7 (cf. [11])**.** Let A be a nonempty Scott closed set of a dcpo P. Then

(i) A is a dcpo.
(ii) For any subset $B \subseteq A$, B is a Scott closed set of dcpo A iff it is a Scott closed set of P. Thus $C_\sigma(A) = \downarrow_{C_\sigma(P)} A = \{B \in C_\sigma(P) : B \subseteq A\}$.

A finite subset F of a dcpo P is way-below an element $a \in P$, denoted by $F \ll a$, if for any directed subset $D \subseteq P$, $a \leq \bigvee D$ implies $D \cap \uparrow F \neq \emptyset$. A dcpo P is quasicontinuous if for any $x \in P$, the family

$$\mathit{fin}(x) = \{F : F \text{ is finite and } F \ll x\}$$

is a directed family (for any $F_1, F_2 \in \mathit{fin}(x)$ there is $F \in \mathit{fin}(x)$ such that $F \subseteq \uparrow F_1 \cap \uparrow F_2$) and for any $x \not\leq y$ there is $F \in \mathit{fin}(x)$ satisfying $y \notin \uparrow F$ (see Definition III-3.2 of [2]). Every continuous dcpo is quasicontinuous.

Every quasicontinuous dcpo is sober (Proposition III-3.7 of [2]). A dcpo P is quasicontinuous iff the Scott open set lattice $\sigma(P)$ of P is hypercontinuous (Theorem VII-3.9 of [2]). Assume that P is a quasicontinuous dcpo and Q is a dcpo such that $C_\sigma(P)$ is isomorphic to $C_\sigma(Q)$. Then $\sigma(P)$ (it is dual to $C_\sigma(P)$) is isomorphic to $\sigma(Q)$(it is dual to $C_\sigma(Q)$, thus $\sigma(Q)$ is also hypercontinuous, implying that Q is quasicontinuous. Thus both ΣP and ΣQ are sober spaces and they have isomorphic closed set lattices, hence by Theorem 3.2, we have $P \cong Q$. From this we obtain the following lemma.

Lemma 3.8. *Every quasicontinuous dcpo is C_σ-unique.*

An element x of a dcpo P is called a quasicontinuous element if the sub-dcpo $\downarrow x$ is a quasicontinuous dcpo.

Theorem 3.9. *Let P be a dcpo. Then P is C_σ-unique if it satisfies the following two conditions:*

(1) *ΣP is bounded sober;*
(2) *every element of P is the supremum of a directed set of quasicontinuous elements.*

Proof. Assume that P is a dcpo satisfying the two conditions. Let Q be a dcpo and $F : C_\sigma(P) \longrightarrow C_\sigma(Q)$ be an isomorphism. Then F restricts to an isomorphism $F : Irr_\sigma(P) \longrightarrow Irr_\sigma(Q)$.

(1) Let $x \in P$ be a quasicontinuous element. Then $F(\downarrow x)$ is in $C_\sigma(Q)$ and, by Remark 3.7, $C_\sigma(\downarrow x) = \{B \in C_\sigma(P) : B \subseteq \downarrow x\} = \downarrow_{C_\sigma(P)} (\downarrow x)$ is isomorphic via F to $\downarrow_{C_\sigma(Q)} F(\downarrow x) = \{E \in C_\sigma(Q) : E \subseteq F(\downarrow x)\} = C_\sigma(F(\downarrow x))$ (all Scott closed sets of $F(\downarrow x)$). Since the dcpo $\downarrow x$ is quasicontinuous, it is C_σ-unique. Hence the dcpo $\downarrow x$ is isomorphic to the dcpo $F(\downarrow x)$, implying that there is a largest element in $F(\downarrow x)$, denoted by $f(x)$. Hence $F(\downarrow x) = \downarrow f(x)$. It is easily observable that the mapping f is well defined on the set of quasicontinuous elements of P, and for any two quasicontinuous elements $x_1, x_2 \in P$, $f(x_1) \leq f(x_2)$ iff $x_1 \leq x_2$.

(2) If $x \in P$ is the supremum of a directed set $\{x_i : i \in I\}$ of quasicontinuous elements x_i, then

$$\begin{aligned} F(\downarrow x) &= F(\sup_{Irr_\sigma(P)}\{\downarrow x_i : i \in I\}) \\ &= \sup_{Irr_\sigma(Q)}\{F(\downarrow x_i) : i \in I\} \\ &= \sup_{Irr_\sigma(Q)}\{\downarrow f(x_i) : i \in I\} \\ &= \downarrow y_x, \end{aligned}$$

where $y_x = \sup_Q\{f(x_i) : i \in I\}$ and $f(x_i)$ is the element in Q defined for quasicontinuous elements x_i in (1). Let $f(x) = y_x$ again.

Thus we have a monotone mapping $f : P \longrightarrow Q$. Following that F is an isomorphism, we have that $f(x_1) \geq f(x_2)$ iff $x_1 \geq x_2$. It remains to show that f is surjective.

(3) If $y \in \downarrow f(P)$, then $\downarrow y \subseteq \downarrow f(x) = F(\downarrow x)$ for some $x \in P$. Since F restricts to an isomorphism between the dcpos $Irr_\sigma(P)$ and $Irr_\sigma(Q)$, there is $H \in Irr_\sigma(P)$ such that $H \subseteq \downarrow x$ and $F(H) = \downarrow y$. But P is bounded-sober, so $H = \downarrow x'$ for some $x' \in P$. It follows that $y = f(x')$, implying $y \in f(P)$. Therefore $f(P)$ is a lower set of Q. Also clearly $f(P)$ is closed under sups of directed set, so it is a Scott closed subset of Q.

(4) Since F is an isomorphism between the lattices $C_\sigma(P)$ and $C_\sigma(Q)$, P and Q are the top elements in the respective lattices, we have that $Q = F(P) = F(\sup_{C_\sigma(P)}\{\downarrow x : x \in P\}) = \sup_{C_\sigma(Q)}\{F(\downarrow x) : x \in P\} = \sup_{C_\sigma(Q)}\{\downarrow f(x) : x \in P\}$.

For each $x \in P$, $\downarrow f(x) \subseteq f(P)$ because $f(P)$ is a Scott closed set of Q, it holds then that $sup_{C_\sigma(Q)}\{\downarrow f(x) : x \in P\} \subseteq f(P)$. Therefore $Q \subseteq f(P)$, which implies $Q = f(P)$. Hence f is also surjective. The proof is thus completed. □

If $x \in P$ is a down-linear element of a dcpo P, then $\downarrow x$ is a chain, so it is continuous (hence quasicontinuous).

Corollary 3.10. *If P is a dcpo satisfying the following conditions, then P is C_σ-unique:*

(1) *P is bounded-sober.*

(2) *every element $a \in P$ is the supremum of a directed set of down-linear elements.*

Example 3.11. In order to answer the question whether every well-filtered dcpo is sober posed by Heckmann [5], Kou [10] constructed another non-sober dcpo P as follows:

Let $X = \{x \in \mathbb{R} : 0 < x \leq 1\}$, $P_0 = \{(k, a, b) \in \mathbb{R} : 0 < k < 1, 0 < b \leq a \leq 1\}$ and

$$P = X \cup P_0.$$

Define the partial order $\sqsubseteq$ on P as follows:

(i) for $x_1, x_2 \in X$, $x_1 \sqsubseteq x_2$ iff $x_1 = x_2$;
(ii) $(k_1, a_1, b_1) \sqsubseteq (k_2, a_2, b_2)$ iff $k_1 \leq k_2, a_1 = a_2$ and $b_1 = b_2$.
(iii) $(k, a, b) \sqsubseteq x$ iff $a = x$ or $kb \leq x < b$.

If $u = (h, a, b) \in P_0$, then $\downarrow u = \{(k, a, b) : k \leq h\}$ is a chain. If $u = x \in P_0$, then $u = \bigvee\{(k, x, x) : 0 < k < 1\}$, where each (k, x, x) is a down-linear element and $\{(k, x, x) : 0 < k < 1\}$ is a chain. Thus P satisfies (2) of Corollary 3.10.

Let F be an irreducible nonempty Scott closed set of P with an upper bound v. If $v = (h, a, b) \in P_0$, then $F \subseteq\downarrow(h, a, b) = \{(k, a, b) : k \leq h\}$. Take $m = \bigvee\{k : (k, a, b) \in F\}$. Then $F =\downarrow(m, a, b)$, is the closure of point (m, a, b).

Now assume that F does not have an upper bound in P_0, then $v = x$ for some $x \in P_0$. If $v \notin F$, then due to the irreducibility of F, there exist a, b such that $F \subseteq \{(k, a, b) : 0 < k < 1\}$, which will imply that F has an upper bound of the form (m, a, b), contradicting the assumption. Therefore $v \in F$, implying that $F =\downarrow v$ (note that $F =\downarrow F$ is a lower set) is the closure of point v. It thus follows that P satisfies (1) as well. By Corollary 3.10, P is C_σ-unique.

Next, we give another class of C_σ-unique dcpos. In [7], Ho and Zhao introduced the following notions.

Definition 3.12. Let L be a poset and $x, y \in L$. The element x is *beneath* y, denoted by $x \prec y$, if for every nonempty Scott-closed set $S \subseteq L$ with $\bigvee S$ existing, $y \leq \bigvee S$ implies $x \in S$. An element x of L is called *C-compact* if $x \prec x$. Let $\kappa(L)$ denote the set of all the C-compact elements of L.

Let P be a poset and $A \subseteq P$ finite. The set $mub(A)$ of minimal upper bounds of A is said to be complete, if for any upper bound x of A, there exists $y \in mub(A)$ such that $y \leq x$. A poset P is said to satisfy the property m, if for all finite sets $A \subseteq P$, $mub(A)$ is complete. A poset P is said to satisfy the property M, if P satisfies the property m and for all finite set $A \subseteq P$, $mub(A)$ is finite.

Remark 3.13. Let L be a complete lattice and $a \in L$ be a C-compact element. If $x, y \in L$ such that $a \leq x \vee y$, then $a \leq \bigvee(\downarrow x \cup \downarrow y)$ and $\downarrow x \cup \downarrow y$ is Scott closed, so $a \in\downarrow x \cup \downarrow y$, implying $a \leq x$ or $a \leq y$. Thus a is $\vee$-irreducible.

Corollary 3.14. *For any dcpo P, $\kappa(C_\sigma(P)) \subseteq Irr_\sigma(P)$. That is, all C-compact Scott closed sets are irreducible.*

Lemma 3.15 [4]**.** *Let P be a dcpo. Then*

(1) *For all $x \in P$, $\downarrow x \in \kappa(C_\sigma(P))$.*
(2) *If P satisfies the property M, then $A \in \kappa(C_\sigma(P))$ iff $A =\downarrow x$ for some $x \in P$.*

Theorem 3.16. *If P is a dcpo satisfying the property M and the condition (2) in Corollary 3.10, then P is C_σ-unique.*

Proof. Let P be a dcpo satisfying the condition (2) in Corollary 3.10 and the property M. Assume that Q is a dcpo and there is an order isomorphism $H : C_\sigma(P) \to C_\sigma(Q)$. Then the restrictions $H : \kappa(C_\sigma(P)) \to \kappa(C_\sigma(Q))$ and $H : Irr_\sigma(P) \to Irr_\sigma(Q)$ are all order isomorphisms.

For all $q \in Q$, by Lemma 3.15(1), $\downarrow q \in \kappa(C_\sigma(Q))$, then $H^{-1}(\downarrow q) = \downarrow x_q$ for a unique $x \in P$ by Lemma 3.15(2). Now define a map $h' : Q \to P$ such that $h'(q) = x_q$ iff $H^{-1}(\downarrow q) = \downarrow x_q$. The mapping h' is monomorphic and order preserving since H^{-1} is. Note that $\kappa(C_\sigma(Q)) \cong \kappa(C_\sigma(P)) \cong P$ is a dcpo.

Now let x be any element of P.

(i) If x is down-linear, then $H(\downarrow x)$ is a linear subset in Q (if $y_1, y_2 \in H(\downarrow x)$, then $h'(y_1), h'(y_2) \in \downarrow x$), and is Scott closed. The supremum $\sup_Q H(\downarrow x)$ exists and is in $H(\downarrow x)$. Thus $H(\downarrow x) = \downarrow q_x$ for some $q_x \in Q$.

(ii) If x is not down-linear, then x is the supremum of a directed set C of down-linear elements. Since H preserves sups in $\kappa(C_\sigma(P))$ and $\kappa(C_\sigma(Q))$, we have that

$$\begin{aligned} H(\downarrow x) &= H(\downarrow \sup C) \\ &= H(\sup_{\kappa(C_\sigma(P))}\{\downarrow c : c \in C\}) \\ &= \sup_{\kappa(C_\sigma(Q))}\{H(\downarrow c) : c \in C\} \\ &= \downarrow \sup_Q \{q_c : \downarrow q_c = H(\downarrow c), c \in C\} \\ &= \downarrow q_x, \end{aligned}$$

for some $q_x \in Q$.

By these facts, we defined a mapping $h : P \to Q$ such that $h(x) = q_x$ iff $F(\downarrow x) = \downarrow q_x$. It is then easy to see that h is monomorphic and order preserving since H is. In addition, it is easy to verify that h' is the inverse of h, hence h is an order isomorphism between P and Q, as desired. □

Note that Kou's and Johnstone's examples of non-sober dcpos do not have the property M.

4. Remarks and some possible further work

We close the paper with some additional remarks and problems for further exploration.

Remark 4.1.

(1) If P is a C_σ-unique dcpo and P^* is the dcpo obtained by adding a top element to P, then one can show that P^* is also C_σ-unique. Let X be the dcpo of Johnstone. Then X^* is C_σ-unique, but X^* is not bounded sober (X is an irreducible Scott closed set of X^* which is not the closure of any point of X^*). Thus a C_σ-unique dcpo need not

be bounded sober. So, bounded sobriety is not a necessary condition for a dcpo to be C_σ-unique.

(2) Recently, Ho, Goubault-Larrecq, Jung and Xi [6] constructed a pair of non-isomorphic dcpos having isomorphic Scott topologies, showing the existence of non-C_σ-unique dcpos. Their counterexample also reveals that sobriety is not a sufficient condition for a dcpo to be C_σ-unique.

In view of the above remarks, to identify larger classes of C_σ-unique dcpos and formulate a full characterization of C_σ-unique dcpos will be our future work.

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Appendix A.

In this part, for reader's convenience we present some details of the proof of a result, essentially due to Drake and Thron, on spaces which are uniquely determined (among all T_0 spaces) by means of their closed set lattices. (This part is requested by one of the referees). In [1], Drake and Thron proved the following result.

Corollary A.1. *Every representation family of a $\mathcal{C}$-lattice $(\Gamma, \geq)$ has exactly one element iff every irreducible element of Γ is strongly irreducible.*

Here a lattice is called a $\mathcal{C}$-lattice if it is isomorphic to the lattice $C(X)$ of all closed sets of a topological space X. An element a of a lattice L is called irreducible (strongly irreducible) iff a can not be expressed as the supremum of a finite (arbitrary) number of elements of L, which are strictly less than a.

By the definition of representation families of $\mathcal{C}$-lattices (see page 58 of [1]) we deduce the following fact, which is equivalent to the above Corollary A.1:

Fact A.2. A T_0 topological space X has the property that (for any T_0 space Y) $C(X)$ isomorphic to $C(Y)$ implies X is homeomorphic to Y if and only if every irreducible closed set in the space is strongly irreducible.

A space X is called a T_D space iff for any $x \in X$, the derived set $d(\{x\}) = cl(\{x\}) - \{x\}$ is a closed set (see Definition 2.1 of [12]). For example, every T_1 space is a T_D space.

Fact A.3. A topological space is both sober and T_D iff every irreducible closed set in the space is strongly irreducible.

Proof. First note that for any $\{A_i : i \in I\} \subseteq C(X)$, the supremum $\bigvee_{C(X)}\{A_i : i \in I\}$ of $A_i's$ in the lattice $C(X)$ equals $cl(\bigcup\{A_i : i \in I\})$.

Assume that the space X is both sober and T_D. Let F be an irreducible element of $C(X)$. Then $F = cl(\{x\})$ for some $x \in X$ because X is sober. Let $F = \bigvee_{C(X)}\{A_i : i \in I\}$ holds, where $A_i \in C(X)(i \in I)$. Then $\bigcup\{A_i : i \in I\} \subseteq F = cl(\{x\})$. Thus $A_i \subseteq cl(\{x\})$ for each i. If $cl(\{x\}) \neq A_i$ for every i, then $A_i \subseteq cl(\{x\}) - \{x\}$, therefore $\bigcup\{A_i : i \in I\} \subseteq cl(\{x\}) - \{x\}$. Since $cl(\{x\}) - \{x\}$ is closed, we have $F = cl(\bigcup\{A_i : i \in I\}) \subseteq cl(\{x\}) - \{x\}$, which contradicts $F = cl(\{x\})$. Hence $F = cl(\{x\}) = A_i$ for i, showing that F is strongly irreducible.

Now assume that every irreducible element of $C(X)$ is strongly irreducible. Let F be a non empty irreducible member of $C(X)$. Then $F = \bigvee_{C(X)}\{cl(\{x\}) : x \in F\}$, so $F = cl(\{x\})$ for some $x \in F$ because F is strongly irreducible. It follows that X is sober. Now let $x \in X$ be any element. Assume that $cl(\{x\}) - \{x\}$ is not closed. Then $cl(\{x\}) - \{x\}$ is a proper subset of $cl(cl(\{x\}) - \{x\})$. But trivially $cl(cl(\{x\}) - \{x\}) \subseteq cl(\{x\})$, thus $cl(cl(\{x\}) - \{x\}) = cl(\{x\})$. Thus $cl(\{x\}) = \bigvee_{C(X)}\{cl(\{y\}) : y \in cl(\{x\}) - \{x\}\}$. Since $cl(\{x\})$ is irreducible, it is strongly irreducible by the assumption, we have $cl(\{x\}) = cl(\{y\})$

for some $y \in cl(\{x\}) - \{x\}$, which is not possible because X is T_0. Therefore $cl(\{x\}) - \{x\}$ must be closed. Hence X is T_D. □

From Fact A.2 and Fact A.3 we derive the following result, first explicitly stated in [13] (page 504 line 11-13) with no proof (where sober spaces are called pc spaces).

Fact A.4. A space X has the property that (for any T_0 space Y) $C(X)$ isomorphic to $C(Y)$ implies X is homeomorphic to Y iff X is both sober and T_D.