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TAYLOR EXPANSION IN LINEAR LOGIC IS INVERTIBLE

DANIEL DE CARVALHO

Innopolis University, Universitetskaya St, 1, Innopolis, Respublika Tatarstan, 420500, Russia
e-mail address: d.carvalho@innopolis.ru

Abstract. Each Multiplicative Exponential Linear Logic (MELL) proof-net can be expanded into a differential net, which is its Taylor expansion. We prove that two different MELL proof-nets have two different Taylor expansions. As a corollary, we prove a completeness result for MELL: We show that the relational model is injective for MELL proof-nets, i.e. the equality between MELL proof-nets in the relational model is exactly axiomatized by cut-elimination.

In the seminal paper by Harvey Friedman [18], it has been shown that equality between simply-typed lambda-terms in the full typed structure $\mathcal{M}_X$ over an infinite set X is completely axiomatized by β and η: for any simply-typed lambda-terms v and u, we have $(\mathcal{M}_X \vDash v = u \Leftrightarrow v \simeq_{\beta\eta} u)$. Some variations, refinements and generalizations of this result have been provided by Gordon Plotkin [30] and Alex Simpson [33]. A natural problem is to know whether a similar result could be obtained for Linear Logic.

Such a result can be seen as a "separation" theorem. To obtain such separation theorems, it is a prerequisite to have a "canonical" syntax. When Jean-Yves Girard introduced Linear Logic (LL) [19], he not only introduced a sequent calculus system but also "proof-nets". Indeed, as for LJ and LK (sequent calculus systems for intuitionnistic and classical logic, respectively), different proofs in LL sequent calculus can represent "morally" the same proof: proof-nets were introduced to find a unique representative for these proofs.

The technology of proof-nets was completely satisfactory for the multiplicative fragment without units.[1] For proof-nets having additives, contractions or weakenings, it was easy to exhibit different proof-nets that should be identified. Despite some flaws, the discovery of proof-nets was striking. In particular, Vincent Danos proved by syntactical means in [6] the confluence of these proof-nets for the Multiplicative Exponential Linear Logic fragment (MELL). For additives, the problem to have a satisfactory notion of proof-net has been addressed in [23]. For MELL, a "new syntax" was introduced in [7]. In the original syntax, the following properties of the weakening and of the contraction did not hold:

- the associativity of the contraction;
- the neutrality of the weakening for the contraction;
- the contraction and the weakening as morphisms of co-algebras.

Key words and phrases: Linear Logic, Denotational semantics, Taylor expansion.

[1]For the multiplicative fragment with units, it has been recently shown [21, 22] that, in some sense, no satisfactory notion of proof-net can exist. Our proof-nets have no jump, so they identify too many sequent calculus proofs, but not more than the relational semantics.

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But they hold in the new syntax; at least for MELL, we got a syntax that was a good candidate to deserve being considered "canonical". Then trying to prove that any two (η-expanded) MELL proof-nets that are equal in some denotational semantics are β-joinable has become sensible and had at least the two following motivations:

- to prove the canonicity of the "new syntax" (if we quotient more normal proof-nets, then we would identify proof-nets having different semantics);
- to prove the confluence by semantic means (if a proof-net reduces to two cut-free proof-nets, then they have the same semantics, so they would be β-joinable, hence equal).

The problem of *injectivity*[2] of the denotational semantics for MELL, which is the question whether equality in the denotational semantics between (η-expanded) MELL proof-nets is exactly axiomatized by cut-elimination or not, can be seen as a study of the separation property with a semantic approach. The first work on the study of this property in the framework of proof-nets is [25] where the authors deal with the translation into LL of the pure λ-calculus; it has been studied more recently for the intuitionistic multiplicative fragment of LL [26] and for differential nets [27]. For Parigot's $\lambda\mu$-calculus, see [8] and [32].

Finally the precise problem of injectivity for MELL has been addressed by Lorenzo Tortora de Falco in his PhD thesis [34] and in [35] for the (multiset based) coherence semantics and the multiset based relational semantics. He gave partial results and counter-examples for the coherence semantics: the (multiset based) coherence semantics is not injective for MELL. Also, it was conjectured that the relational model is injective for MELL. It is worth mentioning that the injectivity of the relational model trivially entails the injectivity of other denotational semantics: non-uniform coherence semantics ([4] and [3]), finiteness spaces [15], weighted sets [1]...

We presented an abstract of a proof of this conjecture in [10]. This result can be seen as

- a semantic separation property in the sense of [18];
- a semantic proof of the confluence property;
- a proof of the "canonicity" of the new syntax of MELL proof-nets;
- a proof of the fact that if the Taylor expansions of two cut-free MELL proof-nets into differential nets coincide, then the two proof-nets coincide.

Differential proof-nets [17] are linear approximations of proof-nets that are meant to allow the expression of the Taylor expansion of proof-nets as infinite series of their linear approximations, which can be seen as a syntactic counterpart of quantitative semantics of Linear Logic (see [16] for an introduction to the topic). Now, in the present paper, we not only provide a fully detailed proof of this result, we also prove a more general result: We show that if the Taylor expansions of *any* two MELL proof-nets into differential nets coincide, then the two proof-nets coincide, i.e. we removed the assumption of the absence of cuts. Then the injectivity of the relational semantics becomes a corollary of this new result. By the way, the proof is essentially the same as before.[3]

[2]The tradition of the lambda-calculus community rather suggests the word "completeness" and the terminology of category theory rather suggests the word "faithfulness", but we follow here the tradition of the Linear Logic community.

[3]The two main differences are the following ones:

(1) The *pseudo-experiments* we consider are not necessarily induced by *experiments* any more, which means that we consider simple differential nets that might reduce to 0 and have no counterpart in the denotational semantics.
(2) The constraints on the basis k of the *k-heterogeneous pseudo-experiments* we consider are stronger.

In [35], a proof of the injectivity of the relational model is given for a weak fragment. But despite many efforts ([34], [35], [2], [28], [27], [29]...), all the attempts to prove the conjecture failed up to now. New progress was made in [13], where it has been proved that the relational semantics is injective for "connected" MELL proof-nets. Even though, there, "connected" is understood as a very strong assumption, the set of "connected" MELL proof-nets contains the fragment of MELL defined by removing weakenings and units. Actually [13] proved a much stronger result: in the full MELL fragment, two cut-free proof-nets R and R' with the same interpretation are the same up to the map associating auxiliary doors with their box (we say that they have the same LPS[4] - for instance, there are exactly four different proof-nets whose LPS is the LPS depicted in Figure 22, p. 22: These four proof-nets are the ones depicted in Figure 23, Figure 24, Figure 25 and Figure 26, p. 38). We wrote: "This result can be expressed in terms of differential nets: two cut-free proof-nets with different LPS have different Taylor expansions. We also believe this work is an essential step towards the proof of the full conjecture." Despite the fact we obtained a very interesting result about *all* the proof-nets (i.e. also for non-"connected" proof-nets[5]), the last sentence was a bit too optimistic, since, in the present paper, which presents a proof of the full conjecture, we could not use any previous result nor any previous technique/idea.

Let us give one more interpretation of its significance. First, notice that a proof of this result should consist in showing that, given two non β-equivalent proof-nets R and R', their respective semantics $[\![R]\!]$ and $[\![R']\!]$ are not equal, i.e. $[\![R]\!] \setminus [\![R']\!] \neq \emptyset$ or $[\![R']\!] \setminus [\![R]\!] \neq \emptyset$.[6] But, actually, we prove something much stronger: We prove that, given a proof-net R, there exist two points α and β such that, for any proof-net R', we have $(\{\alpha, \beta\} \subseteq [\![R']\!] \Leftrightarrow R \simeq_\beta R')$.

Now, the points of the relational model can be seen as non-idempotent intersection types[7] (see [9] and [11] for a correspondence between points of the relational model and System R - System R has also been studied recently in [5]). And the proof given in the present paper uses MELL types only to derive the normalization property; actually we prove the injectivity for cut-free proof-nets in an untyped framework:[8] Substituting the assumption that proof-nets are typed by the assumption that proof-nets are normalizable does not change anything to the proof.[9] In [12], we gave a semantic characterization of normalizable untyped proof-nets and we characterized "head-normalizable" proof-nets as proof-nets having a non-empty interpretation in the relational semantics, while [14] gave a characterization of strongly normalizable untyped proof-nets *via* non-idempotent intersection

[4]The LPS of a cut-free proof-net is the graph obtained by forgetting the outline of the boxes but keeping the trace of the auxiliary doors. The acronym LPS originally stands for "Linear Proof-Structure"; this terminology might be misleading since the LPS is much more informative than the result of an injective 1-experiment but is well-established, so we keep the acronym forgetting what it stood for.

[5]and even adding the MIX rule

[6]The converse, i.e. two β-equivalent proof-nets have the same semantics, holds by soundness.

[7]Idempotency of intersection ($\alpha \cap \alpha = \alpha$) does not hold.

[8]Our proof even works for "non-correct" proof-structures (correctness is the property characterizing nets corresponding in a typed framework with proofs in sequent calculus): we could expect that if the injectivity of the relational semantics holds for proof-nets corresponding with MELL sequent calculus, then it still holds for proof-nets corresponding with MELL+MIX sequent calculus, since the category **Rel** of sets and relations is a compact closed category. The paper [20] assuming correctness substituted in the proof the "bridges" of [13], which are essentially connected components (in the strong sense of the term since the notion of *bridge* ignores boxes - we will consider other "connected components" in our proof), by "empires", which, in contrast, discriminate between the connectors $\otimes$ and $\parr$.

[9]Except that we have to consider the *atomic* subset of the interpretation instead of the full interpretation (see Definition 4.10).

types. Principal typings in untyped λ-calculus are intersection types which allow to recover all the intersection types of some term. If, for instance, we consider the System R of [9] and [11], it is enough to consider some *injective* 1-*point*[10] to obtain the principal typing of an untyped λ-term. But, generally, for normalizable MELL proof-nets, *injective* k-*points*, for any k, are not principal typings; indeed, two cut-free MELL proof-nets having the same LPS have the same injective k-points for any $k \in \mathbb{N}$. In the current paper we show that a 1-point and a k-*heterogeneous point*[11] together allow to recover the interpretation of any normalizable MELL proof-net; by the way, our result cannot be improved in such a way that one point would be already enough for any MELL proof-net (see our Proposition 3.52). So, our work can also be seen as a first attempt to find a right notion of "principal typing" of intersection types in Linear Logic. As a consequence, the introduction of technologies allowing to compute directly by semantic means this principal typing should make possible normalization by evaluation, as in [31] for λ-calculus; that said, the complexity of such a computation is still unclear.

Section 1 formalizes untyped proof-structures (PS's) and typed proof-structures (typed PS's). Taylor expansion is defined in Section 2. Section 3 presents our algorithm leading from the Taylor expansion of R to the rebuilding of R and proves its correctness, which shows the invertibility of Taylor expansion (Corollary 3.51). Section 4 is devoted to show the completeness (the injectivity) of the relational semantics: for any typed PS's R and R', we have ($[\![R]\!] = [\![R']\!] \Leftrightarrow R \simeq_\beta R'$) (Corollary 4.37), where $\simeq_\beta$ is the reflexive symmetric transitive closure of the cut-elimination relation, by showing first that cut-free PS's are characterized by their relational interpretation (Theorem 4.20).

Notations. *We denote by ε the empty sequence.*

For any $n \geq 2$, for any $\alpha_1, \ldots, \alpha_{n+1}$, we define, by induction on n, the $(n+1)$-tuple $(\alpha_1, \ldots, \alpha_{n+1})$ by setting $(\alpha_1, \ldots, \alpha_{n+1}) = (\alpha_1, (\alpha_2, \alpha_3, \ldots, \alpha_{n+1}))$.

For any set E, we denote by $\mathfrak{P}(E)$ the set of subsets of E, by $\mathfrak{P}_{fin}(E)$ the set of finite subsets of E and by $\mathfrak{P}_2(E)$ the set $\{\mathcal{E}_0 \in \mathfrak{P}(E); \mathsf{Card}\,(\mathcal{E}_0) = 2\}$.

A multiset f of elements of some set $\mathcal{E}$ is a function $\mathcal{E} \to \mathbb{N}$; we denote by Supp$(f)$ the support of f *i.e. the set $\{e \in \mathcal{E}; f(e) \neq 0\}$. A multiset f is said to be* finite *if Supp(f) is finite. The set of finite multisets of elements of some set $\mathcal{E}$ is denoted by $\mathcal{M}_{fin}(\mathcal{E})$.*

If f is a function $\mathcal{E} \to \mathcal{E}'$, $x_0 \in \mathcal{E}$ and $y \in \mathcal{E}'$, then we denote by $f[x_0 \mapsto y]$ the function $\mathcal{E} \to \mathcal{E}'$ defined by $f[x_0 \mapsto y](x) = \begin{cases} f(x) & \text{if } x \neq x_0; \\ y & \text{if } x = x_0. \end{cases}$ If f is a function $\mathcal{E} \to \mathcal{E}'$ and $\mathcal{E}_0 \subseteq \mathsf{dom}(f) = \mathcal{E}$, then we denote by $f[\mathcal{E}_0]$ the set $\{f(x); x \in \mathcal{E}_0\}$ and by f_* the function $\mathfrak{P}(E) \to \mathfrak{P}(E)'$ that associates with every $\mathcal{E}_0 \in \mathfrak{P}(E)$ the set $f[\mathcal{E}_0]$.

1. Syntax

1.1. **Differential proof-structures.** We introduce the syntactical objects we are interested in. As recalled in the introduction, correctness does not play any role, that is why we

[10]An *injective k-point* is a point in which all the positive multisets have cardinality k and in which each atom occurring in it occurs exactly twice.

[11]*k-heterogeneous points* are points in which every positive multiset has cardinality k^j for some $j > 0$ and, for any $j > 0$, there is at most one occurrence of a positive multiset having cardinality k^j (see our Definition 4.17).

do not restrict our nets to be correct and we rather consider proof-structures (*PS's*). Since it is convenient to represent formally our proof using differential nets possibly with boxes (*differential PS's*), we define PS's as differential PS's satisfying some conditions (Definition 1.7). More generally, *differential in-PS's* are defined by induction on the depth, which is the maximum level of box nesting: Definition 1.1, Definition 1.2 and Definition 1.4 concern what happens at depth 0, i.e. whenever there is no box; in particular, typed ground-structures allow to represent proofs of the multiplicative fragment (MLL).

We set $\mathfrak{T} = \{\otimes, \parr, 1, \bot, !, ?, ax\}$.

Definition 1.1. A *pre-net* is a 7-tuple $\mathcal{G} = (\mathcal{P}, l, \mathcal{W}, \mathcal{A}, \mathcal{C}, t, \mathcal{L})$, where

- $\mathcal{P}$ is a finite set; the elements of $\mathcal{P}$ are the *ports of* $\mathcal{G}$;
- l is a function $\mathcal{P} \to \mathfrak{T}$; the element $l(p)$ of $\mathfrak{T}$ is the *label of* p *in* $\mathcal{G}$;
- $\mathcal{W}$ is a subset of $\mathcal{P}$; the elements of $\mathcal{W}$ are the *wires of* $\mathcal{G}$;[12]
- $\mathcal{A} \subseteq \mathfrak{P}_2(\mathcal{P})$ is a partition of $\{p \in \mathcal{P}; l(p) = ax\}$; the elements of $\mathcal{A}$ are the *axioms of* $\mathcal{G}$;
- $\mathcal{C}$ is a subset of $\mathfrak{P}_2(\mathcal{P} \setminus \mathcal{W})$ such that $(\forall c, c' \in \mathcal{C})(c \cap c' \neq \emptyset \Rightarrow c = c')$; the elements of $\mathcal{C}$ are the *cuts of* $\mathcal{G}$;
- t is a function $\mathcal{W} \to \{p \in \mathcal{P}; l(p) \notin \{1, \bot, ax\}\}$ such that, for any $p \in \mathcal{P}$, we have $(l(p) \in \{\otimes, \parr\} \Rightarrow \mathsf{Card}(\{w \in \mathcal{W}; t(w) = p\}) = 2)$; if $t(w) = p$, then w is a *premise of* p; the *arity* $a_{\mathcal{G}}(p)$ *of* p is the number of its premises;
- and $\mathcal{L}$ is a subset of $\{w \in \mathcal{W}; l(t(w)) \in \{\otimes, \parr\}\}$ such that $(\forall p \in \mathcal{P})$ $(l(p) \in \{\otimes, \parr\} \Rightarrow \mathsf{Card}(\{w \in \mathcal{L}; t(w) = p\}) = 1)$; if $w \in \mathcal{L}$ such that $t(w) = p$, then w is *the left premise of* p; if $w \in \mathcal{W} \setminus \mathcal{L}$ such that $l(t(w)) \in \{\otimes, \parr\}$, then w is *the right premise of* $t(w)$.

We set $\mathcal{W}(\mathcal{G}) = \mathcal{W}$, $\mathcal{P}(\mathcal{G}) = \mathcal{P}$, $l_{\mathcal{G}} = l$, $t_{\mathcal{G}} = t$, $\mathcal{L}(\mathcal{G}) = \mathcal{L}$, $\mathcal{A}(\mathcal{G}) = \mathcal{A}$ and $\mathcal{C}(\mathcal{G}) = \mathcal{C}$. The set $\mathcal{P}^{\mathsf{f}}(\mathcal{G}) = \mathcal{P} \setminus (\mathcal{W} \cup \bigcup \mathcal{C})$ is the set of *conclusions of* $\mathcal{G}$. For any $t \in \mathfrak{T}$, we set $\mathcal{P}^t(\mathcal{G}) = \{p \in \mathcal{P}; l(p) = t\}$; we set $\mathcal{P}^m(\mathcal{G}) = \mathcal{P}^{\otimes}(\mathcal{G}) \cup \mathcal{P}^{\parr}(\mathcal{G})$; the set $\mathcal{P}^e(\mathcal{G})$ of *exponential ports of* $\mathcal{G}$ is $\mathcal{P}^!(\mathcal{G}) \cup \mathcal{P}^?(\mathcal{G})$.

A *pre-ground-structure* is a pre-net $\mathcal{G}$ such that $\mathsf{im}(t_{\mathcal{G}}) \cap \mathcal{P}^!(\mathcal{G}) = \emptyset$.

Notice that, although we depict cuts as wires[13] (see the content of the box o_3 of the PS R - the third leftmost box at depth 0 of Figure 11, p. 9 - for an example of a cut), cuts are not elements of the set $\mathcal{W}$. A wire $p \in \mathcal{W}$ goes from a port that has the same name p to its target $t(p)$; instead of using arrows in our figures to indicate the direction, we will use the following convention: Unless $l(p) = ax$ (but in this case there is no ambiguity since such a port p can never be the target of any wire), whenever a wire goes from p to some port q, it will be depicted by an edge reaching underneath the vertice corresponding to p.

Definition 1.2. Given a pre-net $\mathcal{G}$, we denote by $\leq_{\mathcal{G}}$ the reflexive transitive closure of the binary relation $P_{\mathcal{G}}$ on $\mathcal{P}(\mathcal{G})$ defined by $(P_{\mathcal{G}}(q, p) \Leftrightarrow t_{\mathcal{G}}(p) = q)$.

A *simple differential net* (resp. a *ground-structure*) is a pre-net $\mathcal{G}$ (resp. a pre-ground-structure) such that the relation $P_{\mathcal{G}}$ is irreflexive and the relation $\leq_{\mathcal{G}}$ is antisymmetric.

Example 1.3. The ground-structure $\mathcal{G}$ of the content of the box o_1 of the PS R (the leftmost box of Figure 11) is defined by: $\mathcal{P}(\mathcal{G}) = \{p_1, p_2, p_3, p_4\}$, $\mathcal{W}(\mathcal{G}) = \{p_2\}$, $l_{\mathcal{G}}(p_1) = \bot = l_{\mathcal{G}}(p_2)$, $l_{\mathcal{G}}(p_3) = ?$, $l_{\mathcal{G}}(p_4) = 1$, $t_{\mathcal{G}}(p_2) = p_3$ and $\mathcal{C}(\mathcal{G}) = \emptyset = \mathcal{A}(\mathcal{G})$.

[12] We identify a wire with its source port.

[13] like wires between principal ports in the formalism of interaction nets [24] (but, in contrast with interaction nets, Definition 1.1 allows axiom-cuts)

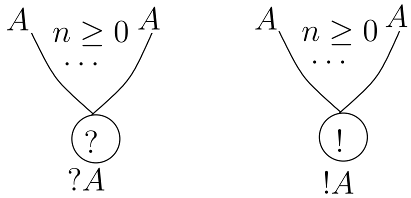


Figure 1: Typing of exponential ports

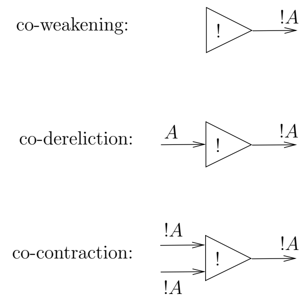


Figure 2: Original cells of differential nets

Types will be used only in Subsection 4.2. We introduce them right now in order to help the reader to see how ground-structures can represent MLL proofs.

We are given a set $\mathcal{X}$ of propositional variables. We set $\mathcal{X}^\perp = \{X^\perp; X \in \mathcal{X}\}$. We define the set $\mathbb{T}$ of *MELL types* as follows: $\mathbb{T} ::= \mathcal{X} \mid \mathcal{X}^\perp \mid 1 \mid \perp \mid (\mathbb{T} \otimes \mathbb{T}) \mid (\mathbb{T} \parr \mathbb{T}) \mid !\mathbb{T} \mid ?\mathbb{T}$. We extend the operator $-^\perp$ from the set $\mathcal{X}$ to the set $\mathbb{T}$ by defining $T^\perp \in \mathbb{T}$ by induction on T, for any $T \in \mathbb{T} \setminus \mathcal{X}$, as follows: $(X^\perp)^\perp = X$ if $X \in \mathcal{X}$; $1^\perp = \perp$; $\perp^\perp = 1$; $(A \otimes B)^\perp = (A^\perp \parr B^\perp)$; $(A \parr B)^\perp = (A^\perp \otimes B^\perp)$; $(!A)^\perp = ?A^\perp$; $(?A)^\perp = !A^\perp$.

Definition 1.4. A *typed simple differential net* (resp. a *typed ground-structure*) is a pair $(\mathcal{G}, \mathsf{T})$ such that $\mathcal{G}$ is a pre-net (resp. a pre-ground-structure) and T is a function $\mathcal{P}(\mathcal{G}) \to \mathbb{T}$ such that

- for any axiom a of $\mathcal{G}$, there exists a propositional variable C such that $\mathsf{T}[a] = \{C, C^\perp\}$;[14]
- for any cut c of $\mathcal{G}$, there exists a MELL type T such that $\mathsf{T}[c] = \{T, T^\perp\}$;

and, for any $p \in \mathcal{P}(\mathcal{G})$, the following properties hold:

- $(l_\mathcal{G}(p) \in \{1, \perp\} \Rightarrow \mathsf{T}(p) = l_\mathcal{G}(p))$;
- if $p \in \mathcal{P}^\otimes(\mathcal{G})$, then $\mathsf{T}(p) = (\mathsf{T}(w_1) \otimes \mathsf{T}(w_2))$, where w_1 (resp. w_2) is the left premise of p (resp. the right premise of p);
- if $p \in \mathcal{P}^\parr(\mathcal{G})$, then $\mathsf{T}(p) = (\mathsf{T}(w_1) \parr \mathsf{T}(w_2))$, where w_1 (resp. w_2) is the left premise of p (resp. the right premise of p);
- and, if p is an exponential port of $\mathcal{G}$, then there exists a MELL type C such that $(\mathsf{T}(p) = l_\mathcal{G}(p)C \wedge (\forall w \in \mathcal{W})(t_\mathcal{G}(w) = p \Rightarrow \mathsf{T}(w) = C))$.

Notice that the ports labelled by "!" are completely symmetric to the ports labelled by "?": They can have any number of premises and the typing rule systematically introduces the connector ! (see Figure 1, while in [17], there were three different kinds of *cells*: co-weakenings (of arity 0) and co-derelictions (of arity 1) that introduce the connector !, and co-contractions (of arity 2) that do not modify the type (see Figure 2).

Fact 1.5. Let $(\mathcal{G}, \mathsf{T})$ be a typed ground-structure (resp. a typed simple differential net). Then $\mathcal{G}$ is a ground-structure (resp. a simple differential net).

Proof. It is enough to notice that, for any $p \in \mathcal{W}(\mathcal{G})$, the size of $\mathsf{T}(t_\mathcal{G}(p))$ is greater than the size of $\mathsf{T}(p)$. □

[14]Our typed proof-structures are η-expanded.

$$\dfrac{\dfrac{\dfrac{\vdash A, A^{\perp} \qquad \vdash B, B^{\perp}}{\vdash (A \otimes B), A^{\perp}, B^{\perp}}\otimes}{\vdash (A \otimes B), (A^{\perp} \parr B^{\perp})}\parr \qquad \vdash \underline{A}, \underline{A}^{\perp}}{\vdash ((A \otimes B) \otimes \underline{A}), (A^{\perp} \parr B^{\perp}), \underline{A}^{\perp}}\otimes$$

Figure 3: Proof π_1

$$\dfrac{\dfrac{\dfrac{\vdash A, A^{\perp} \qquad \vdash B, B^{\perp}}{\vdash (A \otimes B), A^{\perp}, B^{\perp}}\otimes \qquad \vdash \underline{A}, \underline{A}^{\perp}}{\vdash ((A \otimes B) \otimes \underline{A}), A^{\perp}, B^{\perp}, \underline{A}^{\perp}}\otimes}{\vdash ((A \otimes B) \otimes \underline{A}), (A^{\perp} \parr B^{\perp}), \underline{A}^{\perp}}\parr$$

Figure 4: Proof π_2

$$\dfrac{\dfrac{\dfrac{\vdash A, A^{\perp} \qquad \vdash B, B^{\perp}}{\vdash (A \otimes B), A^{\perp}, B^{\perp}}\otimes \qquad \vdash \underline{A}, \underline{A}^{\perp}}{\vdash ((A \otimes B) \otimes \underline{A}), A^{\perp}, B^{\perp}, \underline{A}^{\perp}}\otimes}{\vdash ((A \otimes B) \otimes \underline{A}), (\underline{A}^{\perp} \parr B^{\perp}), A^{\perp}}\parr$$

Figure 5: Proof π_3

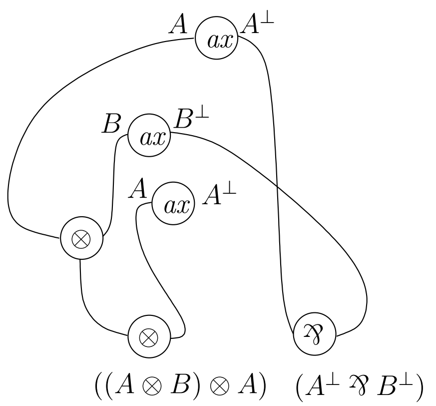


Figure 6: The typed proof-net R'

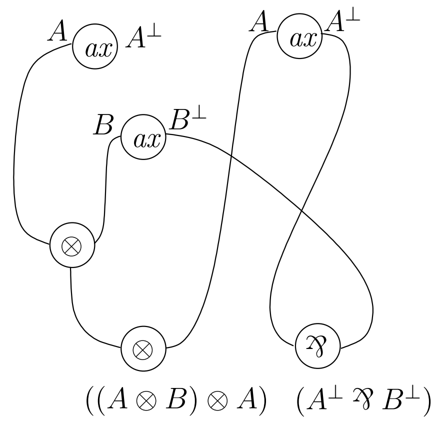


Figure 7: The typed proof-net R''

A ground-structure $\mathcal{G}$ such that $\mathcal{P}^!(\mathcal{G}) = \emptyset$ is essentially a PS of depth 0, so MLL proofs can be represented by typed ground-structures.

Example 1.6. As we wrote in the introduction, the motivation for proof-nets was to have a canonical object to represent different sequent calculus proofs that should be identified. For instance, Figure 3, Figure 4 and Figure 5 are three different sequent calculus proofs of the same sequent,[15] but the two first proofs are two different sequentializations of the same typed proof-net (R', T') depicted in Figure 6, while the third proof is a sequentialization of the typed proof-net (R'', T'') depicted in Figure 7. Let $\mathcal{G}'$ (resp. $\mathcal{G}''$) be the ground-structure that corresponds to the proof-net R' (resp. R'').

We can define $\mathcal{G}'$ and $\mathcal{G}''$ as follows: $\mathcal{P}(\mathcal{G}') = \{p_1, p_2, p_3, p_4, p_5, p_6, p_7, p_8, p_9\} = \mathcal{P}(\mathcal{G}'')$; $\mathcal{A}(\mathcal{G}') = \{\{p_3, p_4\}, \{p_5, p_6\}, \{p_7, p_8\}\} = \mathcal{A}(\mathcal{G}'')$; $l_{\mathcal{G}'} = l_{\mathcal{G}''}$ with $l_{\mathcal{G}'}(p_i) = \begin{cases} ax & \textit{if } 3 \leq i \leq 8; \\ \otimes & \textit{if } i \in \{1, 9\}; \\ \parr & \textit{if } i = 2; \end{cases}$
$\mathcal{W}(\mathcal{G}') = \{p_3,\, p_5,\, p_6,\, p_7,\, p_8,\, p_9\}$ and $\mathcal{W}(\mathcal{G}'') = \{p_3,\, p_5,\, p_6,\, p_7,\, p_4,\, p_9\}$; $\mathcal{L}(\mathcal{G}') = \{p_7, p_8, p_9\}$ and $\mathcal{L}(\mathcal{G}'') = \{p_7, p_4, p_9\}$; and $t_{\mathcal{G}'}(p_3) = p_1 = t_{\mathcal{G}''}(p_3)$, $t_{\mathcal{G}'}(p_6) = p_2 = t_{\mathcal{G}''}(p_6)$, $t_{\mathcal{G}'}(p_7) = p_9 = t_{\mathcal{G}''}(p_7)$, $t_{\mathcal{G}'}(p_5) = p_9 = t_{\mathcal{G}''}(p_5)$, $t_{\mathcal{G}'}(p_9) = p_1 = t_{\mathcal{G}''}(p_9)$, $t_{\mathcal{G}'}(p_8) = p_2$ and $t_{\mathcal{G}''}(p_4) = p_2$.

[15]We underline some occurrences of propositional variables in order to distinguish between different occurrences of the same propositional variable instead of using explicitly the exchange rule.

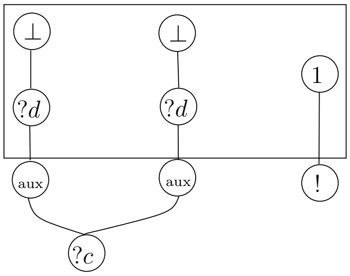


Figure 8: Proof-net in Girard's original syntax

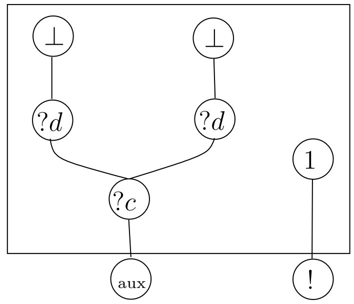


Figure 9: Proof-net in Girard's original syntax

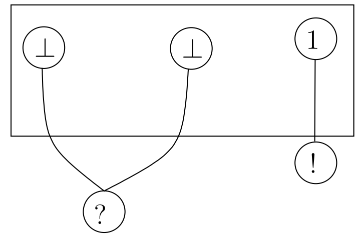


Figure 10: PS (Danos & Regnier's new syntax)

We can now define our notion of *PS*: We recall that this notion formalizes Danos & Regnier's new syntax, and not Girard's original syntax. Figure 8, Figure 9 and Figure 10 illustrate some differences between the two syntaxes: Figure 8 and Figure 9 are two different objects in the original syntax, both of them are represented in the new syntax by the PS that is depicted in Figure 10. In particular, in the new syntax, auxiliary doors of boxes are always premises of contractions. Since between auxiliary doors and contractions several box boundaries might be crossed, we need the auxiliary notion of *(differential) in-PS*. Concerning *differential PS's*, it is worth noticing that the content of each of their boxes is an in-PS, in particular every !-port inside is always the main door of a box.

Definition 1.7. For any $d \in \mathbb{N}$, we define, by induction on d, the set of *differential in-PS's of depth* d (resp. the set of *in-PS's of depth* d) and, for any differential in-PS S of depth d, the sets $\mathcal{P}(S)$ and $\mathcal{P}^{\mathsf{f}}(S) \subseteq \mathcal{P}(S)$. A *differential in-PS of depth* d (resp. an *in-PS of depth* d) is a 4-tuple $S = (\mathcal{G}, \mathcal{B}_0, B_0, t)$ such that

- $\mathcal{G}$ is a simple differential net (resp. a ground-structure); we set $\mathcal{G}(S) = \mathcal{G}$;
- $\mathcal{B}_0 \subseteq \{p \in \mathcal{P}^!(\mathcal{G}); a_{\mathcal{G}}(p) = 0\}$ (resp. $\mathcal{B}_0 = \mathcal{P}^!(\mathcal{G})$) such that $\varepsilon \notin \mathcal{B}_0$ and, for any pair $(p_1, p_2) \in \mathcal{P}(\mathcal{G})$, we have $p_1 \notin \mathcal{B}_0$ and, if p_1 is a pair (p, p') too, then $p \notin \mathcal{B}_0$;[16] the elements of $\mathcal{B}_0$ are the *boxes of* S *at depth* 0;[17]
- B_0 is a function that associates with every $o \in \mathcal{B}_0$ an in-PS of depth $< d$ that enjoys the following property: if $d > 0$, then there exists $o \in \mathcal{B}_0$ such that $B_0(o)$ is an in-PS of depth $d-1$;[18] we set $\mathcal{P}(S) = \mathcal{P}(\mathcal{G}) \cup \bigcup_{o \in \mathcal{B}_0}(\{o\} \times \mathcal{P}(B_0(o)))$; the elements of $\mathcal{P}(S)$ are the *ports of* S;
- t is a partial function $\bigcup_{o \in \mathcal{B}_0}(\{o\} \times \mathcal{P}^{\mathsf{f}}(B_0(o))) \rightharpoonup \mathcal{P}^?(\mathcal{G}) \cup \mathcal{B}_0$ such that, for any $o \in \mathcal{B}_0$, there is a unique $q_o \in \mathcal{P}^{\mathsf{f}}(\mathcal{G}(B_0(o)))$, which we will denote by $!_S(o)$, such that $\{(o, q_o)\} = \{q \in \mathsf{dom}(t); t(q) = o\}$;[19] we set $\mathcal{P}^{\mathsf{f}}(S) = \mathcal{P}^{\mathsf{f}}(\mathcal{G}(S)) \cup \bigcup_{o \in \mathcal{B}_0}\{(o, q); (q \in \mathcal{P}^{\mathsf{f}}(B_0(o)) \wedge (o, q) \notin \mathsf{dom}(t))\}$ and $\mathcal{P}^{\mathsf{f}}_{>0}(S) = \mathcal{P}^{\mathsf{f}}(S) \setminus \mathcal{P}^{\mathsf{f}}(\mathcal{G}(S))$; the elements of $\mathcal{P}^{\mathsf{f}}(S)$ (resp. of $\mathcal{P}^{\mathsf{f}}(\mathcal{G}(S))$, resp. of $\mathcal{P}^{\mathsf{f}}_{>0}(S)$) are the (resp. *shallow*, resp. *non-shallow*) *conclusions of* S.

[16] We cannot simply disallow pairs in $\mathcal{P}(\mathcal{G})$ since in the definition of the differential in-PS $\mathcal{T}_R[i](e)$ (Definition 2.3) we will use pairs to denote copies of ports of the contents of the boxes that have been expanded.

[17] We identify a box with its main door.

[18] The function B_0 maps boxes at depth 0 to their contents.

[19] The function t maps to exponential ports at depth 0 their premises that are doors of boxes.

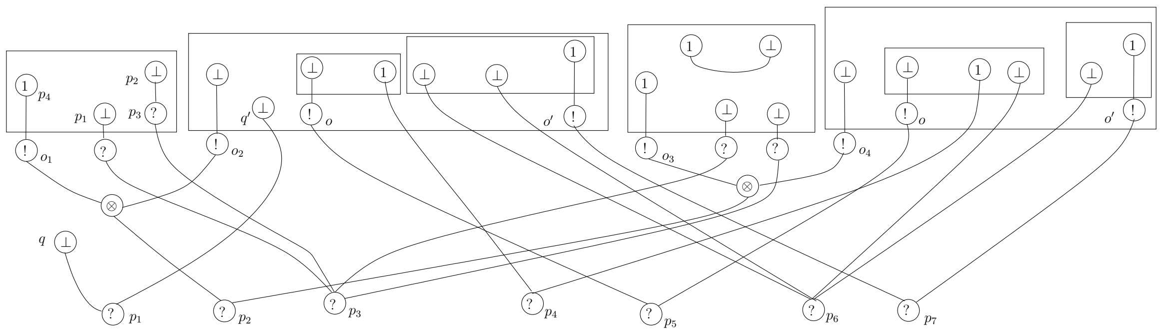


Figure 11: The PS R

We set $\mathcal{P}_0(S) = \mathcal{P}(\mathcal{G}(S))$ (the elements of $\mathcal{P}_0(S)$ are the *ports of* S *at depth* 0) and, for any $l \in \mathfrak{T} \cup \{m, e\}$, we set $\mathcal{P}_0^l(S) = \mathcal{P}^l(\mathcal{G}(S))$. We set $\mathcal{W}_0(S) = \mathcal{W}(\mathcal{G}(S))$, $\mathcal{L}_0(S) = \mathcal{L}(\mathcal{G}(S))$, $\mathcal{A}_0(S) = \mathcal{A}(\mathcal{G}(S))$ and $\mathcal{C}_0(S) = \mathcal{C}(\mathcal{G}(S))$. The function $a_S : \mathcal{P}_0(S) \to \mathbb{N}$ is defined by setting $a_S(p) = a_{\mathcal{G}(S)}(p) + \mathsf{Card}\left(\{q \in \mathsf{dom}(t); t(q) = p\}\right)$ for any $p \in \mathcal{P}_0(S)$. The integer *co-size*(S) is defined by induction on *depth*(S):[20]

$$\textit{co-size}(S) = \sup\left(\{a_S(p); p \in \mathcal{P}_0(S)\} \cup \{\textit{co-size}(B_0(o)); o \in \mathcal{B}_0(S)\}\right)$$

We set $\mathcal{B}_0(S) = \mathcal{B}_0$, $B_S = B_0$ and $t_S = t$. For any $o \in \mathcal{B}_0(S)$, we set $\mathcal{P}_S^{\mathsf{f}}(o) = \{p \in \mathcal{P}^{\mathsf{f}}(B_S(o)); (o, p) \in \mathsf{dom}(t_S)\}$.[21] We denote by t_S the function $\mathcal{W}_0(S) \cup \bigcup_{o \in \mathcal{B}_0(S)}(\{o\} \times \mathcal{P}_S^{\mathsf{f}}(o)) \to \mathcal{P}_0(S)$ that associates with every $p \in \mathcal{W}_0(S)$ the port $t_{\mathcal{G}(S)}(p)$ of $\mathcal{G}(S)$ and with every (o, p), where $o \in \mathcal{B}_0(S)$ and $p \in \mathcal{P}_S^{\mathsf{f}}(o)$, the port $t_S(o, p)$ of $\mathcal{G}(S)$. We set $\mathcal{P}_S^{?}(o) = t_S[\{o\} \times \mathcal{P}_S^{\mathsf{f}}(o)] \setminus \{o\}$ for any $o \in \mathcal{B}_0(S)$. The set $\mathcal{B}(S)$ of *boxes of* S is defined by induction on *depth*(S): $\mathcal{B}(S) = \mathcal{B}_0(S) \cup \bigcup_{o \in \mathcal{B}_0(S)}\{(o, o'); o' \in \mathcal{B}(B_0(o))\}$. For any binary relation $P \in \{\geq, =, <\}$ on $\mathbb{N}$, for any $i \in \mathbb{N}$, we set $\mathcal{B}_0^{Pi}(S) = \{o \in \mathcal{B}_0(S); P(\textit{depth}(B_S(o)), i)\}$ and we define, by induction on *depth*(S), the set $\mathcal{B}^{Pi}(S) \subseteq \mathcal{B}(S)$ as follows: $\mathcal{B}^{Pi}(S) = \mathcal{B}_0^{Pi}(S) \cup \bigcup_{o \in \mathcal{B}_0(S)}\{(o, o'); o' \in \mathcal{B}^{Pi}(B_S(o))\}$. We set $\mathcal{P}_{>i}(S) = \bigcup_{o \in \mathcal{B}_0^{\geq i}(S)}\{(o, q); q \in \mathcal{P}(B_S(o))\}$ and $\mathcal{P}_{\leq i}(S) = \mathcal{P}(S) \setminus \mathcal{P}_{>i}(S)$.

A *differential PS* (resp. a *PS*) is a differential in-PS (resp. an in-PS) S such that $\mathcal{P}^{\mathsf{f}}(S) \subseteq \mathcal{P}^{\mathsf{f}}(\mathcal{G}(S))$.[22]

The set of *cocontractions* of an in-PS S is the set $\mathcal{P}_0^{!}(S) \setminus \mathcal{B}_0(S)$. Notice that an in-PS is a differential in-PS with no co-contraction.

It is worth noticing that the binary relation $\leq_S$ on the set $\mathcal{B}(S) \cup \{\varepsilon\}$ defined by $((o_1, \ldots, o_m) \leq_S (o'_1, \ldots, o'_n) \Leftrightarrow (m \leq n \wedge (o_1, \ldots, o_m) = (o'_1, \ldots, o'_m)))$ defines a tree with ε as the root.

Example 1.8. If R is the PS of depth 2 depicted in Figure 11, then we have $\mathcal{B}_0(R) = \{o_1, o_2, o_3, o_4\}$, $\mathcal{B}(R) = \{o_1, o_2, o_3, o_4, (o_2, o), (o_2, o'), (o_4, o), (o_4, o')\}$, $\mathcal{B}^{=0}(R) = \{o_1, (o_2, o), (o_2, o'), o_3, (o_4, o), (o_4, o')\}$, $\mathcal{B}^{=1}(R) = \{o_2, o_4\}$, $\mathcal{P}^{\mathsf{f}}(R) = \{p_1, p_2, p_3, p_4, p_5, p_6, p_7\}$ and $\mathcal{G}(B_R(o_1))$ is the ground-structure of Example 1.3.

[20] The supremum is taken in $\mathbb{N}$, hence, if S is the empty PS, then *co-size*$(S) = 0$.

[21] Equivalently, $\mathcal{P}_S^{\mathsf{f}}(o) = \{p \in \mathcal{P}^{\mathsf{f}}(B_S(o)); (o, p) \notin \mathcal{P}^{\mathsf{f}}(S)\}$.

[22] Equivalently, a *differential PS* (resp. a *PS*) is a differential in-PS (resp. an in-PS) S such that $(\forall o \in \mathcal{B}_0(S))\mathcal{P}_S^{\mathsf{f}}(o) = \mathcal{P}^{\mathsf{f}}(B_S(o))$.

In the absence of axioms and cuts, our definition of PS through in-PS's is equivalent to our definition of PS in Definition 4 of [10] through $\circ$-PS's. We removed $\circ$-ports because we simplified the proof of Proposition 3.43 and after this simplification they would not play any role any more (actually we introduced a syntactic construction[23] that, roughly speaking, can be seen as a partial recovery of these $\circ$-ports).

Definition 1.9. For any $l \in \{!, ?\}$, for any p, we denote by l_p the PS R of depth 0 such that $\mathcal{P}_0(R) = \{p\}$ and $l_{\mathcal{G}(R)}(p) = l$.

Definition 1.10. For any $d \in \mathbb{N}$, we define, by induction on d, the set of *typed differential in-PS's of depth d* (resp. the set of *typed in-PS's of depth d*): it is the set of pairs (S, T) such that S is a differential in-PS (resp. an in-PS) and T is a function $\mathcal{P}(S) \to \mathbb{T}$ such that:

- $(\mathcal{G}(S), \mathsf{T}|_{\mathcal{P}_0(S)})$ is a typed simple differential net (resp. a typed ground-structure);
- for any $o \in \mathcal{B}_0(S)$, the pair $(B_S(o), \mathsf{T}_o)$ is a typed simple differential net, where T_o is the function $\mathcal{P}(B_S(o)) \to \mathbb{T}$ defined by $\mathsf{T}_o(p) = \mathsf{T}(o, p)$ for any $p \in \mathcal{P}(B_S(o))$;
- and, for any $o \in \mathcal{B}_0(S)$, we have $(\forall q \in \mathcal{P}^{\mathsf{f}}_S(o))(\exists \zeta \in \{?, !\})\mathsf{T}(t_S(o, q)) = \zeta \mathsf{T}(o, q)$.

A *typed differential PS* (resp. a *typed PS*) is a typed differential in-PS (resp. a typed in-PS) (S, T) such that $\mathcal{P}^{\mathsf{f}}(S) \subseteq \mathcal{P}^{\mathsf{f}}(\mathcal{G}(S))$.

1.2. **Isomorphisms.** We want to consider PS's up to the names of the ports, apart from the names of the shallow conclusions. We thus define the equivalence relation $\equiv$ on PS's; this relation is slightly finer than the equivalence relation $\simeq$, which ignores all the names of the ports.

Definition 1.11. For any simple differential nets $\mathcal{G}$ and $\mathcal{G}'$, an isomorphism φ from $\mathcal{G}$ to $\mathcal{G}'$ is a bijection $\mathcal{P}(\mathcal{G}) \simeq \mathcal{P}(\mathcal{G}')$ such that:

- $\varphi[\mathcal{W}(\mathcal{G})] = \mathcal{W}(\mathcal{G}')$
- $\varphi_*[\mathcal{A}(\mathcal{G})] = \mathcal{A}(\mathcal{G}')$
- $\varphi_*[\mathcal{C}(\mathcal{G})] = \mathcal{C}(\mathcal{G}')$
- $\varphi[\mathcal{L}(\mathcal{G})] = \mathcal{L}(\mathcal{G}')$
- $t_{\mathcal{G}'} \circ \varphi_{|\mathcal{W}(\mathcal{G})} = \varphi \circ t_{\mathcal{G}}$
- $l_{\mathcal{G}} = l_{\mathcal{G}'} \circ \varphi$

We write $\varphi : \mathcal{G} \simeq \mathcal{G}'$ to denote that φ is an isomorphism from $\mathcal{G}$ to $\mathcal{G}'$; we write $\mathcal{G} \simeq \mathcal{G}'$ if there exists φ such that $\varphi : \mathcal{G} \simeq \mathcal{G}'$.

Moreover, we write $\varphi : \mathcal{G} \equiv \mathcal{G}'$ to denote that $\varphi : \mathcal{G} \simeq \mathcal{G}'$ and $(\forall p \in \mathcal{P}^{\mathsf{f}}(\mathcal{G}))\varphi(p) = p$; we write $\mathcal{G} \equiv \mathcal{G}'$ if there exists φ such that $\varphi : \mathcal{G} \equiv \mathcal{G}'$.

Definition 1.12. For any differential in-PS S of depth d, for any differential in-PS S', we define, by induction on d, the set of isomorphims from S to S': an isomorphism φ from S to S' is a function $\mathcal{P}(S) \to \mathcal{P}(S')$ such that:

- $(\forall p \in \mathcal{P}_0(S))\varphi(p) \in \mathcal{P}_0(S')$ and the function $\mathcal{G}(\varphi) : \begin{array}{ccc} \mathcal{P}_0(S) & \to & \mathcal{P}_0(S') \\ p & \mapsto & \varphi(p) \end{array}$ is an isomorphism $\mathcal{G}(S) \simeq \mathcal{G}(S')$;
- $\varphi[\mathcal{B}_0(S)] = \mathcal{B}_0(S')$;

[23]See Definition 1.34

- $(\forall o \in \mathcal{B}_0(S))(\forall p \in \mathcal{P}(B_S(o)))(\exists p' \in \mathcal{P}(B_{S'}(\varphi(o))))\varphi(o,p) = (\varphi(o), p')$ and the function
$$\varphi_o : \begin{array}{rcl} \mathcal{P}(B_S(o)) & \to & \mathcal{P}(B_{S'}(\varphi(o))) \\ p & \mapsto & p' \textit{ such that } \varphi(o,p) = (\varphi(o), p') \end{array}$$
is an isomorphism from $B_S(o)$ to $B_{S'}(\varphi(o))$;
- $\mathsf{dom}(t_{S'}) = \varphi[\mathsf{dom}(t_S)]$ and, for any $p \in \mathsf{dom}(t_S)$, we have $(\varphi \circ t_S)(p) = (t_{S'} \circ \varphi)(p)$.

We write $\varphi : S \simeq S'$ to denote that φ is an isomorphism from S to S'; we write $S \simeq S'$ if there exists φ such that $\varphi : S \simeq S'$.

Moreover, we write $\varphi : S \equiv S'$ to denote that $\varphi : S \simeq S'$ and $(\forall p \in \mathcal{P}^{\mathsf{f}}(\mathcal{G}(S)))\varphi(p) = p$; we write $S \equiv S'$ if there exists φ such that $\varphi : S \equiv S'$.

Now, if $\mathcal{T}$ and $\mathcal{T}'$ are two sets of differential in-PS's, we write $\mathcal{T} \equiv \mathcal{T}'$ if there exists a bijection $\varphi : \mathcal{T} \simeq \mathcal{T}'$ such that, for any $T \in \mathcal{T}$, we have $T \equiv \varphi(T)$.

Finally, if (S, T) and (S', T') are two typed differential in-PS's, then we write $(S, \mathsf{T}) \equiv (S', \mathsf{T}')$ if there exists $\varphi : S \equiv S'$ such that $\mathsf{T} = \mathsf{T}' \circ \varphi$.

Fact 1.13. Let (S, T) and (S', T') be two cut-free typed differential in-PS's such that $S \equiv S'$. If $\mathsf{T}|_{\mathcal{P}^{\mathsf{f}}(S)} = \mathsf{T}'|_{\mathcal{P}^{\mathsf{f}}(S')}$, then $(S, \mathsf{T}) \equiv (S', \mathsf{T}')$.

Another variant of the notion of isomorphism will be defined in the next subsection (Definition 1.32). A special case of isomorphism consists in renaming only ports at depth 0:

Definition 1.14. Let S and S' be two differential in-PS's. Let φ be a bijection $\mathcal{P} \simeq \mathcal{P}'$, where $\mathcal{P}' \cap (\mathcal{P}_0(S) \setminus \mathcal{P}) = \emptyset$. We say that S' is obtained from S by renaming the ports via φ and we write $S' = S[\varphi]$ if the following properties hold:

- $\mathcal{P}_0(S') = \overline{\varphi}[\mathcal{P}_0(S)]$
- $\mathcal{W}_0(S') = \overline{\varphi}[\mathcal{W}_0(S)]$
- $\mathcal{A}_0(S') = \{\overline{\varphi}[a]; a \in \mathcal{A}_0(S)\}$
- $\mathcal{C}_0(S') = \{\overline{\varphi}[a]; a \in \mathcal{C}_0(S)\}$
- $\mathcal{L}_0(S') = \overline{\varphi}[\mathcal{L}_0(S)]$
- $l_{\mathcal{G}(S')} \circ \overline{\varphi} = l_{\mathcal{G}(S)}$
- $t_{\mathcal{G}(S')} \circ \overline{\varphi} = \overline{\varphi} \circ t_{\mathcal{G}(S)}$
- $\mathcal{B}_0(S') = \overline{\varphi}[\mathcal{B}_0(S)]$
- $\mathsf{dom}(t_{S'}) = \bigcup_{o \in \mathcal{B}_0(S)} \{(\overline{\varphi}(o), p); p \in \mathcal{P}^{\mathsf{f}}_S(o)\}$ and, for any $(o,p) \in \mathsf{dom}(t_S)$, we have $t_{S'}(\overline{\varphi}(o), p) = t_S(o,p)$
- and $B_{S'} = B_S \circ \overline{\varphi}$,

where $\overline{\varphi}$ is the function $\mathcal{P}_0(S) \to \mathcal{P}_0(S')$ that associates with every $p \in \mathcal{P}_0(S)$ the following port of $\mathcal{G}(S')$: $\begin{cases} p & \textit{if } p \in \mathcal{P}_0(S) \setminus \mathcal{P}; \\ \varphi(p) & \textit{if } p \in \mathcal{P}. \end{cases}$

If $S' = S[\varphi]$ and φ is, for some singleton $\mathcal{E} = \{a\}$, the bijection $\mathcal{P}_0(S) \simeq \mathcal{E} \times \mathcal{P}_0(S)$ that associates with every $p \in \mathcal{P}_0(S)$ the pair (a, p), then S' is denoted by $\langle a, S \rangle$ too.

1.3. **Some operations on differential proof-structures.**

In this subsection, we describe some operations to obtain new PS's from old ones.

For any differential in-PS S, for any integer i, we define a differential in-PS $S^{\leq i}$ of depth $\leq i$, which is obtained from S by removing some boxes:

Definition 1.15. Let S be a differential in-PS and let $i \in \mathbb{N}$. We denote by $S^{\leq i}$ the differential in-PS such that $\mathcal{G}(S^{\leq i}) = \mathcal{G}(S)$, $\mathcal{B}_0(S^{\leq i}) = \mathcal{B}_0^{<i}(S)$, $B_{S^{\leq i}} = B_S|_{\mathcal{B}_0^{<i}(S)}$ and $t_{S^{\leq i}} = t_S|_{\bigcup_{o \in \mathcal{B}_0^{<i}(S)}(\{o\} \times \mathcal{P}_S^{\mathsf{f}}(o))}$.

In particular $S^{\leq 0}$ is essentially the same object as $\mathcal{G}(S)$.

Remark 1.16. If $depth(T) < i$, then $T^{\leq i} = T$.

Remark 1.17. We have $(S^{\leq i})^{\leq i'} = S^{\leq \min\{i,i'\}}$.

Example 1.18. The differential PS $R^{\leq 1}$, where R is the PS depicted in Figure 11, is depicted in Figure 35, p. 54.

We can also erase some ports at depth 0:

Definition 1.19. Let S' and S be two differential in-PS's. Let $\mathcal{Q} \subseteq \mathcal{P}_0(S)$. We write $S' \sqsubseteq_{\mathcal{Q}} S$ to denote that $\mathcal{P}_0(S') \subseteq \mathcal{P}_0(S)$, $\mathcal{W}_0(S') = \{w \in (\mathcal{W}_0(S) \cap \mathcal{P}_0(S')) \setminus (\mathcal{Q} \cap \mathcal{P}_0^e(S)); t_{\mathcal{G}(S)}(w) \in \mathcal{P}_0(S')\}$, $l_{\mathcal{G}(S')} = l_{\mathcal{G}(S)}|_{\mathcal{P}_0(S')}$, $t_{\mathcal{G}(S')} = t_{\mathcal{G}(S)}|_{\mathcal{W}_0(S')}$, $\mathcal{L}(\mathcal{G}(S')) = \mathcal{L}(\mathcal{G}(S)) \cap \{w \in \mathcal{W}_0(S); t_{\mathcal{G}(S)}(w) \in \mathcal{P}_0^m(S')\}$, $\mathcal{A}_0(S') = \{a \in \mathcal{A}_0(S); \bigcup a \subseteq \mathcal{P}_0(S')\}$, $\mathcal{C}_0(S') = \{a \in \mathcal{C}_0(S); \bigcup a \subseteq \mathcal{P}_0(S') \setminus (\mathcal{Q} \cap \mathcal{P}_0^e(S))\}$, $\mathcal{B}_0(S') = \mathcal{B}_0(S) \cap \mathcal{P}_0(S')$, $B_{S'} = B_S|_{\mathcal{B}_0(S')}$ and $t_{S'} = t_S|_{\bigcup_{o \in \mathcal{B}_0(S')}(\{o\} \times \mathcal{P}_S^{\mathsf{f}}(o))}$.

We write $S' \sqsubseteq S$ if there exists $\mathcal{Q}$ such that $S' \sqsubseteq_{\mathcal{Q}} S$.

Remark 1.20. We have $S' \sqsubseteq_{\mathcal{Q}} S$ if and only if $S' \sqsubseteq_{\mathcal{Q} \cap \mathcal{P}_0^e(S)} S$.

Remark 1.21. If $S' \sqsubseteq S$, then $\mathcal{P}_0(S') \cap \mathcal{P}^{\mathsf{f}}(\mathcal{G}(S)) \subseteq \mathcal{P}^{\mathsf{f}}(\mathcal{G}(S'))$.

Remark 1.22. If $S', S'' \sqsubseteq_{\mathcal{Q}} S$ and $\mathcal{P}_0(S') = \mathcal{P}_0(S'')$, then $S' = S''$. So, if, for some $\mathcal{P} \subseteq \mathcal{P}_0(S)$, there exists a differential in-PS S' such that $\mathcal{P}_0(S') = \mathcal{P}$ and $S' \sqsubseteq_{\emptyset} S$, then we can denote by $S_{\restriction \mathcal{P}}$ the unique such differential in-PS S'.

Remark 1.23. We have $S' \sqsubseteq_{\mathcal{Q}} S$ if, and only if, the following properties hold:

- $S'^{\leq 0} \sqsubseteq_{\mathcal{Q}} S^{\leq 0}$
- $\mathcal{B}_0(S') = \mathcal{B}_0(S) \cap \mathcal{P}_0(S')$
- $B_{S'} = B_S|_{\mathcal{B}_0(S')}$
- $t_{S'} = t_S|_{\bigcup_{o \in \mathcal{B}_0(S')}(\{o\} \times \mathcal{P}_S^{\mathsf{f}}(o))}$

Remark 1.24. If $S_1' \sqsubseteq_{\mathcal{Q}} S_1$ and $\varphi : S_1 \simeq S_2$, then there exists a unique $S_2' \sqsubseteq_{\varphi[\mathcal{Q}]} S_2$ such that there exists an isomorphism $S_1' \simeq S_2'$ associating with every port p of S_1' the port $\varphi(p)$ of S_2.

Fact 1.25. Let S, S' and S'' be three differential in-PS's. Let $\mathcal{Q} \subseteq \mathcal{P}_0(S)$ and $\mathcal{Q}' \subseteq \mathcal{P}_0(S')$. If $S'' \sqsubseteq_{\mathcal{Q}'} S'$ and $S' \sqsubseteq_{\mathcal{Q}} S$, then $S'' \sqsubseteq_{\mathcal{Q} \cup \mathcal{Q}'} S$.

Fact 1.26. Let S' and S be two differential in-PS's and let $\mathcal{Q} \subseteq \mathcal{P}_0^e(S)$ such that $S' \sqsubseteq_{\mathcal{Q}} S$. Let $i \in \mathbb{N}$. Then $S'^{\leq i} \sqsubseteq_{\mathcal{Q}} S^{\leq i}$.

The operator $\bigoplus$ glues together several differential in-PS's that share only shallow conclusions that are contractions:[24]

[24]This operation has nothing to do with the additive $\oplus$ of linear logic, it is rather essentially the mix of linear logic.

Definition 1.27. Let $\mathcal{U}$ be a finite set of differential in-PS's. We say that $\mathcal{U}$ is *gluable* if, for any $T, T' \in \mathcal{U}$ such that $T \neq T'$, we have $\mathcal{P}_0(T) \cap \mathcal{P}_0(T') \subseteq \mathcal{P}^{\mathsf{f}}(\mathcal{G}(T)) \cap \mathcal{P}_0^?(T) \cap \mathcal{P}^{\mathsf{f}}(\mathcal{G}(T')) \cap \mathcal{P}_0^?(T')$ and, for any pair $(p_1, p_2) \in \mathcal{P}_0(T)$, we have $p_1 \notin \mathcal{B}_0(T')$ and, if p_1 is a pair (p, p') too, then $p \notin \mathcal{B}_0(T')$.

If $\mathcal{U}$ is gluable, then $\bigoplus \mathcal{U}$ is the differential in-PS such that:

- $\mathcal{P}_0(\bigoplus \mathcal{U}) = \bigcup_{T \in \mathcal{U}} \mathcal{P}_0(T)$
- $\mathcal{W}_0(\bigoplus \mathcal{U}) = \bigcup_{T \in \mathcal{U}} \mathcal{W}_0(T)$
- $l_{\mathcal{G}(\bigoplus \mathcal{U})}(p) = l_{\mathcal{G}(T)}(p)$ for any $p \in \mathcal{P}_0(\bigoplus \mathcal{U})$ and any $T \in \mathcal{U}$ such that $p \in \mathcal{P}_0(T)$
- $\mathcal{A}_0(\bigoplus \mathcal{U}) = \bigcup_{T \in \mathcal{U}} \mathcal{A}_0(T)$
- $\mathcal{C}_0(\bigoplus \mathcal{U}) = \bigcup_{T \in \mathcal{U}} \mathcal{C}_0(T)$
- $\mathcal{L}_0(\bigoplus \mathcal{U}) = \bigcup_{T \in \mathcal{U}} \mathcal{L}_0(T)$
- $t_{\mathcal{G}(\bigoplus \mathcal{U})}(p) = t_{\mathcal{G}(T)}(p)$ for any $p \in \mathcal{W}_0(\bigoplus \mathcal{U})$ and any $T \in \mathcal{U}$ such that $p \in \mathcal{W}_0(T)$;
- $\mathcal{B}_0(\bigoplus \mathcal{U})(p) = \bigcup_{T \in \mathcal{U}} \mathcal{B}_0(T)$
- $B_{\bigoplus \mathcal{U}}(o) = B_T(o)$ for any $o \in \mathcal{B}_0(\bigoplus \mathcal{U})$ and any $T \in \mathcal{U}$ such that $o \in \mathcal{B}_0(T)$;
- $\mathsf{dom}(t_{\bigoplus \mathcal{U}}) = \bigcup_{T \in \mathcal{U}} \mathsf{dom}(t_T)$ and $t_{\bigoplus \mathcal{U}}(p) = t_T(p)$ for any $p \in \mathsf{dom}(t_{\bigoplus \mathcal{U}})$ and any $T \in \mathcal{U}$ such that $p \in \mathsf{dom}(t_T)$.

Remark 1.28. If $\mathcal{U}$ is gluable, then $(\bigoplus \mathcal{U})^{\leq i} = \bigoplus \{U^{\leq i}; U \in \mathcal{U}\}$.

We can add wires:

Definition 1.29. Let S be a differential in-PS. Let $\mathcal{W} \subseteq \mathcal{P}^{\mathsf{f}}(S)$ and $\mathcal{W}' \subseteq \mathcal{P}_0^e(S) \setminus \mathcal{B}_0(S)$ such that $(\forall p \in \mathcal{W} \cap \mathcal{P}_0(S))(\forall p' \in \mathcal{W}') \neg p \leq_{\mathcal{G}(S)} p'$. Let t be a function $\mathcal{W} \to \mathcal{W}'$. Then we denote by $S@t$ the differential in-PS such that

- $\mathsf{t}_{S@t}$ is the extension of t_S such that $\mathsf{dom}(\mathsf{t}_{S@t}) = \mathsf{dom}(\mathsf{t}_S) \cup \mathcal{W}$ and $(\forall p \in \mathcal{W}) \mathsf{t}_{S@t}(p) = t(p)$
- and $\mathcal{P}_0(S@t) = \mathcal{P}_0(S)$, $l_{\mathcal{G}(S@t)} = l_{\mathcal{G}(S)}$, $\mathcal{L}_0(S@t) = \mathcal{L}_0(S)$, $\mathcal{A}_0(S@t) = \mathcal{A}_0(S)$, $\mathcal{C}_0(S@t) = \mathcal{C}_0(S)$, $\mathcal{B}_0(S@t) = \mathcal{B}_0(S)$, $B_{S@t} = B_S$.

Remark 1.30. We have $(S@t)^{\leq i} = (S^{\leq i})@t_{|\mathcal{P}_{\leq i}(S)}$.

We can remove shallow conclusions:

Definition 1.31. Let T be a differential in-PS such that $\mathcal{P}^{\mathsf{f}}(\mathcal{G}(T)) \subseteq \mathcal{P}_0^e(T) \setminus \mathcal{B}_0(T)$. Then $\overline{T}$ is the unique differential in-PS such that

- $\mathcal{P}_0(\overline{T}) = \mathcal{P}_0(T) \setminus \mathcal{P}^{\mathsf{f}}(\mathcal{G}(T))$
- and $T = (\overline{T} \oplus \mathcal{P}^{\mathsf{f}}(\mathcal{G}(T)))@t$, where t is the function that associates with every $p \in \mathsf{dom}(\mathsf{t}_T)$ such that $\mathsf{t}_T(p) \in \mathcal{P}^{\mathsf{f}}(\mathcal{G}(T))$ the port $\mathsf{t}_T(p)$.

If $\mathcal{T}$ is a set of differential in-PS's, then $\overline{\mathcal{T}} = \{\overline{T}; T \in \mathcal{T}\}$.

This operation allows to define the following variant of the notion of isomorphism of differential in-PS's:

Definition 1.32. Let S and U be two differential in-PS's. Let $o \in \mathcal{B}_0(S)$ such that $\mathcal{P}^{\mathsf{f}}(U) \subseteq \mathcal{P}_S^{\mathsf{f}}(o)$. Let T be a differential in-PS such that $\mathcal{P}^{\mathsf{f}}(\mathcal{G}(T)) \subseteq \mathcal{P}_0^e(T) \setminus \mathcal{B}_0(T)$. Then we write $\varphi : U \equiv_{(S,o)} T$ if $\varphi : U \simeq \overline{T}$ such that, for any $p \in \mathcal{P}^{\mathsf{f}}(U)$, we have $t_S(o, p) = \mathsf{t}_T(\varphi(p))$. We write $U \equiv_{(S,o)} T$ if there exists φ such that $\varphi : U \equiv_{(S,o)} T$.

Remark 1.33. Notice that:

- While the relations $\simeq$ and $\equiv$ are symmetric, the relation $\equiv_{(S,o)}$ is *not* symmetric.
- If $\varphi : U \equiv_{(S,o)} T$ and $\psi : U' \equiv U$, then $\varphi \circ \psi : U' \equiv_{(S,o)} T$.

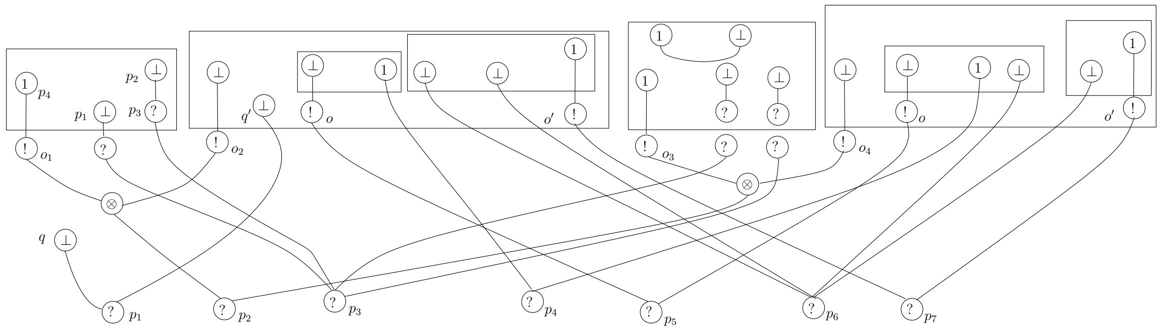


Figure 12: The in-PS $\varphi \cdot_{o_3} R$

- If $\varphi_1 : U \equiv_{(S,o)} T_1$ and $\varphi_2 : U \equiv_{(S,o)} T_2$, then $\psi : T_1 \equiv T_2$, where ψ is the bijection $\mathcal{P}(T_1) \simeq \mathcal{P}(T_2)$ defined by $\psi(p) = \begin{cases} \varphi_2({\varphi_1}^{-1}(p)) & \textit{if } p \notin \mathcal{P}^{\mathsf{f}}(\mathcal{G}(T_1)); \\ p & \textit{otherwise.} \end{cases}$

The following operation consists in adding contractions as shallow conclusions to the content of some box o; the conclusions of the box o that were contracted at depth 0 are now contracted inside the box o. We can then define a complexity measure on in-PS's that decreases with this operation, which allows to prove Proposition 3.43 by induction on this complexity measure.

Definition 1.34. Let R and R_o be two in-PS's. Let $o \in \mathcal{B}_0(R) \cap \mathcal{B}_0(R_o)$. Let $\mathcal{Q}' \subseteq \mathcal{P}^?_0(B_{R_o}(o))$ and let φ be a bijection $\mathcal{P}^?_R(o) \simeq \mathcal{Q}'$. We say that R_o *is obtained from R by adding, according to φ, contractions as shallow conclusions to the content of the box o* and we write $R_o = \varphi \cdot_o R$ if the following properties hold:
- ${R_o}^{\leq 0} = R^{\leq 0}$;
- $\mathcal{B}_0(R_o) = \mathcal{B}_0(R)$;
- $B_{R_o}(o') = \begin{cases} B_R(o') & \textit{if } o' \neq o; \\ R'_o & \textit{if } o' = o; \end{cases}$ with $R'_o = (B_R(o) \oplus \bigoplus_{q' \in \mathcal{Q}'} ?_{q'})@t$, where $\mathcal{Q}'$ is a disjoint set from $\mathcal{P}_0(B_R(o))$ and t is the function $\mathcal{P}^{\mathsf{f}}_R(o) \setminus \{!_R(o)\} \to \mathcal{Q}'$ that associates with every $p \in \mathcal{P}^{\mathsf{f}}_R(o) \setminus \{!_R(o)\}$ the port $\varphi(t_R(o,p))$;
- and $t_{R_o} = t_R|_{\mathsf{dom}(t_R) \setminus (\{o\} \times (\mathcal{P}^{\mathsf{f}}_R(o) \setminus \{!_R(o)\}))}$.

Remark 1.35. For any $o' \in \mathcal{B}_0(B_R(o))$, we have
- $\mathcal{P}^{\mathsf{f}}_{B_R(o)}(o') = \{p \in \mathcal{P}^{\mathsf{f}}_{B_{(\varphi \cdot_o R)}(o)}(o');\ t_{B_{(\varphi \cdot_o R)}(o)}(o', p) \notin \mathcal{Q}'\}$
- and $(\forall p \in \mathcal{P}^{\mathsf{f}}_{B_R(o)}(o'))t_{B_R(o)}(o', p) = t_{B_{(\varphi \cdot_o R)}(o)}(o', p)$.

Moreover, for any $q \in \mathcal{P}^{\mathsf{f}}_R(o)$ such that $t_{B_{(\varphi \cdot_o R)}(o)}(q) \in \mathcal{Q}'$, we have $t_{B_{(\varphi \cdot_o R)}(o)}(q) = \varphi(t_R(o, q))$.

Notice that, if R is a PS, then $B_{(\varphi \cdot_o R)}(o)$ is a PS too (while $B_R(o)$ is not necessarily a PS); we will implicitly use this property in the proofs of Lemma 3.39 and Proposition 3.47.

Example 1.36. The in-PS $\varphi \cdot_{o_3} R$, where φ is some bijection $\mathcal{P}^?_R(o_3) \simeq \mathcal{Q}'$, is depicted in Figure 12.

Example 1.37. The in-PS $\varphi' \cdot_{o_4} R$, where φ' is some bijection $\mathcal{P}^?_R(o_4) \simeq \mathcal{Q}'$, is depicted in Figure 13.

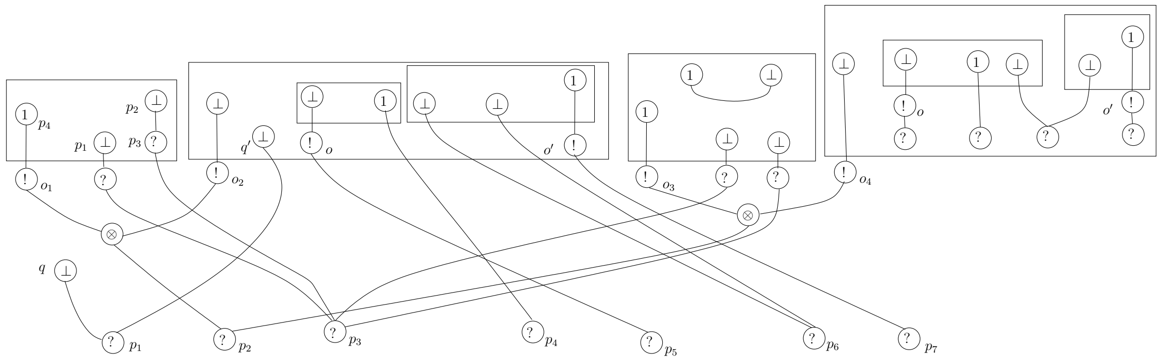


Figure 13: The in-PS $\varphi' \cdot_{o_4} R$

2. Taylor expansion

When Jean-Yves Girard introduced proof-nets in [19], he also introduced *experiments on proof-nets*. Experiments (see our Definition 4.22 in Section 4) are a technology allowing to compute pointwisely the interpretation $\llbracket R \rrbracket$ of a proof-net R in the model directly on the proof-net rather than through some sequent calculus proof obtained from one of its sequentializations: the set of *results* of all the experiments on a given proof-net is its interpretation $\llbracket R \rrbracket$. In an untyped framework, experiments correspond to derivations of intersection types and results correspond to intersection types.

Inspired by this notion, we introduce *pseudo-experiments*.

Definition 2.1. For any differential in-PS R, we define, by induction on $depth(R)$, the set $\mathfrak{E}(R)$ of *pseudo-experiments on* R: it is the set of functions that associate with every $o \in \mathcal{B}_0(R)$ a finite set of pseudo-experiments on $B_R(o)$ and with ε some $m \in \mathbb{N}$.

Given a pseudo-experiment e on a differential in-PS R, we define, by induction on $depth(R)$, the function $e^\# : \mathcal{B}(R) \to \mathfrak{P}_{fin}(\mathbb{N})$ as follows: for any $o \in \mathcal{B}_0(R)$, $e^\#(o) = \{\mathsf{Card}\,(e(o))\}$ and, for any $o' \in \mathcal{B}(B_R(o))$, $e^\#(o, o') = \bigcup_{e_o \in e(o)} {e_o}^\#(o')$.

Our definition is quite *ad hoc*; actually, what we have in mind is the following notion of *canonical pseudo-experiment*: A canonical pseudo-experiment on a differential in-PS R is a function that associates with every $o \in \mathcal{B}_0(R)$ a finite multiset of canonical pseudo-experiments on $B_R(o)$. Thus a canonical pseudo-experiment is an experiment without any labels on the axioms and any constraints on cuts. If we consider pseudo-experiments instead of canonical pseudo-experiments, it is only in order to be less verbose. For instance, if we used canonical pseudo-experiments instead of pseudo-experiments, the differential in-PS S of Definition 2.3 could be defined by setting

$$S = R^{\leq i} \oplus \bigoplus_{o \in \mathcal{B}_0^{\geq i}(R)} \bigoplus_{e_o \in Supp(e(o))} \bigoplus_{z \in \{1,\ldots,e(e_o)\}} \langle o, \langle (e_o, z), \mathcal{T}_{B_R(o)}[i](e_o) \rangle \rangle$$

instead of $S = R^{\leq i} \oplus \bigoplus_{o \in \mathcal{B}_0^{\geq i}(R)} \bigoplus_{e_o \in e(o)} \langle o, \langle e_o, \mathcal{T}_{B_R(o)}[i](e_o) \rangle \rangle$.

Example 2.2. There exists a pseudo-experiment e on the PS R of Figure 11 such that $e^\#(o_1) = \{10^{223}\}$, $e^\#(o_2) = \{10\}$, $e^\#(o_3) = \{10^{224}\}$, $e^\#(o_4) = \{100\}$, $e^\#((o_2, o)) = \{10^3, \ldots, 10^{12}\}$, $e^\#((o_2, o')) = \{10^{13}, \ldots, 10^{22}\}$, $e^\#((o_4, o)) = \{10^{23}, \ldots, 10^{122}\}$ and $e^\#((o_4, o'))$ $= \{10^{123}, \ldots, 10^{222}\}$.

For defining Taylor expansion, we only need to define $\mathcal{T}_R[i](e)$ with $i = 0$, which is the differential in-PS obtained by (fully) expanding the boxes according to e. But a key tool of the proof is the introduction of the differential in-PS's $\mathcal{T}_R[i](e)$ with $i > 0$, which are the differential in-PS's obtained by expanding only the boxes of depth at least i. We recall that the notation $\langle o, R\rangle$ for any differential in-PS R was introduced in Definition 1.14.

Definition 2.3. Let R be an in-PS of depth d. Let e be a pseudo-experiment on R. Let $i \in \mathbb{N}$. We define, by induction on d, a differential in-PS $\mathcal{T}_R[i](e)$ of depth $\min\{i, d\}$ and a function $\kappa_R[i](e) : \mathcal{P}(\mathcal{T}_R[i](e)) \to \mathcal{P}(R)$ as follows: We set $\mathcal{T}_R[i](e) = S@t$, where $S = R^{\leq i} \oplus \bigoplus_{o \in \mathcal{B}_0^{\geq i}(R)} \bigoplus_{e_o \in e(o)} \langle o, \langle e_o, \mathcal{T}_{B_R(o)}[i](e_o)\rangle\rangle$ and t is the function $\mathcal{W}_0 \cup \mathcal{W}_{>0} \to \mathcal{P}_0^e(R)$ with

- $\mathcal{W}_0 = \bigcup_{o \in \mathcal{B}_0^{\geq i}(R)} \bigcup_{e_o \in e(o)} \{(o, (e_o, q)); (q \in \mathcal{P}^{\mathsf{f}}(\mathcal{G}(\mathcal{T}_{B_R(o)}[i](e_o))) \wedge \kappa_{B_R(o)}[i](e_o)(q) \in \mathcal{P}_R^{\mathsf{f}}(o))\}$
- $\mathcal{W}_{>0} = \bigcup_{(o,(e_o,o')) \in \mathcal{B}_0(S) \setminus \mathcal{B}_0(R^{\leq i})} \left\{ ((o, (e_o, o')), q); \begin{array}{l} ((o', q) \in \mathcal{P}_{>0}^{\mathsf{f}}(\mathcal{T}_{B_R(o)}[i](e_o)) \\ \wedge \kappa_{B_R(o)}[i](e_o)(o', q) \in \mathcal{P}_R^{\mathsf{f}}(o)) \end{array} \right\}$
- and $t(p) = \begin{cases} t_R(o, \kappa_{B_R(o)}[i](e_o)(q)) & \textit{if } p = (o, (e_o, q)) \in \mathcal{W}_0; \\ t_R(o, \kappa_{B_R(o)}[i](e_o)(o', q)) & \textit{if } p = ((o, (e_o, o')), q) \in \mathcal{W}_{>0}. \end{cases}$

For any $p \in \mathcal{P}(\mathcal{T}_R[i](e))$, the port $\kappa_R[i](e)(p)$ of R is the following one:

$$\begin{cases} p & \textit{if } p \in \mathcal{P}(R^{\leq i}); \\ (o, \kappa_{B_R(o)}[i](e_o)(p')) & \textit{if } p = (o, (e_o, p')) \in \mathcal{P}_0(\mathcal{T}_R[i](e)) \setminus \mathcal{P}_0(R); \\ (o, \kappa_{B_R(o)}[i](e_o)(o', p')) & \textit{if } p = ((o, (e_o, o')), p') \textit{ and } (o, (e_o, o')) \in \mathcal{B}_0(\mathcal{T}_R[i](e)) \setminus \mathcal{B}_0(R^{\leq i}). \end{cases}$$

If $o \in \mathcal{B}_0^{\geq i}(R)$ and e_o is a pseudo-experiment on $B_R(o)$, then we set

$$R\langle o, i, e_o\rangle = \langle o, \langle e_o, \mathcal{T}_{B_R(o)}[i](e_o)\rangle\rangle.$$

Remark 2.4. We have $\mathcal{W}_0 \subseteq \mathcal{W}_0(\mathcal{T}_R[i](e))$ and $\mathsf{dom}(t_{\mathcal{T}_R[i](e)})$ is the set

$$\mathsf{dom}(t_{R^{\leq i}}) \cup \left(\bigcup_{o \in \mathcal{B}_0^{\geq i}(R)} \bigcup_{e_o \in e(o)} \bigcup_{o' \in \mathcal{B}_0(\mathcal{T}_{B_R(o)}[i](e_o))} \{((o, (e_o, o')), p); p \in \mathcal{P}^{\mathsf{f}}_{\mathcal{T}_{B_R(o)}[i](e_o)}(o')\} \right) \cup \mathcal{W}_{>0}$$

Moreover, for any $p \in \mathsf{dom}(t_{R^{\leq i}})$, we have $t_{\mathcal{T}_R[i](e)}(p) = t_R(p)$.

Notice that if R is cut-free, then $\mathcal{T}_R[i](e)$ is cut-free too. The conclusions are the duplications of the conclusions; in particular, if R is a PS, then $\mathcal{T}_R[i](e)$ is a differential PS:

Fact 2.5. Let R be an in-PS. Let e be a pseudo-experiment on R. Let $i \in \mathbb{N}$. Then we have $(\forall q \in \mathcal{P}(\mathcal{T}_R[i](e)))(\kappa_R[i](e)(q) \in \mathcal{P}^{\mathsf{f}}(R) \Leftrightarrow q \in \mathcal{P}^{\mathsf{f}}(\mathcal{T}_R[i](e)))$.

Proof. By induction on *depth*(R). Let $q \in \mathcal{P}(\mathcal{T}_R[i](e))$ such that $\kappa_R[i](e)(q) \in \mathcal{P}^{\mathsf{f}}(R)$. We distinguish between two cases:

- $\kappa_R[i](e)(q) \in \mathcal{P}^{\mathsf{f}}(R^{\leq i})$: we have $q = \kappa_R[i](e)(q) \in \mathcal{P}^{\mathsf{f}}(R^{\leq i}) \subseteq \mathcal{P}^{\mathsf{f}}(\mathcal{T}_R[i](e))$;
- There exist $o \in \mathcal{B}_0^{\geq i}(R)$ and $p \in \mathcal{P}^{\mathsf{f}}(B_R(o)) \setminus \mathcal{P}_R^{\mathsf{f}}(o)$ such that $\kappa_R[i](e)(q) = (o, p)$: Then
  - either there exist $e_o \in e(o)$ and $q' \in \mathcal{P}_0(\mathcal{T}_{B_R(o)}[i](e_o))$ such that $q = (o, (e_o, q'))$ and $\kappa_{B_R(o)}[i](e_o)(q') = p$, and then, by induction hypothesis, $q' \in \mathcal{P}^{\mathsf{f}}(\mathcal{T}_{B_R(o)}[i](e_o))$, hence $q = (o, (e_o, q')) \in \mathcal{P}^{\mathsf{f}}(R\langle o, i, e_o\rangle)$; since $\kappa_{B_R(o)}[i](e_o)(q') \notin \mathcal{P}_R^{\mathsf{f}}(o)$, we have $q \notin \mathsf{dom}(t_{\mathcal{G}(\mathcal{T}_R[i](e))})$ and we obtain $q \in \mathcal{P}^{\mathsf{f}}(\mathcal{T}_R[i](e))$;

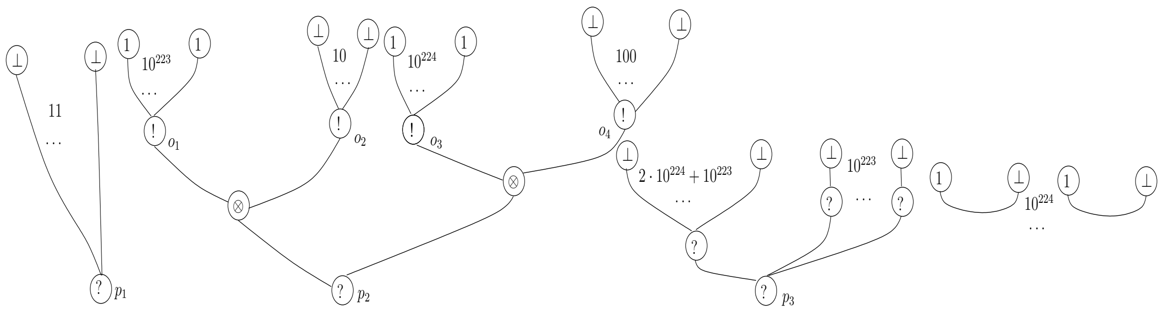


Figure 14: The differential PS $S0_1$

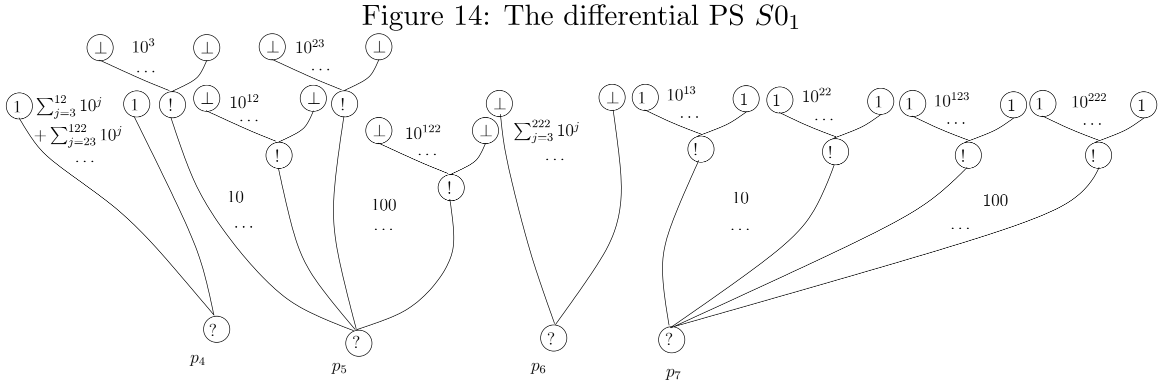


Figure 15: The differential PS $S0_2$

– or there exist $e_o \in e(o)$, $o' \in \mathcal{B}_0(\mathcal{T}_{B_R(o)}[i](e_o))$ and $q' \in \mathcal{P}(B_{\mathcal{T}_{B_R(o)}[i](e_o)}(o'))$ such that $q = ((o,(e_o,o')),q')$ and $\kappa_{B_R(o)}[i](e_o)(o',q') = p$, and then, by induction hypothesis, $(o',q') \in \mathcal{P}^{\mathsf{f}}(\mathcal{T}_{B_R(o)}[i](e_o))$, hence $q = ((o,(e_o,o')),q') \in \mathcal{P}^{\mathsf{f}}(R\langle o,i,e_o\rangle)$; since $\kappa_{B_R(o)}[i](e_o)(o',q') \notin \mathcal{P}^{\mathsf{f}}_R(o)$, we have $q \notin \mathsf{dom}(t_{\mathcal{T}_R[i](e)})$ and we obtain $q \in \mathcal{P}^{\mathsf{f}}(\mathcal{T}_R[i](e))$.

Conversely, let $q \in \mathcal{P}^{\mathsf{f}}(\mathcal{T}_R[i](e))$. We distinguish between three cases:

- $q \in \mathcal{P}^{\mathsf{f}}(R^{\leq i})$: we have $\kappa_R[i](e)(q) = q \in \mathcal{P}^{\mathsf{f}}(R^{\leq i}) \subseteq \mathcal{P}^{\mathsf{f}}(\mathcal{T}_R[i](e))$;
- There exist $o \in \mathcal{B}_0^{\geq i}(R)$, $e_o \in e(o)$ and $q' \in \mathcal{P}^{\mathsf{f}}(\mathcal{G}(\mathcal{T}_{B_R(o)}[i](e_o)))$ such that $\kappa_{B_R(o)}[i](e_o)(q') \notin \mathcal{P}^{\mathsf{f}}_R(o)$ and $q = (o,(e_o,q'))$: by induction hypothesis, we have $\kappa_{B_R(o)}[i](e_o)(q') \in \mathcal{P}^{\mathsf{f}}(B_R(o))$, hence $\kappa_R[i](e)(q) = (o,\kappa_{B_R(o)}[i](e_o)(q'))$ with $\kappa_{B_R(o)}[i](e_o)(q') \in \mathcal{P}^{\mathsf{f}}(B_R(o)) \setminus \mathcal{P}^{\mathsf{f}}_R(o)$; we thus have $\kappa_R[i](e)(q) \in \mathcal{P}^{\mathsf{f}}(R)$.
- There exist $o \in \mathcal{B}_0^{\geq i}(R)$, $e_o \in e(o)$, $o' \in \mathcal{B}_0(\mathcal{T}_{B_R(o)}[i](e_o))$ and $q' \in \mathcal{P}^{\mathsf{f}}(B_{\mathcal{T}_R[i](e)}(o,(e_o,o'))) \setminus \mathcal{P}^{\mathsf{f}}_{\mathcal{T}_R[i](e)}((o,(e_o,o')))$ such that $\kappa_{B_R(o)}[i](e_o)(o',q') \notin \mathcal{P}^{\mathsf{f}}_R(o)$ and $q = ((o,(e_o,o')),q')$: we have $q' \in \mathcal{P}^{\mathsf{f}}(B_{\mathcal{T}_{B_R(o)}[i](e_o)}(o')) \setminus \mathcal{P}^{\mathsf{f}}_{\mathcal{T}_{B_R(o)}[i](e_o)}(o')$, hence $(o',q') \in \mathcal{P}^{\mathsf{f}}(\mathcal{T}_{B_R(o)}[i](e_o))$; by induction hypothesis, we have $\kappa_{B_R(o)}[i](e_o)(o',q') \in \mathcal{P}^{\mathsf{f}}(B_R(o))$; since $\kappa_{B_R(o)}[i](e_o)(o',q') \notin \mathcal{P}^{\mathsf{f}}_R(o)$, we have $\kappa_R[i](e)(q) = (o,\kappa_{B_R(o)}[i](e_o)(o',q')) \in \mathcal{P}^{\mathsf{f}}(R)$. □

Example 2.6. Generally, given a PS R and an integer i, the PS $\mathcal{T}_R[i](e)$ does not depend only on $e^{\#}$ (we can have ${e_1}^{\#} = {e_2}^{\#}$ and not $\mathcal{T}_R[i](e_1) \equiv \mathcal{T}_R[i](e_2)$), but with the PS R depicted in Figure 11, it is not the case: with *this* PS R, we have $(\forall i \in \mathbb{N})(\forall e_1, e_2 \in \mathfrak{E}(R))({e_1}^{\#} = {e_2}^{\#} \Rightarrow \mathcal{T}_R[i](e_1) \equiv \mathcal{T}_R[i](e_2))$. If R is this PS and e some pseudo-experiment such that $e^{\#}$ is as

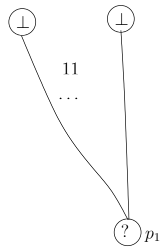


Figure 16: The differential PS $S1_1$

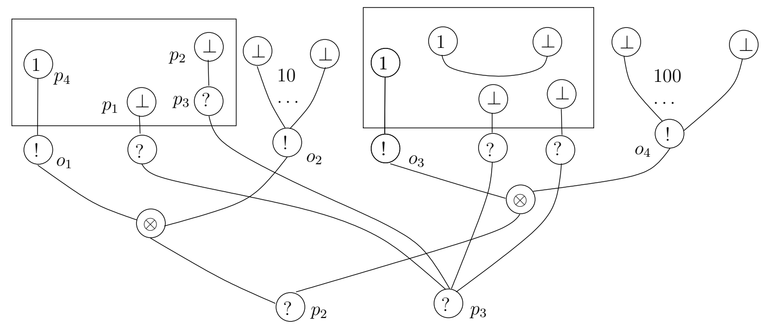


Figure 17: The differential PS $S1_2$

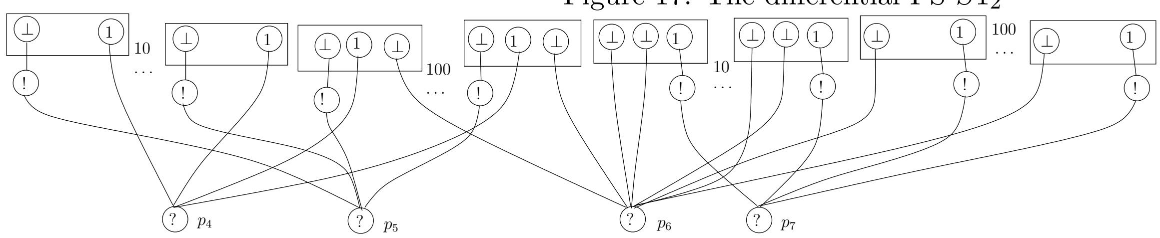


Figure 18: The differential PS $S1_3$

described in Example 2.2, then $\mathcal{T}_R[0](e) = S0_1 \oplus S0_2$, where $S0_1$ and $S0_2$ are the differential PS's depicted in Figures 14 and 15 respectively, and $\mathcal{T}_R[1](e) = S1_1 \oplus S1_2 \oplus S1_3$, where $S1_1$, $S1_2$ and $S1_3$ are the differential PS's depicted in Figures 16, 17 and 18 respectively.

The Taylor expansion we consider has no coefficients (i.e. has coefficients in the Boolean semiring $\mathbb{B} = \{0, 1\}$, where $1 + 1 = 1$). In other words, we consider the support of Taylor expansion with coefficients:

Definition 2.7. Let R be a PS. A *Taylor expansion of* R is a set $\mathcal{T}_R[0]$ of simple differential nets such that $\mathcal{T}_R[0] \equiv \{\mathcal{T}_R[0](e); e \in \mathfrak{E}(R)\}$.

It is clear that two Taylor expansions of R are the same sets of simple differential nets up to the names of the ports that are not conclusions, that is why it makes sense to speak about "the" Taylor expansion of a PS.

An important case for our proof is the partial Taylor expansion of an in-PS of the form $\varphi \cdot_o R$ for some in-PS R, some box o of R at depth 0 and some bijection $\varphi : \mathcal{P}^?_R(o) \simeq \mathcal{Q}'$. Since we have $\mathfrak{E}(\varphi \cdot_o R) = \mathfrak{E}(R)$, we can compare $\mathcal{T}_R[i](e)$ with $\mathcal{T}_{(\varphi \cdot_o R)}[i](e)$. From $B_{B_{(\varphi \cdot_o R)}(o)} = B_{B_R(o)}$, we already deduce $B_{\mathcal{T}_{B_{(\varphi \cdot_o R)}(o)}[i](e)} = B_{\mathcal{T}_{B_R(o)}[i](e)}$; more information is given by the following lemma:

Lemma 2.8. *Let R be an in-PS. Let $o \in \mathcal{B}_0(R)$. Let e_o be a pseudo-experiment on $B_R(o)$. Let φ be some bijection $\mathcal{P}^?_R(o) \simeq \mathcal{Q}'$. Let R_o be an in-PS such that $R_o = \varphi \cdot_o R$. Let $i \in \mathbb{N}$. Then we have:*

- $t_{\mathcal{T}_{B_R(o)}[i](e_o)} = t_{\mathcal{T}_{B_{R_o}(o)}[i](e_o)}\Big|_{\{p \in \mathsf{dom}(t_{\mathcal{T}_{B_{R_o}(o)}[i](e_o)}); t_{\mathcal{T}_{B_{R_o}(o)}[i](e_o)}(p) \notin \mathcal{Q}'\}}$
- *and, for any $o_1 \in \mathcal{B}_0(\mathcal{T}_{B_{R_o}(o)}[i](e_o))$, for any $p \in \mathcal{P}^{\mathsf{f}}_{\mathcal{T}_{B_{R_o}(o)}[i](e_o)}(o_1)$ such that*

$$t_{\mathcal{T}_{B_{R_o}(o)}[i](e_o)}(o_1, p) \in \mathcal{Q}',$$

we have

$$t_{\mathcal{T}_{B_{R_o}(o)}[i](e_o)}(o_1, p) = \varphi(t_{\mathcal{T}_R[i](e)}((o, (e_o, o_1)), p))$$

Proof. We set

$$S = B_R(o)^{\leq i} \oplus \bigoplus_{o' \in \mathcal{B}_0^{\geq i}(B_R(o))} \bigoplus_{e_{o'} \in e_o(o')} \langle o', \langle e_{o'}, \mathcal{T}_{B_{B_R(o)}(o')}[i](e_{o'}) \rangle \rangle$$

and

$$S_o = B_{R_o}(o)^{\leq i} \oplus \bigoplus_{o' \in \mathcal{B}_0^{\geq i}(B_{R_o}(o))} \bigoplus_{e_{o'} \in e_o(o')} \langle o', \langle e_{o'}, \mathcal{T}_{B_{B_{R_o}(o)}(o')}[i](e_{o'}) \rangle \rangle$$

Notice that

$$S_o = B_{R_o}(o)^{\leq i} \oplus \bigoplus_{o' \in \mathcal{B}_0^{\geq i}(B_R(o))} \bigoplus_{e_{o'} \in e_o(o')} \langle o', \langle e_{o'}, \mathcal{T}_{B_{B_R(o)}(o')}[i](e_{o'}) \rangle \rangle$$

and $\mathcal{B}_0(S_o) = \mathcal{B}_0(S)$. Let $p \in \mathsf{dom}(t_{\mathcal{T}_{B_R(o)}[i](e_o)})$ and let us show that $p \in \mathsf{dom}(t_{\mathcal{T}_{B_{R_o}(o)}[i](e_o)})$. We distinguish between two cases:

- $p \in \mathsf{dom}(t_{B_R(o)^{\leq i}})$: Then $p \in \mathsf{dom}(t_{B_{R_o}(o)^{\leq i}}) \subseteq \mathsf{dom}(t_{\mathcal{T}_{B_{R_o}(o)}[i](e_o)})$.
- $p \notin \mathsf{dom}(t_{B_R(o)^{\leq i}})$: Then there exists $(o', (e_{o'}, o'')) \in \mathcal{B}_0(S) \setminus \mathcal{B}_0(B_R(o)^{\leq i}) = \mathcal{B}_0(S_o) \setminus \mathcal{B}_0(B_{R_o}(o)^{\leq i})$ such that p is an element of the set

$$\begin{aligned} & \left\{ ((o', (e_{o'}, o'')), q); \begin{array}{l} ((o'', q) \in \mathcal{P}^{\mathsf{f}}_{>0}(\mathcal{T}_{B_{B_R(o)}(o')}[i](e_{o'})) \\ \wedge \kappa_{B_{B_R(o)}(o')}[i](e_{o'})(o'', q) \in \mathcal{P}^{\mathsf{f}}_{B_R(o)}(o')) \end{array} \right\} \\ = & \left\{ ((o', (e_{o'}, o'')), q); \begin{array}{l} ((o'', q) \in \mathcal{P}^{\mathsf{f}}_{>0}(\mathcal{T}_{B_{B_{R_o}(o)}(o')}[i](e_{o'})) \\ \wedge \kappa_{B_{B_{R_o}(o)}(o')}[i](e_{o'})(o'', q) \in \mathcal{P}^{\mathsf{f}}_{B_R(o)}(o')) \end{array} \right\} \\ \subseteq & \left\{ ((o', (e_{o'}, o'')), q); \begin{array}{l} ((o'', q) \in \mathcal{P}^{\mathsf{f}}_{>0}(\mathcal{T}_{B_{B_{R_o}(o)}(o')}[i](e_{o'})) \\ \wedge \kappa_{B_{B_{R_o}(o)}(o')}[i](e_{o'})(o'', q) \in \mathcal{P}^{\mathsf{f}}_{B_{R_o}(o)}(o')) \end{array} \right\} \\ \subseteq & \ \mathsf{dom}(t_{\mathcal{T}_{B_{R_o}(o)}[i](e_o)}) \end{aligned}$$

Conversely, let $p \in \mathsf{dom}(t_{\mathcal{T}_{B_{R_o}(o)}[i](e_o)})$ such that $t_{\mathcal{T}_{B_{R_o}(o)}[i](e_o)}(p) \notin \mathcal{Q}'$; we will check that $p \in \mathsf{dom}(t_{\mathcal{T}_{B_R(o)}[i](e_o)})$ and $t_{\mathcal{T}_{B_{R_o}(o)}[i](e_o)}(p) = t_{\mathcal{T}_{B_R(o)}[i](e_o)}(p)$. We distinguish between two cases:

- $p \in \mathsf{dom}(t_{B_{R_o}(o)^{\leq i}})$: Then $p \in \mathsf{dom}(t_{(B_R(o) \oplus \bigoplus_{q' \in \mathcal{Q}'} ?_{q'})^{\leq i}}) = \mathsf{dom}(t_{B_R(o)^{\leq i}}) \subseteq \mathsf{dom}(t_{\mathcal{T}_{B_R(o)}[i](e_o)})$. In this case, we have $t_{\mathcal{T}_{B_R(o)}[i](e_o)}(p) = t_{B_R(o)}(p) = t_{B_{R_o}(o)}(p) = t_{\mathcal{T}_{B_{R_o}(o)}[i](e_o)}(p)$.
- $p \notin \mathsf{dom}(t_{B_{R_o}(o)^{\leq i}})$: Then there exists $(o', (e_{o'}, o'')) \in \mathcal{B}_0(S_o) \setminus \mathcal{B}_0(B_{R_o}(o)^{\leq i}) = \mathcal{B}_0(S) \setminus \mathcal{B}_0(B_R(o)^{\leq i})$ such that p is an element of the set

$$\begin{aligned} & \left\{ ((o', (e_{o'}, o'')), q); \begin{array}{l} ((o'', q) \in \mathcal{P}^{\mathsf{f}}_{>0}(\mathcal{T}_{B_{B_{R_o}(o)}(o')}[i](e_{o'})) \\ \wedge \kappa_{B_{B_{R_o}(o)}(o')}[i](e_{o'})(o'', q) \in \mathcal{P}^{\mathsf{f}}_{B_{R_o}(o)}(o')) \end{array} \right\} \\ = & \left\{ ((o', (e_{o'}, o'')), q); \begin{array}{l} ((o'', q) \in \mathcal{P}^{\mathsf{f}}_{>0}(\mathcal{T}_{B_{B_R(o)}(o')}[i](e_{o'})) \\ \wedge \kappa_{B_{B_R(o)}(o')}[i](e_{o'})(o'', q) \in \mathcal{P}^{\mathsf{f}}_{B_{R_o}(o)}(o')) \end{array} \right\} \\ & \textit{(because } B_{B_{R_o}(o)}(o') = B_{B_R(o)}(o')\textit{)} \end{aligned}$$

$$\begin{array}{rl} = & \left\{((o',(e_{o'},o'')),q);\ \begin{array}{l}((o'',q)\in\mathcal{P}^{\mathsf{f}}_{>0}(\mathcal{T}_{B_{B_R(o)}(o')}[i](e_{o'}))\\ \wedge\kappa_{B_{B_R(o)}(o')}[i](e_{o'})(o'',q)\in\mathcal{P}^{\mathsf{f}}_{B_R(o)}(o'))\end{array}\right\}\\ \cup & \left\{((o',(e_{o'},o'')),q);\ \begin{array}{l}((o'',q)\in\mathcal{P}^{\mathsf{f}}_{>0}(\mathcal{T}_{B_{B_R(o)}(o')}[i](e_{o'}))\\ \wedge\kappa_{B_{B_R(o)}(o')}[i](e_{o'})(o'',q)\in\mathcal{P}^{\mathsf{f}}_{B_{R_o}(o)}(o')\\ \wedge t_{B_{R_o}(o)}(o',\kappa_{B_{B_R(o)}(o')}[i](e_{o'})(o'',q))\in\mathcal{Q}')\end{array}\right\}\\ & \textit{(by Remark 1.35, } \mathcal{P}^{\mathsf{f}}_{B_{R_o}(o)}(o')=\mathcal{P}^{\mathsf{f}}_{B_R(o)}(o')\cup\{q'\in\mathcal{P}^{\mathsf{f}}_{B_{R_o}(o)}(o');t_{B_{R_o}(o)}(o',q')\in\mathcal{Q}'\}\textit{)}\\ = & \left\{((o',(e_{o'},o'')),q);\ \begin{array}{l}((o'',q)\in\mathcal{P}^{\mathsf{f}}_{>0}(\mathcal{T}_{B_{B_R(o)}(o')}[i](e_{o'}))\\ \wedge\kappa_{B_{B_R(o)}(o')}[i](e_{o'})(o'',q)\in\mathcal{P}^{\mathsf{f}}_{B_R(o)}(o'))\end{array}\right\}\\ \cup & \left\{((o',(e_{o'},o'')),q);\ \begin{array}{l}((o'',q)\in\mathcal{P}^{\mathsf{f}}_{>0}(\mathcal{T}_{B_{B_R(o)}(o')}[i](e_{o'}))\\ \wedge t_{\mathcal{T}_{B_{R_o}(o)}[i](e_o)}((o',(e_{o'},o'')),q)\in\mathcal{Q}')\end{array}\right\}\\ = & \left\{((o',(e_{o'},o'')),q);\ \begin{array}{l}((o'',q)\in\mathcal{P}^{\mathsf{f}}_{>0}(\mathcal{T}_{B_{B_R(o)}(o')}[i](e_{o'}))\\ \wedge\kappa_{B_{B_R(o)}(o')}[i](e_{o'})(o'',q)\in\mathcal{P}^{\mathsf{f}}_{B_R(o)}(o'))\end{array}\right\}\\ & \textit{(by assumption, we have } t_{\mathcal{T}_{B_{R_o}(o)}[i](e_o)}(p)\notin\mathcal{Q}'\textit{)}\\ \subseteq & \mathsf{dom}(t_{\mathcal{T}_{B_R(o)}[i](e_o)})\end{array}$$

In this case, if q is such that $p=((o',(e_{o'},o'')),q)$, then we have

$$\begin{array}{rcl} t_{\mathcal{T}_{B_R(o)}[i](e_o)}(p) & = & t_{B_R(o)}(o',\kappa_{B_{B_R(o)}(o')}[i](e_{o'})(o'',q))\\ & = & t_{B_{R_o}(o)}(o',\kappa_{B_{B_{R_o}(o)}(o')}[i](e_{o'})(o'',q))\\ & = & t_{\mathcal{T}_{B_{R_o}(o)}[i](e_o)}(p)\end{array}$$

We thus have $t_{\mathcal{T}_{B_R(o)}[i](e_o)}=t_{\mathcal{T}_{B_{R_o}(o)}[i](e_o)}\Big|_{\{p\in\mathsf{dom}(t_{\mathcal{T}_{B_{R_o}(o)}[i](e_o)});t_{\mathcal{T}_{B_{R_o}(o)}[i](e_o)}(p)\notin\mathcal{Q}'\}}$.

Now, let $o_1\in\mathcal{B}_0(\mathcal{T}_{B_{R_o}(o)}[i](e_o))$ and $p\in\mathcal{P}^{\mathsf{f}}_{\mathcal{T}_{B_{R_o}(o)}[i](e_o)}(o_1)$ such that $t_{\mathcal{T}_{B_{R_o}(o)}[i](e_o)}(o_1,p)\in\mathcal{Q}'$.

- If $o_1\in\mathcal{B}_0^{<i}(B_{R_o}(o))$, then $(o_1,p)\in\mathcal{P}^{\mathsf{f}}_R(o)$, hence $t_{\mathcal{T}_{B_{R_o}(o)}[i](e_o)}(o_1,p)=t_{B_{R_o}(o)}(o_1,p)=\varphi(t_R(o,(o_1,p)))=\varphi(t_R(o,\kappa_{B_{R_o}(o)}[i](e_o)(o_1,p)))$;
- if $o_1=(o',(e_{o'},o''))$ with $o'\in\mathcal{B}_0^{\geq i}(B_{R_o}(o))$, then $((o',(e_{o'},o'')),p)\notin\mathcal{P}^{\mathsf{f}}(\mathcal{T}_{B_{R_o}(o)}[i](e_o))$, hence, by Fact 2.5, $(o',\kappa_{B_{B_{R_o}(o)}(o')}[i](e_{o'})(o'',p))=\kappa_{B_{R_o}(o)}[i](e_o)((o',(e_{o'},o'')),p)\notin\mathcal{P}^{\mathsf{f}}(B_{R_o}(o))$. Moreover, since $t_{\mathcal{T}_{B_{R_o}(o)}[i](e_o)}(o_1,p)\in\mathcal{Q}'$, we have $(o'',p)\in\mathcal{P}^{\mathsf{f}}(\mathcal{T}_{B_{B_{R_o}(o)}(o')}[i](e_{o'}))$ (otherwise, $t_{\mathcal{T}_{B_{B_{R_o}(o)}(o')}[i](e_{o'})}(o'',p)\in\mathcal{P}_0(\mathcal{T}_{B_{B_{R_o}(o)}(o')}[i](e_{o'}))=\mathcal{P}_0(\mathcal{T}_{B_{B_R(o)}(o')}[i](e_{o'}))$, which entails $t_{\mathcal{T}_{B_{R_o}(o)}[i](e_o)}(o_1,p)\in\mathcal{P}_0(\mathcal{T}_{B_R(o)}[i](e_o))$, which contradicts $t_{\mathcal{T}_{B_{R_o}(o)}[i](e_o)}(o_1,p)\in\mathcal{Q}'$). By Fact 2.5, we obtain $\kappa_{B_{B_{R_o}(o)}(o')}[i](e_{o'})(o'',p)\in\mathcal{P}^{\mathsf{f}}(B_{B_{R_o}(o)}(o'))$. We showed $\kappa_{B_{B_{R_o}(o)}(o')}[i](e_{o'})(o'',p)\in\mathcal{P}^{\mathsf{f}}_{B_{R_o}(o)}(o')$. We thus have

$$\begin{array}{rcl} t_{\mathcal{T}_{B_{R_o}(o)}[i](e_o)}(o_1,p) & = & t_{B_{R_o}(o)}(o',\kappa_{B_{B_{R_o}(o)}(o')}[i](e_{o'})(o'',p))\\ & = & \varphi(t_R(o,(o',\kappa_{B_{B_{R_o}(o)}(o')}[i](e_{o'})(o'',p))))\\ & = & \varphi(t_R(o,\kappa_{B_{R_o}(o)}[i](e_o)(o_1,p)));\end{array}$$

now, we have $t_R(o, \kappa_{B_{R_o}(o)}[i](e_o)(o_1, p)) = t_R(o, \kappa_{B_R(o)}[i](e_o)(o_1, p)) = t_{\mathcal{T}_R[i](e)}((o,(e_o, o_1)), p)$. □

The rest of this section is devoted to show Proposition 2.12, which shows how one can compute, for any in-PS R, the arity in $\mathcal{T}_R[i](e)$ of a port at depth 0 of R. For that purpose, we introduce the function $b_S^{\geq i}$ that associates with every port p of S at depth greater than i the deepest box of depth at least i that contains p:

Definition 2.9. For any differential in-PS S, for any $i \in \mathbb{N}$, we define, by induction on $depth(S)$, the function $b_S^{\geq i} : \mathcal{P}_{>i}(S) \to \mathcal{B}^{\geq i}(S)$ as follows:

$$\begin{array}{rcl} b_S^{\geq i} : \mathcal{P}_{>i}(S) & \to & \mathcal{B}^{\geq i}(S) \\ (o,p) & \mapsto & \begin{cases} o & \text{if } o \in \mathcal{B}_0^{\geq i}(S) \text{ and } p \in \mathcal{P}_{\leq i}(B_S(o)); \\ (o, b_{B_S(o)}^{\geq i}(p)) & \text{if } o \in \mathcal{B}_0^{\geq i}(S) \text{ and } p \in \mathcal{P}_{>i}(B_S(o)); \end{cases} \end{array}$$

Fact 2.10. Let R be an in-PS. Let e be a pseudo-experiment on R. Let $i \in \mathbb{N}$. Then we have

$$(\forall p \in \mathcal{P}(\mathcal{T}_R[i](e)) \setminus \mathcal{P}_0(R))(p \in \mathcal{P}_0(\mathcal{T}_R[i](e)) \Leftrightarrow (b_R^{\geq 0} \circ \kappa_R[e](i))(p) \in \mathcal{B}^{\geq i}(R))$$

Proof. First notice that, for any differential in-PS S, we have:

$$(\forall p \in \mathcal{P}_{>0}(S))(b_S^{\geq 0}(p) \in \mathcal{B}^{\geq i}(S) \Rightarrow p \in \mathcal{P}_{>i}(S)) \quad (*)$$

Indeed: Let $p \in \mathcal{P}_{>0}(S)$ such that $b_S^{\geq 0}(p) \in \mathcal{B}^{\geq i}(S)$ and let $o \in \mathcal{B}_0(S)$ and $p' \in \mathcal{P}(B_S(o))$ such that $p = (o, p')$; we distinguish between two cases:

- $p' \in \mathcal{P}_0(B_S(o))$: We have $b_S^{\geq 0}(p) = o \in \mathcal{B}_0^{\geq i}(S)$;
- $p' \in \mathcal{P}_{>0}(B_S(o))$: We have $b_S^{\geq 0}(p) = (o, b_{B_S(o)}^{\geq 0}(p')) \in \mathcal{B}_0^{\geq i}(S)$, hence $o \in \mathcal{B}^{\geq i}(S)$;

in both cases we have $o \in \mathcal{B}_0^{\geq i}(S)$, hence $p = (o, p') \in \mathcal{P}_{>i}(S)$.

We prove now the fact by induction on $depth(R)$. If $depth(R) = 0$, then $\mathcal{P}(\mathcal{T}_R[i](e)) \setminus \mathcal{P}_0(R) = \emptyset$. Otherwise, let $p \in \mathcal{P}(\mathcal{T}_R[i](e)) \setminus \mathcal{P}_0(R) \neq \emptyset$:

- If $p \in \mathcal{P}_0(\mathcal{T}_R[i](e))$, then there exist $o \in \mathcal{B}_0^{\geq i}(R)$, $e_o \in e(o)$ and $p' \in \mathcal{P}_0(\mathcal{T}_{B_R(o)}[i](e_o))$ such that $p = (o, (e_o, p'))$; we distinguish between two cases:
  - $p' \in \mathcal{P}_0(B_R(o))$: We have $b_R^{\geq 0}(\kappa_R[i](e)(p)) = b_R^{\geq 0}(o, p') = o \in \mathcal{B}^{\geq i}(R)$.
  - $p' \in \mathcal{P}_{>0}(B_R(o))$: By induction hypothesis, we have $b_{B_R(o)}^{\geq 0}(\kappa_{B_R(o)}[i](e_o)(p')) \in \mathcal{B}^{\geq i}(B_R(o))$, hence, by $(*)$, $\kappa_{B_R(o)}[i](e_o)(p') \in \mathcal{P}_{>i}(B_R(o))$; we thus have
  $$b_R^{\geq 0}(\kappa_R[i](e)(p)) = b_R^{\geq 0}(o, \kappa_{B_R(o)}[i](e_o)(p')) = (o, b_{B_R(o)}^{\geq 0}(\kappa_{B_R(o)}[i](e_o)(p'))) \in \mathcal{B}^{\geq i}(R).$$
- If $p \in \mathcal{P}_{>0}(\mathcal{T}_R[i](e))$, then there exist $o \in \mathcal{B}_0(\mathcal{T}_R[i](e))$ and $p' \in \mathcal{P}(B_{\mathcal{T}_R[i](e)}(o))$ such that $p = (o, p')$; we distinguish between two cases:
  - $o \in \mathcal{B}_0^{<i}(R)$: Then $p' \in \mathcal{P}_{\leq i}(B_R(o))$, hence $b_R^{\geq 0}(\kappa_R[i](e)(p)) = b_R^{\geq 0}(o, p') = o \in \mathcal{B}^{<i}(R)$;
  - $o = (o_1, (e_1, o'))$ with $o_1 \in \mathcal{B}_0^{\geq i}(R)$, $e_1 \in e(o_1)$ and $o' \in \mathcal{B}_0(\mathcal{T}_{B_R(o_1)}[i](e_1))$: By induction hypothesis, we have $b_{B_R(o_1)}^{\geq 0}(\kappa_{B_R(o_1)}[i](e_1)(o', p')) \in \mathcal{B}^{<i}(B_R(o_1))$, hence

$$b_R^{\geq 0}(\kappa_R[i](e)(p)) = b_R^{\geq 0}(o_1, \kappa_{B_R(o_1)}[i](e_1)(o', p')) = (o_1, b_{B_R(o_1)}^{\geq 0}(\kappa_{B_R(o_1)}[i](e_1)(o', p'))) \in \mathcal{B}^{<i}(R).$$

□

If a port q of a PS R is deep enough, then it is duplicated $\sum e^{\#}(b_R^{\geq i}(q))$ times in $\mathcal{T}_R[i](e)$:

Lemma 2.11. *Let R be an in-PS. Let e be a pseudo-experiment on R. Let $i \in \mathbb{N}$. Let $q \in \mathcal{P}_{>i}(R)$. Then we have*

$$\mathsf{Card}\left(\{p \in \mathcal{P}(\mathcal{T}_R[i](e)); \kappa_R[i](e)(p) = q\}\right) = \sum e^{\#}(b_R^{\geq i}(q))$$

Proof. By induction on $depth(R)$. If $depth(R) = 0$, then there is no such q. Otherwise: Let $o \in \mathcal{B}_0^{\geq i}(R)$ and $q' \in \mathcal{P}(B_R(o))$ such that $q = (o, q')$. We distinguish between three cases:

- $q' \in \mathcal{P}_{\leq i}(B_R(o))$: we have $b_R^{\geq 0}(q) = b_R^{\geq 0}(o, q') = o \in \mathcal{B}^{\geq i}(R)$, hence, by Fact 2.10, we have $\{p \in \mathcal{P}(\mathcal{T}_R[i](e)); \kappa_R[i](e)(p) = q\} = \{p \in \mathcal{P}_0(\mathcal{T}_R[i](e)) \setminus \mathcal{P}_0(R); \kappa_R[i](e)(p) = (o, q')\} = \{(o, (e_o, q')); e_o \in e(o)\}$, hence

$$\begin{array}{rcl}\mathsf{Card}\left(\{p \in \mathcal{P}(\mathcal{T}_R[i](e)); \kappa_R[i](e)(p) = q\}\right) & = & \mathsf{Card}\left(e(o)\right) \\ & = & \sum e^{\#}(o) \\ & = & \sum e^{\#}(b_R^{\geq i}(q))\end{array}$$

- $q' \in \mathcal{P}_{>i}(B_R(o))$ and $b_{B_R(o)}^{\geq 0}(q') \in \mathcal{B}^{\geq i}(B_R(o))$: we have $b_R^{\geq 0}(q) = b_R^{\geq 0}(o, q') = (o, b_{B_R(o)}^{\geq 0}(q')) \in \mathcal{B}^{\geq i}(R)$, hence, by Fact 2.10, $\{p \in \mathcal{P}(\mathcal{T}_R[i](e)); \kappa_R[i](e)(p) = q\} = \{p \in \mathcal{P}_0(\mathcal{T}_R[i](e)) \setminus \mathcal{P}_0(R); \kappa_R[i](e)(p) = (o, q')\} = \bigcup_{e_o \in e(o)} \{(o, (e_o, p')); (p' \in \mathcal{P}_0(\mathcal{T}_{B_R(o)}[i](e_o)) \wedge \kappa_{B_R(o)}[i](e_o)(p') = q')\}$; for any $e_o \in e(o)$, by Fact 2.10 again, we have

$$\{p' \in \mathcal{P}(\mathcal{T}_{B_R(o)}[i](e_o)); \kappa_{B_R(o)}[i](e_o)(p') = q'\} = \{p' \in \mathcal{P}_0(\mathcal{T}_{B_R(o)}[i](e_o)); \kappa_{B_R(o)}[i](e_o)(p') = q'\}$$

  and, by induction hypothesis, we have

$$\mathsf{Card}\left(\{p' \in \mathcal{P}(\mathcal{T}_{B_R(o)}[i](e_o)); \kappa_{e_o,i}(p') = q'\}\right) = \sum {e_o}^{\#}(b_{B_R(o)}^{\geq i}(q'))$$

  We thus obtain

$$\begin{array}{rl} & \mathsf{Card}\left(\{p \in \mathcal{P}(\mathcal{T}_R[i](e)); \kappa_R[i](e)(p) = q\}\right) \\ = & \mathsf{Card}\left(\bigcup_{e_o \in e(o)} \{(o, (e_o, p')); (p' \in \mathcal{P}(\mathcal{T}_{B_R(o)}[i](e_o)) \wedge \kappa_{B_R(o)}[i](e_o)(p') = q')\}\right) \\ = & \sum_{e_o \in e(o)} \mathsf{Card}\left(\{p' \in \mathcal{P}(\mathcal{T}_{B_R(o)}[i](e_o)); \kappa_{B_R(o)}[i](e_o)(p') = q'\}\right) \\ = & \sum_{e_o \in e(o)} \sum {e_o}^{\#}(b_{B_R(o)}^{\geq i}(q')) \\ = & \sum e^{\#}(o, b_{B_R(o)}^{\geq i}(q')) \\ = & \sum e^{\#}(b_R^{\geq i}(q))\end{array}$$

- $q' \in \mathcal{P}_{>i}(B_R(o))$ and $b_{B_R(o)}^{\geq 0}(q') \in \mathcal{B}^{<i}(B_R(o))$: we have $b_R^{\geq 0}(q) = b_R^{\geq 0}(o, q') = (o, b_{B_R(o)}^{\geq 0}(q')) \in \mathcal{B}^{<i}(R)$, hence, by Fact 2.10,

$$\{p \in \mathcal{P}(\mathcal{T}_R[i](e)); \kappa_R[i](e)(p) = q\} = \{p \in \mathcal{P}_{>0}(\mathcal{T}_R[i](e)); \kappa_R[i](e)(p) = (o, q')\};$$

  for any $e_o \in e(o)$, by Fact 2.10 again, we have $\{p \in \mathcal{P}(\mathcal{T}_{B_R(o)}[i](e_o)); \kappa_{B_R(o)}[i](e_o)(p) = q'\} = \{p \in \mathcal{P}_{>0}(\mathcal{T}_{B_R(o)}[i](e_o)); \kappa_{B_R(o)}[i](e_o)(p) = q'\}$ and, by induction hypothesis, we have

$$\mathsf{Card}\left(\{p \in \mathcal{P}(\mathcal{T}_{B_R(o)}[i](e_o)); \kappa_{B_R(o)}[i](e_o)(p) = q'\}\right) = \sum {e_o}^{\#}(b_{B_R(o)}^{\geq i}(q'))$$

hence

$$\begin{aligned}
&\mathsf{Card}\left(\{p \in \mathcal{P}(\mathcal{T}_R[i](e)) ; \kappa_R[i](e)(p) = q\}\right)\\
= &\mathsf{Card}\left(\bigcup_{e_o \in e(o)} \{((o,(e_o,o')),p') ; ((o',p') \in \mathcal{P}_{>0}(\mathcal{T}_{B_R(o)}[i](e_o)) \wedge \kappa_{B_R(o)}[i](e_o)(o',p') = q')\}\right)\\
= &\sum_{e_o \in e(o)} \mathsf{Card}\left(\{p \in \mathcal{P}_{>0}(\mathcal{T}_{B_R(o)}[i](e_o)) ; \kappa_{B_R(o)}[i](e_o)(p) = q'\}\right)\\
= &\sum_{e_o \in e(o)} \mathsf{Card}\left(\{p \in \mathcal{P}(\mathcal{T}_{B_R(o)}[i](e_o)) ; \kappa_{B_R(o)}[i](e_o)(p) = q'\}\right)\\
&\textit{(because } (\forall e_o \in e(o))(\forall p \in \mathcal{P}_0(\mathcal{T}_{B_R(o)}[i](e_o)))\kappa_{B_R(o)}[i](e_o)(p) \notin \mathcal{P}_{>i}(B_R(o))\textit{)}\\
= &\sum_{e_o \in e(o)} \sum {e_o}^{\#}(b^{\geq i}_{B_R(o)}(q'))\\
= &\sum e^{\#}(o, b^{\geq i}_{B_R(o)}(q'))\\
= &\sum e^{\#}(b^{\geq i}_R(q))
\end{aligned}$$

□

We finally obtain Proposition 2.12:

Proposition 2.12. *Let R be an in-PS. Let e be a pseudo-experiment on R. Let $i \in \mathbb{N}$. Let $p \in \mathcal{P}_0(R)$. Then we have*

$$a_{\mathcal{T}_R[i](e)}(p) = a_{R^{\leq i}}(p) + \sum_{\substack{p' \in \mathcal{P}_{>i}(R)\\ t_R(p') = p}} \sum e^{\#}(b^{\geq i}_R(p'))$$

Proof. We have

$$\begin{aligned}
&\sum_{o \in \mathcal{B}^{\geq i}_0(R)} \sum_{e_o \in e(o)} \mathsf{Card}\left(\left\{q \in \mathcal{P}^{\mathsf{f}}(\mathcal{G}(\mathcal{T}_{B_R(o)}[i](e_o))) ; \begin{array}{l}(\kappa_{B_R(o)}[i](e_o)(q) \in \mathcal{P}^{\mathsf{f}}_R(o)\\ \wedge t_R(o, \kappa_{B_R(o)}[i](e_o)(q)) = p)\end{array}\right\}\right)\\
= &\sum_{o \in \mathcal{B}^{\geq i}_0(R)} \sum_{e_o \in e(o)} \sum_{\substack{p' \in \mathcal{P}^{\mathsf{f}}_R(o)\\ t_R(o,p') = p}} \mathsf{Card}\left(\left\{q \in \mathcal{P}^{\mathsf{f}}(\mathcal{G}(\mathcal{T}_{B_R(o)}[i](e_o))) ; \kappa_{B_R(o)}[i](e_o)(q) = p'\right\}\right)\\
= &\sum_{o \in \mathcal{B}^{\geq i}_0(R)} \sum_{e_o \in e(o)} \sum_{\substack{p' \in \mathcal{P}^{\mathsf{f}}_R(o)\\ t_R(o,p') = p}} \mathsf{Card}\left(\left\{q \in \mathcal{P}_0(\mathcal{T}_{B_R(o)}[i](e_o)) ; \kappa_{B_R(o)}[i](e_o)(q) = p'\right\}\right)\\
&\textit{(by Fact 2.5)}\\
= &\sum_{o \in \mathcal{B}^{\geq i}_0(R)} \sum_{e_o \in e(o)} \sum_{\substack{p' \in \mathcal{P}^{\mathsf{f}}_R(o)\\ t_R(o,p') = p}} \mathsf{Card}\left(\left\{q \in \mathcal{P}_0(\mathcal{T}_{B_R(o)}[i](e_o)) ; \kappa_R[i](e)(o,(e_o,q)) = (o,p')\right\}\right)
\end{aligned}$$

$$= \sum_{o \in \mathcal{B}_0^{\geq i}(R)} \sum_{\substack{p' \in \mathcal{P}^{\mathsf{f}}_R(o) \\ t_R(o,p')=p}} \sum_{e_o \in e(o)} \mathsf{Card}\left(\{q \in \mathcal{P}_0(\mathcal{T}_{B_R(o)}[i](e_o)); \kappa_R[i](e)(o,(e_o,q)) = (o,p')\}\right)$$

$$= \sum_{\substack{p' \in \mathcal{P}_{>i}(R) \\ t_R(p')=p}} \mathsf{Card}\left(\{q \in \mathcal{P}_0(\mathcal{T}_R[i](e)); \kappa_R[i](e)(q) = p'\}\right)$$

and, for any $o \in \mathcal{B}_0^{\geq i}(R)$, for any $e_o \in e(o)$, we set

$$d_o(e_o) = \sum_{o' \in \mathcal{B}_0(\mathcal{T}_{B_R(o)}[i](e_o))} \mathsf{Card}\left(\left\{(o',q) \in \mathcal{P}^{\mathsf{f}}(\mathcal{T}_{B_R(o)}[i](e_o));\ \begin{array}{l}(\kappa_{B_R(o)}[i](e_o)(o',q) \in \mathcal{P}^{\mathsf{f}}_R(o) \\ \wedge t_R(o,\kappa_{B_R(o)}[i](e_o)(o',q)) = p)\end{array}\right\}\right)$$

we have

$$\begin{aligned} & d_o(e_o) \\ = & \sum_{o' \in \mathcal{B}_0(\mathcal{T}_{B_R(o)}[i](e_o))} \sum_{\substack{q' \in \mathcal{P}^{\mathsf{f}}_R(o) \\ t_R(o,q')=p}} \mathsf{Card}\left(\left\{(o',q) \in \mathcal{P}^{\mathsf{f}}(\mathcal{T}_{B_R(o)}[i](e_o)); \kappa_{B_R(o)}[i](e_o)(o',q) = q'\right\}\right) \\ = & \sum_{o' \in \mathcal{B}_0(\mathcal{T}_{B_R(o)}[i](e_o))} \sum_{\substack{q' \in \mathcal{P}^{\mathsf{f}}_R(o) \\ t_R(o,q')=p}} \mathsf{Card}\left(\{(o',q) \in \mathcal{P}(\mathcal{T}_{B_R(o)}[i](e_o)); \kappa_{B_R(o)}[i](e_o)(o',q) = q'\}\right) \end{aligned}$$

(by Fact 2.5)

hence

$$\begin{aligned} & \sum_{e_o \in e(o)} d_o(e_o) \\ = & \sum_{\substack{q' \in \mathcal{P}^{\mathsf{f}}_R(o) \\ t_R(o,q')=p}} \sum_{e_o \in e(o)} \sum_{o' \in \mathcal{B}_0(\mathcal{T}_{B_R(o)}[i](e_o))} \mathsf{Card}\left(\{(o',q) \in \mathcal{P}(\mathcal{T}_{B_R(o)}[i](e_o)); \kappa_{B_R(o)}[i](e_o)(o',q) = q'\}\right) \\ = & \sum_{\substack{q' \in \mathcal{P}^{\mathsf{f}}_R(o) \\ t_R(o,q')=p}} \sum_{e_o \in e(o)} \sum_{o' \in \mathcal{B}_0(\mathcal{T}_{B_R(o)}[i](e_o))} \mathsf{Card}\left(\{(o',q) \in \mathcal{P}(\mathcal{T}_{B_R(o)}[i](e_o)); \kappa_R[i](e)((o,(e_o,o')),q) = (o,q')\}\right) \end{aligned}$$

and

$$\sum_{o \in \mathcal{B}_0^{\geq i}(R)} \sum_{e_o \in e(o)} d_o(e_o) = \sum_{\substack{p' \in \mathcal{P}_{>i}(R) \\ t_R(p')=p}} \mathsf{Card}\left(\{q \in \mathcal{P}_{>0}(\mathcal{T}_R[i](e)); \kappa_R[i](e)(q) = p'\}\right)$$

We thus have

$$\begin{aligned} & a_{\mathcal{T}_R[i](e)}(p) \\ = & a_{R^{\leq i}}(p) + \sum_{o \in \mathcal{B}_0^{\geq i}(R)} \sum_{e_o \in e(o)} d_o(e_o) \\ & + \sum_{o \in \mathcal{B}_0^{\geq i}(R)} \sum_{e_o \in e(o)} \mathsf{Card}\left(\left\{q \in \mathcal{P}^{\mathsf{f}}(\mathcal{G}(\mathcal{T}_{B_R(o)}[i](e_o)));\ \begin{array}{l}(\kappa_{B_R(o)}[i](e_o)(q) \in \mathcal{P}^{\mathsf{f}}_R(o) \\ \wedge t_R(o,\kappa_{B_R(o)}[i](e_o)(q)) = p)\end{array}\right\}\right) \end{aligned}$$

$$
\begin{aligned}
&= a_{R^{\leq i}}(p) + \sum_{\substack{p' \in \mathcal{P}_{>i}(R)\\ t_R(p')=p}} \mathsf{Card}\left(\{q \in \mathcal{P}_{>0}(\mathcal{T}_R[i](e)); \kappa_R[i](e)(q) = p'\}\right)\\
&+ \sum_{\substack{p' \in \mathcal{P}_{>i}(R)\\ t_R(p')=p}} \mathsf{Card}\left(\{q \in \mathcal{P}_{0}(\mathcal{T}_R[i](e)); \kappa_R[i](e)(q) = p'\}\right)\\
&= a_{R^{\leq i}}(p) + \sum_{\substack{p' \in \mathcal{P}_{>i}(R)\\ t_R(p')=p}} \mathsf{Card}\left(\{q \in \mathcal{P}(\mathcal{T}_R[i](e)); \kappa_R[i](e)(q) = p'\}\right)\\
&= a_{R^{\leq i}}(p) + \sum_{\substack{p' \in \mathcal{P}_{>i}(R)\\ t_R(p')=p}} \sum e^{\#}(b_R^{\geq i}(p'))
\end{aligned}
$$

(by Lemma 2.11) □

Example 2.13. Consider the port p_1 of R at depth 0 (see Figure 11): We have $\{p' \in \mathcal{P}_{>1}(R); t_R(p') = p_1\} = \{(o_2, q')\}$, $b_R^{\geq 1}((o_2, q')) = o_2$ and $a_{R^{\leq 1}}(p_1) = 1$ (see Figure 35, p. 54). Now, if e is a pseudo-experiment as in Example 2.2, then $e^{\#}(o_2) = \{10\}$, hence $a_{R^{\leq 1}}(p_1) + \sum_{\substack{p' \in \mathcal{P}_{>1}(R)\\ t_R(p')=p_1}} \sum e^{\#}(b_R^{\geq 1}(p')) = 1 + \sum\{10\} = 11$. And, indeed, we have $a_{\mathcal{T}_R[1](e)}(p_1) = 11$ (see Figure 16).

Corollary 2.14. *Let R be an in-PS. Let $o \in \mathcal{B}_0(R)$. Let φ be some bijection $\mathcal{P}_R^{?}(o) \simeq \mathcal{Q}'$ and let R_o be an in-PS such that $R_o = \varphi \cdot_o R$. Let $e_o \in e(o)$. Let $i \in \mathbb{N}$. Then, for any $p \in \mathcal{P}_0(\mathcal{T}_{B_R(o)}[i](e_o))$, we have $a_{\mathcal{T}_{B_R(o)}[i](e_o)}(p) = a_{\mathcal{T}_{B_{R_o}(o)}[i](e_o)}(p)$.*

Proof. Let $p \in \mathcal{P}_0(\mathcal{T}_{B_R(o)}[i](e_o))$. If $p \in \mathcal{P}_0(B_R(o))$, then, by Proposition 2.12, we have

$$
\begin{aligned}
a_{\mathcal{T}_{B_R(o)}[i](e_o)}(p) &= a_{B_R(o)^{\leq i}}(p) + \sum_{\substack{p' \in \mathcal{P}_{>i}(B_R(o))\\ t_{B_R(o)}(p')=p}} \sum e_o{}^{\#}(b_{B_R(o)}^{\geq i}(p'))\\
&= a_{B_{R_o}(o)^{\leq i}}(p) + \sum_{\substack{p' \in \mathcal{P}_{>i}(B_{R_o}(o))\\ t_{B_{R_o}(o)}(p')=p}} \sum e_o{}^{\#}(b_{B_{R_o}(o)}^{\geq i}(p'))\\
&= a_{\mathcal{T}_{B_{R_o}(o)}[i](e_o)}(p)
\end{aligned}
$$

If $p \notin \mathcal{P}_0(B_R(o))$, then there exist $o_1 \in \mathcal{B}_0^{\geq i}(B_R(o))$, $e_1 \in e_o(o_1)$ and $p' \in \mathcal{P}_0(\mathcal{T}_{B_{B_R(o)}(o_1)}[i](e_1))$ such that $p = (o_1, (e_1, p'))$; moreover,

$$
\begin{aligned}
a_{\mathcal{T}_{B_R(o)}[i](e_o)}(p) &= a_{\mathcal{T}_{B_{B_R(o)}(o_1)}[i](e_1)}(p')\\
&= a_{\mathcal{T}_{B_{B_{R_o}(o)}(o_1)}[i](e_1)}(p')\\
&= a_{\mathcal{T}_{B_{R_o}(o)}[i](e_o)}(p)
\end{aligned}
$$

□

3. Rebuilding the proof-structure

The Taylor expansion of a PS is an infinite set of simple differential nets (for PS's of depth > 0). It was not known whether from this infinite set is was possible to rebuild the PS;

indeed, *a priori*, two different PS's could have the same Taylor expansion. We will not only show that it is possible to rebuild any PS R from its Taylor expansion $\mathcal{T}_R[0]$, we will also show something much stronger: We are already able to rebuild the PS with only one well-chosen simple differential net that appears in the Taylor expansion, chosen according to a specific information given by a second simple differential net of the Taylor expansion.

The algorithm leading from the simple differential net $\mathcal{T}_R[0](e)$ for some well-chosen pseudo-experiment e on R to the entire rebuilding of R is done in several steps: In the intermediate steps, we obtain a partial rebuilding where some boxes have been recovered but not all of them; a convenient way to represent this information is to use differential PS's, which lie between the purely linear differential proof-nets and the non-linear proof-nets.

The rebuilding of the PS R is done in d steps, where d is the depth of R. We first rebuild the occurrences of the boxes of depth 0 (the deepest ones) and next we rebuild the occurrences of the boxes of depth 1 and so on... This can be formalized using simple differential nets (possibly with boxes) as follows: starting from $\mathcal{T}_R[0](e)$, the first step of the algorithm builds $\mathcal{T}_R[1](e)$, the second step builds $\mathcal{T}_R[2](e)$ from $\mathcal{T}_R[1](e)$, and so on... until $\mathcal{T}_R[\mathit{depth}(R)](e) = R$. We thus reduced the problem of rebuilding the PS to the problem of rebuilding $\mathcal{T}_R[i+1](e)$ from $\mathcal{T}_R[i](e)$ for some well-chosen pseudo-experiment e.

We can hardly obtain much more than the following property for a non-well-chosen pseudo-experiment:

Lemma 3.1. *Let R be an in-PS. Let e be a pseudo-experiment on R. Let $i \in \mathbb{N}$. Then we have:*

- $\mathcal{T}_R[i+1](e)^{\leq i} \sqsubseteq_\emptyset \mathcal{T}_R[i](e)$
- $(\forall o \in \mathcal{B}_0^{<i}(\mathcal{T}_R[i+1](e)))\mathcal{P}^{\mathsf{f}}_{\mathcal{T}_R[i+1](e)}(o) = \mathcal{P}^{\mathsf{f}}_{\mathcal{T}_R[i](e)}(o)$;
- *and* $\kappa_R[i+1](e)|_{\mathcal{P}_0(\mathcal{T}_R[i+1](e))} = \kappa_R[i](e)|_{\mathcal{P}_0(\mathcal{T}_R[i+1](e))}$.

Proof. By induction on $\mathit{depth}(R)$, noticing, by applying Remark 1.30, Remark 1.28 and Remark 1.17, that we have $\mathcal{T}_R[i+1](e)^{\leq i} = S@t$ with

$$S = R^{\leq i} \oplus \bigoplus_{o \in \mathcal{B}_0^{\geq i+1}(R)} \bigoplus_{e_o \in e(o)} \langle o, \langle e_o, \mathcal{T}_{B_R(o)}[i+1](e_o)^{\leq i}\rangle\rangle$$

and t is the function $\mathcal{W}_0 \cup \mathcal{W}_{>0} \to \mathcal{P}_0^e(R)$, where

- $\mathcal{W}_0 = \bigcup_{o \in \mathcal{B}_0^{\geq i+1}(R)} \bigcup_{e_o \in e(o)} \{(o,(e_o,q)); (q \in \mathcal{P}^{\mathsf{f}}(\mathcal{G}(\mathcal{T}_{B_R(o)}[i+1](e_o))) \wedge \kappa_{B_R(o)}[i+1](e_o)(q) \in \mathcal{P}^{\mathsf{f}}_R(o))\}$
- $\mathcal{W}_{>0} = \bigcup_{(o,(e_o,o')) \in \mathcal{B}_0^{<i}(S) \setminus \mathcal{B}_0(R^{\leq i+1})} \left\{ ((o,(e_o,o')),q); \begin{array}{c} ((o',q) \in \mathcal{P}^{\mathsf{f}}_{>0}(\mathcal{T}_{B_R(o)}[i+1](e_o)) \\ \wedge \kappa_{B_R(o)}[i+1](e_o)(o',q) \in \mathcal{P}^{\mathsf{f}}_R(o)) \end{array} \right\}$
- and $t(p) = \begin{cases} t_R(o, \kappa_{B_R(o)}[i+1](e_o)(q)) & \text{if } p = (o,(e_o,q)) \in \mathcal{W}_0; \\ t_R(o, \kappa_{B_R(o)}[i+1](e_o)(o',q)) & \text{if } p = ((o,(e_o,o')),q) \in \mathcal{W}_{>0}. \end{cases}$ □

We thus have to consider some special experiments. As a first requirement, the pseudo-experiments we will consider are *exhaustive*:

Definition 3.2. A pseudo-experiment e of an in-PS R is said to be *exhaustive* if, for any $o \in \mathcal{B}(R)$, we have $0 \notin e^{\#}(o)$.

But this requirement is not strong enough. More specific pseudo-experiments have to be considered.

- In [35], it was shown that given the result of an *injective k-obsessional experiment* (k big enough) of a cut-free proof-net in the fragment $A ::= X|?A \parr A|A \parr ?A|A \otimes A|!A$, one can rebuild the entire experiment and, so, the entire proof-net. We recall that the LPS of a cut-free proof-net forgets the outline of the boxes but keeps the trace of the auxiliary doors (see Figure 22, p. 38 for an example). There, "injective" means that the experiment labels two different axioms with different atoms and "obsessional" means that different copies of the same axiom are labeled by the same atom. Obsessionality entails that the names of the atoms play some role and thus we cannot reduce such experiments to some pseudo-experiments.
- In [13], it was shown that, for any two cut-free MELL proof-nets R and R', we have $LPS(R) = LPS(R')$ if, and only if, for k big enough[25], there exist an *injective k-experiment* on R and an *injective k-experiment* on R' having the same result; as an immediate corollary we obtained the injectivity of the set of (recursively) connected proof-nets. There, "injective" means that not only the experiment labels two different axioms with different atoms, but it labels also different copies of the same axiom by different atoms. Given some proof-net R, there is exactly one injective k-experiment on R up to the names of the atoms. Injectivity allows to reduce such experiments to some pseudo-experiments: it makes sense to define *k-pseudo-experiments* and we could show in the same way that, for any two cut-free proof-nets R and R', the two following statements are equivalent:
  - we have $LPS(R) = LPS(R')$;
  - for any $k \in \mathbb{N}$, for any k-pseudo-experiment e on R, for any k-pseudo-experiment e' on R', we have $\mathcal{T}_R[0](e) = \mathcal{T}_{R'}[0](e')$, where a *$k$-pseudo-experiment* is a pseudo-experiment such that, for any $o \in \mathcal{B}(R)$, we have $e^{\#}(o) = \{k\}$.

  Now, there are many different cut-free PS's with the same LPS (see Figures 22, 23, 24, 25 and 26, p. 38 for an example).

In [13], the interest for *injective* experiments came from the remark that the result of an (atomic) *injective* experiment on a *cut-free* proof-net can be easily identified with a simple differential net of its Taylor expansion in a sum of simple differential nets [17] (it is essentially the content of our Lemma 4.19). Thus any proof using injective experiments can be straightforwardly expressed in terms of simple differential nets and conversely. Since this identification is trivial, besides the idea of considering injective experiments instead of obsessional experiments, the use of the terminology of differential nets does not bring any new insight[26], it just superficially changes the presentation.That is why we decided in [13] to avoid introducing explicitly differential nets. In [10], we made the opposite choice for the following reason: The simple differential net representing the result and the proof-net are both instances of the more general notion of "simple differential nets possibly with boxes", which are used to represent the partial information obtained during the algorithm execution. Moreover, this identification allows to see the injectivity of the relational semantics as a particular case of the invertibility of Taylor expansion.

[25]Interestingly, [20], following the approach of [13], showed that, if these two proof-nets are assumed to be (recursively) connected, then we can take $k = 2$.

[26]For proof-nets with cuts, the situation is completely different: the great novelty of differential nets is that differential nets have a cut-elimination; the simple differential nets appearing in the Taylor expansion of a proof-net with cuts have cuts, while the semantics does not see these cuts. But the proofs of the injectivity only consider cut-free proof-nets.

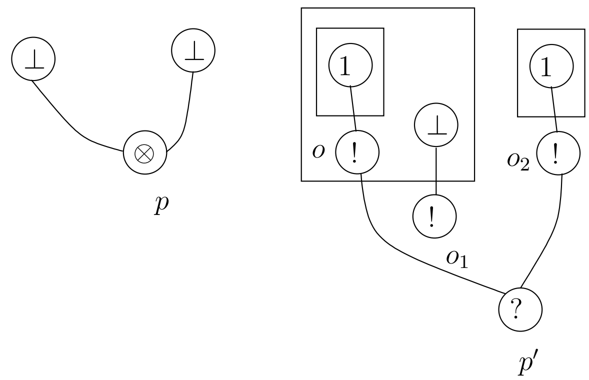


Figure 19: Proof-net S

In the present paper, we introduce the notion of *k-heterogeneous (pseudo-)experiment* (k-heterogeneous pseudo-experiments on *cut-free* PS's are an abstraction of injective k-heterogeneous experiments, where *injective* has the same meaning as in [13]) and the simple differential net we will consider to rebuild entirely the PS is the simple differential net obtained by expanding the boxes according to any *k-heterogeneous pseudo-experiment* (for k big enough): We show that, for any cut-free PS R, given the result α of a *k-heterogeneous experiment* on R for k big enough, if $\alpha \in [\![R']\!]$, where R' is any cut-free PS, then R' is the same PS as R. The constraints on k are given by the result of a 1-experiment, so we show that two (well-chosen) points are enough to determine a PS. The expression "k-heterogeneous" means that, for any two different occurrences of boxes, the experiment never takes the same number of copies: it takes k^{j_1} copies and k^{j_2} copies with $j_1 \neq j_2$ (*a contrario*, in [35] and [13], the experiments always take the same number of copies). As shown by the proof-net S of Figure 19, it is impossible to rebuild the experiment from its result, since there exist five different 4-heterogeneous experiments e_1, e_2, e_3, e_4 and e_5 on S such that, for any $i \in \{1,2,3,4,5\}$, we have $e_i(p) = (*,*)$, $e_i(o_1) = [*,*,*,*]$ and $e_i(p') = [[\underbrace{*,\ldots,*}_{4^2}],\ldots,[\underbrace{*,\ldots,*}_{4^6}]]$: The experiment e_i takes 4 copies of the box o_1 and 4^{i+1} copies of the box o_2.

Actually, more generally, we show that, for *any* PS R, given the simple differential net $\mathcal{T}_R[0](e)$ that belongs to the support of the Taylor expansion and that has been obtained by expanding the boxes according to any (atomic) injective k-heterogeneous pseudo-experiment e on R for k big enough, if $\mathcal{T}_R[0](e)$ belongs to the support of the Taylor expansion of any PS R', then R' is the same PS as R. Notice that in presence of cuts, the k-heterogeneous pseudo-experiment we consider is not necessarily induced (see Definition 4.3) by an experiment.[27]

Definition 3.3. Let $k > 0$. A pseudo-experiment e on an in-PS R is said to be *k-heterogeneous* if

- for any $o \in \mathcal{B}(R)$, for any $m \in e^{\#}(o)$, there exists $j > 0$ such that $m = k^j$;
- for any $o \in \mathcal{B}_0(R)$, for any $o' \in \mathcal{B}(B_R(o))$, we have $(\forall e_1, e_2 \in e(o))\ ({e_1}^{\#}(o') \cap {e_2}^{\#}(o') \neq \emptyset \Rightarrow e_1 = e_2)$;
- and, for any $o_1, o_2 \in \mathcal{B}(R)$, we have $(e^{\#}(o_1) \cap e^{\#}(o_2) \neq \emptyset \Rightarrow o_1 = o_2)$.

[27] In the case there is no such experiment, the simple differential net $\mathcal{T}_R[0](e)$ reduces to 0.

Example 3.4. The pseudo-experiment e of Example 2.2 is a 10-heterogeneous pseudo-experiment.

k-heterogeneous experiments are characterized by (the arities of the co-contractions of) their corresponding terms of the Taylor expansion:

Lemma 3.5. *For any in-PS R, we have $\bigcup_{o \in \mathcal{B}(R)} e^{\#}(o) = a_{\mathcal{T}_R[0](e)}[\mathcal{P}_0^!(\mathcal{T}_R[0](e))]$.*

Proof. By Proposition 2.12, we have $\bigcup_{o \in \mathcal{B}_0(R)} e^{\#}(o) = a_{\mathcal{T}_R[0](e)}[\mathcal{B}_0(R)]$, hence

$$\begin{aligned}
\bigcup_{o \in \mathcal{B}(R)} e^{\#}(o) &= \bigcup_{o \in \mathcal{B}_0(R)} e^{\#}(o) \cup \bigcup_{o \in \mathcal{B}_0(R)} \bigcup_{o' \in \mathcal{B}(B_R(o))} e^{\#}(o, o') \\
&= a_{\mathcal{T}_R[0](e)}[\mathcal{B}_0(R)] \cup \bigcup_{o \in \mathcal{B}_0(R)} \bigcup_{o' \in \mathcal{B}(B_R(o))} \bigcup_{e_o \in e(o)} {e_o}^{\#}(o') \\
&= a_{\mathcal{T}_R[0](e)}[\mathcal{B}_0(R)] \cup \bigcup_{o \in \mathcal{B}_0(R)} \bigcup_{e_o \in e(o)} a_{\mathcal{T}_{B_R(o)}[0](e_o)}[\mathcal{P}_0^!(\mathcal{T}_{B_R(o)}[0](e_o))] \\
&\qquad \textit{(by induction hypothesis)} \\
&= a_{\mathcal{T}_R[0](e)}[\mathcal{B}_0(R)] \cup \bigcup_{o \in \mathcal{B}_0(R)} \bigcup_{e_o \in e(o)} a_{\mathcal{T}_R[0](e)}[\mathcal{P}_0^!(R\langle o, 0, e_o\rangle)] \\
&= a_{\mathcal{T}_R[0](e)}[\mathcal{P}_0^!(\mathcal{T}_R[0](e))]
\end{aligned}$$

□

Corollary 3.6. *Let R be an in-PS. Let e be a pseudo-experiment on R. Let $k > 1$. Then e is a k-heterogeneous pseudo-experiment on R if, and only if, the two following properties hold together:*

(1) $a_{\mathcal{T}_R[0](e)}[\mathcal{P}_0^!(\mathcal{T}_R[0](e))] \subseteq \{k^j ; j > 0\}$
(2) $(\forall p_1, p_2 \in \mathcal{P}_0^!(\mathcal{T}_R[0](e)))(a_{\mathcal{T}_R[0](e)}(p_1) = a_{\mathcal{T}_R[0](e)}(p_2) \Rightarrow p_1 = p_2)$

Proof. For any pseudo-experiment e on R, applying Lemma 3.5, we obtain:

- For any $k > 1$, one has $(\forall o \in \mathcal{B}(R))(\forall m \in e^{\#}(o))(\exists j > 0) m = k^j$ if, and only if, one has $a_{\mathcal{T}_R[0](e)}[\mathcal{P}_0^!(\mathcal{T}_R[0](e))] \subseteq \{k^j ; j > 0\}$.
- One has $(\forall o \in \mathcal{B}_0(R))(\forall o' \in \mathcal{B}(B_R(o)))(\forall e_1, e_2 \in e(o))({e_1}^{\#}(o') \cap {e_2}^{\#}(o') \neq \emptyset \Rightarrow e_1 = e_2)$ if, and only if, one has $(\forall o \in \mathcal{B}_0(R))(\forall e_1, e_2 \in e(o))$ $(a_{\mathcal{T}_R[0](e)}[\mathcal{P}_0^!(R\langle o, 0, e_1\rangle)] \cap a_{\mathcal{T}_R[0](e)}[\mathcal{P}_0^!(R\langle o, 0, e_2\rangle)] \neq \emptyset \Rightarrow e_1 = e_2)$.
- One has $(\forall o_1, o_2 \in \mathcal{B}(R))(e^{\#}(o_1) \cap e^{\#}(o_2) \neq \emptyset \Rightarrow o_1 = o_2)$ if, and only if, one has $(\forall p_1, p_2 \in \mathcal{P}_0^!(\mathcal{T}_R[0](e)))(a_{\mathcal{T}_R[0](e)}(p_1) = a_{\mathcal{T}_R[0](e)}(p_2) \Rightarrow \kappa_R[0](e)(p_1) = \kappa_R[0](e)(p_2))$.

We prove, by induction on *depth*(R), that, for any $k > 1$, the two following properties together:

- $(\forall o \in \mathcal{B}_0(R))(\forall e_1, e_2 \in e(o))$ $(a_{\mathcal{T}_R[0](e)}[\mathcal{P}_0^!(R\langle o, 0, e_1\rangle)] \cap a_{\mathcal{T}_R[0](e)}[\mathcal{P}_0^!(R\langle o, 0, e_2\rangle)] \neq \emptyset \Rightarrow e_1 = e_2)$
- and $(\forall p_1, p_2 \in \mathcal{P}_0^!(\mathcal{T}_R[0](e)))(a_{\mathcal{T}_R[0](e)}(p_1) = a_{\mathcal{T}_R[0](e)}(p_2) \Rightarrow \kappa_R[0](e)(p_1) = \kappa_R[0](e)(p_2))$

imply $(\forall p_1, p_2 \in \mathcal{P}_0^!(\mathcal{T}_R[0](e)))(a_{\mathcal{T}_R[0](e)}(p_1) = a_{\mathcal{T}_R[0](e)}(p_2) \Rightarrow p_1 = p_2)$; the converse is trivial. □

We will rebuild $\mathcal{T}_R[i+1](e)$ from $\mathcal{T}_R[i](e)$ for any k-heterogeneous pseudo-experiment e on R (with $k \geq \beta(R)$, where $\beta(R)$ is an integer[28] provided by any 1-pseudo-experiment

[28]The integer $\beta(R)$ is defined in Definition 3.33.

on R). For this purpose we will introduce our notion of *critical component* (elements of $\mathcal{S}^k_{\mathcal{T}_R[i](e)}(\mathcal{K}_{k,j}(\mathcal{T}_R[i](e)))$ with $j \in \mathcal{N}_i(e)$)[29], which are special connected components of $\mathcal{T}_R[i](e)$, *connected* in a weak sense, and we will consider equivalence classes of critical components of $\mathcal{T}_R[i](e)$ for the relation $\equiv$ that forgets names of ports apart from those of shallow conclusions.[30] We can summarize the three main ideas of our proof as follows:

(1) The simple differential net that corresponds to a k-heterogeneous pseudo-experiments is informative enough to rebuild the entire PS if k is big enough.
(2) We introduce the notion of *partial* Taylor expansion $\mathcal{T}_R[i]$ of a PS R (with $i \in \mathbb{N}$) and reduce the problem of the rebuilding of a PS R to the problem of the rebuilding of the simple differential net *possibly with boxes* $\mathcal{T}_R[i+1](e)$ that corresponds to a k-heterogeneous pseudo-experiment e in the partial Taylor expansion $\mathcal{T}_R[i]$ from the simple differential net *possibly with boxes* $\mathcal{T}_R[i](e)$ that corresponds to the same k-heterogeneous pseudo-experiment e in the partial Taylor expansion $\mathcal{T}_R[i]$.
(3) We consider cardinalities of equivalence classes $\mathcal{S}^k_{\mathcal{T}_R[i](e)}(\mathcal{K}_{k,j}(\mathcal{T}_R[i](e)))/\equiv$ of *critical components* of $\mathcal{T}_R[i](e)$ for the relation $\equiv$ in order to deduce the cardinalities of equivalence classes $\mathcal{S}^k_{\mathcal{T}_R[i+1](e)}(\mathcal{K}_{k,j}(\mathcal{T}_R[i+1](e)))/\equiv$ of *critical components* of $\mathcal{T}_R[i+1](e)$ and the contents of the new boxes of depth i of $\mathcal{T}_R[i+1](e)$.

3.1. **The borders of the boxes.** In this subsection we first show how to recover the set $\bigcup_{o\in\mathcal{B}^{\geq i}(R)}\{\log_k(m); m \in e^{\#}(o)\}$ and, therefore, the set $\mathcal{P}^!_0(\mathcal{T}_R[i](e)) \setminus \mathcal{B}_0(\mathcal{T}_R[i](e))$ of co-contractions of $\mathcal{T}_R[i](e)$ (Lemma 3.11). Next, we show how to determine, from $\mathcal{T}_R[i](e)$, the set $\mathcal{B}^{=i}_0(\mathcal{T}_R[i+1](e))$ of "new" boxes and, for any such "new" box $o \in \mathcal{B}^{=i}_0(\mathcal{T}_R[i+1](e))$, its *border* i.e. the set $t_{\mathcal{T}_R[i+1](e)}[\{o\}\times\mathcal{P}^{\mathsf{f}}_{\mathcal{T}_R[i+1](e)}(o)]$ of exponential ports that are immediately below (Proposition 3.19), which, as shown by Example 3.20, p. 37, is an information that is not provided by the LPS but is still too weak to rebuild the PS (the borders of the boxes do not allow to recover their outlines). In particular, we have $\mathcal{B}^{=i}_0(\mathcal{T}_R[i+1](e)) = !_{e,i}[\mathcal{N}_i(e)]$, where the set $\mathcal{N}_i(e) \subseteq \mathbb{N}$ is defined from the set $\mathcal{M}_0(e)$ of the numbers of copies of boxes taken by the pseudo–experiment e:

Definition 3.7. Let R be a differential in-PS. Let $k > 1$. Let e be a k-heterogeneous pseudo-experiment on R. For any $i \in \mathbb{N}$, we define, by induction on i, $\mathcal{M}_i(e) \subseteq \mathbb{N}\setminus\{0\}$ and $(m_{i,j}(e))_{j\in\mathbb{N}} \in \{0,\ldots,k-1\}^{\mathbb{N}}$ as follows. We set $\mathcal{M}_0(e) = \bigcup_{o\in\mathcal{B}(R)}\{j \in \mathbb{N}; k^j \in e^{\#}(o)\}$ and we write $\mathsf{Card}\,(\mathcal{M}_i(e))$ in base k: $\mathsf{Card}\,(\mathcal{M}_i(e)) = \sum_{j\in\mathbb{N}} m_{i,j}(e)\cdot k^j$; we set $\mathcal{M}_{i+1}(e) = \{j > 0; m_{i,j}(e) \neq 0\}$.

For any $i \in \mathbb{N}$, we set $\mathcal{N}_i(e) = \mathcal{M}_i(e) \setminus \mathcal{M}_{i+1}(e)$.

Notice that all the sets $\mathcal{M}_i(e)$ and $\mathcal{N}_i(e)$ can be computed from $\mathcal{T}_R[0](e)$, since, by Lemma 3.5, we have $\mathcal{M}_0(e) = \log_k[\{a_{\mathcal{T}_R[0](e)}(p); p \in \mathcal{P}^!_0(\mathcal{T}_R[0](e))\}]$.

Example 3.8. If e is a 10-heterogeneous pseudo-experiment as in Example 3.4, then $\mathcal{M}_0(e) = \{1,\ldots,224\}$. We have $\mathsf{Card}\,(\mathcal{M}_0(e)) = 4 + 2\cdot 10^1 + 2\cdot 10^2$, hence $\mathcal{M}_1(e) = \{1,2\}$ and $\mathcal{N}_0(e) = \{3,\ldots,224\}$. We have $\mathsf{Card}\,(\mathcal{M}_1(e)) = 2$, hence $\mathcal{M}_2(e) = \emptyset$ and $\mathcal{N}_1(e) = \{1,2\}$.

[29] The set $\mathcal{N}_i(e)$ is a set of integers that will be defined in Definition 3.7, the set $\mathcal{K}_{k,j}(\mathcal{T}_R[i](e))$ is a set of exponential ports of $\mathcal{T}_R[i](e)$ at depth 0 that will be defined in Definition 3.13 and the sets $\mathcal{S}^k_S(\mathcal{Q})$ of *components T of S that are connected* via *other ports than $\mathcal{Q}$, whose conclusions belong to $\mathcal{Q}$ and with co-size*$(T) < k$ will be defined in Definition 3.25.

[30] This relation has been defined in Definition 1.11.

Lemma 3.9. *Let $k > 1$. Let R be an in-PS. If e is a k-heterogeneous pseudo-experiment on R, then, for any $i \in \mathbb{N}$, there exists a unique bijection*

$$!_{e,i} : \log_k[\bigcup e^{\#}[\mathcal{B}^{\geq i}(R)]] \simeq \mathcal{P}_0^!(\mathcal{T}_R[i](e)) \setminus \mathcal{B}_0(\mathcal{T}_R[i](e))$$

such that, for any $j \in \mathsf{dom}(!_{e,i})$, the following properties hold:

(1) $k^j \in (e^{\#} \circ \kappa_R[i](e) \circ !_{e,i})(j)$
(2) *and* $(a_{\mathcal{T}_R[i](e)} \circ !_{e,i})(j) = k^j$.

Proof. Notice first that, for any $o \in \mathcal{B}_0^{\geq i}(R)$, we have $a_{\mathcal{T}_R[i](e)}(o) = \mathsf{Card}\,(e(o))$, hence the function that associates with every $j \in \bigcup_{o \in \mathcal{B}_0^{\geq i}(R)} \{\log_k(m); m \in e^{\#}(o)\}$ the unique $o_j \in \mathcal{B}_0^{\geq i}(R)$ such that $e^{\#}(o_j) = \{k^j\}$ is a bijection

$$\bigcup_{o \in \mathcal{B}_0^{\geq i}(R)} \{\log_k(m); m \in e^{\#}(o)\} \simeq \mathcal{P}_0^!(R) \setminus \mathcal{B}_0(\mathcal{T}_R[i](e))$$

such that $a_{\mathcal{T}_R[i](e)}(o_j) = \sum e^{\#}(o_j) = k^j$.

Now, we prove the lemma by induction on $depth(R)$. We set

$$!_{e,i}(j) = \begin{cases} o_j & \text{if } j \in \bigcup_{o \in \mathcal{B}_0^{\geq i}(R)} \{\log_k(m); m \in e^{\#}(o)\}; \\ (o, (e_o, !_{e_o,i}(j))) & \text{if } o \in \mathcal{B}_0^{\geq i}(R) \text{ and } k^j \in ({e_o}^{\#} \circ \kappa_{e_o,i} \circ !_{e_o,i})(j). \end{cases}$$

For checking Property 2, notice that, for any $o \in \mathcal{B}_0^{\geq i}(R)$, for any $e_o \in e(o)$, for any $p \in \mathcal{P}_0(\mathcal{T}_{B_R(o)}[i](e_o))$, we have $a_{\mathcal{T}_R[i](e)}(o, (e_o, p)) = a_{\mathcal{T}_{B_R(o)}[i](e_o)}(p)$. ☐

3.1.1. *Identifying the co-contractions that correspond to the new boxes.* This part is devoted to prove Lemma 3.11, which shows that, for any k-heterogeneous pseudo-experiment e on R, for any $i \in \mathbb{N}$, the function $!_{e,i}$ is actually a bijection $\mathcal{M}_i(e) \to \mathcal{P}_0^!(\mathcal{T}_R[i](e)) \setminus \mathcal{B}_0(\mathcal{T}_R[i](e))$ such that, for any $j \in \mathcal{M}_i(e)$, we have $(a_{\mathcal{T}_R[i](e)} \circ !_{e,i})(j) = k^j$.

Lemma 3.10. *Let $k > 1$. For any $i \in \mathbb{N}$, for any in-PS R such that $\mathsf{Card}\,(\mathcal{B}(R)) < k$, for any k-heterogeneous pseudo-experiment e on R, the following properties hold:*

(1) $P_1(R, e, i)$*:* $(\forall o, o' \in \mathcal{B}_0(R))(\forall e_o \in e(o))(\forall j \in \mathcal{M}_i(e_o)) k^j \notin e^{\#}(o')$
(2) $P_2(R, e, i)$*:* $(\forall o, o' \in \mathcal{B}_0(R))(\forall e_o \in e(o))(\forall e_{o'} \in e(o'))(\mathcal{M}_i(e_o) \cap \mathcal{M}_i(e_{o'}) \neq \emptyset \Rightarrow (o, e_o) = (o', e_{o'}))$
(3) $P_3(R, e, i)$*:* $(depth(R) = i \Rightarrow \mathcal{M}_i(e) = \emptyset)$
(4) $P_4(R, e, i)$*:* $m_{i,0}(e) = \mathsf{Card}\left(\mathcal{B}_0^{\geq i}(R)\right)$
(5) $P_5(R, e, i)$*:* $\mathcal{M}_i(e) \subseteq \mathcal{M}_{i-1}(e)$
(6) $P_6(R, e, i)$*:* $\mathcal{M}_i(e) = \log_k[\bigcup e^{\#}[\mathcal{B}_0^{\geq i}(R)]] \cup \bigcup_{o \in \mathcal{B}_0^{\geq i+1}(R)} \bigcup_{e_o \in e(o)} \mathcal{M}_i(e_o)$

Proof. By complete induction on $depth(R) + i$. First, if $i = 0 = depth(R)$, then $\mathcal{M}_{-1} = \emptyset = \mathcal{M}_0$, hence $P_1(R, e, i)$, $P_2(R, e, i)$, $P_3(R, e, i)$, $P_4(R, e, i)$, $P_5(R, e, i)$ and $P_6(R, e, i)$ hold trivially. Now, let $i \in \mathbb{N}$. Notice that if $P_6(R, e, i)$ holds, then $P_3(R, e, i)$ holds; moreover, if, in addition, $P_1(R, e, i)$ and $P_2(R, e, i)$ hold, then

$$\begin{aligned} & \mathsf{Card}\,(\mathcal{M}_i(e)) \\ = \; & \mathsf{Card}\left(\mathcal{B}_0^{\geq i}(R)\right) + \sum_{o \in \mathcal{B}_0^{\geq i+1}(R)} \sum_{e_o \in e(o)} \mathsf{Card}\,(\mathcal{M}_i(e_o)) \end{aligned}$$

$$
\begin{aligned}
&= \mathsf{Card}\left(\mathcal{B}_0^{\geq i}(R)\right) + \sum_{o \in \mathcal{B}_0^{\geq i+1}(R)} \sum_{e_o \in e(o)} \sum_{j \in \mathcal{M}_{i+1}(e_o) \cup \{0\}} m_{i,j}(e_o) \cdot k^j \\
&= \mathsf{Card}\left(\mathcal{B}_0^{\geq i}(R)\right) \\
&+ \sum_{o \in \mathcal{B}_0^{\geq i+1}(R)} \left(\mathsf{Card}\left(\mathcal{B}_0^{\geq i}(B_R(o))\right) \cdot \mathsf{Card}\left(e(o)\right) + \sum_{e_o \in e(o)} \sum_{j \in \mathcal{M}_{i+1}(e_o)} m_{i,j}(e_o) \cdot k^j \right) \\
&\qquad \textit{(by } P_4(B_R(o), e_o, i) \textit{ for each } o \in \mathcal{B}_0^{\geq i+1}(R) \textit{ and each } e_o \in e(o))
\end{aligned}
$$

We thus have $P_4(R, e, i)$. Moreover, if $i > 0$, then by applying $P_6(R, e, i-1)$, $P_6(R, e, i)$ and $P_5(B_R(o), e_o, i)$ for each $o \in \mathcal{B}_0^{\geq i+1}(R)$ and each $e_o \in o$, and by noticing the inclusion $\log_k[\bigcup e^{\#}[\mathcal{B}_0^{\geq i}(R)]] \subseteq \log_k[\bigcup e^{\#}[\mathcal{B}_0^{\geq i-1}(R)]]$, we obtain $P_5(R, e, i)$.

- Let us assume that $i = 0$. For any in-PS R, for any k-heterogeneous pseudo-experiment e on R, since $\bigcup e^{\#}[\mathcal{B}(R)] \subseteq \{k^j; j > 0\}$, we have $\mathsf{Card}\left(\mathcal{M}_{-1}(e)\right) = \sum \bigcup e^{\#}[\mathcal{B}(R)] = \sum_{j \in \log_k[\bigcup e^{\#}[\mathcal{B}(R)]]} k^j$, hence $\mathcal{M}_0(e) = \log_k[\bigcup e^{\#}[\mathcal{B}(R)]]$.

  Let R be an in-PS such that $\mathsf{Card}\left(\mathcal{B}(R)\right) < k$ and let e be a k-heterogeneous experiment on R.

  Let $o, o' \in \mathcal{B}_0(R)$ and let $e_o \in e(o)$. Let $j \in \mathcal{M}_0(e_o)$. There exists $o'' \in \mathcal{B}(B_R(o))$ such that $k^j \in {e_o}^{\#}(o'')$. Assume that $k^j \in e^{\#}(o')$. We have $e^{\#}((o, o'')) \cap e^{\#}(o') \neq \emptyset$; but, by the definition of *k-heterogeneous experiment* (Definition 3.3), this entails that $o' = (o, o'')$; we thus obtain a contradiction with the following requirement of the definition of in-PS's (Definition 1.7): for any $(p_1, p_2) \in \mathcal{P}_0(R)$, we have $p_1 \notin \mathcal{B}_0(R)$. We showed that $P_1(R, e, 0)$ holds.

  Let $o, o' \in \mathcal{B}_0(R)$. Let $e_o \in e(o)$ and $e_{o'} \in e(o')$ such that $\mathcal{M}_0(e_o) \cap \mathcal{M}_0(e_{o'}) \neq \emptyset$. Let $j \in \mathcal{M}_0(e_o) \cap \mathcal{M}_0(e_{o'})$. There exist $o_1 \in \mathcal{B}(B_R(o))$ such that $k^j \in {e_o}^{\#}(o_1) = e^{\#}(o, o_1)$ and $o_2 \in \mathcal{B}(B_R(o'))$ such that $k^j \in {e_{o'}}^{\#}(o_2) = e^{\#}(o', o_2)$. By the definition of *$k$-heterogeneous experiment*, we have $(o, o_1) = (o', o_2)$. We have ${e_o}^{\#}(o_1) \cap {e_{o'}}^{\#}(o_1) \neq \emptyset$, hence, again by the definition of *k-heterogeneous experiment*, we obtain $e_o = e_{o'}$. We showed that $P_2(R, e, 0)$ holds.

  We have

$$
\begin{aligned}
\mathcal{M}_0(e) &= \log_k[\bigcup e^{\#}[\mathcal{B}(R)]] \\
&= \log_k[\bigcup e^{\#}[\mathcal{B}_0(R)]] \cup \bigcup_{o \in \mathcal{B}_0^{\geq 1}(R)} \log_k[\bigcup e^{\#}[\{o\} \times \mathcal{B}_0(B_R(o))]] \\
&= \log_k[\bigcup e^{\#}[\mathcal{B}_0(R)]] \cup \bigcup_{o \in \mathcal{B}_0^{\geq 1}(R)} \bigcup_{e_o \in e(o)} \log_k[\bigcup {e_o}^{\#}[\mathcal{B}_0(B_R(o))]] \\
&= \log_k[\bigcup e^{\#}[\mathcal{B}_0(R)]] \cup \bigcup_{o \in \mathcal{B}_0^{\geq 1}(R)} \bigcup_{e_o \in e(o)} \mathcal{M}_0(e_o)
\end{aligned}
$$

  hence $P_6(R, e, 0)$ holds. We already know that it follows that $P_4(R, e, 0)$ holds.

  Let $j \in \mathcal{M}_0(e)$: We have $k^j \in \bigcup e^{\#}[\mathcal{B}(R)]$, hence $k^j \leq \sum \bigcup e^{\#}[\mathcal{B}(R)]$. Since $j < k^j$, we have $j < \sum \bigcup e^{\#}[\mathcal{B}(R)]$; so $j \in \mathcal{M}_{-1}(e)$. We thus proved $P_5(R, e, 0)$.
- Let us assume that $i > 0$. Let R be an in-PS such that $\mathsf{Card}\left(\mathcal{B}(R)\right) < k$ and let e be a k-heterogeneous experiment on R. By $P_1(R, e, i-1)$ and $P_5(R, e, i-1)$, we have $P_1(R, e, i)$.

By $P_2(R,e,i-1)$ and $P_5(R,e,i-1)$, we have $P_2(R,e,i)$. We have

$$\begin{array}{rl}
& \mathsf{Card}\,(\mathcal{M}_{i-1}(e)) \\
= & \mathsf{Card}\left(\mathcal{B}_0^{\geq i-1}(R)\right) + \sum_{o\in\mathcal{B}_0^{\geq i}(R)} \sum_{e_o\in e(o)} \mathsf{Card}\,(\mathcal{M}_{i-1}(e_o)) \\
& \textit{(by } P_6(R,e,i-1),\ P_1(R,e,i-1) \textit{ and } P_2(R,e,i-1)\textit{)} \\
= & \mathsf{Card}\left(\mathcal{B}_0^{\geq i-1}(R)\right) + \sum_{o\in\mathcal{B}_0^{\geq i}(R)} \sum_{e_o\in e(o)} \left(m_{i-1,0}(e_o) + \sum_{j\in\mathcal{M}_i(e_o)} m_{i-1,j}(e_o)\cdot k^j\right) \\
= & \mathsf{Card}\left(\mathcal{B}_0^{\geq i-1}(R)\right) + \sum_{o\in\mathcal{B}_0^{\geq i}(R)} \sum_{e_o\in e(o)} \left(\mathsf{Card}\left(\mathcal{B}_0^{\geq i-1}(B_R(o))\right) + \sum_{j\in\mathcal{M}_i(e_o)} m_{i-1,j}(e_o)\cdot k^j\right) \\
& \textit{(by } P_4(B_R(o),e_o,i-1) \textit{ for each } o\in\mathcal{B}_0^{\geq i}(R) \textit{ and each } e_o\in e(o)\textit{)} \\
= & \mathsf{Card}\left(\mathcal{B}_0^{\geq i-1}(R)\right) \\
+ & \sum_{o\in\mathcal{B}_0^{\geq i}(R)} \left(\mathsf{Card}\left(\mathcal{B}_0^{\geq i-1}(B_R(o))\right)\cdot \mathsf{Card}\,(e(o)) + \sum_{e_o\in e(o)} \sum_{j\in\mathcal{M}_i(e_o)} m_{i-1,j}(e_o)\cdot k^j\right) \\
= & \mathsf{Card}\left(\mathcal{B}_0^{\geq i-1}(R)\right) + \sum_{o\in\mathcal{B}_0^{\geq i}(R)} \mathsf{Card}\left(\mathcal{B}_0^{\geq i-1}(B_R(o))\right)\cdot \sum e^{\#}(o) \\
& + \sum_{o\in\mathcal{B}_0^{\geq i+1}(R)} \sum_{e_o\in e(o)} \sum_{j\in\mathcal{M}_i(e_o)} m_{i-1,j}(e_o)\cdot k^j \\
& \textit{(by } P_3(B_R(o),e_o,i) \textit{ for each } o\in\mathcal{B}_0^{=i}(R) \textit{ and each } e_o\in e(o)\textit{)}
\end{array}$$

By $P_1(R,e,i)$ and $P_2(R,e,i)$, we have $\{j>0; m_{i-1,j}(e)\neq 0\} = \bigcup_{o\in\mathcal{B}_0^{\geq i}(R)}\{j\in\mathbb{N}; k^j = \sum e^{\#}(o)\} \cup \bigcup_{o\in\mathcal{B}_0^{\geq i+1}(R)} \bigcup_{e_o\in e(o)} \mathcal{M}_i(e_o)$, hence $P_6(R,e,i)$ holds. □

Lemma 3.11. *Let R be an in-PS. Let $k > \mathsf{Card}\,(\mathcal{B}(R))$. For any k-heterogeneous pseudo-experiment e on R, for any $i\in\mathbb{N}$, we have $\mathcal{M}_i(e) = \log_k[\bigcup e^{\#}[\mathcal{B}^{\geq i}(R)]]$, hence $\mathcal{N}_i(e) = \log_k[\bigcup e^{\#}[\mathcal{B}^{=i}(R)]]$ and $(\kappa_R[i](e)\circ !_{e,i})[\mathcal{N}_i(e)] = \mathcal{B}^{=i}(R)$.*

Proof. By induction on *depth*(R). We have

$$\begin{array}{rcl}
\mathcal{M}_i(e) & = & \log_k[\bigcup e^{\#}[\mathcal{B}_0^{\geq i}(R)]] \cup \bigcup_{o\in\mathcal{B}_0^{\geq i+1}(R)} \bigcup_{e_o\in e(o)} \mathcal{M}_i(e_o) \\
& & \textit{(by Lemma 3.10)} \\
& = & \log_k[\bigcup e^{\#}[\mathcal{B}_0^{\geq i}(R)]] \cup \bigcup_{o\in\mathcal{B}_0^{\geq i+1}(R)} \bigcup_{e_o\in e(o)} \log_k[\bigcup {e_o}^{\#}[\mathcal{B}^{\geq i}(B_R(o))]] \\
& & \textit{(by the induction hypothesis)} \\
& = & \log_k[\bigcup e^{\#}[\mathcal{B}_0^{\geq i}(R)] \cup \bigcup_{o\in\mathcal{B}_0^{\geq i+1}(R)} \bigcup_{e_o\in e(o)} \bigcup {e_o}^{\#}[\mathcal{B}^{\geq i}(B_R(o))]]
\end{array}$$

$$\begin{array}{rl} = & \log_k[\bigcup e^{\#}[\mathcal{B}_0^{\geq i}(R)] \cup \bigcup_{o \in \mathcal{B}_0^{\geq i+1}(R)} \bigcup_{e_o \in e(o)} \bigcup e^{\#}[\{o\} \times \mathcal{B}^{\geq i}(B_R(o))]] \\ = & \log_k[\bigcup e^{\#}[\mathcal{B}_0^{\geq i}(R) \cup \bigcup_{o \in \mathcal{B}_0^{\geq i+1}(R)} \bigcup_{e_o \in e(o)} (\{o\} \times \mathcal{B}^{\geq i}(B_R(o)))]] \\ = & \log_k[\bigcup e^{\#}[\mathcal{B}^{\geq i}(R)]] \end{array}$$

Hence

$$\begin{array}{rcl} \mathcal{N}_i(e) & = & \mathcal{M}_i(e) \setminus \mathcal{M}_{i+1}(e) \\ & = & \log_k[\bigcup e^{\#}[\mathcal{B}^{\geq i}(R)]] \setminus \log_k[\bigcup e^{\#}[\mathcal{B}^{\geq i+1}(R)]] \\ & = & \log_k[\bigcup e^{\#}[\mathcal{B}^{=i}(R)]] \end{array}$$

Moreover, since $(\forall j \in \mathcal{N}_i(e))k^j \in (e^{\#} \circ \kappa_R[i](e) \circ !_{e,i})(j)$, we obtain $(\kappa_R[i](e) \circ !_{e,i})[\mathcal{N}_i(e)] = \mathcal{B}^{=i}(R)$. □

Example 3.12. (Continuation of Example 3.8) We thus have $\mathcal{M}_1(e) = \{1, 2\}$ and $\mathcal{P}_0^!(\mathcal{T}_R[i](e)) \setminus \mathcal{B}_0(\mathcal{T}_R[1](e)) = \{o_2, o_4\}$ with $a_{\mathcal{T}_R[1](e)}(o_2) = 10^1$ and $a_{\mathcal{T}_R[1](e)}(o_4) = 10^2$ (see Figures 16, 17 and 18 - we recall that $\mathcal{T}_R[1](e) = S1_1 \oplus S1_2 \oplus S1_3$).

3.1.2. *Determining the contractions immediately below the new boxes.* The set $\mathcal{K}_{k,\mathcal{N}_i(e)}(S)$ of "critical ports" is a set of exponential ports that will play a crucial role in our algorithm.

Definition 3.13. Let S be a differential in-PS. Let $k > 1$. For any $p \in \mathcal{P}_0(S)$, we define the sequence $(m_{k,j}(S)(p))_{j \in \mathbb{N}} \in \{0, \ldots, k-1\}^{\mathbb{N}}$ as follows: $a_S(p) = \sum_{j \in \mathbb{N}} m_{k,j}(S)(p) \cdot k^j$. For any $j \in \mathbb{N}$, we set $\mathcal{K}_{k,j}(S) = \{p \in \mathcal{P}_0(S); m_{k,j}(S)(p) \neq 0\} \cap \mathcal{P}^e(\mathcal{G}(S))$ and, for any $J \subseteq \mathbb{N}$, we set $\mathcal{K}_{k,J}(S) = \bigcup_{j \in J} \mathcal{K}_{k,j}(S)$.

In particular, for any $j \in \mathcal{M}_i(e)$, we have $!_{e,i}(j) \in \mathcal{K}_{k,j}(\mathcal{T}_R[i](e))$.

Example 3.14. We have $\mathcal{K}_{10,1}(\mathcal{T}_R[1](e)) = \{p_1, p_4, p_5, p_6, p_7, o_2\}$ and $\mathcal{K}_{10,2}(\mathcal{T}_R[1](e)) = \{p_4, p_5, p_6, p_7, o_4\}$, where $\mathcal{T}_R[1](e) = S1_1 \oplus S1_2 \oplus S1_3$ with $S1_1$, $S1_2$ and $S1_3$ depicted in Figures 16, 17 and 18 respectively. So we have $\mathcal{K}_{10,\{1,2\}}(\mathcal{T}_R[1](e)) = \{p_1, p_4, p_5, p_6, p_7, o_2, o_4\}$.

Critical ports are defined by their arities. Proposition 3.19 shows that they are exponential ports that are immediately below the "new" boxes.

In particular, this proposition highlights one more essential difference between the k-experiments of [34, 35, 13, 20] and our k-heterogeneous experiments. There, such a k-experiment labelling some contraction p with a multiset of cardinality $\sum_j m_j \cdot k^j$ (where $0 \leq m_j < k$ for any j) gives the information that immediately above the contraction p there are exactly m_{j_0} series of exactly j_0 auxiliary doors. Here, whenever a k-heterogeneous experiment labels some contraction p with a multiset of cardinality $\sum_j m_j \cdot k^j$ (where $0 \leq m_j < k$ for any j), the integer j_0 is not related to the number of auxiliary doors in series anymore; it corresponds, in the case $m_{j_0} > 0$, with the existence of a box that has an occurrence taking k^{j_0} copies of its content, the box having, among all its auxiliary doors, exactly m_{j_0} auxiliary doors that are, each of them, *the first one* (i.e. the deepest one) of a series of auxiliary doors immediately above the contraction p.

By the way, the constraints on the experiments are completely different: The constraints in [34, 35, 13, 20] give a lower bound on the arities of the co-contractions, while the constraints

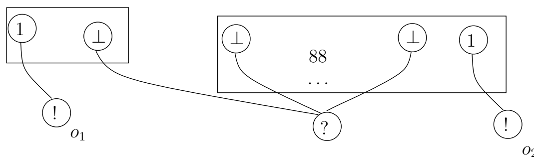


Figure 20: Lower bounding the arities of the co-contractions is not enough

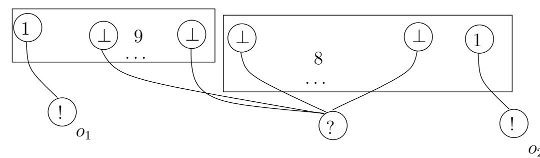


Figure 21: What our decomposition would provide

here are on the *basis* k. For instance, in the case of Figure 20, the co-size is < 100 and there are only 2 boxes, but still it is not enough to consider a 10-heterogeneous experiments with powers ≥ 2: by taking an experiment e with $e^{\#}(o_1) = 10^3$ and $e^{\#}(o_2) = 10^2$, we get a contraction of arity $9800 = 9 \cdot 10^3 + 8 \cdot 10^2$ in the corresponding term of the Taylor expansion, which, following our decomposition, would correspond to the PS of Figure 21.

Example 3.15. (Continuation of Example 3.12) We thus have $!_{f,1}[\mathcal{N}_1(f)] = \{o_2, o_4\}$; and indeed o_2 and o_4 are the boxes of depth 1 at depth 0 of $\mathcal{T}_R[2](e) \equiv R$ (see Figure 11). Moreover we have $\mathcal{K}_{10,1}(\mathcal{T}_R[1](e)) = \{p_1, p_4, p_5, p_6, p_7, o_2\}$ and $\mathcal{K}_{10,2}(\mathcal{T}_R[1](e)) = \{p_4, p_5, p_6, p_7, o_4\}$; and indeed, in Figure 11, we have $t_{\mathcal{T}_R[2](e)}[\{o_2\} \times \mathcal{P}^{\mathsf{f}}_{\mathcal{T}_R[2](e)}(o_2)] = \{p_1, p_4, p_5, p_6, p_7, o_2\}$ and $t_{\mathcal{T}_R[2](e)}[\{o_4\} \times \mathcal{P}^{\mathsf{f}}_{\mathcal{T}_R[2](e)}(o_4)] = \{p_4, p_5, p_6, p_7, o_4\}$.

Definition 3.16. Let R be an in-PS. For any $p \in \mathcal{P}_0(R)$, for any $i \in \mathbb{N}$, we define a subset $\mathcal{B}^{\geq i}_R(p)$ of $\mathcal{B}^{\geq i}(R)$: we set $\mathcal{B}^{\geq i}_R(p) = \{b^{\geq i}_R(q); (q \in \mathcal{P}_{>i}(R) \wedge t_R(q) = p)\}$.

A crucial lemma is the following one:

Lemma 3.17. *Let R be an in-PS. Let $p \in \mathcal{P}_0(R)$. Let $k >$ co-size(R). Let e be a k-heterogeneous pseudo-experiment on R. Let $i \in \mathbb{N}$. Let $p \in \mathcal{P}_0(R)$. Then, for any $j > 0$, we have $p \in \mathcal{K}_{k,j}(\mathcal{T}_R[i](e))$ if, and only if, there exists $o \in \mathcal{B}^{\geq i}_R(p)$ such that $k^j \in e^{\#}(o)$. Moreover we have $a_{\mathcal{T}_R[i](e)}(p)$ mod $k = a_{R^{\leq i}}(p)$.*

Proof. By Proposition 2.12, we have

$$a_{\mathcal{T}_R[i](e)}(p) = a_{R^{\leq i}}(p) + \sum_{\substack{p' \in \mathcal{P}_{>i}(R) \\ t_R(p') = p}} \sum e^{\#}(b^{\geq i}_R(p'))$$

Moreover we have

- $a_{R^{\leq i}}(p) < k$;
- $\mathsf{Card}\,(\{p' \in \mathcal{P}_{>i}(R); t_R(p') = p\}) < k$;
- and $(\forall p' \in \mathcal{P}_{>i}(R))(\forall m \in e^{\#}(b^{\geq i}_R(p')))(\exists j > 0) k^j \in e^{\#}(b^{\leq i}_R(p'))$.

Hence

- $a_{\mathcal{T}_R[i](e)}(p)$ *mod* $k = a_{R^{\leq i}}(p)$
- and $p \in \mathcal{K}_{k,j}(\mathcal{T}_R[i](e))$ if, and only if, $(\exists p' \in \mathcal{P}_{>i}(R))(t_R(p') = p \wedge k^j \in e^{\#}(b^{\geq i}_R(p')))$, i.e. $(\exists o \in \mathcal{B}^{\geq i}_R(p)) k^j \in e^{\#}(o)$. □

Fact 3.18. Let R be an in-PS. Let e be a pseudo-experiment on R. Let $i \in \mathbb{N}$. Let $o \in \mathcal{B}^{=i}_0(\mathcal{T}_R[i+1](e))$. Let $q \in \mathcal{P}_{>i}(R)$. Then we have $b^{\geq i}_R(q) = \kappa_R[i+1](e)(o)$ if, and only if, there exists $q' \in \mathcal{P}(B_{\mathcal{T}_R[i+1](e)}(o))$ such that $q = \kappa_R[i+1](e)(o, q')$.

Proof. By induction on *depth*(R). We assume that $b_R^{\geq i}(q) = \kappa_R[i+1](e)(o)$ and we distinguish between two cases:

- $o \in \mathcal{B}_0^{=i}(R)$: we have $\kappa_R[i+1](e)(o) = o$, hence there exists $q' \in \mathcal{P}_{\leq i}(B_R(o))$ such that $q = (o, q')$; but $B_R(o) = B_{\mathcal{T}_R[i+1](e)}(o)$;
- $o = (o_1, (e_1, o'))$ for some $o_1 \in \mathcal{B}_0^{\geq i}(R)$, $e_1 \in e(o_1)$ and $o' \in \mathcal{B}_0^{=i}(\mathcal{T}_{B_R(o_1)}[e_1](i))$: we have $\kappa_R[i+1](e)(o) = (o_1, \kappa_{B_R(o_1)}[i+1](e_1)(o'))$, hence there exists $q_0 \in \mathcal{P}_{>i}(B_R(o_1))$ such that $q = (o_1, q_0)$ and $b_{B_R(o_1)}^{\geq i}(q_0) = \kappa_{B_R(o_1)}[i+1](e_1)(o')$; by induction hypothesis, there exists $q' \in \mathcal{P}(B_{\mathcal{T}_{B_R(o_1)}[i+1](e_1)}(o'))$ such that $q_0 = \kappa_{B_R(o_1)}[i+1](e_1)(o_1, q')$; but $B_{\mathcal{T}_{B_R(o_1)}[i+1](e_1)}(o') = B_{\mathcal{T}_R[i+1](e)}(o)$.

Conversely, we assume that there exists $q' \in \mathcal{P}(B_{\mathcal{T}_R[i+1](e)}(o))$ such that $q = \kappa_R[i+1](e)(o, q')$ and we distinguish between two cases:

- $o \in \mathcal{B}_0^{=i}(R)$: we have $b_R^{\geq i}(\kappa_R[i+1](e)(o, q')) = b_R^{\geq i}(o, q') = o = \kappa_R[i+1](e)(o)$;
- $o = (o_1, (e_1, o'))$ for some $o_1 \in \mathcal{B}_0^{\geq i}(R)$, $e_1 \in e(o_1)$ and $o' \in \mathcal{B}_0^{=i}(\mathcal{T}_{B_R(o_1)}[e_1](i))$: we have $b_R^{\geq i}(\kappa_R[i+1](e)(o, q')) = b_R^{\geq i}(o_1, (\kappa_{B_R(o_1)}[i+1](e_1)(o', q'))) = (o_1, b_{B_R(o_1)}^{\geq i}(\kappa_{B_R(o_1)}[i+1](e_1)(o', q')))$; since $B_{\mathcal{T}_R[i+1](e)}(o) = B_{\mathcal{T}_{B_R(o_1)}[i+1](e_1)}(o')$, we can apply the induction hypothesis and we thus obtain $b_R^{\geq i}(\kappa_R[i+1](e)(o, q')) = (o_1, \kappa_{B_R(o_1)}[i+1](e_1)(o')) = \kappa_R[i+1](e)(o)$. $\square$

Proposition 3.19. *Let R be an in-PS. Let $k > \mathsf{Card}(\mathcal{B}(R)), \textit{co-size}(R)$. Let e be a k-heterogeneous pseudo-experiment on R and let $i \in \mathbb{N}$. Then we have $\mathcal{B}_0^{=i}(\mathcal{T}_R[i+1](e)) = !_{e,i}[\mathcal{N}_{i,e}]$. Moreover, the set $\mathcal{K}_{k,\mathbb{N}\setminus(\mathcal{M}_i(e)\cup\{0\})}(\mathcal{T}_R[i](e))$ is empty. Furthermore, for any $j \in \mathcal{N}_i(e)$, we have $\mathcal{K}_{k,j}(\mathcal{T}_R[i](e)) = t_{\mathcal{T}_R[i+1](e)}[\{!_{e,i}(j)\} \times \mathcal{P}^{\mathsf{f}}_{\mathcal{T}_R[i+1](e)}(!_{e,i}(j))]$ and, if $!_{e,i}(j) \notin \mathcal{B}_0^{=i}(R)$, then there exist $o \in \mathcal{B}_0^{\geq i+1}(R)$ and $e_o \in e(o)$ such that $j \in \mathcal{N}_i(e_o)$ and $\mathcal{K}_{k,j}(\mathcal{T}_R[i](e)) \setminus \mathcal{P}_0(R) = \{o\} \times (\{e_o\} \times \mathcal{K}_{k,j}(\mathcal{T}_{B_R(o)}[i](e_o)))$. In particular, we have $\mathcal{K}_{k,\mathcal{N}_i(e)}(\mathcal{T}_R[i](e)) \subseteq \mathcal{P}_0^e(\mathcal{T}_R[i+1](e))$.*

Proof. It is trivial to check, by induction on *depth*(R), that we have

$$(\forall o \in \mathcal{B}_0(\mathcal{T}_R[i](e)))(\forall p \in \mathcal{P}(B_{\mathcal{T}_R[i](e)}(o)))\kappa_R[i](e)(o, p) \notin \mathcal{B}^{=i}(R) \quad (*)$$

Now, we prove, by induction on *depth*(R), that $\mathcal{B}_0^{=i}(\mathcal{T}_R[i+1](e)) = !_{e,i}[\mathcal{N}_i(e)]$:

- Let $j \in \mathcal{N}_i(e)$. By Lemma 3.11, there exists $o \in \mathcal{B}^{=i}(R)$ such that $j \in \log_k[e^{\#}(o)]$, hence $k^j \in e^{\#}(o)$. Since $k^j \in e^{\#}((\kappa_R[i](e) \circ !_{e,i})(j))$, we have $(\kappa_R[i](e) \circ !_{e,i})(j) = o \in \mathcal{B}^{=i}(R)$. If $o \in \mathcal{B}_0^{=i}(R)$, then $!_{e,i}(j) = o$. Otherwise, there exist $o_1 \in \mathcal{B}_0(R)$ and $o' \in \mathcal{B}^{=i}(B_R(o_1))$ such that $o = (o_1, o')$: By $(*)$, there exist $e_1 \in e(o_1)$ and $o'' \in \mathcal{P}_0(\mathcal{T}_{B_R(o_1)}[i](e_1))$ such that $\kappa_{B_R(o_1)}[i](e_1)(o'') = o'$ and $!_{e,i}(j) = (o_1, (e_1, o''))$, hence $o'' = !_{e_1,i}(j)$. By induction hypothesis, we have $!_{e_1,i}(j) \in \mathcal{B}_0^{=i}(\mathcal{T}_{B_R(o_1)}[e_1](i+1))$, hence $(o_1, (e_1, o'')) \in \mathcal{B}_0^{=i}(\mathcal{T}_R[e](i+1))$.
- Conversely, let $o \in \mathcal{B}_0^{=i}(\mathcal{T}_R[i+1](e))$. Let $j \in \log_k[e^{\#}(o)]$. By Lemma 3.11, we have $j \in \mathcal{N}_i(e)$. We have $k^j \in e^{\#}(o)$ and $k^j \in e^{\#}((\kappa_R[i](e) \circ !_{e,i})(j))$, hence $(\kappa_R[i](e) \circ !_{e,i})(j) = o$. If $o \in \mathcal{B}_0^{=i}(R)$, then $!_{e,i}(j) = o$. Otherwise, there exist $o_1 \in \mathcal{B}_0^{\geq i+1}(R)$, $e_1 \in e(o_1)$ and $o' \in \mathcal{B}_0^{=i}(B_{\mathcal{T}_{B_R(o_1)}[i+1](e_1)})$ such that $o = (o_1, (e_1, o'))$. By induction hypothesis, there exists $j \in \mathcal{N}_i(e_1)$ such that $!_{e_1,i}(j) = o'$: we have $!_{e,i}(j) = (o_1, (e_1, o'))$.

By Lemma 3.11 and Lemma 3.17, we have $\mathcal{K}_{k,\mathbb{N}\setminus(\mathcal{M}_i(e)\cup\{0\})}(\mathcal{T}_R[i](e)) \cap \mathcal{P}_0(R) = \emptyset$. Moreover, for any $o \in \mathcal{B}_0^{\geq i}(R)$, for any $e_o \in e(o)$, again by Lemma 3.11, we have $\mathcal{M}_i(e_o) \subseteq$

$\mathcal{M}_i(e)$, hence

$$\begin{array}{rl} & \mathcal{K}_{k,\mathbb{N}\setminus(\mathcal{M}_i(e)\cup\{0\})}(\mathcal{T}_R[i](e)) \cap (\{o\} \times (\{e_o\} \times \mathcal{P}_0(\mathcal{T}_{B_R(o)}[i](e_o)))) \\ = & \{o\} \times (\{e_o\} \times \mathcal{K}_{k,\mathbb{N}\setminus(\mathcal{M}_i(e)\cup\{0\})}(\mathcal{T}_{B_R(o)}[i](e_o))) \\ \subseteq & \{o\} \times (\{e_o\} \times \mathcal{K}_{k,\mathbb{N}\setminus(\mathcal{M}_i(e_o)\cup\{0\})}(\mathcal{T}_{B_R(o)}[i](e_o))) \\ = & \{o\} \times (\{e_o\} \times \emptyset) \\ = & \emptyset \textit{ (by induction hypothesis)} \end{array}$$

We showed $\mathcal{K}_{k,\mathbb{N}\setminus(\mathcal{M}_i(e)\cup\{0\})}(\mathcal{T}_R[i](e)) = \emptyset$ $(**)$.

Now, let $j \in \mathcal{N}_i(e)$. We distinguish between two cases:

- $!_{e,i}(j) \in \mathcal{B}_0(R)$: By Lemma 3.17, we have $\mathcal{K}_{k,j}(\mathcal{T}_R[i](e)) \cap \mathcal{P}_0(R) = \{p \in \mathcal{P}_0(R); !_{e,i}(j) \in \mathcal{B}_R^{\geq i}(p)\} = t_R[\{!_{e,i}(j)\} \times \mathcal{P}^{\mathsf{f}}_R(!_{e,i}(j))] = t_{\mathcal{T}_R[i+1](e)}[\{!_{e,i}(j)\} \times \mathcal{P}^{\mathsf{f}}_{\mathcal{T}_R[i+1](e)}(!_{e,i}(j))]$. Moreover, by $(**)$, we have $\mathcal{K}_{k,j}(\mathcal{T}_R[i](e)) \subseteq \mathcal{P}_0(R)$. We thus have $\mathcal{K}_{k,j}(\mathcal{T}_R[i](e)) = t_{\mathcal{T}_R[i+1](e)}[\{!_{e,i}(j)\} \times \mathcal{P}^{\mathsf{f}}_{\mathcal{T}_R[i+1](e)}(!_{e,i}(j))]$.
- $!_{e,i}(j) = (o, (e_o, !_{e_o,i}(j)))$ for some $o \in \mathcal{B}_0^{\geq i}(R)$ and $e_o \in e(o)$: By induction hypothesis, we have $\mathcal{K}_{k,j}(\mathcal{T}_{B_R(o)}[i](e_o)) = t_{\mathcal{T}_{B_R(o)}[i+1](e_o)}[\{!_{e_o,i}(j)\} \times \mathcal{P}^{\mathsf{f}}_{\mathcal{T}_{B_R(o)}[i+1](e_o)}(!_{e_o,i}(j))]$. By $(**)$, we have $\mathcal{K}_{k,j}(\mathcal{T}_R[i](e)) \subseteq \mathcal{P}_0(R) \cup \mathcal{P}_0(R\langle o, i, e_o\rangle)$, hence

$$\begin{array}{rcl} \mathcal{K}_{k,j}(\mathcal{T}_R[i](e)) \setminus \mathcal{P}_0(R) & = & \mathcal{K}_{k,j}(R\langle o, i, e_o\rangle) \\ & = & (\{o\} \times (\{e_o\} \times \mathcal{K}_{k,j}(\mathcal{T}_{B_R(o)}[i](e_o)))) \\ & = & \{o\} \times (\{e_o\} \times t_{\mathcal{T}_{B_R(o)}[i+1](e_o)}[\{!_{e_o,i}(j)\} \times \mathcal{P}^{\mathsf{f}}_{\mathcal{T}_{B_R(o)}[i+1](e_o)}(!_{e_o,i}(j))]) \\ & = & t_{\mathcal{T}_R[i+1](e)}[\{!_{e,i}(j)\} \times \mathcal{P}^{\mathsf{f}}_{\mathcal{T}_R[i+1](e)}(!_{e,i}(j))] \setminus \mathcal{P}_0(R) \end{array}$$

We have

$$\begin{array}{rcl} \mathcal{K}_{k,j}(\mathcal{T}_R[i](e)) \cap \mathcal{P}_0(R) & = & \{p \in \mathcal{P}_0(R); (\exists o \in \mathcal{B}_R^{\geq i}(p)) k^j \in e^{\#}(o)\} \\ & & \textit{(by Lemma 3.17)} \\ & = & \{p \in \mathcal{P}_0(R); \kappa_R[i](e)(!_{e,i}(j)) \in \mathcal{B}_R^{\geq i}(p)\} \\ & & \textit{(by Lemma 3.9)} \\ & = & \{p \in \mathcal{P}_0(R); (\exists q \in \mathcal{P}_{>i}(R))(b_R^{\geq i}(q) = \kappa_R[i](e)(!_{e,i}(j)) \wedge t_R(q) = p)\} \\ & = & \{p \in \mathcal{P}_0(R); (\exists q \in \mathcal{P}_{>i}(R))(b_R^{\geq i}(q) = \kappa_R[i+1](e)(!_{e,i}(j)) \wedge t_R(q) = p)\} \\ & & \textit{(by Lemma 3.1)} \\ & = & \{t_R(\kappa_R[i+1](e)(!_{e,i}(j), q')); q' \in \mathcal{P}(B_{\mathcal{T}_R[i+1](e)}(!_{e,i}(j)))\} \\ & & \textit{(by Fact 3.18)} \\ & = & \{t_R(\kappa_R[i+1](e)(!_{e,i}(j), q')); q' \in \mathcal{P}^{\mathsf{f}}(B_{\mathcal{T}_R[i+1](e)}(!_{e,i}(j)))\} \\ & & \textit{(by Fact 2.5)} \\ & = & t_{\mathcal{T}_R[i+1](e)}[\{!_{e,i}(j)\} \times \mathcal{P}^{\mathsf{f}}_{\mathcal{T}_R[i+1](e)}(!_{e,i}(j))] \cap \mathcal{P}_0(R) \end{array}$$

□

As the following example shows, the information we obtain is already non-trivial, but far away to be strong enough.

Example 3.20. The PS's R_1, R_2, R_3 and R_4 of Figure 23, Figure 24, Figure 25 and Figure 26 respectively have the same LPS, which is depicted in Figure 22. But if we know

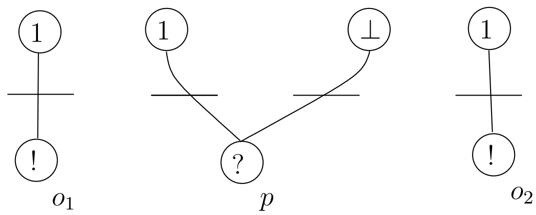


Figure 22: LPS of R_1, R_2, R_3 and R_4

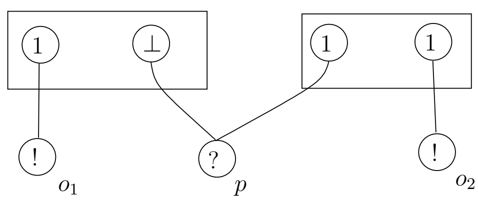


Figure 23: R_1

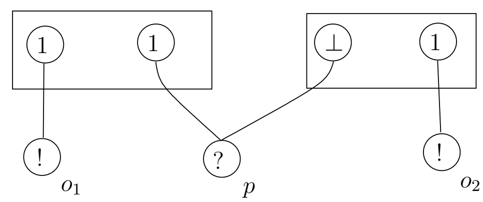


Figure 24: R_2

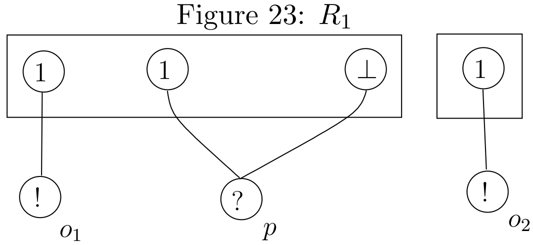


Figure 25: R_3

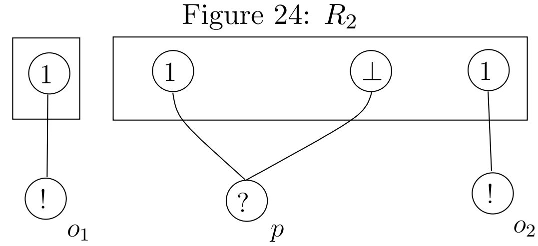


Figure 26: R_4

that $p \in t_R[\{o_1\} \times \mathcal{P}^{\mathsf{f}}_R(o_1)] \cap t_R[\{o_2\} \times \mathcal{P}^{\mathsf{f}}_R(o_2)]$, then we know that $R \neq R_3$ and $R \neq R_4$: The information we already obtain is an information that is not obtained from $LPS(R)$. Still we are not yet able to distinguish between R_1 and R_2. Since $LPS(R_1) = LPS(R_2)$ and $depth(R_1) = 1 = depth(R_2)$, the sets of pseudo-experiments on R_1 and R_2 coincide. Now, for any pseudo-experiment on these two proof-nets R_1 and R_2, we have $a_{\mathcal{T}_{R_1}[0](e)}(o_1) = a_{\mathcal{T}_{R_2}[0](e)}(o_1)$, $a_{\mathcal{T}_{R_1}[0](e)}(o_2) = a_{\mathcal{T}_{R_2}[0](e)}(o_2)$ and $a_{\mathcal{T}_{R_1}[0](e)}(p) = a_{\mathcal{T}_{R_2}[0](e)}(p)$, which shows that the arity of exponential ports in the Taylor expansion is not sufficient to recover the PS.

Corollary 3.21. *Let R be an in-PS. Let $o \in \mathcal{B}_0(R)$. Let φ be some bijection $\mathcal{P}^{?}_R(o) \simeq \mathcal{Q}'$. Let R_o be an in-PS such that $R_o = \varphi \cdot_o R$. Let $k >$ $\mathsf{Card}\,(\mathcal{B}(R))$, co-size$(R)$. Let e be a k-heterogeneous pseudo-experiment on R, let $e_o \in e(o)$, let $i \in \mathbb{N}$ and let $j \in \mathcal{N}_i(e_o)$. Then $\mathcal{K}_{k,j}(\mathcal{T}_{B_{R_o}(o)}[i](e_o)) \cap \mathcal{Q}' \subseteq \varphi[\mathcal{P}^{?}_R(o) \cap \mathcal{K}_{k,j}(\mathcal{T}_R[i](e))]$.*

Proof. Notice first that, since the set $\mathcal{N}_i(e_o)$ is non-empty, by Proposition 3.19, we have $o \in \mathcal{B}_0^{\geq i+1}(R)$. By Proposition 3.19 again, it is enough to show that

$$\begin{aligned} & t_{\mathcal{T}_{B_{R_o}(o)}[i+1](e_o)}[\{!_{e_o,i}(j)\} \times \mathcal{P}^{\mathsf{f}}_{\mathcal{T}_{B_{R_o}(o)}[i+1](e_o)}(!_{e_o,i}(j))] \cap \mathcal{Q}' \\ \subseteq \quad & \varphi[\mathcal{P}^{?}_R(o) \cap t_{\mathcal{T}_R[i+1](e)}[\{!_{e,i}(j)\} \times \mathcal{P}^{\mathsf{f}}_{\mathcal{T}_R[i+1](e)}(!_{e,i}(j))]] \end{aligned}$$

Now, let $p \in \mathcal{P}^{\mathsf{f}}_{\mathcal{T}_{B_{R_o}(o)}[i+1](e_o)}(!_{e_o,i}(j))$ such that $t_{\mathcal{T}_{B_{R_o}(o)}[i+1](e_o)}(!_{e_o,i}(j), p) \in \mathcal{Q}'$. By Lemma 2.8, we have $p \notin \mathcal{P}^{\mathsf{f}}_{\mathcal{T}_{B_{R_o}(o)}[i+1](e_o)}(!_{e_o,i}(j))$ and $t_{\mathcal{T}_{B_{R_o}(o)}[i+1](e_o)}(!_{e_o,i}(j), p) = \varphi(t_R(o, \kappa_{B_{R_o}(o)}[i +$

$1](e_o)(!_{e_o,i}(j),p)))$. Moreover, since $p \in \mathcal{P}^{\mathsf{f}}(B_{\mathcal{T}_{B_R(o)}[i+1](e_o)}(!_{e_o,i}(j)))$, we obtain $(!_{e_o,i}(j),p) \in \mathcal{P}^{\mathsf{f}}(\mathcal{T}_{B_R(o)}[i+1](e_o))$, hence, by Fact 2.5, $\kappa_{B_R(o)}[e_o](i+1)(!_{e_o,i}(j),p) \in \mathcal{P}^{\mathsf{f}}(B_R(o))$: We thus showed $\kappa_{B_R(o)}[i+1](e_o)(!_{e_o,i}(j),p) \in \mathcal{P}^{\mathsf{f}}_R(o)$, which entails $p \in \mathcal{P}^{\mathsf{f}}_{\mathcal{T}_R[i+1](e)}(!_{e,i}(j))$ and $t_R(o, \kappa_{B_{R_o}(o)}[i+1](e_o)(!_{e_o,i}(j),p)) = t_{\mathcal{T}_R[i+1](e)}(!_{e,i}(j),p)$. □

3.2. **Connected components.** In order to achieve the rebuilding of the ground-structure of $\mathcal{T}_R[i+1](e)$ and to recover the content of its boxes, we introduce our notion of *connected component* (Definition 3.25). This subsection is devoted to relate connected components of $\mathcal{T}_R[i](e)$ with connected components of $\mathcal{T}_R[i+1](e)$ (Proposition 3.40 and Proposition 3.43).

The relation $\frown_S$ formalizes the notion of "connectedness" between two ports of S at depth 0. But be aware that, here, "connected" has nothing to do with "connected" in the sense of [13]: here, any two doors of the same box are always "connected".

Definition 3.22. Let T be a differential in-PS. We define the binary relation $\frown_T$ on $\mathcal{P}_0(T)$ as follows: for any $p, p' \in \mathcal{P}_0(T)$, we have $p \frown_T p'$ iff $\{p,p'\} \in \mathcal{A}_0(T) \cup \mathcal{C}_0(T)$ or ($p \in \mathcal{W}_0(T)$ and $p' = t_{\mathcal{G}(T)}(p)$) or ($p' \in \mathcal{W}_0(T)$ and $p = t_{\mathcal{G}(T)}(p')$) or $(\exists o \in \mathcal{B}_0(T))(\exists q, q' \in \mathcal{P}^{\mathsf{f}}_T(o))\{p,p'\} = \{t_T(o,q), t_T(o,q')\}$.

Let $\mathcal{Q} \subseteq \mathcal{P}^e_0(T)$ and let $p, p' \in \mathcal{P}_0(T)$. A *path in T from p to p' without crossing $\mathcal{Q}$* is a finite sequence $(p_0, \ldots, p_n)$ of elements of $\mathcal{P}_0(T)$ such that $p_0 = p$, $p_n = p'$ and, for any $j \in \{0, \ldots, n-1\}$, we have $p_j \frown_T p_{j+1}$ and $(p_j \in \mathcal{Q} \Rightarrow j = 0)$.

We say that T is *connected through ports not in* $\mathcal{Q}$ if, for any $p, p' \in \mathcal{P}_0(T)$, there exists a path in T from p to p' without crossing $\mathcal{Q}$.

Definition 3.23. Let T and S two differential in-PS's and let $\mathcal{Q}$ such that $T \sqsubseteq_{\mathcal{Q}} S$. We write $T \trianglelefteq_{\mathcal{Q}} S$ if the following property holds:

$$(\forall p \in \mathcal{P}_0(T) \setminus \mathcal{Q})(\forall q \in \mathcal{P}_0(S))(p \frown_S q \Rightarrow q \in \mathcal{P}_0(T))$$

As we can expect, the connected components of $\mathcal{T}_R[i+1](e)$ that do not cross any expanded box are exactly the connected components of $\mathcal{T}_R[i](e)$ that do not cross any expanded box:

Fact 3.24. Let R be an in-PS. Let e be an exhaustive pseudo-experiment on R. Let $i \in \mathbb{N}$. Let $\mathcal{Q} \subseteq \mathcal{P}_0(R)$. Let T be a differential in-PS such that $T \sqsubseteq_{\mathcal{Q}} R^{\leq i}$. Then we have $T \trianglelefteq_{\mathcal{Q}} \mathcal{T}_R[i](e)$ if and only if $T \trianglelefteq_{\mathcal{Q}} \mathcal{T}_R[i+1](e)$.

Proof. Assume $T \trianglelefteq_{\mathcal{Q}} \mathcal{T}_R[i](e)$. Let $p \in \mathcal{P}_0(T) \setminus \mathcal{Q}$ and let $q \in \mathcal{P}_0(\mathcal{T}_R[i+1](e))$ such that $p \frown_{\mathcal{T}_R[i+1](e)} q$. Let us show that there is no $o \in \mathcal{B}^{=i}_0(R)$ such that $p \in t_R[\{o\} \times \mathcal{P}^{\mathsf{f}}_R(o)]$: Let us assume that there is such an o, let $e_o \in e(o)$ (such an e_o exists since e is exhaustive) and let $p' \in \mathcal{P}^{\mathsf{f}}_R(o)$ such that $t_R(o,p') = p$; we have $p \frown_{\mathcal{T}_R[i](e)} (o,(e_o,p'))$, hence $(o,(e_o,p')) \in \mathcal{P}_0(T)$, which contradicts $T \sqsubseteq R^{\leq i}$. This entails $p \frown_{\mathcal{T}_R[i](e)} q$.

Conversely, assume that $T \trianglelefteq_{\mathcal{Q}} \mathcal{T}_R[i+1](e)$. Let $p \in \mathcal{P}_0(T) \setminus \mathcal{Q}$ and let $q \in \mathcal{P}_0(\mathcal{T}_R[i](e))$ such that $p \frown_{\mathcal{T}_R[i](e)} q$. If $q = (o,(e_o,q'))$ for some $o \in \mathcal{B}^{=i}_0(R)$, $e_o \in e(o)$ and $q' \in \mathcal{P}^{\mathsf{f}}_R(i)$ such that $t_R(o,q') = q$, then $p \in t_R[\{o\} \times \mathcal{P}^{\mathsf{f}}_R(o)]$, hence $p \frown_{\mathcal{T}_R[i+1](e)} o$; then $o \in \mathcal{P}_0(T)$, which contradicts $T \sqsubseteq R^{\leq i}$. We thus have $q \in \mathcal{P}_0(\mathcal{T}_R[i+1](e))$, hence $p \frown_{\mathcal{T}_R[i+1](e)} q$. □

The sets $\mathcal{S}^k_S(\mathcal{Q})$ of *components T of S that are connected* via *other ports than $\mathcal{Q}$, whose conclusions belong to $\mathcal{Q}$ and with co-size*$(T) < k$ will play a crucial role in the algorithm of the rebuilding of $\mathcal{T}_R[i+1](e)$ from $\mathcal{T}_R[i](e)$, where we will take for $\mathcal{Q}$ a subset of the critical ports $\mathcal{K}_{k,\mathcal{N}_i(e)}(\mathcal{T}_R[i](e))$ that were considered in the previous subsection.[31] The reader already knows that, here, "connected" has nothing to do with the "connected proof-nets" of [13]: there, the crucial tool used was rather the "bridges", which put together two doors of the same copy of some box only if they are connected in the LPS of the proof-net.

Definition 3.25. Let $k \in \mathbb{N}$. Let S be a differential in-PS. Let $\mathcal{Q} \subseteq \mathcal{P}_0^e(S)$. We set

$$\mathcal{S}_S^k(\mathcal{Q}) = \left\{ T \trianglelefteq_{\mathcal{Q}} S;\ \begin{array}{l} (\textit{co-size}(T) < k \textit{ and } \mathcal{P}^{\mathsf{f}}(\mathcal{G}(T)) \subseteq \mathcal{Q} \textit{ and} \\ \mathcal{P}_0(T) \setminus \mathcal{Q} \neq \emptyset \textit{ and } T \textit{ is connected through ports not in } \mathcal{Q}) \end{array} \right\}$$

Remark 3.26. If $T \in \mathcal{S}_S^k(\mathcal{Q})$, then $\mathcal{P}^{\mathsf{f}}(\mathcal{G}(T)) = \mathcal{Q} \cap \mathcal{P}_0(T)$.

A port at depth 0 of S that is not in $\mathcal{Q}$ cannot belong to two different components:

Fact 3.27. Let $k \in \mathbb{N}$. Let S be a differential in-PS. Let $\mathcal{Q} \subseteq \mathcal{P}_0(S)$. Let $T, T' \in \mathcal{S}_S^k(\mathcal{Q})$ such that $(\mathcal{P}_0(T) \cap \mathcal{P}_0(T')) \setminus \mathcal{Q} \neq \emptyset$. Then $T = T'$.

Proof. By Remark 1.22, it is enough to check that $\mathcal{P}_0(T) = \mathcal{P}_0(T')$. □

Example 3.28. We have
- $\mathsf{Card}\left(\mathcal{S}_S^{10}(\{p_1, p_4, p_5, p_6, p_7, o_2\})\right) = 241$
- and $\mathsf{Card}\left(\mathcal{S}_S^{10}(\{p_4, p_5, p_6, p_7, o_4\})\right) = 320$

with $S = S1_1 \oplus S1_2 \oplus S1_3$, where $S1_1$, $S1_2$ and $S1_3$ are the differential PS's of Figures 16, 17 and 18 respectively.

The connected components we consider do not mix several copies of boxes. More precisely:

Lemma 3.29. *Let R be an in-PS. Let $k >$ co-size(R). Let e be a k-heterogeneous pseudo-experiment on R. Let $i \in \mathbb{N}$. Let $\mathcal{P} \subseteq \mathcal{P}_0(\mathcal{T}_R[i](e))$. Let $T \in \mathcal{S}^k_{\mathcal{T}_R[i](e)}(\mathcal{P})$. Let $o \in \mathcal{B}_0^{\geq i}(R)$ and $e_o \in e(o)$ such that $\mathcal{P}_0(T) \cap \mathcal{P}_0(R\langle o, i, e_o\rangle) \neq \emptyset$. Then the following properties hold:*
(1) $\mathcal{P}_0(T) \subseteq (t_R[\{o\} \times \mathcal{P}^{\mathsf{f}}_R(o)] \cap \mathcal{P}) \cup \mathcal{P}_0(R\langle o, i, e_o\rangle)$
(2) $\mathcal{P}_0(T) \cap t_R[\{o\} \times \mathcal{P}^{\mathsf{f}}_R(o)] \subseteq \mathcal{P}^{\mathsf{f}}(\mathcal{G}(T))$
(3) $\mathcal{W}_0(T) \subseteq \mathcal{P}_0(R\langle o, i, e_o\rangle)$

Proof. Notice first that the two following properties hold:
- (i) For any $p \in \mathcal{P}^{\mathsf{f}}_R(o)$, we have $a_{\mathcal{T}_R[i](e)}(t_R(o, p)) \geq k$. Indeed, for any $p \in \mathcal{P}^{\mathsf{f}}_R(o)$, the set $\{p' \in \mathcal{P}_{>i}(R); t_R(p') = p\}$ is non-empty, hence this property is obtained by applying Proposition 2.12.
- (ii) For any $p \in \mathcal{P}_0(T) \setminus \mathcal{P}$, we have $a_T(p) = a_{\mathcal{T}_R[i](e)}(p)$.

By (i) and (ii), we obtain

$$t_R[\{o\} \times \mathcal{P}^{\mathsf{f}}_R(o)] \cap \mathcal{P}_0(T) \subseteq \mathcal{P} \quad (*)$$

Let $p \in \mathcal{P}_0(T) \setminus \mathcal{P}_0(R\langle o, i, e_o\rangle)$. There exist $q_0 \in \mathcal{P}_0(T) \setminus \mathcal{P}$ and a path $(q_0, \ldots, q_n)$ in T from q_0 to $p = q_n$ without crossing $\mathcal{P}$. We set $\iota_0 = \min\{\iota \in \{1, \ldots, n\}; q_\iota \notin \mathcal{P}_0(R\langle o, i, e_o\rangle)\}$. Since $q_{\iota_0} \frown_{\mathcal{T}_R[i](e)} q_{\iota_0+1}$, there exists $p' \in \mathcal{P}^{\mathsf{f}}_R(o)$ such that $q_{\iota_0+1} = t_R(o, p')$, hence, by $(*)$, $q_{\iota_0+1} \in \mathcal{P}$; we thus have $\iota_0 + 1 = n$ and then $p = q_n \in t_R[\{o\} \times \mathcal{P}^{\mathsf{f}}_R(o)] \cap \mathcal{P}$. We showed Property 1.

Property 2 is obtained by applying Property 1 and Remark 3.26.

Finally, Property 3 is an immediate consequence of Properties 1 and 2. □

[31] In presence of cuts, we cannot say any more that these components are "above" $\mathcal{Q}$.

As a consequence, if $T \in \mathcal{S}^k_{\mathcal{T}_R[i](e)}(\mathcal{P})$ for some k-heterogeneous pseudo-experiment e on R, then T has at most one co-contraction, which is necessarily a conclusion of T (hence an element of $\mathcal{P}$) and of arity 1.

A PS R with cuts might have some connected components without any conclusion; copies of such components will occur in the PS $\mathcal{T}_R[i](e)$ and we want to recover from which boxes they come from. That is why we will consider k-heterogeneous experiments with $k > \Theta(R)$, where $\Theta(R)$ is defined as follows:

Definition 3.30. Let R be an in-PS. We set

$$\mathcal{H}(R) = \left\{T \trianglelefteq_\emptyset R;\ (\mathcal{P}_0(T) \neq \emptyset \wedge T \textit{ is connected through ports not in } \emptyset)\ \right\}$$

We denote by H_R the function $\mathcal{P}(R) \to \mathcal{H}(R)$ that associates with every $p \in \mathcal{P}(R)$ the unique $T \in \mathcal{H}(R)$ such that $p \in \mathcal{P}(T)$.

We define, by induction on $depth(R)$, the integer $\Theta(R)$ as follows:

$$\Theta(R) = \mathsf{Card}\left(\{U \in \mathcal{H}(R); \mathcal{P}^{\mathsf{f}}(U) = \emptyset\}\right) + \sum_{o \in \mathcal{B}_0(R)} \Theta(B_R(o))$$

Notice that, if R is a cut-free in-PS, then $\Theta(R) = 0$.

Example 3.31. If R is the PS of Figure 11, then $\mathsf{Card}\,(\mathcal{H}(B_R(o_3))) = 4$ and $\Theta(R) = 1$.

The set $\mathcal{H}(R)$ is an alternative way to describe an in-PS R:

Fact 3.32. Let R be an in-PS. We have $R = \bigoplus \mathcal{H}(R)$.

Definition 3.33. Let R be an in-PS. We set

$$\beta(R) = \max\{\mathsf{Card}\,(\mathcal{B}(R))\,, \textit{co-size}(R), \Theta(R), 1\} + 1$$

Lemma 3.34. *Let R and R_o be two in-PS's. Let $o \in \mathcal{B}_0(R)$. Let φ be a bijection $\mathcal{P}^?_R(o) \simeq \mathcal{Q}'$ such that $R_o = \varphi \cdot_o R$. Let $k >$ co-size(R). There exists a bijection $\theta : \mathcal{S}^k_{B_{R_o}(o)}(\mathcal{Q}') \simeq \mathcal{H}(B_R(o)) \setminus \{H_{B_R(o)}(!_R(o))\}$ such that, for any $T \in \mathcal{S}^k_{B_{R_o}(o)}(\mathcal{Q}')$, we have $\theta(T) = \overline{T}$.*

Proof. Notice first $(\forall p, q \in \mathcal{P}_0(B_R(o)))(p \frown_{B_R(o)} q \Rightarrow p \frown_{B_{R_o}(o)} q)(*)$.

We check that, for any $T \in \mathcal{S}^k_{B_{R_o}(o)}(\mathcal{Q}')$, we have $\overline{T} \in \mathcal{H}(B_R(o)) \setminus \{H_{B_R(o)}(!_R(o))\}$; let $T \in \mathcal{S}^k_{B_{R_o}(o)}(\mathcal{Q}')$:

- By Remark 1.21, $\mathcal{P}_0(\overline{T}) = \mathcal{P}_0(T) \setminus \mathcal{Q}' \subseteq (\mathcal{P}_0(B_{R_o}(o)) \setminus \{!_R(o)\}) \setminus \mathcal{Q}' = \mathcal{P}_0(B_R(o)) \setminus \{!_R(o)\}$;
-

$$\begin{aligned}
\mathcal{W}_0(\overline{T}) &= \{w \in \mathcal{W}_0(T); t_{\mathcal{G}(T)}(w) \notin \mathcal{Q}'\} \\
&= \{w \in (\mathcal{W}_0(B_{R_o}(o)) \cap \mathcal{P}_0(T)) \setminus \mathcal{Q}'; t_{\mathcal{G}(B_{R_o}(o))}(w) \in \mathcal{P}_0(T) \setminus \mathcal{Q}'\} \\
&= \{w \in (\mathcal{W}_0(B_R(o)) \cap \mathcal{P}_0(\overline{T}); t_{\mathcal{G}(B_R(o))}(w) \in \mathcal{P}_0(\overline{T})\}
\end{aligned}$$

- by $(*)$, we have $\overline{T} \trianglelefteq_\emptyset B_R(o)$
- and $(\forall p \in \mathcal{P}_0(T) \setminus \mathcal{Q}')(\forall q \in \mathcal{P}_0(T) \setminus \mathcal{Q}')(p \frown_T q \Rightarrow p \frown_{\overline{T}} q)$, hence $\overline{T}$ is connected through ports not in $\emptyset$.

Now, for any $U \in \mathcal{H}(B_R(o)) \setminus \{H_{B_R(o)}(!_R(o))\}$, we set $T_U = (U \oplus \bigoplus_{p \in \mathcal{P}^{\mathsf{f}}_R(o) \cap \mathcal{P}(U)} ?_{\varphi(t_R(o,p))})@t$, where t is the function $\mathcal{P}^{\mathsf{f}}_R(o) \cap \mathcal{P}(U) \to \varphi[t_R[\{o\} \times (\mathcal{P}^{\mathsf{f}}_R(o) \cap \mathcal{P}(U))]]$ that associates with every $p \in \mathcal{P}^{\mathsf{f}}_R(o) \cap \mathcal{P}(U)$ the port $\varphi(t_R(o,p))$ of $B_{R_o}(o)$, and we check that $T_U \in \mathcal{S}^k_{B_{R_o}(o)}(\mathcal{Q}')$:

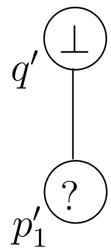


Figure 27: H'_1

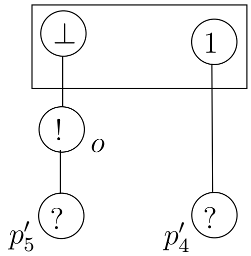


Figure 28: H'_2

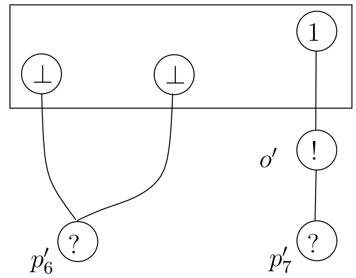


Figure 29: H'_3

- $\mathcal{P}_0(T_U) \subseteq \mathcal{P}_0(U) \cup \mathcal{Q}' \subseteq \mathcal{P}_0(B_R(o)) \cup \mathcal{Q}' = \mathcal{P}_0(B_{R_o}(o))$
-

$$\begin{aligned}
\mathcal{W}_0(T_U) &= \mathcal{W}_0(U) \cup (\mathcal{P}^{\mathsf{f}}_R(o) \cap \mathcal{P}_0(U)) \\
&= \{w \in \mathcal{W}_0(B_R(o)) \cap \mathcal{P}_0(U); t_{\mathcal{G}(B_R(o))}(w) \in \mathcal{P}_0(U)\} \cup (\mathcal{P}^{\mathsf{f}}_R(o) \cap \mathcal{P}_0(U)) \\
&= \{w \in \mathcal{W}_0(B_R(o)) \cap \mathcal{P}_0(U); t_{\mathcal{G}(B_{R_o}(o))}(w) \in \mathcal{P}_0(U)\} \\
&\quad \cup \{w \in \mathcal{P}^{\mathsf{f}}_R(o) \cap \mathcal{P}_0(U); t_{\mathcal{G}(B_{R_o}(o))}(w) \in \mathcal{P}_0(T_U)\} \\
&= \{w \in \mathcal{W}_0(B_{R_o}(o)) \cap \mathcal{P}_0(U); t_{\mathcal{G}(B_{R_o}(o))}(w) \in \mathcal{P}_0(T_U)\} \\
&= \{w \in (\mathcal{W}_0(B_{R_o}(o)) \cap \mathcal{P}_0(U)) \setminus (\mathcal{Q}' \cap \mathcal{P}^e_0(B_{R_o}(o)); t_{\mathcal{G}(B_{R_o}(o))}(w) \in \mathcal{P}_0(T_U)\}
\end{aligned}$$

- $\mathcal{P}^{\mathsf{f}}(\mathcal{G}(T_U)) = \{\varphi(t_R(o,p)); p \in \mathcal{P}^{\mathsf{f}}_R(o) \cap \mathcal{P}(U)\} \subseteq \mathcal{Q}'$
-

$$\begin{aligned}
t_{T_U} &= t_{B_{R_o}(o)}\big|_{\bigcup_{o' \in \mathcal{B}_0(U)}(\{o'\} \times (\mathcal{P}^{\mathsf{f}}_{B_R(o)}(o') \cup \{p \in \mathcal{P}(B_{B_R(o)}(o')); (o',p) \in \mathcal{P}^{\mathsf{f}}_R(o)\}))} \\
&= t_{B_{R_o}(o)}\big|_{\bigcup_{o' \in \mathcal{B}_0(T_U)}(\{o'\} \times (\mathcal{P}^{\mathsf{f}}_{B_R(o)}(o') \cup \{p \in \mathcal{P}(B_{B_R(o)}(o')); (o',p) \in \mathcal{P}^{\mathsf{f}}_R(o)\}))} \\
&= t_{B_{R_o}(o)}\big|_{\bigcup_{o' \in \mathcal{B}_0(T_U)}(\{o'\} \times \mathcal{P}^{\mathsf{f}}_{B_{R_o}(o)}(o'))}
\end{aligned}$$

- by $(*)$, we have $T_U \trianglelefteq_{\mathcal{Q}'} B_{R_o}(o)$ and T_U is connected through ports not in $\mathcal{Q}'$.

Moreover we have $\overline{T_U} = U$, which shows that θ is a surjection and, for any $T \in \mathcal{S}^k_{B_{R_o}(o)}(\mathcal{Q}')$, we have $T = (\overline{T_U} \oplus \bigoplus_{p \in \mathcal{P}^{\mathsf{f}}_R(o) \cap \mathcal{P}(\overline{T_U})} ?_{\varphi(t_R(o,p))})@t$ where t is the function $\mathcal{P}^{\mathsf{f}}_R(o) \cap \mathcal{P}(\overline{T_U}) \to \varphi[t_R[\{o\} \times (\mathcal{P}^{\mathsf{f}}_R(o) \cap \mathcal{P}(\overline{T_U}))]]$ that associates with every $p \in \mathcal{P}^{\mathsf{f}}_R(o) \cap \mathcal{P}(\overline{T_U})$ the port $\varphi(t_R(o,p))$ of $B_{R_o}(o)$, which shows that θ is an injection. □

Example 3.35. Let us consider the PS R of Figure 11. We set $\mathcal{Q}' = \{p'_1, p'_4, p'_5, p'_6, p'_7\}$ and we define the bijection $\varphi : \mathcal{P}^?_R(o_2) \simeq \mathcal{Q}'$ as follows: $\varphi(p_1) = p'_1$; $\varphi(p_4) = p'_4$; $\varphi(p_5) = p'_5$; $\varphi(p_6) = p'_6$; $\varphi(p_7) = p'_7$. If k is big enough, then $\mathcal{S}^k_{(\varphi \cdot_{o_2} R)}(\mathcal{Q}') = \{H'_1, H'_2, H'_3\}$, where H'_1, H'_2 and H'_3 and the differential in-PS's of Figure 27, Figure 28 and Figure 29 respectively.

Lemma 3.36. *Let R be an in-PS. Let $o \in \mathcal{B}_0(R)$. Let $\mathcal{Q}'$ be some set, let φ be some bijection $\mathcal{P}^?_R(o) \simeq \mathcal{Q}'$, let R_o be an in-PS obtained from R by adding, according to φ, contractions as shallow conclusions to the content of the box o. Then $\beta(R_o) \leq \beta(R)$.*

Proof. Let us show that $\Theta(R_o) \leq \Theta(R)$:

- If $\mathcal{P}^{\mathsf{f}}(H_{R_o}(o)) = \emptyset$, then $(\{o\} \times \mathcal{Q}') \cap \mathcal{P}(H_{R_o}(o)) = \emptyset$, hence $\mathcal{Q}' = \emptyset$, which entails that $R_o = R$ and then $H_{R_o}(o) = H_R(o)$.

- Moreover we have $\{T \in \mathcal{H}(B_{R_o}(o)); \mathcal{P}^{\mathsf{f}}(T) = \emptyset\} \subseteq \mathcal{S}^k_{B_{R_o}(o)}(\mathcal{Q}')$, hence, by Lemma 3.34, we have $\{T \in \mathcal{H}(B_{R_o}(o)); \mathcal{P}^{\mathsf{f}}(T) = \emptyset\} \subseteq \mathcal{H}(B_R(o))$. □

Lemma 3.37. *Let R be an in-PS. Let $k >$ co-size(R). Let e be a k-heterogeneous pseudo-experiment on R. Let $i \in \mathbb{N}$. Let $\mathcal{P} \subseteq \mathcal{P}_0(\mathcal{T}_R[i](e))$. Let $o \in \mathcal{B}_0^{\geq i}(R)$. We set $\mathcal{Q} = \mathcal{P}_R^?(o)$. Let φ_o be some bijection $\mathcal{P}_R^?(o) \simeq \mathcal{Q}'$ and let R_o be an in-PS such that $R_o = \varphi_o \cdot_o R$. Then, for any $T \in \mathcal{S}^k_{\mathcal{T}_R[i](e)}(\mathcal{P})$, we have $T \sqsubseteq_{\mathcal{P}} {R_o}^{\leq i}$ if, and only if, $T \sqsubseteq_{\mathcal{P}} R^{\leq i}$.*

Proof. Assume $o \in \mathcal{B}_0^{\geq i}(R)$. Let $T \in \mathcal{S}^k_{\mathcal{T}_R[i](e)}(\mathcal{P})$. Notice that, since $T \sqsubseteq \mathcal{T}_R[i](e)$, we have $o \notin \mathcal{B}_0(T)$.

Assume $T \sqsubseteq_{\mathcal{P}} {R_o}^{\leq i}$. We have $T^{\leq 0} \sqsubseteq_{\mathcal{P}} ({R_o}^{\leq i})^{\leq 0} = {R_o}^{\leq 0} = R^{\leq 0} = (R^{\leq i})^{\leq 0}$. Moreover:
- $\mathcal{B}_0(T) = \mathcal{B}_0({R_o}^{\leq i}) \cap \mathcal{P}_0(T) = \mathcal{B}_0^{<i}(R_o) \cap \mathcal{P}_0(T) = \mathcal{B}_0^{<i}(R) \cap \mathcal{P}_0(T) = \mathcal{B}_0(R^{\leq i}) \cap \mathcal{P}_0(T)$;
- and, since $o \notin \mathcal{B}_0(T)$, we have $B_T = B_R|_{\mathcal{B}_0(T)}$ and $t_T = t_{R^{\leq i}}|_{\bigcup_{o' \in \mathcal{B}_0(T)}(\{o'\} \times \mathcal{P}^{\mathsf{f}}_{R^{\leq i}}(o'))}$.

We obtained $T \sqsubseteq_{\mathcal{P}} R^{\leq i}$.

Conversely, assume $T \sqsubseteq_{\mathcal{P}} R^{\leq i}$. We have $T^{\leq 0} \sqsubseteq_{\mathcal{P}} (R^{\leq i})^{\leq 0} = R^{\leq 0} = {R_o}^{\leq 0} = ({R_o}^{\leq i})^{\leq 0}$. Moreover:
- $\mathcal{B}_0(T) = \mathcal{B}_0(R^{\leq i}) \cap \mathcal{P}_0(T) = \mathcal{B}_0^{<i}(R) \cap \mathcal{P}_0(T) = \mathcal{B}_0^{<i}(R_o) \cap \mathcal{P}_0(T) = \mathcal{B}_0({R_o}^{\leq i}) \cap \mathcal{P}_0(T)$;
- and, since $o \notin \mathcal{B}_0(T)$, we have $B_T = B_{R_o}|_{\mathcal{B}_0(T)}$ and $t_T = t_{{R_o}^{\leq i}}|_{\bigcup_{o' \in \mathcal{B}_0(T)}(\{o'\} \times \mathcal{P}^{\mathsf{f}}_{{R_o}^{\leq i}}(o'))}$.

We obtained $T \sqsubseteq_{\mathcal{P}} {R_o}^{\leq i}$. □

The proof of the following lemma is postponed in the appendix.

Lemma 3.38. *Let R be an in-PS. Let $k >$ co-size(R). Let e be a k-heterogeneous pseudo-experiment on R. Let $i \in \mathbb{N}$. Let $\mathcal{P} \subseteq \mathcal{P}_0(\mathcal{T}_R[i](e))$. Let $o \in \mathcal{B}_0^{\geq i}(R)$. We set $\mathcal{Q} = \mathcal{P}_R^?(o)$. Let φ_o be some bijection $\mathcal{Q} \simeq \mathcal{Q}'$ and let R_o be an in-PS such that $R_o = \varphi_o \cdot_o R$. Let φ_{e_o} be the bijection $\mathcal{Q} \simeq \{o\} \times (\{e_o\} \times \mathcal{Q}')$ defined by $\varphi_{e_o}(p) = (o, (e_o, \varphi_o(p)))$ for any $p \in \mathcal{Q}$. We set $\mathcal{P}_{e_o} = (\mathcal{P} \setminus \mathcal{Q}) \cup \varphi_{e_o}[\mathcal{P} \cap \mathcal{Q}]$. Then:*
1. *for any $T \in \mathcal{S}^k_{\mathcal{T}_R[i](e)}(\mathcal{P})$ such that $\mathcal{P}_0(T) \cap \mathcal{P}_0(R\langle o, i, e_o\rangle) \neq \emptyset$, we have $T[\varphi_{e_o}] \in \mathcal{S}^k_{\mathcal{T}_{R_o}[i](e)}(\mathcal{P}_{e_o})$;*
2. *for any $T \in \mathcal{S}^k_{\mathcal{T}_{R_o}[i](e)}(\mathcal{P}_{e_o})$ such that $\mathcal{P}_0(T) \subseteq \mathcal{P}_0(R_o\langle o, i, e_o\rangle)$, we have $T[{\varphi_{e_o}}^{-1}] \in \mathcal{S}^k_{\mathcal{T}_R[i](e)}(\mathcal{P})$;*
3. *for any $T \in \mathcal{S}^k_{\mathcal{T}_{R_o}[i](e)}(\mathcal{P})$ such that $(\forall e_1 \in e(o))\mathcal{P}_0(T) \cap \mathcal{P}_0(R_o\langle o, i, e_1\rangle) = \emptyset$, we have $T \in \mathcal{S}^k_{\mathcal{T}_R[i](e)}(\mathcal{P})$;*
4. *for any $T \in \mathcal{S}^k_{\mathcal{T}_R[i](e)}(\mathcal{P})$ such that $(\forall e_1 \in e(o))\mathcal{P}_0(T) \cap \mathcal{P}_0(R_o\langle o, i, e_1\rangle) = \emptyset$, we have $T \in \mathcal{S}^k_{\mathcal{T}_{R_o}[i](e)}(\mathcal{P})$;*

The set of "critical components" we will consider in our algorithm is the set

$$\bigcup_{j \in \mathcal{N}_i(e)} \mathcal{S}^k_{\mathcal{T}_R[i](e)}(\mathcal{K}_{k,j}(\mathcal{T}_R[i](e)))$$

From this set we will build the contents of the new boxes. In particular, each port of $\mathcal{T}_R[i](e)$ at depth 0 that goes inside a new box of $\mathcal{T}_R[i+1](e)$ (i.e. each element of $\mathcal{P}_0(\mathcal{T}_R[i](e)) \setminus \mathcal{P}_0(\mathcal{T}_R[i+1](e)))$ belongs to a "critical component":

Lemma 3.39. *Let R be a PS. Let $k > \mathsf{Card}(\mathcal{B}(R)), co\text{-}size(R)$. Let e be a k-heterogeneous experiment on R. Let $i \in \mathbb{N}$. For any $p \in \mathcal{P}_0(\mathcal{T}_R[i](e)) \setminus \mathcal{P}_0(\mathcal{T}_R[i+1](e))$, there exists $T \in \bigcup_{j \in \mathcal{N}_i(e)} \mathcal{S}^k_{\mathcal{T}_R[i](e)}(\mathcal{K}_{k,j}(\mathcal{T}_R[i](e)))$ such that*

$$p \in \mathcal{P}_0(T) \setminus \mathcal{K}_{k,j}(\mathcal{T}_R[i](e)) \subseteq \mathcal{P}_0(\mathcal{T}_R[i](e)) \setminus \mathcal{P}_0(\mathcal{T}_R[i+1](e))$$

Proof. By induction on $depth(R)$. If $depth(R) = 0$, then $\mathcal{P}_0(\mathcal{T}_R[i](e)) \setminus \mathcal{P}_0(\mathcal{T}_R[i+1](e)) = \emptyset$. Assume that $depth(R) > 0$ and let $p \in \mathcal{P}_0(\mathcal{T}_R[i](e)) \setminus \mathcal{P}_0(\mathcal{T}_R[i+1](e))$. There exist $o \in \mathcal{B}_0^{\geq i}(R)$ and $e_o \in e(o)$ such that $p \in \mathcal{P}_0(R\langle o, i, e_o\rangle)$. We distinguish between two cases:

- $o \in \mathcal{B}_0^{=i}(R)$: By Proposition 3.19, there exists $j_0 \in \mathcal{N}_i(e)$ such that $!_{e,i}(j_0) = o$. We have $R\langle o, i, e_o\rangle = \langle o, \langle e_o, B_R(o)\rangle\rangle$, hence there exists $T' \in \mathcal{H}(B_R(o))$ such that $p \in \mathcal{P}_0(\langle o, \langle e_o, T'\rangle\rangle)$. We set $T = (\langle o, \langle e_o, T'\rangle\rangle \oplus \sum_{p \in \mathcal{P}^{\mathsf{f}}(T')} l_{\mathcal{G}(R)}(t_R(o,p))_{t_R(o,p)})@t_R|_{\{o\}\times \mathcal{P}^{\mathsf{f}}(T')}$. We have $\mathcal{P}^{\mathsf{f}}(\mathcal{G}(T)) = t_R[\{o\} \times \mathcal{P}^{\mathsf{f}}(T')] = t_{\mathcal{T}_R[i+1](e)}[\{o\} \times \mathcal{P}^{\mathsf{f}}(T')] \subseteq \mathcal{K}_{k,j_0}(\mathcal{T}_R[i](e))$ (by Proposition 3.19), hence $T \in \mathcal{S}^k_{\mathcal{T}_R[i](e)}(\mathcal{K}_{k,j_0}(\mathcal{T}_R[i](e)))$.
- $o \in \mathcal{B}_0^{\geq i+1}(R)$: There exists $p' \in \mathcal{P}_0(\mathcal{T}_{B_R(o)}[i](e_o)) \setminus \mathcal{P}_0(\mathcal{T}_{B_R(o)}[i+1](e_o))$ such that $p = (o, (e_o, p'))$. Let R_o be an in-PS such that $R_o = \varphi \cdot_o R$, where φ is some bijection $\mathcal{P}^?_R(o) \simeq \mathcal{Q}'$. We have $p' \in \mathcal{P}_0(\mathcal{T}_{B_{R_o}(o)}[i](e_o)) \setminus \mathcal{P}_0(\mathcal{T}_{B_{R_o}(o)}[i+1](e_o))$, hence, by induction hypothesis, there exists $T' \in \bigcup_{j \in \mathcal{N}_i(e_o)} \mathcal{S}^k_{\mathcal{T}_{B_{R_o}(o)}[i](e_o)}(\mathcal{K}_{k,j}(\mathcal{T}_{B_{R_o}(o)}[i](e_o)))$ such that $p' \in \mathcal{P}_0(T')$; let $j_0 \in \mathcal{N}_i(e_o)$ such that $T' \in \mathcal{S}^k_{\mathcal{T}_{B_{R_o}(o)}[i](e_o)}(\mathcal{K}_{k,j_0}(\mathcal{T}_{B_{R_o}(o)}[i](e_o)))$. By Corollary 2.14, we have $T' \in \mathcal{S}^k_{\mathcal{T}_{B_{R_o}(o)}[i](e_o)}((\mathcal{Q}' \cap \mathcal{K}_{k,j_0}(\mathcal{T}_{B_{R_o}(o)}[i](e_o))) \cup \mathcal{K}_{k,j_0}(\mathcal{T}_{B_R(o)}[i](e_o)))$. By Corollary 3.21, we have $T' \in \mathcal{S}^k_{\mathcal{T}_{B_{R_o}(o)}[i](e_o)}(\varphi[\mathcal{P}^?_R(o) \cap \mathcal{K}_{k,j_0}(\mathcal{T}_R[i](e))] \cup \mathcal{K}_{k,j_0}(\mathcal{T}_{B_R(o)}[i](e_o)))$. We denote by φ_{e_o} the bijection $\mathcal{P}^?_R(o) \simeq \{o\} \times (\{e_o\} \times \mathcal{Q}')$ that associates $(o, (e_o, \varphi(q)))$ with every $q \in \mathcal{P}^?_R(o)$. By Proposition 3.19, we have $\langle o, \langle e_o, T'\rangle\rangle \in \mathcal{S}^k_{\mathcal{T}_{R_o}[i](e)}(\varphi_{e_o}[\mathcal{P}^?_R(o) \cap \mathcal{K}_{k,j_0}(\mathcal{T}_R[i](e))] \cup \mathcal{K}_{k,j_0}(\mathcal{T}_R[i](e)))$. We set $T = \langle o, \langle e_o, T'\rangle\rangle[{\varphi_{e_o}}^{-1}]$. By Lemma 3.38 (2), we have $T \in \mathcal{S}^k_{\mathcal{T}_R[i](e)}((\mathcal{P}^?_R(o) \cap \mathcal{K}_{k,j_0}(\mathcal{T}_R[i](e))) \cup \mathcal{K}_{k,j_0}(\mathcal{T}_R[i](e)))$; we thus have $T \in \mathcal{S}^k_{\mathcal{T}_R[i](e)}(\mathcal{K}_{k,j_0}(\mathcal{T}_R[i](e)))$. □

Notice that not each port belonging to a "critical component" goes inside a new box. That is why, for now, we are not yet able to describe exactly the ground-structure of $\mathcal{T}_R[i+1](e)$ but we can only obtain an approximant of $\mathcal{T}_R[i+1](e)^{\leq 0}$:

Proposition 3.40. *Let R be a PS. Let $k > \mathsf{Card}(\mathcal{B}(R)), co\text{-}size(R)$. Let e be a k-heterogeneous pseudo-experiment on R. Let $i \in \mathbb{N}$. We set*

$$\mathcal{P} = \left(\mathcal{P}_0(\mathcal{T}_R[i](e)) \setminus \bigcup_{j \in \mathcal{N}_i(e)} \bigcup_{T \in \mathcal{S}^k_{\mathcal{T}_R[i](e)}(\mathcal{K}_{k,j}(\mathcal{T}_R[i](e)))} \mathcal{P}_0(T) \right) \cup \mathcal{K}_{k,\mathcal{N}_i(e)}(\mathcal{T}_R[i](e))$$

Then we have $\mathcal{T}_R[i](e)^{\leq 0}|_{\mathcal{P}} \sqsubseteq_{\emptyset} \mathcal{T}_R[i+1](e)^{\leq 0}$.

Proof. By Lemma 3.1, we have:

(1) $\mathcal{W}_0(\mathcal{T}_R[i+1](e)) \subseteq \mathcal{W}_0(\mathcal{T}_R[i](e))$ and $t_{\mathcal{G}(\mathcal{T}_R[i+1](e))} = t_{\mathcal{G}(\mathcal{T}_R[i](e))}|_{\mathcal{W}_0(\mathcal{T}_R[i+1](e))}$
(2) $\mathcal{W}_0(\mathcal{T}_R[i](e)) \cap \mathcal{P}_0(\mathcal{T}_R[i+1](e)) \subseteq \mathcal{W}_0(\mathcal{T}_R[i+1](e))$

By 1., we have

$$\begin{array}{rl} & \{w \in \mathcal{W}(\mathcal{G}(\mathcal{T}_R[i+1](e))) \cap \mathcal{P}; t_{\mathcal{G}(\mathcal{T}_R[i+1](e))}(w) \in \mathcal{P}\} \\ \subseteq & \{w \in \mathcal{W}(\mathcal{G}(\mathcal{T}_R[i](e))) \cap \mathcal{P}; t_{\mathcal{G}(\mathcal{T}_R[i](e))}(w) \in \mathcal{P}\} \end{array}$$

By Lemma 3.39 and Proposition 3.19, we have $\mathcal{P} \subseteq \mathcal{P}_0(\mathcal{T}_R[i+1](e))$ $(*)$. By $(*)$ and 2., we have

$$\begin{array}{rl} & \{w \in \mathcal{W}(\mathcal{G}(\mathcal{T}_R[i](e))) \cap \mathcal{P}; t_{\mathcal{G}(\mathcal{T}_R[i](e))}(w) \in \mathcal{P}\} \\ \subseteq & \{w \in \mathcal{W}(\mathcal{G}(\mathcal{T}_R[i+1](e))) \cap \mathcal{P}; t_{\mathcal{G}(\mathcal{T}_R[i+1](e))}(w) \in \mathcal{P}\} \end{array}$$

We thus showed

$$\begin{array}{l} \mathcal{W}(\mathcal{G}(\mathcal{T}_R[i](e))|_{\mathcal{P}}) \\ = \{w \in \mathcal{W}(\mathcal{G}(\mathcal{T}_R[i](e))) \cap \mathcal{P}; t_{\mathcal{G}(\mathcal{T}_R[i](e))}(w) \in \mathcal{P}\} \\ = \{w \in \mathcal{W}(\mathcal{G}(\mathcal{T}_R[i+1](e))) \cap \mathcal{P}; t_{\mathcal{G}(\mathcal{T}_R[i+1](e))}(w) \in \mathcal{P}\} \end{array}$$

□

Example 3.41. (Continuation of Example 3.15) Figure 31 depicts the differential PS $\mathcal{T}_R[1](e)^{\leq 0}|_{\mathcal{P}}$ obtained by applying Proposition 3.40: We have

$$\mathcal{P} = \left(\mathcal{P}_0(\mathcal{T}_R[1](e)) \setminus \bigcup_{j \in \mathcal{N}_1(e)} \bigcup_{T \in \mathcal{S}^k_{\mathcal{T}_R[1](e)}(\mathcal{K}_{10,j}(\mathcal{T}_R[1](e)))} \mathcal{P}_0(T) \right) \cup \mathcal{K}_{10,\mathcal{N}_1(e)}(\mathcal{T}_R[1](e))$$

with $\mathcal{K}_{10,\mathcal{N}_1(e)}(\mathcal{T}_R[i](e)) = \{p_1, p_4, p_5, p_6, p_7, o_2, o_4\}$.

There is no other connected component of $\mathcal{T}_R[i+1](e)$ whose conclusions belong to critical ports of $\mathcal{T}_R[i](e)$ than critical components of $\mathcal{T}_R[i](e)$:

Proposition 3.42. *Let R be an in-PS. Let $k >$ co-size(R). Let e be a k-heterogeneous pseudo-experiment on R. Let $i \in \mathbb{N}$. Then $\mathcal{S}^k_{\mathcal{T}_R[i+1](e)}(\mathcal{K}_{k,\mathcal{N}_i(e)}(\mathcal{T}_R[i](e))) \subseteq \mathcal{S}^k_{\mathcal{T}_R[i](e)}(\mathcal{K}_{k,\mathcal{N}_i(e)}(\mathcal{T}_R[i](e)))$.*

Proof. Let $T \in \mathcal{S}^k_{\mathcal{T}_R[i+1](e)}(\mathcal{K}_{k,\mathcal{N}_i(e)}(\mathcal{T}_R[i](e)))$. Notice that *depth*$(T) < i$. We have

$$T \sqsubseteq_{\mathcal{K}_{k,\mathcal{N}_i(e)}(\mathcal{T}_R[i](e))} \mathcal{T}_R[i+1](e),$$

hence, by Remark 1.16, Lemma 3.1 and Fact 1.25, $T = T^{\leq i} \sqsubseteq_{\mathcal{K}_{k,\mathcal{N}_i(e)}(\mathcal{T}_R[i](e))} \mathcal{T}_R[i](e)$.

If $T \sqsubseteq_{\mathcal{K}_{k,\mathcal{N}_i(e)}(\mathcal{T}_R[i](e))} R^{\leq i}$, then one can apply Fact 3.24.

Otherwise: Let $p \in \mathcal{P}_0(T) \setminus \mathcal{K}_{k,\mathcal{N}_i(e)}(\mathcal{T}_R[i](e))$ and $q \in \mathcal{P}_0(\mathcal{T}_R[i](e)) \setminus \mathcal{P}_0(\mathcal{T}_R[i+1](e))$ such that $p \frown_{\mathcal{T}_R[i](e)} q$. By Lemma 3.39, there exists $T' \in \mathcal{S}^k_{\mathcal{T}_R[i](e)}(\mathcal{K}_{k,\mathcal{N}_i(e)}(\mathcal{T}_R[i](e)))$ such that $p, q \in \mathcal{P}_0(T') \setminus \mathcal{K}_{k,\mathcal{N}_i(e)}(\mathcal{T}_R[i](e)) \subseteq \mathcal{P}_0(\mathcal{T}_R[i](e)) \setminus \mathcal{P}_0(\mathcal{T}_R[i+1](e))$, which contradicts $p \in \mathcal{P}_0(T) \subseteq \mathcal{P}_0(\mathcal{T}_R[i+1](e))$. □

We will apply Proposition 3.43 with $\mathcal{P} = \mathcal{K}_{k,j_0}(\mathcal{T}_R[i](e))$, where $j_0 \in \mathcal{N}_i(e)$, but, since the proof is by induction, we need to slightly generalize it.

Proposition 3.43. *Let R be an in-PS. Let $k \geq \beta(R)$. Let e be a k-heterogeneous pseudo-experiment on R. Let $i \in \mathbb{N}$. Let $\mathcal{P} \subseteq \mathcal{P}^e_0(\mathcal{T}_R[i+1](e)) \setminus \mathcal{B}_0(\mathcal{T}_R[i](e))$. Let $T \in \mathcal{S}^k_{\mathcal{T}_R[i](e)}(\mathcal{P})$ such that $\mathcal{P}^{\mathsf{f}}(T) \subseteq \mathcal{P}^{\mathsf{f}}(\mathcal{G}(T))$. We set*

- $\mathcal{T} = \{T' \in \mathcal{S}^k_{\mathcal{T}_R[i](e)}(\mathcal{P}); T \equiv T'\}$
- $\mathcal{T}' = \{T' \in \mathcal{S}^k_{\mathcal{T}_R[i+1](e)}(\mathcal{P}); T \equiv T'\}$

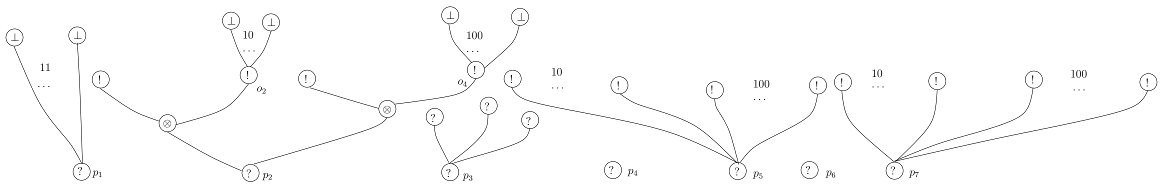


Figure 30: The differential PS $\mathcal{T}_R[1](e)^{\leq 0}$

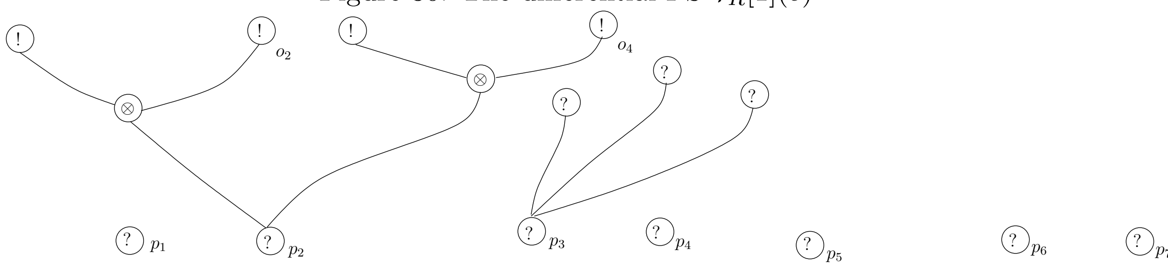


Figure 31: The approximant $\mathcal{T}_R[1](e)^{\leq 0}|_{\mathcal{P}}$ of the differential PS $\mathcal{T}_R[2](e)^{\leq 0}$

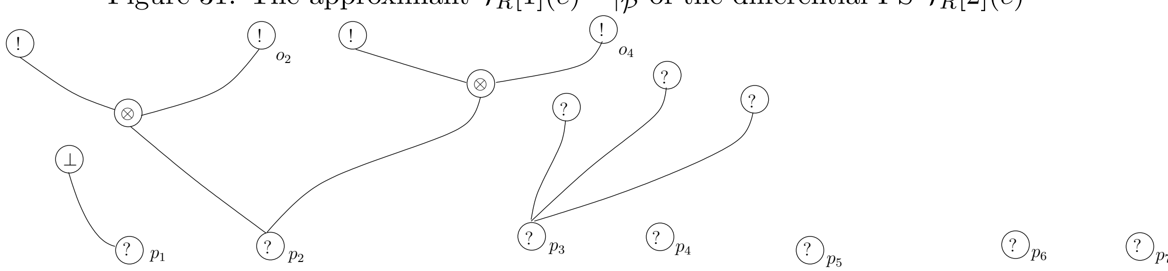


Figure 32: The differential PS $\mathcal{T}_R[2](e)^{\leq 0}$

- $\mathcal{B} = \{o \in \mathcal{B}_0^{\geq i}(R); \mathcal{P}^{\mathsf{f}}(\mathcal{G}(T)) \cap t_R[\{o\} \times \mathcal{P}_R^{\mathsf{f}}(o)] \neq \emptyset\}$
- $\mathcal{B}' = \{o \in \mathcal{B}_0^{\geq i+1}(R); \mathcal{P}^{\mathsf{f}}(\mathcal{G}(T)) \cap t_R[\{o\} \times \mathcal{P}_R^{\mathsf{f}}(o)] \neq \emptyset\}$

Let $(m_j)_{j \in \mathbb{N}}, (m'_j)_{j \in \mathbb{N}} \in \{0, \ldots, k-1\}^{\mathbb{N}}$ *such that* $\mathsf{Card}\,(\mathcal{T}) = \sum_{j \in \mathbb{N}} m_j \cdot k^j$ *and* $\mathsf{Card}\,(\mathcal{T}') = \sum_{j \in \mathbb{N}} m'_j \cdot k^j$*. Then the following properties hold:*

- $\{j \in \mathbb{N} \setminus \{0\}; m_j \neq 0\} \subseteq \mathcal{M}_i(e)$
- $\{j \in \mathbb{N} \setminus \{0\}; m'_j \neq 0\} \subseteq \mathcal{M}_{i+1}(e)$
- $(\forall j \in \mathcal{M}_{i+1}(e)) m'_j = m_j$
- $(\forall j \in \mathcal{N}_i(e)) m_j = \mathsf{Card}\left(\{U \in \mathcal{H}(B_{\mathcal{T}_R[i+1](e)}(!_{e,i}(j))); U \equiv_{(\mathcal{T}_R[i+1](e), !_{e,i}(j))} T\}\right)$

Moreover, if $\mathcal{B} \neq \emptyset$*, then the following properties hold:*

- $!_{e,i}[\{j \in \mathbb{N} \setminus \{0\}; m_j \neq 0\}] \subseteq \mathcal{B} \cup \bigcup_{o \in \mathcal{B}} \bigcup_{e_o \in e(o)} !_{e,i}[\mathcal{M}_i(e_o)]$
- $!_{e,i}[\{j \in \mathbb{N} \setminus \{0\}; m'_j \neq 0\}] \subseteq \mathcal{B}' \cup \bigcup_{o \in \mathcal{B}'} \bigcup_{e_o \in e(o)} !_{e,i}[\mathcal{M}_{i+1}(e_o)]$

Finally, if $\mathcal{P} \subseteq \mathcal{P}_0(R)$*, then* $m_0 = \mathsf{Card}\left(\{T' \in \mathcal{T}; T' \sqsubseteq_{\mathcal{P}} R^{\leq i}\}\right) = m'_0$.

Proof. We prove the proposition by induction on $(depth(R), \mathsf{Card}\,(\mathcal{B}))$ lexicographically ordered. Part I) is devoted to the case where $depth(R) = 0$, Part II) is devoted to the case where $depth(R) > 0$ and $\mathcal{B} = \emptyset$, Part III) is devoted to the case where $depth(R) > 0$, $\mathcal{B} \neq \emptyset$ and $\mathcal{P}^{\mathsf{f}}(\mathcal{G}(T)) \cap \mathcal{B}_0^{\geq i}(R) = \emptyset$, and Part IV) is devoted to the case where $depth(R) > 0$, $\mathcal{B} \neq \emptyset$ and $\mathcal{P}^{\mathsf{f}}(\mathcal{G}(T)) \cap \mathcal{B}_0^{\geq i}(R) \neq \emptyset$.

Part I. $\textit{depth}(R) = 0$: Then $\mathcal{T}_R[i](e) = R = \mathcal{T}_R[i+1](e)$ and $R^{\leq i} = R$, hence $\mathcal{T} = \{T' \in \mathcal{T}; T' \sqsubseteq_{\mathcal{P}} R^{\leq i}\} = \mathcal{T}'$;

- $\mathcal{P}^{\mathsf{f}}(\mathcal{G}(T)) \neq \emptyset$: $\mathsf{Card}\left(\{T' \in \mathcal{T}; T' \sqsubseteq_{\mathcal{P}} R^{\leq i}\}\right) \leq \textit{co-size}(R) < k$;
- $\mathcal{P}^{\mathsf{f}}(\mathcal{G}(T)) = \emptyset$: $\mathsf{Card}\left(\{T' \in \mathcal{T}; T' \sqsubseteq_{\mathcal{P}} R^{\leq i}\}\right) \leq \Theta(R) < k$;

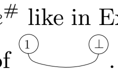
so, in both cases, $\mathsf{Card}\,(\mathcal{T}) = \mathsf{Card}\,(\mathcal{T}') < k$, which entails $m_0 = \mathsf{Card}\,(\mathcal{T}) = \mathsf{Card}\,(\mathcal{T}') = m'_0$ and $\{j \in \mathbb{N} \setminus \{0\}; m_j \neq 0\} = \emptyset = \{j \in \mathbb{N} \setminus \{0\}; m'_j \neq 0\}$; moreover, $\mathcal{M}_{i+1}(e) = \emptyset = \mathcal{N}_i(e)$.

Part II. $\textit{depth}(R) > 0$ and $\mathcal{B} = \emptyset$:[32] Then we distinguish between two cases (Case 1) and Case 2)):

- Case 1) There exist $o \in \mathcal{B}_0^{\geq i}(R)$ and $e_o \in e(o)$ such that the set $\mathcal{P}_0(T) \cap \mathcal{P}_0(R\langle o, i, e_o\rangle)$ is non-empty: By Lemma 3.29, we have $\mathcal{P}_0(T) \subseteq \mathcal{P}_0(R\langle o, i, e_o\rangle)$; we set $\mathcal{P}_o = \{p \in \mathcal{P}_0(\mathcal{T}_{B_R(o)}[i](e_o)); (o, (e_o, p)) \in \mathcal{P}\}$ and $T_0 \in \mathcal{S}^k_{\mathcal{T}_{B_R(o)}[i](e_o)}(\mathcal{P}_o)$ such that $T = \langle o, \langle e_o, T_0\rangle\rangle$; we have $\mathcal{T} = \{\langle o, \langle e_o, T'\rangle\rangle; (T' \in \mathcal{S}^k_{\mathcal{T}_{B_R(o)}[i](e_o)}(\mathcal{P}_o) \wedge T_0 \equiv T')\}$ and $\mathcal{T}' = \{\langle o, \langle e_o, T'\rangle\rangle; (T' \in \mathcal{S}^k_{\mathcal{T}_{B_R(o)}[i+1](e_o)}(\mathcal{P}_o) \wedge T_0 \equiv T')\}$. We have $\beta(B_R(o)) \leq \beta(R)$ and $\mathcal{P}^{\mathsf{f}}(T_0) \subseteq \mathcal{P}^{\mathsf{f}}(\mathcal{G}(T_0))$, hence we can apply the induction hypothesis: we obtain
  - $\{j \in \mathbb{N} \setminus \{0\}; m_j \neq 0\} \subseteq \mathcal{M}_i(e_o) \subseteq \mathcal{M}_i(e)$ (by Lemma 3.11)
  - $\{j \in \mathbb{N} \setminus \{0\}; m'_j \neq 0\} \subseteq \mathcal{M}_{i+1}(e_o) \subseteq \mathcal{M}_{i+1}(e)$ (by Lemma 3.11)
  - and $(\forall j \in \mathcal{M}_{i+1}(e_o)) m'_j = m_j$, which entails $(\forall j \in \mathcal{M}_{i+1}(e)) m'_j = m_j$, since, by Lemma 3.11, we have $\mathcal{M}_{i+1}(e) \cap \mathcal{M}_i(e_o) = \mathcal{M}_{i+1}(e_o)$.

  Now, let $j \in \mathcal{N}_i(e)$, let $U \in \mathcal{H}(B_{\mathcal{T}_R[i+1](e)}(!_{e,i}(j)))$ such that $U \equiv_{(\mathcal{T}_R[i+1](e), !_{e,i}(j))} T$ and let $p \in \mathcal{P}^{\mathsf{f}}(\mathcal{G}(U))$; there exists $p' \in \mathcal{P}^{\mathsf{f}}(\mathcal{G}(T))$ such that $p' = t_{\mathcal{T}_R[i+1](e)}(!_{e,i}(j), p)$; by Proposition 3.19, we have $!_{e,i}(j) \in \mathcal{B}_0^{=i}(R)$ or there exist $o' \in \mathcal{B}_0^{\geq i+1}(R)$ and $e_{o'} \in e(o')$ such that $j \in \mathcal{N}_i(e_{o'})$:
  - $!_{e,i}(j) \in \mathcal{B}_0^{=i}(R)$: then we have $p' \in \mathcal{P}_0(R)$, hence $p' \notin \mathcal{P}$, which is in contradiction with $\mathcal{P}^{\mathsf{f}}(\mathcal{G}(T)) \subseteq \mathcal{P}$;
  - or ($j \in \mathcal{N}_i(e_{o'})$ for some $o' \in \mathcal{B}_0^{\geq i+1}(R)$ and $e_{o'} \in e(o')$): then we have
  $$p' = (o', (e_{o'}, t_{\mathcal{T}_{B_R(o')}[i+1](e_{o'})}(!_{e_{o'},i}(j), p))) \in \mathcal{P}$$
    hence $(o', e_{o'}) = (o, e_o)$.

  This shows that, for any $j \in \mathcal{N}_i(e) \setminus \mathcal{N}_i(e_o)$, we have
  $$\mathsf{Card}\left(\{U \in \mathcal{H}(B_{\mathcal{T}_R[i+1](e)}(!_{e,i}(j))); U \equiv_{(\mathcal{T}_R[i+1](e), !_{e,i}(j))} T\}\right) = 0$$
  but, since $j \notin \mathcal{M}_i(e_o)$, we have $m_j = 0$.

[32] As an instance of this case, take for R the PS that is depicted in Figure 11, take $i = 0$, take for e a 10-heterogeneous experiment with $e^{\#}$ like in Example 2.2, take for $\mathcal{P}$ any subset of $\mathcal{P}_0^e(\mathcal{T}_R[1](e)) \setminus \mathcal{B}_0(\mathcal{T}_R[0](e))$ and for T one of the 10^{224} copies of 

.

For any $j \in \mathcal{N}_i(e_o)$, we have $!_{e,i}(j) = (o, (e_o, !_{e_o,i}(j)))$; applying the induction hypothesis, we obtain

$$\begin{aligned} & \mathsf{Card}\left(\{U \in \mathcal{H}(B_{\mathcal{T}_R[i+1](e)}(!_{e,i}(j))); U \equiv_{(\mathcal{T}_R[i+1](e),!_{e,i}(j))} T\}\right) \\ = \; & \mathsf{Card}\left(\{U \in \mathcal{H}(B_{\mathcal{T}_R[i+1](e)}(o,(e_o,!_{e,i}(j)))); U \equiv_{(\mathcal{T}_R[i+1](e),(o,(e_o,!_{e_o,i}(j))))} T\}\right) \\ = \; & \mathsf{Card}\left(\{U \in \mathcal{H}(B_{\mathcal{T}_{B_R(o)}[i+1](e_o)}(!_{e_o,i}(j))); U \equiv_{(\mathcal{T}_{B_R(o)}[i+1](e_o),!_{e_o,i}(j))} T_0\}\right) \\ = \; & m_j \end{aligned}$$

- Case 2) For any $o \in \mathcal{B}_0^{\geq i}(R)$, for any $e_o \in e(o)$, we have $\mathcal{P}_0(T) \cap \mathcal{P}_0(R\langle o, i, e_o\rangle) = \emptyset$: Notice that then $T \sqsubseteq R$. We set $\mathcal{P}_0 = \mathcal{P} \cap \mathcal{P}_0(R)$. We have $\mathcal{T} = \{T' \in \mathcal{T}; T' \sqsubseteq_{\mathcal{P}_0} R^{\leq i}\}$, $\mathcal{T}' = \{T' \in \mathcal{T}'; T' \sqsubseteq_{\mathcal{P}_0} R^{\leq i}\}$ and $\mathsf{Card}\left(\{T' \in \mathcal{T}; T' \sqsubseteq_{\mathcal{P}_0} R^{\leq i}\}\right) < k$, hence, by Fact 3.24, we have $m_0 = \mathsf{Card}\,(\mathcal{T}) = \mathsf{Card}\,(\mathcal{T}') = m'_0$ and $\{j \in \mathbb{N} \setminus \{0\}; m_j \neq 0\} = \emptyset = \{j \in \mathbb{N} \setminus \{0\}; m'_j \neq 0\}$.

  Since $T \sqsubseteq R$ and $\mathcal{B} = \emptyset$, for any $o \in \mathcal{B}_0^{=i}(\mathcal{T}_R[i+1](e))$, we have

$$\{U \in \mathcal{H}(B_{\mathcal{T}_R[i+1](e)}(o)); U \equiv_{(\mathcal{T}_R[i+1](e),o)} T\} = \emptyset$$

Part III. $depth(R) > 0$, $\mathcal{B} \neq \emptyset$ and $\mathcal{P}^{\mathsf{f}}(\mathcal{G}(T)) \cap \mathcal{B}_0^{\geq i}(R) = \emptyset$:[33] Then let $o \in \mathcal{B}$: Let R_o be an in-PS and φ be some bijection $\mathcal{P}_R^{?}(o) \simeq \mathcal{Q}'$ such that $R_o = \varphi \cdot_o R$.

Roughly speaking, the number $\sum_{j\in\mathbb{N}} m_j \cdot k^j$ (resp. $\sum_{j\in\mathbb{N}} m'_j \cdot k^j$) of components of $\mathcal{T}_R[i](e)$ (resp. $\mathcal{T}_R[i+1](e)$) that are equivalent to the connected component T is the sum of the number $\sum_{j\in\mathbb{N}} p_j \cdot k^j$ (resp. $\sum_{j\in\mathbb{N}} p'_j \cdot k^j$) of such components that come from the expansion of the box o and the number $\sum_{j\in\mathbb{N}} n_j \cdot k^j$ (resp. $\sum_{j\in\mathbb{N}} n'_j \cdot k^j$) of such components that do not come from the expansion of the box o, but we define the sequence $(n_j)_{j\in\mathbb{N}}$ (resp. the sequence $(n'_j)_{j\in\mathbb{N}}$) through $\mathcal{T}_{\varphi\cdot_o R}[i](e)$ (resp. $\mathcal{T}_{\varphi\cdot_o R}[i+1](e)$), and not through $\mathcal{T}_R[i](e)$ (resp. $\mathcal{T}_R[i+1](e)$), in order to be able to apply the induction hypothesis.

Let $j_o \in \mathcal{M}_i(e)$ such that $!_{e,i}(j_o) = o$. We define a subset $\mathcal{N}$ of $\mathcal{N}_i(e)$ as follows: We set $\mathcal{N} = \begin{cases} \{j_o\} & \textit{if } o \in \mathcal{B}_0^{=i}(R); \\ \bigcup_{e_o\in e(o)} \mathcal{N}_i(e_o) & \textit{otherwise.} \end{cases}$ For every $e_o \in e(o)$, let φ_{e_o} be the bijection $\mathcal{P}_R^{?}(o) \simeq \{o\} \times (\{e_o\} \times \mathcal{Q}')$ defined by $\varphi_{e_o}(q) = (o, (e_o, \varphi(q)))$ for any $q \in \mathcal{P}_R^{?}(o)$; we set $\mathcal{P}_{e_o} = (\mathcal{P} \setminus \mathcal{P}_R^{?}(o)) \cup \varphi_{e_o}[\mathcal{P} \cap \mathcal{P}_R^{?}(o)]$.

Let $(p_j)_{j\in\mathbb{N}} \in \{0, \ldots, k-1\}^{\mathbb{N}}$ such that

$$\mathsf{Card}\left(\{T' \in \mathcal{T}; \mathcal{P}_0(T') \cap \bigcup_{e_o\in e(o)} \mathcal{P}_0(R\langle o, i, e_o\rangle) \neq \emptyset\}\right) = \sum_{j\in\mathbb{N}} p_j \cdot k^j$$

For every $e_o \in e(o)$, let $(p_{e_o,j})_{j\in\mathbb{N}} \in \{0, \ldots, k-1\}^{\mathbb{N}}$ such that

$$\mathsf{Card}\left(\{T' \in \mathcal{S}^k_{\mathcal{T}_{R_o}[i](e)}(\mathcal{P}_{e_o}); T' \equiv T[\varphi_{e_o}]\}\right) = \sum_{j\in\mathbb{N}} p_{e_o,j} \cdot k^j$$

Let $(p'_j)_{j\in\mathbb{N}} \in \{0, \ldots, k-1\}^{\mathbb{N}}$ such that

[33] As an instance of this case, take for R the PS that is depicted in Figure 11, take $i = 1$, take for e a 10-heterogeneous experiment with $e^{\#}$ like in Example 2.2, take $\mathcal{P} = \{p_1\}$ and for T the differential PS that is depicted in Figure 33 (see Example 3.44 on page 54).

- if $o \in \mathcal{B}_0^{\geq i+1}(R)$, then

$$\mathsf{Card}\left(\{T' \in \mathcal{T}'; \mathcal{P}_0(T') \cap \bigcup_{e_o \in e(o)} \mathcal{P}_0(R\langle o, i+1, e_o\rangle) \neq \emptyset\}\right) = \sum_{j \in \mathbb{N}} p'_j \cdot k^j$$

- if $o \in \mathcal{B}_0^{=i}(R)$, then $0 = \sum_{j \in \mathbb{N}} p'_j \cdot k^j$.

For any $e_o \in e(o)$, let $(p'_{e_o,j})_{j \in \mathbb{N}} \in \{0, \ldots, k-1\}^{\mathbb{N}}$ such that

- if $o \in \mathcal{B}_0^{\geq i+1}(R)$, then

$$\mathsf{Card}\left(\{T' \in \mathcal{S}^k_{\mathcal{T}_{R_o}[i+1](e)}(\mathcal{P}_{e_o}); T' \equiv T[\varphi_{e_o}]\}\right) = \sum_{j \in \mathbb{N}} p'_{e_o,j} \cdot k^j$$

- if $o \in \mathcal{B}_0^{=i}(R)$, then $0 = \sum_{j \in \mathbb{N}} p'_{e_o,j} \cdot k^j$.

By Lemma 3.29, we have

$$\begin{aligned}&\mathsf{Card}\left(\{T' \in \mathcal{T}; \mathcal{P}_0(T') \cap \bigcup_{e_o \in e(o)} \mathcal{P}_0(R\langle o, i, e_o\rangle) \neq \emptyset\}\right)\\ = &\sum_{e_o \in e_o} \mathsf{Card}\left(\{T' \in \mathcal{T}; \mathcal{P}_0(T') \cap \mathcal{P}_0(R\langle o, i, e_o\rangle) \neq \emptyset\}\right)\end{aligned}$$

hence, by Lemma 3.38 (1) and (2), we have:

- $\sum_{j \in \mathbb{N}} p_j \cdot k^j = \sum_{e_o \in e(o)} \sum_{j \in \mathbb{N}} p_{e_o,j} \cdot k^j$
- $\sum_{j \in \mathbb{N}} p'_j \cdot k^j = \sum_{e_o \in e(o)} \sum_{j \in \mathbb{N}} p'_{e_o,j} \cdot k^j$

Let $(n_j)_{j \in \mathbb{N}} \in \{0, \ldots, k-1\}^{\mathbb{N}}$ such that

$$\mathsf{Card}\left(\{T' \in \mathcal{S}^k_{\mathcal{T}_{R_o}[i](e)}(\mathcal{P}^{\mathsf{f}}(\mathcal{G}(T))); T' \equiv T\}\right) = \sum_{j \in \mathbb{N}} n_j \cdot k^j$$

and let $(n'_j)_{j \in \mathbb{N}} \in \{0, \ldots, k-1\}^{\mathbb{N}}$ such that

$$\mathsf{Card}\left(\{T' \in \mathcal{S}^k_{\mathcal{T}_{R_o}[i+1](e)}(\mathcal{P}^{\mathsf{f}}(\mathcal{G}(T))); T' \equiv T\}\right) = \sum_{j \in \mathbb{N}} n'_j \cdot k^j.$$

By Lemma 3.38 (3) and (4), we have:

- $\sum_{j \in \mathbb{N}} m_j \cdot k^j = \sum_{j \in \mathbb{N}} (p_j + n_j) \cdot k^j$
- $\sum_{j \in \mathbb{N}} m'_j \cdot k^j = \sum_{j \in \mathbb{N}} (p'_j + n'_j) \cdot k^j$

Now, the proof will be in three steps (Step 1), Step 2) and Step 3)):

Step 1). It consists in proving properties about the sequences $(n_j)_{j \in \mathbb{N}}$ and $(n'_j)_{j \in \mathbb{N}}$. We distinguish between two cases:

- $\mathcal{B} = \{o\}$: Then $\{T' \in \mathcal{S}^k_{\mathcal{T}_{R_o}[i](e)}(\mathcal{P}); T' \equiv T\} = \{T' \in \mathcal{S}^k_{\mathcal{T}_{R_o}[i](e)}(\mathcal{P}); (T' \equiv T \wedge T' \sqsubseteq_{\mathcal{P}} {R_o}^{\leq i})\}$ and $\{T' \in \mathcal{S}^k_{\mathcal{T}_{R_o}[i+1](e)}(\mathcal{P}); T' \equiv T\} = \{T' \in \mathcal{S}^k_{\mathcal{T}_{R_o}[i+1](e)}(\mathcal{P}); (T' \equiv T \wedge T' \sqsubseteq_{\mathcal{P}} {R_o}^{\leq i})\}$, hence $\mathsf{Card}\left(\{T' \in \mathcal{S}^k_{\mathcal{T}_{R_o}[i](e)}(\mathcal{P}); T' \equiv T\}\right) < k$ and, by Fact 3.24, we have

$$\mathsf{Card}\left(\{T' \in \mathcal{S}^k_{\mathcal{T}_{R_o}[i](e)}(\mathcal{P}); T' \equiv T\}\right) = \mathsf{Card}\left(\{T' \in \mathcal{S}^k_{\mathcal{T}_{R_o}[i+1](e)}(\mathcal{P}); T' \equiv T\}\right)$$

so $\{j \in \mathbb{N} \setminus \{0\}; n_j \neq 0\} = \emptyset = \{j \in \mathbb{N} \setminus \{0\}; n'_j \neq 0\}$ and

$$n_0 = \mathsf{Card}\left(\{T' \in \mathcal{S}^k_{\mathcal{T}_{R_o}[i](e)}(\mathcal{P}); (T' \equiv T \wedge T' \sqsubseteq_{\mathcal{P}} {R_o}^{\leq i})\}\right) = n'_0$$

(in particular, $(\forall j \in \mathbb{N}) n'_j = n_j$). If $\mathcal{P} \subseteq \mathcal{P}_0(R)$, then, by Lemma 3.37, we have $n_0 = \mathsf{Card}\left(\{T' \in \mathcal{T}; T' \sqsubseteq_{\mathcal{P}} R^{\leq i}\}\right) = n'_0$.

- $\mathcal{B} \setminus \{o\} \neq \emptyset$: By Lemma 3.36, we can apply the induction hypothesis: We have
  - $!_{e,i}[\{j \in \mathbb{N} \setminus \{0\}; n_j \neq 0\}] \subseteq (\mathcal{B} \setminus \{o\}) \cup \bigcup_{o' \in \mathcal{B} \setminus \{o\}} \bigcup_{e_{o'} \in e(o')} !_{e,i}[\mathcal{M}_i(e_{o'})]$
  - $!_{e,i}[\{j \in \mathbb{N} \setminus \{0\}; n'_j \neq 0\}] \subseteq (\mathcal{B}' \setminus \{o\}) \cup \bigcup_{o' \in \mathcal{B}' \setminus \{o\}} \bigcup_{e_{o'} \in e(o')} !_{e,i}[\mathcal{M}_{i+1}(e_{o'})]$
  - $(\forall j \in \mathcal{M}_{i+1}(e)) n'_j = n_j$
  - if $\mathcal{P} \subseteq \mathcal{P}_0(R)$, then $n_0 = \mathsf{Card}\left(\{T' \in \mathcal{S}^k_{\mathcal{T}_{R_o}[i](e)}(\mathcal{P}); (T' \equiv T \wedge T' \sqsubseteq_{\mathcal{P}} {R_o}^{\leq i})\}\right) = n'_0$, hence, by Lemma 3.37, we have $n_0 = \mathsf{Card}\left(\{T' \in \mathcal{T}; T' \sqsubseteq_{\mathcal{P}} R^{\leq i}\}\right) = n'_0$.

  Now, for any $j \in \mathcal{N}_i(e) \setminus \mathcal{N}$, we have $B_{\mathcal{T}_{R_o}[i+1](e)}(!_{e,i}(j)) = B_{\mathcal{T}_R[i+1](e)}(!_{e,i}(j))$, hence

$$\mathsf{Card}\left(\{U \in \mathcal{H}(B_{\mathcal{T}_R[i+1](e)}(!_{e,i}(j))); U \equiv_{(\mathcal{T}_R[i+1](e), !_{e,i}(j))} T\}\right) = n_j$$

Step 2). In order to prove

- $\{j \in \mathbb{N}; p_j \neq 0\} \subseteq \{0, j_o\} \cup \bigcup_{e_o \in e(o)} \mathcal{M}_i(e_o)$
- $(p_0 \neq 0 \Rightarrow n_0 = 0)$
- $\{j \in \mathbb{N}; p'_j \neq 0\} \subseteq \{0, j_o\} \cup \bigcup_{e_o \in e(o)} \mathcal{M}_{i+1}(e_o)$
- $(o \in \mathcal{B}_0^{\geq i+1}(R) \Rightarrow (\forall j \in \{j_o\} \cup \bigcup_{e_o \in e(o)} \mathcal{M}_{i+1}(e_o)) p'_j = p_j)$
- $(p'_0 \neq 0 \Rightarrow n'_0 = 0)$
- $(\forall j \in \mathcal{N}) p_j = \mathsf{Card}\left(\{U \in \mathcal{H}(B_{\mathcal{T}_R[i+1](e)}(!_{e,i}(j))); U \equiv_{(\mathcal{T}_R[i+1](e), !_{e,i}(j))} T\}\right)$

we distinguish between three cases (Case a), Case b) and Case c)):

- Case a) $\mathcal{P}^{\mathsf{f}}(\mathcal{G}(T)) \subseteq \mathcal{P}^?_R(o)$: It is worth noticing that the differential in-PS T has no co-contraction. If there is no $e_1 \in e(o)$ such that $\{T' \in \mathcal{S}^k_{\mathcal{T}_{R_o}[i](e)}(\mathcal{P}_{e_1}); T' \equiv T[\varphi_{e_1}]\} \neq \emptyset$, then, by Lemma 3.38 (1), for any $j \in \mathbb{N}$, we have $p_j = 0 = p'_j$, so there is nothing to prove. From now on, let us assume that there exist $e_1 \in e(o)$ and $T'_1 \in \{T' \in \mathcal{S}^k_{\mathcal{T}_{R_o}[i](e)}(\mathcal{P}_{e_1}); T' \equiv T[\varphi_{e_1}]\}$; let $\zeta : T'_1 \equiv T[\varphi_{e_1}]$ and let $T_1 \in \mathcal{S}^k_{\mathcal{T}_{B_{R_o}(o)}[i](e_1)}(\mathcal{Q}')$ such that $T'_1 = \langle o, \langle e_1, T_1 \rangle \rangle$; notice that, for any $e_o \in e(o)$, we have $T[\varphi_{e_o}] \equiv \langle o, \langle e_o, T_1 \rangle \rangle$, hence

$$\begin{array}{rl} & \{T' \in \mathcal{S}^k_{\mathcal{T}_{R_o}[i](e)}(\mathcal{P}_{e_o}); T' \equiv T[\varphi_{e_o}]\} \\ = & \{\langle o, \langle e_o, T' \rangle \rangle; (T' \in \mathcal{S}^k_{\mathcal{T}_{B_{R_o}(o)}[i](e_o)}(\mathcal{Q}') \wedge T' \equiv T_1)\} \quad (*) \end{array}$$

  we have $\mathcal{P}^{\mathsf{f}}(T_1) \subseteq \mathcal{P}^{\mathsf{f}}(\mathcal{G}(T_1)) \subseteq \mathcal{Q}' \subseteq \mathcal{P}_0(B_{R_o}(o))$, hence, since by Lemma 3.36 we can apply the induction hypothesis, we have

$$p_{e_1,0} = \mathsf{Card}\left(\{T' \in \mathcal{S}^k_{\mathcal{T}_{B_{R_o}(o)}[i](e_1)}(\mathcal{Q}'); (T' \equiv T_1 \wedge T' \sqsubseteq_{\mathcal{Q}'} B_{R_o}(o)^{\leq i})\}\right)$$

and

$$\begin{array}{rl} & \displaystyle\sum_{e_o \in e(o)} \mathsf{Card}\left(\{T' \in \mathcal{S}^k_{\mathcal{T}_{R_o}[i](e)}(\mathcal{P}_{e_o}); T' \equiv T[\varphi_{e_o}]\}\right) \\ = & \displaystyle\sum_{e_o \in e(o)} \mathsf{Card}\left(\{T' \in \mathcal{S}^k_{\mathcal{T}_{B_{R_o}(o)}[i](e_o)}(\mathcal{Q}'); T' \equiv T_1\}\right) \quad \textit{(by } (*)\textit{)} \\ = & \mathsf{Card}\left(\{T' \in \mathcal{S}^k_{\mathcal{T}_{B_{R_o}(o)}[i](e_1)}(\mathcal{Q}'); (T' \equiv T_1 \wedge T' \sqsubseteq_{\mathcal{Q}'} B_{R_o}(o)^{\leq i})\}\right) \cdot \mathsf{Card}\left(e(o)\right) \\ + & \displaystyle\sum_{e_o \in e(o)} \sum_{j \in \mathcal{M}_i(e_o)} p_{e_o,j} \cdot k^j \end{array}$$

hence $!_{e,i}[\{j \in \mathbb{N}; p_j \neq 0\}] \subseteq \{o\} \cup \bigcup_{e_o \in e(o)} !_{e,i}[\mathcal{M}_i(e_o)]$.

If $o \in \mathcal{B}_0^{=i}(R)$, then there exists a bijection $\{T' \in \mathcal{S}^k_{B_{R_o}(o)}(\mathcal{Q}'); T' \equiv T_1\} \simeq \{U \in \mathcal{H}(B_R(o)); U \equiv_{(\mathcal{T}_R[i+1](e),o)} T\}$ that associates with every $T' \in \mathcal{S}^k_{B_{R_o}(o)}(\mathcal{Q}')$ such that $T' \equiv T_1$ the differential in-PS $\overline{T'}$. Indeed:

- By Lemma 3.34, there exists a bijection $\mathcal{S}^k_{B_{R_o}(o)}(\mathcal{Q}') \simeq \mathcal{H}(B_R(o))$ that associates with every $T' \in \mathcal{S}^k_{B_{R_o}(o)}(\mathcal{Q}')$ the differential in-PS $\overline{T'}$;
- If $T' \in \mathcal{S}^k_{B_{R_o}(o)}(\mathcal{Q}')$ and $\psi : T' \equiv T_1$, then the function δ that associates with every $p \in \mathcal{P}(\overline{T'})$ the port $\zeta(o_1, (e_1, \psi(p)))$ of $\overline{T}$ is an isomorphism $\overline{T'} \simeq \overline{T}$ such that, for any $p \in \mathcal{P}^{\mathsf{f}}(\overline{T'})$, we have $\mathsf{t}_{T'}(p) = \mathsf{t}_{B_{R_o}(o)}(p) = \varphi(t_R(o,p))$, hence

$$\begin{array}{rcl} \varphi_{e_1}(\mathsf{t}_T(\delta(p))) & = & \varphi_{e_1}(\mathsf{t}_T(\zeta(o_1,(e_1,\psi(p))))) \\ & = & \mathsf{t}_{T[\varphi_{e_1}]}(\zeta(o_1,(e_1,\psi(p)))) \\ & = & \zeta(\mathsf{t}_{T_1'}(o_1,(e_1,\psi(p)))) \\ & = & \mathsf{t}_{T_1'}(o_1,(e_1,\psi(p))) \\ & = & (o_1,(e_1,\mathsf{t}_{T_1}(\psi(p)))) \\ & = & (o_1,(e_1,\psi(\mathsf{t}_{T'}(p)))) \\ & = & (o_1,(e_1,\mathsf{t}_{T'}(p))) \\ & = & (o_1,(e_1,\varphi(t_R(o,p)))) \\ & = & \varphi_{e_1}(t_R(o,p)) \end{array}$$

So, $t_{\mathcal{T}_R[i+1](e)}(o,p) = t_R(o,p) = \mathsf{t}_T(\delta(p))$.

If $o \in \mathcal{B}_0^{\geq i+1}(R)$, then, by induction hypothesis, $p'_{e_1,0} = \mathsf{Card}\left(\{T' \sqsubseteq_{\mathcal{Q}'} B_{R_o}(o); T' \equiv T_1\}\right)$ and

$$\begin{array}{rl} & \displaystyle\sum_{e_o \in e(o)} \mathsf{Card}\left(\{T' \in \mathcal{S}^k_{\mathcal{T}_{R_o}[i+1](e)}(\mathcal{P}_{e_o}); T' \equiv T[\varphi_{e_o}]\}\right) \\ = & \displaystyle\sum_{e_o \in e(o)} \mathsf{Card}\left(\{T' \in \mathcal{S}^k_{\mathcal{T}_{B_{R_o}(o)}[i+1](e_o)}(\mathcal{Q}'); T' \equiv T_1\}\right) \\ = & \mathsf{Card}\left(\{T' \in \mathcal{S}^k_{\mathcal{T}_{B_{R_o}(o)}[i](e_1)}(\mathcal{Q}'); (T' \equiv T_1 \wedge T' \sqsubseteq_{\mathcal{Q}'} B_{R_o}(o))\}\right) \cdot \mathsf{Card}\left(e(o)\right) \\ + & \displaystyle\sum_{e_o \in e(o)} \sum_{j \in \mathcal{M}_{i+1}(e_o)} p'_{e_o,j} \cdot k^j \end{array}$$

hence $!_{e,i}[\{j \in \mathbb{N}; p'_j \neq 0\} \subseteq \{o\} \cup \bigcup_{e_o \in e(o)} !_{e,i}[\mathcal{M}_{i+1}(e_o)]$; by induction hypothesis, for any $e_o \in e(o)$, we have $(\forall j \in \mathcal{M}_{i+1}(e_o) \cup \{0\}) p'_{e_o,j} = p_{e_o,j}$; moreover, we have

$$p_{j_o} = \mathsf{Card}\left(\{T' \in \mathcal{S}^k_{\mathcal{T}_{B_{R_o}(o)}[i](e_1)}(\mathcal{Q}'); (T' \equiv T_1 \wedge T' \sqsubseteq_{\mathcal{Q}'} B_{R_o}(o))\}\right) = p'_{j_o}$$

now, let $j \in \mathcal{N}$ and let $e_o \in e(o)$ such that $!_{e,i}(j) = (o, (e_o, !_{e_o,i}(j)))$: We have

$$\begin{aligned} & \mathsf{Card}\left(\{U \in \mathcal{H}(B_{\mathcal{T}_R[i+1](e)}(!_{e,i}(j))); U \equiv_{(\mathcal{T}_R[i+1](e), !_{e,i}(j))} T\}\right) \\ = \;& \mathsf{Card}\left(\{U \in \mathcal{H}(B_{\mathcal{T}_{R_o}[i+1](e)}(!_{e,i}(j))); U \equiv_{(\mathcal{T}_{R_o}[i+1](e), !_{e,i}(j))} T[\varphi_{e_o}]\}\right) \\ = \;& p_j \textit{ (by induction hypothesis)} \end{aligned}$$

- Case b) There exists $e_1 \in e(o)$ such that $\mathcal{P}^{\mathsf{f}}(\mathcal{G}(T)) \cap \mathcal{P}_0(R\langle o, i, e_1\rangle) \neq \emptyset$: By Lemma 3.29, we have $\mathcal{P}_0(T) \subseteq \mathcal{P}_0(R\langle o, i, e_1\rangle) \cup t_R[\{o\} \times \mathcal{P}^{\mathsf{f}}_R(o)]$, hence $(\forall j \in \mathbb{N}) n_j = 0 = n'_j$; we set $\mathcal{P}' = \mathcal{Q}' \cup \{p; (o, (e_1, p)) \in \mathcal{P}\}$; let $T_0 \in \mathcal{S}^k_{\mathcal{T}_{B_{R_o}(o)}[i](e_1)}(\mathcal{P}')$ such that $T = \langle o, \langle e_1, T_0\rangle\rangle$; we have $\mathcal{P}^{\mathsf{f}}(T_0) \subseteq \mathcal{P}^{\mathsf{f}}(\mathcal{G}(T_0))$ and, by Lemma 3.36, $\beta(R_o) \leq \beta(R) \leq k$, hence we can apply the induction hypothesis: We obtain

$$\begin{aligned} & \sum_{e_o \in e(o)} \mathsf{Card}\left(\{T' \in \mathcal{T}; \mathcal{P}_0(T') \cap \mathcal{P}_0(R\langle o, i, e_o\rangle) \neq \emptyset\}\right) \\ = \;& \mathsf{Card}\left(\{T' \in \mathcal{T}; \mathcal{P}_0(T') \cap \mathcal{P}_0(R\langle o, i, e_1\rangle) \neq \emptyset\}\right) \\ = \;& \mathsf{Card}\left(\{T' \in \mathcal{S}^k_{\mathcal{T}_{R_o}[i](e)}(\mathcal{P}_{e_1}); T' \equiv T[\varphi_{e_1}]\}\right) \\ = \;& \mathsf{Card}\left(\{T' \in \mathcal{S}^k_{\mathcal{T}_{B_{R_o}(o)}[i](e_1)}(\mathcal{P}'); T' \equiv T_0\}\right) \\ = \;& \sum_{j \in \mathcal{M}_i(e_1) \cup \{0\}} p_{e_1,j} \cdot k^j \end{aligned}$$

hence we have $\{j \in \mathbb{N}; p_j \neq 0\} \subseteq \mathcal{M}_i(e_1) \cup \{0\}$.

If $o \in \mathcal{B}_0^{=i}(R)$ then $\mathcal{M}_i(e_1) = \emptyset$, hence $p_{j_o} = 0$; from the other hand, since $\mathcal{P}^{\mathsf{f}}(\mathcal{G}(T)) \cap \mathcal{P}_0(R\langle o, i, e_1\rangle) \neq \emptyset$, we have $\{U \in \mathcal{H}(B_{\mathcal{T}_R[i+1](e)}(!_{e,i}(j_o))); U \equiv_{(\mathcal{T}_R[i+1](e), !_{e,i}(j))} T\} = \{U \in \mathcal{H}(B_R(o)); U \equiv_{(\mathcal{T}_R[i+1](e), !_{e,i}(j))} T\} = \emptyset$.

If $o \in \mathcal{B}_0^{\geq i+1}(R)$, then, by induction hypothesis, we have

$$\begin{aligned} & \sum_{e_o \in e(o)} \mathsf{Card}\left(\{T' \in \mathcal{T}'; \mathcal{P}_0(T') \cap \mathcal{P}_0(R\langle o, i+1, e_o\rangle) \neq \emptyset\}\right) \\ = \;& \mathsf{Card}\left(\{T' \in \mathcal{T}'; \mathcal{P}_0(T') \cap \mathcal{P}_0(R\langle o, i+1, e_1\rangle) \neq \emptyset\}\right) \\ = \;& \mathsf{Card}\left(\{T' \in \mathcal{S}^k_{\mathcal{T}_{R_o}[i+1](e)}(\mathcal{P}_{e_1}); T' \equiv T[\varphi_{e_1}]\}\right) \\ = \;& \mathsf{Card}\left(\{T' \in \mathcal{S}^k_{\mathcal{T}_{B_{R_o}(o)}[i+1](e_1)}(\mathcal{P}'); T' \equiv T_0\}\right) \\ = \;& \sum_{j \in \mathcal{M}_{i+1}(e_1) \cup \{0\}} p'_{e_1,j} \cdot k^j \end{aligned}$$

and $\{j \in \mathbb{N}; p'_{e_1,j} \neq 0\} \subseteq \mathcal{M}_{i+1}(e_1) \cup \{0\}$ and $(\forall j \in \mathcal{M}_{i+1}(e_1) \cup \{0\}) p'_{e_1,j} = p_{e_1,j}$; let $j \in \mathcal{N}$ and let $e_o \in e(o)$ such that $!_{e,i}(j) = (o, (e_o, !_{e_o,i}(j)))$:

– if $e_o \neq e_1$, then $j \notin \mathcal{M}_i(e_1)$, hence $p_j = 0$; from the other hand, we have

$$\{U \in \mathcal{H}(B_{\mathcal{T}_R[i+1](e)}(!_{e,i}(j))); U \equiv_{(\mathcal{T}_R[i+1](e),!_{e,i}(j))} T\} = \emptyset$$

– if $e_o = e_1$, then we have

$$\begin{aligned} & \mathsf{Card}\left(\{U \in \mathcal{H}(B_{\mathcal{T}_R[i+1](e)}(!_{e,i}(j))); U \equiv_{(\mathcal{T}_R[i+1](e),!_{e,i}(j))} T\}\right) \\ = \; & \mathsf{Card}\left(\{U \in \mathcal{H}(B_{\mathcal{T}_{R_o}[i+1](e)}(!_{e,i}(j))); U \equiv_{(\mathcal{T}_{R_o}[i+1](e),!_{e,i}(j))} T[\varphi_{e_1}]\}\right) \\ = \; & p_j \textit{ (by induction hypothesis)} \end{aligned}$$

- Case c) $\mathcal{P}^{\mathsf{f}}(\mathcal{G}(T)) \cap (\mathcal{P}_0(R) \setminus \{t_R(o,p); p \in \mathcal{P}^{\mathsf{f}}(B_R(o))\}) \neq \emptyset$: By Lemma 3.29, for any $e_o \in e(o)$, we have $\{T' \in \mathcal{T}; \mathcal{P}_0(T') \cap \mathcal{P}_0(R\langle o,i,e_o\rangle) \neq \emptyset\} = \emptyset = \{T' \in \mathcal{T}'; \mathcal{P}_0(T') \cap \mathcal{P}_0(R\langle o,i,e_o\rangle) \neq \emptyset\}$, hence $(\forall j \in \mathbb{N}) p_j = 0 = p'_j$. From the other hand, for any $j \in \mathcal{N}_i(e)$, the set $\{U \in \mathcal{H}(B_{\mathcal{T}_R[i+1](e)}(!_{e,i}(j))); U \equiv_{(\mathcal{T}_R[i+1](e),!_{e,i}(j))} T\}$ is empty.

Step 3). We distinguish between two cases:

- Case a) $o \in \mathcal{B}_0^{=i}(R)$: We have $!_{e,i}[\{j \in \mathbb{N}; p_j \neq 0\} \cap \mathcal{M}_{i+1}(e)] \subseteq !_{e,i}[\{j \in \mathbb{N}; p_j \neq 0\}] \cap !_{e,i}[\mathcal{M}_{i+1}(e)] \subseteq (\{o\} \cup \bigcup_{e_o \in e(o)} !_{e,i}[\mathcal{M}_i(e_o)]) \cap !_{e,i}[\mathcal{M}_{i+1}(e)] = (\{o\} \cap !_{e,i}[\mathcal{M}_{i+1}(e)]) \cup \bigcup_{e_o \in e(o)} (!_{e,i}[\mathcal{M}_i(e_o)] \cap !_{e,i}[\mathcal{M}_{i+1}(e)]) \subseteq \emptyset \cup \bigcup_{e_o \in e(o)} !_{e,i}[\mathcal{M}_i(e_o) \cap \mathcal{M}_{i+1}(e)] = \emptyset$, hence, for any $j \in \mathbb{N}$, $m_j = n_j$; by Lemma 3.38 (3) and(4), we have

$$\mathsf{Card}\left(\mathcal{T}'\right) = \mathsf{Card}\left(\{T' \in \mathcal{S}^k_{\mathcal{T}_{R_o}[i+1](e)}(\mathcal{P}^{\mathsf{f}}(\mathcal{G}(T))); T' \equiv T\}\right)$$

  hence, for any $j \in \mathbb{N}$, we have $m'_j = n'_j$; since, for any $j \in \mathcal{M}_{i+1}(e)$, we have $n'_j = n_j$, we obtain $(\forall j \in \mathcal{M}_{i+1}(e)) m'_j = m_j$.
- Case b) $o \in \mathcal{B}_0^{\geq i+1}(R)$: We have $\{j \in \mathbb{N}; m'_j \neq 0\} = \{j \in \mathbb{N}; n'_j \neq 0\} \cup \{j \in \mathbb{N}; p'_j \neq 0\}$, hence $!_{e,i}[\{j \in \mathbb{N} \setminus \{0\}; m'_j \neq 0\}] \subseteq \mathcal{B}' \cup \bigcup_{o \in \mathcal{B}'} \bigcup_{e_o \in e(o)} !_{e,i}[\mathcal{M}_{i+1}(e_o)]$. For any $j \in \{j_o\} \cup \bigcup_{e_o \in e(o)} \mathcal{M}_{i+1}(e_o)$, we have $n'_j = n_j$ and $p'_j = p_j$, hence $m'_j = m_j$; for any $j \in \mathcal{M}_{i+1}(e) \setminus (\{j_o\} \cup \bigcup_{e_o \in e(o)} \mathcal{M}_{i+1}(e_o))$, we have $n'_j = n_j$ and $p'_j = 0 = p_0$, hence $m'_j = m_j$.

Part IV. $\mathit{depth}(R) > 0$, $\mathcal{B} \neq \emptyset$ and $\mathcal{P}^{\mathsf{f}}(\mathcal{G}(T)) \cap \mathcal{B}_0^{\geq i}(R) \neq \emptyset$: Let $o \in \mathcal{P}^{\mathsf{f}}(\mathcal{G}(T)) \cap \mathcal{B}_0^{\geq i}(R)$. There exists $e_o \in e(o)$ such that $(o,(e_o,!_R(o))) \in \mathcal{P}_0(T)$ and $t_T(o,(e_o,!_R(o))) = o$. Notice that o is a co-contraction of T such that $a_{\mathcal{G}(T)}(o) > 0$, which entails that the set $\{T' \in \mathcal{T}; T' \sqsubseteq_{\mathcal{P}} R^{\leq i}\}$ is empty. We distinguish between two cases: Case a), where $\mathcal{P}^{\mathsf{f}}(\mathcal{G}(T)) \subseteq t_R[\{o\} \times \mathcal{P}^{\mathsf{f}}_R(o)]$ and Case b), where $\mathcal{P}^{\mathsf{f}}(\mathcal{G}(T)) \setminus t_R[\{o\} \times \mathcal{P}^{\mathsf{f}}_R(o)] \neq \emptyset$.

- Case a) $\mathcal{P}^{\mathsf{f}}(\mathcal{G}(T)) \subseteq t_R[\{o\} \times \mathcal{P}^{\mathsf{f}}_R(o)]$: Let $j_o \in \mathcal{M}_i(e)$ such that $!_{e,i}(j_o) = o$.
  - We have $\mathsf{Card}\left(\mathcal{T}\right) = k^{j_o}$.
  - We have $\mathsf{Card}\left(\mathcal{T}'\right) = \begin{cases} k^{j_o} & \textit{if } o \in \mathcal{B}_0^{\geq i+1}(R); \\ 0 & \textit{otherwise.} \end{cases}$
  - We have $\{U \in \mathcal{H}(B_{\mathcal{T}_R[i+1](e)}(!_{e,i}(j_o))); U \equiv_{(\mathcal{T}_R[i+1](e),!_{e,i}(j_o))} T\} = H_{B_R(o)}(!_R(o))$, hence, for any $j \in \mathcal{N}_i(e)$,

$$\begin{aligned} & \mathsf{Card}\left(\{U \in \mathcal{H}(B_{\mathcal{T}_R[i+1](e)}(!_{e,i}(j))); U \equiv_{(\mathcal{T}_R[i+1](e),!_{e,i}(j))} T\}\right) \\ = \; & \begin{cases} 1 & \textit{if } j = j_o \in \mathcal{N}_i(e); \\ 0 & \textit{otherwise.} \end{cases} \end{aligned}$$

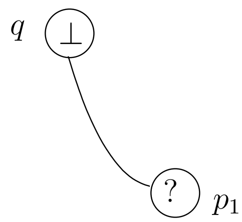


Figure 33: The differential in-PS $T \in \mathcal{S}^{10}_{\mathcal{T}_R[1](e)}(\{p_1\})$

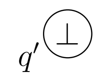


Figure 34: The in-PS $U_0 \in \mathcal{H}(B_{\mathcal{T}_R[2](e)}(!_{e,1}(1)))$

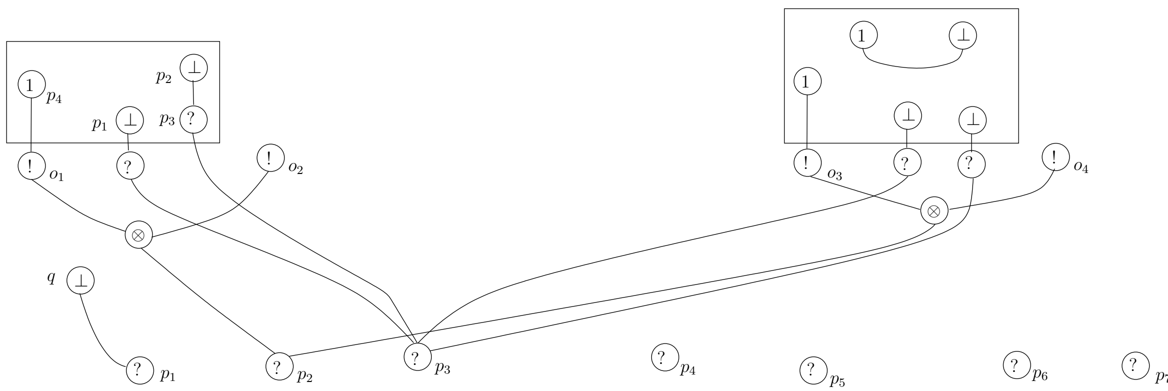


Figure 35: The differential PS $R^{\leq 1}$

- Case b) $\mathcal{P}^{\mathsf{f}}(\mathcal{G}(T)) \setminus t_R[\{o\} \times \mathcal{P}^{\mathsf{f}}_R(o)] \neq \emptyset$: By Lemma 3.29, $\mathcal{P}_0(T) \setminus t_R[\{o\} \times \mathcal{P}^{\mathsf{f}}_R(o)] \subseteq \mathcal{P}_0(R\langle o, i, e_o\rangle)$, hence $\mathcal{P} \subseteq \mathcal{P}_0(R)$ does not hold.
  - For any $T' \in \mathcal{T} \cup \mathcal{T}'$, we have $(o, (e_o, !_R(o))) \in \mathcal{P}_0(T')$, hence $\mathsf{Card}\,(\mathcal{T}) \leq 1$ and $\mathsf{Card}\,(\mathcal{T}') \leq 1$.
  - For any $j \in \mathcal{N}_i(e)$, we have
$$\{U \in \mathcal{H}(B_{\mathcal{T}_R[i+1](e)}(!_{e,i}(j))); U \equiv_{(\mathcal{T}_R[i+1](e), !_{e,i}(j))} T\} = \emptyset \qquad \square$$

Example 3.44. (Continuation of Example 3.41) Let T be the differential in-PS depicted in Figure 33. We have $T \in \mathcal{S}^{10}_{\mathcal{T}_R[1](e)}(\{p_1\})$, $\mathsf{Card}\,(\mathcal{T}) = 11$ and $\mathsf{Card}\,(\mathcal{T}') = 1$, $\mathcal{B} = \{o_2\}$ and $\mathcal{B}' = \emptyset$. We thus have:

- $m_0 = 1 = m_1$ and $m_j = 0$ for any $j \geq 2$
- and $m'_0 = 1$ and $m'_j = 0$ for any $j \geq 1$.

We recall (see Example 3.8) that $\mathcal{M}_1(e) = \{1, 2\} = \mathcal{N}_1(e)$ and $\mathcal{M}_2(e) = \emptyset$. Notice that we have $\{U \in \mathcal{H}(B_{\mathcal{T}_R[2](e)}(!_{e,1}(1))); U \equiv_{(\mathcal{T}_R[i+1](e), !_{e,i}(j))} T\} = \{U_0\}$, where U_0 is the in-PS depicted in Figure 34.

The differential PS $R^{\leq 1}$ is depicted in Figure 35: We have $\{T' \in \mathcal{T}; T' \sqsubseteq_{\{p_1\}} R^{\leq 1}\} = \{T\}$.

3.3. The content of the boxes.

Lemma 3.45. *Let S be a differential in-PS. Let $o \in \mathcal{B}_0(S)$. Let $\mathcal{T}$ be a set of differential in-PS's that is gluable. Assume that there exists a bijection $\gamma : \mathcal{H}(B_S(o)) \simeq \mathcal{T}$ such that, for any $V \in \mathcal{H}(B_S(o))$, we have $V \equiv_{(S,o)} \gamma(V)$. Then $B_S(o) \equiv_{(S,o)} \bigoplus \mathcal{T}$.*

Proof. For each $V \in \mathcal{H}(B_S(o))$, we are given $\varphi_V : V \equiv_{(S,o)} \gamma(V)$. Now, we define $\varphi : \bigoplus \mathcal{H}(B_S(o)) \equiv_{(S,o)} \bigoplus \mathcal{T}$ as follows: for any $p \in \mathcal{P}(\bigoplus \mathcal{H}(B_S(o)))$, we set $\varphi(p) = \varphi_{H_{B_S(o)}(p)}(p)$. But, by Fact 3.32, we have $\bigoplus \mathcal{H}(B_S(o)) = B_S(o)$. □

Fact 3.46. Let R be an in-PS. Let $o \in \mathcal{B}_0^{\geq i}(R)$. Let $e_o \in e(o)$. Let $i \in \mathbb{N}$. Let $o_1 \in \mathcal{B}_0(\mathcal{T}_{B_R(o)}[i](e_o))$. Let $V \in \mathcal{H}(B_{\mathcal{T}_{B_R(o)}[i](e_o)}(o_1))$ and let T be an in-PS such that $V \equiv_{(\mathcal{T}_{B_R(o)}[i](e_o),o_1)} T$. Then $V \equiv_{(\mathcal{T}_R[i](e),(o,(e_o,o_1)))} \langle o, \langle e_o, T\rangle\rangle$.

Proof. Let $\varphi : V \equiv_{(\mathcal{T}_{B_R(o)}[i](e_o),o_1)} T$. Let φ' be the function $\mathcal{P}(V) \to \mathcal{P}(\langle o, \langle e_o, T\rangle\rangle)$ defined by $\varphi'(p) = (o, (e_o, \varphi(p)))$ for any p. For any $p \in \mathcal{P}^{\mathsf{f}}(V)$, we have

$$\begin{aligned} t_{\mathcal{T}_R[i](e)}((o,(e_o,o_1)),p) &= (o,(e_o,t_{\mathcal{T}_{B_R(o)}[i](e_o)}(o_1,p))) \\ &= (o,(e_o,t_T(\varphi(p)))) \\ &= t_{\langle o,\langle e_o,T\rangle\rangle}(\varphi'(p)) \end{aligned}$$

We thus have $\varphi' : V \equiv_{(\mathcal{T}_R[i](e),(o,(e_o,o_1)))} \langle o, \langle e_o, T\rangle\rangle$. □

The rebuilding of the content of the boxes of $\mathcal{T}_R[i+1](e)$ is achieved by the following proposition:

Proposition 3.47. *Let R be a PS. Let $k \geq \beta(R)$. Let e be a k-heterogeneous pseudo-experiment on R. Let $i \in \mathbb{N}$. Let $j_0 \in \mathcal{N}_i(e)$. We set $\mathfrak{T} = \mathcal{S}^k_{\mathcal{T}_R[i](e)}(\mathcal{K}_{k,j_0}(\mathcal{T}_R[i](e)))/\equiv$. For any $\mathcal{T} \in \mathfrak{T}$, let $(m_j^{\mathcal{T}})_{j\in\mathbb{N}} \in \{0,\ldots,k-1\}^{\mathbb{N}}$ such that $\mathsf{Card}\,(\mathcal{T}) = \sum_{j\in\mathbb{N}} m_j^{\mathcal{T}} \cdot k^j$. Let $\mathcal{U} \subseteq \mathcal{S}^k_{\mathcal{T}_R[i](e)}(\mathcal{K}_{k,j_0}(\mathcal{T}_R[i](e)))$ such that, for any $\mathcal{T} \in \mathfrak{T}$, we have $\mathsf{Card}\,(\mathcal{U} \cap \mathcal{T}) = m_{j_0}^{\mathcal{T}}$. Then we have*

$$B_{\mathcal{T}_R[i+1](e)}(!_{e,i}(j_0)) \equiv_{(\mathcal{T}_R[i+1](e),!_{e,i}(j_0))} \bigoplus \mathcal{U}$$

Proof. We first prove, by induction on *depth*(R), that, for any $j \in \mathcal{N}_i(e)$, there exists an injection $\xi : \mathcal{H}(B_{\mathcal{T}_R[i+1](e)}(!_{e,i}(j))) \to \mathcal{S}^k_{\mathcal{T}_R[i](e)}(\mathcal{K}_{k,j}(\mathcal{T}_R[i](e)))$ such that, for any $V \in \mathcal{H}(B_{\mathcal{T}_R[i+1](e)}(!_{e,i}(j)))$, we have $V \equiv_{(\mathcal{T}_R[i+1](e),!_{e,i}(j))} \xi(V)$. $(*)$

Let $j \in \mathcal{N}_i(e)$ and let $V \in \mathcal{H}(B_{\mathcal{T}_R[i+1](e)}(!_{e,i}(j)))$.

If $!_{e,i}(j) \in \mathcal{B}_0^{=i}(R)$, then $V \in \mathcal{H}(B_R(!_{e,i}(j)))$: Let $e_1 \in e(!_{e,i}(j))$. Let $\xi(V)$ be the following differential PS:

$$\xi(V) = (\langle !_{e,i}(j), \langle e_1, V\rangle\rangle \oplus \bigoplus_{p\in\mathcal{P}^{\mathsf{f}}(B_R(!_{e,i}(j)))\cap\mathcal{P}(V)} l_{\mathcal{G}(R)}(t_R(!_{e,i}(j),p))_{t_R(!_{e,i}(j),p)})@t$$

where t is the function $\{!_{e,i}(j)\} \times (\{e_1\} \times (\mathcal{P}^{\mathsf{f}}(B_R(!_{e,i}(j))) \cap \mathcal{P}(V))) \to t_R[\{!_{e,i}(j)\} \times (\mathcal{P}^{\mathsf{f}}(B_R(!_{e,i}(j)))\cap\mathcal{P}(V))]$ that associates with every $(!_{e,i}(j),(e_1,p))$ for some $p \in \mathcal{P}^{\mathsf{f}}(B_R(!_{e,i}(j)))\cap\mathcal{P}(V)$ the port $t_R(!_{e,i}(j),p)$ of $\mathcal{G}(R)$. By Proposition 3.19, we have

$$\begin{aligned} \mathcal{P}^{\mathsf{f}}(\mathcal{G}(\xi(V))) &= t_R[\{!_{e,i}(j)\} \times (\mathcal{P}^{\mathsf{f}}(B_R(!_{e,i}(j))) \cap \mathcal{P}(V))] \\ &\subseteq t_R[\{!_{e,i}(j)\} \times \mathcal{P}^{\mathsf{f}}(B_R(!_{e,i}(j)))] \\ &= t_R[\{!_{e,i}(j)\} \times \mathcal{P}^{\mathsf{f}}_R(!_{e,i}(j))] \\ &= t_{\mathcal{T}_R[i+1](e)}[\{!_{e,i}(j)\} \times \mathcal{P}^{\mathsf{f}}_{\mathcal{T}_R[i+1](e)}(!_{e,i}(j))] \\ &= \mathcal{K}_{k,j}(\mathcal{T}_R[i](e)) \end{aligned}$$

hence $\xi(V) \in \mathcal{S}^k_{\mathcal{T}_R[i](e)}(\mathcal{K}_{k,j}(\mathcal{T}_R[i](e)))$. Moreover, we have $\overline{\xi(V)} = \langle !_{e,i}(j), \langle e_1, V\rangle\rangle$ and $V \equiv_{(\mathcal{T}_R[i+1](e),!_{e,i}(j))} \xi(V)$.

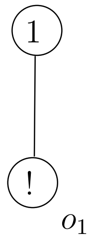


Figure 36: The differential PS V_1

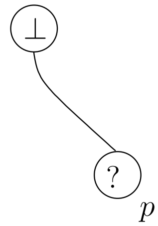


Figure 37: The differential PS V_2

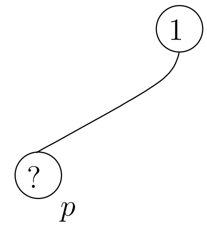


Figure 38: The differential PS V_3

Now, if $!_{e,i}(j) = (o, (e_o, !_{e_o,i}(j)))$ for some $o \in \mathcal{B}_0^{\geq i+1}(R)$ and $e_o \in e(o)$, then we have $V \in \mathcal{H}(B_{\mathcal{T}_{B_R(o)}[i+1](e_o)}(!_{e_o,i}(j)))$: Let φ be some bijection $\mathcal{P}_R^?(o) \simeq \mathcal{Q}'$ and let R_o be an in-PS such that $R_o = \varphi \cdot_o R$; we have $V \in \mathcal{H}(B_{\mathcal{T}_{B_{R_o}(o)}[i+1](e_o)}(!_{e_o,i}(j)))$, hence, by induction hypothesis, there exists $T'_V \in \mathcal{S}^k_{\mathcal{T}_{B_{R_o}(o)}[i](e_o)}(\mathcal{K}_{k,j}(\mathcal{T}_{B_{R_o}(o)}[i](e_o)))$ such that $V \equiv_{(\mathcal{T}_{B_{R_o}(o)}[i+1](e_o), !_{e_o,i}(j))} T'_V$. Let φ_{e_o} be the bijection $\mathcal{P}_R^?(o) \simeq \{o\} \times (\{e_o\} \times \mathcal{Q}')$ defined by $\varphi_{e_o}(p) = (o, (e_o, \varphi(p)))$ for any $p \in \mathcal{P}_R^?(o)$. We have $\langle o, \langle e_o, T'_V \rangle\rangle \in \mathcal{S}^k_{\mathcal{T}_{R_o}[i](e)}((\mathcal{K}_{k,j}(\mathcal{T}_R[i](e)) \setminus \mathcal{P}_R^?(o)) \cup \varphi_{e_o}[\mathcal{K}_{k,j}(\mathcal{T}_R[i](e)) \cap \mathcal{P}_R^?(o)])$, hence, by Lemma 3.38 (2), $\langle o, \langle e_o, T'_V \rangle\rangle[{\varphi_{e_o}}^{-1}] \in \mathcal{S}^k_{\mathcal{T}_R[i](e)}(\mathcal{K}_{k,j}(\mathcal{T}_R[i](e)))$. Moreover, by Fact 3.46, we have $V \equiv_{(\mathcal{T}_{R_o}[i+1](e), !_{e,i}(j))} \langle o, \langle e_o, T'_V \rangle\rangle$, hence $V \equiv_{(\mathcal{T}_R[i+1](e), !_{e,i}(j))} \langle o, \langle e_o, T'_V \rangle\rangle[{\varphi_{e_o}}^{-1}]$. We can thus set $\xi(V) = \langle o, \langle e_o, T'_V \rangle\rangle[{\varphi_{e_o}}^{-1}]$.

We proved $(*)$. Now, let us show that there exists a bijection

$$\gamma : \mathcal{H}(B_{\mathcal{T}_R[i+1](e)}(!_{e,i}(j_0))) \simeq \mathcal{U}$$

such that, for any $V \in \mathcal{H}(B_{\mathcal{T}_R[i+1](e)}(!_{e,i}(j_0)))$, we have $V \equiv_{(\mathcal{T}_R[i+1](e), !_{e,i}(j_0))} \gamma(V)$: For any $T \in \mathsf{im}(\xi)$, by Proposition 3.43, there exists a bijection

$$\gamma_T : \{U \in \mathcal{H}(B_{\mathcal{T}_R[i+1](e)}(!_{e,i}(j_0))); U \equiv_{\mathcal{T}_R[i+1](e), !_{e,i}(j_0)} T\} \simeq \{T' \in \mathcal{U}; T' \equiv T\}$$

We define the function γ by setting $\gamma(V) = \gamma_{\xi(V)}(V)$ for any V. Since ξ is an injection, γ is an injection. Let us check that γ is a surjection too: Let $T \in \mathcal{U}$; by Proposition 3.43, there exists $V \in \mathcal{H}(B_{\mathcal{T}_R[i+1](e)}(!_{e,i}(j_0)))$ such that $V \equiv_{(\mathcal{T}_R[i+1](e), !_{e,i}(j))} T$; by Remark 1.33, we have $T \in \mathsf{im}(\gamma_{\xi(V)})$.

Finally, we can apply Lemma 3.45 to obtain $B_{\mathcal{T}_R[i+1](e)}(!_{e,i}(j_0)) \equiv_{(\mathcal{T}_R[i+1](e), !_{e,i}(j_0))} \bigoplus \mathcal{U}$. □

Example 3.48. Consider the 3-heterogeneous experiment e on the PS's R_1 and R_2 (which are depicted in Figure 23 and in Figure 24 respectively) such that $e^{\#}(o_1) = \{3\}$ and $e^{\#}(o_2) = \{9\}$. We set $\mathfrak{T}_1 = \mathcal{S}^3_{\mathcal{T}_{R_1}[0](e)}(\mathcal{K}_{3,1}(\mathcal{T}_{R_1}[0](e)))/\equiv$ and $\mathfrak{T}_2 = \mathcal{S}^3_{\mathcal{T}_{R_2}[0](e)}(\mathcal{K}_{3,1}(\mathcal{T}_{R_2}[0](e)))/\equiv$. We have $\mathsf{Card}\,(\mathfrak{T}_1) = 3 = \mathsf{Card}\,(\mathfrak{T}_2)$. We have

- $\mathsf{Card}\,(\{V \in \bigcup \mathfrak{T}_1; V \equiv V_1\}) = 3 = \mathsf{Card}\,(\{V \in \bigcup \mathfrak{T}_2; V \equiv V_1\})$,
- $\mathsf{Card}\,(\{V \in \bigcup \mathfrak{T}_1; V \equiv V_2\}) = 3 = \mathsf{Card}\,(\{V \in \bigcup \mathfrak{T}_2; V \equiv V_3\})$
- and $\mathsf{Card}\,(\{V \in \bigcup \mathfrak{T}_1; V \equiv V_3\}) = 9 = \mathsf{Card}\,(\{V \in \bigcup \mathfrak{T}_2; V \equiv V_2\})$,

where V_1, V_2 and V_3 are the differential PS's that are depicted in Figure 36, Figure 37 and Figure 38 respectively.

3.4. **Characterizing a PS by some finite subset of its Taylor expansion.** The proof of the following proposition contains a full description of the algorithm leading from $\mathcal{T}_R[i](e)$ to $\mathcal{T}_R[i+1](e)$:

Proposition 3.49. *Let R and R' be two PS's. Let $k \geq \max\{\beta(R), \beta(R')\}$. Let e be a k-heterogeneous pseudo-experiment on R and let e' be a k-heterogeneous pseudo-experiment on R' such that $\mathcal{T}_R[0](e) \equiv \mathcal{T}_{R'}[0](e')$. Then, for any $i \in \mathbb{N}$, we have $\mathcal{T}_R[i](e) \equiv \mathcal{T}_{R'}[i](e')$.*

Proof. By induction on i. We assume that we are given $\varphi : \mathcal{T}_R[i](e) \equiv \mathcal{T}_{R'}[i](e')$.

We set $\mathcal{M} = \mathcal{M}_i(e)$ (resp. $\mathcal{M}' = \mathcal{M}_i(e')$) and $\mathcal{N} = \mathcal{N}_i(e)$ (resp. $\mathcal{N}' = \mathcal{N}_i(e')$). There is a bijection $! : \mathcal{M} \simeq \mathcal{P}_0^!(\mathcal{T}_R[i](e)) \setminus \mathcal{B}_0(\mathcal{T}_R[i](e))$ (resp. a bijection $!' : \mathcal{M}' \simeq \mathcal{P}_0^!(\mathcal{T}_{R'}[i](e')) \setminus \mathcal{B}_0(\mathcal{T}_{R'}[i](e'))$) such that, for any $j \in \mathcal{M}$, we have $(a_{\mathcal{T}_R[i](e)} \circ !)(j) = k^j$ (resp. $(a_{\mathcal{T}_{R'}[i](e')} \circ !')(j) = k^j$). For any $j \in \mathcal{N}$, we set $\mathcal{T}_j = \mathcal{S}^k_{\mathcal{T}_R[i](e)}(\mathcal{K}_{k,j}(\mathcal{T}_R[i](e)))$ (resp. $\mathcal{T}'_j = \mathcal{S}^k_{\mathcal{T}_{R'}[i](e')}(\mathcal{K}_{k,j}(\mathcal{T}_{R'}[i](e'))))$. We set $\mathfrak{T} = \bigcup_{j \in \mathcal{N}}(\mathcal{T}_j/\equiv)$ (resp. $\mathfrak{T}' = \bigcup_{j \in \mathcal{N}'}(\mathcal{T}'_j/\equiv)$). We set $\mathcal{P} = (\mathcal{P}_0(\mathcal{T}_R[i](e)) \setminus \bigcup_{j \in \mathcal{N}} \bigcup_{T \in \mathcal{T}_j} \mathcal{P}_0(T)) \cup \mathcal{K}_{k,\mathcal{N}}(\mathcal{T}_R[i](e))$ (resp. $\mathcal{P}' = (\mathcal{P}_0(\mathcal{T}_{R'}[i](e')) \setminus \bigcup_{j \in \mathcal{N}'} \bigcup_{T' \in \mathcal{T}'_j} \mathcal{P}_0(T')) \cup \mathcal{K}_{k,\mathcal{N}'}(\mathcal{T}_{R'}[i](e'))$). For any $\mathcal{T} \in \mathfrak{T}$ (resp. $\mathcal{T}' \in \mathfrak{T}'$), we define $(m_j^{\mathcal{T}}) \in \{0, \ldots, k-1\}^{\mathbb{N}}$ (resp. $(n_j^{\mathcal{T}'}) \in \{0, \ldots, k-1\}^{\mathbb{N}}$) as follows: $\mathsf{Card}\,(\mathcal{T}) = \sum_{j \in \mathbb{N}} m_j^{\mathcal{T}} \cdot k^j$ (resp. $\mathsf{Card}\,(\mathcal{T}') = \sum_{j \in \mathbb{N}} n_j^{\mathcal{T}'} \cdot k^j$).

Notice that $\mathcal{M} = \mathcal{M}'$, $\mathcal{N} = \mathcal{N}'$ and, for any $j \in \mathcal{N}$, $\varphi[\mathcal{K}_{k,j}(\mathcal{T}_R[i](e))] = \mathcal{K}_{k,j}(\mathcal{T}_{R'}[i](e'))$. Moreover, there exists a bijection $\sigma : \bigcup_{j \in \mathcal{N}} \mathcal{T}_j \simeq \bigcup_{j \in \mathcal{N}} \mathcal{T}'_j$ such that, for any $T \in \bigcup_{j \in \mathcal{N}} \mathcal{T}_j$, there is an isomorphism $T \simeq \sigma(T)$ associating with every port p of T the port $\varphi(p)$ of $\mathcal{T}_{R'}[i](e')$, hence $\varphi[\mathcal{P}] = \mathcal{P}'$. Also, notice that, for any $T_1, T_2 \in \bigcup_{j \in \mathcal{N}} \mathcal{T}_j$, we have $(T_1 \equiv T_2 \Leftrightarrow \sigma(T_1) \equiv \sigma(T_2))$, which entails that there exists a bijection $\overline{\sigma} : \mathfrak{T} \simeq \mathfrak{T}'$ such that, for any $\mathcal{T} \in \mathfrak{T}$, for any $\mathcal{T}' \in \mathfrak{T}'$, for any $T \in \mathcal{T}$, for any $T' \in \mathcal{T}'$, we have $(\sigma(T) = T' \Rightarrow \overline{\sigma}(\mathcal{T}) = \mathcal{T}')$; moreover, for any $\mathcal{T} \in \mathfrak{T}$, we have $\mathsf{Card}\,(\mathcal{T}) = \mathsf{Card}\,(\overline{\sigma}(\mathcal{T}))$. For any $\mathcal{T} \in \mathfrak{T}$, we thus have $(n_j^{\overline{\sigma}(\mathcal{T})})_{j \in \mathbb{N}} = (m_j^{\mathcal{T}})_{j \in \mathbb{N}}$.

For any $j \in \mathcal{N}$, we set $\mathcal{V}_j = \mathcal{S}^k_{\mathcal{T}_R[i+1](e)}(\mathcal{K}_{k,j}(\mathcal{T}_R[i](e)))$. We set $\mathfrak{V} = \bigcup_{j \in \mathcal{N}}(\mathcal{V}_j/\equiv)$. For any $j \in \mathcal{N}$, we are given $\mathcal{U}_j \subseteq \mathcal{T}_j$ such that, for any $\mathcal{T} \in \mathcal{T}_j/\equiv$, we have $\mathsf{Card}\,(\mathcal{U}_j \cap \mathcal{T}) = m_j^{\mathcal{T}}$. Then one can describe the differential PS $\mathcal{T}_R[i+1](e)$ as follows:

- $\mathcal{T}_R[i](e)^{\leq 0}|_{\mathcal{P}} \sqsubseteq_{\emptyset} \mathcal{T}_R[i+1](e)^{\leq 0}$ (by Proposition 3.40);
- $\mathcal{T}_R[i+1](e)^{\leq i} \sqsubseteq_{\emptyset} \mathcal{T}_R[i](e)$ (by Lemma 3.1);
- $(\forall j \in \mathcal{N})\mathcal{V}_j \subseteq \mathcal{T}_j$ (by Proposition 3.42);
- for any $\mathcal{V} \in \mathfrak{V}$, we have $\mathsf{Card}\,(\mathcal{V}) = \sum_{j \notin \mathcal{N}} m_j^{\mathcal{T}} \cdot k^j$, where $\mathcal{T} \in \mathfrak{T}$ such that $\mathcal{V} \subseteq \mathcal{T}$ (by Proposition 3.43);
- $\mathcal{B}_0(\mathcal{T}_R[i+1](e)) = (\mathcal{B}_0(\mathcal{T}_R[i](e)) \cap \mathcal{P}_0(\mathcal{T}_R[i+1](e))) \cup ![\mathcal{N}]$ (by Lemma 3.1 and Proposition 3.19);
- and, for any $j \in \mathcal{N}$, there exists $\varphi_j : B_{\mathcal{T}_R[i+1](e)}(!(j)) \equiv_{(\mathcal{T}_R[i+1](e), !(j))} \bigoplus \mathcal{U}_j$ (by Proposition 3.47).

In the same way: For any $j \in \mathcal{N}$, we set $\mathcal{V}'_j = \mathcal{S}^k_{\mathcal{T}_{R'}[i+1](e')}(\mathcal{K}_{k,j}(\mathcal{T}_{R'}[i](e')))$. We set $\mathfrak{V}' = \bigcup_{j \in \mathcal{N}}(\mathcal{V}'_j/\equiv)$. For any $j \in \mathcal{N}$, we set $\mathcal{U}'_j = \sigma[\mathcal{U}_j]$. Then one can describe the differential PS $\mathcal{T}_{R'}[i+1](e')$ as follows:

- $\mathcal{T}_{R'}[i](e')^{\leq 0}|_{\mathcal{P}'} \sqsubseteq_{\emptyset} \mathcal{T}_{R'}[i+1](e')^{\leq 0}$ (by Proposition 3.40);
- $\mathcal{T}_{R'}[i+1](e')^{\leq i} \sqsubseteq_{\emptyset} \mathcal{T}_{R'}[i](e')$ (by Lemma 3.1);
- $(\forall j \in \mathcal{N}')\mathcal{V}'_j \subseteq \mathcal{T}'_j$ (by Proposition 3.42);

- for any $\mathcal{V}' \in \mathfrak{V}'$, we have $\mathsf{Card}\,(\mathcal{V}') = \sum_{j \notin \mathcal{N}'} n_j^{\mathcal{T}'} \cdot k^j$, where $\mathcal{T}' \in \mathfrak{T}'$ such that $\mathcal{V}' \subseteq \mathcal{T}'$ (by Proposition 3.43);
- $\mathcal{B}_0(\mathcal{T}_{R'}[i+1](e')) = (\mathcal{B}_0(\mathcal{T}_{R'}[i](e')) \cap \mathcal{P}_0(\mathcal{T}_{R'}[i+1](e'))) \cup !'[\mathcal{N}']$ (by Lemma 3.1 and Proposition 3.19);
- and, for any $j \in \mathcal{N}'$, there exists $\varphi'_j : B_{\mathcal{T}_{R'}[i+1](e')}(!'(j)) \equiv_{(\mathcal{T}_{R'}[i+1](e'), !'(j))} \bigoplus \mathcal{U}'_j$ (by Proposition 3.47).

So, for any $\mathcal{T} \in \mathfrak{T}$, there exists a bijection $\tau_\mathcal{T} : \mathcal{T} \simeq \overline{\sigma}(\mathcal{T})$ such that

$$\tau_\mathcal{T}[\mathcal{T} \cap (\bigcup_{j \in \mathcal{N}} \mathcal{S}^k_{\mathcal{T}_R[i+1](e)}(\mathcal{K}_{k,j}(\mathcal{T}_R[i](e))))] = \overline{\sigma}(\mathcal{T}) \cap (\bigcup_{j \in \mathcal{N}'} \mathcal{S}^k_{\mathcal{T}_{R'}[i+1](e')}(\mathcal{K}_{k,j}(\mathcal{T}_{R'}[i](e'))))$$

hence there exist a bijection $\tau : \bigcup_{j \in \mathcal{N}} \mathcal{T}_j \simeq \bigcup_{j \in \mathcal{N}} \mathcal{T}'_j$ and a sequence $(\psi_T)_{T \in \bigcup_{j \in \mathcal{N}} \mathcal{T}_j}$ such that $(\forall T \in \bigcup_{j \in \mathcal{N}} \mathcal{T}_j) \psi_T : \sigma(T) \equiv \tau(T)$ and

$$\tau[\bigcup_{j \in \mathcal{N}} \mathcal{S}^k_{\mathcal{T}_R[i+1](e)}(\mathcal{K}_{k,j}(\mathcal{T}_R[i](e)))] = \bigcup_{j \in \mathcal{N}'} \mathcal{S}^k_{\mathcal{T}_{R'}[i+1](e')}(\mathcal{K}_{k,j}(\mathcal{T}_{R'}[i](e')))$$

For any $p \in \mathcal{P}_0(\mathcal{T}_{R'}[i](e')) \setminus \mathcal{P}'$, let $H(p)$ be the unique $T' \in \bigcup_{j \in \mathcal{N}} \mathcal{T}'_j$ such that $p \in \mathcal{P}_0(T')$.

We can thus define a bijection $\psi : \mathcal{P}(\mathcal{T}_R[i+1](e)) \simeq \mathcal{P}(\mathcal{T}_{R'}[i+1](e'))$ such that:

- for any $p \in \mathcal{P}$, we have $\psi(p) = \varphi(p) \in \mathcal{P}'$;
- for any $o \in \mathcal{B}_0(\mathcal{T}_R[i+1](e)) \cap \mathcal{P}$, for any $p \in \mathcal{P}(B_{\mathcal{T}_R[i+1](e)}(o))$, we have $\psi(o,p) = \varphi(o,p)$;
- for any $p \in \mathcal{P}_0(\mathcal{T}_R[i+1](e)) \setminus \mathcal{P}$, we have $\psi(p) = \psi_{H(\varphi(p))}(\varphi(p))$;
- for any $o \in \mathcal{B}_0^{<i}(\mathcal{T}_R[i+1](e)) \setminus \mathcal{P}$, for any $p \in \mathcal{P}(B_{\mathcal{T}_R[i+1](e)}(o))$, we have $\psi(o,p) = \psi_{H(\varphi(o))}(\varphi(o,p))$;
- and, for any $j \in \mathcal{N}$, for any $p \in \mathcal{P}(B_{\mathcal{T}_R[i+1](e)}(!(j)))$, we have $\psi(!(j),p) = (\varphi(!(j)), ({\varphi'_j}^{-1} \circ \varphi \circ \varphi_j)(p))$.

It is straightforward to check that $\psi : \mathcal{T}_R[i+1](e) \equiv \mathcal{T}_{R'}[i+1](e')$. In particular:

- for any $p \in \mathcal{W}_0(\mathcal{T}_R[i+1](e)) \setminus \mathcal{P}$ such that $t_{\mathcal{G}(\mathcal{T}_R[i+1](e))}(p) \in \mathcal{P}$, we have $\varphi(t_{\mathcal{G}(\mathcal{T}_R[i](e))}(p)) \in \mathcal{P}' \cap \mathcal{P}_0(H(\varphi(p))) = \mathcal{P}^{\mathsf{f}}(H(\varphi(p)))$, hence

$$\begin{array}{rcl} \psi(t_{\mathcal{G}(\mathcal{T}_R[i+1](e))}(p)) & = & \varphi(t_{\mathcal{G}(\mathcal{T}_R[i+1](e))}(p)) \\ & = & \varphi(t_{\mathcal{G}(\mathcal{T}_R[i](e))}(p)) \\ & = & \psi_{H(\varphi(p))}(\varphi(t_{\mathcal{G}(\mathcal{T}_R[i](e))}(p))) \\ & = & \psi_{H(\varphi(p))}(t_{\mathcal{G}(\mathcal{T}_{R'}[i](e'))}(\varphi(p))) \\ & = & t_{\mathcal{G}(\mathcal{T}_{R'}[i](e'))}(\psi_{H(\varphi(p))}(\varphi(p))) \\ & = & t_{\mathcal{G}(\mathcal{T}_{R'}[i+1](e'))}(\psi_{H(\varphi(p))}(\varphi(p))) \\ & = & t_{\mathcal{G}(\mathcal{T}_{R'}[i+1](e'))}(\psi(p)) \end{array}$$

- we have

$$\begin{array}{rcl} \psi[\mathcal{B}_0(\mathcal{T}_R[i+1](e))] & = & \psi[\mathcal{B}_0(\mathcal{T}_R[i+1](e)) \cap \mathcal{P}] \cup \psi[\mathcal{B}_0(\mathcal{T}_R[i+1](e)) \setminus \mathcal{P}] \\ & = & \varphi[\mathcal{B}_0(\mathcal{T}_R[i+1](e)) \cap \mathcal{P}] \cup \{\psi_{H(\varphi(o))}(\varphi(o)); o \in \mathcal{B}_0(\mathcal{T}_R[i+1](e)) \setminus \mathcal{P}\} \\ & = & (\mathcal{B}_0(\mathcal{T}_{R'}[i+1](e')) \cap \mathcal{P}') \cup (\mathcal{B}_0(\mathcal{T}_{R'}[i+1](e')) \setminus \mathcal{P}') \\ & = & \mathcal{B}_0(\mathcal{T}_{R'}[i+1](e')) \end{array}$$

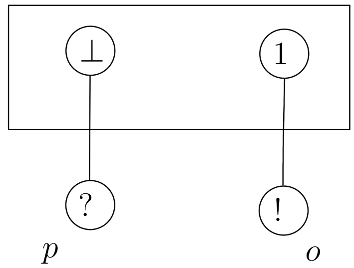


Figure 39: The PS R of the proof of Proposition 3.52

- For any $j \in \mathcal{N}$, we have an isomorphism $\varphi_{!(j)} : \bigoplus \mathcal{U}_j \simeq \bigoplus \mathcal{U}'_j$ that associates with every $p \in \mathcal{P}(\bigoplus \mathcal{U}_j)$ the port $p \in \mathcal{P}(\bigoplus \mathcal{U}'_j)$, hence $(\varphi'_j)^{-1} \circ \varphi_{!(j)} \circ \varphi_j$ is an isomorphism $B_{\mathcal{T}_R[i+1](e)}(!(j)) \simeq B_{\mathcal{T}_{R'}[i+1](e')}(!'(j))$. □

A set of two well-chosen terms of the Taylor expansion are already enough to characterize a PS:

Theorem 3.50. *For any PS R having $\mathcal{T}$ as Taylor expansion, there exists a finite subset $\mathcal{T}_0$ of $\mathcal{T}$ with $\mathsf{Card}\,(\mathcal{T}_0) = 2$ such that, for any PS R' having $\mathcal{T}'$ as Taylor expansion, for any $\mathcal{T}'_0 \subseteq \mathcal{T}'$, we have $(\mathcal{T}_0 \equiv \mathcal{T}'_0 \Rightarrow R \equiv R')$.*

Proof. From $\mathcal{T}_R[0](e_1)$ for e_1 a 1-experiment, one can compute $\beta(R)$. Indeed, one can easily check by induction on *depth*(R) that we have:
- *co-size*(R) = *co-size*$(\mathcal{T}_R[0](e_1))$
- $\mathsf{Card}\,(\mathcal{B}(R)) = \mathsf{Card}\,(\mathcal{B}(\mathcal{T}_R[0](e_1)))$
- $\Theta(R) = \Theta(\mathcal{T}_R[0](e_1))$

One can then take for $\mathcal{T}_0$ the set $\{\mathcal{T}_R[0](e_1), \mathcal{T}_R[0](e)\}$, where e is any k-heterogeneous pseudo-experiment on R with $k \geq \beta(R)$. Indeed:

Let R' be a PS having $\mathcal{T}'$ as Taylor expansion and let $\{T'_1, T'_2\} \subseteq \mathcal{T}'$ such that $T'_1 \equiv \mathcal{T}_R[0](e_1)$ and $T'_2 \equiv \mathcal{T}_R[0](e)$. There exist a pseudo-experiment e'_1 on R' such that $T'_1 = \mathcal{T}_{R'}[0](e'_1)$ and a pseudo-experiment e' on R' such that $T'_2 = \mathcal{T}_{R'}[0](e')$. By Corollary 3.6, the pseudo-experiment e' is a k-heterogeneous experiment on R'. So we can apply Proposition 3.49. □

In particular, we obtain the invertibility of Taylor expansion:

Corollary 3.51. *Let R_1 and R_2 be two PS's having respectively $\mathcal{T}_1$ and $\mathcal{T}_2$ as Taylor expansions. If $\mathcal{T}_1 \equiv \mathcal{T}_2$, then $R_1 \equiv R_2$.*

The finite subset $\mathcal{T}_0$ of Theorem 3.50 has cardinality 2. A natural question is to ask if the theorem could be refined in such a way that one could have a singleton $\mathcal{T}_0$ (for any PS). The answer is: no, it is not possible.

Proposition 3.52. *There exists a PS R having $\mathcal{T}$ as Taylor expansion such that, for any $T \in \mathcal{T}$, there exist a PS R' having $\mathcal{T}'$ as Taylor expansion and $T' \in \mathcal{T}'$ such that $T \equiv T'$ holds but $R \equiv R'$ does not hold.*

Proof. For R, we take the PS of depth 1 depicted in Figure 39. The PS R has as a Taylor expansion the set $\{T_n; n \in \mathbb{N}\}$, where, for any $n \in \mathbb{N}$, the differential PS T_n is as depicted in Figure 40. For any $n \in \mathbb{N}$, let R'_n be the PS as depicted in Figure 41 having $\mathcal{T}'_n$ as Taylor

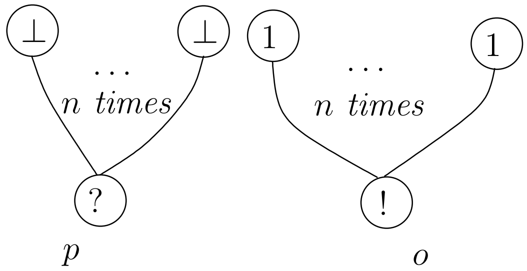


Figure 40: The differential PS's T_n of Proposition 3.52

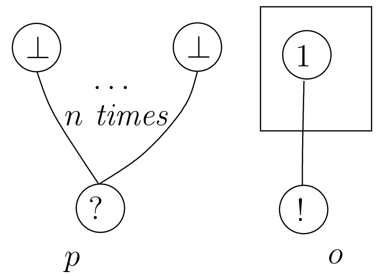


Figure 41: The PS's R'_n of Proposition 3.52

expansion; there exists $T' \in \mathcal{T}'_n$ such that $T' \equiv T_n$. □

4. Relational semantics

4.1. **Untyped framework.** For the semantics of PS's in the multiset based relational semantics, we are given a set $\mathcal{A}$ that does not contain any couple nor any 3-tuple and such that $* \notin \mathcal{A}$. We define, by induction on n, the sets $D_{\mathcal{A},n}$ for any $n \in \mathbb{N}$:

- $D_{\mathcal{A},0} = \{+,-\} \times (\mathcal{A} \cup \{*\})$
- $D_{\mathcal{A},n+1} = D_{\mathcal{A},0} \cup (\{+,-\} \times D_{\mathcal{A},n} \times D_{\mathcal{A},n}) \cup (\{+,-\} \times \mathcal{M}_{\mathit{fin}}(D_{\mathcal{A},n}))$

We set $D_{\mathcal{A}} = \bigcup_{n\in\mathbb{N}} D_{\mathcal{A},n}$.

- For any $\alpha \in D_{\mathcal{A}}$, we denote by $\mathit{height}(\alpha)$ the integer $\min\{n \in \mathbb{N}; \alpha \in D_{\mathcal{A},n}\}$. For any finite set $\mathcal{P}$, for any function $x : \mathcal{P} \to D_{\mathcal{A}}$, we set $\mathit{height}(x) = \max\{\mathit{height}(x(p)); p \in \mathcal{P}\}$.
- For any $\alpha \in D_{\mathcal{A}}$, we define the integer $\mathit{size}(\alpha)$ by induction on $\mathit{height}(\alpha)$: for any $\delta \in \{+,-\}$, we set $\mathit{size}((\delta,\gamma)) = 1$ if $\gamma \in \mathcal{A} \cup \{*\}$, $\mathit{size}((\delta,\alpha_1,\alpha_2)) = 1 + \mathit{size}(\alpha_1) + \mathit{size}(\alpha_2)$ if $\alpha_1, \alpha_2 \in D_{\mathcal{A}}$, and $\mathit{size}((\delta,[\alpha_1,\ldots,\alpha_m])) = 1 + \sum_{j=1}^{m} \mathit{size}(\alpha_j)$. For any finite set $\mathcal{P}$, for any function $x : \mathcal{P} \to D_{\mathcal{A}}$, we set $\mathit{size}(x) = \sum_{p\in\mathcal{P}} \mathit{size}(x(p))$.
- For any $\alpha \in D_{\mathcal{A}}$, we denote by $\mathit{At}(\alpha)$ the set of $\gamma \in \mathcal{A}$ that occur in α. For any finite set $\mathcal{P}$, for any function $x : \mathcal{P} \to D_{\mathcal{A}}$, we set $\mathit{At}(x) = \bigcup_{\alpha\in\mathsf{im}(x)} \mathit{At}(\alpha)$.

Definition 4.1. For any $\alpha \in D_{\mathcal{A}}$, we define $\alpha^\perp \in D_{\mathcal{A}}$ as follows:

- if $\alpha \in \mathcal{A} \cup \{*\}$ and $\delta \in \{+,-\}$, then $(\delta,\alpha)^\perp = (\delta^\perp,\alpha)$;
- if $\alpha = (\delta,\alpha_1,\alpha_2)$ with $\delta \in \{+,-\}$ and $\alpha_1,\alpha_2 \in D_{\mathcal{A}}$, then $\alpha^\perp = (\delta^\perp, {\alpha_1}^\perp, {\alpha_2}^\perp)$;
- if $\alpha = (\delta,[\alpha_1,\ldots,\alpha_m])$ with $\delta \in \{+,-\}$ and $\alpha_1,\ldots,\alpha_m \in D_{\mathcal{A}}$, then $\alpha^\perp = (\delta^\perp,[{\alpha_1}^\perp,\ldots,{\alpha_m}^\perp])$;

where $+^\perp = -$ and $-^\perp = +$.

The following definition is an adaptation of the original definition in [19] to our framework. If e is an experiment on some PS S, then $\mathcal{P}(e)|_{\mathcal{P}^{\mathsf{f}}(S)}$ is its *result*. Then the interpretation of a PS is the set of results of its experiments:

Definition 4.2. Let S be a differential in-PS. We define, by induction on $\mathit{depth}(S)$, the set of *experiments on* S: it is the set of pairs $e = (\mathcal{P}(e),\mathcal{B}(e))$, where

- $\mathcal{P}(e)$ is a function that associates with every $p \in \mathcal{P}_0(S)$ an element of $D_{\mathcal{A}}$ and with every $p \in \mathcal{P}_{>0}(S)$ an element of $\mathcal{M}_{\mathit{fin}}(D_{\mathcal{A}})$,

- and $\mathcal{B}(e)$ is a function which associates with every $o \in \mathcal{B}_0(S)$ a finite multiset of experiments on $B_S(o)$

such that

- for any $p \in \mathcal{P}_0^m(S)$, for any $w_1, w_2 \in \mathcal{W}_0(S)$ such that $t_{\mathcal{G}(S)}(w_1) = p = t_{\mathcal{G}(S)}(w_2)$, $w_1 \in \mathcal{L}(\mathcal{G}(S))$ and $w_2 \notin \mathcal{L}(\mathcal{G}(S))$, we have $\mathcal{P}(e)(p) = (\delta, \mathcal{P}(e)(w_1), \mathcal{P}(e)(w_2))$ with $\delta \in \{+,-\}$ and $(l_{\mathcal{G}(S)}(p) = \otimes \Leftrightarrow \delta = +)$;
- for any $\{p, p'\} \in \mathcal{A}_0(S) \cup \mathcal{C}_0(S)$, we have $\mathcal{P}(e)(p) = \mathcal{P}(e)(p')^\perp$;
- for any $p \in \mathcal{P}_0^e(S)$, we have $\mathcal{P}(e)(p) = (\delta, \sum_{\substack{w \in \mathcal{W}_0(S) \\ t_{\mathcal{G}(S)}(w) = p}} [\mathcal{P}(e)(w)] + \sum_{\substack{q \in \mathcal{P}_{>0}(S) \\ t_S(q) = p}} \mathcal{P}(e)(q))$ and $\delta \in \{+,-\}$ and $(l_{\mathcal{G}(S)}(p) = \,! \Leftrightarrow \delta = +)$;
- for any $p \in \mathcal{P}_0^1(S)$, we have $\mathcal{P}(e)(p) = (+, *)$ and, for any $p \in \mathcal{P}_0^\perp(S)$, we have $\mathcal{P}(e)(p) = (-, *)$;
- for any $o \in \mathcal{B}_0(S)$, for any $p \in \mathcal{P}_0(B_S(o))$, we have $\mathcal{P}(e)(p) = \sum_{e_o \in Supp(\mathcal{B}(e)(o))} \mathcal{B}(e)(o)(e_o) \cdot [\mathcal{P}(e_o)(p)]$;
- for any $o \in \mathcal{B}_0(S)$, for any $p \in \mathcal{P}_{>0}(B_S(o))$, we have $\mathcal{P}(e)(p) = \sum_{e_o \in Supp(\mathcal{B}(e)(o))} \mathcal{B}(e)(o)(e_o) \cdot \mathcal{P}(e_o)(p)$.

If S is a PS, then we set $[\![S]\!] = \{\mathcal{P}(e)|_{\mathcal{P}^{\mathsf{f}}(S)};\ e \textit{ is an experiment on } S\}$.

Every experiment induces a pseudo-experiment:

Definition 4.3. Let S be an in-PS. Let e be an experiment on S. Then we define, by induction on *depth*(S), a pseudo-experiment $\overline{e}$ on S as follows:

- $\overline{e}(\varepsilon) = 0$
- and, for any $o \in \mathcal{B}_0(S)$, $\overline{e}(o) = \bigcup_{e_o \in Supp(e(o))} \{\overline{e_o}[\varepsilon \mapsto i]; 1 \leq i \leq \mathcal{B}(e)(e_o)\}$.

Definition 4.4. Let $r \in \mathcal{M}_{fin}(D_\mathcal{A})$. We say that r is $\mathcal{A}$-*injective* if, for any $(\delta, \gamma) \in \{+,-\} \times \mathcal{A}$, there is at most one occurrence of (δ, γ) in r.

For any set $\mathcal{P}$, for any function $x : \mathcal{P} \to D_\mathcal{A}$, we say that x is $\mathcal{A}$-*injective* if $\sum_{p \in \mathcal{P}} [x(p)]$ is $\mathcal{A}$-injective; moreover a subset D_0 of $(D_\mathcal{A})^\mathcal{P}$ is said to be $\mathcal{A}$-injective if, for any $x \in D_0$, the function x is $\mathcal{A}$-injective.

An experiment e on a differential PS S is said to be *injective* if $\mathcal{P}(e)|_{\mathcal{P}^{\mathsf{f}}(R)}$ is $\mathcal{A}$-injective.

Definition 4.5. Let $\sigma : \mathcal{A} \to D_\mathcal{A}$. For any $\alpha \in D_\mathcal{A}$, we define $\sigma \cdot \alpha \in D_\mathcal{A}$ as follows:

- if $\alpha \in \mathcal{A}$, then $\sigma \cdot (+, \alpha) = \sigma(\alpha)$ and $\sigma \cdot (-, \alpha) = \sigma(\alpha)^\perp$;
- if $\alpha = (\delta, *)$ for some $\delta \in \{+,-\}$, then $\sigma \cdot \alpha = \alpha$;
- if $\delta \in \{+,-\}$ and $\alpha_1, \alpha_2 \in D_\mathcal{A}$, then $\sigma \cdot (\delta, \alpha_1, \alpha_2) = (\delta, \sigma \cdot \alpha_1, \sigma \cdot \alpha_2)$;
- if $\delta \in \{+,-\}$ and $\alpha_1, \ldots, \alpha_m \in D_\mathcal{A}$, then $\sigma \cdot (\delta, [\alpha_1, \ldots, \alpha_m]) = (\delta, [\sigma \cdot \alpha_1, \ldots, \sigma \cdot \alpha_m])$.

For any set $\mathcal{P}$, for any function $x : \mathcal{P} \to D_\mathcal{A}$, we define a function $\sigma \cdot x : \mathcal{P} \to D_\mathcal{A}$ by setting: $(\sigma \cdot x)(p) = \sigma \cdot x(p)$ for any $p \in \mathcal{P}$.

Fact 4.6. For any $\sigma : A \to D_\mathcal{A}$, for any $\alpha \in D_\mathcal{A}$, we have $\sigma \cdot \alpha^\perp = (\sigma \cdot \alpha)^\perp$.

Proof. By induction on α. □

Lemma 4.7. *For any functions $\sigma, \sigma' : \mathcal{A} \to D_\mathcal{A}$, for any $\alpha \in D_\mathcal{A}$, we have $\sigma \cdot (\sigma' \cdot \alpha) = (\sigma \cdot \sigma') \cdot \alpha$.*

Proof. By induction on α, applying Fact 4.6. □

Definition 4.8. Let S be a differential in-PS. Let e be an experiment on S. Let $\sigma : \mathcal{A} \to D_{\mathcal{A}}$. We define, by induction of *depth*(S), an experiment $\sigma \cdot e$ on S by setting

- $\mathcal{P}(\sigma \cdot e) = \sigma \cdot \mathcal{P}(e)$
- $\mathcal{B}(\sigma \cdot e)(o) = \sum_{e_o \in Supp(\mathcal{B}(e)(o))} \mathcal{B}(e)(o)(e_o) \cdot [\sigma \cdot e_o]$ for any $o \in \mathcal{B}_0(S)$.

Since we deal with untyped proof-nets, we cannot assume that the proof-nets are η-expanded and that experiments label the axioms only by atoms. That is why we introduce the notion of *atomic experiment*:

Definition 4.9. For any differential in-PS S, we define, by induction on *depth*(S), the set of *atomic experiments on* S: it is the set of experiments e on S such that

- for any $\{p, q\} \in \mathcal{A}(\mathcal{G}(S))$, we have $\mathcal{P}(e)(p), \mathcal{P}(e)(q) \in \{+, -\} \times \mathcal{A}$;
- and, for any $o \in \mathcal{B}_0(S)$, the multiset $\mathcal{B}(e)(o)$ is a multiset of atomic experiments on $B_S(o)$.

Definition 4.10. Let $\mathcal{P}$ be a set.

Let $x \in (D_{\mathcal{A}})^{\mathcal{P}}$. A *renaming of* x is a function $\sigma : \mathcal{A} \to D_{\mathcal{A}}$ such that $(\forall \gamma \in At(x))\sigma(\gamma) \in \{+, -\} \times \mathcal{A}$.

Let $\mathcal{D} \subseteq (D_{\mathcal{A}})^{\mathcal{P}}$. Let $x \in \mathcal{D}$, we say that x is $\mathcal{D}$-*atomic* if we have

$$(\forall \sigma \in (D_{\mathcal{A}})^{\mathcal{A}})(\forall y \in \mathcal{D})(\sigma \cdot y = x \Rightarrow \sigma \in \mathfrak{R}(y))$$

We denote by $\mathcal{D}_{At}$ the subset of $\mathcal{D}$ consisting of the $\mathcal{D}$-atomic elements of $\mathcal{D}$.

Fact 4.11. Let $x \in D_{\mathcal{A}}$. Let σ and τ be two applications $\mathcal{A} \to D_{\mathcal{A}}$. Then we have $(\sigma \cdot \tau \in \mathfrak{R}(x) \Rightarrow (\tau \in \mathfrak{R}(x) \wedge \sigma \in \mathfrak{R}(\tau \cdot x)))$.

Proof. Let us assume that $\sigma \cdot \tau \in \mathfrak{R}(x)$. By Lemma 4.7, we have $(\sigma \cdot \tau) \cdot x = \sigma \cdot (\tau \cdot x)$. So, we have $size(\tau \cdot x) \leq size(\sigma \cdot (\tau \cdot x)) = size((\sigma \cdot \tau) \cdot x) = size(x) \leq size(\tau \cdot x) \leq size(\sigma \cdot (\tau \cdot x))$, hence $size(\tau \cdot x) = size(x)$ and $size(\sigma \cdot (\tau \cdot x)) = size(\tau \cdot x)$, which entails that $\tau \in \mathfrak{R}(x)$ and $\sigma \in \mathfrak{R}(\tau \cdot x)$. □

Fact 4.12. Let S and S' be two cut-free differential PS's of depth 0. Let e and e' be two atomic experiments on S and S' respectively such that $\mathcal{P}(e)|_{\mathcal{P}^{\mathsf{f}}(S)} = \mathcal{P}(e)|_{\mathcal{P}^{\mathsf{f}}(S')}$ is $\mathcal{A}$-injective. Then $S \equiv S'$.

Proof. By induction on $\mathsf{Card}\,(\mathcal{P}_0(S))$. □

For any PS R, any $[\![R]\!]$-atomic injective point is the result of some atomic experiment on R:

Fact 4.13. Let R be a cut-free in-PS. Let $x \in [\![R]\!]_{At}$. We assume that the set $\mathcal{A}$ is infinite or x is $\mathcal{A}$-injective. Then there exists an atomic experiment e on R such that $e_{|\mathcal{P}^{\mathsf{f}}(R)} = x$.

Proof. We prove, by induction on $(depth(R), \mathsf{Card}\,(\mathcal{P}_0(R)))$ lexicographically ordered, that, for any non-atomic experiment e' on R, there exist an experiment e on R, a function $\sigma : \mathcal{A} \to D_{\mathcal{A}}$ such that $\sigma \cdot e = e'$ and $\gamma \in At(e_{|\mathcal{P}^{\mathsf{f}}(R)})$ such that $\sigma(\gamma) \notin \{+, -\} \times \mathcal{A}$. □

The converse does not necessarily hold (but see Lemma 4.29 about *typable* cut-free PS's): for some cut-free PS's R, there are results of atomic injective experiments on R that are not $[\![R]\!]$-atomic. Indeed, consider Figure 42. There exists an atomic injective experiment e on R'' such that

- $\mathcal{P}(e)(p_1) = (-, [(+, \gamma_1), \ldots, (+, \gamma_7), (+, (+, \gamma_8), (+, \gamma_9)), \ldots, (+, (+, \gamma_{22}), (+, \gamma_{23}))])$,
- $\mathcal{P}(e)(p_2) = (-, [(-, \gamma_1), \ldots, (-, \gamma_7), (-, (-, \gamma_8), (-, \gamma_9)), \ldots, (-, (-, \gamma_{22}), (-, \gamma_{23}))])$

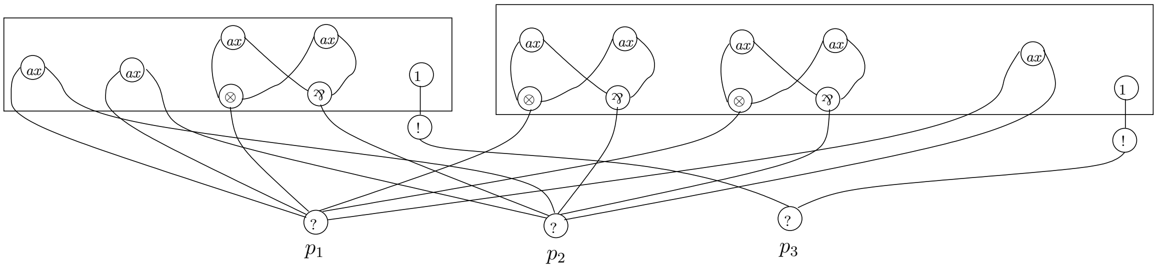


Figure 42: PS R''

- and $\mathcal{P}(e)(p_3) = (-, [(+, [(+, *), (+, *)]), (+, [(+, *), (+, *), (+, *)])])$,

where $\{\gamma_1, \ldots, \gamma_{23}\} \subseteq \mathcal{A}$. But $e_{|\{p_1,p_2,p_3\}}$ is not in $(\llbracket R'' \rrbracket)_{At}$: there exists an atomic injective experiment e' on R' such that

- $\mathcal{P}(e')(p_1) = (-, [(+, \gamma_1), \ldots, (+, \gamma_8), (+, (+, \gamma_{10}), (+, \gamma_{11})), \ldots, (+, (+, \gamma_{22}), (+, \gamma_{23}))])$,
- $\mathcal{P}(e')(p_2) = (-, [(-, \gamma_1), \ldots, (-, \gamma_8), (-, (-, \gamma_{10}), (-, \gamma_{11})), \ldots, (-, (-, \gamma_{22}), (-, \gamma_{23}))])$
- and $\mathcal{P}(e')(p_3) = (-, [(+, [(+, *), (+, *)]), (+, [(+, *), (+, *), (+, *)])])$;

we set $\sigma(\gamma) = \begin{cases} (+, \gamma) & \textit{if } \gamma \in \mathcal{A} \setminus \{\gamma_8\}; \\ (+, (+, \gamma_8), (+, \gamma_9)) & \textit{if } \gamma = \gamma_8; \end{cases}$ - we have $\sigma \cdot e'_{|\{p_1,p_2,p_3\}} = e_{|\{p_1,p_2,p_3\}}$.

But it does not matter, because yet atomic points are enough to generate all the points:

Fact 4.14. Let R be an in-PS. For any $y \in \llbracket R \rrbracket$, there exist $x \in \llbracket R \rrbracket_{At}$ and $\sigma : \mathcal{A} \to D_{\mathcal{A}}$ such that $\sigma \cdot x = y$.

Proof. By induction on $size(\sum_{p \in \mathcal{P}^{\mathrm{f}}(R)} [y(p)])$: if $y \in \llbracket R \rrbracket_{At}$, then we can set $x = y$ and $\sigma = id_{\mathcal{A}}$; if $y \notin \llbracket R \rrbracket_{At}$, then there exist a function $\sigma' : \mathcal{A} \to D_{\mathcal{A}}$, $y' \in \llbracket R \rrbracket$ such that $\sigma' \cdot y' = y$ and $\gamma \in At(y')$ such that $\sigma'(\gamma) \notin \mathcal{A}$, hence $size(\sum_{p \in \mathcal{P}^{\mathrm{f}}(R)} [y'(p)]) < size(\sum_{p \in \mathcal{P}^{\mathrm{f}}(R)} [y(p)])$. By induction hypothesis, there exist $x \in \llbracket R \rrbracket_{At}$ and $\sigma'' : \mathcal{A} \to D_{\mathcal{A}}$ such that $\sigma'' \cdot x = y'$. We set $\sigma = \sigma' \cdot \sigma''$: we have $\sigma \cdot x = (\sigma' \cdot \sigma'') \cdot x = \sigma' \cdot (\sigma'' \cdot x) = \sigma' \cdot y' = y$. □

Lemma 4.15. *Let R be an in-PS. Let e be an (resp. atomic) experiment on R. Then there exists an (resp. atomic) experiment $\mathcal{T}_R(e)$ on $\mathcal{T}_R[0](\overline{e})$ such that $\mathcal{P}(\mathcal{T}_R(e))_{|\mathcal{P}^{\mathrm{f}}(\mathcal{T}_R[0](\overline{e}))} = \mathcal{P}(e)_{|\mathcal{P}^{\mathrm{f}}(R)}$.*

Proof. We prove, by induction on $depth(R)$, that, for any in-PS R, for any experiment e on R, there exists an experiment $\mathcal{T}_R(e)$ on $\mathcal{T}_R[0](\overline{e})$ such that

- for any $p \in \mathcal{P}_0(R)$, we have $\mathcal{P}(e)(p) = \mathcal{P}(\mathcal{T}_R(e))(p)$;
- and, for any $p \in \mathcal{P}_{>0}(R)$, we have $\mathcal{P}(e)(p) = \sum_{\substack{q \in \mathcal{P}(\mathcal{T}_R[0](\overline{e})) \\ \kappa_R[0](e)(q) = p}} [\mathcal{P}(\mathcal{T}_R(e))(q)]$.

One can take for $\mathcal{T}_R(e)$ the following experiment on $\mathcal{T}_R[0](\overline{e})$:

- for any $p \in \mathcal{P}_0(R)$, $\mathcal{P}(\mathcal{T}_R(e))(p) = \mathcal{P}(e)(p)$;
- for any $o \in \mathcal{B}_0(R)$, for any $e_o \in Supp(\mathcal{B}(e)(o))$, for any $p \in \mathcal{P}_0(\mathcal{T}_{B_R(o)}[0](\overline{e_o}))$,

$$\mathcal{P}(\mathcal{T}_R(e))((o, (e_o, p)) = \mathcal{P}(\mathcal{T}_{B_R(o)}(e_o))(p).$$ □

Definition 4.16. Let $k > 1$. An experiment e on some in-PS R is said to be *k-heterogeneous* if the pseudo-experiment $\overline{e}$ is k-heterogeneous.

Definition 4.17. Let $k > 1$. For any function $x : \mathcal{P} \to D_{\mathcal{A}}$, where $\mathcal{P}$ is any set, we say that x is *k-heterogeneous* if the following properties hold:

- for any multiset a, the pair $(+, a)$ occurs at most once in $\mathsf{im}(x)$ and, if it occurs, then there exists $j > 0$ such that the cardinality of a is k^j;
- for any multisets a_1 and a_2 having the same cardinality such that the pairs $(+, a_1)$ and $(+, a_2)$ occur in $\mathsf{im}(x)$, we have $a_1 = a_2$.

Lemma 4.18. *Let R be a cut-free in-PS. Let $k > 1$. For any experiment e on R, if $\mathcal{P}(e)|_{\mathcal{P}^{\mathsf{f}}(R)}$ is k-heterogeneous, then e is k-heterogeneous. Conversely, if e is an atomic k-heterogeneous experiment on R, then $\mathcal{P}(e)|_{\mathcal{P}^{\mathsf{f}}(R)}$ is k-heterogeneous.*

Proof. Let e be an experiment on R such that $\mathcal{P}(e)|_{\mathcal{P}^{\mathsf{f}}(R)}$ is k-heterogeneous. By Lemma 4.15, there exists an experiment $\mathcal{T}_R(e)$ on $\mathcal{T}_R[0](\overline{e})$ such that $\mathcal{P}(\mathcal{T}_R(e))|_{\mathcal{P}^{\mathsf{f}}(\mathcal{T}_R[0](\overline{e}))} = \mathcal{P}(e)|_{\mathcal{P}^{\mathsf{f}}(R)}$. Since $\mathcal{T}_R[0](\overline{e})$ is cut-free, we have:

(1) $a_{\mathcal{T}_R[0](\overline{e})}[\mathcal{P}_0^!(\mathcal{T}_R[0](\overline{e}))] \subseteq \{k^j; j > 0\}$

(2) $(\forall p_1, p_2 \in \mathcal{P}_0^!(\mathcal{T}_R[0](\overline{e})))(a_{\mathcal{T}_R[0](\overline{e})}(p_1) = a_{\mathcal{T}_R[0](\overline{e})}(p_2) \Rightarrow p_1 = p_2)$

By Corollary 3.6, the pseudo-experiment $\overline{e}$ is k-heterogeneous.

Conversely, let e be an atomic k-heterogeneous experiment on R. Then, by Lemma 4.15, there exists an atomic experiment $\mathcal{T}_R(e)$ of $\mathcal{T}_R[0](\overline{e})$ such that $\mathcal{P}(\mathcal{T}_R(e))|_{\mathcal{P}^{\mathsf{f}}(\mathcal{T}_R[0](\overline{e}))} = \mathcal{P}(e)|_{\mathcal{P}^{\mathsf{f}}(R)}$. By Corollary 3.6, we have:

(1) $a_{\mathcal{T}_R[0](\overline{e})}[\mathcal{P}_0^!(\mathcal{T}_R[0](\overline{e}))] \subseteq \{k^j; j > 0\}$

(2) $(\forall p_1, p_2 \in \mathcal{P}_0^!(\mathcal{T}_R[0](\overline{e})))(a_{\mathcal{T}_R[0](\overline{e})}(p_1) = a_{\mathcal{T}_R[0](\overline{e})}(p_2) \Rightarrow p_1 = p_2)$

Since $\mathcal{T}_R(e)$ is atomic, $\mathcal{P}(\mathcal{T}_R(e))|_{\mathcal{P}^{\mathsf{f}}(\mathcal{T}_R[0](\overline{e}))}$ is k-heterogeneous. □

Lemma 4.19. *Let R and R' be two cut-free PS's such that $\mathcal{P}^{\mathsf{f}}(R) = \mathcal{P}^{\mathsf{f}}(R')$. Let e be an injective atomic experiment on R and let e' be an injective atomic experiment on R' such that $\mathcal{P}(e)|_{\mathcal{P}^{\mathsf{f}}(R)} = \mathcal{P}(e')|_{\mathcal{P}^{\mathsf{f}}(R')}$. Then $\mathcal{T}_R[0](\overline{e}) \equiv \mathcal{T}_{R'}[0](\overline{e'})$.*

Proof. By Lemma 4.15, there exists an atomic experiment $\mathcal{T}_R(e)$ on $\mathcal{T}_R[0](\overline{e})$ such that $\mathcal{P}(\mathcal{T}_R(e))|_{\mathcal{P}^{\mathsf{f}}(\mathcal{T}_R[0](\overline{e}))} = \mathcal{P}(e)|_{\mathcal{P}^{\mathsf{f}}(R)}$ and an atomic experiment $\mathcal{T}_{R'}(e')$ on $\mathcal{T}_{R'}[0](\overline{e'})$ such that $\mathcal{P}(\mathcal{T}_{R'}(e'))|_{\mathcal{P}^{\mathsf{f}}(\mathcal{T}_{R'}[0](\overline{e'}))} = \mathcal{P}(e')|_{\mathcal{P}^{\mathsf{f}}(R')}$. We apply Fact 4.12. □

Theorem 4.20. *Let R be a cut-free PS. If the set $\mathcal{A}$ is infinite, then there exists an $\mathcal{A}$-injective subset D_0 of $[\![R]\!]_{At}$ with $\mathsf{Card}\,(D_0) = 2$ such that, for any cut-free PS R' with $\mathcal{P}^{\mathsf{f}}(R) = \mathcal{P}^{\mathsf{f}}(R')$, one has $(D_0 \subseteq [\![R']\!]_{At} \Rightarrow R \equiv R')$.*

Proof. Let f be an injective 1-experiment on R (its existence is ensured by the assumption that the set $\mathcal{A}$ is infinite). By Fact 4.14, there exists $x_1 \in [\![R]\!]_{At}$ and $\sigma_1 : \mathcal{A} \to D_{\mathcal{A}}$ such that $\sigma_1 \cdot x_1 = f|_{\mathcal{P}^{\mathsf{f}}(R)}$; the point x_1 is $\mathcal{A}$-injective too. Let $k_1 \in \mathbb{N}$ be the greatest cardinality of the negative multisets that occur in $\mathsf{im}(x_1)$. Let $k_2 \in \mathbb{N}$ be the number of occurrences of positive multisets in $\mathsf{im}(x_1)$. Let $k > \max\{k_1, k_2\}$ and let e be an injective k-heterogeneous experiment on R (its existence is ensured by the assumption that the set $\mathcal{A}$ is infinite). By Fact 4.14, there exist $x \in [\![R]\!]_{At}$ and $\sigma : \mathcal{A} \to D_{\mathcal{A}}$ such that $\sigma \cdot x = e_{|\mathcal{P}^{\mathsf{f}}(R)}$. We can take $D_0 = \{x_1, x\}$. Indeed:

Since the point $e_{|\mathcal{P}^{\mathsf{f}}(R)}$ is $\mathcal{A}$-injective, the point x is $\mathcal{A}$-injective too. By Lemma 4.18, $e_{|\mathcal{P}^{\mathsf{f}}(R)}$ is k-heterogeneous, hence x too. Let R' be a cut-free PS such that $\mathcal{P}^{\mathsf{f}}(R) = \mathcal{P}^{\mathsf{f}}(R')$ and $D_0 \subseteq [\![R']\!]_{At}$. By Fact 4.13, there exist an atomic experiment f_0 (resp. f_0') on R (resp. of R') such that $\mathcal{P}(f_0)|_{\mathcal{P}^{\mathsf{f}}(R)} = x_1$ (resp. $\mathcal{P}(f_0')|_{\mathcal{P}^{\mathsf{f}}(R')} = x_1$) and an atomic experiment e_0 (resp.

e_0') on R (resp. of R') such that $\mathcal{P}(e_0)|_{\mathcal{P}^{\mathsf{f}}(R)} = x$ (resp. $\mathcal{P}(e_0')|_{\mathcal{P}^{\mathsf{f}}(R')} = x$). By Lemma 4.19, we have $\mathcal{T}_R[0](\overline{e_0}) \equiv \mathcal{T}_{R'}[0](\overline{e_0'})$.

Moreover, the experiments f_0 and f_0' are 1-experiments on R and R' respectively (all the positive multisets of x_1 have cardinality 1). We have:

- $\textit{co-size}(R) = k_1 = \textit{co-size}(R')$;
- $\mathsf{Card}\,(\mathcal{B}(R)) = k_2 = \mathsf{Card}\,(\mathcal{B}(R'))$;
- and $\Theta(R) = 0 = \Theta(R)$ (because R and R' are cut-free).

We thus have $k \geq \max\{\beta(R), \beta(R')\}$. Finally, by Lemma 4.18, the experiments e_0 and e_0' are k-heterogeneous. We can then apply Proposition 3.49 to obtain $R \equiv R'$. □

Corollary 4.21. *Let R and R' be two cut-free PS's such that $\mathcal{P}^{\mathsf{f}}(R) = \mathcal{P}^{\mathsf{f}}(R')$. If the set $\mathcal{A}$ is infinite, then one has ($R \equiv R' \Leftrightarrow [\![R]\!]_{At} = [\![R']\!]_{At}$).*

4.2. **Typed framework.** We want to apply Theorem 4.20 to obtain a similar result for typed PS's (Theorem 4.36). For that, we need to relate $[\![(R, \mathsf{T})]\!]$ to $[\![R]\!]_{At}$; it is the role of Lemma 4.30.

Definition 4.22. We assume that we are given a set $[\![X]\!]$ for each $X \in \mathcal{X}$. Then, for any $C \in \mathbb{T}$, we define, by induction on C, the set $[\![C]\!]$ as follows: $[\![1]\!] = \{*\} = [\![\bot]\!]$; $[\![(C_1 \otimes C_2)]\!] = [\![C_1]\!] \times [\![C_2]\!] = [\![(C_1 \parr C_2)]\!]$; $[\![!C]\!] = \mathcal{M}_{\mathit{fin}}([\![C]\!]) = [\![?C]\!]$.

Let (R, T) be a typed differential in-PS. We define, by induction on $\mathit{depth}(R)$, the set of *experiments on* (R, T): it is the set of pairs $e = (\mathcal{P}(e), \mathcal{B}(e))$, where

- $\mathcal{P}(e)$ is a function that associates with every $p \in \mathcal{P}_0(R)$ an element of $[\![\mathsf{T}(p)]\!]$ and with every $p \in \mathcal{P}_{>0}(R)$ an element of $\mathcal{M}_{\mathit{fin}}([\![\mathsf{T}(p)]\!])$,
- and $\mathcal{B}(e)$ is a function which associates with every $o \in \mathcal{B}_0(R)$ a finite multiset of experiments on $(B_R(o), \mathsf{T}|_{\mathcal{P}(B_R(o))})$

such that

- for any $p \in \mathcal{P}_0^m(R)$, for any $w_1, w_2 \in \mathcal{W}_0(R)$ such that $t_{\mathcal{G}(R)}(w_1) = p = t_{\mathcal{G}(R)}(w_2)$, $w_1 \in \mathcal{L}(\mathcal{G}(R))$ and $w_2 \notin \mathcal{L}(\mathcal{G}(R))$, we have $\mathcal{P}(e)(p) = (\mathcal{P}(e)(w_1), \mathcal{P}(e)(w_2))$;
- for any $\{p, p'\} \in \mathcal{A}_0(R) \cup \mathcal{C}_0(R)$, we have $\mathcal{P}(e)(p) = \mathcal{P}(e)(p')$;
- for any $p \in \mathcal{P}_0^e(R)$, we have $\mathcal{P}(e)(p) = \sum_{\substack{w \in \mathcal{W}_0(R) \\ t_{\mathcal{G}(R)}(w) = p}} [\mathcal{P}(e)(w)] + \sum_{o \in \mathcal{B}_0(R)} \sum_{e_o \in \mathit{Supp}(\mathcal{B}(e)(o))} \sum_{\substack{q \in \mathcal{P}^{\mathsf{f}}(B_R(o)) \\ t_R(o,q) = p}} \mathcal{B}(e)(o)(e_o) \cdot \mathcal{P}(e_o)(q)$;
- for any $o \in \mathcal{B}_0(R)$, for any $p \in \mathcal{P}_0(B_R(o))$, we have $\mathcal{P}(e)(p) = \sum_{e_o \in \mathit{Supp}(\mathcal{B}(e)(o))} \mathcal{B}(e)(o)(e_o) \cdot [\mathcal{P}(e_o)(p)]$;
- for any $o \in \mathcal{B}_0(R)$, for any $p \in \mathcal{P}_{>0}(B_R(o))$, we have $\mathcal{P}(e)(p) = \sum_{e_o \in \mathit{Supp}(\mathcal{B}(e)(o))} \mathcal{B}(e)(o)(e_o) \cdot \mathcal{P}(e_o)(p)$.

For any experiment $e = ((R, \mathsf{T}), \mathcal{P}(e), \mathcal{B}(e))$, we set $\mathcal{P}(e) = \mathcal{P}(e)$ and $\mathcal{B}(e) = \mathcal{B}(e)$. We set $[\![(R, \mathsf{T})]\!] = \{\mathcal{P}(e)|_{\mathcal{P}^{\mathsf{f}}(R)};\ e \textit{ is an experiment on } (R, \mathsf{T})\}$.

From now on, we assume that, for any $X \in \mathcal{X}$, the set $[\![X]\!]$ does not contain any couple nor any 3-tuple and $* \notin \mathcal{A}$ and we assume that $\mathcal{A} = \bigcup_{X \in \mathcal{X}} [\![X]\!]$. We define, by induction on n, the sets $\overline{D}_{\mathcal{A},n}$ for any $n \in \mathbb{N}$:

- $\overline{D}_{\mathcal{A},0} = \mathcal{A} \cup \{*\}$
- $\overline{D}_{\mathcal{A},n+1} = \overline{D}_{\mathcal{A},0} \cup (\overline{D}_{\mathcal{A},n} \times \overline{D}_{\mathcal{A},n}) \cup \mathcal{M}_{\mathit{fin}}(\overline{D}_{\mathcal{A},n})$

We set $\overline{D}_{\mathcal{A}} = \bigcup_{n\in\mathbb{N}} \overline{D}_{\mathcal{A},n}$. We define the function $U : D_{\mathcal{A}} \to \overline{D}_{\mathcal{A}}$ as follows:
- if $\alpha = (\delta, \gamma)$ with $\delta \in \{+,-\}$ and $\gamma \in \mathcal{A} \cup \{*\}$, then $U(\alpha) = \gamma$;
- if $\alpha = (\delta, \alpha_1, \alpha_2)$ with $\delta \in \{+,-\}$ and $\alpha_1, \alpha_2 \in D$, then $U(\delta, \alpha_1, \alpha_2) = (U(\alpha_1), U(\alpha_2))$;
- if $\alpha = (\delta, \alpha_0)$ with $\delta \in \{+,-\}$ and $\alpha_0 \in D$, then $U(\delta, \alpha_0) = U(\alpha_0)$.

Definition 4.23. Let $\sigma : \mathcal{A} \to \mathcal{A}$. For any $\alpha \in \overline{D}_{\mathcal{A}}$, we define $\sigma \cdot \alpha \in \overline{D}_{\mathcal{A}}$ as follows:
- if $\alpha \in \mathcal{A}$, then $\sigma \cdot \alpha = \sigma(\alpha)$;
- if $\alpha = *$, then $\sigma \cdot \alpha = \alpha$;
- if $\alpha_1, \alpha_2 \in \overline{D}_{\mathcal{A}}$, then $\sigma \cdot (\alpha_1, \alpha_2) = (\sigma \cdot \alpha_1, \sigma \cdot \alpha_2)$;
- if $\alpha_1, \ldots, \alpha_m \in \overline{D}_{\mathcal{A}}$, then $\sigma \cdot [\alpha_1, \ldots, \alpha_m] = [\sigma \cdot \alpha_1, \ldots, \sigma \cdot \alpha_m]$.

For any set $\mathcal{P}$, for any function $x : \mathcal{P} \to \overline{D}_{\mathcal{A}}$, we define a function $\sigma \cdot x : \mathcal{P} \to \overline{D}_{\mathcal{A}}$ by setting: $(\sigma \cdot x)(p) = \sigma \cdot x(p)$ for any $p \in \mathcal{P}$.

For any function $\sigma : \mathcal{A} \to D_{\mathcal{A}}$, we define the function $U(\sigma) : \mathcal{A} \to \mathcal{A}$ as follows:

$$U(\sigma)(\gamma) = \begin{cases} U(\sigma(\gamma)) & \textit{if } \sigma(\gamma) \in \{+,-\} \times \mathcal{A}; \\ \gamma & \textit{otherwise.} \end{cases}$$

Fact 4.24. For any $\alpha \in D_{\mathcal{A}}$, for any $\sigma \in \mathfrak{R}(\alpha)$, we have $U(\sigma \cdot \alpha) = U(\sigma) \cdot U(\alpha)$.

Fact 4.25. Let (R, T) be a typed in-PS. Then, for any experiment e on R, for any $p \in \mathcal{P}_0(R)$, we have $\textit{height}(e(p)) \geq \textit{height}(\mathsf{T}(p))$.

Definition 4.26. Any $\alpha \in D_{\mathcal{A}}$ is said to be *uniform* if, for any occurrence of any multiset a that occurs in α, for any $\beta, \beta' \in \textit{Supp}(a)$, we have $\textit{height}(\beta) = \textit{height}(\beta')$.

For any finite set $\mathcal{P}$, any function $x : \mathcal{P} \to D_{\mathcal{A}}$ is said to be *uniform* if, for any $p \in \mathcal{P}$, $x(p)$ is uniform.

Fact 4.27. Let (R, T) be a typed in-PS. Then, for any atomic experiment e on R, for any $p \in \mathcal{P}_0(R)$, $e(p)$ is uniform and we have $\textit{height}(e(p)) = \textit{height}(\mathsf{T}(p))$.

Fact 4.28. Let $\alpha, \alpha' \in D_{\mathcal{A}}$ and $\sigma : \mathcal{A} \to D_{\mathcal{A}}$ such that α' is uniform and $\sigma \cdot \alpha' = \alpha$. We have $\textit{height}(\alpha) > \textit{height}(\alpha')$ if, and only if, $\sigma \notin \mathfrak{R}(\alpha')$.

Proof. By induction on $\textit{height}(\alpha)$. □

Lemma 4.29. *We assume that, for any $X \in \mathcal{X}$, the set $[\![X]\!]$ is infinite. Let (R, T) be a cut-free typed PS. Then, for any atomic experiment e on R, we have $e_{|\mathcal{P}^{\mathsf{f}}(R)} \in [\![R]\!]_{At}$.*

Proof. Let e be an atomic experiment on R. Let $x' \in [\![R]\!]$ and let $\sigma : \mathcal{A} \to D_{\mathcal{A}}$ such that $\sigma \cdot x' = e_{|\mathcal{P}^{\mathsf{f}}(R)}$. By Fact 4.14, there exist $x \in [\![R]\!]_{At}$ and $\tau : \mathcal{A} \to D_{\mathcal{A}}$ such that $\tau \cdot x = x'$. By Fact 4.13 and Fact 4.27, the point x is uniform and $(\forall p \in \mathcal{P}^{\mathsf{f}}(R))\textit{height}(x(p)) = \textit{height}(\mathsf{T}(p)) = e(p)$. By Lemma 4.7, we have $(\sigma \cdot \tau) \cdot x = \sigma \cdot (\tau \cdot x) = \sigma \cdot x' = e_{|\mathcal{P}^{\mathsf{f}}(R)}$. By Fact 4.28, we have $\sigma \cdot \tau \in \mathfrak{R}(x)$, hence, by Fact 4.11, $\sigma \in \mathfrak{R}(\tau \cdot x) = \mathfrak{R}(x')$, which entails $e_{|\mathcal{P}^{\mathsf{f}}(R)} \in [\![R]\!]_{At}$. □

Lemma 4.30. *Let us assume that, for any $X \in \mathcal{X}$, the set $[\![X]\!]$ is infinite. Let (R, T) be a cut-free typed PS. Then $\{U \circ x; x \in [\![R]\!]_{At}\} = \bigcup_{x \in [\![(R,\mathsf{T})]\!]} \{\sigma \cdot x; \sigma \in \mathcal{A}^{\mathcal{A}}\}$.*

Proof. By Lemma 4.29, we have $[\![(R, \mathsf{T})]\!] \subseteq \{U \circ x; x \in [\![R]\!]_{At}\}$. Let $x \in [\![(R, \mathsf{T})]\!]$. Now, for any $x' \in [\![R]\!]_{At}$, for any $\sigma \in \mathfrak{R}(x')$, we have $\sigma \cdot x' \in [\![R]\!]_{At}$; indeed, let $x \in [\![R]\!]$ and $\tau : \mathcal{A} \to D_{\mathcal{A}}$ such that $\tau \cdot x = \sigma \cdot x'$; by Fact 4.13 and Fact 4.27, we have $\textit{height}(x) = \textit{height}(x')$, hence, by

Fact 4.28, $\tau \in \mathfrak{R}(x)$. So, by Fact 4.24, we have $\bigcup_{x \in [\![(R,\mathsf{T})]\!]} \{\sigma \cdot x; \sigma \in \mathcal{A}^{\mathcal{A}}\} \subseteq \{U(\sigma) \cdot (U \circ x); (x \in [\![R]\!]_{At} \wedge \sigma \in \mathcal{A}^{\mathcal{A}})\} = \{U \circ (\sigma \cdot x); (x \in [\![R]\!]_{At} \wedge \sigma \in \mathfrak{R}(x))\} = \{U \circ x; x \in [\![R]\!]_{At}\}$.

Conversely, let $x \in [\![R]\!]_{At}$. If $\mathcal{X} = \emptyset$, then $U \circ x \in [\![(R,\mathsf{T})]\!]$. Otherwise, the set $\mathcal{A}$ is infinite, hence there exist an injective atomic experiment e on R and $\sigma \in \mathfrak{R}(x)$ such that $x = \sigma \cdot e_{|\mathcal{P}^{\mathsf{f}}(R)}$. Let $\tau : \mathcal{A} \to D_{\mathcal{A}}$ such that:

- for any $p \in \mathcal{P}_0^{ax}(R)$, for any $\gamma \in \mathcal{A}$, for any $\delta \in \{+,-\}$ such that $e(p) = (\delta, \gamma)$, we have $U(\tau)(\gamma) \in [\![\mathsf{T}(p)]\!]$;
- and, for any $p \in \mathcal{P}^{ax}(R) \cap \mathcal{P}_{>0}(R)$, for any $\gamma \in \mathcal{A}$, for any $\delta \in \{+,-\}$ such that $(\delta, \gamma) \in \mathit{Supp}(e(p))$, we have $U(\tau)(\gamma) \in [\![\mathsf{T}(p)]\!]$.

We have $U(\tau) \cdot (U \circ e_{|\mathcal{P}^{\mathsf{f}}(R)}) \in [\![(R,\mathsf{T})]\!]$. It is clear that there exists $\sigma' \in \mathfrak{R}(\tau \cdot e_{|\mathcal{P}^{\mathsf{f}}(R)})$ such that $\sigma' \cdot (\tau \cdot e_{|\mathcal{P}^{\mathsf{f}}(R)}) = \sigma \cdot e_{|\mathcal{P}^{\mathsf{f}}(R)}$. We have $U \circ x = U \circ (\sigma \cdot e_{|\mathcal{P}^{\mathsf{f}}(R)}) = U \circ (\sigma' \cdot (\tau \cdot e_{|\mathcal{P}^{\mathsf{f}}(R)})) = U(\sigma') \cdot (U \circ (\tau \cdot e_{|\mathcal{P}^{\mathsf{f}}(R)})) = U(\sigma') \cdot (U(\tau) \cdot (U \circ e_{|\mathcal{P}^{\mathsf{f}}(R)}))$ (by applying Fact 4.24 twice). □

Definition 4.31. Let $\mathcal{D} \subseteq (D_{\mathcal{A}})^{\mathcal{P}}$ for some set $\mathcal{P}$. We say that $\mathcal{D}$ *reflects renamings* if the following property holds: $(\forall x \in \mathcal{D})(\forall y \in D_{\mathcal{A}})(\forall \sigma \in \mathfrak{R}(y))(\sigma \cdot y = x \Rightarrow y \in \mathcal{D})$

Definition 4.32. For any finite set $\mathcal{P}$, any function $x : \mathcal{P} \to D_{\mathcal{A}}$ is said to be *balanced* if, for any $\gamma \in \mathcal{A}$, there are as many occurrences of $(+,\gamma)$ in $\sum_{p \in \mathcal{P}}[x(p)]$ as occurrences of $(-,\gamma)$ in $\sum_{p \in \mathcal{P}}[x(p)]$.

Fact 4.33. Let S be a differential in-PS. Then, for any $x \in [\![S]\!]$, x is balanced.

Proof. By induction on $(\mathit{depth}(S), \mathsf{Card}(\mathcal{B}(S)), \mathsf{Card}(\mathcal{P}_0(S)), \mathsf{Card}(\mathcal{C}_0(S)))$ lexicographically ordered. □

Fact 4.34. Let $\mathcal{P}$ be some finite set and let T be a function $\mathcal{P} \to \mathbb{T}$. Let $x_1, x_2 \in (D_{\mathcal{A}})^{\mathcal{P}}$ $\mathcal{A}$-injective and balanced such that $U \circ x_1 = U \circ x_2$ and, for any $p \in \mathcal{P}$, we have $U(x_1(p)) \in [\![\mathsf{T}(p)]\!]$. Then there exists $\sigma \in \mathfrak{R}(x_1)$ such that $\sigma \cdot x_1 = x_2$.

Proof. By induction on $\sum_{p \in \mathcal{P}} \mathit{size}(\mathsf{T}(p))$. □

Lemma 4.35. *Let $\mathcal{P}$ be some finite set. Let $\mathcal{D} \subseteq (D_{\mathcal{A}})^{\mathcal{P}}$. Let $x \in \mathcal{D}$ and let $\sigma \in \mathfrak{R}(x)$ such that $\sigma \cdot x \in \mathcal{D}_{At}$. Then $x \in \mathcal{D}_{At}$.*

Proof. Let $y \in \mathcal{D}$ and let $\tau : \mathcal{A} \to D_{\mathcal{A}}$ such that $\tau \cdot y = x$. Then, by Lemma 4.7, we have $(\sigma \cdot \tau) \cdot y = \sigma \cdot (\tau \cdot y) = \sigma \cdot x$. We have $\sigma \cdot \tau \in \mathfrak{R}(y)$, hence, by Fact 4.11, $\tau \in \mathfrak{R}(y)$. □

Theorem 4.36 is similar to Theorem 4.20:

Theorem 4.36. *Let (R,T) be a cut-free typed PS. If, for any $X \in \mathcal{X}$, the set $[\![X]\!]$ is infinite, then there exists $D_0 \subseteq [\![(R,\mathsf{T})]\!]$ with $\mathsf{Card}(D_0) = 2$ such that, for any cut-free typed PS (R',T') with $\mathcal{P}^{\mathsf{f}}(R) = \mathcal{P}^{\mathsf{f}}(R')$ and $\mathsf{T}_{|\mathcal{P}^{\mathsf{f}}(R)} = \mathsf{T}'_{|\mathcal{P}^{\mathsf{f}}(R')}$, we have $(D_0 \subseteq [\![(R',\mathsf{T}')]\!] \Rightarrow (R,\mathsf{T}) \equiv (R',\mathsf{T}'))$.*

Proof. By Theorem 4.20, there exists an $\mathcal{A}$-injective subset $D_0' = \{y, z\}$ of $[\![R]\!]_{At}$ with $\mathsf{Card}(D_0) = 2$ such that, for any cut-free PS R', we have $(D_0' \subseteq [\![R']\!]_{At} \Rightarrow R \equiv R')$. We can take $D_0 = \{U \circ x; x \in D_0'\}$. Indeed:

Let (R',T') be a typed PS such that $\mathcal{P}^{\mathsf{f}}(R) = \mathcal{P}^{\mathsf{f}}(R')$, $\mathsf{T}_{|\mathcal{P}^{\mathsf{f}}(R)} = \mathsf{T}'_{|\mathcal{P}^{\mathsf{f}}(R')}$ and $D_0 \subseteq [\![(R',\mathsf{T}')]\!]$. By Lemma 4.30, there exist $y', z' \in [\![R']\!]_{At}$ such that $U \circ y = U \circ y'$ and $U \circ z = U \circ z'$. By Fact 4.33, we can apply Fact 4.34 to obtain that there exist $\sigma_y \in \mathfrak{R}(y)$ and $\sigma_z \in \mathfrak{R}(z)$ such that $\sigma_y \cdot y = y'$ and $\sigma_z \cdot z = z'$. By Lemma 4.35, we thus have $D_0' \subseteq [\![R']\!]_{At}$, hence $R \equiv R'$. By Fact 1.13, we obtain $(R,\mathsf{T}) \equiv (R',\mathsf{T}')$. □

Now, we take advantage of the normalization property of the typed PS's:

Corollary 4.37. *Let* (R_1, T_1) *and* (R_2, T_2) *be two typed PS's such that* $\mathcal{P}^{\mathsf{f}}(R_1) = \mathcal{P}^{\mathsf{f}}(R_2)$ *and* $\mathsf{T}_1|_{\mathcal{P}^{\mathsf{f}}(R_1)} = \mathsf{T}_2|_{\mathcal{P}^{\mathsf{f}}(R_2)}$. *If, for any* $X \in \mathcal{X}$, *the set* $[\![X]\!]$ *is infinite, then we have* $((R_1, \mathsf{T}_1) \simeq_\beta (R_2, \mathsf{T}_2) \Leftrightarrow [\![(R_1, \mathsf{T}_1)]\!] = [\![(R_2, \mathsf{T}_2)]\!])$.

Proof. Let us assume that $[\![(R_1, \mathsf{T}_1)]\!] = [\![(R_2, \mathsf{T}_2)]\!]$. There exist two cut-free typed PS's $(R'_1, \mathsf{T'}_1)$ and $(R'_2, \mathsf{T'}_2)$ such that $(R_1, \mathsf{T}_1) \simeq_\beta (R'_1, \mathsf{T'}_1)$ and $(R_2, \mathsf{T}_2) \simeq_\beta (R'_2, \mathsf{T'}_2)$. We have $\mathcal{P}^{\mathsf{f}}(R'_1) = \mathcal{P}^{\mathsf{f}}(R_1) = \mathcal{P}^{\mathsf{f}}(R_2) = \mathcal{P}^{\mathsf{f}}(R'_2)$ and $[\![(R'_1, \mathsf{T'}_1)]\!] = [\![(R_1, \mathsf{T}_1)]\!] = [\![(R_2, \mathsf{T}_2)]\!] = [\![(R'_2, \mathsf{T'}_2)]\!]$. By Theorem 4.36, we have $(R'_1, \mathsf{T'}_1) \equiv (R'_2, \mathsf{T'}_2)$, hence $(R_1, \mathsf{T}_1) \simeq_\beta (R_2, \mathsf{T}_2)$. □

Conclusion

The aim of this work was to prove theorems that relate the syntax of the proof-nets with their relational semantics firstly and with their Taylor expansion secondly. But incidentally we proved an interesting intrinsic semantical result: We showed that the entire relational semantics of any normalizable proof-net can be rebuilt from two well-chosen points (and that it is impossible to strengthen this result by rebuilding any proof-net from only one well-chosen point). So, in some way, these two points together could be seen as a principal typing of intersection types for the given proof-net. With the algorithm we described, we can first rebuild the normal form of the proof-net from this "principal typing" and then compute the semantics. Now, some technology could probably be developed to compute the semantics directly from these points without rebuilding the syntactic object, like in the case of untyped lambda-calculus (expansion, substitution...).

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5. Appendix: Proof of Lemma 3.38

Proof. With Lemma 2.8 in mind, we can describe $\mathcal{T}_{R_o}[i](e)$:

- $\mathcal{P}_0(\mathcal{T}_{R_o}[i](e)) = \mathcal{P}_0(\mathcal{T}_R[i](e)) \cup \bigcup_{e_1 \in e(o)} \{(o,(e_1,p)); p \in \mathcal{Q}'\}$;
- $l_{\mathcal{G}(\mathcal{T}_{R_o}[i](e))}(p) = \begin{cases} l_{\mathcal{G}(\mathcal{T}_R[i](e))}(p) & \text{if } p \in \mathcal{P}_0(\mathcal{T}_R[i](e)); \\ ? & \text{otherwise;} \end{cases}$
- $\kappa_{R_o}[i](e)(p) = \begin{cases} \kappa_R[i](e)(p) & \text{if } p \in \mathcal{P}_0(\mathcal{T}_R[i](e)); \\ (o, \varphi_o(q')) & \text{if } p = \varphi_{e_1}(q') \text{ for some } e_1 \in e(o),\ q' \in \mathcal{Q}; \\ q & \text{if } p = q; \end{cases}$
- $\mathcal{W}_0(\mathcal{T}_{R_o}[i](e)) = \mathcal{W}_0(\mathcal{T}_R[i](e)) \cup \bigcup_{e_1 \in e(o)} \{(o,(e_1,p)); p \in \mathcal{Q}'\}$;
- $t_{\mathcal{G}(\mathcal{T}_{R_o}[i](e))}(p)$ is the following port of $\mathcal{G}(\mathcal{T}_{R_o}[i](e))$:
  $\begin{cases} (o,(e_1,\varphi_o(t_{\mathcal{G}(\mathcal{T}_R[i](e))}(p)))) & \text{if } p = (o,(e_1,p')) \text{ with } e_1 \in e(o) \text{ and } t_{\mathcal{G}(\mathcal{T}_R[i](e))}(p) \in \mathcal{Q}; \\ t_{\mathcal{G}(\mathcal{T}_R[i](e))}(p) & \text{otherwise.} \end{cases}$
- $\mathcal{C}_0(\mathcal{T}_{R_o}[i](e)) = \mathcal{C}_0(\mathcal{T}_R[i](e))$ and $\mathcal{A}_0(\mathcal{T}_{R_o}[i](e)) = \mathcal{A}_0(\mathcal{T}_R[i](e))$
- $\mathcal{B}_0(\mathcal{T}_{R_o}[i](e)) = \mathcal{B}_0(\mathcal{T}_R[i](e))$
- $B_{\mathcal{T}_{R_o}[i](e)} = B_{\mathcal{T}_R[i](e)}$
- $\mathsf{dom}(t_{\mathcal{T}_{R_o}[i](e)}) = \mathsf{dom}(t_{\mathcal{T}_R[i](e)})$ and $t_{\mathcal{T}_{R_o}[i](e)}(o',p) =$
  $\begin{cases} (o,(e_1,\varphi_o(t_{\mathcal{T}_R[i](e)}(o',p)))) & \begin{array}{l}\text{if } t_{\mathcal{T}_R[i](e)}(o',p) \in \mathcal{Q} \text{ and } o' = (o,(e_1,o'')) \\ \text{for some } e_1 \in e(o),\ o'' \in \mathcal{B}_0(\mathcal{T}_{B_R(o)}[i](e_1));\end{array} \\ t_{\mathcal{T}_R[i](e)}(o',p) & \text{otherwise;} \end{cases}$

For any $p, p' \in \mathcal{P}_0(R\langle o,i,e_o\rangle) \cup \mathcal{Q} \cup \varphi_{e_o}[\mathcal{Q}]$ such that $\mathsf{Card}(\{p,p'\} \cap \mathcal{Q}) \leq 1$, we have $p \frown_{\mathcal{T}_{R_o}[i](e)} p'$ if, and only if, one of the following properties holds:

- $p, p' \in \mathcal{P}_0(\mathcal{T}_R[i](e)) \setminus \mathcal{Q}$ and $p \frown_{\mathcal{T}_R[i](e)} p'$
- $\{p,p'\} \subseteq \{(o,(e_o,p'')), \varphi_{e_o}(t_R(o,p''))\}$ for some $p'' \in \mathcal{P}^{\mathsf{f}}_R(o)$
- $\{p,p'\} \cap \mathcal{Q}$ is some singleton $\{p''\}$ with $p'' \in \mathcal{Q}$ and $\{p,p'\} \subseteq \{p'', \varphi_{e_o}(p'')\}$

1) Let $T \in \mathcal{S}^k_{\mathcal{T}_R[i](e)}(\mathcal{P})$ such that $\mathcal{P}_0(T) \cap \mathcal{P}_0(R\langle o,i,e_o\rangle) \neq \emptyset$. We have $\mathcal{P}_0(T[\varphi_{e_o}]) = (\mathcal{P}_0(T) \setminus \mathcal{Q}) \cup \varphi_{e_o}[\mathcal{Q} \cap \mathcal{P}_0(T)]$, hence, by Lemma 3.29, $\mathcal{P}_0(T[\varphi_{e_o}]) \subseteq \mathcal{P}_0(R\langle o,i,e_o\rangle) \cup \varphi_{e_o}[\mathcal{Q} \cap \mathcal{P}_0(T)]$ $(*)$. Again, by Lemma 3.29, we have $\mathcal{P}_0(T) \cap \mathcal{Q} \subseteq \mathcal{P}$ $(**)$.

Let $w \in \mathcal{W}_0(T)$: We have $w \notin \mathcal{P}$ (hence, by $(**)$, $w \in \mathcal{P}_0(T[\varphi_{e_o}]) \setminus \mathcal{P}_{e_o}$). Again by Lemma 3.29, there exists $p \in \mathcal{P}_0(\mathcal{T}_{B_R(o)}[i](e_o))$ such that $w = (o,(e_o,p))$. We distinguish between two cases:

- $p \in \mathcal{P}^{\mathsf{f}}_R(o)$: we have $t_{\mathcal{G}(\mathcal{T}_R[i](e))}(o,(e_o,p)) = t_R(o,p) \in \mathcal{P}_0(T) \cap \mathcal{Q}$, hence $t_{\mathcal{G}(\mathcal{T}_{R_o}[i](e))}(o,(e_o,p)) = (o,(e_o,\varphi_o(t_R(o,p)))) \in \mathcal{P}_0(T[\varphi_{e_o}])$;
- $p \notin \mathcal{P}^{\mathsf{f}}_R(o)$: we have $t_{\mathcal{G}(\mathcal{T}_{R_o}[i](e))}(o,(e_o,p)) = t_{\mathcal{G}(\mathcal{T}_R[i](e))}(o,(e_o,p)) \in \mathcal{P}_0(T) \setminus \mathcal{Q} \subseteq \mathcal{P}_0(T[\varphi_{e_o}])$.

We thus have

$$\begin{array}{ll} & \mathcal{W}_0(T) \\ \subseteq & \{w \in (\mathcal{W}_0(\mathcal{T}_{R_o}[i](e)) \cap \mathcal{P}_0(T[\varphi_{e_o}])) \setminus \mathcal{P}_{e_o}; t_{\mathcal{G}(\mathcal{T}_{R_o}[i](e))}(w) \in \mathcal{P}_0(T[\varphi_{e_o}])\} \end{array}$$

Conversely, let $w \in (\mathcal{W}_0(\mathcal{T}_{R_o}[i](e)) \cap \mathcal{P}_0(T[\varphi_{e_o}])) \setminus \mathcal{P}_{e_o}$ such that $t_{\mathcal{G}(\mathcal{T}_{R_o}[i](e))}(w) \in \mathcal{P}_0(T[\varphi_{e_o}])$. By $(*)$, there exists $p \in \mathcal{P}_0(\mathcal{T}_{B_R(o)}[i](e_o))$ such that $w = (o,(e_o,p))$. We distinguish between two cases:

- $p \in \mathcal{P}^{\mathsf{f}}_R(o)$: we have $t_{\mathcal{G}(\mathcal{T}_{R_o}[i](e))}(o,(e_o,p)) = (o,(e_o,\varphi_o(t_R(o,p)))) \in \mathcal{P}_0(T[\varphi_{e_o}])$, hence $(o,(e_o,\varphi_o(t_R(o,p)))) \in \varphi_{e_o}[\mathcal{Q} \cap \mathcal{P}_0(T)]$; so, $t_{\mathcal{G}(\mathcal{T}_R[i](e))}(o,(e_o,p)) = t_R(o,p) \in \mathcal{P}_0(T)$, which shows that $(o,(e_o,p)) \in \mathcal{W}_0(T)$;

- $p \notin \mathcal{P}^{\mathsf{f}}_R(o)$: we have $t_{\mathcal{G}(\mathcal{T}_R[i](e))}(o,(e_o,p)) = t_{\mathcal{G}(\mathcal{T}_{R_o}[i](e))}(o,(e_o,p)) \in \mathcal{P}_0(T[\varphi_{e_o}]) \setminus \varphi_{e_o}[\mathcal{Q}] \subseteq \mathcal{P}_0(T)$.

We thus have

$$\begin{array}{rl} & \{w \in (\mathcal{W}_0(\mathcal{T}_{R_o}[i](e)) \cap \mathcal{P}_0(T[\varphi_{e_o}])) \setminus \mathcal{P}_{e_o}; t_{\mathcal{G}(\mathcal{T}_{R_o}[i](e))}(w) \in \mathcal{P}_0(T[\varphi_{e_o}])\} \\ \subseteq & \mathcal{W}_0(T) \end{array}$$

Moreover, $\mathcal{W}_0(T[\varphi_{e_o}]) = \mathcal{W}_0(T)$; we thus showed:

$$\begin{array}{rl} & \mathcal{W}_0(T[\varphi_{e_o}]) \\ = & \{w \in (\mathcal{W}_0(\mathcal{T}_{R_o}[i](e)) \cap \mathcal{P}_0(T[\varphi_{e_o}])) \setminus \mathcal{P}_{e_o}; t_{\mathcal{G}(\mathcal{T}_{R_o}[i](e))}(w) \in \mathcal{P}_0(T[\varphi_{e_o}])\} \end{array}$$

So, $T[\varphi_{e_o}] \sqsubseteq_{\mathcal{P}_{e_o}} \mathcal{T}_{R_o}[i](e)$.

Since T is connected through ports not in $\mathcal{P}$, $T[\varphi_{e_o}]$ is connected through ports not in $\varphi_{e_o}[\mathcal{P} \cap \mathsf{dom}(\varphi_{e_o})] \cup (\mathcal{P} \setminus \mathsf{dom}(\varphi_{e_o})) = \mathcal{P}_{e_o}$.

Finally, $\textit{co-size}(T[\varphi_{e_o}]) = \textit{co-size}(T)$ and $\mathcal{P}^{\mathsf{f}}(\mathcal{G}(T[\varphi_{e_o}])) = \mathcal{P}_0(T[\varphi_{e_o}]) \setminus (\mathcal{W}_0(T[\varphi_{e_o}]) \cup \bigcup \mathcal{C}_0(T[\varphi_{e_o}])) = ((\mathcal{P}_0(T) \setminus \mathcal{Q}) \cup \varphi_{e_o}[\mathcal{Q} \cap \mathcal{P}_0(T)]) \setminus (\mathcal{W}_0(T) \cup \bigcup \mathcal{C}_0(T)) = (\mathcal{P}^{\mathsf{f}}(\mathcal{G}(T)) \setminus \mathcal{Q}) \cup \varphi_{e_o}[\mathcal{Q} \cap \mathcal{P}_0(T)] \subseteq (\mathcal{P} \setminus \mathcal{Q}) \cup \varphi_{e_o}[\mathcal{Q} \cap \mathcal{P}_0(T)] = \mathcal{P}_{e_o}$. Let $p \in \mathcal{P}_0(T[\varphi_{e_o}])$ and $p' \in \mathcal{P}_0(\mathcal{T}_{R_o}[i](e))$ such that $p \frown_{\mathcal{T}_{R_o}[i](e)} p'$ and $p' \notin \mathcal{P}_0(T[\varphi_{e_o}])$ (hence $p \notin \mathcal{Q}$, so $\mathsf{Card}\,(\{p, p'\} \cap \mathcal{Q}) \leq 1$): We distinguish between three cases:

- $p, p' \in \mathcal{P}_0(\mathcal{T}_R[i](e)) \setminus \mathcal{Q}$ and $p \frown_{\mathcal{T}_R[i](e)} p'$: we have $p \in \mathcal{P} \setminus \mathcal{Q} \subseteq \mathcal{P}_{e_o}$;
- $\{p, p'\} \subseteq \{(o,(e_o,p'')), \varphi_{e_o}(t_R(o,p''))\}$ for some $p'' \in \mathcal{P}^{\mathsf{f}}_R(o)$: either $p = \varphi_{e_o}(t_R(o,p'))$ (hence $p \in \varphi_{e_o}[\mathcal{Q}] \subseteq \mathcal{P}_{e_o}$) or $p = (o,(e_o,p''))$. From now on, let us assume that $p = (o,(e_o,p''))$. We have $t_R(o,p'') \in \mathcal{Q}$. If $t_R(o,p'') \in \mathcal{P}_0(T)$, then $p' = \varphi_{e_o}(t_R(o,p'')) \in \mathcal{P}_0(T)[\varphi_{e_o}]$, which contradicts $p' \notin \mathcal{P}_0(T)[\varphi_{e_o}]$; we thus have $t_R(o,p'') \notin \mathcal{P}_0(T)$, which entails $p \in \mathcal{P} \setminus \mathcal{Q} \subseteq \mathcal{P}_{e_o}$.
- $\{p, p'\} \cap \mathcal{Q}$ is some singleton $\{p''\}$ with $p'' \in \mathcal{Q}$ and $\{p, p'\} \subseteq \{p'', \varphi_{e_o}(p'')\}$: since $p \notin \mathcal{Q}$, we have $p' = p''$ and $p = \varphi_{e_o}(p') \in \mathcal{P}_0(T[\varphi_{e_o}])$, hence $p' \in \mathcal{P}_0(T)$ and, finally, $p = \varphi_{e_o}(p'') \in \varphi_{e_o}[\mathcal{P} \cap \mathcal{Q}] \subseteq \mathcal{P}_{e_o}$.

In the three cases, we have $p \in \mathcal{P}_{e_o}$.

So, we showed $T[\varphi_{e_o}] \in \mathcal{S}^k_{\mathcal{T}_{R_o}[i](e)}(\mathcal{P}_{e_o})$.

2) Let $T \in \mathcal{S}^k_{\mathcal{T}_{R_o}[i](e)}(\mathcal{P}_{e_o})$ such that $\mathcal{P}_0(T) \subseteq \mathcal{P}_0(R_o\langle o, i, e_o\rangle)$. We have $\mathcal{P}_0(T[{\varphi_{e_o}}^{-1}]) = (\mathcal{P}_0(T) \setminus \varphi_{e_o}[\mathcal{Q}]) \cup \{q' \in \mathcal{Q}; \varphi_{e_o}(q') \in \mathcal{P}_0(T)\}$.

Since $t_{\mathcal{G}(\mathcal{T}_{R_o}[i](e))}[\varphi_{e_o}[\mathcal{Q}]] = \mathcal{Q}$ and $\mathcal{Q} \cap \mathcal{P}_0(T) = \emptyset$, we have

$$\begin{array}{rcl} \mathcal{W}_0(T) & = & \{w \in (\mathcal{W}_0(\mathcal{T}_{R_o}[i](e)) \cap \mathcal{P}_0(T)) \setminus \mathcal{P}_{e_o}; t_{\mathcal{G}(\mathcal{T}_{R_o}[i](e))}(w) \in \mathcal{P}_0(T)\} \\ & = & \{w \in (\mathcal{W}_0(\mathcal{T}_{R_o}[i](e)) \cap \mathcal{P}_0(T)) \setminus \mathcal{P}_{e_o}; t_{\mathcal{G}(\mathcal{T}_R[i](e))}(w) \in \mathcal{P}_0(T)\} \\ & = & \{w \in (\mathcal{W}_0(\mathcal{T}_R[i](e)) \cap \mathcal{P}_0(T)) \setminus \mathcal{P}; t_{\mathcal{G}(\mathcal{T}_R[i](e))}(w) \in \mathcal{P}_0(T)\} \end{array}$$

Moreover, we have $\mathcal{W}_0(T[{\varphi_{e_o}}^{-1}]) = \mathcal{W}_0(T)$. Hence

$$\mathcal{W}_0(T[{\varphi_{e_o}}^{-1}]) = \{w \in (\mathcal{W}_0(\mathcal{T}_R[i](e)) \cap \mathcal{P}_0(T)) \setminus \mathcal{P}; t_{\mathcal{G}(\mathcal{T}_R[i](e))}(w) \in \mathcal{P}_0(T)\}$$

Since T is connected modulo $\mathcal{P}_{e_o}$, $T[{\varphi_{e_o}}^{-1}]$ is connected modulo ${\varphi_{e_o}}^{-1}[\mathcal{P}_{e_o} \cap \mathsf{dom}({\varphi_{e_o}}^{-1})] \cup (\mathcal{P}_{e_o} \setminus \mathsf{dom}({\varphi_{e_o}}^{-1})) = \mathcal{P}$.

Finally, $\textit{co-size}(T[{\varphi_{e_o}}^{-1}]) = \textit{co-size}(T)$ and $\mathcal{P}^{\mathsf{f}}(\mathcal{G}(T[{\varphi_{e_o}}^{-1}])) = \mathcal{P}_0(T[{\varphi_{e_o}}^{-1}]) \setminus (\mathcal{W}_0(T[{\varphi_{e_o}}^{-1}]) \cup \bigcup \mathcal{C}_0(T[{\varphi_{e_o}}^{-1}])) = ((\mathcal{P}_0(T) \setminus \varphi_{e_o}[\mathcal{Q}]) \cup \{q' \in \mathcal{Q}; \varphi_{e_o}(q') \in \mathcal{P}_0(T)\}) \setminus (\mathcal{W}_0(T) \cup \bigcup \mathcal{C}_0(T)) = (\mathcal{P}^{\mathsf{f}}(\mathcal{G}(T)) \setminus \varphi_{e_o}[\mathcal{Q}]) \cup \{q' \in \mathcal{Q}; \varphi_{e_o}(q') \in \mathcal{P}_0(T)\} \subseteq (\mathcal{P}_{e_o} \setminus \varphi_{e_o}[\mathcal{Q}]) \cup \{q' \in \mathcal{Q}; \varphi_{e_o}(q') \in \mathcal{P}_0(T)\} = (((\mathcal{P} \setminus \mathcal{Q}) \cup \varphi_{e_o}[\mathcal{P} \cap \mathcal{Q}]) \setminus \varphi_{e_o}[\mathcal{Q}]) \cup \{q' \in \mathcal{Q}; \varphi_{e_o}(q') \in \mathcal{P}_0(T)\} \subseteq \mathcal{P}$. Let $p \in \mathcal{P}_0(T[{\varphi_{e_o}}^{-1}])$

and $p' \in \mathcal{P}_0(\mathcal{T}_R[i](e))$ such that $p \frown_{\mathcal{T}_R[i](e)} p'$ and $p' \notin \mathcal{P}_0(T[{\varphi_{e_o}}^{-1}])$. We distinguish between three cases:

- $p, p' \notin \mathcal{Q}$: we have $p \in \mathcal{P}_0(T)$, $p' \notin \mathcal{P}_0(T)$ and $p \frown_{\mathcal{T}_{R_o}[i](e)} p'$, hence, since $T \trianglelefteq_{\mathcal{P}_{e_o}} \mathcal{T}_{R_o}[i](e)$, we have $p \in \mathcal{P}_{e_o}$; but $\mathcal{P}_{e_o} \cap \mathcal{P}_0(\mathcal{T}_R[i](e)) \subseteq \mathcal{P}$;
- $p \in \mathcal{Q}$: we have $\varphi_{e_o}(p) \in \mathcal{P}^{\mathsf{f}}(\mathcal{G}(T)) \subseteq \mathcal{P}_{e_o}$, hence $p \in \mathcal{P}$;
- $p \notin \mathcal{Q}$ and $p' \in \mathcal{Q}$: we have $p \in \mathcal{P}^{\mathsf{f}}(\mathcal{G}(T)) \subseteq \mathcal{P}_{e_o}$; but $\mathcal{P}_{e_o} \cap \mathcal{P}_0(\mathcal{T}_R[i](e)) \subseteq \mathcal{P}$.

In the three cases, we have $p \in \mathcal{P}$.

So, we showed $T[{\varphi_{e_o}}^{-1}] \in \mathcal{S}^k_{\mathcal{T}_R[i](e)}(\mathcal{P})$.

3) Let $T \in \mathcal{S}^k_{\mathcal{T}_{R_o}[i](e)}(\mathcal{P})$ such that $(\forall e_1 \in e(o))\mathcal{P}_0(T) \cap \mathcal{P}_0(R_o\langle o, i, e_1\rangle) = \emptyset$. Notice that $q \notin \mathcal{P}_0(T)$; indeed, since $q \in \mathcal{P}^{\mathsf{f}}(\mathcal{G}(\mathcal{T}_{R_o}[i](e)))$, we have $(q \in \mathcal{P}_0(T) \Rightarrow q \in \mathcal{P})$, but $q \notin \mathcal{P}$.

Since $(\forall w \in \mathcal{P}_0(T)) w \notin \bigcup_{e_1 \in e(o)} \{(o, (e_1, p)); p \in \mathcal{Q}'\}$, we have:

- $\mathcal{W}_0(\mathcal{T}_R[i](e)) \cap \mathcal{P}_0(T) = \mathcal{W}_0(\mathcal{T}_{R_o}[i](e)) \cap \mathcal{P}_0(T)$
- and $(\forall w \in \mathcal{W}_0(\mathcal{T}_R[i](e)) \cap \mathcal{P}_0(T)) t_{\mathcal{G}(\mathcal{T}_R[i](e))}(w) = t_{\mathcal{G}(\mathcal{T}_{R_o}[i](e))}(w)$.

Hence

$$\begin{aligned}\mathcal{W}_0(T) &= \{w \in (\mathcal{W}_0(\mathcal{T}_{R_o}[i](e)) \cap \mathcal{P}_0(T)) \setminus \mathcal{P}; t_{\mathcal{G}(\mathcal{T}_{R_o}[i](e))}(w) \in \mathcal{P}_0(T)\} \\ &= \{w \in (\mathcal{W}_0(\mathcal{T}_R[i](e)) \cap \mathcal{P}_0(T)) \setminus \mathcal{P}; t_{\mathcal{G}(\mathcal{T}_R[i](e))}(w) \in \mathcal{P}_0(T)\}\end{aligned}$$

So, $T \sqsubseteq_{\mathcal{P}} \mathcal{T}_R[i](e)$.

Let $p \in \mathcal{P}_0(T)$ and $q' \in \bigcup_{e_1 \in e(o)} \mathcal{P}_0(R\langle o, i, e_1\rangle)$ such that $p \frown_{\mathcal{T}_R[i](e)} q'$: We have $p \in \mathcal{Q}$. Since from $T \trianglelefteq_{\mathcal{P}} \mathcal{T}_{R_o}[i](e)$ and $(\forall e_1 \in e(o))\mathcal{P}_0(T) \cap \mathcal{P}_0(R_o\langle o, i, e_1\rangle) = \emptyset$ we deduce $\mathcal{P}_0(T) \cap \mathcal{Q} \subseteq \mathcal{P}$, we obtain $p \in \mathcal{P}$. We thus have $T \trianglelefteq_{\mathcal{P}} \mathcal{T}_R[i](e)$.

We showed $T \in \mathcal{S}^k_{\mathcal{T}_R[i](e)}(\mathcal{P})$.

4) Let $T \in \mathcal{S}^k_{\mathcal{T}_R[i](e)}(\mathcal{P})$ such that $(\forall e_1 \in e(o))\mathcal{P}_0(T) \cap \mathcal{P}_0(R\langle o, i, e_1\rangle) = \emptyset$.

Let $p \in \mathcal{P}_0(T)$ and $q' \in \bigcup_{e_1 \in e(o)} \mathcal{P}_0(R\langle o, i, e_1\rangle)$ such that $p \frown_{\mathcal{T}_{R_o}[i](e)} q'$: We have $p \in \mathcal{Q}$. Since from $T \trianglelefteq_{\mathcal{P}} \mathcal{T}_R[i](e)$ and $(\forall e_1 \in e(o))\mathcal{P}_0(T) \cap \mathcal{P}_0(R\langle o, i, e_1\rangle) = \emptyset$ we deduce $\mathcal{P}_0(T) \cap \mathcal{Q} \subseteq \mathcal{P}$, we obtain $p \in \mathcal{P}$. We thus have $T \trianglelefteq_{\mathcal{P}} \mathcal{T}_{R_o}[i](e)$.

We showed $T \in \mathcal{S}^k_{\mathcal{T}_{R_o}[i](e)}(\mathcal{P})$. □