

## A GHOST AT $\omega_1$

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**ABSTRACT.** In the final chain of the countable powerset functor, we show that the set at index  $\omega_1$ , regarded as a transition system, is not strongly extensional because it contains a “ghost” element that has no successor even though its component at each successor index is inhabited. The method, adapted from a construction of Forti and Honsell, also gives ghosts at larger ordinals in the final chain of other subfunctors of the powerset functor. This leads to a precise description of which sets in these final chains are strongly extensional.

*For Jirka Adámek on his 70th birthday, with thanks for his many contributions and the inspiration he has given to our community.*

### 1. INTRODUCTION

Initial algebras and final coalgebras of endofunctors are important in many areas of mathematics and computer science. One versatile way of constructing an initial algebra of  $F$  is to form the *initial chain* [Adá74], a transfinite sequence of objects  $(\mu^{(i)}F)_{i \in \text{Ord}}$  where  $\text{Ord}$  denotes the class of ordinals. We form the initial chain by applying  $F$  at each successor ordinal and taking a colimit at each limit ordinal. If it stabilizes, it yields an initial algebra  $\mu F$ . Dually [Bar93], we form the *final chain*  $(\nu^{(i)}F)_{i \in \text{Ord}}$ , by applying  $F$  at each successor ordinal and taking a limit at each limit ordinal. If it stabilizes, it yields a final coalgebra  $\nu F$ .

These constructions make sense for any endofunctor on any category, provided the relevant colimits or limits exist. But certain endofunctors on **Set** have received particular attention: the powerset functor ( $\mathcal{P}$ ) and its subfunctors, notably finite powerset ( $\mathcal{P}_f$ ) and countable powerset ( $\mathcal{P}_c$ ), which send a set to its set of finite or countable subsets respectively. That is because these functors have many applications, e.g. nondeterministic automata, the semantics of nondeterministic programs and the foundations of set theory. For cardinality reasons, the powerset functor has no initial algebra or final coalgebra; when we refer to  $\mu\mathcal{P}$  or  $\nu\mathcal{P}$ , these are proper classes. By contrast,  $\mathcal{P}_f$  and  $\mathcal{P}_c$  do have a (small) initial algebra and final coalgebra.

Now the initial chains of  $\mathcal{P}_f$  and  $\mathcal{P}_c$  are easy to understand: each is an increasing sequence of subsets of the initial algebra. But the final chains are more subtle. Their form was established in [ALM<sup>+</sup>15, Wor05].

- $\nu^{(\omega)}\mathcal{P}_f$  consists of the final coalgebra and some extra elements. The next  $\omega$  steps in the final chain of  $\mathcal{P}_f$  remove these extra elements, and the chain stabilizes at  $\omega + \omega$ .

- $\nu^{(\omega_1)}\mathcal{P}_c$  consists of the final coalgebra and some extra elements. The next  $\omega$  steps in the final chain of  $\mathcal{P}_c$  remove these extra elements, and the chain stabilizes at  $\omega_1 + \omega$ .

These descriptions may give the impression that the final chains of  $\mathcal{P}_f$  and  $\mathcal{P}_c$  are essentially similar. However, they tell only part of the story. For  $\nu^{(\omega)}\mathcal{P}_f$ , as well as being a *superset* of  $\nu\mathcal{P}_f$ , is also a *subset* of  $\nu\mathcal{P}$ . This is because it can be endowed with a transition relation, setting  $x \rightsquigarrow y$  when  $y_n \in x_{n+1}$  for all  $n \in \mathbb{N}$ , that makes it into a *strongly extensional* transition system, i.e. one where bisimilarity is equality. Specifically, Worrell [Wor05] characterized  $\nu^{(\omega)}\mathcal{P}_f$  as the set of *compactly branching* elements<sup>1</sup> of  $\nu\mathcal{P}$ , also studied in [Abr05, KKV04]. While all these elements are compactly branching, they are not all finitely (or even countably) branching, and the extra  $\omega$  steps are required to remove those that are not, so that only the finitely branching elements remain.

Is there a similar story for  $\nu^{(\omega_1)}\mathcal{P}_c$ ? The question is asked in [ALM<sup>+</sup>15]. This set too can be endowed with a transition relation, setting  $x \rightsquigarrow y$  when  $y_i \in x_{i+1}$  for all  $i < \omega_1$ . But the resulting transition system is not strongly extensional. We demonstrate this by giving two distinct elements that are “dead” in the sense of having no successor. One of these is the expected dead element: each component at a successor index is empty. The other’s components at successor index are all inhabited. Since the latter “appears to be alive”, yet is dead, we call it a *ghost*.

Thus we see a significant difference between  $\nu^{(\omega)}\mathcal{P}_f$  and  $\nu^{(\omega_1)}\mathcal{P}_c$ . An element of  $\nu^{(\omega)}\mathcal{P}_f$  may lie outside the final coalgebra of  $\mathcal{P}_f$ , but only because it is not finitely branching. By contrast, an element of  $\nu^{(\omega_1)}\mathcal{P}_c$  may lie outside the final coalgebra of  $\mathcal{P}_c$  despite having no successors at all.

**Structure of Paper.** Before introducing the final chain, Section 2 gives preliminaries on transition systems, cardinals, and the sequence of approximants to bisimilarity. Section 3 introduces the final chain of a general functor, and in particular of the powerset and restricted powerset functors. It also introduces the notions of channel and range that help us to understand these final chains.

The main narrative begins in Section 4, which endows the final chain of  $\mathcal{P}$  with the structure of a transition system and describe its basic properties. The rest of the paper is devoted to studying this system.

In Section 5 our focus is on those properties that hold specifically at *countable* ordinals. This leads us in Section 6 to the question of whether these properties—most importantly, strong extensionality—hold at  $\omega_1$ . We prove that there is a ghost and deduce that these properties do not hold.

The ghost is obtained by adapting a method in [FH89]. The next two sections give further results that showcase the power of this method.

- Section 7 answers the question: how many successors may an element of the final chain of  $\mathcal{P}_c$  have?
- Section 8 looks at larger subfunctors of  $\mathcal{P}$  corresponding to subsets of larger cardinality, and their final chain beyond  $\omega_1$ . Remarkably, we see a sharp division: at each limit ordinal  $i$ , either the  $i$ th set is strongly extensional or there is a ghost. We thus revisit each of the properties we initially proved at countable ordinals, including surjectivity of connecting maps, and completely characterize the ordinals at which they hold.

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<sup>1</sup>Elements of  $\nu\mathcal{P}$  are represented in [ALM<sup>+</sup>15, Sch10, Wor05] as strongly extensional trees modulo isomorphism. Those trees are unrelated to the trees used in this paper.

We end with a discussion of related work in Section 9 and conclusions in Section 10.

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## 2. PRELIMINARIES

**2.1. Pointed Systems.** A *transition system* consists of a set  $X$  and relation  $\rightsquigarrow \subseteq X \times X$ . When  $x \rightsquigarrow y$  we say that  $y$  is a *successor* of  $x$ . A subset  $U \subseteq X$  is a *subsystem* when, for all  $x \in U$ , every successor of  $x$  is in  $U$ . An *embedding* of transition systems  $e: M \rightarrow N$  is an isomorphism from  $M$  to a subsystem of  $N$ .

A *pointed system* consists of a transition system  $M = (X, \rightsquigarrow)$  and an element  $x \in X$ . Given a family of pointed systems  $((M_i, x_i))_{i \in I}$ , a *parent* of the family is a pointed system  $(N, y)$  equipped with embeddings  $(e_i: M_i \rightarrow N)_{i \in I}$  such that the successors of  $y$  are listed without repetition as  $(e_i(x_i))_{i \in I}$ . A parent may be constructed as follows: if each  $M_i$  is  $(X_i, \rightsquigarrow_i)$ , define

$$\begin{aligned} N &\stackrel{\text{def}}{=} \left(1 + \sum_{i \in I} X_i, \rightsquigarrow\right) \quad \text{with } \rightsquigarrow \text{ given in the evident way} \\ y &\stackrel{\text{def}}{=} \text{inl } * \\ e_i &: z \mapsto \text{inr}(i, z) \end{aligned}$$

We generalize transition systems to coalgebras. Let  $F$  be an endofunctor on **Set**, e.g. the powerset functor  $\mathcal{P}$ , which sends a set  $X$  to  $\mathcal{P}X$  and a function  $f: A \rightarrow B$  to  $U \mapsto \{f(a) \mid a \in U\}$ . An  *$F$ -coalgebra* consists of a set  $X$  and map  $X \rightarrow FX$ . Thus a transition system corresponds to a  $\mathcal{P}$ -coalgebra. An  *$F$ -coalgebra morphism*  $(X, \zeta) \rightarrow (Y, \xi)$  is a map  $f: X \rightarrow Y$  such that

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow \zeta & & \downarrow \xi \\ FX & \xrightarrow{Ff} & FY \end{array}$$

commutes. For example, an embedding of transition systems corresponds to an injective  $\mathcal{P}$ -coalgebra morphism. A *pointed  $F$ -coalgebra* consists of an  $F$ -coalgebra  $M = (X, \zeta)$  together with an element  $x \in X$ .

We often write a pointed system or pointed coalgebra  $(M, x)$  as just  $x$ , leaving  $M$  implicit.

**2.2. Cardinals.** As usual, we identify a cardinal with the least ordinal of that cardinality. Thus we identify  $\aleph_0$  with  $\omega$  and  $\aleph_1$  with  $\omega_1$ —the least uncountable ordinal. The special symbol  $\infty$  is treated as greater than every cardinal and ordinal.

Let  $\kappa$  be either a cardinal or  $\infty$ . For a set  $X$ , we write  $\mathcal{P}_{<\kappa}X$  for the set of its  $<\kappa$ -sized subsets. This gives a subfunctor  $\mathcal{P}_{<\kappa}$  of  $\mathcal{P}$ . In particular:

$$\begin{aligned} \mathcal{P}_f &= \mathcal{P}_{<\aleph_0} \\ \mathcal{P}_c &= \mathcal{P}_{<\aleph_1} \\ \mathcal{P} &= \mathcal{P}_{<\infty} \end{aligned}$$

We also refer to the functor  $\mathcal{P}_{<\kappa}^+$ , that sends a set  $X$  to the set of  $<\kappa$ -sized subsets that are inhabited.

A pointed system  $(M, x)$  is  $<\kappa$ -branching when  $x \rightsquigarrow^* y$  implies that  $y$  has  $<\kappa$ -many successors. This is equivalent to  $(M', x)$  being a pointed  $\mathcal{P}_{<\kappa}$ -coalgebra, for some subsystem  $M'$  of  $M$ .

Henceforth we assume that  $\kappa$  is *infinite and regular*. (In particular  $\aleph_0$  is regular, as are infinite successor cardinals such as  $\aleph_1$ . But  $\aleph_\omega$ , for example, is not. We deem  $\infty$  to be regular.) This makes  $\mathcal{P}_{<\kappa}$  a submonad of  $\mathcal{P}$ , and it also gives the following.

**Proposition 2.1.** *For a  $<\kappa$ -sized set  $A$ , any decreasing sequence of subsets  $(X_i)_{i<\kappa}$  is eventually constant. In particular, if each  $X_i$  is inhabited then so is  $\bigcap_{i<\kappa} X_i$ .*

**2.3. Approximants to bisimilarity.** The class of pointed systems is equipped with a decreasing sequence  $(\sim_i)_{i \in \text{Ord}}$  of equivalence relations, defined as follows [Mal76, Mil89].

- $x \sim_{i+1} x'$  when for every  $x \rightsquigarrow y$  there is  $x' \rightsquigarrow y'$  such that  $y \sim_i y'$ , and *vice versa*.
- If  $i$  is a limit, then  $x \sim_i x'$  when for all  $j < i$  we have  $x \sim_j x'$ .
- We deem 0 a limit, so  $x \sim_0 x'$  always.

We say that  $x$  and  $x'$  are *bisimilar*, written  $x \sim x'$ , when for all  $i \in \text{Ord}$  we have  $x \sim_i x'$ . In particular, for any embedding  $e: M \rightarrow N$  we have  $(N, e(x)) \sim (M, x)$ . The following is a key tool for analyzing these relations.

**Definition 2.2.** Pointed systems  $(v_i)_{i \in \text{Ord}}$  are defined as follows:  $v_i$  is a parent of  $(v_j)_{j < i}$ .

Essentially Definition 2.2 is the von Neumann encoding of the ordinals. It is useful because of the following facts.

**Proposition 2.3.** *Let  $i$  and  $j$  be ordinals.*

- (1) *For an ordinal  $k$ , we have  $v_i \sim_k v_j$  iff either  $i = j$  or  $k \leq i, j$ .*
- (2) *We have  $v_i \sim v_j$  iff  $i = j$ .*

*Proof.* Part (1) is by induction on  $k$  and part (2) follows. □

The approximants to bisimilarity have the following properties.

**Proposition 2.4.**

- (1) *Let  $x$  be an  $<\kappa$ -branching pointed system and  $y$  a pointed system. Then  $x \sim y$  iff, for all  $i < \kappa$ , we have  $x \sim_i y$ .*
- (2) *For  $i < \kappa$ , there are  $<\kappa$ -branching pointed systems  $x$  and  $y$  such that  $x \sim_i y$  but  $x \not\sim_{i+1} y$ .*

*Proof.*

- (1) See [BM96][Lemma 11.13] following [vG87], and also [AILS12][Theorem B.1(3)].
- (2) Take  $v_i$  and  $v_{i+1}$ . □

**Proposition 2.5.** *Let  $x$  be an  $<\kappa$ -branching pointed system and  $y$  a pointed system. Then  $x$  has a successor  $z$  such that  $z \sim y$  iff, for all  $i < \kappa$ , it has a successor  $z$  such that  $z \sim_i y$ .*

*Proof.*  $(\Rightarrow)$  is obvious. For  $(\Leftarrow)$ , by Proposition 2.1,  $x$  has a successor  $z$  such that  $z \sim_j y$  for all  $j < \kappa$ , i.e.  $z \sim_\kappa y$ . Since  $z$  is  $<\kappa$ -branching, Proposition 2.4 (1) gives  $z \sim y$ . □

### 3. THE FINAL CHAIN

**3.1. Constructing the Final Chain.** Our treatment of the final chain relies on the following. Let  $I$  be a well-ordered set.

**Definition 3.1.** An *inverse  $I$ -chain* is a functor  $D : I^{\text{op}} \rightarrow \mathbf{Set}$ . Explicitly, it consists of

- for all  $i \in I$ , a set  $D_i$ , called the  $i$ -th level
- for all  $j \leq i \in I$ , a *connecting map*  $D_{i \geq j} : D_j \rightarrow D_i$

such that  $D_{i \geq i} = \text{id}_{D_i}$  for all  $i \in I$ , and  $D_i \xrightarrow{D_{i \geq j}} D_j$  commutes for all  $k \leq j \leq i \in I$ .

$$\begin{array}{ccc} D_i & \xrightarrow{D_{i \geq j}} & D_j \\ & \searrow^{D_{i \geq k}} & \downarrow^{D_{j \geq k}} \\ & & D_k \end{array}$$

An inverse  $I$ -chain is a kind of tree [Jec03, BMSS12, BBM14]. It may be represented as a poset  $A$  in which, for every  $x \in A$ , the set  $\{y \in A \mid x < y\}$  is well-ordered with order-type less than that of  $I$ . This intuitive picture gives rise to the following notions.

**Definition 3.2.** Let  $D$  be an inverse  $I$ -chain.

- (1) If  $I$  has a least element  $0$ , then an element of  $D_0$  is a *root*.
- (2) For  $j \leq i \in I$  and  $x \in D_j$ , an  *$i$ -development* of  $x$  is an element  $y \in D_i$  such that  $D_{i \geq j}y = x$ .
- (3) A *full branch* of  $D$  is an indexed tuple  $(x_i)_{i \in I}$  with  $x_i \in D_i$  for all  $i \in I$  and  $D_{i \geq j}x_i = x_j$  for all  $j \leq i \in I$ . (If  $I$  is empty, the empty tuple  $()$  is the sole full branch.)

We now proceed to define the final chain of an endofunctor  $F$  on  $\mathbf{Set}$ .

**Definition 3.3.** [Adá74, Bar93] The *final chain* of  $F$  is an inverse  $\text{Ord}$ -chain, with  $i$ th level written  $\nu^{(i)}F$  and connecting map written  $\nu^{(i \geq j)}F : \nu^{(i)}F \rightarrow \nu^{(j)}F$  or  $\nu^{(i \geq j)}$  for short. These sets and maps are given as follows.

- $\nu^{(i+1)}F = F\nu^{(i)}F$ .
- $\nu^{(i+1 \geq j+1)} = F\nu^{(i \geq j)}$  for  $j \leq i$ .
- If  $i$  is a limit then  $\nu^{(i)}F$  is the limit of  $(\nu^{(j)}F)_{j < i}$ , i.e. the set of full branches. For  $j < i$ , the map  $\nu^{(i \geq j)}$  is  $\pi_j : (x_j)_{j < i} \mapsto x_j$ .
- Since we deem  $0$  a limit ordinal, we have  $\nu^{(0)}F = \{()\}$ .

Thus

$$\nu^{(i+1 \geq j+1)}\mathcal{P}_{<\kappa} : a \mapsto \{\nu^{(i \geq j)}b \mid b \in a\}$$

We define a full branch  $(0_i)_{i \in \text{Ord}}$  through the final chain of  $\mathcal{P}_{<\kappa}$ , as follows:

- $0_{i+1} \stackrel{\text{def}}{=} \emptyset$ .
- If  $i$  is a limit then  $0_i \stackrel{\text{def}}{=} (0_j)_{j < i}$ .

For  $i > 0$ , note that  $0_i$  is the unique  $i$ -development of  $0_1$ .

**3.2. Coalgebra projections.** Again let  $F$  be an endofunctor on  $\mathbf{Set}$ .

**Definition 3.4.** For an  $F$ -coalgebra  $(X, \zeta)$  and ordinal  $i$ , the  $i$ th *coalgebra projection*, written  $p_i^{(X, \zeta)}$  or just  $p_i$ , is the map  $X \rightarrow \nu^{(i)}F$  given as follows<sup>2</sup>.

<sup>2</sup>In the language of [CUV09], the pair  $(\nu^{(i)}F, \nu^{(i+1 \geq i)})$  is a *corecursive  $F$ -algebra*, and  $p_i^{(X, \zeta)}$  is the unique map from  $(X, \zeta)$  to it. The connecting maps are  $F$ -algebra morphisms.

- The map  $p_{i+1}$  is  $X \xrightarrow{\zeta} FX \xrightarrow{Fp_i} F\nu^{(i)}F$
- For a limit  $i$ , the map  $p_i$  is  $(p_j)_{j<i}$ .

We may summarize as follows.

- Each pointed  $F$ -coalgebra  $x$  gives rise to a full branch  $(p_i x)_{i \in \text{Ord}}$  through the final chain of  $F$ .
- For any  $F$ -coalgebra morphism  $f: (X, \zeta) \rightarrow (Y, \xi)$  and  $x \in X$ , the full branches  $(p_i x)_{i \in \text{Ord}}$  and  $(p_i f(x))_{i \in \text{Ord}}$  are equal.

For a pointed system  $x$ , we have  $p_{i+1}x = \{p_i y \mid x \rightsquigarrow y\}$ . The coalgebra projections are related to bisimilarity and its approximants as follows.

**Proposition 3.5.** *Let  $x$  and  $y$  be pointed systems.*

- (1) *For any ordinal  $i$ , we have  $x \sim_i y$  iff  $p_i x = p_i y$ .*
- (2) *We have  $x \sim y$  iff the full branches  $(p_i x)_{i \in \text{Ord}}$  and  $(p_i y)_{i \in \text{Ord}}$  are equal.*

*Proof.* Part (1) is by induction on  $i$  and part (2) follows. □

**Definition 3.6.** An element  $a \in \nu^{(i)}\mathcal{P}_f$  is said to be *F-coalgebraic* when it is of the form  $p_i x$  for a pointed  $F$ -coalgebra  $x$ . We write  $\nu_{\text{coalg}}^{(i)}F$  for the set of  $F$ -coalgebraic elements of  $\nu^{(i)}F$ .

Thus  $\nu_{\text{coalg}}^{(i)}\mathcal{P}_{<\kappa}$  corresponds to the class of  $<\kappa$ -branching pointed systems modulo  $\sim_i$ . Note that  $a \in \nu^{(i+1)}\mathcal{P}_{<\kappa}$  is  $\mathcal{P}_{<\kappa}$ -coalgebraic iff all its elements are, by the parent construction.

**Definition 3.7.** Let  $i$  be a limit. An element of  $\nu^{(i)}F$  is *F-Cauchy* when all of its components are  $F$ -coalgebraic.

Thus the set of  $F$ -Cauchy elements is  $\lim_{j<i} \nu_{\text{coalg}}^{(j)}F$ . Note that  $F$ -coalgebraic implies  $F$ -Cauchy, but the converse need not be the case. (For example, for  $F: X \mapsto 0$ , the element  $() \in \nu^{(0)}F$  is  $F$ -Cauchy but not  $F$ -coalgebraic.) As explained in Section 9.3, the “Cauchy” terminology comes from the notion of Cauchy sequence.

**3.3. Channels and Ranges.** The following notions are key to understanding the final chain of  $\mathcal{P}_{<\kappa}$ .

**Definition 3.8.** Let  $I$  be a well-ordered set, and  $D$  an inverse  $I$ -chain.

- (1) A *channel* through  $D$  consists of a subset  $C_i \subseteq D_i$  for all  $i \in I$ , such that for all  $j \leq i \in I$ 
  - if  $x \in C_i$  then  $D_{i \geq j} x \in C_j$
  - every  $y \in C_j$  has an  $i$ -development in  $C_i$ .
- (2) For any set  $U$  of full branches of  $D$ , the *range* of  $U$  is the channel whose  $i$ th level is  $\{x_i \mid x \in U\}$ .

Note the following:

- Any channel through  $D$  with a root has all levels inhabited.
- $\lim_{i \in I} \mathcal{P}_{<\kappa} D_i$  is the set of channels through  $D$  with all levels  $<\kappa$ -sized.
- The function  $(\mathcal{P}_{<\kappa} \pi_i)_{i \in I}: \mathcal{P}_{<\kappa} \lim_{i \in I} D_i \rightarrow \lim_{i \in I} \mathcal{P}_{<\kappa} D_i$  sends a  $<\kappa$ -sized set of full branches to its range.

For general  $F$ , we have the following.

**Proposition 3.9.** *Let  $i$  be an ordinal.*

- (1) The map  $\Psi_i \stackrel{\text{def}}{=} (\nu^{(i \geq j)})_{j < i}: \nu^{(i)}F \cong \lim_{j < i} F\nu^{(j)}F$  is an isomorphism (bijection).  
(2) If  $i$  is a limit, the following commutes:

$$\begin{array}{ccccc}
\nu^{(i)}F & = & \lim_{j < i} \nu^{(j)}F & \xleftarrow{\nu^{(i+1 \geq i)}} & F \lim_{j < i} \nu^{(j)}F & = & \nu^{(i+1)}F \\
& & \Psi_i \downarrow \cong & & \swarrow (F\pi_j)_{j < i} & & \\
& & \lim_{j < i} F\nu^{(j)}F & & & & 
\end{array}$$

Consider the case  $F = \mathcal{P}_{<\kappa}$ . For a limit  $i$ , the elements of  $\nu^{(i)}\mathcal{P}_{<\kappa}$  are, by definition, the *full branches* of the inverse chain  $(\nu^{(j)}\mathcal{P}_{<\kappa})_{j < i}$ . But they correspond via  $\Psi_i$  to the *channels* through the same inverse chain. We repeatedly move between these two viewpoints—full branches and channels—in this paper. For example, the map  $\nu^{(i+1 \geq i)}$  is hard to grasp directly, but by Proposition 3.9(2) it corresponds to  $(\mathcal{P}_{<\kappa}\pi_j)_{j < i}$ , which sends a set of full branches to its range.

**3.4. Injectivity of Connecting Maps.** A key question in the study of final chains (as with initial chains) is stabilization [TAKR75, AK95, AT11, Wor05, ALM<sup>+</sup>15, AP15].

**Proposition 3.10.** *Suppose the final chain of  $F$  is stable at  $i$ , i.e.  $\nu^{(i+1 \geq i)}$  is bijective.*

- $\nu^{(k \geq j)}$  is bijective for all  $k \geq j \geq i$ .
- The  $F$ -coalgebra  $(\nu^{(i)}F, (\nu^{(i+1 \geq i)})^{-1})$  is final, with the unique coalgebra map from any coalgebra  $M$  given by  $p_i^M$ .
- The  $j$ th projection from this final coalgebra is given by  $\nu^{(i \geq j)}$  for  $j \leq i$ , and by  $(\nu^{(j \geq i)})^{-1}$  for  $j \geq i$ .

By Proposition 3.9(2), for any limit  $i$ , if  $F$  preserves limits of inverse  $i$ -chains then its final chain is stable at  $i$ . This fact is useful for a *polynomial* functor  $F$ , i.e. one of the form  $X \mapsto \sum_{i \in I} X^{A_i}$ . Since it preserves limits of connected diagrams, its final chain is stable at  $\omega$  [Bar93]. However,  $\mathcal{P}_{<\kappa}$  requires a more subtle analysis, given in [Wor05], that we now reprise.

**Proposition 3.11.** *Let  $F$  be an endofunctor on **Set** that preserves injections and intersections (such as  $\mathcal{P}_{<\kappa}$  or  $\mathcal{P}_{<\kappa}^+$ ). Suppose  $\nu^{(i+1 \geq i)}F$  is an injection. Then  $\nu^{(k \geq j)}F$  is an injection for all  $k \geq j \geq i$ , and the final chain is stable at  $i + \omega$ .*

A functor  $F$  is said to preserve the limit of an inverse  $I$ -chain  $D$  up to an injection (surjection) when the map

$$(F\pi_i)_{i \in I}: F \lim_{i \in I} D_i \rightarrow \lim_{i \in I} FD_i$$

is injective (surjective).

**Proposition 3.12.** [Wor05] *Let  $\kappa < \infty$ . Then  $\mathcal{P}_{<\kappa}$  and  $\mathcal{P}_{<\kappa}^+$  preserve, up to an injection, the limits of any inverse  $\kappa$ -chain  $D$ . Explicitly: if  $U$  and  $V$  are  $<\kappa$ -sized sets of full branches through  $D$  and have the same range, then  $U = V$ .*

*Proof.* Let  $a \in U$ . For  $i < \kappa$ , let  $V(i)$  be the set of  $b \in V$  such that  $b_i = a_i$ . Since  $V$  is  $<\kappa$ -sized,  $\bigcap_{i < \kappa} V(i)$  is inhabited by Proposition 2.1, so  $a \in V$ .  $\square$

Thus  $\nu^{(\kappa+1 \geq \kappa)}(\mathcal{P}_{<\kappa})$  is injective and the final chain of  $\mathcal{P}_{<\kappa}$  is stable at  $\kappa + \omega$ . As proved in [ALM<sup>+</sup>15] and reprised below (Proposition 4.13), it is not stable at any smaller ordinal.

We can now say precisely which connecting maps are injective.

**Proposition 3.13.** *For ordinals  $j \leq i$ , the following are equivalent.*

- (1) *The connecting map  $\nu^{(i \geq j)} \mathcal{P}_{< \kappa}$  is an injection.*
- (2) *Either  $j = i$  or  $j \geq \kappa$  (for  $\kappa < \infty$ ).*

*Proof.* (2)  $\Rightarrow$  (1) follows from Propositions 3.12 and 3.11. For the converse, if  $j < \kappa$ , then Proposition 2.4(2) shows that  $\nu^{(j+1 \geq j)} \mathcal{P}_{< \kappa}$  is not injective.  $\square$

### 3.5. Surjectivity of connecting maps.

**Proposition 3.14.** *For any ordinal  $i$ , the following are equivalent.*

- (1) *The map  $\nu^{(i+1 \geq i)} F$  is surjective.*
- (2) *For all  $h \geq i$ , the map  $\nu^{(h \geq i)} F$  is surjective.*
- (3) *All elements of  $\nu^{(i)} F$  are  $F$ -coalgebraic.*

*Proof.* (3)  $\Rightarrow$  (2)  $\Rightarrow$  (1): Trivial.

(1)  $\Rightarrow$  (3): By the Axiom of Choice,  $\nu^{(i+1 \geq i)}$  has a section  $\zeta : \nu^{(i)} F \rightarrow \nu^{(i+1)} F$ . Put  $M = (\nu^{(i)} F, \zeta)$ . For all  $j \leq i$ , we show  $p_j^M = \nu^{(i \geq j)}$  by induction on  $j$ . The limit case is trivial. For the successor case,  $p_j^M = \nu^{(i \geq j)}$  implies  $Fp_j^M = F\nu^{(i \geq j)} = \nu^{(i+1 \geq j+1)}$  and hence

$$\begin{array}{ccccc}
 \nu^{(i)} F & & & & \nu^{(j+1)} F \\
 & \searrow^{p_{j+1}^M} & & & \nearrow \\
 & & \nu^{(i+1)} F & \xrightarrow{Fp_j^M} & \\
 & \swarrow_{\zeta} & & & \\
 \nu^{(i)} F & & & & \nu^{(j+1)} F \\
 & \searrow_{\nu^{(i+1 \geq i)}} & & & \nearrow_{\nu^{(i \geq j+1)}} \\
 \nu^{(i)} F & & & & \nu^{(j+1)} F \\
 & \swarrow_{\text{id}} & & & \nwarrow
 \end{array}$$

commutes. We conclude that  $p_i^M$  is the identity.  $\square$

**Proposition 3.15.** *Let  $\kappa > \aleph_0$ . Then  $\mathcal{P}_{< \kappa}$  and  $\mathcal{P}_{< \kappa}^+$  preserve, up to a surjection, the limit of any inverse  $\omega$ -chain  $D$ . Explicitly: any channel  $C$  through  $D$  with all levels  $< \kappa$ -sized is the range of some  $< \kappa$ -sized set of full branches.*

*Proof.* For  $n \in \mathbb{N}$ , each  $x \in C_n$  extends by dependent choice to a full branch through  $D$ . For each  $n \in \mathbb{N}$  and  $x \in C_n$ , choose such a branch  $\theta_n(x)$ , by the Axiom of Choice. Then the set  $\{\theta_n(x) \mid n \in \mathbb{N}, x \in C_n\}$  is  $< \kappa$ -sized because  $< \kappa$  is uncountable and regular, and has range  $C$ .  $\square$

**Corollary 3.16.** *Let  $\kappa > \aleph_0$ .*

- (1) *For  $i < \omega_1$  and  $h \geq i$ , the connecting map  $\nu^{(h \geq i)} \mathcal{P}_{< \kappa}$  is surjective.*
- (2) *For  $i < \omega_1$ , every element of  $\nu^{(i)} \mathcal{P}_{< \kappa}$  is  $\mathcal{P}_{< \kappa}$ -coalgebraic.*
- (3) *Every element of  $\nu^{(\omega_1)} \mathcal{P}_{< \kappa}$  is  $\mathcal{P}_{< \kappa}$ -Cauchy.*

*Proof.* We just need to prove  $\nu^{(i+1 \geq i)} : \nu^{(i)} \mathcal{P}_{< \kappa} \rightarrow \nu^{(i+1)} \mathcal{P}_{< \kappa}$  surjective, for  $i < \omega_1$ . The case  $i = 0$  is evident. If  $i$  is a positive limit, take a strictly increasing sequence  $(i_n)_{n \in \mathbb{N}}$  with supremum  $i$ , then apply Proposition 3.15. The successor case follows from the fact that  $\mathcal{P}_{< \kappa}$  preserves surjections.  $\square$



## 4. THE FINAL CHAIN AS A TRANSITION SYSTEM

**4.1. Full powerset.** We shall now explain how the final chain of  $\mathcal{P}$  constitutes a transition system.

**Definition 4.1.** The *predecessor* of an ordinal is defined as follows:

- $\text{pred}(i+1) \stackrel{\text{def}}{=} i$ .
- If  $i$  is a limit,  $\text{pred}(i) \stackrel{\text{def}}{=} i$ .
- Since we deem 0 a limit,  $\text{pred}(0) = 0$ .

In general,  $\text{pred}(i)$  is the supremum of all ordinals less than  $i$ .

Note, by the way, that  $\text{pred}$  is the left adjoint of the successor function. We come to the main definition:

**Definition 4.2.** The transition relation  $\rightsquigarrow_i$  from  $\nu^{(i)}\mathcal{P}$  to  $\nu^{(\text{pred}(i))}\mathcal{P}$  is defined as follows.

- For  $a \in \nu^{(i+1)}\mathcal{P}$  and  $b \in \nu^{(i)}\mathcal{P}$ , we set  $a \rightsquigarrow_{i+1} b$  when  $b \in a$ .
- If  $i$  is a limit, then for  $a, b \in \nu^{(i)}\mathcal{P}$ , we set  $a \rightsquigarrow_i b$  when  $b_j \in a_{j+1}$  for all  $j < i$ .
- Since we deem 0 a limit,  $() \rightsquigarrow_0 ()$ .

In general,  $a \rightsquigarrow_i b$  when for all  $j < i$  we have  $\nu^{(\text{pred}(i) \geq j)}b \in \nu^{(i \geq j+1)}a$ .

Note that connecting maps and coalgebra projections preserve transition:

**Proposition 4.3.**

- (1) For  $j \leq i$  and  $a \in \nu^{(i)}\mathcal{P}$ , if  $a \rightsquigarrow_i b$  then  $\nu^{(i \geq j)}a \rightsquigarrow_j \nu^{(\text{pred}(i) \geq \text{pred}(j))}b$ .
- (2) For a pointed system  $x$  and ordinal  $i$ , if  $x \rightsquigarrow y$  then  $p_i x \rightsquigarrow_i p_{\text{pred}(i)} x$ .

Let us translate Definition 4.2 into the language of Section 3.3. For general  $F$ , the map

$$\Phi_i \stackrel{\text{def}}{=} (\nu^{(\text{pred}(i) \geq j})}_{j < i} : \nu^{(\text{pred}(i))}F \rightarrow \lim_{j < i} \nu^{(j)}F$$

is an isomorphism (bijection)—indeed the identity if  $i$  is a limit. For  $a \in \nu^{(i)}\mathcal{P}$  and  $b \in \nu^{(\text{pred}(i))}\mathcal{P}$  we note that  $\Psi_i(a)$  is a channel, and  $\Phi_i(b)$  a full branch, through  $(\nu^{(j)}\mathcal{P})_{j < i}$ . We have  $a \rightsquigarrow_i b$  when  $\Phi_i(b)$  is a full branch through  $\Psi_i(a)$ .

**4.2. Restricted powerset.** The final chain of  $\mathcal{P}_{<\kappa}$  forms a subsystem of the final chain of  $\mathcal{P}$ , in the following sense.

**Proposition 4.4.** Let  $a \in \nu^{(i)}\mathcal{P}_{<\kappa}$  and  $a \rightsquigarrow_i b$  then  $b \in \nu^{(\text{pred}(i))}\mathcal{P}_{<\kappa}$ .

*Proof.* Since  $\Psi_i(a)$  is a channel through  $(\nu^{(j)}\mathcal{P}_{<\kappa})_{j < i}$  (with all levels  $<\kappa$ -sized), every full branch through it is a full branch through  $(\nu^{(j)}\mathcal{P}_{<\kappa})_{j < i}$ .  $\square$

As we shall see, the subsystem  $(\nu^{(i)}\mathcal{P}_{<\kappa})_{i \in \text{Ord}}$  is not  $<\kappa$ -branching (Proposition 4.12 below). Furthermore, the  $\mathcal{P}_{<\kappa}$ -coalgebraic elements need not form a subsystem (Proposition 8.22 below). However, the  $\mathcal{P}_{<\kappa}$ -Cauchy elements do form a subsystem.

Suppose now that  $\kappa < \infty$ . Beyond  $\kappa$ , we recall that connecting maps are injections (Proposition 3.13). They are moreover transition system embeddings in the following sense.

**Proposition 4.5.** For  $i \geq j \geq \kappa$  and  $a \in \nu^{(i)}\mathcal{P}_{<\kappa}$ , the map  $\nu^{(\text{pred}(i) \geq \text{pred}(j))}$  is a bijection from the  $\rightsquigarrow_i$ -successors of  $a$  to the  $\rightsquigarrow_j$ -successors of  $\nu^{(i \geq j)}a$ .

*Proof.* This says that, if  $C$  is a channel through  $(\nu^{(k)}\mathcal{P}_{<\kappa})_{k<i}$  with all levels  $<\kappa$ -sized, then each full branch  $b$  of  $C \upharpoonright_j$  extends uniquely to a full branch  $a$  of  $C$ . For  $k < i$  we define  $a_k$  as follows.

- If  $k \geq \kappa$ , then for each  $l < \kappa$  the set  $R_l$  of  $a \in C_k$  such that  $a_l = b_l$  is inhabited, so  $\bigcap_{l \in \kappa} R_l$  has an element by Proposition 2.1, unique since  $\nu^{(k \geq \kappa)}$  is injective. Let  $a_k$  be this element.
- If  $k < \kappa$ , set  $a_k \stackrel{\text{def}}{=} b_k$ . □

Hence, beyond  $\kappa + \omega$ , the transition system agrees with the final coalgebra structure given by Proposition 3.10:

**Corollary 4.6.** *For  $i \geq \kappa + \omega$  and  $a \in \nu^{(i)}\mathcal{P}_{<\kappa}$ , the map  $\nu^{(i \geq \text{pred}(i))}$  is a bijection from the elements of  $(\nu^{(i+1 \geq i)})^{-1}a$  to the  $\rightsquigarrow_i$ -successors of  $a$ .*

**4.3. Examples.** The following result is adapted from [FH89].

**Proposition 4.7.** *Let  $x$  and  $y$  be pointed systems and  $i \in \text{Ord}$ . We have  $p_i x \rightsquigarrow_i p_{\text{pred}(i)} y$  iff, for every  $j < i$ , the pointed system  $x$  has a successor  $z$  such that  $z \sim_j y$ . (Cf. Proposition 2.5.)*

*Proof.* Because  $p_i x \rightsquigarrow_i p_{\text{pred}(i)} y$  iff for every  $j < i$  we have  $p_j y \in p_{j+1} x$ , and the latter is  $\{p_j z \mid x \rightsquigarrow z\}$ . □

In several cases we shall precisely describe the  $\rightsquigarrow_i$ -successors of an element.

**Proposition 4.8.** *For any  $i > 0$ , the element  $0_i$  has no successors.*

*Proof.* If  $0_i \rightsquigarrow_i b$  then  $b_0 \in 0_1 = \emptyset$ . □

**Proposition 4.9.** *Let  $\kappa < \infty$  and let  $x$  be an  $<\kappa$ -branching pointed system. For  $i \geq \kappa$ , the set of successors of  $p_i x$  is  $\{p_{\text{pred}(i)} y \mid x \rightsquigarrow y\}$ .*

*Proof.* Let  $j$  be the maximum of  $i$  and  $\kappa + \omega$ . If  $p_i x \rightsquigarrow b$ , then, by Proposition 4.5,  $p_{j+1} x$  has a unique element  $c$  such that  $\nu^{(j \geq \text{pred}(i))} c = b$ . By the final coalgebra property,  $x$  has a successor  $y$  such that  $p_j y = c$  and hence  $p_{\text{pred}(i)} y = b$ . □

**Proposition 4.10.** *Let  $x$  be a pointed system with a unique successor  $y$ . For any ordinal  $i$ , the element  $p_i x$  has unique successor  $p_{\text{pred}(i)} y$ .*

*Proof.* If  $p_i x \rightsquigarrow b$  then for all  $j < i$  we have  $b_j \in p_{j+1} x = \{p_j y\}$  so  $b_j = p_j y$ . Thus  $b = p_{\text{pred}(i)} y$ . □

We next consider the von Neumann ordinals.

**Proposition 4.11.** *For ordinals  $j \leq i$ , the successors of  $p_i v_j$  are listed without repetition*

- as  $(p_{\text{pred}(i)} v_k)_{k < j}$ , if  $j < i$
- as  $(p_{\text{pred}(i)} v_k)_{k \leq \text{pred}(i)}$ , if  $j = i$ .

*Proof.* Proposition 4.7 tells us that these are indeed successors. Uniqueness is by Proposition 2.3.

Let  $j \leq i$  and  $p_i v_j \rightsquigarrow x$ . For every  $k < i$ , we have  $x_k \in p_{k+1} v_j$ , which is  $\{p_k y \mid v_j \rightsquigarrow y\}$ . Thus  $x_k$  is expressible as  $p_k y$  for some successor  $y$  of  $v_j$ , i.e.  $y = v_l$  for some  $l < j$ . There is a unique such  $l$  that is  $\leq k$ ; we call it  $g(k)$ .

Let  $C$  be the set of  $k < i$  such that  $g(k) = k$  and  $D$  its complement, i.e. the set of  $k < i$  such that  $g(k) < k$ .

For any  $k \leq l < i$  we have  $p_k v_{g(k)} = p_k v_{g(l)}$  i.e.  $t_{g(k)} \sim_k t_{g(l)}$ . This can be unpacked as follows.

- (1) If  $k \in C$  then  $g(l) \geq k$ .
  - (2) if  $k \in D$  then  $g(l) = g(k)$ .
- (2) implies that  $g(l) = g(k) < k \leq l$ , so  $l \in D$ . Thus  $D$  is upper and  $C$  is lower.

Let  $k$  be the supremum of  $C$ . Since  $\text{pred}(i)$  is an upper bound for  $C$  we have  $k \leq \text{pred}(i)$ .

We show that  $x = p_{\text{pred}(i)} v_k$ , i.e. that for any  $l < i$  we have  $g(l) = l \sqcap k$ . If  $l < k$  this holds because  $l \in C$ . If  $l \geq k$ , we must show  $g(l) = k$ . Firstly, (1) says that  $g(l)$  is an upper bound for  $C$ , hence  $g(l) \geq k$ . If  $l = k$  we are done. If  $l > k$  then  $k + 1 \leq l < i$ , and  $k \leq g(k + 1) \leq k + 1$ . Since  $k + 1 \notin C$ , we have  $g(k + 1) = k$  and hence by (2),  $g(l) = k$ .

Since  $j$  is an upper bound for  $C$ , we have  $k \leq j$ . If  $j < i$ , then  $j \in D$  but  $k \in C$  so  $k < j$ .  $\square$

**Proposition 4.12.** *For  $\kappa < \infty$ , the set  $\nu^{(\kappa)} \mathcal{P}_{<\kappa}$  has a  $\mathcal{P}_{<\kappa}$ -Cauchy element with precisely  $\kappa$  successors.*

*Proof.* Put  $e \stackrel{\text{def}}{=} p_\kappa v_\kappa$ . For  $i < \kappa$ , the component  $e_i$  is  $p_i v_\kappa = p_i v_i$ , hence  $\mathcal{P}_{<\kappa}$ -coalgebraic. By Proposition 4.11, the successors of  $e$  are listed without repetition as  $(p_\kappa v_i)_{i < \kappa}$ , so  $e$  has  $\kappa$ -many successors.  $\square$

Thus we may characterize which connecting maps are bijective.

**Proposition 4.13.** [ALM<sup>+</sup>15] *For ordinals  $j \leq i$ , the following are equivalent:*

- (1) *The connecting map  $\nu^{(i \geq j)} \mathcal{P}_{<\kappa}$  is bijective.*
- (2) *Either  $j = i$  or  $j \geq \kappa + \omega$  (for  $\kappa < \infty$ ).*

*Proof.* For (2) $\Rightarrow$ (1), we saw in Section 3.4 that the final chain is stable at  $\kappa + \omega$ . For the converse, assuming  $j < i$ , Proposition 3.13 covers the case where  $j < \kappa$ . For  $j = \kappa + n$ , where  $n \in \mathbb{N}$ , take  $a \in \nu^{(\kappa)} \mathcal{P}_{<\kappa}$  with  $\kappa$  successors. Then  $\{-\}^n a$  is not in the range of  $\nu^{(\kappa+n+1 \geq \kappa+n)}$ , because for every  $b \in \nu^{(\kappa+n+1)} \mathcal{P}_{<\kappa}$ , each  $n$ -step descendant of  $b$  has  $<\kappa$ -successors.  $\square$

## 5. BEFORE $\omega_1$

This section reprises from [ALM<sup>+</sup>15] several properties that the final chain enjoys at countable ordinals. (We have already seen one such property—surjectivity of connecting maps.) We first mention some properties enjoyed at *finite* ordinals:

**Proposition 5.1.** *Let  $n \in \mathbb{N}$ .*

- (1) *For  $a, b \in \nu^{(n)} \mathcal{P}$ , if  $a \sim_n b$  then  $a = b$ .*
- (2) *For any pointed system  $x$ , we can characterize  $p_n x$  as the unique  $b \in \nu^{(n)} \mathcal{P}$  such that  $b \sim_n x$ .*
- (3) *For any ordinal  $i$  and  $a \in \nu^{(i+n)} \mathcal{P}$ , we have  $p_n a = \nu^{(i+n \geq n)} a$ , and it can be characterized as the unique  $b \in \nu^{(n)} \mathcal{P}$  such that  $b \sim_n a$ .*

*Proof.* By induction on  $n$ .  $\square$

The properties enjoyed at countable ordinals derive from the following fact.

**Proposition 5.2.** *For  $j < i < \omega_1$  and all  $a \in \nu^{(i)} \mathcal{P}$  we have*

$$\nu^{(i \geq j+1)} a = \{ \nu^{(i \geq j)} b \mid a \rightsquigarrow_i b \}$$

*Proof.* Trivial if  $i$  is a successor or 0. If  $i$  is a positive limit, take a strictly increasing sequence  $(i_n)_{n \in \mathbb{N}}$  with supremum  $i$ . Any  $b \in \nu^{(i \geq j+1)}a$ , is an element of the  $j$ th level of the channel  $\Psi_i(a)$  through  $(\nu^{(k)}\mathcal{P}_{<\kappa})_{k < i}$ , and it extends by dependent choice to a full branch.  $\square$

**Lemma 5.3.** *Let  $j \leq i < \omega_1$ . For  $a \in \nu^{(i)}\mathcal{P}$  we have  $p_j a = \nu^{(i \geq j)}a$ .*

*Proof.* By induction on  $j$ , using Proposition 5.2 for the successor case.  $\square$

**Proposition 5.4.** *Let  $i < \omega_1$ .*

- (1) *For  $a \in \nu^{(i)}\mathcal{P}$  we have  $p_i a = a$ .*
- (2) *Every element of  $\nu^{(i)}\mathcal{P}$  is  $\mathcal{P}$ -coalgebraic.*
- (3) *For  $a, b \in \nu^{(i)}\mathcal{P}$ , if  $a \sim_i b$  then  $a = b$ .*
- (4) *For  $a, b \in \nu^{(i)}\mathcal{P}$ , if  $a \sim b$  then  $a = b$ .*

*Proof.*

- (1) By Lemma 5.3.
- (2) Follows from part (1).
- (3)  $a = p_i a = p_i b = b$ .
- (4) Follows from part (3).  $\square$

**Corollary 5.5.**

- (1) *Let  $i < \omega_1$ . For any pointed system  $x$ , we may characterize  $p_i x$  as the unique  $b \in \nu^{(i)}\mathcal{P}$  such that  $b \sim_i x$ .*
- (2) *Let  $j \leq i < \omega_1$ . For any  $a \in \nu^{(i)}\mathcal{P}$  we may characterize  $\nu^{(i \geq j)}a$  as the unique  $b \in \nu^{(j)}\mathcal{P}$  such that  $b \sim_j a$ .*

*Proof.*

- (1)  $p_i p_i x = p_i x$ , by Proposition 5.4(1), so  $p_i x \sim_i x$ . Uniqueness is by Proposition 5.4(3).
- (2)  $p_j \nu^{(i \geq j)}a = \nu^{(i \geq j)}a = \nu^{(i \geq j)}p_i a = p_j a$ , so  $\nu^{(i \geq j)}a \sim_j a$ . Uniqueness is by Proposition 5.4(3).  $\square$

A pointed system  $a$  is  $<\kappa$ -branching at depth  $< i$  when every  $b$  such that  $a \rightsquigarrow^m b$ , for a natural number  $m < i$ , has  $<\kappa$  successors. If  $i \geq \omega$ , this just says that  $a$  is  $<\kappa$ -branching.

**Proposition 5.6.** *Assume  $\kappa < \infty$ .*

- (1) *Let  $j$  be an ordinal. Every element of  $\nu^{(\kappa+j)}\mathcal{P}_{<\kappa}$  that is  $\mathcal{P}_{<\kappa}$ -coalgebraic is  $<\kappa$ -branching, and conversely if  $\kappa = \aleph_0$ .*
- (2) *Let  $j \leq i$  be ordinals. Every element of  $\nu^{(\kappa+j)}\mathcal{P}_{<\kappa}$  that is in the range of  $\nu^{(\kappa+i \geq \kappa+j)}$  is  $<\kappa$ -branching at depth  $< i$ , and conversely if  $\kappa = \aleph_0$ .*

*Proof.*

- (1)  $(\Rightarrow)$  follows from Proposition 4.9. For  $(\Leftarrow)$ , if  $a \in \nu^{(\omega+j)}\mathcal{P}_f$  is finitely branching, then it is  $\mathcal{P}_f$ -coalgebraic since  $p_{\omega+j}a = a$ .
- (2)  $(\Rightarrow)$  follows from Proposition 4.5, since every element of  $\nu^{(\kappa+i)}\mathcal{P}_{<\kappa}$  is  $<\kappa$ -branching at depth  $< i$ . For  $(\Leftarrow)$ , let  $a \in \nu^{(\kappa+j)}\mathcal{P}_{<\kappa}$  be  $<\kappa$ -branching at depth  $< i$ . Since  $b \stackrel{\text{def}}{=} p_{\omega+i}a$  is sent by  $\nu^{(\omega+i \geq \omega+j)}$  to  $a$ , it suffices to show that  $b \in \nu^{(\omega+i)}\mathcal{P}_f$ . If  $i \geq \omega$  this holds because  $a$  is finitely branching; we prove the case  $i < \omega$  by induction on  $i$ . For  $i = 0$  we have  $j = i$  and  $b = a \in \nu^{(\kappa)}\mathcal{P}_f$ . For  $i = i' + 1$  we have  $b = \{p_{\omega+i'}c \mid a \rightsquigarrow c\}$ , which is in  $\nu^{(\omega+i)}\mathcal{P}_f$  because  $a$  has finitely many successors and each of them is finitely branching at depth  $< i'$ .  $\square$

6. A GHOST AT  $\omega_1$ 

The previous section has established properties of  $\nu^{(i)}\mathcal{P}_{<\kappa}$  for  $i < \omega_1$ . But for  $i = \omega_1$ , many questions remain unresolved.

- For  $i < \omega_1$ , every element of  $\nu^{(i)}\mathcal{P}_{<\kappa}$  is  $\mathcal{P}_{<\kappa}$ -coalgebraic (for  $\kappa > \aleph_0$ ). Is every element of  $\nu^{(\omega_1)}\mathcal{P}_c$  at least  $\mathcal{P}$ -coalgebraic?
- For  $i < \omega_1$ , the set  $\nu^{(i)}\mathcal{P}$  is strongly extensional, i.e. bisimilar elements are equal. What about  $\nu^{(\omega_1)}\mathcal{P}_c$ ?
- For  $j < i < \omega_1$  we saw (Proposition 5.2) that  $\nu^{(i \geq j+1)}\mathcal{P}$  sends  $a$  to  $\{\nu^{(\text{pred}(i) \geq j)}b \mid a \rightsquigarrow b\}$ . Is this true for  $\nu^{(\omega_1 \geq j+1)}\mathcal{P}_c$ , where  $j < \omega_1$ ?
- The finitely branching elements of  $\nu^{(\omega)}\mathcal{P}_f$  are the  $\mathcal{P}_f$ -coalgebraic ones. Are the countably branching elements of  $\nu^{(\omega_1)}\mathcal{P}_c$  all  $\mathcal{P}_c$ -coalgebraic?

The following result will provide a negative answer to all the above questions.

**Proposition 6.1.** *The set  $\nu^{(\omega_1)}\mathcal{P}_c$  has an element  $a$ , distinct from  $0(\omega_1)$ , that has no successor.*

Such an element  $a$  is called a *ghost* because it “appears to be alive” (for all  $j < \omega_1$ , the component  $a_{j+1}$  is inhabited) yet is “dead” (has no successors). The negative answers are deduced as follows.

- For a pointed system  $x$ , if  $a = p_{\omega_1}x$ , then  $() \in a_1 = p_1x = \{p_0y \mid x \rightsquigarrow y\}$ . So  $x$  has a successor  $y$ , so  $a = p_{\omega_1}x \rightsquigarrow p_{\omega_1}y$ , contradiction.
- $a$  and  $0_{\omega_1}$  are bisimilar (having no successors) yet distinct. So  $\nu^{(\omega_1)}\mathcal{P}_c$  is not strongly extensional.
- For  $j < \omega_1$ , the set  $\nu^{(i \geq j+1)}a = a_{j+1}$  has an element  $b$ , but  $b$  is not of the form  $\nu^{(i \geq j)}c$  for a successor  $c$  of  $a$ .
- The element  $a$  is countably branching but not  $\mathcal{P}_c$ -coalgebraic.

A ghost corresponds via  $\Psi_{\omega_1}$  to a channel through  $(\nu^{(i)}\mathcal{P}_c)_{i < \omega_1}$ , with all levels countable, that has a root but no full branch. The rest of the section is devoted to proving that such a channel exists. Our proof involves three steps:

- (1) obtaining an “Aronszajn tree”
- (2) embedding it into the complete binary tree
- (3) embedding the complete binary tree into the final chain.

We begin with the following notions.

**Definition 6.2.** Let  $I$  be a well-ordered set with least element. A *tidy  $I$ -tree*<sup>3</sup> is an inverse  $I$ -chain  $D$  with the following properties.

- For any  $j \leq i \in I$ , the connecting map  $D_{i \geq j}$  is surjective, i.e. every  $a \in D_j$  has an  $i$ -development.
- For any limit  $i \in I$ , every full branch through  $(D_j)_{j < i}$  has at most one extension to a full branch through  $(D_j)_{j \leq i}$ .
- $D$  has a (necessarily unique) root.

Note that a tidy  $I$ -tree is a channel through itself.

It is important to know whether a tidy tree is guaranteed to have a full branch. The following two results [HS54, Kur36] show that this is not always the case. Our presentation follows [Ber].

<sup>3</sup>The set-theoretic literature commonly uses a more general notion of tree. See Section 9.1 for a comparison.

**Proposition 6.3.** *For regular  $\lambda > \aleph_0$ , there is a tidy  $\lambda$ -tree that has all levels  $\leq \lambda$ -sized and has no full branch.*

*Proof.* For each  $i < \lambda$ , let  $D_i$  be the set of strictly increasing sequences  $0 = x_0 < \dots < x_n < \lambda$  of ordinals, where  $x_n \geq i$  but  $x_m < i$  for all  $m < n$ . Clearly  $D_i$  is  $\leq \lambda$ -sized. For  $j \leq i < \lambda$ , let  $D_{i \geq j} : D_i \rightarrow D_j$  send  $x \in D_i$  to its unique prefix in  $D_j$ . Then  $(D_i)_{i < \lambda}$  is a tidy  $\lambda$ -tree, and any full branch would give a cofinal  $\omega$ -sequence in  $\lambda$ , contradicting regularity.  $\square$

**Theorem 6.4.** *There is a tidy  $\omega_1$ -tree that has all levels countable and no full branch. (Such a tree is said to be Aronszajn.)*

*Proof.* For  $i < \omega_1$ , let  $E_i$  be the set of strictly increasing  $i$ -sequences  $x = (x_j)_{j < i}$  of nonnegative rationals. The supremum of  $x \in E_i$ —taken to be 0 if  $x$  is empty—is a nonnegative real or  $\infty$ , and written  $\sup(x)$ .

We shall define a countable subset  $D_i \subseteq E_i$  for all  $i < \omega_1$ , with the following properties.

- (1) If  $x \in D_i$  then its supremum is (finite and) rational.
- (2) If  $j \leq i < \omega_1$  and  $x \in D_i$  then the  $j$ -sequence prefix of  $x$  is in  $D_j$ .
- (3) If  $j < i < \omega_1$ , then for any  $x \in D_j$  and rational  $r > \sup(x)$ ,  $x$  has an extension in  $D_i$  with supremum  $r$ .

Suppose that we have  $D_j$  for all  $j < i$ , with these properties. We define  $D_i$  as follows; properties (1)–(3) are easily verified in each case.

- For  $i = 0$ , let  $D_i$  consist of the empty sequence.
- Suppose  $i = j + 1$ . Let  $D_i$  consist of all the extended sequences  $xr$  for  $x \in D_j$  and rational  $r > \sup(x)$ .
- Suppose  $i > 0$  is a limit. By the Axiom of Choice, choose, for each  $j < i$  and  $x \in D_j$  and rational  $r > \sup(x)$ , a strictly increasing sequence of ordinals  $(k_n)_{n \in \mathbb{N}}$ , where  $k_0 = j$ , with supremum  $i$ ; and a strictly increasing sequence of rationals  $(s_n)_{n \in \mathbb{N}}$ , where  $s_0 = \sup(x)$ , with supremum  $r$ . (For example, take  $s_n = 2^{-n}\sup(x) + (1 - 2^{-n})r$ .) We define  $P_n(j, x, r) \in D_{k_n}$  with supremum  $s_n$  by induction on  $n \in \mathbb{N}$ .
  - Let  $x_0$  be  $x$ .
  - Let  $x_{n+1}$  be an extension of  $x_n$  in  $D_{k_n}$  with supremum  $s_{n+1}$ .

Then the  $i$ -sequence  $\bigsqcup_{n \in \mathbb{N}} P_n(j, x, r)$  has supremum  $r$  and extends  $x$ . Let  $D_i$  be the set of all these.

We take the inverse chain  $(D_i)_{i < \omega_1}$ , with connecting map  $D_{i \geq j}$  sending  $x \in D_i$  to its  $j$ -sequence prefix. The limit property is clear. The connecting map  $D_{i \geq j}$  is surjective because, in the nontrivial case  $j < i$ , every  $x \in D_j$  has an extension in  $D_i$  with supremum  $\sup(x) + 1$ . The levels are countable by construction. Finally, any full branch of  $(D_i)_{i < \omega_1}$  would give a strictly increasing  $\omega_1$ -sequence of nonnegative rationals, which does not exist.  $\square$

This completes our first step. Before describing the second, we must give a suitable notion of embedding.

**Definition 6.5.** A *cofinal embedding* of inverse chains  $\alpha : (D_i)_{i \in I} \rightarrow (E_j)_{j \in J}$  consists of the following.

- A monotone map  $f : I \rightarrow J$  (the *index map*) that is cofinal, i.e. for any  $j \in J$  there is  $i \in I$  such that  $fi \geq j$ .
- For each  $i \in I$ , an injection  $\alpha_i : D_i \rightarrow E_{fi}$  (the  *$i$ -th level map*) that is natural in  $i$ , i.e. for  $j \leq i$  in  $I$ , if  $y \in D_i$  is an  $i$ -development of  $x \in D_j$  then  $\alpha_i y$  must be an  $fi$ -development of  $\alpha_j x$ .

For any channel  $B$  through  $D$ , its  $\alpha$ -image is the unique channel  $C$  through  $E$  such that, for all  $i \in I$ , the set  $C_{f_i}$  is  $\{\alpha_i x \mid x \in B_i\}$ .

**Proposition 6.6.** *Let  $\alpha: (D_i)_{i \in I} \rightarrow (E_j)_{j \in J}$  be a cofinal embedding of inverse chains. Let  $B$  be a channel through  $D$ , with  $\alpha$ -image  $C$ .*

- (1) *(Assuming  $I$  has a least element.) If  $B$  has a root, then  $C$  does too.*
- (2) *If  $B$  has all levels  $< \kappa$ -sized, then so does  $C$ .*
- (3) *The map  $(y_j)_{j \in J} \mapsto (x_{\alpha_i})_{i \in I}$  is a bijection from the full branches of  $C$  to those of  $B$ .*

By part (3), if  $B$  has no full branch, then neither does  $C$ . Our second step involves the following tidy tree.

**Definition 6.7.**

- (1) Let  $\{0, 1\}^i$  be the set of  $i$ -sequences  $(d_j)_{j < i}$  of bits.
- (2) For  $j \leq i$  and  $x \in \{0, 1\}^i$  we write  $x \upharpoonright_j$  for the restriction of  $x$  to  $j$ .
- (3) For a positive limit  $i$ , the *complete binary  $i$ -tree* is  $(\{0, 1\}^j)_{j < i}$ , with connecting maps given by restriction.

**Proposition 6.8.** *Let  $D$  be a tidy  $\omega_1$ -tree with all levels countable. Then there is a cofinal embedding  $\alpha$  of  $D$  into the complete binary  $\omega_1$ -tree.*

*Proof.* The index map is  $i \mapsto \omega \times i$ , which is well-defined and cofinal because  $i < \omega_1$  implies  $i \leq \omega \times i < \omega_1$ . The injection  $\alpha_i: D_i \rightarrow \{0, 1\}^{\omega \times i}$  is defined by induction on  $i$ , ensuring naturality wrt all  $j \leq i$ . For the successor case, we choose an injection from the countably many  $i + 1$ -developments of each  $x \in D_i$  to the  $2^{\aleph_0}$ -many  $\omega \times (i + 1)$  developments of  $\alpha_i x \in \{0, 1\}^{\omega \times i}$ . The case where  $i$  is a limit is uniquely defined: each  $x \in D_i$  is mapped to the unique  $i$ -sequence that, for all  $j < i$ , extends  $\alpha_j D_{i \geq j} x$ . This is injective because  $x$  is determined by  $(D_{i \geq j})_{j < i}$ .  $\square$

**Corollary 6.9.** *There is a channel through the complete binary  $\omega_1$ -tree, with all levels countable, that has a root but no full branch.*

*Proof.* Proposition 6.4 gives an Aronszajn tidy tree. Embed it into the complete binary  $\omega_1$ -tree and take its image.  $\square$

It remains to cofinally embed the complete binary  $\omega_1$ -tree into  $(\nu^{(i)}\mathcal{P}_c)_{i < \omega_1}$ . For this we encode a sequence of bits as a set of ordinals, following [FH89][Lemmas 2.2 and 2.3].

**Definition 6.10.** For any ordinal  $i$ , and  $c \in \{0, 1\}^i$  we define

$$\bar{c} = \{j < i \mid c_j = 1\} \cup \{i\}$$

For each set of ordinals  $R$ , let  $u_R$  be a parent of  $(v_i)_{i \in R}$ .

**Lemma 6.11.** *Let  $j \leq i$  and  $c \in \{0, 1\}^j$  and  $d \in \{0, 1\}^i$ . Then  $c = d \upharpoonright_j$  iff  $u_{\bar{c}} \sim_{j+1} u_{\bar{d}}$ .*

**Corollary 6.12.** *There is a cofinal embedding  $\beta$  from the complete binary  $\omega_1$ -tree into  $(\nu^{(i)}\mathcal{P}_c)_{i < \omega_1}$ , with index map  $i \mapsto i + 1$ .*

*Proof.* The injection  $\beta_j: \{0, 1\}^j \rightarrow \nu^{(j+1)}\mathcal{P}_{< \kappa}$  sends  $c$  to  $p_{j+1}u_{\bar{c}}$ . Injectivity and naturality follow from Lemma 6.11.  $\square$

We now complete our proof. Corollary 6.9 gives a channel  $E$  through the complete binary  $\omega_1$ -tree, with all levels countable, that has a root but no full branch. Its  $\beta$ -image is a channel through  $(\nu^{(i)}\mathcal{P}_c)_{i < \omega_1}$  with the same properties.

## 7. THE NUMBER OF SUCCESSORS

We further consider the final chain of  $\mathcal{P}_c$ . Proposition 4.12 tells us that some element of  $\nu^{(\omega)}\mathcal{P}_c$  has  $\aleph_0$  successors (since  $\nu^{(\omega)}\mathcal{P}_c = \nu^{(\omega)}\mathcal{P}_f$ ), and some element of  $\nu^{(\omega_1)}\mathcal{P}_c$  has  $\aleph_1$  successors. Are there elements with more successors than this? Although this question is a digression from our main narrative, it provides an application of the infrastructure we have assembled.

**Proposition 7.1.** *The size of  $\nu^{(i)}\mathcal{P}_c$  is*

- finite and positive, if  $i$  is finite
- $2^{\aleph_0}$ , if  $\omega \leq i < \omega_1$
- $2^{\aleph_1}$ , if  $i \geq \omega_1$ .

*Proof.* Induction on  $i$  gives the upper bound, since  $\mathcal{P}_c$  and countable limit preserves  $\leq 2^{\aleph_0}$ -sizedness, and  $\omega_1$ -limit preserves  $\leq 2^{\aleph_1}$ -sizedness. From  $\omega_1$  onwards the sets cannot get bigger. The lower bound is proved as follows. For a set of ordinals  $R$ , we write  $u_R$  for a parent of  $(v_i)_{i \in R}$ .

- For  $i \geq \omega$  we have a family  $(p_i u_R)_{R \subseteq \omega}$  of  $2^{\aleph_0}$  distinct elements of  $\mathcal{P}_c i$ . For distinctness, let  $R, S \subseteq \omega$  with  $n \in R \setminus S$ . Then  $u_R \not\sim_{n+2} u_S$  because there is no  $m \in S$  such that  $v_n \sim_{n+1} v_m$ .
- For  $i \geq \omega_1$  we have a family  $(p_i u_J)_{J \subseteq \omega_1}$  of  $2^{\aleph_1}$  distinct elements of  $\mathcal{P}_c i$ , by the same argument.  $\square$

**Proposition 7.2.** *For every positive limit  $i < \omega_1$ , some element of  $\nu^{(i)}\mathcal{P}_c$  has  $2^{\aleph_0}$  successors.*

*Proof.* Let  $(i_n)_{n \in \mathbb{N}}$  be a strictly increasing sequence with supremum  $i$ . We define a cofinal embedding  $\alpha$  of the complete binary  $\omega$ -tree into the complete binary  $i$ -tree with index map  $n \mapsto i_n$  as follows: the injection  $\alpha_n$  sends  $(c_m)_{m < n}$  to  $(d_j)_{j < i_n}$ , where  $d_j$  is  $c_m$  if  $j = i_m$  for some (unique)  $m < n$  and 0 otherwise. Corollary 6.12 gives a cofinal embedding  $\beta$  of the complete binary  $i$ -tree into  $(\nu^{(j)}\mathcal{P}_c)_{j < i}$ . The  $\beta\alpha$ -image of the complete binary  $\omega$ -tree is a channel through  $(\nu^{(j)}\mathcal{P}_c)_{j < i}$  with  $2^{\aleph_0}$  full branches, corresponding across  $\Psi_i$  to an element of  $\nu^{(i)}\mathcal{P}_c$  with  $2^{\aleph_0}$  successors.  $\square$

At  $\omega_1$  the question is harder to answer.

**Proposition 7.3.** *For any cardinal  $\lambda$ , the following are equivalent.*

- (1) *Some element of  $\nu^{(\omega_1)}\mathcal{P}_c$  has precisely  $\lambda$  successors.*
- (2) *There is a tidy  $\omega_1$ -tree, with all levels countable, that has precisely  $\lambda$  full branches.*

*Proof.* For (1) $\Rightarrow$ (2), any such element corresponds across  $\Psi_{\omega_1}$  to the desired tree—unless it is  $0_{\omega_1}$ , but in that case  $\lambda = 0$  and Theorem 6.4 gives (2).

For (2) $\Rightarrow$ (1), let  $D$  be such a tree. We embed it cofinally into the complete binary  $\omega_1$ -tree (Proposition 6.8), and embed this in turn into  $(\nu^{(i)}\mathcal{P}_c)_{i < \omega_1}$  (Corollary 6.12), and the image corresponds via  $\Psi_{\omega_1}$  to the desired element.  $\square$

Thus the existence of an element of  $\nu^{(\omega_1)}\mathcal{P}_c$  with more than  $\aleph_1$  successors is equivalent to the existence of a tidy  $\omega_1$ -tree, with all levels countable, that has more than  $\aleph_1$  full branches. Such a tree is said to be *Kurepa* and its existence is independent of ZFC, under certain assumptions<sup>4</sup>.

<sup>4</sup>ZFC cannot prove that a Kurepa tree exists, if ZFC + an inaccessible is consistent [Sil71]. ZFC cannot prove that a Kurepa tree does not exist, if ZFC is consistent (Solovay).



8. BEYOND  $\omega_1$ 

**8.1. Properties of Ordinals and Cardinals.** We have now completed our study of the final chain of the countable powerset functor. But for  $\mathcal{P}_{<\kappa}$  we can continue, by asking what happens in the final chain beyond  $\omega_1$ . We shall see that, surprisingly, there is a sharp division between those limit ordinals where strong extensionality holds and those where there is a ghost. This is our main result, Theorem 8.13 below. The same method of [FH89] is adapted to construct all these ghosts.

Our classification relies on the following properties.

**Definition 8.1.** An ordinal is  $\omega$ -*cofinal* when it is the supremum of a strictly increasing sequence  $(i_n)_{n \in \mathbb{N}}$ .

Thus every countable limit is either 0 or  $\omega$ -cofinal.

**Definition 8.2.** Let  $\lambda$  be a regular infinite cardinal.

- (1)  $\lambda$  has the *tree property* when every tidy  $\lambda$ -tree with all levels  $<\lambda$ -sized has a full branch.
- (2)  $\lambda$  is *strongly inaccessible* when, for every  $<\lambda$ -sized set  $A$ , the set  $\mathcal{P}A$  is also  $<\lambda$ -sized.
- (3)  $\lambda$  is *weakly compact* when it is strongly inaccessible and has the tree property.

Note that in Definition 8.2(1), it is essential for the levels to be  $<\lambda$ -sized, because of Proposition 6.3. Only in the case  $\lambda = \aleph_0$  is a full branch guaranteed to exist without this condition. By Theorem 6.4,  $\aleph_1$  does not have the tree property. Whether  $\aleph_2$  has it is independent of ZFC under certain assumptions<sup>5</sup>.

We deem  $\aleph_0$  to be strongly inaccessible (and hence weakly compact). Successor cardinals such as  $\aleph_1$  are not inaccessible. The importance of strong inaccessibility in our story comes from the following fact.

**Proposition 8.3.** *If  $\lambda$  is strongly inaccessible, then for all  $i < \lambda$  the set  $\nu^{(i)}\mathcal{P}$  is  $<\lambda$ -sized.*

*Proof.* By induction on  $i$ . The successor case is by the definition of strong inaccessibility. For a limit  $i < \lambda$ , we use the fact that if  $(A_j)_{j < i}$  is an inverse chain of  $<\lambda$ -sized sets then its limit is  $<\lambda$ -sized.  $\square$

**8.2. Connecting map to a successor.** In Section 5 we derived several properties of countable ordinals from Proposition 5.2. Accordingly, we shall begin by analyzing at which ordinals the property described there holds.

**Definition 8.4.** The following limit ordinals are said to be  $\kappa$ -*extensible*:

- 0.
- Any  $\omega$ -cofinal ordinal.
- Any weakly compact cardinal.
- Any limit ordinal  $\geq \kappa + \omega$ , if  $\kappa < \infty$ .
- $\kappa$ , if  $\kappa < \infty$  and  $\kappa$  has the tree property.

We complete Proposition 5.2 as follows.

**Proposition 8.5.** *For ordinals  $j < i$  the following are equivalent.*

<sup>5</sup>ZFC cannot prove that  $\aleph_2$  has the tree property, if ZFC is consistent [Spe49]. ZFC cannot prove that  $\aleph_2$  lacks the tree property, if ZFC + a weakly compact cardinal is consistent [Mit72].

- (1) For all  $a \in \nu^{(i)}\mathcal{P}_{<\kappa}$  we have  $\nu^{(i \geq j+1)}a = \{\nu^{(i \geq j)}b \mid a \rightsquigarrow_i b\}$ .  
(2)  $i$  is either a successor or a  $\kappa$ -extensible limit.

*Proof.* of (2)  $\Rightarrow$  (1). The successor and zero cases are trivial, and the case  $i \geq \kappa + \omega$  follows from Corollary 4.6. Let us consider the other cases. For  $b \in \nu^{(i \geq j+1)}$ , we know that  $b$  is an element of the  $j$ th level of the channel  $\Psi_i(a)$  through  $(\nu^{(k)}\mathcal{P}_{<\kappa})_{k < i}$ , and we want to extend it to a full branch. We may reformulate this problem by expressing  $i$  as  $j + i'$ , and defining the tidy  $i'$ -tree  $(A_k)_{k < i'}$  where  $A_k$  is the set of  $j + k$ -developments of  $b$ . We want a full branch for this tree.

- Suppose that  $i$  is  $\omega$ -cofinal, so  $i'$  is too. Let  $(i_n)_{n \in \mathbb{N}}$  be a strictly increasing sequence with supremum  $i'$ . We obtain a full branch by dependent choice. (The size of the sets  $A_k$  is immaterial in this case.)
- Suppose that  $\kappa$  has the tree property and  $i = \kappa$ , so  $i' = \kappa$ . For  $k < \kappa$  the set  $A_k$ , being a subset of  $\nu^{(j+k)}\mathcal{P}_{<\kappa}$ , is  $<\kappa$ -sized. So the tree property gives a full branch through  $(A_k)_{k < \kappa}$ .
- Suppose that  $i$ , and therefore  $i'$ , is a weakly compact cardinal  $\lambda < \kappa$ . For  $k < \lambda$ , we have also  $j + k < \lambda$ , and so  $A_k$ , being a subset of  $\nu^{(j+k)}\mathcal{P}_{<\kappa}$ , is  $<\lambda$ -sized by Proposition 8.3. So the tree property gives a full branch through  $(A_k)_{k < \lambda}$ .  $\square$

To prove the converse, we use the following notion of ghost.

**Definition 8.6.** Let  $i$  be a limit. A  $\mathcal{P}_{<\kappa}$ -ghost at  $i$  is a  $\mathcal{P}_{<\kappa}$ -Cauchy element of  $\nu^{(i)}\mathcal{P}_{<\kappa}$ , distinct from  $0(i)$ , that has no successor.

The Cauchy condition was omitted in Proposition 6.1 because every element of  $\nu^{(\omega_1)}\mathcal{P}_{<\kappa}$  is  $\mathcal{P}_{<\kappa}$ -Cauchy (Corollary 3.16(3).)

Any  $\mathcal{P}_{<\kappa}$ -ghost  $a$  at  $i$  does not satisfy the equation in (1), because the LHS is  $a_{j+1}$ , which is inhabited, but the RHS is empty. So there cannot be a  $\mathcal{P}_{<\kappa}$ -ghost at a  $\kappa$ -extensible ordinals. The following two results establish that, at all other limits, a  $\mathcal{P}_{<\kappa}$ -ghost does exist, giving (1)  $\Rightarrow$  (2) as required.

Again we use channels: a  $\mathcal{P}_{<\kappa}$ -ghost at  $i$  corresponds via  $\Psi_i$  to a channel through  $(\nu_{\text{coalg}}^{(j)}\mathcal{P}_{<\kappa})_{j < i}$ , with all levels  $<\kappa$ -sized, that has a root but no full branch.

**Proposition 8.7.** Let  $\kappa < \infty$  not have the tree property. Then there is a  $\mathcal{P}_{<\kappa}$ -ghost at  $\kappa$ .

*Proof.* Since  $\kappa$  does not have the tree property, there is a tidy  $\kappa$ -tree  $D$  with no full branch. (Such a tree is said to be  $\kappa$ -Aronszajn.)

We give a cofinal embedding  $\alpha$  of  $D$  into the complete binary  $\kappa$ -tree as follows. The index map is  $f : i \mapsto \sum_{j < i} |D_j|$ . This is well-defined and cofinal because  $i < \kappa$  implies  $i \leq fi < \kappa$ , since  $1 \leq |D_j| < \kappa$  and  $\kappa$  is regular. We give the  $i$ -th level map  $\alpha_i : D_i \rightarrow \{0, 1\}^{fi}$  by induction on  $i$ , ensuring naturality wrt all  $j \leq i$ . For the successor case, we have  $f(i+1) = fi + |D_i|$ , so we take an injection from the  $\leq |D_i|$ -many  $i+1$ -developments of each  $x \in D_i$  to the  $2^{|D_i|}$ -many  $f(i+1)$ -developments of  $\alpha_i x \in \{0, 1\}^{fi}$ . The case where  $i$  is a limit is uniquely defined, as in the proof of Proposition 6.8.

The image  $E$  of  $D$  is a channel through the complete binary  $\kappa$ -tree with a root but no full branch. As in Corollary 6.12, we define a cofinal embedding  $\beta$  of the complete binary  $\kappa$ -tree into  $(\nu_{\text{coalg}}^{(j)}\mathcal{P}_{<\kappa})_{j < \kappa}$ , with index map  $j \mapsto j + 1$ . The  $\beta$ -image of  $E$  is the desired channel.  $\square$

**Proposition 8.8.** Let  $i < \kappa$  be a positive limit that is neither  $\omega$ -cofinal nor weakly compact. Then there is a  $\mathcal{P}_{<\kappa}$ -ghost at  $i$ .

*Proof.* Let  $\lambda$  be the cofinality of  $i$ . Then  $\lambda$  is a regular cardinal  $> \aleph_0$ . There are three possibilities:

- (1)  $i$  is not a regular cardinal, i.e.  $i > \lambda$ .
- (2)  $i = \lambda$  and  $\lambda$  is not strongly inaccessible.
- (3)  $i = \lambda$  and  $\lambda$  does not have the tree property.

For case (1), Proposition 6.3 gives a tidy  $\lambda$ -tree  $D$ , with every level  $\leq \lambda$ -sized, that has no full branch. Evidently there is a cofinal embedding  $\alpha$  of  $D$  into the complete  $\lambda$ -ary  $\lambda$ -tree  $(\lambda^j)_{j < \lambda}$ , with index map  $j \mapsto j$ . We shall give a cofinal embedding  $\beta$  of the complete  $\lambda$ -ary  $\lambda$ -tree into  $(\nu_{\text{coalg}}^{(j)} \mathcal{P}_{< \kappa})_{j < i}$ . Then the  $\beta\alpha$ -image of  $D$  is a channel through  $(\nu_{\text{coalg}}^{(j)} \mathcal{P}_{< \kappa})_{j < i}$ , with every level  $\leq \lambda$ -sized, that has a root but no full branch. Whereas we previously encoded a sequence of bits as a set of ordinals, we shall now encode a sequence of elements of  $\lambda$  as a set of sets of ordinals.

Since  $i > \lambda$ , express  $i$  as the supremum of a strictly increasing  $\lambda$ -sequence  $(\eta_j)_{j < \lambda}$  in  $\eta$ , with  $\eta_0 \geq \lambda$ . The idea is to encode an ordered pair  $(k, l)$ , where  $k, l < \lambda$ , as  $\{\eta_k, l\}$ . Clearly, for  $k, k' < j < \lambda$  and  $l, l' < \lambda$ , we have

$$u_{\{\eta_k, l\}} \sim_{\eta_{j+1}} u_{\{\eta_{k'}, l'\}} \iff k = k' \text{ and } l = l' \quad (8.1)$$

For  $j < \lambda$  and  $c \in \lambda^j$ , we set

$$\bar{c} \stackrel{\text{def}}{=} \{\{\eta_k, c_k\} \mid k < j\} \cup \{\{\eta_j, l\} \mid l < \lambda\}$$

For a set of ordinals  $R$  we write  $u_R$  for a parent of  $(v_j)_{j \in R}$ , and for a set of sets of ordinals  $S$  we write  $t_S$  for a parent of  $(u_R)_{R \in S}$ . For  $k \leq j < \lambda$  and  $c \in \lambda^k$  and  $d \in \lambda^j$ , we deduce from (8.1) that

$$c = d \upharpoonright_k \iff t_{\bar{c}} \sim_{\eta_{k+2}} t_{\bar{d}} \quad (8.2)$$

The cofinal embedding  $\beta$  of the complete  $\lambda$ -ary  $\lambda$ -tree into  $(\nu_{\text{coalg}}^{(j)} \mathcal{P}_{< \kappa})_{j < i}$  has index map  $j \mapsto \eta_j + 2$ . The  $j$ -th level map  $\beta_j \lambda^j \rightarrow \nu_{\text{coalg}}^{(\eta_j+2)} \mathcal{P}_{< \kappa}$  sends  $c$  to  $p_{\eta_j+2} t_{\bar{c}}$ . Injectivity and naturality follow from 8.2.

For case (2), Proposition 6.3 gives a tidy  $\lambda$ -tree  $D$ , with all levels  $\leq \lambda$ -sized, that has no full branch. We give a cofinal embedding  $\alpha$  of  $D$  in the complete binary  $\lambda$ -tree as follows. Since  $\lambda$  is not strongly inaccessible, there is a cardinal  $\mu < \lambda$  such that  $2^\mu \geq \lambda$ . The index map is  $j \mapsto \mu \times j$ . This is well-defined and cofinal because  $j < \lambda$  implies  $j \leq \mu \times j < \lambda$ , since  $1 \leq \mu < \lambda$  and  $\lambda$  is regular. We give the  $j$ -th level map  $\alpha_j : D_j \rightarrow \{0, 1\}^{\mu \times j}$  by induction on  $j$ , ensuring naturality wrt all  $k \leq j$ . For the successor case, we take an injection from the  $\leq \lambda$ -many  $j+1$ -developments of each  $x \in D_j$  to the  $2^\mu$ -many  $\mu \times (j+1)$  developments of  $\alpha_j x \in \{0, 1\}^{\mu \times j}$ . The case where  $j$  is a limit is uniquely defined, as in the proof of Proposition 6.8.

Next we obtain an embedding  $\beta$  of the complete binary  $\lambda$ -tree into  $(\nu_{\text{coalg}}^{(j)} \mathcal{P}_{< \kappa})_{j < \lambda}$ , with index map  $j \mapsto j+1$ , as in Corollary 6.12. The  $\beta\alpha$ -image of  $D$  is a channel through  $(\nu_{\text{coalg}}^{(j)} \mathcal{P}_{< \kappa})_{j < \lambda}$ , with all levels  $\leq \lambda$ -sized, that has a root but no full branch.

For case (3), Proposition 8.7 gives a  $\mathcal{P}_{< \lambda}$ -ghost at  $\lambda$ , which is also a  $\mathcal{P}_{< \kappa}$ -ghost at  $\lambda$ .  $\square$

**8.3. Consequences of the ghosts.** Our aim is to revisit each of the properties listed in Section 5, to see at which ordinals it holds. So we want to obtain many negative results from a ghost. We begin with some methods for doing so.

**Lemma 8.9.** *A  $\mathcal{P}$ -ghost at limit  $i$  is not in the image of  $\nu^{(i+1 \geq i)}\mathcal{P}$ .*

*Proof.* Let  $\nu^{(i+1 \geq i)}b$ , for  $b \in \nu^{(i+1)}\mathcal{P}$ , be a  $\mathcal{P}$ -ghost at  $i$ . It corresponds across  $\Psi_i$  to the range of  $b$ . Any element of  $b$  would be a full branch of the range of  $b$ , which does not exist, so  $b$  is empty. But then its range is empty, contradiction.  $\square$

We shall use the singleton operation  $\{-\}$  applied  $n$  times to a ghost.

**Lemma 8.10.** *Let  $a$  a  $\mathcal{P}$ -ghost at positive limit  $i$ , and  $n \in \mathbb{N}$ . Then  $\{-\}^n a \in \nu^{(i+n)}\mathcal{P}$  has the following properties.*

- (1)  $p_{n+1}\{-\}^n a \neq \nu^{(i+n \geq n+1)}\{-\}^n a$ .
- (2)  $\{-\}^n a$  is not in the image of  $\nu^{(i+n+1 \geq i+n)}\mathcal{P}$ .
- (3)  $\{-\}^n a$  is not  $\mathcal{P}$ -coalgebraic.
- (4)  $\{-\}^n a \sim \{-\}^n 0(i)$  but  $\{-\}^n a \neq \{-\}^n 0(i)$ .
- (5)  $\{-\}^n a \not\sim_{n+1} \nu^{(i+n \geq j)}\{-\}^n a$  for any  $j < i$  such that  $j > n$ .

*Proof.*

- (1) Induction on  $n$  gives  $p_{n+1}\{-\}^n a = \{-\}^n \emptyset$  and  $\nu^{(i+n \geq n+1)}\{-\}^n a = \{-\}^n \{\emptyset\}$ .
- (2) Induction on  $n$ . The case  $n = 0$  is Lemma 8.9. The inductive step follows from  $\nu^{(i+n+2 \geq i+n+1)}b = \{\nu^{(i+n+1 \geq i+n)}c \mid c \in b\}$ .
- (3) Follows from part (2).
- (4) Each part is by induction on  $n$ .
- (5) We have  $\nu^{(i+n \geq j)}\{-\}^n a \rightsquigarrow^n \nu^{(i \geq \text{pred}^n(j))}a$ . Since  $n < j < i$  we have  $0 < \text{pred}^n(j) < i$ , so  $\nu^{(i \geq \text{pred}^n(j))}a$  has a successor. But if  $\{-\}^n a \rightsquigarrow^n b$ , then  $b = a$ , which has no successor.  $\square$

For our results, we use the following terminology.

**Definition 8.11.** For an ordinal  $i$ , let  $i = i' + m$  where  $i'$  is a limit and  $m \in \mathbb{N}$ . Then we write

$$\begin{aligned} \text{LP}(i) &\stackrel{\text{def}}{=} i', & \text{the limit part of } i. \\ \text{FP}(i) &\stackrel{\text{def}}{=} m, & \text{the finite part of } i. \end{aligned}$$

The following completes Lemma 5.3.

**Theorem 8.12.** *For  $j \leq i$ , the following are equivalent.*

- (1) For all  $a \in \nu^{(i)}\mathcal{P}_{<\kappa}$  we have  $p_i a = \nu^{(i \geq j)}a$ .
- (2) Either  $j \leq \text{FP}(i)$  or  $\text{LP}(i)$  is  $\kappa$ -extensible.

*Proof.* For (2) $\Rightarrow$ (1), the case ( $j \leq \text{FP}(i)$ ) is Proposition 5.1(3), and the case of  $\kappa$ -extensible  $\text{LP}(i)$  is proved the same way as Lemma 5.3.

For the converse, let  $a$  be a  $\mathcal{P}_{<\kappa}$ -ghost  $a$  at  $\text{LP}(i)$ . If  $p_j\{-\}^{\text{FP}(i)}a = \nu^{(i \geq j)}\{-\}^{\text{FP}(i)}a$  then (since  $\text{FP}(i) + 1 \leq j$ ) we have  $p_{\text{FP}(i)+1}\{-\}^{\text{FP}(i)}a = \nu^{(i \geq \text{FP}(i)+1)}\{-\}^{\text{FP}(i)}a$ , contradicting Lemma 8.10(1).  $\square$

We come to the main result of the paper, which completes Proposition 5.4.

**Theorem 8.13.** *For any ordinal  $i$ , the following are equivalent.*

- (1) For all  $a \in \nu^{(i)}\mathcal{P}_{<\kappa}$  we have  $p_i a = a$ .
- (2) Every  $a \in \nu^{(i)}\mathcal{P}_{<\kappa}$  is  $\mathcal{P}$ -coalgebraic.
- (3) For  $a, b \in \nu^{(i)}\mathcal{P}_{<\kappa}$ , if  $a \sim_i b$  then  $a = b$ .
- (4) For  $a, b \in \nu^{(i)}\mathcal{P}_{<\kappa}$ , if  $a \sim b$  then  $a = b$ .
- (5)  $\text{LP}(i)$  is  $\kappa$ -extensible.

*Proof.* Proposition 8.12 gives (5) $\Leftrightarrow$ (1). Parts (2)–(4) follow just as in Proposition 5.4.

If  $\text{LP}(i)$  is not  $\kappa$ -extensible, let  $a$  be a  $\mathcal{P}_{<\kappa}$ -ghost at  $\text{LP}(i)$ . By Lemma 8.10(3),  $\{-\}^{\text{FP}(i)}a$  is not  $\mathcal{P}$ -coalgebraic, and is bisimilar to but distinct from  $0(i)$ .  $\square$

Another way of obtaining negative results from a ghost is the following.

**Lemma 8.14.** *Let  $i < \kappa$  be a positive limit, and let  $a$  be a  $\mathcal{P}_{<\kappa}$ -ghost at  $i$ . There is a sequence  $(x_m)_{m \in \mathbb{N}}$  of  $<\kappa$ -branching pointed systems such that*

- $p_i x_0 \rightsquigarrow a$
- every successor of  $x_0$  has a successor
- $x_{m+1}$  is a parent of just  $x_m$ .

*Proof.* The only difficulty is to obtain  $x_0$ , for we can then obtain  $x_m$  by induction on  $m$ . Since  $a$  is  $\mathcal{P}_{<\kappa}$ -Cauchy, we may choose, for each positive  $j < i$ , a  $<\kappa$ -branching pointed system  $y_j$  such that  $a_{j+1} = p_{j+1}y_j$  (and hence  $a_j = p_j y_j$ ). Let  $x_0$  be a parent of  $(y_j)_{j < i}$ . Then for  $j < i$ , we have  $(p_i x_0)_{j+1} = p_{j+1}x_0 = \{p_j y \mid x_0 \rightsquigarrow y\} \ni a_j$ , so  $p_i x_0 \rightsquigarrow a$ . Any successor  $y$  of  $x_0$  is, for some  $j < i$ , an embedding applied to  $y_j$ . Since  $p_{j+1}y_j$  is inhabited,  $y_j$  and hence  $y$  must have a successor.  $\square$

We shall complete Corollary 5.5, which characterizes  $p_i x$  and  $\nu^{(i \geq j)}a$ , in two parts: firstly considering when these elements have the required property (Proposition 8.15), and secondly considering when no other element has it (Proposition 8.16).

**Proposition 8.15.**

- (1) For an ordinal  $j$ , the following are equivalent.
  - (a) For any  $<\kappa$ -branching pointed system  $x$  we have  $x \sim_j p_j x$ .
  - (b) Either  $j \geq \kappa$  (for  $\kappa < \infty$ ) or  $\text{LP}(j)$  is  $\kappa$ -extensible.
- (2) For ordinals  $j \leq i$ , the following are equivalent.
  - (a) For any  $a \in \nu^{(i)}\mathcal{P}_{<\kappa}$ , we have  $\nu^{(i \geq j)}a \sim_j a$ .
  - (b) Either  $j = i$  or  $j \leq \text{FP}(i)$  or  $j \geq \kappa$  (for  $\kappa < \infty$ ) or both  $\text{LP}(j)$  and  $\text{LP}(i)$  are  $\kappa$ -extensible.

*Proof.* For (1b) $\Rightarrow$ (1a), the  $\kappa$ -extensible case is proved as in Corollary 5.5(1 and the case  $j \geq \kappa$  holds by Proposition 4.9. For (2b) $\Rightarrow$ (2a), the  $\kappa$ -extensible case is proved as in Corollary 5.5(2, the case  $j \geq \kappa$  is by Proposition 4.5 and the case  $j \leq \text{FP}(i)$  follows from Proposition 5.1(3).

For the converse we proceed as follows.

- (1) If  $j < \kappa$  and  $\text{LP}(j)$  is not  $\kappa$ -extensible, let  $a$  be a  $\mathcal{P}_{<\kappa}$ -ghost at  $\text{LP}(j)$  and form the sequence  $(x_m)_{m \in \mathbb{N}}$  as in Lemma 8.14. Put  $n \stackrel{\text{def}}{=} \text{FP}(j)$ . Then  $p_j x_n \rightsquigarrow^n p_j x_0 \rightsquigarrow a$  and  $a$  has no successor, whereas if  $x_n \rightsquigarrow^{n+1} y$  then  $y$  has a successor. So  $p_j x_n \not\rightsquigarrow_{n+2} x_n$ .
- (2) Suppose  $\text{FP}(i) < j < i, \kappa$ , and  $\text{LP}(j)$  and  $\text{LP}(i)$  are not both  $\kappa$ -extensible. We consider three cases.

- If  $\text{LP}(j) = \text{LP}(i)$ , put  $L = \text{LP}(j)$  and  $j = L + n$  and  $i = L + m + n + 1$ . Let  $a$  be a  $\mathcal{P}_{<\kappa}$ -ghost at  $L$  and form the sequence  $(x_m)_{m \in \mathbb{N}}$  as in Lemma 8.14. Put  $b \stackrel{\text{def}}{=} p_i x_n$ . Then  $\nu^{(i \geq j)} b = p_{L+n} x_n \rightsquigarrow^{n+1} a$ , which has no successor. On the other hand, if  $b \rightsquigarrow^{n+1} c$  then  $c = p_{L+m} y$ , for some successor  $y$  of  $x_0$ , and since  $y$  has a successor,  $c$  does too. So  $b \not\rightsquigarrow_{n+2} \nu^{(i \geq j)} b$ .
- If  $\text{LP}(i)$  is not  $\kappa$ -extensible and  $j < \text{LP}(i)$ , let  $a$  be a  $\mathcal{P}_{<\kappa}$ -ghost at  $\text{LP}(i)$ . Put  $n \stackrel{\text{def}}{=} \text{FP}(i)$  and apply Lemma 8.10(5).
- If  $\text{LP}(i)$  is  $\kappa$ -extensible then  $\text{LP}(j)$  is not, so let  $a$  be a  $\mathcal{P}_{<\kappa}$ -ghost at  $\text{LP}(j)$  and form the sequence  $(x_m)_{m \in \mathbb{N}}$  as in Lemma 8.14. Put  $n \stackrel{\text{def}}{=} \text{FP}(j)$  and  $b \stackrel{\text{def}}{=} p_i x_n$ . Then  $b \sim_i x_n$  but  $\nu^{(i \geq j)} b = p_j x_n \not\rightsquigarrow_{n+2} x_n$ , as we saw in the proof of part 1. So  $b \not\rightsquigarrow_{n+1} \nu^{(i \geq j)} b$ .  $\square$

**Proposition 8.16.**

- (1) For any ordinal  $j$ , the following are equivalent.
  - (a) For any  $<\kappa$ -branching pointed system  $x$ , any  $b \in \nu^{(j)} \mathcal{P}_{<\kappa}$  such that  $b \sim_j x$  is  $p_j x$ .
  - (b) For any  $<\kappa$ -branching pointed system  $x$ , we can characterize  $p_j x$  as the unique  $b \in \nu^{(j)} \mathcal{P}_{<\kappa}$  such that  $b \sim_j x$ .
  - (c)  $\text{LP}(j)$  is  $\kappa$ -extensible.
- (2) For any ordinals  $j \leq i$ , the following are equivalent.
  - (a) For any  $a \in \nu^{(i)} \mathcal{P}_{<\kappa}$ , any  $b \in \nu^{(j)} \mathcal{P}_{<\kappa}$  such that  $b \sim_j a$  is  $\nu^{(i \geq j)} a$ .
  - (b) For any  $a \in \nu^{(i)} \mathcal{P}_{<\kappa}$ , we can characterize  $\nu^{(i \geq j)} a$  as the unique  $b \in \nu^{(j)} \mathcal{P}_{<\kappa}$  such that  $b \sim_j a$ .
  - (c) Either  $j \leq \text{FP}(i)$  or both  $\text{LP}(j)$  and  $\text{LP}(i)$  are  $\kappa$ -extensible.

*Proof.* The proof of (1c) $\Rightarrow$ (1b) and (2c) $\Rightarrow$ (2b) is proved the same way as Corollary 5.5. For the converse parts, we form a sequence  $(z_m)_{m \in \mathbb{N}}$  where  $z_0$  is a parent of nothing, i.e. a pointed system with no successor, and  $z_{m+1}$  a parent of just  $z_m$ . (Essentially this is the Zermelo encoding of the natural numbers.)

In both parts, if  $\text{LP}(j)$  is not  $\kappa$ -extensible, let  $a$  be a  $\mathcal{P}_{<\kappa}$ -ghost at  $\text{LP}(j)$  and put  $n \stackrel{\text{def}}{=} \text{FP}(j)$ . Then  $p_j z_n = \{-\}^n 0_{\text{LP}(j)}$  (lemma:  $p_{k+n} z_n = \{-\}^n 0_k$ , by induction on  $n$ ), which is bisimilar to but distinct from  $\{-\}^n a$ , so (1a) is false. And, in part 2, this is also  $\nu^{(i \geq j)} p_i z_n$ , so (2a) is false.

In part (2), if  $\text{LP}(i)$  is not  $\kappa$ -extensible and  $j > m \stackrel{\text{def}}{=} \text{FP}(i)$ , let  $a$  be a  $\mathcal{P}_{<\kappa}$ -ghost at  $\text{LP}(i)$ . Then  $p_j z_m$  is bisimilar to  $z_m$  (this is a lemma for all  $j > m$ , by induction on  $m$ ) and hence to  $\{-\}^m a$ . But  $p_j z_m = \nu^{(i \geq j)} \{-\}^m a$  would imply, by applying  $\nu^{(j \geq m+1)}$ , that  $p_{m+1} z_m = \nu^{(i \geq m+1)} \{-\}^m a$  i.e.  $\{-\}^m = \{-\}^m()$ , contradiction. So (2a) is false.  $\square$

We complete Proposition 5.6 as follows.

**Proposition 8.17.** Assume  $\kappa < \infty$ .

- (1) Let  $j$  be an ordinal. Every element of  $\nu^{(\kappa+j)} \mathcal{P}_{<\kappa}$  that is  $\mathcal{P}_{<\kappa}$ -coalgebraic is  $<\kappa$ -branching, and conversely iff either  $j \geq \omega$  or  $\kappa$  has the tree property.
- (2) Let  $j \leq i$  be ordinals. Every element of  $\nu^{(\kappa+j)} \mathcal{P}_{<\kappa}$  that is in the range of  $\nu^{(\kappa+i \geq \kappa+j)}$  is  $<\kappa$ -branching at depth  $< i$ , and conversely iff either  $j = i$  or  $j \geq \omega$  or  $\kappa$  has the tree property.

*Proof.* We only need to prove the converse parts. If  $j \geq \omega$  then every element of  $\nu^{(\kappa+j)} \mathcal{P}_{<\kappa}$  is  $<\kappa$ -branching. If  $\kappa$  has the tree property, we proceed as for  $\aleph_0$ . Otherwise, let  $a$  be a

$\mathcal{P}_{<\kappa}$ -ghost at  $\kappa$ . The element  $\{-\}^j a$  is  $<\kappa$ -branching but not  $\mathcal{P}_{<\kappa}$ -coalegebraic and not in the image of  $\nu^{(\kappa+j+1 \geq \kappa+j)}$ , by Lemma 8.10(2)–(3).  $\square$

**8.4. Surjectivity and  $\mathcal{P}_{<\kappa}$ -coalegebraicity.** We next consider which connecting maps are surjective.

**Proposition 8.18.** *Let  $\lambda$  be a regular infinite cardinal with the tree property that is less than  $\kappa$ . Then  $\mathcal{P}_{<\kappa}$  and  $\mathcal{P}_{<\kappa}^+$  preserve, up to a surjection, the limit of any inverse  $\lambda$ -chain  $D$  with all levels  $<\lambda$ -sized. Explicitly, any channel  $C$  through  $D$  is the range of a  $<\kappa$ -sized set of full branches.*

*Proof.* It suffices to prove the case where  $D_i = \lim_{j < i} D_j$  for each limit  $i < \lambda$ . For  $i < \lambda$ , each  $x \in C_i$  extends by the tree property to a full branch through  $D$ , because all levels are  $<\lambda$ -sized. For each  $i < \lambda$  and  $x \in C_i$ , choose such a branch  $\theta_i(x)$ , by the Axiom of Choice. The set  $\{\theta_i(x) \mid i < \lambda, x \in C_i\}$  has size  $\leq \lambda \times \lambda = \lambda < \kappa$  and has range  $C$ .  $\square$

**Proposition 8.19.** *For ordinals  $j \leq i$ , the following are equivalent.*

- (1) *The connecting map  $\nu^{(i \geq j)} \mathcal{P}_{<\kappa}$  is an surjection.*
- (2) *Either  $j = i$  or  $\text{LP}(j)$  is either 0 or  $\omega$ -cofinal or weakly compact or  $\geq \kappa + \omega$  (for  $\kappa < \infty$ ).*

*Proof.* By Proposition 3.14 we may assume  $i = j + 1$ . The case  $j \geq \kappa$  has been treated (Proposition 3.13 and 4.13), so we assume  $j < \kappa$ . Since  $\mathcal{P}_{<\kappa}$  preserves and reflects surjectivity, we may assume  $j$  is a limit.

For (2) $\Rightarrow$ (1), it suffices by Proposition 3.9 for  $\mathcal{P}_{<\kappa}$  to preserve the limit of any inverse  $j$ -chain up to a surjection. The  $j = 0$  case is trivial, the  $\omega$ -cofinal case is by Proposition 3.15, and the weakly compact case is by Proposition 8.18. For the converse, if  $j < \kappa$  and  $j$  is not  $\kappa$ -extensible then there is a  $\mathcal{P}_{<\kappa}$ -ghost at  $j$ , which is not even  $\mathcal{P}$ -coalegebraic.  $\square$

**Corollary 8.20.** *For an ordinal  $i$ , the following are equivalent.*

- *Every element of  $\nu^{(i)} \mathcal{P}_{<\kappa}$  is  $\mathcal{P}_{<\kappa}$ -coalegebraic.*
- *$\text{LP}(i)$  is either 0 or  $\omega$ -cofinal or weakly compact or  $\geq \kappa + \omega$  (for  $\kappa < \infty$ ).*

*Proof.* Follows from Proposition 8.19 by Proposition 3.14.  $\square$

We next see that  $\mathcal{P}_{<\kappa}$ -Cauchy is weaker than  $\mathcal{P}_{<\kappa}$ -coalegebraic.

**Proposition 8.21.** *For a limit ordinal  $i$ , the following are equivalent.*

- (1) *Every  $\mathcal{P}_{<\kappa}$ -Cauchy element of  $\nu^{(i)} \mathcal{P}_{<\kappa}$  is  $\mathcal{P}_{<\kappa}$ -coalegebraic.*
- (2)  *$i$  is either 0 or  $\omega$ -cofinal or weakly compact or  $\geq \kappa + \omega$  (for  $\kappa < \infty$ ).*

*Proof.* For (2) $\Rightarrow$ (1), every element of  $\nu^{(i)} \mathcal{P}_{<\kappa}$  is  $\mathcal{P}_{<\kappa}$ -coalegebraic. For the converse: if  $i < \kappa$  and is neither 0 nor  $\omega$ -cofinal nor weakly compact, let  $a$  be a  $\mathcal{P}_{<\kappa}$ -ghost at  $i$ . It is  $\mathcal{P}_{<\kappa}$ -Cauchy but not  $\mathcal{P}$ -coalegebraic. For  $i = \kappa$ , Proposition 4.12 gives a  $\mathcal{P}_{<\kappa}$ -Cauchy element of  $\nu^{(\kappa)} \mathcal{P}_{<\kappa}$  that has  $\kappa$  successors and so is not  $\mathcal{P}_{<\kappa}$ -coalegebraic.  $\square$

We substantiate our claim in Section 4.2 that the coalegebraic elements need not form a subsystem of the final chain.

**Proposition 8.22.** *For a limit ordinal  $i$ , the following are equivalent.*

- (1) *The  $\mathcal{P}_{<\kappa}$ -coalegebraic elements form a subsystem of  $\nu^{(i)} \mathcal{P}_{<\kappa}$ .*
- (2)  *$i$  is either 0 or  $\omega$ -cofinal or weakly compact or  $\geq \kappa$  (for  $\kappa < \infty$ ).*

*Proof.* For (2) $\Leftrightarrow$ (1), if  $i \geq \kappa$ , we apply Proposition 4.9. In the other cases, all elements are  $\mathcal{P}_{<\kappa}$ -coalgebraic. For the converse, if the conclusion is false let  $a$  be a  $\mathcal{P}_{<\kappa}$ -ghost at  $i$ . It is not  $\mathcal{P}$ -coalgebraic, and by Lemma 8.14 it is a successor of a  $\mathcal{P}_{<\kappa}$ -coalgebraic element.  $\square$

## 9. RELATED WORK

**9.1. Trees vs tidy trees.** Our formulation of the Aronszajn and Kurepa properties used the notion of *tidy* tree. This suited our purposes, but to avoid confusion it must be compared with the following more general notion that commonly appears in the set-theoretic literature.

**Definition 9.1.** Let  $I$  be a well-ordered set. An  $I$ -tree is an inverse  $I$ -chain where every level is inhabited.

If  $I$  has a least element, then clearly every tidy  $I$ -tree is an  $I$ -tree; but there is also a kind of converse, as follows. (Cf. [Jec03][Lemma 9.13].)

**Proposition 9.2.** *Let  $\lambda$  be a regular infinite cardinal. For any  $\lambda$ -tree  $D$  with all levels  $<\lambda$ -sized, there is a tidy  $\lambda$ -tree  $E$  with all levels  $<\lambda$ -sized and a bijection from the full branches of  $E$  to those of  $D$ .*

*Proof.* We first construct a channel through  $D$ , with all levels inhabited, that contains all the full branches. For each  $i < \lambda$ , let  $C_i$  be the set of all  $x \in D_i$  that have a  $k$ -development for all  $k < \lambda$  such that  $k \geq i$ . To see that  $C$  is a channel, let  $j \leq i < \lambda$  and  $x \in C_j$ . For each  $k < \lambda$  such that  $k \geq i$ , let  $R(k)$  be the set of  $i$ -developments of  $x$  that has an  $k$ -development in  $C$ . Since  $C_i$  is  $<\lambda$ -sized the subset  $\bigcap_{i \leq k < \lambda} R(k)$  has an element, which is an  $i$ -development of  $x$  in  $C_i$ . We likewise prove for all  $i < \lambda$  that  $C_i$  is inhabited. Evidently every full branch of  $D$  is a full branch of  $C$ .

Next we form the inverse chain  $E$ , whose  $i$ th level, for  $i < \lambda$ , is  $\lim_{j < i} D_j$ , and whose connecting maps are given by restriction. Let  $\alpha: D \rightarrow E$  be the cofinal embedding with index map  $i \mapsto i + 1$  and  $i$ -th level map sending  $x \in D_i$  to  $(D_{i \geq j} x)_{j \leq i} \in E_{i+1}$ . The  $\alpha$ -image of  $C$  is a tidy tree, and by Proposition 6.6 has the required properties.  $\square$

**Corollary 9.3.**

- (1) *There is a Kurepa tree (an  $\omega_1$ -tree with all levels countable that has more than  $\aleph_1$  full branches) iff there is a tidy one.*
- (2) *Let  $\lambda$  be a regular infinite cardinal. There is a  $\lambda$ -Aronszajn tree (a  $\lambda$ -tree with all levels  $<\lambda$ -sized that has no full branch) iff there is a tidy one.*

So our use of tidy trees is not a significant change from the usual formulation.

**9.2. Saturation.** In [ALM<sup>+</sup>15], an account is given of the final chain of  $\mathcal{P}$  at countable ordinals. Elements of the final chain are observed to be *saturated*. At limit ordinals, this property may be expressed as follows.

**Proposition 9.4.** *Let  $i$  be a  $\kappa$ -extensible (e.g. countable) limit. For any  $a \in \nu^{(i)}\mathcal{P}$  and pointed system  $x$ , if for all  $j < i$ ,  $a$  has a  $\rightsquigarrow_i$ -successor  $y$  such that  $y \sim_j x$ , then  $a$  has a  $\rightsquigarrow_i$ -successor  $y$  such that  $y \sim_i x$ .*

*Proof.* By Theorem 8.13,  $a = p_i a$ , and transitions to  $p_i x$  by Proposition 4.7 since  $\text{pred}(i) = i$ . Lastly,  $p_i x \sim_i x$  by Proposition 8.15(1).  $\square$



It is not known whether this property holds in the case  $i = \omega_1$ . We conjecture that it does not, even in the case where  $a \in \nu^{(\omega_1)}\mathcal{P}_c$  and  $x$  is countably branching.

**9.3. Cauchy completeness.** To enable comparison with [FH89, LR96], let us formulate Proposition 8.21 in terms of Cauchy sequences.

- An  $i$ -sequence of pointed systems  $(x_j)_{j < i}$  is *Cauchy* when for all  $j \leq k < i$  we have  $x_j \sim_j x_k$ .
- A *limit* for a Cauchy  $i$ -sequence  $(x_j)_{j < i}$  is a pointed system  $y$  such that for all  $j < i$  we have  $x_j \sim_i y$ .

We then have the following.

**Proposition 9.5.** *For an ordinal  $i$ , the following are equivalent.*

- (1) *Every Cauchy  $i$ -sequence of  $<\kappa$ -branching pointed systems has a  $<\kappa$ -branching limit.*
- (2)  *$i$  is either a successor or 0 or  $\omega$ -cofinal or weakly compact or  $\geq \kappa + \omega$  (for  $\kappa < \infty$ ).*

*Proof.* The successor case is evident. For the limit case, (1) corresponds to Proposition 8.21(1). To see this, note that a Cauchy sequence  $(x_j)_{j < i}$  gives a  $\mathcal{P}_{<\kappa}$ -Cauchy element  $a = (p_j x)_{j < i} \in \nu^{(i)}\mathcal{P}_{<\kappa}$ , and every  $\mathcal{P}_{<\kappa}$ -Cauchy element arises in this way. Moreover,  $y$  is a limit for  $(x_j)_{j < i}$  iff  $a = p_i y$ , so  $(x_j)_{j < i}$  has a limit iff  $a$  is  $\mathcal{P}_{<\kappa}$ -coalgebraic.  $\square$

**9.4. Comparison.** As stated, the work of [FH89] provides the basis of the method we have used to obtain ghosts. Because that paper is concerned with models of set theories, it treats primarily the full powerset functor. It studies systems that resemble the final chain system in satisfying Proposition 4.7, but differ from it by including only coalgebraic elements. As we have seen—Proposition 8.22—these do not form a subsystem of the final chain, so the difference is considerable.

The work of [LR96] treats not only systems of coalgebraic elements but also systems of Cauchy elements (and the latter do form a subsystem of the final chain), specifically for the full powerset functor. A more elaborate proof of Cauchy incompleteness is given in order to obtain additional negative results. Relating that work to the present paper is a matter for future research.

## 10. CONCLUSIONS

We have investigated several properties that the final chain of  $\mathcal{P}_{<\kappa}$ , viewed as a transition system, enjoys at countable ordinals. In particular, the *strong extensionality* property: bisimilar elements are equal. We have seen that these properties do not hold at  $\omega_1$ , because of an element distinct from  $0_{\omega_1}$  that has no successor—a “ghost”. Using the same method, we have precisely identified those ordinals at which each of these properties does hold.

## REFERENCES

- [Abr05] S. Abramsky. A Cook’s tour of the finitary non-well-founded sets. In Sergei N. Artëmov, Howard Barringer, Artur S. d’Avila Garcez, Luís C. Lamb, and John Woods, editors, *We Will Show Them! Essays in Honour of Dov Gabbay, Volume One*, pages 1–18. College Publications, 2005.
- [Adá74] J. Adámek. Free algebras and automata realizations in the language of categories. *Comment. Math. Univ. Carolinae*, 15(3):589–609, 1974.
- [AILS12] L. Aceto, A. Ingólfssdóttir, P. B. Levy, and J. Sack. Characteristic formulae: A general approach. *Mathematical Structures in Computer Science*, 22(2):125–173, 2012. special issue devoted to papers from SOS 2009.
- [AK95] J. Adamek and V. Koubek. On the greatest fixed point of a set functor. *TCS: Theoretical Computer Science*, 150, 1995.
- [ALM<sup>+</sup>15] J. Adámek, P. B. Levy, S. Milius, L. S. Moss, and L. Sousa. On final coalgebras of power-set functors and saturated trees - to George Janelidze on the occasion of his sixtieth birthday. *Applied Categorical Structures*, 23(4):609–641, 2015.
- [AP15] J. Adámek and T. Palm. How many steps does a terminal coalgebra need? Talk given at Domains XII, Cork, Ireland, 2015.
- [AT11] J. Adámek and V. Trnková. Initial algebras and terminal coalgebras in many-sorted sets. *Mathematical Structures in Computer Science*, 21(2):481–509, 2011.
- [Bar93] M. Barr. Terminal coalgebras in well-founded set theory. *Theoretical Computer Science*, 114(2):299–315, 21 June 1993.
- [BBM14] A. Bizjak, L. Birkedal, and M. Miculan. A model of countable nondeterminism in guarded type theory. In Gilles Dowek, editor, *Rewriting and Typed Lambda Calculi - Joint International Conference, RTA-TLCA 2014, Held as Part of the Vienna Summer of Logic, VSL 2014, Vienna, Austria, July 14-17, 2014. Proceedings*, volume 8560 of *Lecture Notes in Computer Science*, pages 108–123. Springer, 2014.
- [Ber] G. Bergman. Some empty inverse limits. Manuscript available at <http://math.berkeley.edu/~gbergman/papers/unpub/emptylim.pdf>.
- [BM96] J. Barwise and L. S. Moss. *Vicious Circles*. CSLI, 1996.
- [BMSS12] L. Birkedal, R. E. Møgelberg, J. Schwinghammer, and K. Støvring. First steps in synthetic guarded domain theory: step-indexing in the topos of trees. *Logical Methods in Computer Science*, 8(4), 2012.
- [CUV09] V. Capretta, T. Uustalu, and V. Vene. Corecursive algebras: A study of general structured corecursion. In M. Oliveira and J. Woodcock, editors, *SBMF*, volume 5902 of *LNCS*, pages 84–100. Springer, 2009.
- [FH89] M. Forti and F. Honsell. Models of self-descriptive set theories. In *Partial Differential Equations and the Calculus of Variations: Essays in Honor of Ennio De Giorgi*, pages 473–518. Birkhäuser Boston, 1989.
- [HS54] G. Higman and A. H. Stone. On inverse systems with trivial limits. *J. London Math. Soc.*, 29:233–236, 1954.
- [Jec03] T. Jech. *Set Theory, the Third Millennium Edition, Revised and Expanded*. Springer, 2003.
- [KKV04] C. Kupke, A. Kurz, and Y. Venema. Stone coalgebras. *Theor. Comput. Sci*, 327(1-2):109–134, 2004.
- [Kur36] G. Kurepa. Ensembles linéaires et une classe de tableaux ramifiés (tableaux ramifiés de M. Aronszajn). *Publ. Math. Univ. Belgrade*, 6:129–160, 1936. <http://elib.mi.sanu.ac.rs/files/journals/publ/6/14.pdf>.
- [LR96] R. S. Lazic and A. W. Roscoe. On transition systems and non-well-founded sets. In *Papers on General Topology and Applications: 11th Summer Conference at the University of Southern Maine, 1995 Annals of the New York Academy of Sciences*, volume 806, pages 238–264, 1996.
- [Mal76] R. J. Malitz. *Set theory in which the axiom of foundations fails*. PhD thesis, UCLA, 1976. (unpublished).
- [Mil89] R. Milner. *Communication and Concurrency*. Prentice-Hall, 1989.
- [Mit72] W. Mitchell. Aronszajn trees and the independence of the transfer property. *Annals of Mathematical Logic*, 5(1):21–46, 1972.
- [Sch10] Daniel Schwencke. Coequational logic for accessible functors. *Information and Computation*, 208(12):1469–1489, 2010.

- [Sil71] Jack Silver. The independence of Kurepa's conjecture and two-cardinal conjectures in model theory. In D. S. Scott, editor, *Axiomatic Set Theory*, volume 13 of *Proceedings of Symposia in Pure Mathematics*, pages 383–390. AMS, 1971.
- [Spe49] E. Specker. Sur un problème de Sikorski. *Colloquium Mathematicae*, 2(1):9–12, 1949.
- [TAKR75] V. Trnková, J. Adámek, V. Koubek, and J. Reiterman. Free algebras, input processes and free monads. *Commentationes Mathematicae Universitatis Carolinae*, 016(2):339–351, 1975.
- [vG87] R. J. van Glabbeek. Bounded nondeterminism and the approximation induction principle in process algebra. In F. J. Brandenburg, G. Vidal-Naquet, and M. Wirsing, editors, *Proceedings of the 4th Annual Symposium on Theoretical Aspects of Computer Science, STACS'87 (Passau, Germany, February 19-21, 1987)*, volume 247 of *LNCS*, pages 336–347. Springer-Verlag, Berlin-Heidelberg-New York-London-Paris-Tokyo-Hong Kong, 1987.
- [Wor05] J. Worrell. On the final sequence of a finitary set functor. *Theoretical Computer Science*, 338(1–3):184–199, June 2005.