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COALGEBRAIC INFINITE TRACES AND KLEISLI SIMULATIONS

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Abstract. Kleisli simulation is a categorical notion introduced by Hasuo to verify finite trace inclusion. They allow us to give definitions of *forward and backward simulation* for various types of systems. A generic categorical theory behind Kleisli simulation has been developed and it guarantees the soundness of those simulations with respect to *finite* trace semantics. Moreover, those simulations can be aided by *forward partial execution* (FPE)—a categorical transformation of systems previously introduced by the authors.

In this paper, we give Kleisli simulation a theoretical foundation that assures its soundness also with respect to *infinitary* traces. There, following Jacobs' work, infinitary trace semantics is characterized as the "largest homomorphism." It turns out that soundness of forward simulations is rather straightforward; that of backward simulation holds too, although it requires certain additional conditions and its proof is more involved. We also show that FPE can be successfully employed in the infinitary trace setting to enhance the applicability of Kleisli simulations as witnesses of trace inclusion. Our framework is parameterized in the monad for branching as well as in the functor for linear-time behaviors; for the former we mainly use the powerset monad (for nondeterminism), the sub-Giry monad (for probability), and the lift monad (for exception).

1. Introduction

1.1. **Language Inclusion.** *Language inclusion* of transition systems is an important problem in both qualitative and quantitative verification. In a qualitative setting the problem is concretely as follows: for given two nondeterministic systems $\mathcal{X}$ and $\mathcal{Y}$, check if $L(\mathcal{X}) \subseteq L(\mathcal{Y})$—that is, if the set of words generated by $\mathcal{X}$ is included in the set of words generated by $\mathcal{Y}$. In a typical usage scenario, $\mathcal{X}$ is a model of the *implementation* in question while $\mathcal{Y}$ is a model that represents the *specification* of $\mathcal{X}$. More concretely, $\mathcal{Y}$ is a system such that $L(\mathcal{Y})$ is easily seen not to contain anything "dangerous"—therefore the language inclusion $L(\mathcal{X}) \subseteq L(\mathcal{Y})$ immediately implies that $L(\mathcal{X})$ contains no dangerous output, either. Such a

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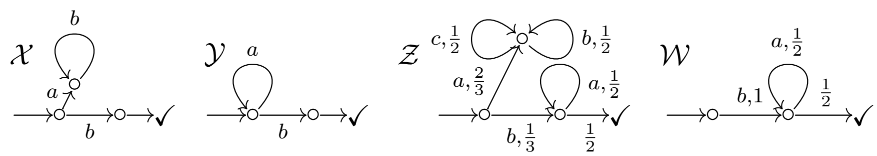


Figure 1: Examples of nondeterministic and probabilistic automata

situation can also arise in a *quantitative setting* where a specification is about *probability*, *reward*, and so on.

Example 1.1. In Figure 1 are four examples of transition systems: $\mathcal{X}$ and $\mathcal{Y}$ are qualitative/nondeterministic; $\mathcal{Z}$ and $\mathcal{W}$ exhibit probabilistic branching. We shall denote the *finite* language of a system $\mathcal{A}$ by $L^*(\mathcal{A})$ and the *infinitary*[1] one by $L^\infty(\mathcal{A})$. We define that a generated finite word is one with a run that ends with the termination symbol ✓.

In the nondeterministic setting, languages are *sets* of words. We have $L^*(\mathcal{X}) = \{b\} \subseteq \{b, ab, aab, \ldots\} = L^*(\mathcal{Y})$, i.e. *finite* language inclusion from $\mathcal{X}$ to $\mathcal{Y}$. However $abb \ldots \in L^\infty(\mathcal{X})$ while $abb \ldots \notin L^\infty(\mathcal{Y})$, hence *infinitary* language inclusion fails.

In the probabilistic setting, languages are naturally *probability distributions* over words; and language inclusion refers to the pointwise order between probabilities. For example $L^*(\mathcal{Z}) = [b \mapsto \frac{1}{6}, ba \mapsto \frac{1}{12}, baa \mapsto \frac{1}{24}, \ldots]$ and $L^*(\mathcal{W}) = [b \mapsto \frac{1}{2}, ba \mapsto \frac{1}{4}, baa \mapsto \frac{1}{8}, \ldots]$; since the former assigns no greater probabilities to all the words, we say that the finite language of $\mathcal{Z}$ is *included* in that of $\mathcal{W}$. This quantitative notion of trace inclusion is also useful in verification: it gives e.g. an upper bound for the probability for something bad.

Finally, the *infinitary* languages for probabilistic systems call for measure-theoretic machinery since, in most of the cases, any infinite word gets assigned the probability 0 (which is also the case in $\mathcal{Z}$ and $\mathcal{W}$). Here it is standard to assign probabilities to *cylinder sets* rather than to individual words; see e.g. [BK08]. An example of a cylinder set is $\{aw \mid w \in \{a, b, c\}^\omega\}$. The language $L^\infty(\mathcal{Z})$ assigns $\frac{2}{3}$ to it, while $L^\infty(\mathcal{W})$ assigns 0; therefore we do not have *infinitary* language inclusion from $\mathcal{Z}$ to $\mathcal{W}$.

1.2. **Simulation.**

There are many known algorithms for checking language inclusion. A well-known one for NFA is a complete one that reduces the problem to emptiness check; however it involves complementation, hence determinization, that incurs an exponential blowup.

One of the alternative approaches to language inclusion is by *simulation*. In the simulation-based verification we look for a simulation, that is, a witness for *stepwise* language inclusion. The notion of simulation is commonly defined so that it implies (proper, global) language inclusion—a property called *soundness*. Although its converse (*completeness*) fails in many settings, such simulation-based approaches tend to have an advantage in computational cost. One prototype example of such simulation notions is *forward* and *backward simulation* by Lynch and Vaandrager [LV95], for nondeterministic automata. They are shown in [LV95] to satisfy soundness with respect to finite traces: explicitly, existence of a forward (or backward) simulation from $\mathcal{X}$ to $\mathcal{Y}$ implies $L^*(\mathcal{X}) \subseteq L^*(\mathcal{Y})$ where the

[1] Note that in this paper, the word "*infinitary*" means "*possibly infinite*" and does not mean all the behaviors have an infinite length. For example, in Figure 1, $L^\infty(\mathcal{X})$ includes a finite-length word b and $L^\infty(\mathcal{X})$ assigns a probability $\frac{1}{6}$ to b.

languages $L^*(\mathcal{X})$ and $L^*(\mathcal{Y})$ collects all the *finite* words generated. The simulations are also shown in [LV95] to be *partly* sound with respect to *infinite* traces: i.e. existence of a forward (or backward, under the additional assumptions of *image-finite* and *total*) simulation from $\mathcal{X}$ to $\mathcal{Y}$ implies $L^\infty(\mathcal{X}) \subseteq L^\infty(\mathcal{Y})$ where the languages $L^\infty(\mathcal{X})$ and $L^\infty(\mathcal{Y})$ collects all the *infinitary* words.

1.3. **Kleisli Simulation and Finite Trace.** *Kleisli simulation* [Has06, Has10, UH14, UH17] is a categorical generalization of these notions of forward and backward simulation by Lynch and Vaandrager. It builds upon the use of coalgebras in a *Kleisli category*, in [HJS07, Jac04, PT99], where they are used to characterize finite traces. Specifically:

- A branching system $\mathcal{X}$ is represented as an *F-coalgebra* $c : X \nrightarrow FX$ in the Kleisli category $\mathcal{K}\ell(T)$, for a suitable choice of a functor F and a monad T. Here F and T are parameters that determine the (linear-time) *transition type* and the *branching type*, respectively, of the system $\mathcal{X}$. Examples are:
  - $F = 1 + \Sigma \times (\_)$ (terminate, or (output and continue)) and the *powerset monad* $T = \mathcal{P}$ on **Sets** (nondeterminism), if $\mathcal{X}$ is a nondeterministic automaton (with explicit termination);
  - the same functor $F = 1 + \Sigma \times (\_)$ and the *sub-Giry monad* $T = \mathcal{G}$ [Gir82] on the category **Meas** of measurable spaces and measurable functions, for their probabilistic variant; and
  - the same functor $F = 1 + \Sigma \times (\_)$ and the *lift monad* $T = \mathcal{L}$ on **Sets** for automata with exception.
- In [HJS07], under certain conditions on F and T, it is shown that a *final F-coalgebra* in $\mathcal{K}\ell(T)$ arises as a lifting of an initial F-algebra in **Sets**. Moreover, it is observed that the natural notion of "finite trace semantics" or "(finite) languages" is captured by a unique homomorphism via finality. This works uniformly for a wide variety of systems, by changing F and T.

It is shown in [Has06] that, with respect to this categorical modeling of finite traces [HJS07], both forward and backward Kleisli simulation are indeed sound. This categorical background allows us to instantiate Kleisli simulation for various concrete systems—including both qualitative and quantitative ones—and obtain simulation notions whose soundness with respect to *finite* traces comes for free [Has06, Has10]. Like many other notions of simulation, the resulting simulation notion sometimes fails to be complete. This drawback of incompleteness with respect to finite traces can be partly mended by *forward partial execution* (FPE), a transformation of systems introduced in [UH14] (and its extended version [UH17]) that potentially increases the likelihood of existence of simulations.

1.4. **Contributions.** While the automata-theoretic simulations in [LV95] are known to be useful for checking both finite and infinitary trace inclusion, their coalgebraic generalization (Kleisli simulation) has been applied only to the finite trace setting. In this paper we continue our series of work [Has06, Has10, UH14, UH17] and study the relationship between Kleisli simulations and *infinitary* traces. This turns out to be more complicated than we had expected, a principal reason being that *infinitary* traces are less well-behaved than finite traces (the latter being characterized simply by finality).

For a suitable coalgebraic modeling of infinitary traces we principally follow [Jac04]—also relying on observations in [Cîr10, KK13]—and characterize infinitary traces in terms of

branching type	monad T	finite trace [Has06]	infinitary trace [current work]
nondeterministic	$\mathcal{P}$	fwd. sim. bwd. sim.	fwd. sim. TIF-bwd. sim.
probabilistic	$\mathcal{G}$	fwd. sim. bwd. sim.	fwd. sim. total bwd. sim.
with exception	$\mathcal{L}$	fwd. sim. bwd. sim.	fwd. sim. total bwd. sim.

Table 1: Three different settings

largest homomorphisms. More specifically, we lift a final F-coalgebra in **Sets** to the Kleisli category $\mathcal{K}\ell(T)$ and prove that the latter admits a largest homomorphism. In this paper we (principally) work with: the powerset monad $\mathcal{P}$ (on **Sets**), the sub-Giry monad $\mathcal{G}$ (on **Meas**), and the lift monad $\mathcal{L}$ (on **Sets**) as a monad T for branching (see Table 1); and a polynomial functor F for linear-time behaviors.

Here are our concrete contributions. For each of the above combinations of T and F:

- We show that forward Kleisli simulations are sound with respect to inclusion of *infinitary* languages. The proof of this general result is not hard, exploiting the above coalgebraic modeling of infinitary languages as largest homomorphisms.
- We show that backward simulations are sound too, although here we have to impose suitable restrictions, namely *totality* and *image-finiteness*. The soundness proofs are much more involved, too, and call for careful inspection of the construction of infinitary trace semantics. The proofs are given separately for $T = \mathcal{P}$, and for $T = \mathcal{G}$ and $\mathcal{L}$, because of differences in the relevant constructions (see Remark 5.1).
- We show that *forward partial execution* (FPE)—a transformation from [UH14, UH17] that aids discovery of forward or backward simulations—is applicable also to the current setting of *infinitary* trace inclusion. More specifically we prove: *soundness* of FPE (discovery of a simulation after FPE indeed witnesses infinitary language inclusion); and its *adequacy* (FPE does not destroy simulations that are already there). Suitable restrictions, *totality* and *image-finiteness*, are imposed on the simulating system in order to ensure the adequacy with respect to backward simulation.

Organization. Section 2 is devoted to categorical preliminaries; we fix notations there. In Section 3 we review the previous works that we rely on, namely coalgebraic infinitary trace semantics [Jac04], Kleisli simulation [Has06, Has10, UH14, UH17], and FPE [UH14, UH17]. Our technical contributions are in the subsequent sections: in Section 4 we study the nondeterministic setting (i.e. the powerset monad $\mathcal{P}$ on **Sets** and a polynomial functor F); Section 5 is for the probabilistic setting (where the monad T is the sub-Giry monad $\mathcal{G}$); and Section 6 is for systems with exception (where T is the lift monad $\mathcal{L}$).

Some definitions and results in Sections 4–5 are marked with †. Those marked ones are essentially results for specific settings (namely $T = \mathcal{P}$ and $T = \mathcal{G}$) but formulated in general terms with an arbitrary T subject to certain axioms. We do so in the hope that the axioms thus identified will help to discover new instances. Indeed, the results in Section 6 for $T = \mathcal{L}$ are derived from the general results developed in Section 5.

Compared to the earlier version [UH15] of this paper, the current version additionally contains the following materials.

- Section 2.2 is added, where preliminaries about ranked alphabet and infinitary trees are given.
- We added sections where coincidence between the coalgebraic definition and the automata-theoretic definition of infinitary language is presented. Namely: Section 4.4 is for nondeterministic setting, Section 5.4 is for probabilistic setting, and Section 6.4 for the setting where the system can abort with an exception.
- In [UH15] we mainly used two monads—$\mathcal{P}$ and $\mathcal{G}$. Now one additional monad—the *lift monad* $\mathcal{L}$—is also discussed in this paper. Moreover, a brief discussion about the *subdistribution monad* $\mathcal{D}$ is added, too.
- We added some examples that are absent in [UH15].
- We added proofs that are omitted in [UH15].

2. Preliminaries

2.1. Categorical Preliminaries.

Definition 2.1. A *polynomial functor* F on **Sets** is defined by the following BNF notation: $F ::= \mathrm{id} \mid A \mid F_1 \times F_2 \mid \coprod_{i\in I} F_i$. Here $A \in$ **Sets** and I is a countable set.

The notion of polynomial functor can be also defined for **Meas**—the category of measurable spaces and measurable functions between them.

Definition 2.2. A *(standard Borel) polynomial functor* F on **Meas** is defined by the following BNF notation: $F ::= \mathrm{id} \mid (A, \mathfrak{F}_A) \mid F_1 \times F_2 \mid \coprod_{i\in I} F_i$. Here I is a countable set; and we require that $(A, \mathfrak{F}_A) \in$ **Meas** is a *standard Borel space* (see e.g. [Doo94]). The σ-algebra $\mathfrak{F}_{FX}$ associated to FX is defined in the obvious manner. Namely: for $F = \mathrm{id}$, $\mathfrak{F}_{FX} = \mathfrak{F}_X$; for $F = (A, \mathfrak{F}_A)$, $\mathfrak{F}_{FX} = \mathfrak{F}_A$; for $F = F_1 \times F_2$, $\mathfrak{F}_{FX}$ is the smallest σ-algebra that contains $A_1 \times A_2$ for all $A_1 \in \mathfrak{F}_{F_1X}$ and $A_2 \in \mathfrak{F}_{F_2X}$; for for $F = \coprod_{i\in I} F_i$, $\mathfrak{F}_{FX} = \{\coprod_{i\in I} A_i \mid A_i \in \mathfrak{F}_{F_iX}\}$.

On arrows, F acts in the same manner as a polynomial functor on **Sets**.

In what follows, a standard Borel polynomial functor is often called simply a *polynomial functor*.

The technical requirement of being standard Borel in the above will be used in the probabilistic setting of Section 5 (it is also assumed and exploited in [Cîr10] and in [Sch09] that we rely on). A standard Borel space is a measurable space induced by a Polish space; for further details see e.g. [Doo94].

We go on to introduce monads T for branching. We principally use three monads—the *powerset monad* $\mathcal{P}$ on **Sets**, the *sub-Giry* monad $\mathcal{G}$ on **Meas**, and the *lift monad* $\mathcal{L}$ on **Sets**. The monad $\mathcal{G}$ is an adaptation of the *Giry monad* [Gir82] and inherits most of its structure from the Giry monad; see Remark 2.7.

Definition 2.3 (monads $\mathcal{P}$, $\mathcal{G}$ and $\mathcal{L}$)**.** The *powerset monad* is the monad $(\mathcal{P}, \eta^{\mathcal{P}}, \mu^{\mathcal{P}})$ on **Sets** such that $\mathcal{P}X = \{A \subseteq X\}$ and $\mathcal{P}f(A) = \{f(x) \mid x \in A\}$. Its unit is given by the singleton set $\eta^{\mathcal{P}}_X(x) = \{x\}$ and its multiplication is given by $\mu^{\mathcal{P}}_X(M) = \bigcup_{A\in M} A$.

The *sub-Giry monad* is the monad $(\mathcal{G}, \eta^{\mathcal{G}}, \mu^{\mathcal{G}})$ on **Meas** such that

- $\mathcal{G}(X, \mathfrak{F}_X) = (\mathcal{G}X, \mathfrak{F}_{\mathcal{G}X})$, where the underling set $\mathcal{G}X$ is the set of all *subprobability measures* on $(X, \mathfrak{F}_X)$. The latter means those measures which assign to the whole space X a value in the unit interval $[0, 1]$.

- The σ-algebra $\mathfrak{F}_{\mathcal{G}X}$ on $\mathcal{G}X$ is the smallest σ-algebra such that, for all $S \in \mathfrak{F}_X$, the function $\mathrm{ev}_S : \mathcal{G}X \to [0,1]$ defined by $\mathrm{ev}_S(P) = P(S)$ is measurable.
- $\mathcal{G}f(\nu)(S) = \nu(f^{-1}(S))$ where $f : (X, \mathfrak{F}_X) \to (Y, \mathfrak{F}_Y)$ is measurable, $\nu \in \mathcal{G}X$, and $S \in \mathfrak{F}_Y$.
- $\eta^{\mathcal{G}}_{(X,\mathfrak{F}_X)(x)}$ is given by the *Dirac measure*: $\eta^{\mathcal{G}}_{(X,\mathfrak{F}_X)}(x)(S)$ is 1 if $x \in S$ and 0 otherwise.
- $\mu^{\mathcal{G}}_{(X,\mathfrak{F}_X)}(\Psi)(S) = \int_{\mathcal{G}(X,\mathfrak{F}_X)} \mathrm{ev}_S \, d\Psi$ where $\Psi \in \mathcal{G}^2X$, $S \in \mathfrak{F}_X$ and ev_S is defined as above.

The *lift monad* is a monad $(\mathcal{L}, \eta^{\mathcal{L}}, \mu^{\mathcal{L}})$ on **Sets** such that $\mathcal{L}X = \{\bot\} + X$, $\mathcal{L}f(y) = f(y)$ if $y \neq \bot$ and $\mathcal{L}f(y) = \bot$ otherwise. Its unit is given by $\eta^{\mathcal{L}}_X(x) = x$ and its multiplication is given by $\mu^{\mathcal{L}}_X(z) = z$.

A monad gives rise to a category called its *Kleisli category* (see e.g. [Mac78]).

Definition 2.4 (Kleisli category $\mathcal{K}\ell(T)$)**.** Given a monad (T, η, μ) on a category $\mathbb{C}$, the *Kleisli category* for T is the category $\mathcal{K}\ell(T)$ whose objects are the same as those of $\mathbb{C}$, and for each pair of objects X, Y, the homset $\mathcal{K}\ell(T)(X,Y)$ is given by $\mathbb{C}(X, TY)$. An arrow in $\mathcal{K}\ell(T)$ is referred to as a *Kleisli arrow*, and depicted by $X \nrightarrow Y$ for distinction from $\mathbb{C}$. Note that it is nothing but an arrow $X \to TY$ in the base category $\mathbb{C}$.

Moreover, for two sequential Kleisli arrows $f : X \nrightarrow Y$ and $g : Y \nrightarrow Z$, their composition is given by $\mu_Z \circ Tg \circ f$ and denoted by $g \odot f$. The *Kleisli inclusion functor* is the functor $J : \mathbb{C} \to \mathcal{K}\ell(T)$ such that $JX = X$ and $Jf = \eta_Y \circ f$ for $f : X \to Y$ in $\mathbb{C}$.

It is known that a functor $F : \mathbb{C} \to \mathbb{C}$ canonically lifts to a functor $\overline{F} : \mathcal{K}\ell(T) \to \mathcal{K}\ell(T)$, given that there exists a natural transformation $\lambda : FT \Rightarrow TF$ that is compatible with the unit and the multiplication of T.

Lemma 2.5 (distributive law, [Mul93])**.** *Let T be a monad and F be an endofunctor on a category $\mathbb{C}$. The following conditions are equivalent*

(1) *The functor F can be lifted to the Kleisli category $\mathcal{K}\ell(T)$: namely, there exists a functor $\overline{F} : \mathcal{K}\ell(T) \to \mathcal{K}\ell(T)$ such that $\overline{F} \circ J = J \circ F$.*
(2) *There exists a natural transformation $\lambda : FT \Rightarrow TF$ such that the following diagrams commute for all $X \in \mathbb{C}$.*

$$\begin{array}{ccc} FX & \xrightarrow{F\eta_X} & FTX \\ & \searrow^{\eta_{FX}} & \downarrow{\scriptstyle \lambda_X} \\ & & TFX \end{array} \qquad \begin{array}{ccccc} FT^2X & \xrightarrow{\lambda_{TX}} & TFTX & \xrightarrow{T\lambda_X} & T^2FX \\ \downarrow{\scriptstyle F\mu_X} & & & & \downarrow{\scriptstyle \mu_{FX}} \\ FTX & & \xrightarrow{\lambda_X} & & TFX \end{array}$$

Such λ is called a distributive law*.* □

Throughout this paper, we fix the orders on homsets of $\mathcal{K}\ell(\mathcal{P})$, $\mathcal{K}\ell(\mathcal{G})$ and $\mathcal{K}\ell(\mathcal{L})$ as follows.

Definition 2.6 (order enrichment of $\mathcal{K}\ell(\mathcal{P})$, $\mathcal{K}\ell(\mathcal{G})$ and $\mathcal{K}\ell(\mathcal{L})$)**.** We define an order on $\mathcal{K}\ell(\mathcal{P})(X,Y)$ by: $f \sqsubseteq g$ if and only if $\forall x \in X.\, f(x) \subseteq g(x)$. For $\mathcal{K}\ell(\mathcal{G})(X,Y)$ we define: $f \sqsubseteq g$ if and only if $\forall x \in X. \forall A \in \mathfrak{F}_Y.\, f(x)(A) \leq g(x)(A)$. Here the last $\leq$ is the usual order in the unit interval $[0,1]$. We define an order on $\mathcal{K}\ell(\mathcal{L})(X,Y)$ by $f \sqsubseteq g$ if and only if $\forall x \in X.\, f(x) = \bot$ or $f(x) = g(x)$.

Remark 2.7. The sub-Giry monad $\mathcal{G}$ is an adaptation of the *Giry monad* from [Gir82]; in the original Giry monad one only allows (proper) *probability measures*, i.e. measures that map the whole space to 1. We work with the sub-Giry monad because, without this relaxation from probability to subprobability, the order structure in Definition 2.6 is reduced to the equality.

2.2. **Ranked Alphabet and Infinitary Trees.** There is a natural correspondence between polynomial functors and *ranked alphabets*. In this paper a functor F for the (linear-time) transition type is restricted to a polynomial one; this means that we are dealing with (T-branching) systems that generate *trees* over some ranked alphabet. Here we collect some standard notions and notations on (the conventional presentation of) such finite/infinite trees; they will be used later in showing that our coalgebraic infinitary traces indeed capture infinitary tree languages of such systems.

Trees are labeled with letters from an alphabet.

Definition 2.8. A *ranked alphabet* is a family $\Sigma = (\Sigma_n)_{n\in\omega}$ of sets. A *standard Borel ranked alphabet* is a family $\Sigma = \big((\Sigma_n, \mathfrak{F}_n)\big)_{n\in\omega}$ of standard Borel spaces. The index $n \in \omega$ is called an *arity*.

For the definition of infinitary trees we follow [Cou83]. Each node labeled with a letter $a \in \Sigma_n$ of arity n has n children. The idea of k-prefix trees, presented below, is introduced in [Ell71]. It can be regarded as a finite tree of depth k that is obtained by truncating an infinitary tree.

Definition 2.9. Let $\Sigma = (\Sigma_n)_{n\in\omega}$ be a ranked alphabet. A Σ-*labeled infinitary tree* is a pair (D, l) of a *domain* $D \subseteq \mathbb{N}^*$ and a *labeling function* $l : D \to \bigcup_{n\in\omega} \Sigma_n$ such that:

- the domain D is prefix-closed (i.e. $\forall \alpha \in \mathbb{N}^*. \forall i \in \mathbb{N}.\ \alpha i \in D \Rightarrow \alpha \in D$), nonempty, and downward-closed (i.e. $\forall \alpha \in \mathbb{N}^*. \forall i \in \mathbb{N}.\ \alpha i \in D \Rightarrow \forall j \le i.\ \alpha j \in D$); and
- for all $\alpha \in D$ such that $l(\alpha) \in \Sigma_n$ and $i \in \mathbb{N}$, $\alpha i \in D$ if and only if $0 \le i \le n-1$.

For $k \in \omega$, a Σ-*labeled k-prefix tree* is a pair (D, l) of a *domain* D and a *labeling function* $l : D \to \bigcup_{n\in\omega} \Sigma_n$ such that:

- if $k = 0$ then D is an empty set and if $k > 0$ then $D \subseteq \bigcup_{i\le k-1} \mathbb{N}^i$;
- the domain D is prefix-closed, nonempty, and downward-closed; and
- for $\alpha \in D$ such that $|\alpha| < k-1$ and $l(\alpha) \in \Sigma_n$ and $i \in \mathbb{N}$, $\alpha i \in D$ if and only if $0 \le i \le n-1$.

Here k is called the *depth* of (D, l). We write $\mathrm{Tree}_\infty(\Sigma)$ and $\mathrm{Tree}^k(\Sigma)$ for the sets of all Σ-labeled infinitary trees and Σ-labeled k-prefix trees, respectively.

A k-prefix tree $t = (D, l) \in \mathrm{Tree}^k(\Sigma)$ is called a *prefix* of a tree $t' = (D', l') \in \mathrm{Tree}_\infty(\Sigma) \cup \bigcup_{k'\ge k} \mathrm{Tree}^{k'}(\Sigma)$ if $D \subseteq D'$ and $l(\alpha) = l'(\alpha)$ holds for all $\alpha \in D$. We write $t \preceq t'$ if t is a prefix of t'. For $t \in \mathrm{Tree}^k(\Sigma)$, a *cylinder set induced by* t is the set $\mathrm{cyl}(t) \subseteq \mathrm{Tree}_\infty(\Sigma)$ that is defined by $\mathrm{cyl}(t) = \{t' \in \mathrm{Tree}_\infty(\Sigma) \mid t \preceq t'\}$.

For $t = (D, l) \in \mathrm{Tree}_\infty(\Sigma)$ and $\alpha \in D$, *the α-th subtree of* t is a Σ-labeled infinitary tree $t_\alpha = (D_\alpha, l_\alpha) \in \mathrm{Tree}_\infty(\Sigma)$ where $D_\alpha = \{\beta \in \mathbb{N}^* \mid \alpha\beta \in D\}$ and $l_\alpha(\beta) = l(\alpha\beta)$.

In later sections we will use an *infinitary tree automaton*—an automaton that generates Σ-labeled trees. Note that an infinitary tree automaton might output a finite-depth tree whose leaves are labeled with 0-ary letters $a \in \Sigma_0$ (cf. footnote 1 on page 2). Especially when $\Sigma_0 = \{\checkmark\}$ and $\Sigma_i = \emptyset$ for all $i \ge 2$, an infinitary tree automaton can be regarded as an automaton that generates *words* instead of trees. We call such an automaton *infinitary automaton* (suppressing the word "tree").

Remark 2.10. As described above, we regard infinitary tree automata as *generative* ones that *output* trees throughout this paper. Another characterization of behaviors of infinitary tree automata is to regard them as *reactive* ones that *accept* trees. These two characterizations coincide in the nondeterministic setting. In contrast, they are distinguished in the probabilistic setting: in the former case, an automaton randomly outputs a tree while in the latter case, an automaton assigns a probability where the tree is accepted to each tree.

A system that generates Σ-labeled infinitary trees will later be represented as an $\overline{F_\Sigma}$-coalgebra on the Kleisli category of some monad where $\overline{F_\Sigma}$ is a lifting of F_Σ. Here the functor F_Σ is a polynomial functor defined as follows.

Definition 2.11. For a ranked alphabet $\Sigma = (\Sigma_n)_{n\in\omega}$, we define $F_\Sigma : \mathbf{Sets} \to \mathbf{Sets}$ by $F_\Sigma = \coprod_{n\in\omega} \Sigma_n \times (\_)^n$. For a standard Borel ranked alphabet $\Sigma = \big((\Sigma_n, \mathfrak{F}_n)\big)_{n\in\omega}$, we define $F_\Sigma : \mathbf{Meas} \to \mathbf{Meas}$ by $F_\Sigma = \coprod_{n\in\omega} (\Sigma_n, \mathfrak{F}_n) \times (\_)^n$.

3. Infinite Traces, Kleisli Simulations and Coalgebras in $\mathcal{K}\ell(T)$

In this section we review some categorical constructs, the relationship among which lies at the heart of this paper. They are namely: coalgebraic infinitary trace semantics [Jac04], Kleisli simulation [Has06, Has10, UH14, UH17] and forward partial execution (FPE) [UH14, UH17].

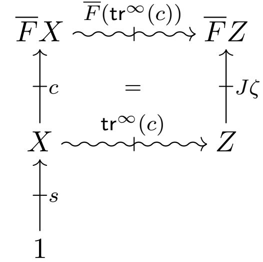
The following situation is identified in [Jac04] (see also Section 4.4, Section 5.4 and Section 6.4): the largest homomorphism to a certain coalgebra (that we describe below) coincides with the standard, conventionally defined notion of infinitary language, for a variety of systems. An instance of it is shown to arise, in [Jac04], when $\mathbb{C} = \mathbf{Sets}$, $T = \mathcal{P}$ and F is a polynomial functor. In Section 4 we will give another proof for this fact; the new proof will serve our goal of showing soundness of backward simulations.

Definition 3.1 (infinitary trace situation)**.** Let F be an endofunctor and T be a monad on a category $\mathbb{C}$. We assume that each homset of the Kleisli category $\mathcal{K}\ell(T)$ carries an order $\sqsubseteq$. The functor F and the monad T constitute an *infinitary trace situation* with respect to $\sqsubseteq$ if they satisfy the following conditions.

- There exists a final F-coalgebra $\zeta : Z \to FZ$ in $\mathbb{C}$.
- There exists a distributive law $\lambda : FT \Rightarrow TF$, yielding a lifting $\overline{F}$ on $\mathcal{K}\ell(T)$ of F by Lemma 2.5.
- For each coalgebra $c : X \nrightarrow \overline{F}X$ in $\mathcal{K}\ell(T)$, the lifting $J\zeta : Z \nrightarrow \overline{F}Z$ of ζ admits the largest homomorphism. That is, there exists a homomorphism $\mathsf{tr}^\infty(c) : X \nrightarrow Z$ from c to $J\zeta$ such that, for any homomorphism f from c to $J\zeta$, $f \sqsubseteq \mathsf{tr}^\infty(c)$ holds.

In [Has06, Has10, UH14, UH17] we augment a coalgebra with an explicit arrow for initial states. The resulting notion is called a (T, F)-system.

Definition 3.2 ((T, F)-systems and their infinitary trace semantics, [HJS07, Jac04])**.** Let $\mathbb{C}$ be a category with a final object $1 \in \mathbb{C}$. A (T, F)-*system* is a triple $\mathcal{X} = (X, s, c)$ consisting of a *state space* $X \in \mathbb{C}$, a Kleisli arrow $s : 1 \nrightarrow X$ for *initial states*, and $c : X \nrightarrow \overline{F}X$ for *transition*.

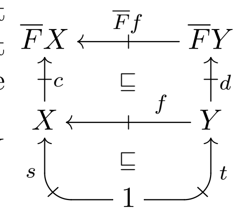


Let us assume that the endofunctor F and the monad T on $\mathbb{C}$ constitute an infinitary trace situation. The *coalgebraic infinitary trace semantics* of a (T, F)-system $\mathcal{X} = (X, s, c)$ is the Kleisli arrow $\mathsf{tr}^\infty(c) \odot s : 1 \nrightarrow Z$ where $\mathsf{tr}^\infty(c)$ is the largest homomorphism in Definition 3.1 (see the diagram, in $\mathcal{K}\ell(T)$, on the right). Recall that $\odot$ denotes composition in $\mathcal{K}\ell(T)$ (Definition 2.4).

Suppose that we are given two (T, F)-systems $\mathcal{X} = (X, s, c)$ and $\mathcal{Y} = (Y, t, d)$. Let us say we aim to prove the inclusion between infinitary trace semantics, that is, to show $\mathsf{tr}^{\infty}(c) \odot s \sqsubseteq \mathsf{tr}^{\infty}(d) \odot t$ with respect to the order in the homset $\mathcal{K}\ell(T)(1, Z)$. Our goal in this paper is to offer *Kleisli simulations* as a sound means to do so.

The notions of *forward* and *backward Kleisli simulation* are introduced in [Has06] as a categorical generalization of forward or backward simulations in [LV95]. They are defined as Kleisli arrows between (the state spaces of) two (T, F)-system that are subject to certain inequalities—in short they are *lax/oplax coalgebra homomorphisms*. In [Has06] they are shown to be sound with respect to *finite* trace semantics—the languages of finite words, concretely, and the unique arrow to a lifted initial algebra (that is a final coalgebra, see [HJS07] and the introduction), abstractly. In this paper we are interested in their relation to *infinitary* trace semantics.

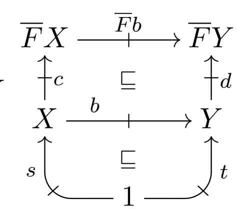
Definition 3.3 (forward and backward Kleisli simulation, [Has06])**.** Let F be an endofunctor and T be a monad on $\mathbb{C}$ such that each homset of $\mathcal{K}\ell(T)$ carries an order $\sqsubseteq$. Let $\mathcal{X} = (X, s, c)$ and $\mathcal{Y} = (Y, t, d)$ be (T, F)-systems.



A *forward Kleisli simulation* from $\mathcal{X}$ to $\mathcal{Y}$ is a Kleisli arrow $f : Y \nrightarrow X$ that satisfies the following conditions (see the diagram):

$$s \sqsubseteq f \odot t, \quad \text{and} \quad c \odot f \sqsubseteq \overline{F}f \odot d.$$

We write $\mathcal{X} \sqsubseteq_{\mathbf{F}} \mathcal{Y}$ if there exists a forward simulation from $\mathcal{X}$ to $\mathcal{Y}$.



A *backward Kleisli simulation* from $\mathcal{X}$ to $\mathcal{Y}$ is a Kleisli arrow $b : X \nrightarrow Y$ that satisfies the following conditions (see the diagram):

$$b \odot s \sqsubseteq t, \quad \text{and} \quad \overline{F}b \odot c \sqsubseteq d \odot b.$$

We write $\mathcal{X} \sqsubseteq_{\mathbf{B}} \mathcal{Y}$ if there exists a backward simulation from $\mathcal{X}$ to $\mathcal{Y}$.

Forward partial execution (FPE) is a transformation of (T, F)-systems introduced in [UH14] (and its extended version [UH17]) for the purpose of aiding discovery of Kleisli simulations. Intuitively, it "executes" the given system by one step.

Definition 3.4 (FPE, [UH14, UH17])**.** Let F be an endofunctor and T be a monad on $\mathbb{C}$. *Forward partial execution* (FPE) is a transformation that takes a (T, F)-system $\mathcal{X} = (X, s : 1 \nrightarrow X, c : X \nrightarrow \overline{F}X)$ as input, and returns a (T, F)-system $\mathcal{X}_{\mathsf{FPE}} = (\overline{F}X, c \odot s : 1 \nrightarrow \overline{F}X, \overline{F}c : \overline{F}X \nrightarrow \overline{F}^2 X)$ as output.

It is shown in [UH17] that FPE is a valid technique for establishing inclusion of *finite* trace semantics, in the technical senses of *soundness* and *adequacy*. Soundness asserts that discovery of a Kleisli simulation after applying FPE indeed witnesses trace inclusion between the original systems; adequacy asserts that if there is a Kleisli simulation between the original systems, then there is one, too, between the transformed systems. In this paper, naturally, we wish to establish the same results for *infinitary* trace semantics.

4. Systems with Nondeterministic Branching

In the rest of the paper we develop a coalgebraic theory of infinitary traces and (Kleisli) simulations—the main contribution of the paper. We do so separately for the nondeterministic setting ($T = \mathcal{P}$), for the probabilistic setting ($T = \mathcal{G}$), and for the setting where the system can abort with an exception ($T = \mathcal{L}$). This is because of the difference in the constructions of infinitary traces, and consequently in the soundness proofs.

In this section we focus on the nondeterministic setting; we assume that F is a polynomial functor on **Sets**.

4.1. Construction of Infinitary Traces.

We start with showing that the combination of polynomial F and $T = \mathcal{P}$ constitute an infinitary trace situation (Definition 3.1). This is already known from [Jac04]. The proof in [Jac04] combines fibrational intuitions with some constructions that are specific to **Sets**. Here we present a different proof. It exploits an order-theoretic structure of the Kleisli category $\mathcal{K}\ell(\mathcal{P})$; this will be useful later in showing soundness of (restricted) backward simulations. Our proof also paves the way to the probabilistic case in Section 5.

In fact, our proof for infinitary trace situation is stated axiomatically, in the form of the following proposition. (Recall that statements marked with † are axiomatic ones.) This is potentially useful in identifying new examples other than the combination of polynomial F and $T = \mathcal{P}$ (although we have not yet managed to do so). Its proof is essentially the construction of a greatest fixed point by transfinite induction [CC79].

Proposition 4.1.† *Let $\mathbb{C}$ be a category, F be an endofunctor on $\mathbb{C}$, and T be a monad on $\mathbb{C}$. Assume the following conditions.*

1. *There exists a final F-coalgebra $\zeta : Z \to FZ$ in $\mathbb{C}$.*
2. *There exists a distributive law $\lambda : FT \Rightarrow TF$, yielding a lifting $\overline{F}$ on $\mathcal{K}\ell(T)$ of F.*
3. *For each $X, Y \in \mathcal{K}\ell(T)$, the homset $\mathcal{K}\ell(T)(X, Y)$ carries a partial order $\sqsubseteq$. Moreover, $\overline{F}$'s action on arrows, as well as composition of arrows in $\mathcal{K}\ell(T)$, is monotone with respect to this order.*
4. *For each $X, Y \in \mathcal{K}\ell(T)$, every (possibly transfinite) decreasing sequence in $\mathcal{K}\ell(T)(X, Y)$ has the greatest lower bound. That is: let $\mathfrak{a}$ be a ordinal and $(g_{\mathfrak{i}} : X \nrightarrow Y)_{\mathfrak{i}<\mathfrak{a}}$ be a family of arrows such that $\mathfrak{i} \le \mathfrak{j}$ implies $g_{\mathfrak{i}} \sqsupseteq g_{\mathfrak{j}}$. Then their infimum $\bigsqcap_{\mathfrak{i}<\mathfrak{a}} g_{\mathfrak{i}}$ exists.*
5. *For each $X \in \mathbb{C}$, the homset $\mathcal{K}\ell(T)(X, Z)$ has the largest element $\top_{X,Z}$.*

Then T and F constitute an infinitary trace situation with respect to $\sqsubseteq$.

Proof. Let $c : X \nrightarrow \overline{F}X$ be an $\overline{F}$-coalgebra in $\mathcal{K}\ell(T)$. We shall construct the largest homomorphism $\mathsf{tr}^{\infty}(c) : X \nrightarrow Z$ from c to $J\zeta$, by transfinite induction.

We define an endofunction $\Phi_c : \mathcal{K}\ell(T)(X, Z) \to \mathcal{K}\ell(T)(X, Z)$ by $\Phi_c(f) = J\zeta^{-1} \odot \overline{F}f \odot c$. By the monotonicity of Kleisli composition $\odot$ and the functor $\overline{F}$ (Assumption (3)), Φ_c is also monotone. For each ordinal $\mathfrak{a}$, we define $\Phi_c^{\mathfrak{a}}(\top_{X,Z}) \in \mathcal{K}\ell(T)(X, Z)$ by transfinite induction on $\mathfrak{a}$ as follows:

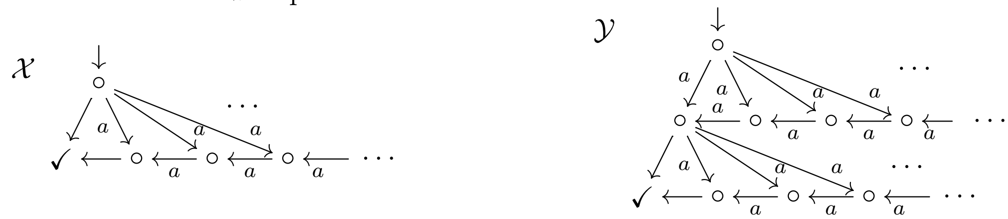
- $\Phi_c^0(\top_{X,Z}) = \top_{X,Z}$;
- For a successor ordinal $\mathfrak{a}$, $\Phi_c^{\mathfrak{a}}(\top_{X,Z}) = \Phi_c(\Phi_c^{\mathfrak{a}-1}(\top_{X,Z}))$; and
- For a limit ordinal $\mathfrak{a}$, $\Phi_c^{\mathfrak{a}}(\top_{X,Z}) = \bigsqcap_{\mathfrak{i}<\mathfrak{a}} \Phi_c^{\mathfrak{i}}(\top_{X,Z})$ (cf. Assumption (4)).

We define $\mathfrak{l}$ to be the smallest ordinal such that the cardinality of $\mathfrak{l}$ is greater than that of $\mathcal{K}\ell(T)(X, Z)$. Then from [CC79], $\Phi_c^{\mathfrak{l}}(\top_{X,Z})$ is the greatest fixed point of Φ_c. Note that a Kleisli arrow is a homomorphism from c to $J\zeta$ if and only if it is a fixed point of Φ_c. Therefore $\Phi_c^{\mathfrak{l}}(\top_{X,Z})$ is the largest homomorphism from c to $J\zeta$. □

Note that the ω^{op}-continuity—preservation of the greatest lower bound of a decreasing sequence—of composition $\odot$ in $\mathcal{K}\ell(T)$ is not assumed. This is because $\mathcal{P}$—our choice for T

in this section—does not satisfy it, while it satisfies ω-continuity—preservation of the least upper bound of an increasing sequence. Indeed, consider $f : X \nrightarrow Y$ and a decreasing sequence $(g_i : Y \nrightarrow Z)_{i\in\omega}$, both in $\mathcal{K}\ell(\mathcal{P})$. Then we have $\left(\left(\bigsqcap_{i\in\omega} g_i\right) \odot f\right)(x) = \bigcup_{y\in f(x)} \bigcap_{i\in\omega} g_i(y)$ while $\left(\bigsqcap_{i\in\omega}(g_i \odot f)\right)(x) = \bigcap_{i\in\omega} \bigcup_{y\in f(x)} g_i(y)$, and these two are not equal in general. This failure of ω^{op}-continuity prevents us from applying the (simpler) *Kleene fixed-point theorem*, in which induction terminates after ω steps. The following examples show that there does exist a nondeterministic automaton for which the largest homomorphism is obtained after steps bigger than ω.

Example 4.2. In the construction of the largest homomorphism in Proposition 4.1, we need $\omega + 1$ steps for the nondeterministic automaton $\mathcal{X}$ on the left below. We need $2\omega + 1$ steps for $\mathcal{Y}$ on the right below. In a similar manner, for an arbitrary ordinal $\mathfrak{a}$, we can construct an automaton where $\mathfrak{a}$ steps are needed.



It is easy to check that all the assumptions in Proposition 4.1 are satisfied by polynomial F and $T = \mathcal{P}$. Therefore we have the following result that is the main theorem of this section.

Theorem 4.3. *The combination of polynomial F and $T = \mathcal{P}$ constitute an infinitary trace situation.*

Proof. We show that F and $\mathcal{P}$ satisfy Assumptions (1)–(5) in Proposition 4.1.

It is known that Assumption (1) is satisfied [AK79]. It is known that Assumption (2) is also satisfied [HJS07, Lemma 2.4]. It is easy to see that F and $\mathcal{P}$ on **Sets** satisfy the Assumptions (3) and (4) where the infimum is given by intersection. It is also easy to see that Assumption (5) is satisfied; $\top_{X,Z} : X \nrightarrow Z$ is given by $\top_{X,Z}(x) = Z$ for all $x \in X$.

Therefore by Proposition 4.1, F and $\mathcal{P}$ constitute an infinitary trace situation. □

4.2. **Kleisli Simulations for Nondeterministic Systems.** In this section we prove that forward and backward Kleisli simulations can be used to witness *infinitary* trace inclusion. This fact is already shown in [LV95] for nondeterministic word automata. The coalgebraic theory developed in this section is a generalization of the results in [LV95] because it is applicable also to nondeterministic *tree* automata.

4.2.1. *Forward Simulations.* Soundness of forward simulation is not hard; we do not have to go into the construction in Proposition 4.1.

Theorem 4.4. *Given two $(\mathcal{P}, F)$-systems $\mathcal{X} = (X, s, c)$ and $\mathcal{Y} = (Y, t, d)$, $\mathcal{X} \sqsubseteq_{\mathbf{F}} \mathcal{Y}$ implies* $\mathsf{tr}^{\infty}(c) \odot s \sqsubseteq \mathsf{tr}^{\infty}(d) \odot t$. □

The proof, much like Proposition 4.1, is formulated as a general result, singling out some sufficient axioms.

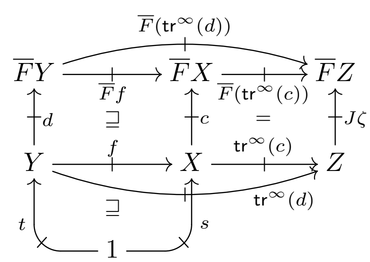
Lemma 4.5.† *Let F be an endofunctor and T be a monad on $\mathbb{C}$; assume further that they constitute an infinitary trace situation (with respect to $\sqsubseteq$). We assume the following conditions.*

(1) *Each homset of $\mathcal{K}\ell(T)$ is ω-complete, that is, each increasing ω-sequence in it has the least upper bound.*
(2) *Composition $\odot$ of arrows in $\mathcal{K}\ell(T)$ and $\overline{F}$'s action on arrows are both ω-continuous (i.e. they preserve the least upper bound of an increasing ω-sequence).*

For two (T, F)-systems $\mathcal{X} = (X, s, c)$ and $\mathcal{Y} = (Y, t, d)$, if $f : Y \nrightarrow X$ is a forward simulation from $\mathcal{X}$ to $\mathcal{Y}$, then $\mathsf{tr}^\infty(c) \odot f \sqsubseteq \mathsf{tr}^\infty(d)$. As a consequence we have $\mathsf{tr}^\infty(c) \odot s \sqsubseteq \mathsf{tr}^\infty(d) \odot t$.

Proof. Let $\zeta : Z \to FZ$ be a final F-coalgebra in $\mathbb{C}$. We define a function $\Phi_d : \mathcal{K}\ell(T)(Y, Z) \to \mathcal{K}\ell(T)(Y, Z)$ by $\Phi_d(g) = J\zeta^{-1} \odot \overline{F}g \odot d$; note that ζ is a final coalgebra and hence an isomorphism. Then

$$\begin{aligned}
\mathsf{tr}^\infty(c) \odot f &= J\zeta^{-1} \odot \overline{F}(\mathsf{tr}^\infty(c)) \odot c \odot f && (\mathsf{tr}^\infty(c) \text{ is a homomorphism})\\
&\sqsubseteq J\zeta^{-1} \odot \overline{F}(\mathsf{tr}^\infty(c)) \odot \overline{F}f \odot d && (f \text{ is a forward simulation})\\
&= \Phi_d(\mathsf{tr}^\infty(c) \odot f) && (\text{by definition of } \Phi_Y).
\end{aligned}$$



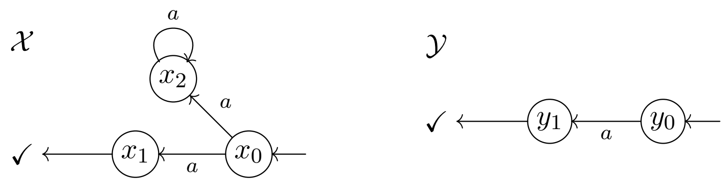
By the assumption that $\overline{F}$ and the composition are monotone, Φ_d is also monotone. Therefore by repeatedly applying Φ_d to the both sides of the above inequality, we obtain an increasing sequence $\mathsf{tr}^\infty(c) \odot f \sqsubseteq \Phi_d(\mathsf{tr}^\infty(c) \odot f) \sqsubseteq \Phi_d^2(\mathsf{tr}^\infty(c) \odot f) \sqsubseteq \cdots$ in $\mathcal{K}\ell(T)(Y, Z)$.

As $\mathcal{K}\ell(T)(Y, Z)$ is ω-complete, the least upper bound $\bigsqcup_{i<\omega} \Phi^i(\mathsf{tr}^\infty(c) \odot f)$ exists. By the assumption that $\overline{F}$ and $\odot$ are both ω-continuous, Φ_d is also ω-continuous. Therefore we have $\Phi(\bigsqcup_{i<\omega} \Phi^i(\mathsf{tr}^\infty(c) \odot f)) = \bigsqcup_{i<\omega} \Phi^{i+1}(\mathsf{tr}^\infty(c) \odot f) = \bigsqcup_{i<\omega} \Phi^i(\mathsf{tr}_z^\infty(c) \odot f)$. This means that $\bigsqcup_{i<\omega} \Phi^i(\mathsf{tr}^\infty(c) \odot f)$ is a fixed point of Φ_Y, hence a homomorphism from d to $J\zeta$. As $\mathsf{tr}^\infty(d)$ is the largest homomorphism from d to $J\zeta$, this implies $\mathsf{tr}^\infty(c) \odot f \sqsubseteq \bigsqcup_{i<\omega} \Phi^i(\mathsf{tr}^\infty(c) \odot f) \sqsubseteq \mathsf{tr}^\infty(d)$. Combining with the assumption that f is a forward simulation (specifically its condition on initial states), we have $\mathsf{tr}^\infty(c) \odot s \sqsubseteq \mathsf{tr}^\infty(c) \odot f \odot t \sqsubseteq \mathsf{tr}^\infty(d) \odot t$. □

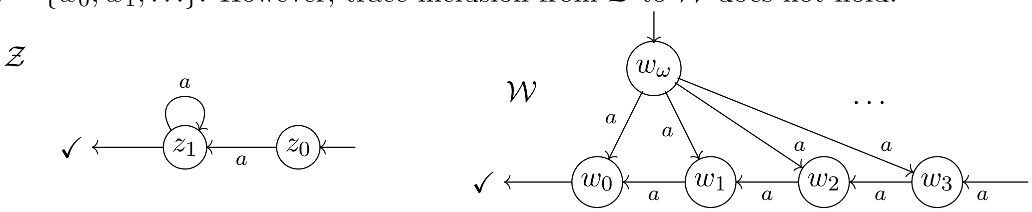
It is known from [HJS07] that the combination of polynomial F and $T = \mathcal{P}$ satisfy the conditions of Lemma 4.5. Hence we obtain Theorem 4.4, i.e. soundness of forward simulation in the nondeterministic setting.

4.2.2. *Backward Simulations.* Next we wish to establish soundness of *backward* Kleisli simulations with respect to *infinitary* traces (for finite traces it is shown in [Has06]). In fact, the desired soundness fails in general: here are counterexamples.

Example 4.6. There exists a (not total) backward simulation from the nondeterministic automaton $\mathcal{X}$ to $\mathcal{Y}$ below. Concretely, the backward simulation $b : \{x_0, x_1, x_2\} \to \mathcal{P}(\{y_0, y_1\})$ is given by $b(x_0) = \{y_0\}$, $b(x_1) = \{y_1\}$ and $b(x_2) = \emptyset$. However, the simulated automaton $\mathcal{X}$ outputs an infinite word $aaa\ldots$ while $\mathcal{Y}$ does not. Therefore the infinitary traces of $\mathcal{X}$ are not included in those of $\mathcal{Y}$.



There exists a (not image-finite) backward simulation from $\mathcal{Z}$ to $\mathcal{W}$ below. Concretely, the backward simulation $b : \{z_0, z_1\} \to \mathcal{P}(\{w_\omega, w_0, w_1, \ldots\})$ is given by $b(z_0) = \{w_\omega\}$ and $b(z_1) = \{w_0, w_1, \ldots\}$. However, trace inclusion from $\mathcal{Z}$ to $\mathcal{W}$ does not hold.



Nevertheless, for nondeterministic word automata, it is known that imposing certain restrictions (*totality* and *image-finiteness*) leads to soundness of backward simulation [LV95]. In this section, for $(\mathcal{P}, F)$-system with polynomial F, we prove a similar result in general coalgebraic terms. This allows us to use (restricted) backward simulation to check language inclusion between not only nondeterministic *word* automata but also nondeterministic *tree* automata. Moreover, this also gives us an idea about how we should impose restriction to make backward simulation sound in the probabilistic setting (Section 5.2.2).

Definition 4.7 (totality, image-finiteness, TIF-backward simulation)**.** Let $\mathcal{X} = (X, s, c)$ and $\mathcal{Y} = (Y, t, d)$ be $(\mathcal{P}, F)$-systems. A backward simulation $b : X \nrightarrow Y$ from $\mathcal{X}$ to $\mathcal{Y}$ is *total* if $b(x) \neq \emptyset$ for all $x \in X$; it is *image-finite* if $b(x) \subseteq Y$ is finite for all $x \in X$. If b satisfies both of the two conditions, it is called a *TIF-backward simulation*. We write $\mathcal{X} \sqsubseteq_{\mathbf{B}}^{\mathrm{TIF}} \mathcal{Y}$ if there exists a TIF-backward simulation from $\mathcal{X}$ to $\mathcal{Y}$.

Theorem 4.8 (soundness of $\sqsubseteq_{\mathbf{B}}^{\mathrm{TIF}}$)**.** *For two $(\mathcal{P}, F)$-systems $\mathcal{X} = (X, s, c)$ and $\mathcal{Y} = (Y, t, d)$, $\mathcal{X} \sqsubseteq_{\mathbf{B}}^{\mathrm{TIF}} \mathcal{Y}$ implies $\mathsf{tr}^\infty(c) \odot s \sqsubseteq \mathsf{tr}^\infty(d) \odot t$.*

The proof of Theorem 4.8 is, yet again, via the following axiomatic development. We first characterize *totality* and *image-finiteness* using categorical terms.

Definition 4.9 (TIF-backward simulation, generally)**.**† Let F be an endofunctor and T be a monad on $\mathbb{C}$ that satisfy the conditions in Proposition 4.1 with respect to $\sqsubseteq$. For two (T, F)-systems $\mathcal{X} = (X, s, c)$ and $\mathcal{Y} = (Y, t, d)$, a *TIF-backward simulation* from $\mathcal{X}$ to $\mathcal{Y}$ is a backward simulation $b : X \nrightarrow Y$ that satisfies the following conditions.

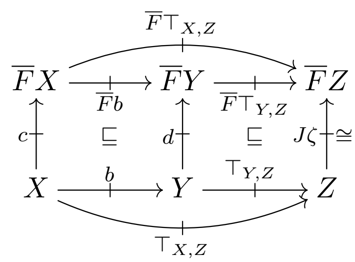
(1) The arrow $b : X \nrightarrow Y$ satisfies $\top_{Y,Z} \odot b = \top_{X,Z}$ for any $Z \in \mathcal{K}\ell(T)$.
(2) Precomposing $b : X \nrightarrow Y$ preserves the greatest lower bound of any decreasing transfinite sequence. That is, let $A \in \mathcal{K}\ell(T)$, $\mathfrak{a}$ be an ordinal, and $(g_\mathfrak{i} : Y \nrightarrow A)_{\mathfrak{i} < \mathfrak{a}}$ be a family of Kleisli arrows such that $\mathfrak{i} \leq \mathfrak{j}$ implies $g_\mathfrak{i} \sqsupseteq g_\mathfrak{j}$. Then we have $\bigsqcap_{\mathfrak{i} \in \mathfrak{a}} (g_\mathfrak{i} \odot b) = (\bigsqcap_{\mathfrak{i} \in \mathfrak{a}} g_\mathfrak{i}) \odot b$.

We write $\mathcal{X} \sqsubseteq_{\mathbf{B}}^{\mathrm{TIF}} \mathcal{Y}$ if there exists a TIF-backward simulation from $\mathcal{X}$ to $\mathcal{Y}$.

Assumption (2) of Definition 4.9 resembles how “finiteness” is formulated in category theory, e.g. in the definition of *finitely presented* objects.

This general TIF-backward simulation satisfies soundness. For its proof we have to look into the inductive construction of the largest homomorphism in Proposition 4.1.

Lemma 4.10.† *Let F and T be as in Proposition 4.1. For two (T,F)-systems $\mathcal{X}=(X,s,c)$ and $\mathcal{Y}=(Y,t,d)$, $\mathcal{X}\sqsubseteq^{\mathrm{TIF}}_{\mathbf{B}}\mathcal{Y}$ (in the sense of Definition 4.9) implies $\mathsf{tr}^{\infty}(c)\sqsubseteq\mathsf{tr}^{\infty}(d)\odot b$. Furthermore it follows that $\mathsf{tr}^{\infty}(c)\odot s\sqsubseteq\mathsf{tr}^{\infty}(d)\odot t$.*



Proof. Let $\zeta:Z\to FZ$ be a final F-coalgebra in $\mathbb{C}$. We define $\Phi_c:\mathcal{K}\ell(T)(X,Z)\to\mathcal{K}\ell(T)(X,Z)$ and $\Phi_d:\mathcal{K}\ell(T)(Y,Z)\to\mathcal{K}\ell(T)(Y,Z)$ as in the proof of Proposition 4.1. Moreover, in the same manner as in the proof of Proposition 4.1, for each ordinal $\mathfrak{a}$, we define $\Phi_c^{\mathfrak{a}}(\top_{X,Z}):X\mathrel{\ooalign{$\to$\cr\hidewidth$+$\hidewidth}} Z$ and $\Phi_d^{\mathfrak{a}}(\top_{Y,Z}):Y\mathrel{\ooalign{$\to$\cr\hidewidth$+$\hidewidth}} Z$ by transfinite induction on $\mathfrak{a}$. As we have seen in the proof of Proposition 4.1, there exist ordinals $\mathfrak{l}_c$ and $\mathfrak{l}_d$ such that $\mathsf{tr}^{\infty}(c)=\Phi_c^{\mathfrak{l}_c}(\top_{X,Z})$ and $\mathsf{tr}^{\infty}(d)=\Phi_d^{\mathfrak{l}_d}(\top_{Y,Z})$. Let $\mathfrak{l}=\max(\mathfrak{l}_c,\mathfrak{l}_d)$. We shall now prove by transfinite induction that, for each $\mathfrak{a}$, we have $\Phi_c^{\mathfrak{a}}(\top_{X,Z})\sqsubseteq\Phi_d^{\mathfrak{a}}(\top_{Y,Z})\odot b$; this will yield our goal by taking $\mathfrak{a}=\mathfrak{l}$.

For $\mathfrak{a}=0$, from Assumption (1) of Definition 4.9, we have $\Phi_c^{\mathfrak{a}}(\top_{X,Z})=\top_{X,Z}=\top_{Y,Z}\odot b=\Phi_d^{\mathfrak{a}}(\top_{Y,Z})\odot b$.

Assume that $\mathfrak{a}$ is a successor ordinal and $\Phi_c^{\mathfrak{a}-1}(\top_{X,Z})\sqsubseteq\Phi_d^{\mathfrak{a}-1}(\top_{Y,Z})\odot b$. Then

$$\begin{aligned}\Phi_c^{\mathfrak{a}}(\top_{X,Z})&\sqsubseteq J\zeta^{-1}\odot\overline{F}(\Phi_d^{\mathfrak{a}-1}(\top_{Y,Z}))\odot\overline{F}b\odot c &&\text{(by induction hypothesis)}\\ &\sqsubseteq J\zeta^{-1}\odot\overline{F}(\Phi_d^{\mathfrak{a}-1}(\top_{Y,Z}))\odot d\odot b &&\text{(b is a backward simulation)}\\ &=\Phi_d^{\mathfrak{a}}(\top_{Y,Z})\odot b &&\text{(by definition)}\,.\end{aligned}$$

Let $\mathfrak{a}$ be a limit ordinal and assume that $\Phi_c^{\mathfrak{i}}(\top_{X,Z})\sqsubseteq\Phi_d^{\mathfrak{i}}(\top_{Y,Z})\odot b$ for all $\mathfrak{i}<\mathfrak{a}$. Then

$$\begin{aligned}\Phi_c^{\mathfrak{a}}(\top_{X,Z})&\sqsubseteq \textstyle\bigsqcap_{\mathfrak{i}<\mathfrak{a}}\bigl(\Phi_Y^{\mathfrak{i}}(\top_{Y,Z})\odot b\bigr) &&\text{(by induction hypothesis)}\\ &=\Phi_d^{\mathfrak{a}}(\top_{Y,Z})\odot b &&\text{(by Assumption (2) of Definition 4.9)}\,.\end{aligned}$$

Thus $\mathsf{tr}^{\infty}(c)\sqsubseteq\mathsf{tr}^{\infty}(d)\odot b$. The last claim follows from b's condition on initial states. □

Next we show that a TIF-backward simulation in the specific sense of Definition 4.7 is also a TIF-backward simulation in the general sense of Definition 4.9. To this end, we first prove the following "pigeon-hole" sublemma.

Sublemma 4.11. *Let $\mathfrak{a}$ be a limit ordinal, C be a finite set and $f:\mathfrak{a}\to C$. Then*

$$\exists c\in C.\ \forall\mathfrak{i}<\mathfrak{a}.\ \exists\mathfrak{b}\geq\mathfrak{i}.\ f(\mathfrak{b})=c\,.$$

Proof. For each $c\in C$, we define $A_c\subseteq\mathfrak{a}$ by $A_c=\{\mathfrak{d}\in\mathfrak{a}\mid f(\mathfrak{d})=c\}$. We prove the statement by contradiction. Assume the negation of the claim, that is,

$$\forall c\in C.\ \exists\mathfrak{i}_c<\mathfrak{a}.\ \forall\mathfrak{b}\geq\mathfrak{i}_c.\ f(\mathfrak{b})\neq c\,.$$

This is equivalent to assuming that for all $c\in C$, there exists $\mathfrak{i}_c<\mathfrak{a}$ such that for all $\mathfrak{j}\in A_c$, $\mathfrak{j}<\mathfrak{i}_c$ holds. Then $\bigsqcup_{\mathfrak{j}\in A_c}\mathfrak{j}\leq\mathfrak{i}_c<\mathfrak{a}$ for all $c\in C$. As C is finite and $\bigcup_{c\in C}A_c=\mathfrak{a}$, this implies $\mathfrak{a}=\bigsqcup_{\mathfrak{j}<\mathfrak{a}}\mathfrak{j}=\bigsqcup_{c\in C}\bigsqcup_{\mathfrak{j}\in A_c}\mathfrak{j}\leq\bigsqcup_{c\in C}\mathfrak{i}_c<\mathfrak{a}$. This contradicts and the statement is proved. □

Lemma 4.12. *In Definition 4.9, let $T=\mathcal{P}$ and F be a polynomial functor. Assumption (1) is satisfied if $b(x)\neq\emptyset$ for each $x\in X$; Assumption (2) is satisfied if $b(x)$ is finite for each $x\in X$.*

Proof. Assume that $b(x) \neq \emptyset$ for all $x \in X$. To show that Assumption (1) in Definition 4.9 is satisfied, it suffices to prove $z \in \top_{Y,Z} \odot b(x)$ for all $z \in Z$ and $x \in X$. By the assumption, there exists $y \in Y$ such that $y \in b(x)$. Therefore for all $z \in Z$, $z \in \top_{Y,Z}(y) \subseteq \top_{Y,Z} \odot b(x)$.

Assume that $b(x)$ is finite for all $x \in X$. Note that $\bigsqcap_{\mathfrak{i}<\mathfrak{a}}(g_{\mathfrak{a}} \odot b)(x) = \bigcap_{\mathfrak{i}<\mathfrak{a}} \bigcup_{y \in b(x)} g_{\mathfrak{i}}(y)$ while $\left(\bigsqcap_{\mathfrak{i}<\mathfrak{a}} g_{\mathfrak{i}}\right) \odot b(x) = \bigcup_{y \in b(x)} \bigcap_{\mathfrak{i}<\mathfrak{a}} g_{\mathfrak{i}}(y)$. As it is easily shown that the latter is always included in the former, it suffices to prove that $z \in \bigcap_{\mathfrak{i}<\mathfrak{a}} \bigcup_{y \in b(x)} g_{\mathfrak{i}}(y)$ implies $z \in \bigcup_{y \in b(x)} \bigcap_{\mathfrak{i}<\mathfrak{a}} g_{\mathfrak{i}}(y)$.

If $z \in \bigcap_{\mathfrak{i}<\mathfrak{a}} \bigcup_{y \in b(x)} g_{\mathfrak{i}}(y)$, then for all $\mathfrak{i} < \mathfrak{a}$, there exists $y_{\mathfrak{i}} \in b(x)$ such that $z \in g_{\mathfrak{i}}(y_{\mathfrak{i}})$. As $b(x)$ is assumed to be finite, from Sublemma 4.11, we have

$$\exists y \in b(x).\ \forall \mathfrak{i} < \mathfrak{a}.\ \exists \mathfrak{j} \geq \mathfrak{i}.\ z \in g_{\mathfrak{j}}(y)\,.$$

As $\mathfrak{i} \leq \mathfrak{j}$ implies $g_{\mathfrak{i}} \sqsupseteq g_{\mathfrak{j}}$, $z \in g_{\mathfrak{j}}(y)$ implies $z \in g_{\mathfrak{i}}(y)$. Therefore $z \in \bigcup_{y \in b(x)} \bigcap_{\mathfrak{i}<\mathfrak{a}} g_{\mathfrak{i}}(y)$ holds. □

Proof of Theorem 4.8. Immediate from Lemma 4.10 and Lemma 4.12. □

Even with the additional constraints of totality and image-finiteness, backward Kleisli simulations seems to be a viable method for establishing infinitary trace inclusion, because there exists a pair of nondeterministic automata such that a TIF-backward simulation can prove trace inclusion between them but a forward simulation cannot. An example of such a pair of automata is shown below.

Example 4.13. The infinitary traces of the nondeterministic automata $\mathcal{X}$ below are included in the infinitary traces of $\mathcal{Y}$. There exists no forward simulation from $\mathcal{X}$ to $\mathcal{Y}$ while a TIF-backward simulation does exist.

4.3. **Forward Partial Execution for Nondeterministic Systems.** Recall that in Definition 3.4, we have reviewed *forward partial execution* (FPE) [UH14, UH17]—a transformation of coalgebraic systems that potentially increases the likelihood of existence of simulations. We now apply FPE in the current setting of nondeterminism and infinitary traces. We follow the setting in [UH17] for the *finite* traces, and formulate FPE's "correctness" in the following theorem.

Theorem 4.14. *Let F be a polynomial functor on* **Sets**. *For $(\mathcal{P}, F)$-systems $\mathcal{X} = (X, s, c)$ and $\mathcal{Y} = (Y, t, d)$, the following hold.*

(1) (a) *(soundness of FPE for forward simulation)* $\mathcal{X}_{\mathsf{FPE}} \sqsubseteq_{\mathbf{F}} \mathcal{Y}$ *implies* $\mathsf{tr}^{\infty}(c) \odot s \sqsubseteq \mathsf{tr}^{\infty}(d) \odot t$.
   (b) *(adequacy of FPE for forward simulation)* $\mathcal{X} \sqsubseteq_{\mathbf{F}} \mathcal{Y}$ *implies* $\mathcal{X}_{\mathsf{FPE}} \sqsubseteq_{\mathbf{F}} \mathcal{Y}$.

(2) (a) *(soundness of FPE for backward simulation)* $\mathcal{X} \sqsubseteq_{\mathbf{B}}^{\mathrm{TIF}} \mathcal{Y}_{\mathsf{FPE}}$ *implies* $\mathsf{tr}^{\infty}(c) \odot s \sqsubseteq \mathsf{tr}^{\infty}(d) \odot t$.
   (b) *(adequacy of FPE for backward simulation)* $\mathcal{X} \sqsubseteq_{\mathbf{B}}^{\mathrm{TIF}} \mathcal{Y}$ *implies* $\mathcal{X} \sqsubseteq_{\mathbf{B}}^{\mathrm{TIF}} \mathcal{Y}_{\mathsf{FPE}}$, *assuming that the following hold.*
      (i) $d(y) \neq \emptyset$ *for all* $y \in Y$.
      (ii) $d(y)$ *is finite for all* $y \in Y$.

Informally: *soundness* means that discovery of a simulation after applying FPE still witnesses the trace inclusion between the original systems; and *adequacy* means that the relationship $\sqsubseteq_{\mathbf{F}}$ (or $\sqsubseteq_{\mathbf{B}}^{\mathrm{TIF}}$) is not destroyed by application of FPE. The theorem also implies that FPE must be applied to the "correct side" of the desired trace inclusion $L^\infty(\mathcal{X}) \sqsubseteq L^\infty(\mathcal{Y})$: $\mathcal{X}$ in the search for a forward simulation; and $\mathcal{Y}$ in the search for a backward one.

Note that the adequacy property is independent from the choice of trace semantics (finite or infinitary). Therefore the statement (1b) of Theorem 4.14 is the same as its counterpart in [UH17]. For the statement (2b), however, we have to check that the TIF restriction (that is absent in [UH17]) is indeed carried over.

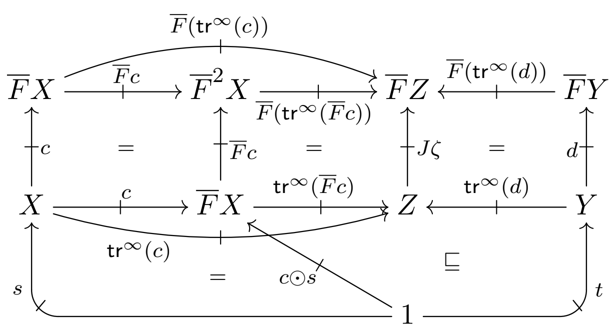
In [UH17] it is shown that FPE can indeed create a simulation that does not exist between the original systems. Its practical use is witnessed by experimental results in [UH14, UH17], where FPE was used in verifications of security protocols. It would not be hard to observe the same in the current setting for *infinitary* traces.

For the proof of Theorem 4.14, once again, we turn to an axiomatic development.

Theorem 4.15 (FPE and forward simulation).[†] *Let F be an endofunctor and T be a monad on $\mathbb{C}$, as in Lemma 4.5 (that is, they constitute an infinitary trace situation and satisfy the two additional assumptions.) Let $\mathcal{X} = (X, s, c)$ and $\mathcal{Y} = (Y, t, d)$ be (T, F)-systems. Then*

(1) *(soundness for forward simulation) $\mathcal{X}_{\mathsf{FPE}} \sqsubseteq_{\mathbf{F}} \mathcal{Y}$ implies $\mathsf{tr}^\infty(c) \odot s \sqsubseteq \mathsf{tr}^\infty(d) \odot t$.*
(2) *(adequacy for forward simulation) $\mathcal{X} \sqsubseteq_{\mathbf{F}} \mathcal{Y}$ implies $\mathcal{X}_{\mathsf{FPE}} \sqsubseteq_{\mathbf{F}} \mathcal{Y}$.*

Proof. **(1)(soundness).**



Let $\zeta : Z \to FZ$ be a final F-coalgebra. By definition, $\mathcal{X}_{\mathsf{FPE}} = (\overline{F}X, c \odot s, \overline{F}c)$. Assume $\mathcal{X}_{\mathsf{FPE}} \sqsubseteq_{\mathbf{F}} \mathcal{Y}$. Then by soundness of a forward Kleisli simulation, we have:

$$\mathsf{tr}^\infty(\overline{F}c) \odot (c \odot s) \sqsubseteq \mathsf{tr}^\infty(d) \odot t \,. \tag{4.1}$$

As $\mathsf{tr}^\infty(c)$ is a homomorphism from c to $J\zeta$, we have:

$$\mathsf{tr}^\infty(c) = (J\zeta)^{-1} \odot \overline{F}(\mathsf{tr}^\infty(c)) \odot c \,. \tag{4.2}$$

Here $(J\zeta)^{-1} \odot \overline{F}(\mathsf{tr}^\infty(c))$ is a homomorphism from an $\overline{F}$-coalgebra $\overline{F}c : \overline{F}X \nrightarrow \overline{F}^2X$ to $J\zeta$ because of the following equation.

$$\begin{aligned} J\zeta \odot \big((J\zeta)^{-1} \odot \overline{F}(\mathsf{tr}^\infty(c))\big) &= \overline{F}(\mathsf{tr}^\infty(c)) \\ &= \overline{F}\big((J\zeta)^{-1} \odot \overline{F}(\mathsf{tr}^\infty(c)) \odot c\big) && \text{(by (4.2))} \\ &= \overline{F}\big((J\zeta)^{-1} \odot \overline{F}(\mathsf{tr}^\infty(c))\big) \odot \overline{F}(c) \end{aligned}$$

As $\mathsf{tr}^\infty(\overline{F}c)$ is the largest homomorphism from $\overline{F}c$ to $J\zeta$, we have:

$$(J\zeta)^{-1} \odot \overline{F}(\mathsf{tr}^\infty(c)) \sqsubseteq \mathsf{tr}^\infty(\overline{F}c) \,. \tag{4.3}$$

From the equations (4.1–4.3), $\mathsf{tr}^\infty(c)\odot s \sqsubseteq \mathsf{tr}^\infty(d)\odot t$ follows.

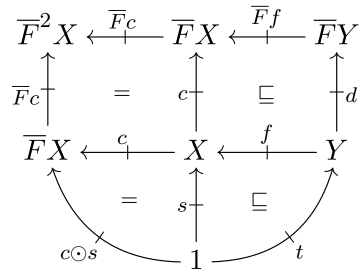
(2)(adequacy). Let $f : Y \nrightarrow X$ be a forward Kleisli simulation from $\mathcal{X}$ to $\mathcal{Y}$. Then we have:

$$\overline{F}c\odot(c\odot f) \sqsubseteq \overline{F}c\odot(\overline{F}f\odot d) = \overline{F}(c\odot f)\odot d$$

and

$$c\odot s \sqsubseteq c\odot(f\odot t) = (c\odot f)\odot t\,.$$

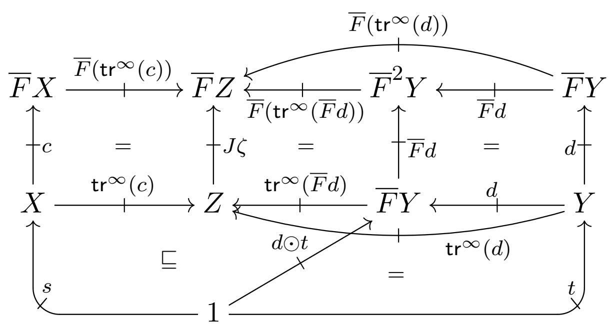
Hence $c\odot f : Y \nrightarrow \overline{F}X$ is a forward simulation from $\mathcal{X}_{\mathsf{FPE}}$ to $\mathcal{Y}$. □



Theorem 4.16 (FPE and backward simulation).† *Let F be an endofunctor and T be a monad on $\mathbb{C}$ that satisfy the conditions in Proposition 4.1 (hence those in Lemma 4.10). Let $\mathcal{X} = (X, s, c)$ and $\mathcal{Y} = (Y, t, d)$ be (T, F)-systems.*

1. *(soundness for backward simulation) $\mathcal{X} \sqsubseteq^{\mathrm{TIF}}_{\mathbf{B}} \mathcal{Y}_{\mathsf{FPE}}$ implies $\mathsf{tr}^\infty(c)\odot s \sqsubseteq \mathsf{tr}^\infty(d)\odot t$.*
2. *(adequacy for backward simulation) $\mathcal{X} \sqsubseteq^{\mathrm{TIF}}_{\mathbf{B}} \mathcal{Y}$ implies $\mathcal{X} \sqsubseteq^{\mathrm{TIF}}_{\mathbf{B}} \mathcal{Y}_{\mathsf{FPE}}$ if the following conditions are satisfied.*
   (a) *The coalgebra $d : Y \nrightarrow \overline{F}Y$ satisfies $\top_{\overline{F}Y,Z}\odot d = \top_{Y,Z}$.*
   (b) *Precomposing d preserves the greatest lower bound of a (possibly transfinite) decreasing sequence.*

Proof. **(1) (soundness).**



Let $b : X \nrightarrow \overline{F}Y$ be a TIF-backward simulation from $\mathcal{X}$ to $\mathcal{Y}_{\mathsf{FPE}}$. Then by soundness of TIF-backward simulation, we have:

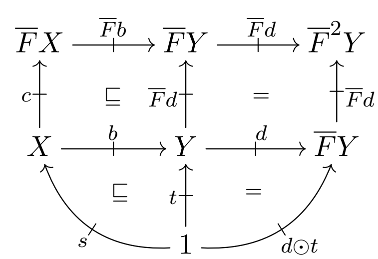
$$\mathsf{tr}^\infty(c)\odot s \sqsubseteq \mathsf{tr}^\infty(\overline{F}d)\odot(d\odot t)\,. \tag{4.4}$$

It is easy to see that $d : Y \nrightarrow \overline{F}Y$ is a forward simulation from $\mathcal{Y}_{\mathsf{FPE}}$ to $\mathcal{Y}$. Therefore by soundness of forward simulation, we have:

$$\mathsf{tr}^\infty(\overline{F}d)\odot(d\odot t) \sqsubseteq \mathsf{tr}^\infty(d)\odot t\,. \tag{4.5}$$

From the inequalities (4.4) and (4.5), we have $\mathsf{tr}^\infty(c)\odot s \sqsubseteq \mathsf{tr}^\infty(d)\odot t$.

(2) (adequacy). Let $b : X \nrightarrow Y$ be a TIF-backward simulation from $\mathcal{X}$ to $\mathcal{Y}$. In a similar manner to the proof of Theorem 4.15 (1), we can prove that $d\odot b : X \nrightarrow \overline{F}Y$ is a backward simulation from $\mathcal{X}$ to $\mathcal{Y}_{\mathsf{FPE}}$. Moreover, the Assumptions (2a) and (2b) imply that $d\odot b$ satisfy Assumptions (1) and (2) in Definition 4.9. Therefore $d\odot b$ is a TIF-backward simulation from $\mathcal{X}$ to $\mathcal{Y}_{\mathsf{FPE}}$. □



Proof of Theorem 4.14. (1) is immediate from Theorem 4.15. In a similar manner to Lemma 4.12, we can prove (2) using Theorem 4.16. □

4.4. **Coincidence between Automata-theoretic and Coalgebraic Infinitary Trace Semantics.** In this section we give a sanity-check result for the coalgebraic infinitary trace semantics that is defined in Section 4.1. Namely, for nondeterministic systems, we show a coincidence between: the coalgebraic infinitary trace semantics formalized in the previous section; and the (infinitary) tree language that is defined using automata-theoretic terms. Here we formalize the latter as follows (Recall the notations from Section 2.2).

Definition 4.17. Let Σ be a ranked alphabet. A $(\mathcal{P}, F_\Sigma)$-system $\mathcal{X} = (X, s, c)$ is called a *Σ-labeled nondeterministic tree automaton.* For a Σ-labeled infinitary tree $t = (D, l)$ and a state $x \in X$, a *run tree of $\mathcal{X}$ from x that generates t* is a $(X)_{n\in\omega}$-labeled[2] infinitary tree $t_r = (D, l_r)$, with the same domain as t, such that:

- $l_r(\varepsilon) = x$; and
- for any element $\alpha \in D$ of the common domain, assume that $l(\alpha) = a \in \Sigma_n$, $l_r(\alpha) = y$ and $l_r(\alpha i) = y_i$ for each $i \in \{0, \ldots, n-1\}$. Then $(a, y_0, \ldots, y_{n-1}) \in c(y)$ holds.

For a state $x \in X$, the *infinitary language of $\mathcal{X}$ from x* is the set $L^\infty(\mathcal{X}, x) \subseteq \mathrm{Tree}_\infty(\Sigma)$ that is defined by $L^\infty(\mathcal{X}, x) = \{t \in \mathrm{Tree}_\infty(\Sigma) \mid \text{there is a run tree of } \mathcal{X} \text{ from } x \text{ that generates } t\}$. The *infinitary language of $\mathcal{X}$* is the set $L^\infty(\mathcal{X}) = \bigcup_{x\in s(*)} L^\infty(\mathcal{X}, x)$, where $*$ denotes the unique element of a singleton 1.

The following is the main result of this section.

Proposition 4.18. *Let Σ be a ranked alphabet. The carrier set of a final F_Σ-coalgebra is given by $\mathrm{Tree}_\infty(\Sigma)$. Moreover for a Σ-labeled nondeterministic tree automaton $\mathcal{X} = (X, s, c)$, we have $\mathsf{tr}^\infty(c)(x) = L^\infty(\mathcal{X}, x)$ for all $x \in X$, and hence $\mathsf{tr}^\infty(c) \odot s(*) = L^\infty(\mathcal{X})$.*

Proof. We define an arrow $\zeta : \mathrm{Tree}_\infty(\Sigma) \to F_\Sigma\,(\mathrm{Tree}_\infty(\Sigma))$ in **Sets** by $\zeta(t) = (a, (t_0, \ldots, t_{n-1}))$ for each $t \in \mathrm{Tree}_\infty(\Sigma)$ such that $t = (D, l)$, $a = l(\varepsilon) \in \Sigma_n$, and the i-th subtree of t is t_i. It is known that ζ is a final F_Σ-coalgebra (see e.g. [RT93]). We show that $L^\infty(\mathcal{X}, \_) : X \nrightarrow \mathrm{Tree}_\infty(\Sigma)$ is the largest homomorphism from c to $J\zeta$.

We first show that $L^\infty(\mathcal{X}, \_)$ is a homomorphism. For $x \in X$, we have:

$$
\begin{aligned}
&(\overline{F_\Sigma} L^\infty(\mathcal{X}, \_)) \odot c(x) \\
&= (\overline{F_\Sigma} L^\infty(\mathcal{X}, \_)) \left(\left\{ (a, x_0, \ldots, x_{n-1}) \;\middle|\; \begin{array}{l} n \in \omega, a \in \Sigma_n, x_0, \ldots, x_{n-1} \in X, \\ (a, x_0, \ldots, x_{n-1}) \in c(x) \end{array} \right\} \right) \\
&= \left\{ (a, t_0, \ldots, t_{n-1}) \;\middle|\; \begin{array}{l} n \in \omega,\ a \in \Sigma_n, \\ \exists x_0, \ldots, x_{n-1} \in X. \left(\begin{array}{l} (a, x_0, \ldots, x_{n-1}) \in c(x), \\ t_i \in L^\infty(\mathcal{X}, x_i) \text{ for each } i \end{array} \right) \end{array} \right\} \\
&= J\zeta \left(\left\{ (D, l) \in \mathrm{Tree}_\infty(\Sigma) \;\middle|\; \begin{array}{l} n \in \omega, l(\varepsilon) \in \Sigma_n, \\ \exists x_0, \ldots, x_{n-1} \in X. \left(\begin{array}{l} (l(\varepsilon), x_0, \ldots, x_{n-1}) \in c(x), \\ l(i) \in L^\infty(\mathcal{X}, x_i) \text{ for each } i \end{array} \right) \end{array} \right\} \right)
\end{aligned}
$$

[2]Note that $(X)_{n\in\omega}$ is the ranked alphabet $\Sigma' = (\Sigma'_n)_{n\in\omega}$ such that $\Sigma'_n = X$ for each $n \in \omega$.

$$= J\zeta\left(\left\{(D,l)\in \mathrm{Tree}_\infty(\Sigma)\;\middle|\; \begin{array}{l} n\in\omega, l(\varepsilon)\in\Sigma_n, \\ \exists x_0,\dots,x_{n-1}\in X. \left(\begin{array}{l}(l(\varepsilon),x_0,\dots,x_{n-1})\in c(x),\\ \text{for each } i \text{, there is a run tree}\\ \quad t_{r,i} \text{ of } \mathcal{X} \text{ from } x_i \text{ that}\\ \quad\text{generates } i\text{-th subtree } t_i \text{ of } t\end{array}\right)\end{array}\right\}\right)$$

$$= J\zeta\big(\{t\in \mathrm{Tree}_\infty(\Sigma)\mid \text{there exists a run tree } t_r \text{ of } \mathcal{X} \text{ from } x \text{ that generates } t\}\big)$$

$$= \big(J\zeta\odot L^\infty(\mathcal{X},\_)\big)(x)\,.$$

Therefore $L^\infty(\mathcal{X},\_)$ is a homomorphism from c to $J\zeta$.

It remains to prove that $L^\infty(\mathcal{X},\_)$ is the largest homomorphism. Let $g : X \nrightarrow \mathrm{Tree}_\infty(\Sigma)$ be a homomorphism from c to $J\zeta$. We fix $x \in X$ and $t = (D,l) \in g(x)$, and show that $t \in L^\infty(\mathcal{X},x)$. To this end, it suffices to construct a run tree $t_r = (D,l_r)$ of $\mathcal{X}$ from x that generates t. For each $\alpha\in D$, we define a state $l_r(\alpha)\in X$ such that $t_\alpha \in g(l_r(\alpha))$ by induction on the length of α as follows.

- If $\alpha=\varepsilon$, we define it by $l_r(\alpha)=x$. By the assumption, $t_\alpha = t\in g(x) = g(l_r(\alpha))$.
- Let $l(\alpha)\in\Sigma_n$ and assume that $t_\alpha\in g(l_r(\alpha))$. As g is a homomorphism from c to $J\zeta$, $t_\alpha \in g(l_r(\alpha)) = J\zeta^{-1}\odot \overline{F_\Sigma} g\odot c(l_r(\alpha))$. By the definition of ζ, this means that there exist a family of states $x_0,\dots,x_{n-1}$ such that $(l(\alpha),x_0,\dots,x_{n-1})\in c(l_r(\alpha))$ and $t_{\alpha i}\in g(x_i)$ for each $i\in\{0,\dots,n-1\}$. We define $l_r(\alpha i)$ by $l_r(\alpha i) = x_i$.

By the axiom of dependent choice, this l_r is well-defined. Moreover, by its construction, (D,l_r) is a run tree of $\mathcal{X}$ from x that generates t. Therefore $t\in L^\infty(\mathcal{X},x)$.

Therefore $g\sqsubseteq L^\infty(\mathcal{X},\_)$ holds, and $L^\infty(\mathcal{X},\_)$ is the largest homomorphism from c to $J\zeta$. Hence we have $\mathsf{tr}^\infty(c) = L^\infty(\mathcal{X},\_)$. This immediately implies $\mathsf{tr}^\infty(c)\odot s(*) = L^\infty(\mathcal{X})$. □

Hence the coalgebraic definition of infinitary trace semantics in Definition 3.2 indeed characterizes the languages of Σ-labeled nondeterministic tree automata in Definition 4.17.

5. Systems with Probabilistic Branching

We now turn to probabilistic systems. They are modeled as $(\mathcal{G},F)$-systems in the category **Meas**. Here we establish largely the same statements as in Section 4, but many constructions and proofs are different. Throughout this section F is assumed to be a (standard Borel) polynomial functor on **Meas** (Definition 2.2).

5.1. Construction of Infinitary Traces.

In this section, like in Section 4.1, we establish an infinitary trace situation.

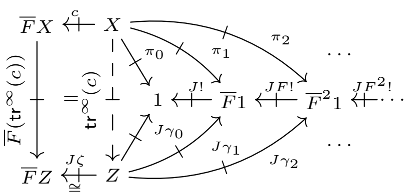
Our goal is to construct the largest homomorphism from an $\overline{F}$-coalgebra c in to a lifted final coalgebra $J\zeta : Z\nrightarrow \overline{F}Z$ in $\mathcal{K}\ell(\mathcal{G})$; we do so inductively, much like in the nondeterministic setting (Section 4.1), starting from the top element and going down along a decreasing sequence.

Remark 5.1. Compared to the nondeterministic case ($T=\mathcal{P}$), major differences are as follows.

- Composition of Kleisli arrows is ω^{op}-continuous in $\mathcal{K}\ell(\mathcal{G})$ (see Theorem 5.3 later). This is an advantage, because we can appeal to the Kleene fixed point theorem and we only need inductive construction up-to ω steps (while, for $\mathcal{P}$, we needed transfinite induction).

- A big disadvantage, however, is the absence of the top element $\top_{X,Z}$ in $\mathcal{K}\ell(T)(X,Z)$. One can imagine a top element $\top_{X,Z}$ to assign 1 to every event—this is however not a (probability) measure.

To cope with the latter challenge, we turn to the *final F-sequence* in **Meas** that yields a final F-coalgebra as its limit. Instead of using a sequence like $\top \sqsupseteq \Phi(\top) \sqsupseteq \cdots$ in $\mathcal{K}\ell(T)(X,Z)$ (where the largest element $\top$ does not exist anyway), we use a decreasing sequence that goes along the final sequence, which is known to yield a final F-coalgebra [Sch09]. Once the largest element along the sequence is obtained, we can construct the largest homomorphism from it in a similar manner to [Cîr10].



The precise construction is found in the proof of the following proposition.

Proposition 5.2.† *Let $\mathbb{C}$ be a category, F be an endofunctor on $\mathbb{C}$, and T be a monad on $\mathbb{C}$ where each homset of $\mathcal{K}\ell(T)$ carries an order $\sqsubseteq$. We assume the following conditions.*

(1) *The category $\mathbb{C}$ has a final object 1; the final ω^{op}-sequence $1 \xleftarrow{!_{F1}} F1 \xleftarrow{F!_{F1}} F^2 1 \xleftarrow{F^2!_{F1}} \cdots$ has a limit $(Z,(\gamma_i : Z \to F^i 1)_{i\in\omega})$; and moreover, F preserves this limit. Hence the limit carries a final F-coalgebra* [AK79].
(2) *There exists a distributive law $\lambda : FT \Rightarrow TF$, yielding a lifting $\overline{F}$ on $\mathcal{K}\ell(T)$ of F.*
(3) *For $X,Y \in \mathcal{K}\ell(T)$, every decreasing ω^{op}-sequence $f_0 \sqsupseteq f_1 \sqsupseteq \ldots$ in $\mathcal{K}\ell(T)(X,Y)$ has the greatest lower bound $\bigsqcap_{i\in\omega} f_i$. Moreover, composition of arrows in $\mathcal{K}\ell(T)$ and $\overline{F}$'s action on arrows are both ω^{op}-continuous. That is, for each $g : Z \nrightarrow X$ and $h : Y \nrightarrow W$, we have $g \odot (\bigsqcap_{i\in\omega} f_i) = \bigsqcap_{i\in\omega}(g \odot f_i)$, $(\bigsqcap_{i\in\omega} f_i) \odot h = \bigsqcap_{i\in\omega}(f_i \odot h)$, and $\overline{F}(\bigsqcap_{i\in\omega} f_i) = \bigsqcap_{i\in\omega}(\overline{F} f_i)$.*
(4) *The lifting $J(!_X)$ of the unique arrow to 1 is the largest element of $\mathcal{K}\ell(T)(X,1)$.*
(5) *The functor J lifts the limit in Assumption (1) to a 2-limit. Namely, for any cone $(X,(\pi_i : X \nrightarrow F^i 1)_{i\in\omega})$ over the sequence $1 \overset{J!_{F1}}{\nleftarrow} \overline{F}1 \overset{J F!_{F1}}{\nleftarrow} \overline{F}^2 1 \overset{JF^2!_{F1}}{\nleftarrow} \cdots$, there uniquely exists $l : X \nrightarrow Z$ such that $\pi_i = J\gamma_i \odot l$ holds for each $i \in \omega$. Moreover, if $l' : X \nrightarrow Z$ satisfies $J\gamma_i \odot l' \sqsubseteq J\gamma_i \odot l$ for each $i \in \omega$, then $l' \sqsubseteq l$ holds.*

Then F and T constitute an infinitary trace situation with respect to $\sqsubseteq$.

In more elementary terms, Assumption (5) of Proposition 5.2 asserts that: J lifts the limit Z; and the lifted limit satisfies a stronger condition of preserving the order between cones to the order between mediating maps.

Intuitively, Assumptions 1, 3 and 5 together ensure that we can "transfer" the greatest element $J(!_X)$ in $\mathcal{K}\ell(T)(X,1)$ (Assumption 4) to the greatest homomorphism from c to $J\zeta$.

Proof. Let $c : X \nrightarrow \overline{F}X$ be a $\overline{F}$-coalgebra in $\mathcal{K}\ell(T)$.

$$\begin{array}{ccc} \overline{F}X & \xrightarrow{\overline{F}f} & \overline{F}1 \\ c \uparrow & & \downarrow J!_{F1} \\ X & \xrightarrow{f} & 1 \end{array}$$

We first construct a cone $(X,(\alpha_i : X \nrightarrow \overline{F}^i 1)_{i\in\omega})$ over the sequence $1 \overset{J!_{F1}}{\nleftarrow} \overline{F}1 \overset{JF!_{F1}}{\nleftarrow} \overline{F}^2 1 \overset{JF^2!_{F1}}{\nleftarrow} \cdots$. To this end, we start with defining an arrow $\alpha_0 : X \nrightarrow 1$. Let us define a function $\Psi_c : \mathcal{K}\ell(T)(X,1) \to \mathcal{K}\ell(T)(X,1)$ by $\Psi_c(f) = J!_{F1} \odot \overline{F}f \odot c$ (see the diagram on the right). As composition in $\mathcal{K}\ell(T)$ and $\overline{F}$'s action on arrows are both monotone (by Assumption (3)), Ψ_c is also monotone. Moreover, as $J!_X$ is the largest element in $\mathcal{K}\ell(T)(X,1)$ (Assumption (4)), we have $J!_X \sqsupseteq \Psi_c(J!_X)$. Therefore by repeatedly applying Ψ_c to the both sides of the inequality, we obtain a decreasing sequence $J!_X \sqsupseteq \Psi_c(J!_X) \sqsupseteq \Psi_c^2(J!_X) \sqsupseteq \cdots$.

By Assumption (3), their greatest lower bound $\bigsqcap_{i\in\omega} \Psi_c^i(J!_X) : X \nrightarrow 1$ exists; we write $\Psi_c^\omega(J!_X)$ for the greatest lower bound. Here, as composition of arrows in $\mathcal{K}\ell(T)$ and $\overline{F}$'s action on arrows are both ω^{op}-continuous, Ψ_c is also ω^{op}-continuous. Therefore by the Kleene fixed point theorem, $\Psi_c^\omega(J!_X)$ is the greatest fixed point of Ψ_c.

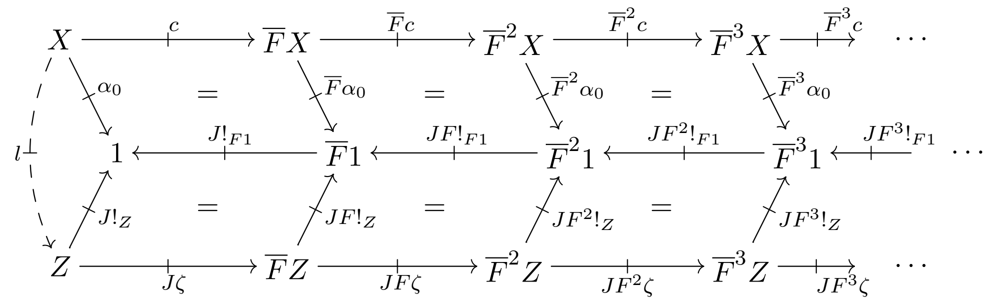
Using this greatest fixed point, for each $i < \omega$, we define an arrow $\alpha_i : X \nrightarrow \overline{F}^i 1$ inductively as follows:

$$\alpha_0 = \Psi_c^\omega(J!_X) \quad \text{and} \quad \alpha_{i+1} = \overline{F}\alpha_i \odot c\,. \tag{5.1}$$

Then we can prove $\alpha_i = \overline{F}^i J!_{F1} \odot \alpha_{i+1}$ for all $i \in \omega$ by induction on i as follows. For $i = 0$, we have:

$$\begin{aligned}
J!_{F1} \odot \alpha_1 &= J!_{F1} \odot \overline{F}(\Psi_c^\omega(J!_X)) \odot c && \text{(by definition)}\\
&= \Psi_c(\Psi_c^\omega(J!_X)) && \text{(by definition)}\\
&= \Psi_c^\omega(J!_X) && (\Psi_c^\omega(J!_X) \text{ is a fixed point})\\
&= \alpha_0 && \text{(by definition).}
\end{aligned}$$

For the step case, assume that $\alpha_i = \overline{F}^i J!_{F1} \odot \alpha_{i+1}$. Applying $\overline{F}$ and composing c from the right, we have $\alpha_{i+1} = \overline{F}^{i+1} J!_1 \odot \alpha_{i+2}$. Hence we have $\alpha_i = \overline{F}^i J!_{F1} \odot \alpha_{i+1}$ for all $i \in \omega$. This means that $(X, (\alpha_i : X \nrightarrow \overline{F}^i 1)_{i\in\omega})$ is a cone over the sequence $1 \overset{J!_{F1}}{\nleftarrow} \overline{F}1 \overset{JF!_{F1}}{\nleftarrow} \overline{F}^2 1 \overset{JF^2!_{F1}}{\nleftarrow} \cdots$. Therefore by Assumption (5), there exists a unique mediating arrow $l : X \nrightarrow Z$ from the cone $(X, (\alpha_i)_{i\in\omega})$ to the $(Z, (J\gamma_i)_{i\in\omega})$, on the one hand.



On the other hand, $J\zeta^{-1} \odot \overline{F}l \odot c$ is also a mediating arrow from $(X, (\alpha_i)_{i\in\omega})$ to $(Z, (J\gamma_i)_{i\in\omega})$. Indeed, for all $i \in \omega$, we have:

$$\begin{aligned}
J\gamma_i \odot \big(J\zeta^{-1} \odot \overline{F}l \odot c\big) &= \overline{F}^{i+1} J!_{F1} \odot \overline{F}J\gamma_i \odot \overline{F}l \odot c && (\zeta \text{ is a mediating arrow})\\
&= \overline{F}^{i+1} J!_{F1} \odot \overline{F}\alpha_i \odot c && (l \text{ is a mediating arrow})\\
&= \overline{F}^{i+1} J!_{F1} \odot \alpha_{i+1} && (\text{by definition of } \alpha_{i+1})\\
&= \alpha_i && ((X, (\alpha_i)_{i\in\omega}) \text{ is a cone})\,.
\end{aligned}$$

Hence by the uniqueness of the mediating arrow, we have $l = J\zeta^{-1} \odot \overline{F}l \odot c$ and l is a homomorphism from c to $j\zeta$.

To conclude the proof, we have to show that l the largest homomorphism from c to $J\zeta$. Let $g : X \nrightarrow Z$ be a homomorphism from c to $J\zeta$. We construct a cone $(X, (\beta_i : X \nrightarrow \overline{F}^i 1)_{i\in\omega})$ over the sequence $1 \overset{J!_{F1}}{\nleftarrow} \overline{F}1 \overset{JF!_{F1}}{\nleftarrow} \overline{F}^2 1 \overset{JF^2!_{F1}}{\nleftarrow} \cdots$ by $\beta_i = J\gamma_i \odot g$. Then we can prove that $\beta_i \sqsubseteq \alpha_i$ for all $i \in \omega$ by induction on i as follows.

For $i = 0$, we have:

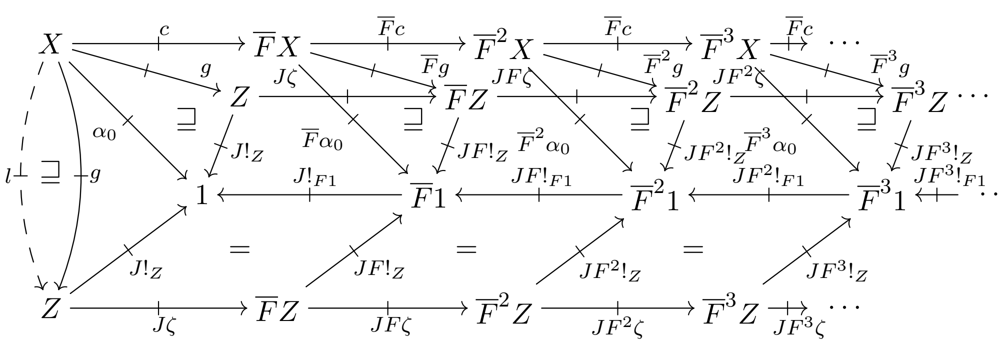
$$\begin{aligned}
\Psi_c(\beta_0) &= J!_{F1} \odot \overline{F} J\gamma_0 \odot \overline{F} g \odot c && \text{(by definition)}\\
&= J!_{F1} \odot \overline{F} J\gamma_0 \odot J\zeta \odot g && \text{(g is a homomorphism)}\\
&= J!_{F1} \odot JF\gamma_0 \odot J\zeta \odot g && \text{($\overline{F}$ is a lifting of F)}\\
&= J\gamma_0 \odot g && \text{(1 is a final object in $\mathbb{C}$)}\\
&= \beta_0 && \text{(by definition).}
\end{aligned}$$

Therefore β_0 is a fixed point of Ψ_c. As $\alpha_0 = \Psi_c^{\omega}(J!_X)$ is the greatest fixed point of Ψ_c, we have $\beta_0 \sqsubseteq \alpha_0$.

Assume $\beta_i \sqsubseteq \alpha_i$. Then

$$\begin{aligned}
\beta_{i+1} &= J\gamma_{i+1} \odot g && \text{(by definition)}\\
&= JF\gamma_i \odot J\zeta \odot g && \text{(ζ is a mediating arrow)}\\
&= \overline{F} J\gamma_i \odot J\zeta \odot g && \text{($\overline{F}$ is a lifting of F)}\\
&= \overline{F} J\gamma_i \odot \overline{F} g \odot c && \text{(g is a homomorphism)}\\
&= \overline{F}\beta_i \odot c && \text{($(X, (\beta_i)_{i\in\omega})$ is a cone)}\\
&\sqsubseteq \overline{F}\alpha_i \odot c && \text{(by the induction hypothesis and that $\overline{F}$ is monotone))}\\
&= \alpha_{i+1} && \text{(by definition)}\,.
\end{aligned}$$

Hence $\beta_i \sqsubseteq \alpha_i$ holds for all $i \in \omega$. This implies $J\gamma_i \odot g \sqsubseteq J\gamma_i \odot l$ for all $i \in \omega$. As $(Z, (J\gamma_i : Z \nrightarrow \overline{F}^i 1)_{i\in\omega})$ is a 2-limit (Assumption (5)), we have $g \sqsubseteq l$.



Therefore l is the largest homomorphism from c to $J\zeta$. □

Now we show that polynomial F and $T = \mathcal{G}$ constitute an infinitary trace situation. To this end, we have to check that polynomial F and $T = \mathcal{G}$ satisfy the assumptions in Proposition 5.2. The most nontrivial is Assumption (5); there we rely on results in [Sch09] for the fact that a limit is lifted to a limit. That the latter is indeed a 2-limit is not hard, exploiting suitable monotonicity.

Theorem 5.3. *The combination of polynomial F and $T = \mathcal{G}$ constitute an infinitary trace situation.*

Proof. We show that F and $\mathcal{G}$ satisfy Assumptions (1)–(5) in Proposition 5.2.

It is known that Assumption (1) is satisfied [Sch09].

It is also known that a distributive law $\lambda : F\mathcal{G} \Rightarrow \mathcal{G}F$ exists [Cîr10]. Therefore Assumption (2) is satisfied.

Now we prove that Assumption (3) is satisfied. Assume that a family $(f_i : (X, \mathfrak{F}_X) \nrightarrow (Y, \mathfrak{F}_Y))_{i\in\omega}$ of Kleisli arrows constitutes a decreasing sequence. We can define their greatest lower bound $\bigsqcap_{i\in\omega} f_i : X \nrightarrow Y$ in a pointwise manner: for $x \in X$ and $A \in \mathfrak{F}_Y$,

$$(\bigsqcap_{i\in\omega} f_i)(x)(A) = \lim_{i\to\infty} (f_i(x)(A)) .$$

It is easy to see that polynomial F preserves this pointwise greatest lower bound. Measurability of $\bigsqcap_{i\in\omega} f_i$ and ω^{op}-continuity of Kleisli composition can be proved in a similar manner to the proof of [BMP14, Proposition 9].

It is easy to see that Assumption (4) is satisfied.

Finally, we prove that Assumption (5) is satisfied. If $F1$ is empty, then the limit Z is also empty and Assumption (5) is satisfied. Assume that $F1$ is not empty. It is known that the sub-Giry monad $\mathcal{G}$ preserves a limit over an ω^{op}-sequence consisting of standard Borel spaces and surjective measurable functions [Sch09, Corollary 1]. In the current setting, F^i1 is a standard Borel space for all $i \in \omega$ because: 1 is standard Borel; Σ_n is standard Borel for each $n \in \omega$; and standard Borel spaces are closed under countable coproducts and countable limits [Kec95, 12.B]. Moreover, it is easy to see that $F^i!_{F1}$ is surjective for each $i \in \omega$ because $!_{F1} : F1 \to 1$ is a surjective function and the polynomial functor F preserves epimorphisms. Therefore by [Sch09, Corollary 1], the limit $(Z, (\gamma_i : Z \to F^i1)_{i\in\omega})$ over the final ω^{op}-sequence $1 \stackrel{!_{F1}}{\leftarrow} F1 \stackrel{F!_{F1}}{\leftarrow} F^21 \stackrel{F^2!_{F1}}{\leftarrow} \cdots$ is preserved by $\mathcal{G}$. This immediately implies that $J : \mathbf{Meas} \to \mathcal{K}\ell(\mathcal{G})$ preserves the limit. It is easy to see that the resulting limit is a 2-limit.

Hence by Proposition 5.2, F and $\mathcal{G}$ constitute an infinitary trace situation. □

We will later discuss another pair of a functor and a monad that can also model probabilistic systems in Section 5.5. The proof that the pair constitutes an infinitary trace situation is very different from the one for polynomial F and $T = \mathcal{G}$ above, because the axiomatic results in Proposition 4.1 and Proposition 5.2 are not applicable.

5.2. Kleisli Simulations for Probabilistic Systems.

5.2.1. *Forward Simulations.* Soundness of forward simulation, in the current probabilistic setting, follows immediately from the the axiomatic development in Lemma 4.5.

Theorem 5.4. *Given two $(\mathcal{G}, F)$-systems $\mathcal{X} = (X, s, c)$ and $\mathcal{Y} = (Y, t, d)$, $\mathcal{X} \sqsubseteq_{\mathbf{F}} \mathcal{Y}$ implies $\mathsf{tr}^\infty(c) \odot s \sqsubseteq \mathsf{tr}^\infty(d) \odot t$.*

Proof. In a similar manner to the proof of Theorem 5.3, we can show that F and $\mathcal{G}$ satisfy the assumptions in Lemma 4.5. Therefore the statement is immediate from Lemma 4.5. □

Example 5.5. We define a ranked alphabet $\Sigma = (\Sigma_n)_{n\in\omega}$ by $\Sigma_0 = \{a\}$, $\Sigma_2 = \{b\}$ and $\Sigma_i = \emptyset$ for each $i \in \mathbb{N} \setminus \{0, 2\}$. We define $(\mathcal{G}, F_\Sigma)$-systems $\mathcal{X} = (X, s, c)$ and $\mathcal{Y} = (Y, t, d)$ as follows:

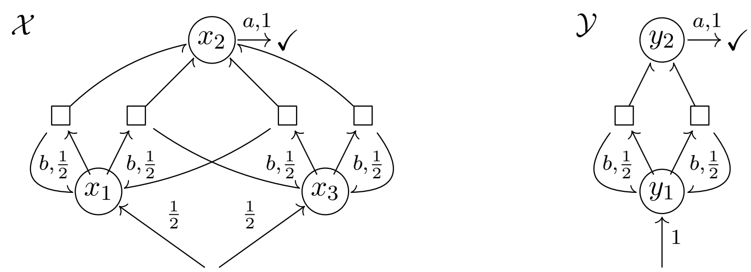
- $X = (\{x_1, x_2, x_3\}, \mathcal{P}\{x_1, x_2, x_3\})$ and $Y = (\{y_1, y_2\}, \mathcal{P}\{y_1, y_2\})$.
- $s(\{x_1\}) = s(\{x_3\}) = \frac{1}{2}$, $s(\{x_2\}) = 0$, $t(\{y_1\}) = 1$ and $t(\{y_2\}) = 0$.

- $c(x_1)(\{a\}) = c(x_3)(\{a\}) = 0$, $c(x_2)(\{a\}) = 1$, $d(y_1)(\{a\}) = 0$, $d(y_2)(\{a\}) = 1$,

$$c(x)(\{(b, x', x'')\}) = \begin{cases} \frac{1}{2} & \big((x, x', x'') \in \{(x_1, x_1, x_2), (x_1, x_3, x_2)\}\big) \\ \frac{1}{2} & \big((x, x', x'') \in \{(x_3, x_2, x_1), (x_3, x_2, x_3)\}\big) \\ 0 & \text{(otherwise)} \end{cases} \quad \text{and}$$

$$d(y)(\{(b, y', y'')\}) = \begin{cases} \frac{1}{2} & \big((y, y', y'') \in \{(y_1, y_1, y_2), (y_1, y_2, y_1)\}\big) \\ 0 & \text{(otherwise)}\,. \end{cases}$$

We can illustrate $\mathcal{X}$ and $\mathcal{Y}$ as follows. Here $z \xrightarrow{a,p} \checkmark$ means $c(z)(\{a\}) = p$ or $d(z)(\{a\}) = p$, and $z \xrightarrow{b,p} \Box \rightrightarrows^{z_1}_{z_2}$ means $c(z)(\{(b, z_1, z_2)\}) = p$ or $d(z)(\{(b, z_1, z_2)\}) = p$.



We define a Kleisli arrow $f : Y \nrightarrow X$ in $\mathcal{K}\ell(\mathcal{G})$ as follows:

$$f(y_1)(\{x\}) = \begin{cases} \frac{1}{2} & (x \in \{x_1, x_3\}) \\ 0 & \text{(otherwise)} \end{cases} \quad \text{and} \quad f(y_2)(\{x\}) = \begin{cases} 1 & (x = x_2) \\ 0 & \text{(otherwise)}\,. \end{cases}$$

Then f is a forward simulation from $\mathcal{X}$ to $\mathcal{Y}$. By Theorem 5.4, we have $\mathsf{tr}^\infty(c) \odot s \sqsubseteq \mathsf{tr}^\infty(d) \odot t$.

5.2.2. *Backward Simulations.* We turn to backward simulations. Similarly to the nondeterministic setting (Section 4.2.2), we have to impose a certain restriction on backward Kleisli simulations to ensure soundness. By the feature of $\mathcal{G}$ that composition in $\mathcal{K}\ell(\mathcal{G})$ is ω^{op}-continuous—a feature absent in $\mathcal{K}\ell(\mathcal{P})$—the image-finiteness condition is no longer needed.

Definition 5.6 (total backward simulation)**.** Let $\mathcal{X} = (X, s, c)$ and $\mathcal{Y} = (Y, t, d)$ be $(\mathcal{G}, F)$-systems. A backward simulation $b : X \nrightarrow Y$ from $\mathcal{X}$ to $\mathcal{Y}$ is *total* if $b(x)(Y) = 1$ for all $x \in X$. We write $\mathcal{X} \sqsubseteq^{\mathrm{T}}_{\mathbf{B}} \mathcal{Y}$ if there exists a total backward simulation from $\mathcal{X}$ to $\mathcal{Y}$.

Theorem 5.7 (soundness of $\sqsubseteq^{\mathrm{T}}_{\mathbf{B}}$)**.** *For two $(\mathcal{G}, F)$-systems $\mathcal{X} = (X, s, c)$ and $\mathcal{Y} = (Y, t, d)$, $\mathcal{X} \sqsubseteq^{\mathrm{T}}_{\mathbf{B}} \mathcal{Y}$ implies* $\mathsf{tr}^\infty(c) \odot s \sqsubseteq \mathsf{tr}^\infty(d) \odot t$. □

The proof of Theorem 5.7 is via the following axiomatic development.

Definition 5.8 (total backward simulation, generally)**.**[†] Let F be an endofunctor and T be a monad on $\mathbb{C}$ that satisfy the conditions in Proposition 5.2 with respect to $\sqsubseteq$. For two (T, F)-systems $\mathcal{X} = (X, s, c)$ and $\mathcal{Y} = (Y, t, d)$, a *total backward simulation* from $\mathcal{X}$ to $\mathcal{Y}$ is a backward simulation $b : X \nrightarrow Y$ that satisfies the following condition:

(1) The arrow $b : X \nrightarrow Y$ satisfies $J!_Y \odot b = J!_X$. Here $!_Y : Y \to 1$ is the unique function.

We write $\mathcal{X} \sqsubseteq^{\mathrm{T}}_{\mathbf{B}} \mathcal{Y}$ if there exists a total backward simulation from $\mathcal{X}$ to $\mathcal{Y}$.

This general total backward simulation satisfies soundness. For its proof we have to look into the inductive construction of the largest homomorphism in Section 5.1 (Proposition 5.2).

Lemma 5.9.† *Let F and T be as in Proposition 5.2. For two (T,F)-systems $\mathcal{X}=(X,s,c)$ and $\mathcal{Y}=(Y,t,d)$, $\mathcal{X}\sqsubseteq^{\mathrm{T}}_{\mathbf{B}}\mathcal{Y}$ (in the sense of Definition 5.8) implies $\mathsf{tr}^{\infty}(c)\sqsubseteq\mathsf{tr}^{\infty}(d)\odot b$. Furthermore it follows that $\mathsf{tr}^{\infty}(c)\odot s\sqsubseteq\mathsf{tr}^{\infty}(d)\odot t$.*

Proof. We prove $\mathsf{tr}^{\infty}(c)\sqsubseteq\mathsf{tr}^{\infty}(d)\odot b$ along the construction of $\mathsf{tr}^{\infty}(c)$ and $\mathsf{tr}^{\infty}(d)$ in the proof of Proposition 5.2.

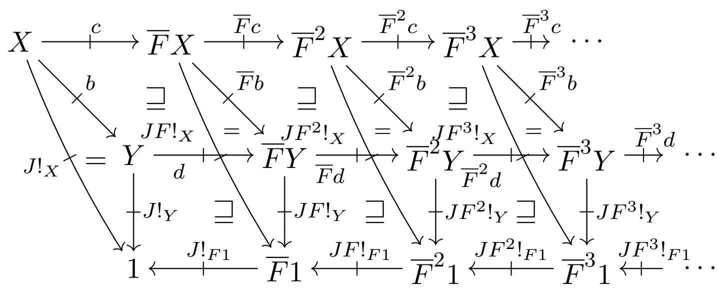
For each $i\in\omega$, we define $\Psi^i_c(J!_X):X\nrightarrow 1$ and $\Psi^i_d(J!_Y):Y\nrightarrow 1$ as in the proof of Proposition 5.2. We first prove $\Psi^i_c(J!_X)\sqsubseteq\Psi^i_d(J!_Y)\odot b$ for all $i\in\omega$ by induction on i as follows.

If $i=0$, by Assumption (1) in Definition 5.8, we have $\Psi^0_X(J!_X)=\Psi^0_Y(J!_Y)\odot b$.

Let $i>0$ and assume that $\Psi^{i-1}_c(J!_X)\sqsubseteq\Psi^{i-1}_d(J!_Y)\odot b$. Then

$$\begin{aligned}\Psi^i_c(J!_X)&=J!_{F1}\odot\overline{F}(\Psi^{i-1}_c(J!_X))\odot d &&\text{(by definition of }\Psi_c\text{)}\\ &\sqsubseteq J!_{F1}\odot\overline{F}(\Psi^{i-1}_d(J!_Y))\odot\overline{F}b\odot c &&\text{(by induction hypothesis)}\\ &\sqsubseteq J!_{F1}\odot\overline{F}(\Psi^{i-1}_d(J!_Y))\odot d\odot b &&\text{(}b\text{ is a backward simulation)}\\ &=\Psi^i_d(J!_Y)\odot b &&\text{(by definition of }\Psi_d\text{)}.\end{aligned}\tag{5.2}$$

Therefore we have $\Psi^i_c(J!_X)\sqsubseteq\Psi^i_d(J!_Y)\odot b$ for all $i\in\omega$.



Now let $(X,(\alpha^X_i:X\nrightarrow\overline{F}^i1)_{i\in\omega})$ and $(Y,(\alpha^Y_i:Y\nrightarrow\overline{F}^i1)_{i\in\omega})$ be cones over the sequence $1\overset{J!_{F1}}{\nleftarrow}\overline{F}1\overset{JF!_{F1}}{\nleftarrow}\overline{F}^21\overset{JF^2!_{F1}}{\nleftarrow}\cdots$; they are defined by the equation (5.1) in the proof of Proposition 5.2. Recall that $\mathsf{tr}^{\infty}(c):X\nrightarrow Z$ is the unique mediating arrow from the cone $(X,(\alpha^X_i:X\nrightarrow\overline{F}^i1)_{i\in\omega})$ to the 2-limit $(Z,(J\gamma_i:Z\nrightarrow\overline{F}^i1)_{i\in\omega})$, and similarly $\mathsf{tr}^{\infty}(d):Y\nrightarrow Z$ is the unique mediating arrow from $(Y,(\alpha^Y_i:Y\nrightarrow\overline{F}^i1)_{i\in\omega})$ to the same 2-limit. Note here that for each $i\in\omega$, we have:

$$J\gamma_i\odot(\mathsf{tr}^{\infty}(d)\odot b)\;=\;(J\gamma_i\odot\mathsf{tr}^{\infty}(d))\odot b\;=\;\alpha^Y_i\odot b\,.$$

Therefore $\mathsf{tr}^{\infty}(d)\odot b:X\nrightarrow Z$ is the unique mediating arrow from a cone $(X,(\alpha^Y_i\odot b:X\nrightarrow\overline{F}^i1)_{i\in\omega})$ to $(Z,(J\gamma_i:Z\nrightarrow\overline{F}^i1)_{i\in\omega})$. We prove $\alpha^X_i\sqsubseteq\alpha^Y_i\odot b$ for all $i\in\omega$ by induction on i as follows:

For $i = 0$, we have:

$$\begin{aligned}
\alpha_0^X &= \prod_{i\in\omega} \Psi_c^i(J!_X) && \text{(by definition)}\\
&\sqsubseteq \prod_{i\in\omega} (\Psi_d^i(J!_Y) \odot b) && \text{(by (5.2))}\\
&= \big(\prod_{i\in\omega} \Psi_d^i(J!_Y)\big) \odot b && \text{(by Assumption (3) of Proposition 5.2)}\\
&= \alpha_0^Y \odot b && \text{(by definition)}\,.
\end{aligned}$$

Let $i > 0$ and assume that $\alpha_{i-1}^X \sqsubseteq \alpha_{i-1}^Y \odot b$. Then

$$\begin{aligned}
\alpha_i^X &= \overline{F}\alpha_{i-1}^X \odot c && \text{(by definition)}\\
&\sqsubseteq \overline{F}\alpha_{i-1}^Y \odot \overline{F}b \odot c && \text{(by induction hypothesis and the monotonicity of } \overline{F})\\
&\sqsubseteq \overline{F}\alpha_{i-1}^Y \odot d \odot b && (b \text{ is a backward simulation})\\
&= \alpha_i^Y \odot b && \text{(by definition).}
\end{aligned}$$

Therefore we have $\alpha_i^X \sqsubseteq \alpha_i^Y \odot b$ for all $i \in \omega$. As $(Z, (J\gamma_i : Z \nrightarrow \overline{F}^i 1)_{i\in\omega})$ is a 2-limit, this implies $\mathsf{tr}^\infty(c) \sqsubseteq \mathsf{tr}^\infty(d) \odot b$.

The last claim follows from b's condition on initial states. □

Lemma 5.10. *In Definition 5.8, if $T = \mathcal{G}$ and F is a polynomial functor, Assumption (1) is satisfied if $b(x)(Y) = 1$ for each $x \in X$.*

Proof. We assume that $b(x)(Y) = 1$ for each $x \in X$. By the definition of multiplication $\mu^{\mathcal{G}}$ of the sub-Giry monad (see Definition 2.3), for $x \in X$, we have:

$$\big(J!_Y \odot b\big)(x)(1) = b(x)\big(!_Y^{-1}(1)\big) = b(x)(Y) = 1 = J!_X(x)(1)\,.$$

Therefore Assumption (1) is satisfied. □

Proof of Theorem 5.7. In Lemma 5.10 we prove that a total backward simulation in the specific sense of Definition 5.6 is also a total backward simulation in the general sense of Definition 5.6. Therefore Lemma 5.9 yields trace inclusion. □

Example 5.11. We continue Example 5.5. The arrow $f : Y \nrightarrow X$ in Example 5.5 is also a total backward simulation from $\mathcal{Y}$ to $\mathcal{X}$. By Theorem 5.7, this implies $\mathsf{tr}^\infty(c) \odot s \sqsupseteq \mathsf{tr}^\infty(d) \odot t$, and therefore together with the forward simulation in Example 5.5, we have $\mathsf{tr}^\infty(c) \odot s = \mathsf{tr}^\infty(d) \odot t$.

5.3. **Forward Partial Execution for Probabilistic Systems.**

Theorem 5.12. *Let F be a polynomial functor on* **Meas**. *For $(\mathcal{G}, F)$-systems $\mathcal{X} = (X, s, c)$ and $\mathcal{Y} = (Y, t, d)$, the following hold.*

(1) (a) *(soundness of FPE for forward simulation) $\mathcal{X}_{\mathsf{FPE}} \sqsubseteq_{\mathbf{F}} \mathcal{Y}$ implies* $\mathsf{tr}^\infty(c) \odot s \sqsubseteq \mathsf{tr}^\infty(d) \odot t$.
  (b) *(adequacy of FPE for forward simulation) $\mathcal{X} \sqsubseteq_{\mathbf{F}} \mathcal{Y}$ implies $\mathcal{X}_{\mathsf{FPE}} \sqsubseteq_{\mathbf{F}} \mathcal{Y}$.*

(2) (a) *(soundness of FPE for backward simulation) $\mathcal{X} \sqsubseteq_{\mathbf{B}}^{\mathrm{T}} \mathcal{Y}_{\mathsf{FPE}}$ implies* $\mathsf{tr}^\infty(c) \odot s \sqsubseteq \mathsf{tr}^\infty(d) \odot t$.

(b) *(adequacy of FPE for backward simulation)* $\mathcal{X} \sqsubseteq^{\mathrm{T}}_{\mathbf{B}} \mathcal{Y}$ *implies* $\mathcal{X} \sqsubseteq^{\mathrm{T}}_{\mathbf{B}} \mathcal{Y}_{\mathsf{FPE}}$, *assuming that:* $d(y)(FY) = 1$ *for all* $y \in Y$. □

The item (1) for forward simulations follows immediately from Theorem 4.15. For the relationship to backward simulations, we develop the following general result that can be proved in a similar manner to the proof of Theorem 4.14.

Theorem 5.13 (FPE and backward simulation).† *Let F be an endofunctor and T be a monad on $\mathbb{C}$ that satisfy the conditions in Proposition 5.2 (hence those in Lemma 5.9). Let $\mathcal{X} = (X, s, c)$ and $\mathcal{Y} = (Y, t, d)$ be (T, F)-systems.*

(1) *(soundness for backward simulation)* $\mathcal{X} \sqsubseteq^{\mathrm{T}}_{\mathbf{B}} \mathcal{Y}_{\mathsf{FPE}}$ *implies* $\mathsf{tr}^\infty(c) \odot s \sqsubseteq \mathsf{tr}^\infty(d) \odot t$.
(2) *(adequacy for backward simulation)* $\mathcal{X} \sqsubseteq^{\mathrm{T}}_{\mathbf{B}} \mathcal{Y}$ *implies* $\mathcal{X} \sqsubseteq^{\mathrm{T}}_{\mathbf{B}} \mathcal{Y}_{\mathsf{FPE}}$, *assuming that: the coalgebra* $d : Y \to\!\!\!\!\!\to \overline{F}Y$ *satisfies* $J!_{FY} \odot d = J!_Y$. □

5.4. **Coincidence between Automata-theoretic and Coalgebraic Infinitary Trace Semantics.** In this section we again give a sanity-check result like in Section 4.4. Namely, we show a coincidence between coalgebraic infinitary trace semantics that is defined in Section 5.1, and infinitary language that is defined using automata-theoretic terms. Although its statement is a sanity-check, the result requires somewhat delicate treatment of measure-theoretic structures. We first describe the definition of the latter.

In this section, for simplicity, we assume that both the state space $(X, \mathfrak{F}_X)$ and all components $(\Sigma_n, \mathfrak{F}_{\Sigma_n})$ in the ranked alphabet $\Sigma = \big((\Sigma_n, \mathfrak{F}_{\Sigma_n})\big)_{n\in\omega}$ are countable sets with the discrete σ-algebras. It is not difficult to generalize the results in this section for automata labeled with a general standard Borel ranked alphabet Σ.

Definition 5.14. Let $\Sigma = \big((\Sigma_n, \mathcal{P}\Sigma_n)\big)_{n\in\omega}$ be a standard Borel ranked alphabet such that all Σ_n are countable sets equipped with the discrete σ-algebras. A *Σ-labeled probabilistic tree automaton* is a $(\mathcal{G}, F_\Sigma)$-system $\mathcal{X} = (X, s, c)$ where X is a countable set equipped with the discrete σ-algebra.

For a given Σ-labeled probabilistic tree automaton $\mathcal{X} = (X, s, c)$, we now define its automata-theoretic (infinitary) language. It is defined as a probability measure $L^\infty(\mathcal{X})$ on a set $\mathrm{Tree}_\infty(\Sigma)$ of infinitary trees. The definitions are all as usual.

Definition 5.15. For a standard Borel ranked alphabet $\Sigma = ((\Sigma_n, \mathcal{P}\Sigma_n))_{n\in\omega}$, we define a set $\mathcal{S}_\Sigma \subseteq \mathcal{P}(\mathrm{Tree}_\infty(\Sigma))$ by $\mathcal{S}_\Sigma = \{\mathrm{cyl}(t) \mid k \in \omega, t \in \mathrm{Tree}^k(\Sigma)\}$, where $\mathrm{cyl}(t)$ is from Definition 2.9. A σ-algebra $\mathfrak{F}_\infty$ on $\mathrm{Tree}_\infty(\Sigma)$ is the smallest σ-algebra that contains $\mathcal{S}_\Sigma$.

To define a probability measure $L^\infty(\mathcal{X})$ on a measurable space $(\mathrm{Tree}_\infty(\Sigma), \mathfrak{F}_\infty)$, we have to fix a value $L^\infty(\mathcal{X})(A)$ for each $A \in \mathfrak{F}_\infty$. As is standard, by Carathéodory's extension theorem (see e.g. [ADD00]), it suffices to fix a value $L^\infty(\mathcal{X})(A)$ for all cylinders $A = \mathrm{cyl}(t)$ in a "compatible" manner.

To this end, we first review the notion of *branching process* (see e.g. [Har64]). It is used to fix the value $L^\infty(\mathcal{X})(\mathrm{Tree}_\infty(\Sigma))$ (note that $\mathrm{Tree}_\infty(\Sigma)$ is a cylinder set induced by the 0-prefix tree). Intuitively the value is probability with which the probabilistic automaton does not abort.

Definition 5.16. A *branching process* is a pair $\Delta = (\Gamma, \tau)$ consisting of a finite set Γ of *types*, and a *transition function* $\tau : \Gamma \times \Gamma^* \to [0, 1]$ such that $\sum_{\alpha\in\Gamma^*}(x, \alpha) = 1$ for all $x \in \Gamma$.

A branching process Δ and an *initial process* $x_0 \in \Gamma$ give rise to a Markov chain $\mathcal{M}_{\Delta,x_0}$ such that: the state space is a set Γ^* of *population of processes*; the transition function $\tau_{\mathcal{M}} : \Gamma^* \times \Gamma^* \to [0,1]$ is given by

$$\tau_{\mathcal{M}}\big(\langle x_0 \dots x_{n-1}\rangle, \beta\big) = \sum_{\substack{\beta_0,\dots,\beta_{n-1} \in \Gamma^* \\ \text{s.t. } \beta = \beta_0 \dots \beta_{n-1}}} \prod_{1 \le i \le n-1} \tau(x_i, \beta_i);$$

and the initial state is a singleton tuple $\langle x_0 \rangle$. Here juxtaposition $\beta_0 \dots \beta_{n-1}$ denotes the concatenation of tuples. For a pair of types $x_0, x \in \Gamma$, the *probability of reaching x from x_0* is the value $\mathrm{Reach}(\Delta, x_0, x) \in [0,1]$ with which a population that has a type x in it is reached in $\mathcal{M}_{\Delta,x_0}$.

Intuitively, in every transition of a branching process, each process in the population gives birth to child processes randomly. The probability that a process x gives birth to children represented by a population $\alpha \in \Gamma^*$ is given by $\tau(x, \alpha)$.

From a given Σ-labeled probabilistic tree automaton $\mathcal{X}$, we can obtain a branching process $\Delta_{\mathcal{X}}$ by adding a new process $\perp$ that means aborting of the system, and by "forgetting" the labels on transitions.

Definition 5.17. For a Σ-labeled probabilistic tree automaton $\mathcal{X} = (X, s, c)$, its *skeleton* is a branching process $\Delta_{\mathcal{X}} = (\Gamma_{\mathcal{X}}, \tau_{\mathcal{X}})$ where $\Gamma_{\mathcal{X}} = X + \{\perp\}$ and $\tau_{\mathcal{X}}$ is defined as follows.

$$\tau_{\mathcal{X}}(x, \alpha) = \begin{cases} \sum_{a \in \Sigma_n} c(x)\big(\{(a, x_0, \dots, x_{n-1})\}\big) & (x \in X, \alpha = \langle x_0, \dots, x_{n-1}\rangle \in X^*) \\ 1 - \sum_{\beta \in X^*} \tau(x, \beta) & (x \in X, \alpha = \langle \perp \rangle) \\ 1 & (x \in \{\perp\}, \alpha = \langle \perp \rangle) \\ 0 & (\text{otherwise}) \end{cases}$$

Now we are ready to define a value $L^\infty(\mathcal{X})(\mathrm{cyl}(t))$ for each prefix tree t. In particular, the value $L^\infty(\mathcal{X})(\mathrm{Tree}_\infty(\Sigma))$ is defined as the probability with which the state $\perp$ is not reached from x in the skeleton $\Delta_{\mathcal{X}}$.

Proposition 5.18. *Let $\mathcal{X} = (X, s, c)$ be a Σ-labeled probabilistic tree automaton. For a state $x \in X$, a natural number $k \in \omega$ and a k-prefix tree $t \in \mathit{Tree}^k(\Sigma)$, we define the value $\nu_x(t) \in [0,1]$ by induction on k as follows.*

- *If $k = 0$, then $\nu_x(t) = 1 - \mathit{Reach}(\Delta_{\mathcal{X}}, x, \perp)$.*
- *Let $k > 0$. If $t = (D, l)$, $l(\varepsilon) = a \in \Sigma_n$ and t_i is the i-th subtree of t, then:*

$$\nu_x(t) = \sum_{x_0,\dots,x_{n-1} \in X} \Big(c(x)\big(\{(a, x_0, \dots, x_{n-1})\}\big) \cdot \prod_{i=0}^{n-1} \nu_{x_i}(t_i) \Big).$$

Then for each $x \in X$, there exists a unique probability measure $L^\infty(\mathcal{X}, x)$ on $(\mathit{Tree}_\infty(\Sigma), \mathfrak{F}_\infty)$ such that $L^\infty(\mathcal{X}, x)(\mathit{cyl}(t)) = \nu_x(t)$.

This proposition is proved using Carathéodory's extension theorem and the following "compatibility" lemma.

Lemma 5.19. *In Proposition 5.18, for all $k \in \omega$ and $t \in \mathit{Tree}^k(\Sigma)$, we have:*

$$\sum_{\substack{s \in \mathit{Tree}^{k+1}(\Sigma) \\ \text{s.t. } t \preceq s}} \nu_x(s) = \nu_x(t). \tag{5.3}$$

Proof. For a $(k+1)$-prefix tree $s = (D_s, l_s) \in \mathrm{Tree}^{k+1}(\Sigma)$, we write a_s for $l_s(\varepsilon)$ and n_s for the arity of a_s (i.e. $a_s \in \Sigma_{n_s}$). We prove the equation (5.3) by induction on k.

If $k = 0$, as t and all the subtrees of s except for s itself are 0-prefix trees, we have:

$$
\begin{aligned}
&\sum_{\substack{s\in\mathrm{Tree}^{k+1}(\Sigma)\\ \text{s.t. } t\preceq s}} \nu_x(s)\\
&= \sum_{s\in\mathrm{Tree}^{1}(\Sigma)} \sum_{x_0,\dots,x_{n_s-1}\in X} c(x)\big(\{(a_s,x_0,\dots,x_{n_s-1})\}\big)\cdot \prod_{i=0}^{n_s-1}\nu_{x_i}(s_i)\\
&\hspace{20em}\text{(by definition of } \nu_x(s))\\
&= \sum_{n=0}^{\infty}\sum_{a\in\Sigma_n}\sum_{x_0,\dots,x_{n-1}\in X} c(x)\big(\{(a,x_0,\dots,x_{n-1})\}\big)\cdot \prod_{i=0}^{n-1}\nu_{x_i}(s_i)\\
&= \sum_{n=0}^{\infty}\sum_{a\in\Sigma_n}\sum_{x_0,\dots,x_{n-1}\in X} c(x)\big(\{(a,x_0,\dots,x_{n-1})\}\big)\cdot \prod_{i=0}^{n-1}(1-\mathrm{Reach}(\Delta_{\mathcal{X}},x_i,\bot))\\
&\hspace{20em}(s_i\in\mathrm{Tree}^0(\Sigma)\text{ for each } i)\\
&= \sum_{n=0}^{\infty}\sum_{x_0,\dots,x_{n-1}\in X} \tau_{\mathcal{X}}(x,\langle x_0,\dots,x_{n-1}\rangle)\cdot \prod_{i=0}^{n-1}(1-\mathrm{Reach}(\Delta_{\mathcal{X}},x_i,\bot))\\
&\hspace{20em}\text{(by definition of } \tau_{\mathcal{X}} \text{ of } \Delta_{\mathcal{X}})\\
&= 1-\mathrm{Reach}(\Delta_{\mathcal{X}},x,\bot) \qquad\qquad \text{(by definition of branching process)}\\
&= \nu_x(t) \qquad\qquad \text{(by definition of } \nu_x(t))\,.
\end{aligned}
$$

Next, let $k > 0$ and assume that $\sum_{v\in\mathrm{Tree}^k(\Sigma)\text{ s.t. } u\preceq v}\nu_x(v) = \nu_x(u)$ holds for all $u \in \mathrm{Tree}^{k-1}(\Sigma)$ and $x \in X$. Let $t = (D,l) \in \mathrm{Tree}^k(\Sigma)$, $a = l(\varepsilon)$ and $a \in \Sigma_n$. Moreover, let t_i be the i-th subtrees of $t \in \mathrm{Tree}^k(\Sigma)$. Then:

$$
\begin{aligned}
&\sum_{\substack{s\in\mathrm{Tree}^{k+1}(\Sigma)\\ \text{s.t. } t\preceq s}} \nu_x(s)\\
&= \sum_{\substack{s\in\mathrm{Tree}^{k+1}(\Sigma)\\ \text{s.t. } t\preceq s}} \sum_{x_0,\dots,x_{n-1}\in X}\left(c(x)\big(\{(a,x_0,\dots,x_{n-1})\}\big)\cdot \prod_{i=0}^{n-1}\nu_{x_i}(s_i)\right)\\
&\hspace{20em}\text{(by definition of } \nu_x(s))\\
&= \sum_{x_0,\dots,x_{n-1}\in X}\left(c(x)\big(\{(a,x_0,\dots,x_{n-1})\}\big)\cdot \left(\sum_{\substack{s\in\mathrm{Tree}^{k+1}(\Sigma)\\ \text{s.t. } t\preceq s}}\prod_{i=0}^{n-1}\nu_{x_i}(s_i)\right)\right)\\
&= \sum_{x_0,\dots,x_{n-1}\in X}\left(c(x)\big(\{(a,x_0,\dots,x_{n-1})\}\big)\cdot \left(\sum_{\substack{s_0\in\mathrm{Tree}^{k}(\Sigma)\\ \text{s.t. } t_0\preceq s_0}}\cdots\sum_{\substack{s_{n-1}\in\mathrm{Tree}^{k}(\Sigma)\\ \text{s.t. } t_{n-1}\preceq s_{n-1}}}\prod_{i=0}^{n-1}\nu_{x_i}(s_i)\right)\right)
\end{aligned}
$$

$$\begin{aligned}
&\qquad\qquad\qquad\qquad (\{s \in \mathrm{Tree}^{k+1}(\Sigma) \mid t \preceq s\} \cong \textstyle\prod_{i=0}^{n-1}\{s_i \in \mathrm{Tree}^{k}(\Sigma) \mid t_i \preceq s_i\})\\
&= \sum_{x_0,\dots,x_{n-1}\in X}\left(c(x)\big(\{(a,x_0,\dots,x_{n-1})\}\big)\cdot \left(\prod_{i=0}^{n-1} \sum_{\substack{s_i\in \mathrm{Tree}^k(\Sigma)\\ \text{s.t. } t_i\preceq s_i}} \nu_{x_i}(s_i)\right)\right)\\
&\qquad\qquad\qquad\qquad\qquad\qquad (s_i \text{ does not appear in } \nu_{x_j}(s_j) \text{ if } i\neq j)\\
&= \sum_{x_0,\dots,x_{n-1}\in X}\Big(c(x)\big(\{(a,x_0,\dots,x_{n-1})\}\big)\cdot \prod_{i=0}^{n-1}\nu_{x_i}(t_i)\Big) \qquad \text{(by induction hypothesis)}\\
&= \nu_x(t) \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(by definition of } \nu_x(t)).
\end{aligned}$$

Therefore $\sum_{s\in \mathrm{Tree}^{k+1}(\Sigma) \text{ s.t. } t\preceq s} \nu_x(s) = \nu_x(t)$ holds for all $k \in \omega$ and $t \in \mathrm{Tree}^k(t)$. □

Proof of Proposition 5.18. Immediate from Carathéodory's extension theorem [ADD00] and Lemma 5.19. □

Definition 5.20. Let $\mathcal{X} = (X, s, c)$ be a Σ-labeled probabilistic tree automaton. For a state $x \in X$, the *infinitary language of $\mathcal{X}$ from x* is a probability measure $L^\infty(\mathcal{X}, x)$ on $(\mathrm{Tree}_\infty(\Sigma), \mathfrak{F}_\infty)$ in Proposition 5.18. The *infinitary language of $\mathcal{X}$* is a probability measure $L^\infty(\mathcal{X})$ on a $(\mathrm{Tree}_\infty(\Sigma), \mathfrak{F}_\infty)$ that is defined by $L^\infty(\mathcal{X})(A) = \sum_{x\in X} s(*)(x)\cdot L^\infty(\mathcal{X}, x)(A)$ for $A \in \mathfrak{F}_\infty$.

The following is the main result of this section.

Proposition 5.21. *The carrier of a final F_Σ-coalgebra in* **Meas** *is isomorphic to $(\mathit{Tree}_\infty(\Sigma), \mathfrak{F}_\infty)$, and for a Σ-labeled probabilistic tree automaton $\mathcal{X} = (X, s, c)$, $\mathsf{tr}^\infty(c)(x) = L^\infty(\mathcal{X}, x)$ holds for all $x \in X$. Furthermore it follows that $\mathsf{tr}^\infty \odot s(c)(*) = L^\infty(\mathcal{X})$.*

To prove this proposition, we use the lemma below that states that the unreachable probability of a branching process can be calculated as the greatest fixed point of a certain function. It is a direct consequence of the well-known result that the reachability probability of a Markov chain can be calculated as the least fixed point of a certain function (see e.g. [BK08, Theorem 10.15]). A generalized statement of the following lemma (one for *branching Markov decision processes*) is given in [ESY15]. In the rest of this section, for a vector $\mathbf{v} \in [0,1]^X$ and $x \in X$, $\mathbf{v}_x$ denotes the x-th element of $\mathbf{v}$.

Lemma 5.22. *Let $\Delta = (\Gamma, \tau)$ be a branching process and $y \in \Gamma$. We define a function $P_y : [0,1]^\Gamma \to [0,1]^\Gamma$ as follows:*

$$\big(P_y(\mathbf{v})\big)_x = \sum_{\substack{n\in\omega,\ x_0,\dots,x_{n-1}\in\Gamma \text{ s.t.}\\ y\notin\{x_0,\dots,x_{n-1}\}}} \tau\big(x, \langle x_0 \dots x_{n-1}\rangle\big)\cdot \prod_{i=0}^{n-1}\mathbf{v}_{x_i}\,.$$

As P_y is a monotone function, P_y has the greatest fixed point $\mathbf{v}^{y,\max} \in [0,1]^\Gamma$. Then

$$1 - \mathit{Reach}(\Delta, x, y) = (\mathbf{v}^{y,\max})_x\,.$$ □

Proof of Proposition 5.21. We define an arrow $\zeta : (\mathrm{Tree}_\infty(\Sigma), \mathfrak{F}_\infty) \to F_\Sigma(\mathrm{Tree}_\infty(\Sigma), \mathfrak{F}_\infty)$ in **Meas** in a similar manner to the final F_Σ-coalgebra in **Sets** (see Proposition 4.18): namely, $\zeta(t) = (a, (t_0, \dots, t_{n-1}))$. (Here $t = (D, l)$, $a = l(\varepsilon) \in \Sigma_n$, and for each $i \in \{0, \dots, n-1\}$, $t_i = (D_i, l_i)$ where $D_i = \{\alpha \in \mathbb{N}^* \mid i\alpha \in D\}$ and $l_i(\alpha) = l(i\alpha)$.) It is easy to see that ζ

is a measurable function. It is also easy to see that ζ is a final F_Σ-coalgebra: the unique homomorphism from a coalgebra is given in the same way as the final F_Σ-coalgebra in **Sets**.

Next we show that a function $L^\infty(\mathcal{X},\_) : X \nrightarrow \mathrm{Tree}_\infty(\Sigma)$ in Definition 5.20 is the largest homomorphism from c to $J\zeta$. As X is equipped with the discrete σ-algebra, $L^\infty(\mathcal{X},\_)$ is indeed an arrow in **Meas**.

Let $\mathbf{v}^{\max} \in [0,1]^X$ be the greatest fixed point of a function $P : [0,1]^X \to [0,1]^X$ that is defined as follows (much like in Lemma 5.22):

$$\big(P(\mathbf{v})\big)_x = \sum_{n=0}^{\infty} \sum_{x_0,\dots,x_{n-1}\in X} \left(\sum_{a\in\Sigma_n} c(x)\big(\{(a,x_0,\dots,x_{n-1}\}\big)\right) \cdot \prod_{i=0}^{n-1} \mathbf{v}_{x_i}\,. \tag{5.4}$$

Recall that $L^\infty(\mathcal{X},x)(\mathrm{Tree}_\infty(\Sigma))$ is defined by $L^\infty(\mathcal{X},x)(\mathrm{Tree}_\infty(\Sigma)) = 1 - \mathrm{Reach}(\Delta_\mathcal{X},x,\perp)$. Therefore by Lemma 5.22, we have $L^\infty(\mathcal{X},x)(\mathrm{Tree}_\infty(\Sigma)) = (\mathbf{v}^{\max})_x$.

We first show that $L^\infty(\mathcal{X},\_)$ is a homomorphism. By Carathéodory's extension theorem, it suffices to prove the following equation for all $k \in \omega$ and $t \in \mathrm{Tree}^k(\Sigma)$.

$$\big(J\zeta^{-1} \odot \overline{F_\Sigma} L^\infty(\mathcal{X},\_) \odot c\big)(x)\big(\mathrm{cyl}(t)\big) = L^\infty(\mathcal{X},x)(\mathrm{cyl}(t))$$

We prove this equation by induction on k.

If $k = 0$, then as $\mathrm{cyl}(t) = \mathrm{Tree}_\infty(\Sigma)$, we have:

$$\begin{aligned}
&\big(J\zeta^{-1} \odot \overline{F_\Sigma}(L^\infty(\mathcal{X},_)) \odot c\big)(x)\big(\mathrm{cyl}(t)\big)\\
&= \sum_{n=0}^{\infty} \sum_{x_0,\dots,x_{n-1}\in X} \sum_{a\in\Sigma_n} \left(c(x)\big(\{(a,x_0,\dots,x_{n-1}\}\big) \cdot \prod_{i=0}^{n-1} L^\infty(\mathcal{X},x_i)\big(\mathrm{Tree}_\infty(\Sigma)\big)\right)\\
&= \sum_{n=0}^{\infty} \sum_{x_0,\dots,x_{n-1}\in X} \sum_{a\in\Sigma_n} \left(c(x)\big(\{(a,x_0,\dots,x_{n-1}\}\big) \cdot \prod_{i=0}^{n-1} (\mathbf{v}^{\max})_{x_i}\right)\\
&= \big(P(\mathbf{v}^{\max})\big)_x && \text{(by definition of } P)\\
&= (\mathbf{v}^{\max})_x && \text{(as } \mathbf{v}_{\max} \text{ is a fixed point of } P),
\end{aligned}$$

on the one hand. On the other hand, we have shown that

$$L^\infty(\mathcal{X},x)\big(\mathrm{cyl}(t)\big) = (\mathbf{v}^{\max})_x\,.$$

Therefore we have $\big(J\zeta^{-1} \odot \overline{F_\Sigma} L^\infty(\mathcal{X},\_) \odot c\big)(x)(\mathrm{cyl}(t)) = L^\infty(\mathcal{X},x)(\mathrm{cyl}(t))$ for $t \in \mathrm{Tree}^0(\Sigma)$.

Let $k > 0$ and $t = (D,l) \in \mathrm{Tree}^{k+1}(\Sigma)$ where $l(\varepsilon) = a \in \Sigma_n$. Moreover, let t_i be the i-th subtree of t where $0 \le i \le n-1$. Then

$$\begin{aligned}
&\Big(J\zeta^{-1} \odot \overline{F_\Sigma}\big(L^\infty(\mathcal{X},_)\big) \odot c\Big)(x)\big(\mathrm{cyl}(t)\big)\\
&= \sum_{x_0,\dots,x_{n-1}\in X} c(x)\big(\{(a,x_0,\dots,x_{n-1})\}\big) \cdot \prod_{i=0}^{n-1} L^\infty(\mathcal{X},x_i)\big(\mathrm{cyl}(t_i)\big) && \text{(by definition of } \zeta)\\
&= L^\infty(\mathcal{X},x)(\mathrm{cyl}(t)) && \text{(by definition of } L^\infty(\mathcal{X},x))\,.
\end{aligned}$$

Therefore we have $\big(J\zeta^{-1} \odot \overline{F} L^\infty(\mathcal{X},\_) \odot c\big)(x)(\mathrm{cyl}(t)) = L^\infty(\mathcal{X},x)(\mathrm{cyl}(t))$ for $t \in \mathrm{Tree}^{k+1}(\Sigma)$.

Hence $L^\infty(\mathcal{X},\_)$ is a homomorphism from c to $J\zeta$.

It remains to show that $L^\infty(\mathcal{X},\_)$ is the largest homomorphism. Let $g : X \nrightarrow \mathrm{Tree}_\infty(\Sigma)$ be a homomorphism from c to $J\zeta$. By monotonicity of the extension of a measure, it suffices to prove $g(x)\big(\mathrm{cyl}(t)\big) \le L^\infty(\mathcal{X},x)\big(\mathrm{cyl}(t)\big)$ for all $x \in X$, $k \in \omega$ and $t \in \mathrm{Tree}^k(\Sigma)$. We prove this by induction on k.

If $k = 0$, then $\mathrm{cyl}(t) = \mathrm{Tree}_\infty(\Sigma)$. Hence we have:

$$\begin{aligned}
&g(x)\big(\mathrm{cyl}(t)\big)\\
&= \big(J\zeta^{-1} \odot \overline{F_\Sigma} g \odot c\big)(x)\big(\mathrm{Tree}_\infty(\Sigma)\big) && (g \text{ is a homomorphism})\\
&= \sum_{n=0}^{\infty} \sum_{x_0,\dots,x_{n-1}\in X} \left(\sum_{a\in\Sigma_n} c(x)\big(\{(a,x_0,\dots,x_{n-1})\}\big)\right) \cdot \prod_{i=0}^{n-1} g(x_i)\big(\mathrm{Tree}_\infty(\Sigma)\big)\,.
\end{aligned}$$

Here we define a vector $\mathbf{w} \in [0,1]^X$ by $\mathbf{w}_x = g(x)(\mathrm{Tree}_\infty(\Sigma))$ for each $x \in X$. The equation above implies that $\mathbf{w}$ is a fixed point of P defined in (5.4). As $\mathbf{v}^{\max}$ is the greatest fixed point of P, we have $g(x)(\mathrm{Tree}_\infty(\Sigma)) = \mathbf{w}_x \le (\mathbf{v}^{\max})_x = L^\infty(\mathcal{X},x)(\mathrm{Tree}_\infty(\Sigma))$.

Let $k > 0$ and assume that $g(x)(\mathrm{cyl}(s)) \le L^\infty(\mathcal{X},x)(\mathrm{cyl}(s))$ holds for all $x \in X$ and $s \in \mathrm{Tree}^{k-1}(\Sigma)$. Let $t = (D,l) \in \mathrm{Tree}^k(\Sigma)$ and $l(\varepsilon) = a \in \Sigma_n$. We write t_i for the i-th subtree of t. Then

$$\begin{aligned}
&g(x)(\mathrm{cyl}(t))\\
&= \big(J\zeta^{-1} \odot \overline{F_\Sigma} g \odot c\big)(x)(\mathrm{cyl}(t)) && (g \text{ is a homomorphism})\\
&= \sum_{x_0,\dots,x_{n-1}} \left(c(x)\big(\{(a,x_0,\dots,x_{n-1})\}\big) \cdot \prod_{i=0}^{n-1} g(x_i)(\mathrm{cyl}(t_i)) \right)\\
&\le \sum_{x_0,\dots,x_{n-1}} \left(c(x)\big(\{(a,x_0,\dots,x_{n-1})\}\big) \cdot \prod_{i=0}^{n-1} L^\infty(\mathcal{X},x_i)(\mathrm{cyl}(t_i)) \right)\\
& && (\text{by induction hypothesis})\\
&= \big(J\zeta^{-1} \odot \overline{F_\Sigma}(L^\infty(\mathcal{X},_) \odot c\big)(x)(\mathrm{cyl}(t)) && (\text{by definition of } \overline{F_\Sigma})\\
&= L^\infty(\mathcal{X},x)(\mathrm{cyl}(t)) && (L^\infty(\mathcal{X},_) \text{ is a homomorphism})\,.
\end{aligned}$$

Therefore we have $g(x)(\mathrm{cyl}(t)) \le L^\infty(\mathcal{X},x)(\mathrm{cyl}(t))$ for $t \in \mathrm{Tree}^k(\Sigma)$.

Hence $L^\infty(\mathcal{X},\_)$ is the largest homomorphism from c to $J\zeta$ and therefore $\mathsf{tr}^\infty(c) = L^\infty(\mathcal{X},\_)$. This immediately implies $\mathsf{tr}^\infty(c) \odot s(*) = L^\infty(\mathcal{X})$. □

5.5. **Another Modeling of Probabilistic Branching: Subdistribution Monad.** In the previous sections we used the sub-Giry monad $\mathcal{G}$ and a (standard Borel) polynomial functor F on **Meas** to model probabilistic systems. In this section, we discuss another pair—a polynomial functor F and the subdistribution monad $\mathcal{D}$ on **Sets**—that can also model probabilistic systems. For a given set $X \in$ **Sets**, $\mathcal{D}X$ is the set of (discrete) subdistributions over X.

Definition 5.23 (subdistribution monad)**.** A *subdistribution monad* is a monad $(\mathcal{D},\eta^{\mathcal{D}},\mu^{\mathcal{D}})$ on **Sets** such that

- $\mathcal{D}X = \{p : X \to [0,1] \mid \sum_{x\in X} p(x) \le 1\}$,
- $\mathcal{D}f(p)(y) = \sum_{x\in f^{-1}(y)} p(x)$,
- $\eta^{\mathcal{D}}_X(x)(y) = \begin{cases} 1 & (y = x)\\ 0 & (\text{otherwise}), \end{cases}$ and

- $\mu^{\mathcal{D}}_X(\Phi)(x) = \sum_{p\in\mathcal{D}X} \Phi(p)\cdot p(x)$.

Definition 5.24 (order enrichment of $\mathcal{K}\ell(\mathcal{D})$)**.** We define an order on $\mathcal{K}\ell(\mathcal{D})(X,Y)$ by $f \sqsubseteq g$ if and only if $\forall x \in X.\, \forall y \in Y.\, f(x)(y) \le g(x)(y)$.

Next, we show that F_Σ in Definition 2.11 and the subdistribution monad $\mathcal{D}$ constitute an infinitary trace situation by giving an explicit definition of the largest homomorphism.

Proposition 5.25. *Let Σ be a ranked alphabet and F_Σ be the functor on* **Sets** *defined in Definition 2.11. Then F_Σ and the subdistribution monad $\mathcal{D}$ constitute an infinitary trace situation.*

Proof. Let $\zeta : \mathrm{Tree}_\infty(\Sigma) \to F_\Sigma\left(\mathrm{Tree}_\infty(\Sigma)\right)$ be a final F_Σ-coalgebra in **Sets** that we defined in the proof of Proposition 4.18. For an $\overline{F_\Sigma}$-coalgebra $c : X \nrightarrow \overline{F_\Sigma}X$, we construct the largest homomorphism $h : X \nrightarrow \mathrm{Tree}_\infty(\Sigma)$ from c to $J\zeta$.

To this end, for $x \in X$, an integer $k \in \omega$ and a k-prefix tree $t \in \mathrm{Tree}^k(\Sigma)$, we first define a value $\xi_x(t^k) \in [0,1]$ by induction on k as follows:

- for $k = 0$, $\xi_x(t) = 1$, and
- for $k > 0$,

$$\xi_x(t) = \sum_{x_0,\dots,x_{n-1}\in X} \left(c(x)(a, x_0, \dots, x_{n-1}) \cdot \prod_{i=0}^{n-1} \xi_{x_i}(t_i)\right), \tag{5.5}$$

where $t = (D,l) \in \mathrm{Tree}^k(\Sigma)$, $a = l^k(\varepsilon) \in \Sigma_n$, and t_i is the i-th subtree of t.

For $t' \in \mathrm{Tree}_\infty(\Sigma)$ and $k \in \omega$, let $\mathrm{prefix}_k(t') = (\mathrm{prefix}_k(D'), \mathrm{prefix}_k(l'))$ be the unique k-prefix tree that is a prefix of t'. We define $h : X \nrightarrow \mathrm{Tree}_\infty(\Sigma)$ by $h(x)(t) = \lim_{k\to\infty} \xi_x(\mathrm{prefix}_k(t))$. As $\sum_{n\in\omega}\sum_{a\in\Sigma_n}\sum_{x_0,\dots,x_{n-1}\in X} c(x)(a, x_0, \dots, x_{n-1}) \le 1$, the sequence $\left(\xi_x(\mathrm{prefix}_k(t))\right)_{k\in\omega}$ is decreasing with respect to k. Therefore this h is well-defined.

We first show that this h is a homomorphism. For all $x \in X$, $n \in \omega$ and $t = (D,l) \in \mathrm{Tree}_\infty(\Sigma)$ such that $l(\varepsilon) = a \in \Sigma_n$ and i-th subtree of t is t_i, we have:

$$\begin{aligned}
&\left(J\zeta^{-1} \odot \overline{F_\Sigma}h \odot c\right)(x)(t) \\
&= \sum_{x_0,\dots,x_{n-1}\in X} c(x)(a, x_0, \dots, x_{n-1}) \cdot \prod_{i=0}^{n-1} h(x_i)(t_i) && \text{(by definition of } \zeta) \\
&= \sum_{x_0,\dots,x_{n-1}\in X} c(x)(a, x_0, \dots, x_{n-1}) \cdot \prod_{i=0}^{n-1} \lim_{k\to\infty} \xi_{x_i}(\mathrm{prefix}_k(t_i)) && \text{(by definition of } h) \\
&= \lim_{k\to\infty} \sum_{x_0,\dots,x_{n-1}\in X} c(x)(a, x_0, \dots, x_{n-1}) \cdot \prod_{i=0}^{n-1} \xi_{x_i}(\mathrm{prefix}_k(t_i)) \\
&= \lim_{k\to\infty} \xi_x(\mathrm{prefix}_k(t)) && \text{(by definition of } \xi) \\
&= h(x)(t) && \text{(by definition of } h)\,.
\end{aligned}$$

To conclude the proof, we show that h is the largest homomorphism. Let $g : X \nrightarrow \mathrm{Tree}_\infty(\Sigma)$ be a homomorphism from c to $J\zeta$. We prove $g(x)(t) \le h(x)(t)$ for all $x \in X$ and $t \in \mathrm{Tree}_\infty(\Sigma)$. To this end, we first prove $g(x)(t) \le \xi_x(\mathrm{prefix}_k(t))$ for all $k \in \omega$, $x \in X$ and $t \in \mathrm{Tree}_\infty(\Sigma)$ by induction on k.

If $k = 0$ then for all x and t, we have $g(x)(t) \le 1 = \xi_x(\mathrm{prefix}_k(t))$.

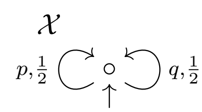
Let $k > 0$ and assume that $g(x)(t) \leq \xi_x(\mathrm{prefix}_{k-1}(t))$ for all x and t. Then

$$\begin{aligned}
&g(x)(t)\\
&= \bigl(J\zeta^{-1} \odot \overline{F_\Sigma} g \odot c\bigr)(x)(t) && (g \text{ is a homomorphism})\\
&= \sum_{x_0,\dots,x_{n-1}\in X} c(x)(a,x_0,\dots,x_{n-1}) \cdot \prod_{i=0}^{n-1} g(x_i)(t_i) && (\text{by definition of } \zeta)\\
&\leq \sum_{x_0,\dots,x_{n-1}\in X} c(x)(a,x_0,\dots,x_{n-1}) \cdot \prod_{i=0}^{n-1} \xi_{x_i}(\mathrm{prefix}_{k-1}(t_i)) && (\text{by induction hypothesis})\\
&= \xi_x(t) && (\text{by definition of } \xi)
\end{aligned}$$

Hence for all x and t, we have $g(x)(t) \leq \lim_{k\to\infty} \xi_x(\mathrm{prefix}_k(t)) = h(x)(t)$. □

In the proof above, we have constructed infinitary traces for $T = \mathcal{D}$ in concrete terms. It is rather different from the axiomatic proofs for $T = \mathcal{P}$ (Theorem 4.3) and $T = \mathcal{G}$ (Theorem 5.3): It is because infinitary traces for $T = \mathcal{D}$ does not follow from either of our general axiomatic results (Proposition 4.1 or Proposition 5.2).

It is easy to see that there exists X and Z in **Sets** such that $\mathcal{K}\ell(\mathcal{D})(X,Z)$ does not have $\top_{X,Z}$. Therefore we cannot construct the largest homomorphism by using Proposition 4.1. Neither can we use Proposition 5.2. In fact Assumption (5) fails. Indeed, let F be an endofunctor on **Sets** that is defined by $F(\_) = \{p,q\} \times (\_)$. Then the limit of the final ω^{op}-sequence $1 \overset{!_{F1}}{\leftarrow} F1 \overset{F!_{F1}}{\leftarrow} F^2 1 \overset{F^2!_{F1}}{\leftarrow} \cdots$ is given by $(Z, (\gamma_i : Z \to F^i 1)_{i\in\omega})$ where $Z = \{p,q\}^\omega$ and $\gamma_i(a_0 a_1 \dots) = a_0 a_1 \dots a_{i-1}$. Hence the carrier of the final F-coalgebra ζ is given by Z. We define $X \in \mathcal{K}\ell(\mathcal{D})$ and $c : X \rightarrow\!\!\!\!\!\!+\; \overline{F}X$ by $X = \{*\}$ and $c(*)(a,*) = \frac{1}{2}$ where $a \in \{p,q\}$. It is not so hard to see that the largest homomorphism $\mathsf{tr}^\infty(c) : X \rightarrow\!\!\!\!\!\!+\; Z$ from c to $J\zeta$ is given by $\mathsf{tr}^\infty(c)(x)(w) = 0$ for each $w \in Z = \{p,q\}^\omega$. However, we cannot obtain this $\mathsf{tr}^\infty(c)$ with the procedure in the proof of Proposition 5.2.



For each $i \in \omega$, we inductively define $\alpha_i : X \rightarrow\!\!\!\!\!\!+\; \overline{F}^i 1$ by $\alpha_0 = J!_X$ and $\alpha_{i+1} = \overline{F}\alpha_i \odot c$. It is easy to see that $(X, (\alpha_i)_{i\in\omega})$ is a cone over a sequence $1 \overset{J!_{F1}}{\leftarrow\!\!\!\!\!\!+} \overline{F}1 \overset{JF!_{F1}}{\leftarrow\!\!\!\!\!\!+} \overline{F}^2 1 \overset{JF^2!_{F1}}{\leftarrow\!\!\!\!\!\!+} \cdots$. However, it is also easy to see that there does not exist $f : X \rightarrow\!\!\!\!\!\!+\; Z$ such that $J\gamma_i \odot f = \alpha_i$.

As a consequence, we can construct the largest homomorphism from c to $J\zeta$ neither by using the construction in Proposition 4.1 nor Proposition 5.2. This prevents us from applying the general theories for Kleisli simulations in Sections 4–5.

The resulting infinitary trace semantics in Proposition 5.25 has limited use, however, due to the discrete nature of an arrow $X \to \mathcal{D}(\mathrm{Tree}_\infty(\Sigma))$. That is, it assigns a probability to each single tree, and the probability is most of the time 0 (see Example 1.1).

6. Systems with Exception

In this section, we focus on systems that possibly abort with exception. They are modeled as $(\mathcal{L}, F)$-systems in the category **Sets**, where $\mathcal{L}$ is from Definition 2.3. Here the categorical/axiomatic results in Section 5 are applicable. Therefore, much like for $\mathcal{G}$, forward or total backward simulations (see Section 5.2) witness trace inclusion. In this section, we assume that F is a polynomial functor on **Sets**.

6.1. **Construction of Infinitary Traces.** In this section, we prove the same results as Sections 4.1 and 5.1 for $T = \mathcal{L}$. To this end, we rely on Proposition 5.2 (but not Proposition 4.1, since $\mathcal{L}X$ does not have the greatest element).

Theorem 6.1. *The combination of polynomial F and $T = \mathcal{L}$ constitute an infinitary trace situation.*

Proof. We show that A polynomial functor F on **Sets** and a lift monad $\mathcal{L}$ satisfy Assumptions (1)–(5) in Proposition 5.2 with respect to the order in Definition 2.6.

It is easy to see that F and $\mathcal{L}$ on **Sets** satisfy the Assumptions (1) and (4).

It is known that Assumption (2) is satisfied [HJS07, Lemma 2.4].

To prove that Assumption (3) is satisfied, it suffices to show that for all $x \in X$, $\prod_{i\in\omega}(g_i \odot b)(x) = \bot$ if and only if $(\prod_{i\in\omega} g_i) \odot b(x) = \bot$. If $b(x) = \bot$, then we have $\prod_{i\in\omega}(g_i \odot b)(x) = (\prod_{i\in\omega} g_i) \odot b(x) = \bot$. If $b(x) \neq \bot$, we have:

$$\prod_{i\in\omega}(g_i \odot b)(x) = \bot \Leftrightarrow \exists i \in \omega.\, g_i(b(x)) = \bot \Leftrightarrow \prod_{i\in\omega} g_i(b(x)) = \bot \Leftrightarrow (\prod_{i\in\omega} g_i \odot b)(x) = \bot\,.$$

Hence Assumption (3) is satisfied in both cases.

As a connected limit and a coproduct commute in **Sets** [ABLR02], the Kleisli inclusion functor $J : \mathbf{Sets} \to \mathcal{K}\ell(T)$ preserves ω^{op}-limit. It is easy to see that this limit is a 2-limit. Therefore Assumption (5) is satisfied. □

6.2. **Kleisli Simulation for Systems with Exception.** It is known that a polynomial F and $\mathcal{L}$ satisfy the assumptions of Lemma 4.5 [Has06]. Hence we can use forward Kleisli simulation to check infinitary trace inclusion between tree automata with exception.

For an $(\mathcal{L}, F_\Sigma)$-system, as we have seen in Theorem 6.1, the largest homomorphism can be constructed using Proposition 5.2. Therefore from Lemma 5.9, we can use total backward simulation (Definition 5.8) to check infinitary trace inclusion between $(\mathcal{L}, F_\Sigma)$-systems. For $(\mathcal{L}, F_\Sigma)$-systems the totality means the following.

Proposition 6.2. *In Definition 5.8, if $T = \mathcal{L}$ and F is a polynomial functor, then Assumption (1) is satisfied if $b(x) \neq \bot$ for all $x \in X$.*

Proof. Let $*$ be the unique element of the final object 1. By the assumption we have $b(x) \neq \bot$. Therefore we have $* = !_Y(b(x)) = J!_Y \odot b(x)$ for all $x \in X$. This concludes the proof. □

6.3. **Forward Partial Execution for Systems with Exception.** From Theorem 4.15, soundness and adequacy of FPE for forward simulation hold for $(\mathcal{L}, F)$-systems.

By the construction of the largest homomorphism, soundness and adequacy of FPE for backward simulation hold if the simulating automaton satisfies the assumptions in Theorem 5.13(2). It is easy to see that the assumptions can be described as follows.

Proposition 6.3. *If $T = \mathcal{L}$ and F is a polynomial functor, the assumption in Theorem 5.13(2) is satisfied if $d(y) \neq \bot$ for each $y \in Y$.* □

6.4. **Coincidence between Automata-theoretic and Coalgebraic Infinitary Trace Semantics.** The same results as in Sections 4.4 and 5.4 can be shown also for $T = \mathcal{L}$.

Automata-theoretically, an $(\mathcal{L}, F_\Sigma)$-system $\mathcal{X} = (X, s, c)$ can be regarded almost as a deterministic automaton that outputs infinite trees, except that at each stage of its behavior, the transition function $c : X \to \mathcal{L}F_\Sigma X$ can output $\bot$ and abort. In that aborting case, the output of $\mathcal{X}$ is undefined; this automata-theoretic characterization of $\mathcal{X}$ naturally induces a function $L^\infty(\mathcal{X}, \_) : X \to \{\bot\} + \mathrm{Tree}_\infty(\Sigma)$. It is straightforward to see that this function coincides with the largest homomorphism $\mathsf{tr}^\infty(c) : X \to \mathcal{L}\mathrm{Tree}_\infty(\Sigma)$ from c to $J\zeta$, which defines coalgebraic infinitary trace semantics.

7. Related Work

In this paper, for a coalgebraic modeling of infinitary traces, we followed [Jac04] where a *Kleisli* category is used. In [JSS12], an approach towards a coalgebraic characterization of *finite* traces via an *Eilenberg-Moore* category is introduced. We used "Kleisli approach" because of its potential for an extension to Büchi or parity automata. The extension seems difficult for Eilenberg-Moore approach because the approach is based on (generalized) *determinization* of systems, and it is well-known that determinization of Büchi automata strictly decreases its expressive power.

The construction of the largest homomorphism given in Proposition 5.2 is based on the one in [Cîr10]. The latter imposes some technical conditions on a monad T, including a "totality" condition that excludes $T = \mathcal{P}$ from its instances (while the nonempty powerset monad is an instance). Our assumption of lifting to a 2-limit (Assumption (5) in Proposition 5.2) is inspired by a condition in [Cîr10], namely that the limit Z is lifted to a *weak* limit in $\mathcal{K}\ell(T)$. It is not the case that Proposition 5.2 subsumes the construction in [Cîr10]: the former does not apply to the nonempty powerset monad (but our Proposition 4.1 does apply to it).

In [KK13], an explicit description of a (proper, not weakly) final $\overline{F}$-coalgebra is given for $F \in \{\, \Sigma \times (\_),\ 1 + \Sigma \times (\_)\, \}$ and $T \in \{\mathcal{G}, \mathcal{G}_{=1}\}$. Here $\mathcal{G}_{=1}$ is the *Giry monad* and restricts $\mathcal{G}$ to proper, not sub-, distributions. We do not use their (proper finality) results for modeling of infinitary traces, because: 1) if $T = \mathcal{G}$ then the final coalgebras do not coincide with the set of infinitary words; and 2) if $T = \mathcal{G}_{=1}$ then language inclusion is reduced to the equality. We are skeptical about the value of developing simulation-based methods for the latter degenerate case, one reason being that trace equivalence is often much easier than trace inclusion. For example, finite trace inclusion for probabilistic systems is undecidable [BC03] while trace equivalence is decidable [KMO$^+$11].

In [Sch09], it is shown that: a limit of an ω^{op}-sequence consisting of standard Borel spaces and surjective measurable functions is preserved by a polynomial functor F (where constants are restricted to standard Borel spaces), and also by $\mathcal{G}$. It is also shown there that such a polynomial functor F preserves standard Borel spaces, and so does $\mathcal{G}$. These facts imply the existence of a final $\mathcal{G}F$-coalgebra in **Meas** for every polynomial functor F. Note however that this final $\mathcal{G}F$-coalgebra captures (probabilistic) bisimilarity, not trace semantics.

8. Conclusions and Future Work

We have shown that the technique forward and backward Kleisli simulations [Has06] and that of FPE [UH14, UH17]—techniques originally developed for witnessing *finite* trace inclusion—are also applicable to *infinitary* trace semantics. We followed [Jac04] (and also [Cîr10, KK13]) to characterize infinitary trace semantics in coalgebraic terms, on which we established properties of Kleisli simulations such as soundness. We developed our theory for three classes of instances: nondeterministic systems, probabilistic ones and ones with exception. These three turn out to result from two categorical principles that are rather different (see Remark 5.1).

There are some directions for future work. In [UH14] (and its extended version [UH17]), in addition to FPE, a transformation called *backward partial execution* (BPE) is introduced. Similarly to FPE, BPE can also aid forward and backward Kleisli simulation for *finite* traces in the sense that it satisfy soundness and adequacy. However, BPE is only defined for word automata (with T-branching) and not generally for (T, F)-systems. Defining BPE categorically and proving its soundness and adequacy with respect to infinitary traces, possibly restricting to word automata, is one of the future work.

In this paper, we used Kleisli simulation to compare simple automata where an infinite-depth tree is accepted if it only has an infinite path on the automata. More complex automata for infinite-length words have been introduced, such as Büchi automata and parity automata. Extending the notion of Kleisli simulation so that such automata with complex accepted conditions can be compared is one of the directions of future work.

Another direction is implementation and experiments. As forward and backward Kleisli simulations in this paper are defined in almost the same way as [UH14, UH17], we can use the implementation already developed there.

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