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A TOPOLOGICAL INTERPRETATION OF THREE LEIBNIZIAN PRINCIPLES WITHIN THE FUNCTIONAL EXTENSIONS

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Abstract. Three philosophical principles are often quoted in connection with Leibniz: "objects sharing the same properties are the same object" (*Identity of indiscernibles*), "everything can possibly exist, unless it yields contradiction" (*Possibility as consistency*), and "the ideal elements correctly determine the real things" (*Transfer*).

Here we give a precise logico-mathematical formulation of these principles within the framework of the Functional Extensions, mathematical structures that generalize at once compactifications, completions, and elementary extensions of models. In this context, the above Leibnizian principles appear as topological or algebraic properties, namely: a property of separation, a property of compactness, and a property of directeness, respectively.

Abiding by this interpretation, we obtain the somehow surprising conclusion that these Leibnizian principles *may be fulfilled in pairs, but not all three together.*

Introduction

The great philosopher, mathematician and logician Gottfried Wilhelm von Leibniz, in many of his philosophical writings, has been inspired by, and consequently has given inspiration to several important mathematical ideas. In this article we consider, within a mathematical framework, three philosophical principles that are often quoted in connection with Leibniz.[1] We cannot discuss and articulate the philosophical aspects of these principles: we simply give here a few quotes in order to justify and explain our mathematical interpretation of these Leibnizian principles.

Identity of indiscernibles

"objects sharing the same properties are the same object"

> There are never in nature two beings which are perfectly identical to each other, and in which it is impossibile to find any internal difference ...
> (*Monadology*)

Key words and phrases: transfer principle, indiscernibles, nonstandard models, functional extensions.

It is a great pleasure to dedicate this paper to Furio: how many joint papers in the Eighties and Nineties deal with mathematical structures arising from fundamental philosophical principles!

[1]For the Leibniz's quotes see www.leibnizedition.de/.

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> Two indiscernible individuals cannot exist. [...] To put two indiscernible things is to put the same thing under two names.
>
> (*Fourth letter to Clarke*)

> [...] dans les choses sensibles on n'en trouve jamais deux indiscernables [...] (*Fifth letter to Clarke*)

Possibility as consistency

"everything can possibly exist, unless it yields contradiction"

- *Impossible* is what yields an absurdity.
- *Possible* is not impossible.
- *Necessary* is that, whose opposite is impossible.
- *Contingent* is what is not necessary.

(unpublished*, 1680 ca.*)

> [...] nothing is absolutely necessary, when the contrary is possible. [...] Absolutely necessary is [...] that whose opposite yields a contradiction.
>
> (*Dialogue between Theofile and Polydore*)

Transfer principle

"the ideal elements correctly determine the real things"

> Perhaps *the infinite and infinitely small* [numbers] that we conceive *are imaginary, nevertheless* [they are] *suitable to determine the real things*, as usually do the imaginary roots. They are situated in the ideal regions, from where things are ruled by laws, even though they do not lie in the part of matter. (Letter to Johann Bernoulli*, 1698*)

In this paper, we try and give a precise mathematical formulation of these principles in the context of the *Functional Extensions* of [11], structures which generalize at once compactifications and completions of topological spaces, and nonstandard extensions of models (see also [4]). Given a set M, a *functional extension* of M is a superset ${}^*M$ of M such that every function $f : M \to M$ has a distinguished extension ${}^*f : {}^*M \to {}^*M$ that preserves compositions and equalizers. Moreover, assuming that the so called Puritz preorder of ${}^*M$ is directed, the operator $*$ can be appropriately defined so as to provide also all properties P and all relations R on M with distinguished extensions ${}^*P$ and ${}^*R$ on ${}^*M$.

Following the basic idea that the elements of the ["standard"] set M are the "real objects" of the "actual world", whereas the ["nonstandard"] extension ${}^*M$ contains also the "ideal elements" of all "possible worlds", an appropriate interpretation of the Leibniz's principles in the context of functional extensions might be the following:

Ind: *different elements of* ${}^*M$ *are* separated *by the extension* ${}^*P$ *of some "real" property* P *of* M*;*

Poss: *a family of "real" properties of* M *that are not* contradictory *in* M *has extensions to* ${}^*M$ *that are* all simultaneously satisfied *in* ${}^*M$;

Tran: *a statement involving "real" elements, properties and relations of* M *is* true in M *if and only if the corresponding statement about their "ideal" extensions is* true in ${}^*M$. (Clearly here one has to admit only "first order" statements, so as to avoid trivial inconsistencies.)

We shall see below that the "real" properties of M, when extended to ${}^*M$, may be taken as the *clopen* subsets of a topology on ${}^*M$ (the so called *S-topology*), and so the above principles Ind and Poss turn out to be respectively a property of *separation* and a property of *compactness* of the S-topology of ${}^*M$. On the other hand, the principle Tran requires *all* the conditions postulated for the $*$-extension in [11], including the order-theoretic property of directeness, so as to take care of properties and relations of arbitrary arities.

On the basis of results similar to those of [8, 4], we obtain the somehow surprising consequence that these Leibnizian principles *can be fulfilled in pairs, but not all three together*. Grounding on the mathematical results of this paper, and, chiefly, on the philosophical writings of Leibniz himself, *e.g.*

> Les parties du temps ou du lieu [...] sont des choses ideales, ainsi elles se rassemblent parfaitement comme deux unités abstraites. Mais il n'est pas de même de deux Uns concrets [...] c'est à dire veritablement actuels.[2] (*Fifth letter to Clarke*)

we decide to call *Leibnizian* those functional extensions that satisfy Poss together with Tran, letting hold the principle Ind only inside the "standard" universe.

The paper is organized as follows:

- In Section 1, we recall the precise definition of *functional extension* together with the main properties stated in [11]. In particular, we establish the fundamental result that the functional extensions are *all and only the complete elementary* (=nonstandard) extensions of any model M.
- In Section 2 we introduce the S-topology on functional extensions, and we determine which functional extensions satisfy the transfer principle Tran together with either Poss, or Ind, according to the properties of the S-topology. Consequently, in Subsection 2.1 we see how to obtain functional extensions where the principles Poss and Tran hold together, whereas in Subsection 2.2 we examine the structure of the extensions verifying Ind and Tran together. The *impossibility of satisfying simultaneously the three Leibnizian principles* follows immediately.
- In Section 3 we consider which set theoretic hypotheses are needed in order to get functional extensions satisfying the principle Ind. Leaving apart these conditions, in Subsection 3.1 we motivate by his own quotes the (supposed) preference of Leibniz himself for dropping off Ind while maintaining the remaining principles Poss and Tran.
- In Section 4 we suggest how a slight weakening of the notion of functional extension allows for admitting the simultaneous holding of the principles Ind and Poss, still preserving a large amount of Tran, (by dropping off only the incompatible"analytic" part of the condition (equ) that forces everywhere different functions on X to have everywhere different extensions to ${}^*X$).
- A few concluding remarks and three connected set theoretic open questions can be found in the final Section 5.

In general, we refer to [10] for all the topological notions and facts used in this paper, and to [6] for definitions and facts concerning ultrapowers, ultrafilters, and nonstandard models. General references for nonstandard Analysis could be [13, 1]; specific for our "elementary" approach is [4].

[2] The parts of time or place ... are ideal things, so they perfectly resemble like two abstract unities. But it is not so with two concrete Ones, ... that is truly actual [things].

1. Functional Extensions and the transfer principle

The Transfer Principle Tran is the very ground of the usefulness of the nonstandard methods in mathematics. It allows for obtaining correct results about, say, the real numbers by using *ideal elements*, like actual *infinitesimal* or *infinite* numbers. In this section we review the main features of the *functional extensions* introduced in the paper [11] (see also [4]), with the goal of characterizing all *nonstandard* (= complete elementary) extensions by means of a few simple properties of an operation $*$ that assigns an appropriate extension ${}^*f : {}^*X \to {}^*X$ to each function $f : X \to X$.

These structures are a sort of *compactifications or completions of discrete spaces* that turn out to comprehend *all and only* the *nonstandard extensions of models*.

Recall that the *Puritz (pre)ordering* $\lesssim_P$ of ${}^*X$ is defined by $\eta \lesssim_P \xi$ if there exists $f : X \to X$ such that ${}^*f(\xi) = \eta$ (see [16, 15]). Then we set our fundamental definition as follows:

Definition 1.1. A *functional extension* of X is a proper superset ${}^*X$ of X where a *distinguished extension* ${}^*f : {}^*X \to {}^*X$ is assigned to each function $f : X \to X$, so as to satisfy the following conditions for all $f, g : X \to X$.

(comp)**: Preservation of compositions:** ${}^*(g \circ f) = {}^*g \circ {}^*f$
i.e. ${}^*g({}^*f(\xi)) = {}^*(g \circ f)(\xi)$ for all $\xi \in {}^*X$;

(equ)**: Preservation of equalizers:**[3] ${}^*(\chi_{fg}) = \chi_{{}^*f{}^*g}$
where $\chi_{\phi\psi}$ is the characteristic function of the equalizer $Eq(\phi, \psi)$
of the functions ϕ and ψ, *i.e.* $\chi_{\phi\psi}(z) = \begin{cases} 1 & if \quad \phi(z) = \psi(z), \\ 0 & otherwise; \end{cases}$

(dir)**: Directness of the Puritz order:** for all $\xi, \eta \in {}^*X$ there exist
$p_1, p_2 : X \to X$ and $\zeta \in {}^*X$ such that $\xi = {}^*p_1(\zeta)$ and $\eta = {}^*p_2(\zeta)$,
so that $\xi, \eta \lesssim_P \zeta$.

The importance of the property (dir), called *coherence* in [8], is due to the fact that, by providing an "internal coding of pairs", it allows for extending multivariate functions "parametrically": this possibility is essential in order to get the full principle Tran, which involves properties, relations, and functions of any arities. More precisely, the following facts that hold in every functional extension ${}^*X$ of X allow for considering only *unary* functions (see Subsection 3.2 of [11]):

- *For all* $\xi_1, \ldots, \xi_n \in {}^*X$ *there exist* $p_1, \ldots, p_n : X \to X$ *and* $\zeta \in {}^*X$ *such that* ${}^*p_i(\zeta) = \xi_i$.
- *If* $p_1, \ldots, p_n, q_1, \ldots, q_n : X \to X$ *and* $\xi, \eta \in {}^*X$ *satisfy* ${}^*p_i(\xi) = {}^*q_i(\eta)$*, then* ${}^*(F \circ (p_1, \ldots, p_n))(\xi) = {}^*(F \circ (q_1, \ldots, q_n))(\eta)$ *for all* $F : X^n \to X$.

It follows that there is a unique way of assigning an extension ${}^*F$ to every function $F : X^n \to X$ in such a way that all compositions are preserved. Then, by using the characteristic functions in n variables, one can assign an extension ${}^*R$ also to all n-ary relations R on X.

Several important cases of the transfer principle are easy to deduce: *e.g.*, if f is constant, or injective, or surjective, or characteristic, then so is ${}^*f$. In particular the extension ${}^*\chi_A$ of the characteristic function χ_A of a subset $A \subseteq X$, can be taken as the characteristic function of the $*$-*extension* ${}^*A$ of A in ${}^*X$, thus obtaining a *Boolean isomorphism* between the field $\mathcal{P}(X)$ of all subsets of X and a field $\mathcal{C}\ell({}^*X)$ of subsets of ${}^*X$ (see Theorem 1.2 of [11]).

[3] In [11], this property is denoted by (diag), and called *preservation of the diagonal*, in view of the fact that $\chi_{fg}(x) = \chi_\Delta(f(x), g(x))$, where χ_Δ is the characteristic function of the diagonal $\Delta \subseteq X \times X$. (For convenience we always assume that $0, 1 \in X$.)

While both properties (comp) and (equ) of Definition 1.1 are clear instances of the transfer principle, as they correspond to the statements

$$\forall x \in X \,.\, f(g(x)) = (f \circ g)(x) \quad \text{and} \quad \forall x \in X \,.\, f(x) = g(x) \Longleftrightarrow \chi_{fg}(x) = 1,$$

respectively, the same is not apparent for the third property (dir), which is given in a *second order* formulation. On the contrary, an even stronger *uniform version* of (dir) can be obtained by Tran: simply take p_1, p_2 to be the compositions of a fixed bijection $\delta : X \to X \times X$ with the ordinary projections $\pi_1, \pi_2 : X \times X \to X$, and apply Tran to the statement

$$\forall x, y \in X \,.\, \exists z \in X \,.\, p_1(z) = x,\ p_2(z) = y.$$

Hence, when ${}^*X$ is a nonstandard extension of X, the three defining properties (comp), (equ), and (dir) of functional extensions are fulfilled by hypothesis. Conversely, we devoted [11] to prove the fact (partly anticipated in [4]) that the combination of *three natural, simple instances* of the transfer principle, namely (comp), (equ), and (dir), makes any functional extension ${}^*X$ a *limit ultrapower* of X, thus providing the full *Transfer Principle* Tran. So the functional extensions are exactly the *hyper-extensions* in the sense of [4] (*i.e.* all nonstandard models).

We state without proof the corresponding theorem, and we direct the reader to [11], where a *purely algebraic* proof is given in all details, and two more (a *topological* and a *purely logical* one) are outlined.

Theorem 1.2 (Thm. 2.2 and Cor. 2.3 of [11]). *Any nonstandard extension ${}^*X$ of X is a functional extension of X, and conversely any functional extension ${}^*X$ of X satisfies the transfer principle* Tran. □

2. The S-topology and the principles Ind and Poss.

In order to study our formalizations of the Leibnizian principles Ind and Poss within the functional extensions, it is natural to consider on ${}^*X$ a topology that corresponds to the classical *S-topology* of Nonstandard Analysis, already considered since [17] (S stands for *Standard*). In this topology, the *closure* in ${}^*X$ of a subset $A \subseteq X$ is given by its extension ${}^*A$, so the field of subsets $\mathcal{C}\ell({}^*X) = \{{}^*A \subseteq {}^*X \mid A \subseteq X\}$ is both an *open basis* of the S-topology and the *field of all its clopen* (=closed and open) subsets (since ${}^*(X \setminus A) = {}^*X \setminus {}^*A$).

Remark that all functions ${}^*f$ are *continuous* with respect to the S-topology, because ${}^*f^{-1}({}^*A) = {}^*(f^{-1}(A))$ for all $A \subseteq X$, so the inverse images of clopen sets are clopen. Hence for ${}^*X$ being a *topological extension* in the sense of [8] it needs only that the S-topology be T_1 (and so, according to the next theorem, Hausdorff).

We begin by characterizing the *separation properties* of the S-topology in the same way as in Theorem 1.4 of [8]:

Theorem 2.1. *Let ${}^*X$ be a functional extension of X, and put on ${}^*X$ the S-topology. Then, as a topological space, ${}^*X$ is 0-dimensional, and either totally disconnected or not T_0. It follows that ${}^*X$ with the S-topology is Hausdorff if and only if it is T_0.*

Proof. The S-topology has a clopen basis by definition. In this topology the closure of the point ξ is $M_\xi = \bigcap_{\xi \in A} {}^*A$, an intersection of clopen sets. If $M_\xi = \{\xi\}$ for all $\xi \in {}^*X$, then different points have disjoint clopen neighborhoods, hence the S-topology is regular (and totally disconnected).

Assume instead that there exist points $\eta \neq \xi$ such that $\eta \in M_\xi$. Then the S-topology is not T_0, because η belongs to the same basic open (clopen) sets as ξ. In fact, given $A \subseteq X$, $\xi \in {}^*A$ implies $\eta \in {}^*A$, by the choice of η. Similarly $\xi \notin {}^*A$ implies $\xi \in {}^*(X \setminus A)$, hence $\eta \in {}^*(X \setminus A)$ and so $\eta \notin {}^*A$.

In particular the S-topology is Hausdorff (in fact regular and totally disconnected) whenever it is T_0. □

Now the principle Ind simply means that, given $\xi \neq \eta \in {}^*X$ there exist disjoint subsets $A, B \subseteq X$ such that $\xi \in {}^*A$ and $\eta \in {}^*B$, *i.e.* that the S-topology of ${}^*X$ is Hausdorff.

On the other hand, the principle Poss states that all the $*$-extensions of a family of finitely compatible properties of X are simultaneously satisfied in ${}^*X$. So the corresponding family $\mathcal{F}$ of subsets of X has the finite intersection property, hence the intersection $\bigcap_{A \in \mathcal{F}} {}^*A$ of the corresponding $*$-extensions has to be nonempty. But requiring this is equivalent to require that every *proper filter of clopen sets* in ${}^*X$ has *nonempty intersection*, *i.e.* that the S-topology of ${}^*X$ is *quasi-compact*. (Following [10], we call *compact* only Hausdorff spaces.)

So we can completely determine, from the S-topology, when the principles Ind and Poss hold in a functional extension:

Corollary 2.2. *Let ${}^*X$ be a functional extension of X with the S-topology. Then*
(1) *the principle* **Ind** *holds if and only if ${}^*X$ is Hausdorff;*
(2) *the principle* **Poss** *holds if and only if ${}^*X$ is quasi-compact; hence*
(3) *the principles* **Ind** *and* **Poss** *hold simultaneously in ${}^*X$ if and only if it is compact.* □

2.1. Combining Tran with Poss.

In order to combine the principle Tran with Poss, we recall that a nonstandard model whose S-topology is quasi-compact is commonly called an *enlargement*. It is well known that every structure has *arbitrarily saturated* elementary extensions (see *e.g.* [6]), and any $(2^{|X|})^+$-*saturated elementary extension* of X is easily seen to be a *nonstandard enlargement* (see *e.g.* [1] or [4]). Therefore Corollary 2.2 yields

Theorem 2.3. *The nonstandard enlargements of X are exactly those functional extension of X that satisfy both principles* Tran *and* Poss*.* □

Thus we get a lot of functional extensions satisfying simultaneously Tran and Poss.

2.2. Combining Tran with Ind.

We pass now to characterize all those functional extensions of X that satisfy simultaneously the principles Tran and Ind.

The S-topology of these extensions is Hausdorff, according to Corollary 2.2, so we know that the intersection $M_\xi = \bigcap_{\xi \in A} {}^*A$ is the singleton $\{\xi\}$, hence the filter $\mathcal{U}_\xi$ of the subsets A of X such that ${}^*A$ contains ξ uniquely determines the point $\xi \in {}^*X$. Moreover, the filter $\mathcal{U}_\xi$ is actually an *ultrafilter* on X such that

$$\mathcal{U}_{{}^*f(\xi)} = \overline{f}(\mathcal{U}_\xi) = \{A \subseteq X \mid f^{-1}(A) \in \mathcal{U}_\xi\} \quad \textit{for all } f : X \to X.$$

So, when ${}^*X$ is Hausdorff, we must have that

$$\overline{f}(\mathcal{U}_\xi) = \overline{g}(\mathcal{U}_\xi) \Longleftrightarrow {}^*f(\xi) = {}^*g(\xi)$$

for all $\xi \in {}^*X$ and all $f, g : X \to X$.

Now ${}^*f(\xi) = {}^*g(\xi)$ holds if and only if $\chi_{{}^*f{}^*g}(\xi) = {}^*(\chi_{fg})(\xi) = 1$, or equivalently if and only if the equalizer $E(f, g)$ of f and g belongs to $\mathcal{U}_\xi$.

Recall that an ultrafilter $\mathcal{U}$ on X is called *Hausdorff*[4] if the condition

$$(\mathsf{H}) \qquad \overline{f}(\mathcal{U}) = \overline{g}(\mathcal{U}) \iff \{\, x \in X \mid f(x) = g(x)\} \in \mathcal{U}.$$

holds for all $f, g : X \to X$.

So the condition (H) has to be verified by all ultrafilters $\mathcal{U}_\xi$ in order that ${}^*X$ be Hausdorff. We have thus proved the following

Theorem 2.4. *A functional extension ${}^*X$ of X satisfies simultaneously the principles* Ind *and* Tran *if and only if all the ultrafilters $\mathcal{U}_\xi$ on X associated to the points $\xi \in {}^*X$ are Hausdorff.* □

We shall deal in the next section with the set theoretic strength of the combination of Ind with Tran. By now we simply recall that there are plenty of non-Hausdorff ultrafilters on X: *e.g.* all *tensor products* $\mathcal{V} = \mathcal{U} \otimes \mathcal{U}$ of an ultrafilter $\mathcal{U}$ with itself contradict (H), because the "projections" p_1, p_2 of the property (dir) give $\overline{p}_1(\mathcal{V}) = \mathcal{U} = \overline{p}_2(\mathcal{V})$.

Thus we can easily state what we announced in the introduction, namely

Theorem 2.5. *No extension satisfies at once the three Leibnizian principles* Ind, Poss, *and* Tran. □

3. Set theoretic problems in combining Ind with Tran.

As shown by Theorem 2.4, combining Ind with Tran requires special ultrafilters, named *Hausdorff* in Subsection 2.2. Despite the apparent weakness of their defining property (H), which is actually true whenever any of the involved functions is i*njective* (or *constant*), not much is known about Hausdorff ultrafilters.

On *countable* sets, the property (H) is satisfied by *selective* ultrafilters as well as by *products of pairwise non-isomorphic selective* ultrafilters (see [9]), but their existence in pure ZFC is still unproved. However any hypothesis providing infinitely many non-isomorphic selective ultrafilters over $\mathbb{N}$, like the Continuum Hypothesis CH or Martin's Axiom MA, provides any *countable* set with infinitely many non-isomorphic functional extensions that satisfy Ind.

On *uncountable* sets the situation is highly problematic: it is proved in [9] that Hausdorff ultrafilters on sets of size greater than or equal to $\mathfrak{u}$ (the least size of an ultrafilter basis on $\mathbb{N}$) cannot be *regular*. All what is provable in ZFC about the size of $\mathfrak{u}$ is that $\aleph_1 \leq \mathfrak{u} \leq 2^{\aleph_0}$ (see e.g. [5]). In particular, the existence of functional extensions satisfying Ind with uniform ultrafilters, even on $\mathbb{R}$, would imply that of inner models with measurable cardinals. (Such ultrafilters have been obtained by forcing only assuming much stronger hypotheses, see [12]).

Be it as it may, as far as we do not abide ZFC as our foundational theory, *the existence of functional extensions without indiscernibles, although consistent, cannot be proved.*

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[4] The property (H) has been introduced in [7] under the name (C). Hausdorff ultrafilters are studied in [12, 9, 2].

3.1. **The choice of Gottfried Wilhelm von Leibniz.** We have seen that (at least) one of the three Leibnizian principles that we have investigated has to be left out. The most reasonable choice seems to be that of dropping Ind. In fact, even if one neglects the set theoretic problems that have been outlined above, one should pay attention to what Leibniz, whose logico-mathematical insight into philosophical questions cannot be overestimated, wrote about it.

> [...] cette supposition de deux indiscernables [...] paroist possible en termes abstraits, mais elle n'est point compatible avec l'ordre des choses [...]
>
> Quand je nie qu'il y ait [...] deux corps indiscernables, je ne dis point qu'il soit impossible absolument d'en poser, mais que c'est une chose contraire a la sagesse divine [...]
>
> Les parties du temps ou du lieu [...] sont des choses ideales, ainsi elles se rassemblent parfaitement comme deux unités abstraites. Mais il n'est pas de même de deux Uns concrets [...] c'est à dire veritablement actuels.
>
> Je ne dis pas que deux points de l'Espace sont un meme point, ny que deux instans du temps sont un meme instant comme il semble qu'on m'impute [...][5] (*Fifth letter to Clarke*, [14], pp. 131-135)

From these quotes, it appears that Leibniz himself considered the identity of indiscernibles as a "physical" rather than a "logical" principle: it may be actually true, but its negation is non-contradictory in principle, so *it could fail in some possible world.* Moreover only "properties of the real world" M are considered in this principle: so it seems natural, and not absurd, to assume that objects indiscernible by these "real" properties may be separated by some abstract, "ideal" property of $^*M$.

On this ground we finally decide to call *Leibnizian* a functional extension that satisfies both **Poss** and **Tran**, and so necessarily not **Ind**. Topologically, this choice means that the S-topologies of the Leibnizian extensions $^*M$ are *quasi-compact*, but only their restrictions to the "standard' model M are obviously Hausdorff (actually *discrete*). Thus the existence of plenty of *Leibnizian extensions* is granted by the final results of Section 2, without any need of supplementary set theoretic hypotheses.

4. Combining Ind with Poss.

By Theorems 1.2 and 2.5, the principles Ind and Poss cannot hold simultaneously in a functional extension, where Tran always holds. So, if we want to justify our initial assertion that the three Leibnizian principles may be verified in pairs, we have to relax the properties of the $*$-extension given in Definition 1.1, so as to allow the S-topology to be *compact*.

Of course, such a weakening should maintain most preservation properties of the functional extensions, although necessarily losing part of the transfer principle. In particular, by compactness, *all* ultrafilters on X have to be realized as associated to a *unique* point

[5] ... this supposition of two indiscernibles ... seems abstractly possible, but it is incompatible with the order of things...
When I deny that there are ... two indiscernible bodies, I do not say that [this existence] is absolutely impossible to assume, but that it is a thing contrary to divine wisdom ...
The parts of time or place ... are ideal things, so they perfectly resemble like two abstract unities. But it is not so with two concrete Ones,... that is truly actual [things].
I don't say that two points of Space are one same point, neither that two instants of time are one same instant as it seems that one imputes to me ...

$\xi \in {}^*X$. It follows that the typical "analytic" property of the *nonstandard* extension of functions, namely that

everywhere different functions have everywhere different extensions,

has to *fail* in the presence of non-Hausdorff ultrafilters.

Fortunately, such a weakening may be realized still maintaining all the typical preservation properties of the *continuous* extensions of functions, which are allowed to reach equality *only at limit points*. In fact the crucial property of Subsection 2.2

(H) $\quad \overline{f}(\mathcal{U}) = \overline{g}(\mathcal{U}) \iff \{x \in X \mid f(x) = g(x)\} \in \mathcal{U}.$

holds for *arbitrary* ultrafilters whenever at least one of the functions $f, g : X \to X$ either is *injective* or has *finite range*. So we may maintain the conditions (comp) and (dir), and weaken (equ) by postulating it only in those cases.

Therefore we define a *weak functional extension* as follows:

Definition 4.1. A *weak functional extension* of X is a superset ${}^*X$ of X where a *distinguished extension* ${}^*f : {}^*X \to {}^*X$ is assigned to each function $f : X \to X$, so as to satisfy the following conditions

(comp)**: Preservation of compositions:** ${}^*(g \circ f) = {}^*g \circ {}^*f$ for all $f, g : X \to X$;

(wequ)**: Weak preservation of equalizers:** ${}^*(\chi_{fg}) = \chi_{{}^*f{}^*g}$ if at least one of the functions $f, g : X \to X$ either is injective or has finite range;

(dir)**: Directness of the Puritz order:** for all $\xi, \eta \in {}^*X$ there exists $\zeta \in {}^*X$ such that $\xi, \eta \lesssim_P \zeta$.

Also in this weaker context, the extension ${}^*\chi_A$ of the characteristic function χ_A of a subset $A \subseteq X$ can be taken as the characteristic function of the $*$-*extension* ${}^*A$ of A in ${}^*X$, thus obtaining again a *Boolean isomorphism* between the field $\mathcal{P}(X)$ of all subsets of X and a field $\mathcal{C}\ell({}^*X)$ of subsets of ${}^*X$. So the latter can again generate the S-topology, with all the properties stated in Section 2; in particular the $*$-extended functions are continuous (and unique when Ind holds), and many important instances of Tran can be deduced.

A detailed study of the weak functional extensions lies outside the scope of this article. We simply recall that Ind and Poss together imply that the S-topology is compact, and that the map $\xi \mapsto \mathcal{U}_\xi$ establishes a *biunique correspondence* between the points of the *compact* extension ${}^*X$ and the set of all ultrafilters over X. The latter set can be naturally identified with the Stone-Čech compactification βX of the discrete space X, with its usual topology having as basis $\{\mathcal{O}_A \mid A \in \mathcal{P}(X)\}$, where $\mathcal{O}_A$ is the set of all ultrafilters containing A. (The embedding $e : X \to \beta X$ being given by the principal ultrafilters.)

In fact it turns out that the Stone-Čech compactification is essentially the unique weak functional extension of X that satisfies both principles Ind and Poss, according to the following theorem, whose proof specializes that of Theorem 2.1 of [8].

Theorem 4.2. *Let ${}^*X$ be a weak functional extension of X satisfying both principles* Ind *and* Poss*, endowed with the S-topology, and identify the Stone-Čech compactification βX of the discrete space X with the set of all ultrafilters on X. Then the map $\upsilon : {}^*X \to \beta X$ defined by*

$$\upsilon(\xi) = \mathcal{U}_\xi = \{A \subseteq X \mid \xi \in {}^*A\}$$

establishes a homeomorphism between ${}^*X$ with the S-topology and βX with its compact topology. Moreover, for all $f : X \to X$, one has

$$\overline{f} \circ \upsilon = \upsilon \circ {}^*f,$$

where $\overline{f}$ is the unique continuous extension of f to βX.

Proof. For all $x \in X$, $\mathcal{U}_x$ is the principal ultrafilter generated by x, hence υ induces the canonical embedding of X into βX.

Moreover, for all $A \subseteq X$, the map υ induces a mapping of the clopen subset ${}^*A$ of ${}^*X$ onto the basic open set $\mathcal{O}_A$ of βX. Therefore υ is continuous and open with respect to the S-topology of ${}^*X$, *i.e.* υ is bicontinuous between the S-topology of ${}^*X$ and the compact topology of βX.

On the other hand, the map υ is injective because the S-topology is Hausdorff, and surjective because it is quasi-compact. So, being bicontinuous, the map υ is a homeomorphism.

Finally, we have $\xi \in {}^*A \iff {}^*f(\xi) \in {}^\star(f(A))$, for all $\xi \in {}^*X$, or equivalently $A \in \mathcal{U}_\xi \iff f(A) \in \mathcal{U}_{{}^*f(\xi)}$. It follows that $\overline{f} \circ \upsilon = \upsilon \circ {}^*f$, and the last assertion of the theorem is proved. $\square$

5. Final Remarks and Open Questions

Remark that we have proved that all Hausdorff (weak) functional extensions use the same "function-extending mechanism", namely that arising from the Stone-Čech compactification. Therefore, in the Hausdorff case, the choice of the $*$-extensions of functions is forced by the unique topological compactification of X.

Also remark that the conditions (equ) and (dir) are independent, even when Ind holds, provided that Hausdorff ultrafilters exist. Call *invariant* a subspace Y of ${}^*X$ (or of the Stone-Čech compactification βX of X) if

$${}^*f(\xi) \in Y \ \text{(respectively } \overline{f}(\mathcal{U}_\xi) \in Y) \text{ for all } f : X \to X \text{ and all } \xi \in Y.$$

It is easily seen that any invariant subspace Y of a (weak) functional extension ${}^*X$ is itself a (weak) functional extension of X, by taking the restrictions to Y of the functions ${}^*f$ for all $f : X \to X$; and clearly the corresponding ultrafilters $\mathcal{U}_\xi$, $\xi \in Y$ constitute an invariant subspace of βX.

Now there are invariant subspaces of βX where (equ) holds whereas (dir) fails and *vice versa*, as well as invariant subspaces where both fail or hold. In fact, for $\mathcal{U} \in \beta X$ let $Y_\mathcal{U} = \{\overline{f}(\mathcal{U}) \mid f : X \to X\}$ be the invariant subspace generated by $\mathcal{U}$. Clearly $Y_\mathcal{U}$ is directed, so (dir) holds in $Y_\mathcal{U}$ for all ultrafilters $\mathcal{U}$, whereas (equ) holds if and only if $\mathcal{U}$ is Hausdorff. On the other hand, let $\mathcal{U}$ and $\mathcal{V}$ be ultrafilters such that neither of them belongs to the invariant subspace generated by the other one: then $Y_\mathcal{U} \cup Y_\mathcal{V}$ is an invariant subspace where (dir) fails, while (equ) holds if and only if both $\mathcal{U}$ and $\mathcal{V}$ are Hausdorff.

5.1. Some open questions.

We conclude the paper with three open questions that involve special ultrafilters, and so should be of independent set theoretic interest.

(1) Is the existence of functional extensions of $\mathbb{N}$ *without indiscernibles* provable in ZFC, or at least derivable from set-theoretic hypotheses weaker than those providing *selective ultrafilters*? *E.g.* from $\mathfrak{r} = \mathfrak{c}$, where $\mathfrak{r}$ is a cardinal invariant of the continuum not dominated by $\mathbf{cov}(\mathcal{B})$ (see [5])?

(2) Is it consistent with ZFC that there exist nonstandard real lines ${}^*\mathbb{R}$ *without indiscernibles* where all points correspond to *uniform* ultrafilters?

(3) Is the existence of *countably compact functional extensions* consistent with ZFC? (These extensions would be of great interest, because they would verify Ind, Tran, and the weakened version of Poss that considers only *sequences* of properties.)

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