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INFINITE AND BI-INFINITE WORDS WITH DECIDABLE MONADIC THEORIES

DIETRICH KUSKE, JIAMOU LIU, AND ANASTASIA MOSKVINA

Institut für Theoretische Informatik, TU Ilmenau, Germany
e-mail address: dietrich.kuske@tu-ilmenau.de

The University of Auckland, New Zealand
e-mail address: jiamou.liu@auckland.ac.nz

Auckland University of Technology, New Zealand
e-mail address: anastasia.moskvina@aut.ac.nz

Abstract. We study word structures of the form $(D, <, P)$ where D is either $\mathbb{N}$ or $\mathbb{Z}$, $<$ is the natural linear ordering on D and $P \subseteq D$ is a predicate on D. In particular we show:

(a) The set of recursive ω-words with decidable monadic second order theories is Σ_3-complete.
(b) Known characterisations of the ω-words with decidable monadic second order theories are transfered to the corresponding question for bi-infinite words.
(c) We show that such "tame" predicates P exist in every Turing degree.
(d) We determine, for $P \subseteq \mathbb{Z}$, the number of predicates $Q \subseteq \mathbb{Z}$ such that $(\mathbb{Z}, \leq, P)$ and $(\mathbb{Z}, \leq, Q)$ are indistinguishable by monadic second order formulas.

Through these results we demonstrate similarities and differences between logical properties of infinite and bi-infinite words.

1. Introduction

The decision problem for logical theories of linear structures and their expansions has been an important question in theoretical computer science. Büchi in [4] proved that the monadic second order theory (henceforth "MSO-theory") of the linear ordering $(\mathbb{N}, \leq)$ is decidable. Expanding the structure $(\mathbb{N}, \leq)$ by unary functions or binary relations typically leads to undecidable MSO-theories. Hence numerous works have been focusing on structures of the form $(\mathbb{N}, \leq, P)$ where P is a unary predicate. Elgot and Rabin [7] showed that for many natural unary predicates P, such as the set of factorial numbers, the set of powers of k, and the set of kth powers (for fixed k), the structure $(\mathbb{N}, \leq, P)$ has decidable MSO-theory; on the other hand, there are structures $(\mathbb{N}, \leq, P)$ whose MSO-theory is undecidable [5]. Many subsequent works further expanded the field [16, 6, 13, 14, 11, 10].

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(1) Semenov observed that, in order to decide the MSO-theory of an ω-word $\alpha = (\mathbb{N}, \leq, P)$, it suffices to determine whether a given regular language has a factor in α beyond every position and, if not, determine some position beyond which α has no factor from that language. This idea is formalised by the notion of an "indicator of recurrence". In [13], he provided a full characterisation: $(\mathbb{N}, \leq, P)$ has decidable MSO-theory if and only if P is recursive and there is a recursive indicator of recurrence for P.

(2) Rabinovich and Thomas observed that, in order to decide the MSO-theory of an ω-word $\alpha = (\mathbb{N}, \leq, P)$, it suffices to have some computable factorisation of α such that late factors cannot be distinguished by formulas of bounded quantifier depth. This idea is formalised by the notion of a "uniformly homogeneous set". In [11], Rabinovich and Thomas provided a full characterisation: $(\mathbb{N}, \leq, P)$ has a decidable MSO-theory if and only if P is recursive and there is a recursive uniformly homogeneous set.

    They also observed that it suffices to be able to compute, from a quantifier depth k, a pair of k-types (u, v) such that α can be factorised in such a way that all factors (except the first factor that has type u) have type v. In [11], they also showed that the MSO-theory of $(\mathbb{N}, \leq, P)$ is decidable if and only if there is a "recursive type-function".

This paper has three general goals: The first is to compare these characterisations in some precise sense. The second is to investigate the above results in the context of *bi-infinite words*, which are structures of the form $(\mathbb{Z}, \leq, P)$. The third is to compare the logical properties of infinite words and bi-infinite words. More specifically, the paper discusses the following questions:

(a) In Section 4, we analyze the recursion-theoretical bound of the set of all computable predicates $P \subseteq \mathbb{N}$ where $(\mathbb{N}, \leq, P)$ has a decidable MSO-theory. It is noted that the second characterisation by Rabinovich and Thomas turns out to be a Σ_5-statement. Differently the characterisation by Semenov and the first characterisation by Rabinovich and Thomas both consist of Σ_3-statements, and hence deciding if a given $(\mathbb{N}, \leq, P)$ has decidable MSO-theory is in Σ_3. We show that the problem is in fact Σ_3-complete. Hence these two characterisations are optimal in terms of their recursion-theoretical complexity.

(b) If the MSO-theory of $(\mathbb{N}, \leq, P)$ is decidable, then P is recursive. For bi-infinite words of the form $(\mathbb{Z}, \leq, P)$, this turns out not to be necessary. To the contrary, Theorem 5.4 demonstrates that every m-degree contains some bi-infinite word with a decidable theory. Even more, Corollary 5.9 shows that every decidable MSO-theory of a recurrent bi-infinite word is realised in every Turing degree.

(c) In the rest of Section 5, we then investigate which of the three characterisations can be lifted to bi-infinite words, i.e., structures of the form $(\mathbb{Z}, \leq, P)$ with $P \subseteq \mathbb{Z}$. It turns out that this is nicely possible for Semenov's characterisation and for the second characterisation by Rabinovich and Thomas, but not for their first one.

(d) The final Section 6 investigates how many bi-infinite words are indistinguishable from $(\mathbb{Z}, \leq, P)$. It turns out that this depends on the periodicity properties of P: if P is periodic, there are only finitely many equivalent bi-infinite words, if P is recurrent and non-periodic, there are $2^{\aleph_0}$ many, and if P is not recurrent, then there are $\aleph_0$ many.

2. Preliminaries

2.1. **Words.** We use $\mathbb{N}$, $\widetilde{\mathbb{N}}$ and $\mathbb{Z}$ to denote the set of natural numbers (including 0), negative integers (not containing 0), and integers, respectively.

A *finite word* is a mapping $u\colon \{0,1,\dots,n-1\} \to \{0,1\}$ with $n \in \mathbb{N}$, it is usually written $u(0)u(1)u(2)\cdots u(n-1)$. The set of positions of u is $\{0,1,\dots,n-1\}$, its length $|u|$ is n. The unique finite word of length 0 (and therefore with empty set of positions) is denoted ε. The set of all finite words is $\{0,1\}^*$, the set of all non-empty finite words is $\{0,1\}^+$.

An ω-*word* is a mapping from $\mathbb{N}$ into $\{0,1\}$. Often, an ω-word α is written as the sequence $\alpha(0)\alpha(1)\alpha(2)\cdots$. Its set of positions is $\mathbb{N}$; $\{0,1\}^\omega$ is the set of ω-words. An ω^*-*word* is a mapping α from $\widetilde{\mathbb{N}}$ into $\{0,1\}$, it is usually written as the sequence $\cdots\alpha(-3)\alpha(-2)\alpha(-1)$. Its set of positions is $\widetilde{\mathbb{N}}$; $\{0,1\}^{\omega^*}$ is the set of ω^*-words.

Finally, a *bi-infinite word* ξ is a mapping from $\mathbb{Z}$ into $\{0,1\}$, written as the sequence

$$\cdots\xi(-2)\xi(-1)\xi(0)\xi(1)\xi(2)\cdots$$

(this notation has to be taken with care since, *e.g.*, the bi-infinite words $\xi_i\colon \mathbb{Z} \to \{0,1\}\colon n \mapsto (|n|+i) \bmod 2$ with $i \in \{0,1\}$ are both described as $\cdots 0101010\cdots$, but they are different). The set of positions of a bi-infinite word is $\mathbb{Z}$.

Shift-equivalence and period will be important notions in this context: two bi-infinite words ξ and ζ are *shift-equivalent* if there is $p \in \mathbb{N}$ with $\xi(n) = \zeta(n+p)$ for all $n \in \mathbb{Z}$. Furthermore, the period of the bi-infinite word ξ is the least natural number $p > 0$ with $\xi(n) = \xi(n+p)$ for all $n \in \mathbb{N}$ – clearly, the period needs not exist: $0^{\omega^*}1^\omega$ is not periodic.

When saying "word", we mean "a finite, an ω-, an ω^*- or a bi-infinite word", "infinite word" means "ω- or ω^*-word".

For two finite words u and v, the concatenation uv is again a finite word of length $|u|+|v|$ with $uv(i) = u(i)$ if $0 \le i < |u|$ and $uv(i) = v(i-|u|)$ for $|u| \le i < |uv|$. More generally, and in a similar way, we can also concatenate a finite or ω^*-word u and a finite or ω-word v giving rise to some word uv. Similarly, we can concatenate infinitely many finite nonempty words u_i giving an ω-word $u_0u_1u_2\cdots$, an ω^*-word $\cdots u_{-2}u_{-1}u_0$, and a bi-infinite word $\cdots u_{-2}u_{-1}u_0u_1u_2\cdots$ (where the position 0 is the first position of u_0). As usual, u^ω denotes the ω-word $uuuu\cdots$ for $u \in \{0,1\}^+$, analogously, $u^{\omega^*} = \cdots uuu$.

Let w be some word and i,j be two positions with $i \le j$. Then we write $w[i,j]$ for the finite word $w(i)w(i+1)\cdots w(j) \in \{0,1\}^+$. A finite word u is a *factor* of w if there are $i,j \in D$ with $i \le j$ and $w[i,j] = u$ or if u is the empty word ε. The set of factors of w is $F(w)$. If β is an ω- or a bi-infinite word and i is a position in w, then $\beta[i,\infty)$ is the ω-word $\beta(i)\beta(i+1)\beta(i+2)\cdots$. Symmetrically, if β is an ω^*- or a bi-infinite word and i is a position in β, then $\beta(-\infty,i]$ is the ω^*-word $\cdots\beta(i-2)\beta(i-1)\beta(i)$.

Let u be some finite word. Then u^R is the *reversal* of u, i.e., the finite word of length $|u|$ with $u^R(i) = u(|u|-i-1)$ for all $0 \le i < |u|$. The reversal of an ω-word α is the ω^*-word α^R with $\alpha^R(i) = \alpha(-i-1)$ for all $i \in \widetilde{\mathbb{N}}$. The reversal of an ω^*-word α is the ω-word α^R with $\alpha^R(i) = \alpha(-i-1)$ for all $i \in \mathbb{N}$. Finally, the reversal of a bi-infinite word ξ is the bi-infinite word ξ^R with $\xi^R(i) = \xi(-i)$ for all $i \in \mathbb{Z}$.

An ω-word β is *recurrent* if any factor of β appears infinitely often in β, i.e., $F(\beta) = F(\beta[i,\infty))$ for all $i \in \mathbb{N}$. A bi-infinite word ξ is *recurrent* if $F(\xi) = F(\xi(-\infty,i]) = F(\xi[i,\infty))$ for all $i \in \mathbb{Z}$. Note that $0^{\omega^*}1^\omega$ is not recurrent despite the fact that it is build from two recurrent ω-words. Semenov [13] calls recurrent bi-infinite words *homogeneous*.

2.2. **Logic.** With any word w, we associate a relational structure $M_w = (D, \leq, P)$ where D is the set of positions of w (i.e., a subset of $\mathbb{Z}$), $\leq$ is the restriction of the natural linear order on $\mathbb{Z}$ to D, and $P = \{n \in D \mid w(n) = 1\} = w^{-1}(1)$. Relational structures of this form are called labeled linear orders.

We use the standard logical system over the signature of labeled linear orders. Hence first order logic FO has relational symbols $\leq$ and P. Monadic second order logic MSO extends FO by allowing unary second order variables $X, Y, \ldots$, their corresponding atomic predicates (*e.g.* $X(y)$), and quantification over set variables. By Sent, we denote the set of sentences of the logic MSO.

For a word w and an MSO-sentence φ, we write $w \models \varphi$ for "the sentence φ holds in the relational structure M_w". The MSO-*theory* of the word w is the set $\mathsf{MTh}(w)$ of all MSO-sentences φ that are true in w.

Example 2.1. We use $\mathrm{succ}(x, z)$ as shorthand for the formula $x < z \wedge \neg\exists y \colon x < y < z$. Let $n \in \mathbb{N}$ and $\varphi(x)$ be a formula with a free variable x. Then the formula

$$\exists x_0, x_1, \ldots, x_n \colon \bigwedge_{0 \leq i < n} \mathrm{succ}(x_i, x_{i+1}) \wedge x_n = x \wedge \varphi(x_0)$$

expresses that φ holds for $x - n$. We will abbreviate this as $\varphi(x - n)$.

Let $n \in \mathbb{N}$ and consider the following formula:

$$\varphi(x, y) = \exists X \colon \forall z \colon (X(z) \Leftrightarrow z = x \vee (x < z \wedge X(z - n))) \wedge X(y)$$

If w is some word with two positions i and j, then $w \models \varphi(i, j)$ if and only if $i \leq j$ and $n \mid j - i$.

With any MSO-formula φ, we associate its *quantifier rank* $\mathsf{qr}(\varphi) \in \mathbb{N}$: the atomic formulas have quantifier rank 0; $\mathsf{qr}(\varphi_1 \wedge \varphi_2) = \mathsf{qr}(\varphi_1 \vee \varphi_2) = \max\{\mathsf{qr}(\varphi_1), \mathsf{qr}(\varphi_2)\}$; $\mathsf{qr}(\neg\varphi) = \mathsf{qr}(\varphi)$; and $\mathsf{qr}(\exists X \colon \varphi) = \mathsf{qr}(\forall X \colon \varphi) = \mathsf{qr}(\varphi) + 1$ where X is a first- or second-order variable.

Definition 2.2. Let $k \in \mathbb{N}$. Two words w_1 and w_2 are *k-equivalent* (denoted $w_1 \equiv_k w_2$) if $w_1 \models \varphi$ if and only if $w_2 \models \varphi$ for all MSO-sentences φ with $\mathsf{qr}(\varphi) \leq k$. Equivalence classes of this equivalence relation are called *k-types*.

The words w_1 and w_2 are MSO-*equivalent* (denoted $w_1 \equiv w_2$) if $w_1 \equiv_k w_2$ for all $k \in \mathbb{N}$. Equivalence classes of this equivalence relation are called *types*.

Let $k \geq 2$ and u, v be two words with $u \equiv_k v$. If u is finite, then it satisfies the sentence $(\exists x \forall y \colon x \leq y) \wedge (\exists x \forall y \colon x \geq y)$. Consequently, also v is finite. Analogously, u is an ω-word iff v is an ω-word etc. We will therefore speak of a "k-type of finite words" when we mean a k-type that contains some finite word (and analogously for ω-words etc).

Often, we will use the following known results without mentioning them again. They follow from the well-understood relation between monadic second-order logic and automata (*cf.* [18, 8]).

Theorem 2.3.

(1) *Let $k \geq 2$.*
- *For any ω-word α, there exist finite words x and y with $xy \equiv_k x$, $yy \equiv_k y$ and $\alpha \equiv_k xy^\omega$. Any such pair (x, y) is* a representative of the k-type of α.
- *For any ω^*-word α, there exist finite words x and y with $xy \equiv_k y$, $xx \equiv_k x$ and $\alpha \equiv_k x^{\omega^*} y$. Any such pair (x, y) is* a representative of the k-type of α.

- *For any bi-infinite word ξ, there exist finite words x, y and z with $xy \equiv_k yz \equiv_k y$, $xx \equiv_k x$, $zz \equiv_k z$, and $\xi \equiv_k x^{\omega^*} yz^{\omega}$. Any such triple (x, y, z) is* a representative of the k-type of ξ.

(2) *The following sets are decidable:*
- $\{\varphi \in \mathsf{Sent} \mid \forall u \in \{0,1\}^* \colon u \models \varphi\}$
- $\{(u, \varphi) \mid u \in \{0,1\}^*, \varphi \in \mathsf{Sent}, u \models \varphi\}$
- $\{(u, v, \varphi) \mid u, v \in \{0,1\}^*, v \neq \varepsilon, \varphi \in \mathsf{Sent}, uv^{\omega} \models \varphi\}$
- $\{(u, v, w, \varphi) \mid u, v, w \in \{0,1\}^*, u, w \neq \varepsilon, \varphi \in \mathsf{Sent}, u^{\omega^*} vw^{\omega} \models \varphi\}$
- $\{(u, v, k) \mid u, v \in \{0,1\}^*, k \in \mathbb{N}, u \equiv_k v\}$. *This means that it is decidable whether u and v represent the same k-type of finite words.*
- *Similarly, it is decidable whether two pairs of finite words represent the same k-type of ω-words (of ω^*-words, resp). It is also decidable whether two triples of finite words represent the same k-type of bi-infinite words.*

(3) $\equiv_k$ *is a congruence for concatenation: If $u, v \in \{0,1\}^* \cup \{0,1\}^{\omega^*}$ and $u', v' \in \{0,1\}^* \cup \{0,1\}^{\omega}$ with $u \equiv_k v$ and $u' \equiv_k v'$, then $uu' \equiv_k vv'$. From representatives of the k-types of u and v, one can compute a representative of the k-type of uv.*

(4) $\equiv_k$ *is even a congruence for infinite concatenations: If $u_i, v_i \in \{0,1\}^+$ with $u_i \equiv_k v_i$ for all $i \in \mathbb{Z}$, then the following hold:*

$$u_0 u_1 \cdots \equiv_k v_0 v_1 \cdots$$
$$\cdots u_{-1} u_0 \equiv_k \cdots v_{-1} v_0$$
$$\cdots u_{-1} u_0 u_1 \cdots \equiv_k \cdots v_{-1} v_0 v_1 \cdots$$

(5) *If $u \in \{0,1\}^* \cup \{0,1\}^{\omega^*}$ and $v \in \{0,1\}^* \cup \{0,1\}^{\omega}$ such that $\mathsf{MTh}(u)$ and $\mathsf{MTh}(v)$ are both decidable, then $\mathsf{MTh}(uv)$ is decidable* [15].

2.3. **Recursion theoretic notions.** This paper makes use of standard notions in recursion theory; the reader is referred to [12, 17] for a thorough introduction. We assume a canonical effective enumeration $\Phi_0, \Phi_1, \Phi_2, \ldots$ of all partial recursive functions on the natural numbers. The set W_e is the domain $\mathsf{dom}(\Phi_e)$ and is the e^{th} *recursively enumerable set.* Let $\mathsf{TOT} \subseteq \mathbb{N}$ be the set of natural numbers e such that Φ_e is total, i.e., $W_e = \mathbb{N}$. Furthermore, let $\mathsf{REC} \subseteq \mathbb{N}$ be the set of natural numbers e such that W_e is decidable. We will also use the notion of many-one reductions (or m-reductions) and of Turing-reductions: $A \leq_m B$ denotes the existence of an m-reduction of A to B, $A \leq_T B$ that of a Turing-reduction. These relations are transitive and reflexive, the induced equivalence relations are denoted $\equiv_m$ and $\equiv_T$, respectively. An equivalence class of $\equiv_m$ is an *m-degree*, an equivalence class of $\equiv_T$ is a *Turing-degree*. Recall that the class of Turing-degrees, ordered by $\leq_T/\equiv_T$ is an upper semilattice. For two sets of integers A and B, the supremum of their Turing-degrees is the Turing-degree of $A \oplus B = \{2a \mid a \in A\} \cup \{2b+1 \mid b \in B\}$.

We will, when appropriate, understand an ω-word α as the set $\alpha^{-1}(1)$. Then, it makes sense to say "α is recursive" meaning "$\alpha^{-1}(1)$ is recursive" or to speak about the degree of α. Similarly, an ω^*-word α is identified with the set $-\alpha^{-1}(1)$ and a bi-infinite word ξ with the set $\{2i \mid i \geq 0, \xi(i) = 1\} \cup \{2i+1 \mid i > 0, \xi(-i) = 1\}$. We also assume an effective enumeration of all finite words, so that sets of finite words can be understood as subsets of $\mathbb{N}$, hence the notions "degree of a set of finite words" and $\xi \oplus F$ with $\xi \in \{0,1\}^{\mathbb{Z}}$ and $F \subseteq \{0,1\}^*$ make sense.

A set $A \subseteq \mathbb{N}$ belongs to Π_2 (the second universal level of the arithmetical hierarchy) if there exists a decidable set $P \subseteq \mathbb{N} \times \mathbb{N}^m \times \mathbb{N}^n$ such that A is the set of natural numbers a satisfying

$$\forall x_1, \dots, x_m \exists y_1, \dots y_n \colon P(a, \bar{x}, \bar{y}) \, .$$

A set $B \subseteq \mathbb{N}$ is Π_2-*hard* if, for every $A \in \Pi_2$, there exists an m-reduction from A to B; the set B is Π_2-*complete* if, in addition, $B \in \Pi_2$.

Similarly, $A \subseteq \mathbb{N}$ belongs to Σ_3 (the third existential level of the arithmetical hierarchy) if there exists a decidable set $P \subseteq \mathbb{N} \times \mathbb{N}^\ell \times \mathbb{N}^m \times \mathbb{N}^n$ such that A is the set of natural numbers a satisfying

$$\exists x_1, \dots, x_\ell \forall y_1, \dots, y_m \exists z_1, \dots z_n \colon P(a, \bar{x}, \bar{y}, \bar{z}) \, .$$

The notions Σ_3-hard and Σ_3-complete are defined similarly to the corresponding notions for Π_2. For our purposes, it is important that the set TOT is Π_2-complete and the set REC is Σ_3-complete [17].

3. When is the MSO-theory of an ω-word decidable?

In this section, we recall the answers by Semenov [13] and by Rabinovich and Thomas [11] and we present a slight strengthening of Semenov's answer.

3.1. Semenov's characterization.

The first characterisation is provided by Semenov in [13]. He observed that, in order to decide the MSO-theory of some ω-word α, it is necessary and sufficient to determine whether words from a given regular set recur in α (and, if not, from what point on no factor of α belongs to the regular set). This led to the definition of an "indicator of recurrence":

Definition 3.1. Let α be some ω-word. An *indicator of recurrence* for α is a function $\mathrm{rec} \colon \mathsf{Sent} \to \mathbb{N} \cup \{\top\}$ such that, for every MSO-sentence φ, the following hold:

- if $\mathrm{rec}(\varphi) = \top$, then $\forall k \, \exists j \geq i \geq k \colon \alpha[i,j] \models \varphi$
- if $\mathrm{rec}(\varphi) \neq \top$, then $\forall j \geq i \geq \mathrm{rec}(\varphi) \colon \alpha[i,j] \models \neg\varphi$

Formally, Semenov's formulation from [13] uses the class of regular languages instead of Sent. He actually means any effective representation, *e.g.*, as regular expressions or finite automata. Here, we use the effective representation by MSO-sentences as logic is the main focus of this paper.

Theorem 3.2 (Semenov's Characterisation [13])**.** *Let α be an ω-word. Then $\mathsf{MTh}(\alpha)$ is decidable if and only if there is a recursive indicator of recurrence for α and the ω-word α is recursive.*

Note that an ω-word can have many recursive indicators of recurrence: if rec is such an indicator, then also $\varphi \mapsto 2 \cdot \mathrm{rec}(\varphi)$ (with $\top = 2 \cdot \top$) is an indicator of recurrence. Differently, there is only one weak indicator of recurrence of any ω-word α:

Definition 3.3. Let α be some ω-word. The *weak indicator of recurrence* for α is the function $\mathrm{rec}'\colon \mathsf{Sent} \to \{0,1,\top\}$ defined as follows:

$$\mathrm{rec}'(\varphi) = \begin{cases} 0 & \text{no factor of } \alpha \text{ satisfies } \varphi \\ \top & \text{there are infinitely many } i \in \mathbb{N} \\ & \qquad \text{such that there exists } j \geq i \text{ with } \alpha[i,j] \models \varphi \\ 1 & \text{otherwise} \end{cases}$$

Note that $\mathrm{rec}'(\varphi) = 1$ does not imply that there are only finitely many factors of α satisfying φ: let $\alpha = 10^\omega$ and $\varphi = \exists x\colon P(x)$. Then φ is satisfied by all factors of the form $\alpha[0,j]$, but by no factor $w[i,j]$ with $i > 0$. Hence $\mathrm{rec}'(\varphi) = 1$ in this case.

Corollary 3.4. *Let α be an ω-word. Then $\mathsf{MTh}(\alpha)$ is decidable if and only if the weak indicator of recurrence of α is recursive and the ω-word α is recursive as well.*

Proof. Suppose that $\mathsf{MTh}(\alpha)$ is decidable such that, by Theorem 3.2, α is recursive and there exists a recursive indicator of recurrence rec. Let rec' be the weak indicator of recurrence for α. For $\varphi \in \mathsf{Sent}$, consider the sentence

$$\psi_\varphi = \exists x, y\colon (x \leq y \wedge \varphi_{x,y})$$

(here $\varphi_{x,y}$ results from φ by restricting all quantifiers to the interval $[x,y]$). Then we have

$$\mathrm{rec}'(\varphi) = \begin{cases} 0 & \text{if } \alpha \models \neg\psi_\varphi \\ 1 & \text{if } \alpha \models \psi_\varphi \text{ and } \mathrm{rec}(\varphi) \in \mathbb{N} \\ \top & \text{otherwise.} \end{cases}$$

Since validity of ψ_φ in α is decidable, the function rec' is recursive.

For the other direction, suppose α is recursive and rec' is the recursive weak indicator of recurrence. We construct a recursive indicator of recurrence as follows: If $\mathrm{rec}'(\varphi) = \top$, then set $\mathrm{rec}(\varphi) = \top$. Now suppose $\mathrm{rec}'(\varphi) \in \{0,1\}$. For $n \in \mathbb{N}$, consider the sentence

$$\psi_n = \exists x, y\colon (n \leq x \leq y \wedge \varphi_{x,y})\,.$$

Since $\mathrm{rec}'(\varphi) \neq \top$, there is a minimal natural number n with $\alpha \models \neg\psi_n$. Setting $\mathrm{rec}(\varphi) = n$ ensures that $\mathrm{rec}\colon \mathsf{Sent} \to \mathbb{N} \cup \{\top\}$ is an indicator of recurrence. Note that $\mathrm{rec}(\varphi)$ is minimal among all those numbers n satisfying $\mathrm{rec}'(\psi_n) = 0$. Hence the function rec is recursive implying, by Theorem 3.2, that $\mathsf{MTh}(\alpha)$ is decidable. □

3.2. **Rabinovich and Thomas' characterization.** Two other characterisations are given by Rabinovich and Thomas in [11]. The idea is to decompose an ω-word into infinitely many finite sections all of which (except possibly the first one) have the same k-type.

Definition 3.5. Let $\alpha \in \{0,1\}^\omega$, $u, v \in \{0,1\}^+$, $k \in \mathbb{N}$, and $H \subseteq \mathbb{N}$ be infinite.

- The set H is a *k-homogeneous factorisation of α into (u,v)* if $\alpha[0,i-1] \equiv_k u$ and $\alpha[i,j-1] \equiv_k v$ for all $i,j \in H$ with $i < j$.
- The set H is *k-homogeneous for α* if it is a k-homogeneous factorisation of α into some finite words (u,v).
- Let $H = \{h_i \mid i \in \mathbb{N}\}$ with $h_0 < h_1 < \dots$. The set H is *uniformly homogeneous for α* if, for all $k \in \mathbb{N}$, the set $\{h_i \mid i \geq k\}$ is k-homogeneous for α.

As with indicators of recurrence, any ω-word has many uniformly homogeneous sets: the existence of at least one follows by a repeated and standard application of Ramsey's theorem, and there are infinitely many since any infinite subset of a uniformly homogeneous set is again uniformly homogeneous.

Theorem 3.6 (1st Rabinovich-Thomas' Characterisation [11]). *Let α be an ω-word. Then $\mathsf{MTh}(\alpha)$ is decidable if and only if there exists a recursive uniformly homogeneous set for α and the ω-word α is recursive.*

The idea of the second characterisation by Rabinovich and Thomas is to compute, from $k \in \mathbb{N}$, a representative of the k-type of the ω-word α. This is formalised as follows:

Definition 3.7. Let α be some ω-word and $\mathrm{tp}\colon \mathbb{N} \to \{0,1\}^+ \times \{0,1\}^+$. The function tp is a *type-function for* α if, for all $k \in \mathbb{N}$, the ω-word α has a k-homogeneous factorisation into $\mathrm{tp}(k) = (u, v)$.

Let tp be a type-function for the ω-word α and let $k \in \mathbb{N}$. Then there exists a k-homogeneous factorisation H of α into $\mathrm{tp}(k) = (u, v)$. Let $H = \{h_i \mid i \in \mathbb{N}\}$ such that $h_0 < h_1 < \dots$. Then we get

$$\begin{aligned} \alpha &= \alpha[0, h_0 - 1]\, \alpha[h_0, h_1 - 1]\, \alpha[h_1, h_2 - 1] \cdots \\ &\equiv_k uv^\omega\,. \end{aligned}$$

Furthermore, $v \equiv_k \alpha[h_0, h_2 - 1] = \alpha[h_0, h_1 - 1]\, \alpha[h_1, h_2 - 1] \equiv_k vv$. Consequently, $\mathrm{tp}(k)$ is a representative of the k-type of α. Recall that validity in α of a sentence of quantifier depth k can be determined from any representative of the k-type of α. Hence, to decide the MSO-theory of α, it suffices to have a recursive type-function. The converse implication of the following theorem holds since the "minimal type function" can be expressed in MSO (*cf.* proof of Theorem 5.22).

Theorem 3.8 (2nd Rabinovich-Thomas' Characterisation [11]). *Let α be an ω-word. Then $\mathsf{MTh}(\alpha)$ is decidable if and only if α has a recursive type-function.*

Note that, differently from Theorem 3.6, this theorem does not mention that α is recursive. But this recursiveness is implicit: Let tp be a recursive type-function and $k \in \mathbb{N}$. Then one can write down a first-order sentence of quantifier-depth $k + 2$ expressing that $\alpha(k) = 1$. Let $\mathrm{tp}(k + 2) = (u, v)$. Then $\alpha \equiv_{k+2} uv^\omega$ implies $\alpha(k) = uv^k(k)$, hence $\alpha(k)$ is computable from k, i.e., $\alpha^{-1}(1)$ is recursive.

4. How difficult is it to tell whether the MSO-theory of an ω-word is decidable?

In this section we show that the question whether $\mathsf{MTh}(\alpha)$ is decidable for a recursive ω-word α is Σ_3-complete.

Technically, we will consider the following two sets:

$$\mathsf{DecTh}^{\mathsf{MSO}}_{\mathbb{N}} = \{e \in \mathsf{REC} \mid \mathsf{MTh}(\mathbb{N}, \leq, W_e) \text{ is decidable}\}$$

$$\mathsf{UndecTh}^{\mathsf{MSO}}_{\mathbb{N}} = \{e \in \mathsf{REC} \mid \mathsf{MTh}(\mathbb{N}, \leq, W_e) \text{ is undecidable}\}$$

Note that $(\mathbb{N}, \leq, W_e)$ is the labeled linear order M_w associated to the characteristic ω-word α of the e^{th} recursively enumerable set W_e. We will prove that the first set is in Σ_3 and that any separator of the two sets (i.e., any set containing $\mathsf{DecTh}^{\mathsf{MSO}}_{\mathbb{N}}$ and disjoint from $\mathsf{UndecTh}^{\mathsf{MSO}}_{\mathbb{N}}$) is Σ_3-hard.

Lemma 4.1. *The set* $\mathsf{DecTh}_{\mathbb{N}}^{\mathsf{MSO}}$ *belongs to* Σ_3.

We present two proofs of this lemma, one based on the first Rabinovich-Thomas characterisation, the second one based on the Semenov characterization.

Proof. (based on Theorem 3.6) Let α be some recursive ω-word.

Recall that a set $H \subseteq \mathbb{N}$ is infinite and recursive iff there exists a total computable and strictly monotone function f such that $H = \{f(n) \mid n \in \mathbb{N}\}$. Now consider the following statement:

$$\begin{aligned}\exists e\,\forall k, i, j, i', j'\colon\ & e \in \mathsf{TOT} \wedge \\ & i < j \Rightarrow \Phi_e(i) < \Phi_e(j) \wedge \\ & (k \le i \le j \wedge k \le i' \le j' \Rightarrow \alpha[\Phi_e(i), \Phi_e(j)] \equiv_k \alpha[\Phi_e(i'), \Phi_e(j')]\end{aligned}$$

It expresses that there exists a total computable function (namely Φ_e) that is strictly monotone. Its image then consists of the numbers

$$\Phi_e(0) < \Phi_e(1) < \Phi_e(2) < \dots .$$

The last line expresses that this image is uniformly homogeneous for α. Hence this statement says that there exists a recursive uniformly homogeneous set for α, i.e., that $\mathsf{MTh}(\alpha)$ is decidable by Theorem 3.6.

Let $k, i, i', j, j' \in \mathbb{N}$ with $k \le i \le j$ and $k \le i' \le j'$. Then we can compute the finite words $\alpha[\Phi_e(i), \Phi_e(j)]$ and $\alpha[\Phi_e(i'), \Phi_e(j')]$ since α is recursive. Hence it is decidable whether

$$\alpha[\Phi_e(i), \Phi_e(j)] \equiv_k \alpha[\Phi_e(i'), \Phi_e(j')]$$

holds.

Since $\mathsf{TOT} \in \Pi_2$, the whole statement is consequently in Σ_3. $\square$

Proof. (based on Theorem 3.2) We enumerate the set Sent of MSO-sentences in any effective way as $\varphi_0, \varphi_1, \dots$. Let $e \in \mathsf{TOT}$ and consider the function $\mathrm{rec}\colon \mathsf{Sent} \to \mathbb{N} \cup \{\top\}$ defined by

$$\varphi_i \mapsto \begin{cases} \Phi_e(i) - 1 & \text{if } \Phi_e(i) > 0 \\ \top & \text{if } \Phi_e(i) = 0\,. \end{cases}$$

This function is an indicator of recurrence for the ω-word α if and only if the following holds:

$$\begin{aligned}\forall \varphi \in \mathsf{Sent}\colon\ & \mathrm{rec}(\varphi) \ne \top \Rightarrow \forall k \ge j \ge \mathrm{rec}(\varphi)\colon \alpha[j,k] \models \neg\varphi \wedge \\ & \mathrm{rec}(\varphi) = \top \Rightarrow \forall j \exists \ell \ge k \ge j\colon \alpha[k,\ell] \models \varphi\end{aligned}$$

Given the definition of rec, this is equivalent to saying

$$\begin{aligned}\forall i\colon\ & \Phi_e(i) > 0 \Rightarrow \forall k \ge j \ge \Phi_e(i)\colon \alpha[j,k] \models \neg\varphi_i \wedge \\ & \Phi_e(i) = 0 \Rightarrow \forall j \exists \ell \ge k \ge j\colon \alpha[k,\ell] \models \varphi_i\,.\end{aligned}$$

If α is recursive, this is a Π_2-statement. Consequently, also the existence of a recursive indicator of recurrence is a Σ_3-statement. $\square$

We could present a third proof based on Corollary 3.4: For any $\varphi \in \mathsf{Sent}$, let φ'_n denote the sentence $\exists y \ge x \ge n\colon \varphi_{x,y}$ where $\varphi_{x,y}$ results from φ by restricting all quantifiers to the interval $[x, y]$. Then the proof can be constructed in the same way as the proof above, except using the following Π_2 statement:

$$\begin{aligned}\forall \varphi \in \mathsf{Sent}\colon\ & \mathrm{rec}(\varphi) = 0 \Rightarrow \forall k \ge j \ge 0\colon \alpha[j,k] \models \neg\varphi \wedge \\ & \mathrm{rec}(\varphi) = \top \Rightarrow \forall j \exists \ell \ge k \ge j\colon \alpha[k,\ell] \models \varphi \wedge \\ & \mathrm{rec}(\varphi) \notin \{0, \top\} \Rightarrow \mathrm{rec}(\varphi) = 1\end{aligned}$$

Remark 4.2. From the second characterisation by Rabinovich and Thomas (Theorem 3.8), we can only infer that $\mathsf{DecTh}_{\mathbb{N}}^{\mathsf{MSO}}$ is in Σ_5:

Let α be some recursive ω-word and $u, v \in \{0,1\}^+$. Then, by the proof of [11, Proposition 7], there exists a k-homogeneous factorisation of α into (u, v), if the following Σ_3-statement $\varphi(u, v)$ holds:

$$\exists x \forall y \exists z, z' \colon (\alpha[0, x-1] \equiv_k u \wedge y < z < z' \wedge \alpha[x, z-1] \equiv_k \alpha[z, z'-1] \equiv_k v)$$

Hence a function $\mathrm{tp} \colon \mathbb{N} \to \{0,1\}^+ \times \{0,1\}^+$ is a type-function for α iff the Π_4-statement $\forall k \in \mathbb{N} \colon \varphi(\mathrm{tp}(k))$ holds. Consequently, there is a recursive type-function iff we have

$$\exists e \colon e \in \mathsf{TOT} \wedge \forall k \colon \varphi(\mathrm{pair}(\Phi_e(k)))$$

where $\mathrm{pair} \colon \mathbb{N} \to \{0,1\}^+ \times \{0,1\}^+$ is a computable surjection. Since this statement is an Σ_5-statement, the claim follows.

Remark 4.3. Recall that any MSO-sentence can be translated into a deterministic parity automaton that accepts precisely those words that satisfy the sentence (*cf.* [8]). Hence, $\mathsf{MTh}(\alpha)$ is decidable if and only if the set of deterministic parity automata accepting α is decidable. This statement is a Σ_4-statement.

Three (out of five) characterisations of the decidable recursive ω-words result in the same recursion-theoretic upper bound Σ_3 of the set $\mathsf{DecTh}_{\mathbb{N}}^{\mathsf{MSO}}$. It is therefore natural to ask if these characterisations are "optimal". Namely, if one can separate $\mathsf{DecTh}_{\mathbb{N}}^{\mathsf{MSO}}$ from $\mathsf{UndecTh}_{\mathbb{N}}^{\mathsf{MSO}}$ using a simpler statement. We now prepare a negative answer to this question (which is an affirmative answer to the optimality question posed first).

Lemma 4.4. *From $k \in \mathbb{N}$, one can compute $\ell \in \mathbb{N}$ such that $0^\ell \equiv_k 0^{2\ell}$.*

Proof. Up to logical equivalence, there are only finitely many MSO-sentences of quantifier-rank at most k. Hence there are only finitely many $\equiv_k$-equivalence classes. Consequently, there are $i, j \geq 1$ with $0^i \equiv_k 0^{i+j}$. Even more, we can effectively find such a pair by simply checking all pairs (i, j) (since k-equivalence of finite words is decidable).

With $\ell = ij$, we then get

$$0^\ell = 0^i 0^{\ell-i} \equiv_k 0^{i+ij} 0^{\ell-i} = 0^{2\ell}$$

where $0^i 0^{\ell-i} \equiv_k 0^{i+ij} 0^{\ell-i}$ follows from $0^i \equiv_k 0^{i+j}$. □

We now construct an m-reduction from REC to any separator of the sets $\mathsf{DecTh}_{\mathbb{N}}^{\mathsf{MSO}}$ and $\mathsf{UndecTh}_{\mathbb{N}}^{\mathsf{MSO}}$: Let $e \in \mathbb{N}$. Then the sets $\{2a \mid a \in W_e\}$ and $2\mathbb{N}+1$ are both (effectively) recursively enumerable and so is their union. Hence, by [12, Corollary 5.V(d)(i)], one can compute $f \in \mathbb{N}$ such that Φ_f is total and injective and

$$\{2a \mid a \in W_e\} \cup (2\mathbb{N}+1) = \{\Phi_f(i) \mid i \in \mathbb{N}\}.$$

For $i \in \mathbb{N}$, set

$$x_i = 2^{\Phi_f(i)} \cdot \prod_{0 \leq j \leq i} (2j+1) \tag{4.1}$$

and consider the ω-word $\alpha_e = 10^{x_0} 10^{x_1} 10^{x_2} \cdots$. Since Φ_f is total, this ω-word is recursive.

Lemma 4.5. *Let $e \in \mathbb{N}$. The MSO-theory of the ω-word α_e is decidable if and only if the e^{th} recursively enumerable set W_e is recursive, i.e., $e \in \mathsf{REC}$.*

Proof. First suppose that the MSO-theory of α_e is decidable. For $a \in \mathbb{N}$, we have $a \in W_e$ iff there exists $i \geq 0$ with $2a = \Phi_f(i)$ iff there exists $i \geq 0$ such that 2^{2a} is the greatest power of 2 that divides x_i. Consequently, $a \in W_e$ iff the ω-word α_e satisfies

$$\begin{aligned}\exists x, y \in P\colon \quad & x < y \wedge \forall z\colon (x < z < y \Rightarrow z \notin P) \wedge \\ & 2^{2a} \mid y - x - 1 \wedge 2^{2a+1} \not\mid y - x - 1\end{aligned} \tag{4.2}$$

By Example 2.1, $n \mid y - x - 1$ is expressible by an MSO-formula, i.e., the above formula can be written as an MSO-sentence. Since validity in α_e of the resulting MSO-sentence is decidable, the set W_e is recursive.

Conversely, let W_e be recursive. To show that the MSO-theory of α_e is decidable, let φ be some MSO-sentence. Let $k = \mathsf{qr}(\varphi)$ be the quantifier-rank of φ. To decide whether $\alpha_e \models \varphi$, we proceed as follows:

- First, compute $\ell > 0$ such that $0^\ell \equiv_k 0^{2\ell}$. This is possible by Lemma 4.4.
- Next determine $a, b \in \mathbb{N}$ such that $\ell = 2^a(2b+1)$.
- Then compute $i \geq b$ such that $\Phi_f(j) > a$ for all $j > i$: to this aim, first determine $A = \{n \leq a \mid n \in W_e \text{ or } a \text{ odd}\}$ which is possible since W_e is decidable. Then compute the least $i \geq b$ such that $A \subseteq \{\Phi_f(j) \mid j \leq i\}$. Since Φ_f is injective, we get $\Phi_f(j) > a$ for all $j > i$.
- Decide whether $10^{x_0}10^{x_1}\dots10^{x_i}(10^\ell)^\omega$ satisfies φ which is possible since this ω-word is ultimately periodic.

Let $j > i$. Then $\Phi_f(j) > a$ and $j > i \geq b$ imply that x_j is a multiple of ℓ. Consequently $0^{x_j} \equiv_k 0^\ell$. We therefore obtain

$$\alpha_e \equiv_k 10^{x_1}10^{x_2}\cdots10^{x_i}(10^\ell)^\omega\,.$$

Hence the above algorithm is correct. □

Since REC is Σ_3-complete [12, Theorem 14.XVI], Lemma 4.5 and Lemma 4.1 imply that the problem of deciding whether a recursive ω-word has a decidable MSO-theory is Σ_3-complete:

Theorem 4.6.
- $\mathsf{DecTh}^{\mathsf{MSO}}_{\mathbb{N}}$ *is in* Σ_3.
- *Any set containing* $\mathsf{DecTh}^{\mathsf{MSO}}_{\mathbb{N}}$ *and disjoint from* $\mathsf{UndecTh}^{\mathsf{MSO}}_{\mathbb{N}}$ *is* Σ_3*-hard.*

Remark. Theorem 3.6 is proved in [11] not only for the logic MSO, but also for the weaker logic FO and for the intermediate logic $\mathsf{FO}+\mathsf{MOD}$ that extends FO by modulo-counting quantifiers. Consequently, Lemma 4.1 also holds for the logics FO and $\mathsf{FO}+\mathsf{MOD}$ *mutatis mutandis.*

On the other hand, (4.2) in the proof of Lemma 4.5 can easily be expressed in $\mathsf{FO}+\mathsf{MOD}$ implying that also Lemma 4.5 holds for this logic. Furthermore, one may use a very similar reduction to prove the same Σ_3-bound for the FO-theory: replace the definition of x_i from (4.1) by $x_i = \Phi_f(j)$ (and $0^\ell \equiv_k 0^{2\ell}$ by $0^\ell \equiv^{\mathsf{FO}}_k 0^{\ell+1}$ in Lemma 4.4). A similar argument as in Lemma 4.5 proves that W_e is recursive if and only if the ω-word α_e obtained this way has a decidable FO-theory.

Thus, the above Theorem 4.6 also holds for the logics FO and $\mathsf{FO}+\mathsf{MOD}$.

5. When is the MSO-theory of a bi-infinite word decidable?

In this section, we investigate whether the characterisations from Theorems 3.2, 3.6, and 3.8 and from Corollary 3.4 can be lifted from ω- to bi-infinite words.

A crucial notion will be that of the theory of a language: Let $L \subseteq \{0,1\}^*$ be a language. Its MSO-*theory* $\mathsf{MTh}(L)$ is the set of sentences $\varphi \in \mathsf{Sent}$ such that $w \models \varphi$ for all $w \in L$, i.e., $\mathsf{MTh}(L) = \bigcap_{w\in L} \mathsf{MTh}(w)$.

In [13, pages 602-603], Semenov proves the following characterizations:

Theorem 5.1 ([13])**.** *Let ξ be a bi-infinite word.*

(1) *If ξ is not recurrent, then $\mathsf{MTh}(\xi)$ is decidable if and only if $\mathsf{MTh}(\xi(-\infty,-1])$ and $\mathsf{MTh}(\xi[0,\infty))$ are both decidable.*
(2) *If ξ is recurrent, then $\mathsf{MTh}(\xi)$ and $\mathsf{MTh}(F(\xi))$ are interreducible. In particular, in this case $\mathsf{MTh}(\xi)$ is decidable if and only if $\mathsf{MTh}(F(\xi))$ is decidable.*

5.1. Complicated bi-infinite words with decidable MSO-theory.

We first demonstrate that, in the second statement of Theorem 5.1, we cannot replace the decidability of $\mathsf{MTh}(F(\xi))$ by that of $F(\xi)$.

Lemma 5.2. *Let $L \subseteq \{0,1\}^*$ be a set of finite words. Then the following are equivalent:*

- *There exists a recurrent bi-infinite word ξ with $F(\xi) = L$.*
- (a) *L contains a non-empty word.*
  (b) *If $uvw \in L$, then $v \in L$.*
  (c) *For any $u, w \in L$, there is a finite word v such that $uvw \in L$*

In addition, ξ can be chosen recursive iff L is recursively enumerable.

Proof. First suppose $L = F(\xi)$ for some recurrent bi-infinite word ξ. Then (a), (b), and (c) are obvious. If, in addition, ξ is recursive, then its set of factors L is recursively enumerable.

Conversely, suppose (a), (b), and (c) hold and L is recursively enumerable (the proof for non-recursively enumerable sets L can be extracted easily from this one). By (a), there exists a non-empty word $u \in L$. From (c), we obtain that L is infinite. Let $f\colon \mathbb{N} \to \{0,1\}^*$ be a computable and total function with $L = \{f(i) \mid i \in \mathbb{N}\}$. We will write u_i for the word $f(i)$. Inductively, we construct two sequences $(x_i)_{i>0}$ and $(y_i)_{i>0}$ of words from L such that, for all $i \in \mathbb{N}$, the finite word

$$w_i = u_i x_i u_{i-1} x_{i-1} \dots u_1 x_1 u_0 y_1 u_1 y_2 u_2 \dots y_i u_i$$

belongs to L.

Let $i > 0$ and suppose we already defined the words x_j and y_j for $j < i$ such that $w_{i-1} \in L$. To extend w_{i-1} to the left, let $j \in \mathbb{N}$ be the minimal index with $f(j) \in u_i\{0,1\}^* w_{i-1}$ (such a number j exists by (b)). Choose $x_i \in \{0,1\}^*$ with $f(j) = u_i x_i w_{i-1}$. Next we extend this word from L symmetrically to the right: let $k \in \mathbb{N}$ be minimal with $f(k) \in u_i x_i w_{i-1}\{0,1\}^* u_i$ and choose $y_i \in \{0,1\}^*$ such that $f(k) = u_i x_i w_{i-1} y_i u_i$.

Then the bi-infinite word

$$\xi = \cdots u_3 x_3\, u_2 x_2\, u_1 x_1\, u_0\, y_1 u_1\, y_2 u_2\, y_3 u_3 \cdots$$

satisfies $L \subseteq F(\xi)$.

Let $v \in \{0,1\}^*$ be some factor of ξ. Then there is $i \in \mathbb{N}$ such that v is a factor of w_i. Since $w_i \in L$, condition (b) implies $v \in L$. Hence $F(\xi) = L$.

Now let $v \in F(\xi) = L$. By (c), there are infinitely many $i \in \mathbb{N}$ such that v is a factor of u_i. Hence ξ is recurrent. It is also recursive since the word w_i is computable from w_{i-1}. □

Theorem 5.3. *There exists a recurrent and recursive bi-infinite word ξ whose set of factors is decidable, but* $\mathsf{MTh}(\xi)$ *is undecidable.*

Proof. Let $f\colon \mathbb{N} \to \mathbb{N}$ be some recursive and total function such that $\{f(i) \mid i \in \mathbb{N}\}$ is not recursive. Let $L \subseteq \{0,1\}^*$ be the set of all finite words u with the following property: If $10^{2i+1}10^{2j}1$ is a factor of u, then $j = f(i)$. This set is clearly recursive, contains a non-empty word, and satisfies conditions (a), (b), and (c) from Lemma 5.2. Hence there exists a recurrent and recursive bi-infinite word ξ with $F(\xi) = L$.

For $j \in \mathbb{N}$, consider the following sentence:

$$\begin{aligned} \exists x < y\colon \quad & P(x) \wedge P(y+2j) \wedge \neg\, 2 \mid y - x - 1 \\ \wedge \quad & \forall z\colon (x < z < y + 2j \wedge P(z) \to z = y) \end{aligned}$$

It expresses that the language $1(00)^*010^{2j}1$ contains a factor of ξ. But this is the case iff it contains a factor of some word from L iff there exists $i \in \mathbb{N}$ with $j = f(i)$. Since this is undecidable, the MSO-theory of ξ is undecidable. □

Suppose ξ is not recurrent with decidable MSO-theory. Then by the first statement of Theorem 5.1, the MSO-theories of the two "halves" of ξ are decidable. Hence these two halves are recursive implying that ξ is recursive as well.

Our next two theorems show that the situation is "in some sense more exotic" (as Semenov puts it [14, page 165]) when we consider recurrent bi-infinite words. Namely, we construct non-recursive bi-infinite words with decidable MSO-theories whose "halves" have undecidable MSO-theories.

Theorem 5.4. *There exists a recursive and recurrent bi-infinite word ξ with decidable* MSO*-theory, such that every nontrivial m-degree $\mathbf{a}$ contains some bi-infinite word $\xi_{\mathbf{a}}$ with* $\mathsf{MTh}(\xi) = \mathsf{MSO}(\xi_{\mathbf{a}})$*. Furthermore, $\xi_{\mathbf{a}}(-\infty,-1]$ and $\xi_{\mathbf{a}}[0,\infty)$ both belong to $\mathbf{a}$.*

Proof. Let $g\colon \mathbb{N} \to \{0,1\}^*$ be a computable surjection and define

$$\xi = \dots g(2)\, g(1)\, g(0)\, g(1)\, g(2) \dots .$$

Clearly, ξ is recursive and recurrent with $F(\xi) = \{0,1\}^*$. Since $\mathsf{MTh}(\{0,1\}^*)$ is decidable, the MSO-theory of the recurrent bi-infinite word ξ is decidable by Theorem 5.1.

Since ξ is recursive, we can set $\xi_{\mathbf{a}} = \xi$ for the minimal nontrivial m-degree $\mathbf{a}$ of all nontrivial recursive sets.

Now let $\mathbf{a}$ be some m-degree above the m-degree of all nontrivial recursive sets. Furthermore, let $A \in \mathbf{a}$ be an arbitrary set in the m-degree $\mathbf{a}$. Since $\mathbf{a}$ is nontrivial, we get $\emptyset \neq A \neq \mathbb{N}$. We denote the characteristic function of A by χ_A. Then let $\beta_{\mathbf{a}}$ be the ω-word

$$\beta_{\mathbf{a}} = \chi_A(0)g(0)\chi_A(1)g(1)\chi_A(2)g(2)\cdots$$

and set $\xi_{\mathbf{a}} = \beta_{\mathbf{a}}^R\, \beta_{\mathbf{a}}$.

Since g is recursive, we get $\beta_{\mathbf{a}} \leq_m A$. Conversely, $n \in A$ iff the ω-word $\beta_{\mathbf{a}}$ carries 1 at position $\sum_{0 \leq i < n}(1 + |g(i)|)$, i.e., $A \leq_m \beta_{\mathbf{a}}$. This proves $\beta_{\mathbf{a}} \in \mathbf{a}$. It follows that also $\xi_{\mathbf{a}} \in \mathbf{a}$.

Note that also the bi-infinite word $\xi_{\mathbf{a}}$ is recurrent with $F(\xi_{\mathbf{a}}) = \{0,1\}^*$. Hence, by Theorem 5.1, $\mathsf{MTh}(\xi) = \mathsf{MTh}(\xi_{\mathbf{a}})$. □

The above theorem provides us with a theory $\mathsf{MTh}(\xi)$ that is realised (by some bi-infinite word) in every non-trivial m-degree $\mathbf{a}$. We next ask to what extend this holds for every MSO-theory. First, if ξ is non-recurrent or periodic, then ξ is computable from its MSO-theory. Hence, all realisations of $\mathsf{MTh}(\xi)$ are computable in $\mathsf{MTh}(\xi)$ and are therefore of bounded complexity. It remains to consider the recurrent, non-periodic case. We therefore first demonstrate some facts about the factor set $F(\xi)$ of a recurrent bi-infinite word ξ.

Definition 5.5. Let $L \subseteq \{0,1\}^*$ be a language. A word $u \in L$ is *left-determining in L* if for every $k \in \mathbb{N}$ there is exactly one word $vu \in L$ with $|v| = k$. Similarly, u is *right-determining in L* if for every $k \in \mathbb{N}$ there is exactly one word $uv \in L$ with $|v| = k$. The word $u \in L$ is *determining in L* if it is both left- and right-determining.

Intuitively a word $w \in L$ is left-determining (right-determining) in L if it can be extended on the left (right) in a unique way.

Lemma 5.6. *Let ξ be a recurrent bi-infinite word. The following are equivalent:*
(1) *ξ is periodic.*
(2) *$F(\xi)$ contains a determining word.*
(3) *$F(\xi)$ contains a right-determining word.*
(3') *$F(\xi)$ contains a left-determining word.*

Proof. For (1)$\to$(2), let $\xi = u^{\omega^*}u^{\omega}$ be a periodic word. Then u is determining in $F(\xi)$. The direction (2)$\to$(3) is trivial by the very definition.

For (3)$\to$(1), suppose u is a right-determining word in $F(\xi)$. Choose $i < j$ such that $\xi[i, i+|u|-1] = \xi[j, j+|u|-1] = u$ (such a pair $i < j$ exists since ξ is recurrent). With $p = j-i$, we claim $\xi(n) = \xi(n+p)$ for all $n \in \mathbb{Z}$: First let $n \geq j+|u|$. Then $\xi[i, n]$ and $\xi[j, n+p]$ are two words from $F(\xi)$ that both start with u. We have $|\xi[i, n]| = n-i-1 = n+p-j-1 = |\xi[j, n+p]|$. Since u is right-determining, this implies $\xi[i, n] = \xi[j, n+p]$ and therefore $\xi(n) = \xi(n+p)$. Consequently, $\xi[j+|u|, \infty) = \xi[j+|u|, j+|u|+p]^{\omega}$. Next let $n < j+|u|$. Since ξ is recurrent, there is $k < n$ with $\xi[k, k+|u|-1] = u$. Since u is right-determining, this implies $\xi[k, \infty) = \xi[j+|u|, \infty) = \xi[j+|u|, j+|u|+p]^{\omega}$ and therefore in particular $\xi(n) = \xi(n+p)$.

The implications (2)$\to$(3')$\to$(1) are shown analogously. $\square$

Lemma 5.6 states that a recurrent non-periodic bi-infinite word does not contain any left-determining or right-determining factor, and thus can be extended in both directions (left and right) in at least two ways without changing the set of its factors. This observation allows to prove the following:

Lemma 5.7. *Let ξ be a recurrent non-periodic bi-infinite word and let $f_{\xi} \colon \mathbb{N} \to F(\xi)$ be a surjection (that we identify with the relation $\{(n, f_{\xi}(n)) \mid n \in \mathbb{N}\}$). For any set $A \subseteq \mathbb{N}$, there is a recurrent bi-infinite word ξ_A such that $F(\xi) = F(\xi_A)$ and $\xi_A(-\infty, -1] \oplus f_{\xi} \equiv_T \xi_A[0, \infty) \oplus f_{\xi} \equiv_T A \oplus f_{\xi}$.*

Proof. In the following, we write w_s for the factor $f_\xi(s)$ of ξ.

Now let $A \subseteq \mathbb{N}$ be arbitrary. We will construct a sequence of tuples

$$t_s = (u_s, v_s, x_s, y_s) \in (\{0,1\}^*)^4$$

such that, for all $s \in \mathbb{N}$, the finite word

$$\begin{aligned} z_s &= w_s y_s v_s \, z_{s-1} \, u_s x_s w_s \qquad (\text{with } z_{-1} = \varepsilon) \\ &= w_s y_s v_s \, w_{s-1} y_{s-1} v_{s-1} \dots w_0 y_0 v_0 \; u_0 x_0 w_0 \dots u_{s-1} x_{s-1} w_{s-1} \, u_s x_s w_s \end{aligned}$$

is a factor from $F(\xi)$ (the bi-infinite word ξ_A will be the "limit" of these words).

To start with $s = 0$ note the following: since ξ is recurrent and $w_0 \in F(\xi)$, the bi-infinite word ξ contains a factor from $w_0\{0,1\}^* w_0$. Let $n \in \mathbb{N}$ be minimal with $f_\xi(n) \in w_0\{0,1\}^* w_0$. Choose $y_0 \in \{0,1\}^*$ such that $f_\xi(n) = w_0 \, y_0 \, w_0$ and set $u_0 = v_0 = x_0 = \varepsilon$.

For the induction step, assume that we constructed the tuple t_s and that z_s is a factor of ξ. Since ξ is recurrent but not periodic, the word z_s is not right-determining in $F(\xi)$ by Lemma 5.6. Hence there are two distinct finite words u and u' of the same length such that $z_s u, z_s u' \in F(\xi)$. Let $(k, \ell) \in \mathbb{N}^2$ be the lexicographically minimal pair with $f_\xi(k), f_\xi(\ell) \in z_s\{0,1\}^*$, $|f_\xi(k)| = |f_\xi(\ell)|$, and $f_\xi(k) \neq f_\xi(\ell)$. Choose the word u_{s+1} such that

$$z_s u_{s+1} = \begin{cases} f_\xi(k) & \text{if } s \in A \\ f_\xi(\ell) & \text{otherwise.} \end{cases}$$

Now the word $z_s u_{s+1}$ is a factor of ξ. Since ξ is recurrent, it has some factor from $z_s u_{s+1}\{0,1\}^* w_{s+1}$. Choose $m \in \mathbb{N}$ minimal such that $f_\xi(m)$ belongs to this set and let $x_{s+1} \in \{0,1\}^*$ such that $f_\xi(m) = z_s u_{s+1} x_{s+1} w_{s+1}$.

To choose v_{s+1} and y_{s+1}, we proceed symmetrically to the left: The word $z'_s = z_s u_{s+1} x_{s+1} w_{s+1}$ is a factor of ξ that is not left-determining. Hence there exists a pair of distinct words v and v' of the same length with $v z'_s, v' z'_s \in F(\xi)$. Choose the pair $(k', \ell') \in \mathbb{N}^2$ lexicographically minimal with $f_\xi(k'), f_\xi(\ell') \in \{0,1\}^* z'_s$ of the same length but distinct. Then choose the word v_{s+1} such that

$$v_{s+1} z'_s = \begin{cases} f_\xi(k') & \text{if } s \in A \\ f_\xi(\ell') & \text{otherwise.} \end{cases}$$

Since the word $v_{s+1} z'_s$ is a factor of the recurrent word ξ, we can choose $m' \in \mathbb{N}$ minimal with $f_\xi(m) \in w_{s+1}\{0,1\}^* v_{s+1} z'_s$. Then let $y_{s+1} \in \{0,1\}^*$ such that $f_\xi(m) = w_{s+1} y_{s+1} v_{s+1} z'_s$. This completes the construction of the tuple t_{s+1} and therefore the inductive construction of all the tuples t_s.

Now set $\xi_A = \cdots w_1 y_1 v_1 \, w_0 y_0 v_0 \; u_0 x_0 w_0 \, u_1 x_1 w_1 \cdots$.

Observe the following:

- Note that $F(\xi) = \{w_s \mid s \in \mathbb{N}\} \subseteq F(\xi_A)$.
- Let $u \in F(\xi_A)$. There exists $s \in \mathbb{N}$ such that $u \in F(z_s)$ implying $F(\xi_A) \subseteq F(\xi)$ since $z_s \in F(\xi)$.

Hence $F(\xi) = F(\xi_A)$. To show that ξ_A is recurrent, let $u \in F(\xi_A)$. Then there are infinitely many $s \in \mathbb{N}$ with $u \in F(w_s)$. Hence u appears in ξ_A before and beyond every position, i.e., ξ_A is indeed recurrent.

Since the above describes how to compute the bi-infinite word ξ_A using the oracles A and f_ξ (technically, the oracle $A \oplus f_\xi$), we get $\xi_A \leq_T A \oplus f_\xi$ and therefore

$$\xi_A[0,\infty) \oplus f_\xi \leq_T \xi_A \oplus f_\xi \leq_T (A \oplus f_\xi) \oplus f_\xi \equiv_T A \oplus f_\xi \, .$$

We next show $A \leq_T \xi_A[0,\infty) \oplus f_\xi$: To determine whether $s \in A$ suppose we already know which of the natural numbers $i < s$ belong to A. Then the construction of ξ_A above allows to build t_s using the oracle f_ξ. Now construct t_{s+1} assuming $s \in A$ again using the oracle f_ξ. If the resulting word

$$u_0 x_0 w_0\, u_1 x_1 w_1 \dots u_{s+1} x_{s+1} w_{s+1}$$

(i.e., the "second half" of z_{s+1}) is a prefix of $\xi_A[0,\infty)$, then $s \in A$. Otherwise, $s \notin A$. Hence, indeed, $A \leq_T \xi_A[0,\infty) \oplus f_\xi$ and therefore

$$A \oplus f_\xi \leq_T (f_A[0,\infty) \oplus f_\xi) \oplus f_\xi \equiv_T f_A[0,\infty) \oplus f_\xi\,.$$

In summary, we showed $A \oplus f_\xi \equiv_T f_A[0,\infty) \oplus f_\xi$, the equivalence $A \oplus f_\xi \equiv_T f_A(-\infty,-1] \oplus f_\xi$ follows similarly. □

The following is the outcome of our attempt to relativise Theorem 5.4. It differs in several aspects from that theorem in that it talks about Turing-degrees as opposed to m-degrees and that not every (large enough) degree contains some MSO-equivalent bi-infinite word, but is the join of the degree of such a word and an enumeration of the factor set of ξ.

Theorem 5.8. *Let ξ be a recurrent non-periodic bi-infinite word, let $f_\xi\colon \mathbb{N} \to F(\xi)$ be a surjection, and let $\mathbf{a}$ be a Turing-degree above the degree of f_ξ. Then there exists a bi-infinite word ξ_A with $\mathsf{MTh}(\xi_A) = \mathsf{MTh}(\xi)$ such that $\xi_A(-\infty,-1] \oplus f_\xi \equiv_T \xi_A[0,\infty) \oplus f_\xi \in \mathbf{a}$.*

Proof. Let $A \in \mathbf{a}$ be arbitrary and consider the bi-infinite word ξ_A from Lemma 5.7. From $F(\xi) = F(\xi_A)$ and Theorem 5.1, we get $\mathsf{MTh}(\xi) = \mathsf{MTh}(\xi_A)$. Furthermore, $f_\xi \leq_T A$ implies $A \equiv_T A \oplus f_\xi$ and therefore $\xi_A(-\infty,-1] \oplus f_\xi \equiv_T \xi_A[0,\infty) \oplus f_\xi \equiv_T A \in \mathbf{a}$. □

As a consequence, we obtain that Theorem 5.4 holds for any recurrent and non-periodic bi-infinite word with a decidable theory (since the decidability of $\mathsf{MTh}(\xi)$ implies that $F(\xi)$ is decidable and therefore recursively enumerable):

Corollary 5.9. *Let ξ be a recurrent and non-periodic bi-infinite word such that $F(\xi)$ is recursively enumerable. Then every non-trivial Turing-degree $\mathbf{a}$ contains some bi-infinite word $\xi_{\mathbf{a}}$ with $\mathsf{MTh}(\xi) = \mathsf{MTh}(\xi_{\mathbf{a}})$ and $\xi_{\mathbf{a}}(-\infty,-1] \equiv_T \xi_{\mathbf{a}}[0,\infty) \in \mathbf{a}$.*

Proof. By the assumption on the set $F(\xi)$, there is a recursive surjection $f_\xi\colon \mathbb{N} \to F(\xi)$. Now the bi-infinite word ξ_A from Theorem 5.8 satisfies $\mathsf{MTh}(\xi_A) = \mathsf{MTh}(\xi)$ and $\xi_A[0,\infty) \equiv_T \xi_A[0,\infty) \oplus f_\xi \in \mathbf{a}$ as well as $\xi_A(-\infty,-1] \equiv_T \xi_A(-\infty,-1] \oplus f_\xi \in \mathbf{a}$ since f_ξ is decidable. □

Remark 5.10. Bès and Cégielski construct labeled linear orders λ of order type $\omega \cdot (\mathbb{Z},\leq)$ with $\mathsf{MTh}(\lambda)$ decidable such that the first-order theory of (λ, x) is undecidable for any $x \in \lambda$ (such structures are called "weakly maximal decidable").

Let $\xi_{\mathbf{a}}$ be the bi-infinite word from the above corollary. Since $\xi_{\mathbf{a}}[i,\infty)$ and $\xi_{\mathbf{a}}[0,\infty)$ differ in a finite word, only, we get that also $\xi_{\mathbf{a}}[i,\infty)$ belongs to $\mathbf{a}$ for any $i \in \mathbb{Z}$. Consequently, the Turing-degree of the first-order theory of $\xi_{\mathbf{a}}[i,\infty)$, and therefore of $(\xi_{\mathbf{a}}, i)$ is, for any $i \in \mathbb{Z}$, at least $\mathbf{a}$. Consequently, any decidable MSO-theory $\mathsf{MTh}(\xi)$ of a recurrent bi-infinite word is realised by some weakly maximal decidable structure that is a labeled linear order of order type $(\mathbb{Z},\leq)$.

5.2. **A characterization à la Semenov I.**

Definition 5.11. Let ξ be a bi-infinite word. A pair of functions $(\mathrm{rec}_{\leftarrow}, \mathrm{rec}_{\rightarrow})$ with $\mathrm{rec}_{\leftarrow}, \mathrm{rec}_{\rightarrow} \colon \mathsf{Sent} \to \mathbb{Z} \cup \{\top\}$ is an *indicator of recurrence for* ξ if the following hold for any sentence $\varphi \in \mathsf{Sent}$:

- if $\mathrm{rec}_{\leftarrow}(\varphi) = \top$, then $\forall k \in \mathbb{Z}\, \exists i \leq j \leq k \colon \xi[i,j] \models \varphi$
- if $\mathrm{rec}_{\leftarrow}(\varphi) \neq \top$, then $\forall i \leq j \leq \mathrm{rec}_{\leftarrow}(\varphi) \colon \xi[i,j] \models \neg\varphi$
- if $\mathrm{rec}_{\rightarrow}(\varphi) = \top$, then $\forall k \in \mathbb{Z}\, \exists j \geq i \geq k \colon \xi[i,j] \models \varphi$
- if $\mathrm{rec}_{\rightarrow}(\varphi) \neq \top$, then $\forall j \geq i \geq \mathrm{rec}_{\rightarrow}(\varphi) \colon \xi[i,j] \models \neg\varphi$

A bi-infinite word ξ "consists" of an ω^*-word $\xi_{\leftarrow}$ and an ω-word $\xi_{\rightarrow}$. Then, roughly speaking, an indicator of recurrence for the *bi-infinite* word ξ consists of a pair of indicators of recurrence, one for $\xi_{\leftarrow}$ and one for $\xi_{\rightarrow}$ (which is not quite true since $\mathrm{rec}_{\leftarrow}(\varphi) > \mathrm{rec}_{\rightarrow}(\varphi)$ is possible).

Therefore, the following characterisation is very similar to Theorem 3.2, the only difference is that the condition "ξ is recursive" is replaced by the more general one "ξ is recursive or recurrent".

Theorem 5.12. *Let ξ be a bi-infinite word. Then $\mathsf{MTh}(\xi)$ is decidable if and only if ξ has a recursive indicator of recurrence and the bi-infinite word ξ is recursive or recurrent.*

This theorem is an immediate consequence of Proposition 5.13 and 5.14 below. We first consider the case that ξ is non-recurrent where the full analogy to Theorem 3.2 holds:

Proposition 5.13. *Let ξ be a non-recurrent bi-infinite word. Then $\mathsf{MTh}(\xi)$ is decidable if and only if ξ has a recursive indicator of recurrence and the bi-infinite word ξ is recursive.*

Proof. Let $\xi_{\leftarrow} = \xi(-\infty, -1]$ and $\xi_{\rightarrow} = \xi[0, \infty)$.

First suppose that the MSO-theory of ξ is decidable. Then, by Theorem 5.1, the MSO-theories of $\xi_{\leftarrow}$ and $\xi_{\rightarrow}$ are decidable. From Theorem 3.2, we learn that the two ω-words $\xi_{\leftarrow}^R$ and $\xi_{\rightarrow}$ are recursive and have recursive indicators of recurrence. Consequently, the bi-infinite word ξ is recursive and has a recursive indicator of recurrence.

Conversely suppose that ξ is recursive and $(\mathrm{rec}_{\leftarrow}, \mathrm{rec}_{\rightarrow})$ is a recursive indicator of recurrence. Clearly, the infinite words $\xi_{\leftarrow}$ and $\xi_{\rightarrow}$ are recursive. Furthermore, the function

$$\mathrm{rec} \colon \mathsf{Sent} \to \mathbb{N} \colon \varphi \mapsto \begin{cases} \top & \text{if } \mathrm{rec}_{\rightarrow}(\varphi) = \top \\ \max(0, \mathrm{rec}_{\rightarrow}(\varphi)) & \text{otherwise} \end{cases}$$

is a recursive indicator of recurrence for the ω-word $\xi_{\rightarrow}$. Hence, by Theorem 3.2, $\mathsf{MTh}(\xi_{\rightarrow})$ is decidable; the decidability of $\mathsf{MTh}(\xi_{\leftarrow})$ can be shown analogously. Since $\xi = \xi_{\leftarrow}\xi_{\rightarrow}$, the MSO-theory of ξ is decidable. □

Proposition 5.14. *Let ξ be a recurrent bi-infinite word. Then $\mathsf{MTh}(\xi)$ is decidable if and only if ξ has a recursive indicator of recurrence.*

Proof. First suppose $\mathsf{MTh}(\xi)$ is decidable. Consider the function

$$f \colon \mathsf{Sent} \to \mathbb{N} \cup \{\top\} \colon \varphi \mapsto \begin{cases} \top & \text{if } \exists w \in F(\xi) \colon w \models \varphi \\ 0 & \text{otherwise.} \end{cases}$$

Since ξ is recurrent, (f, f) is an indicator of recurrence for ξ. From the decidability of $\mathsf{MTh}(\xi)$, we get that $\mathsf{MTh}(F(\xi))$ is decidable. But this implies that f is computable. Hence (f, f) is a recursive indicator of recurrence.

Conversely, suppose $(\mathrm{rec}_\leftarrow, \mathrm{rec}_\rightarrow)$ is a recursive indicator of recurrence for ξ. Then, for $\varphi \in \mathsf{Sent}$, we can decide whether there exists $w \in F(\xi)$ with $w \models \varphi$ (since ξ is recurrent, this is the case if and only if $\mathrm{rec}_\leftarrow(\varphi) = \top$). Thus, $\mathsf{MTh}(F(\xi))$ is decidable implying, by Theorem 5.1, that $\mathsf{MTh}(\xi)$ is decidable. □

Since Theorem 5.12 follows from the two above propositions, its proof is completed. □

Let $\mathbf{a}$ be some m-degree above REC and let $\xi_{\mathbf{a}}$ be the word from Theorem 5.4 with decidable MSO-theory. Then $\xi_{\mathbf{a}}[0,\infty) \in \mathbf{a}$ is not recursive. Consequently, the MSO-theory of $\xi_{\mathbf{a}}[0,\infty)$ is above $\mathbf{a}$ and therefore undecidable. In other words, we have a bi-infinite word with decidable MSO-theory such that the MSO-theory of its positive part is undecidable. As a consequence to Theorem 5.12, we now show that this cannot happen for recursive bi-infinite words.

Corollary 5.15. *Let ξ be a recursive bi-infinite word with a decidable* MSO*-theory. Then the* MSO*-theories of $\xi_\leftarrow = \xi(-\infty,-1]$ and of $\xi_\rightarrow = \xi[0,\infty)$ are both decidable.*

Proof. By Theorem 5.12, ξ has a recursive indicator of recurrence $(\mathrm{rec}_\leftarrow, \mathrm{rec}_\rightarrow)$. Define the functions $f, g\colon \mathsf{Sent} \to \mathbb{N} \cup \{\top\}$ as follows:

$$f(\varphi) = \begin{cases} \top & \text{if } \mathrm{rec}_\leftarrow(\varphi) = \top \\ 0 & \text{if } \mathrm{rec}_\leftarrow(\varphi) \geq 0 \\ |\mathrm{rec}_\leftarrow(\varphi)| - 1 & \text{otherwise} \end{cases}$$

$$g(\varphi) = \begin{cases} \top & \text{if } \mathrm{rec}_\rightarrow(\varphi) = \top \\ 0 & \text{if } \mathrm{rec}_\rightarrow(\varphi) < 0 \\ \mathrm{rec}_\rightarrow(\varphi) & \text{otherwise} \end{cases}$$

We claim that f and g are indicators of recurrence for the two ω-words $\xi_\leftarrow^R$ and $\xi_\rightarrow$ (for notational simplicity, we only prove it for the ω-word $\xi_\rightarrow$): Let $\varphi \in \mathsf{Sent}$.

- If $g(\varphi) = \top$, then $\mathrm{rec}_\rightarrow(\varphi) = \top$. Hence, for all $k \geq 0$ there exist $j \geq i \geq k$ with $\xi[i,j] \models \varphi$. But ξ and $\xi_\rightarrow$ agree in the interval $[i,j]$.
- Suppose $g(\varphi) \neq \top$, i.e., $\mathrm{rec}_\rightarrow(\varphi) \in \mathbb{Z}$. Hence, for all natural numbers $j \geq i \geq \mathrm{rec}_\rightarrow(\varphi)$, we have $\xi[i,j] \models \neg\varphi$. This implies (as above) that all $j \geq i \geq g(\varphi)$ satisfy $\xi_\rightarrow[i,j] \models \neg\varphi$.

Note that $\xi_\leftarrow^R$ and $\xi_\rightarrow$ are recursive ω-words (this is the only place where we use that ξ is recursive). Hence, by Theorem 3.2, the MSO-theories of $\xi_\leftarrow^R$ and of $\xi_\rightarrow$ are both decidable. □

5.3. **A characterization à la Semenov II.** We return to the question when the MSO-theory of a recurrent bi-infinite word is decidable.

Definition 5.16. Let ξ be a bi-infinite word. A pair of functions $(\mathrm{rec}'_\leftarrow, \mathrm{rec}'_\rightarrow)$ with $\mathrm{rec}_\leftarrow, \mathrm{rec}_\rightarrow\colon \mathsf{Sent} \to \{0,1,\top\}$ is a *weak indicator of recurrence for* ξ if there exist $x, y \in \mathbb{Z}$ such that $\mathrm{rec}'_\leftarrow$ is the weak indicator of recurrence for $\xi(-\infty,x]^R$ and $\mathrm{rec}'_\rightarrow$ is the weak indicator of recurrence for $\xi[y,\infty)$.

Differently from ω-words, a bi-infinite word can have more than one weak indicator of recurrence based on different reference points x and y.

Theorem 5.17. *Let ξ be a bi-infinite word. Then* $\mathsf{MTh}(\xi)$ *is decidable if and only if ξ has a recursive weak indicator of recurrence and ξ is recursive or recurrent.*

Proof. Again, we have to handle the cases of recurrent and of non-recurrent words separately.

So first let ξ be non-recurrent. Suppose that $\mathsf{MTh}(\xi)$ is decidable. Then, by Theorem 5.1, the infinite words $\xi_\leftarrow = \xi(-\infty,-1]$ and $\xi_\rightarrow = \xi[0,\infty)$ have decidable MSO-theories. From Corollary 3.4, we learn that $\xi_\leftarrow^R$ and $\xi_\rightarrow$ (and therefore ξ) are recursive with recursive weak indicators of recurrence $\mathrm{rec}'_\leftarrow$ and $\mathrm{rec}'_\rightarrow$. Hence the pair $(\mathrm{rec}'_\leftarrow, \mathrm{rec}'_\rightarrow)$ is a recursive weak indicator of recurrence for ξ and the bi-infinite word ξ is recursive.

Suppose, conversely, that ξ is recursive and $(\mathrm{rec}'_\leftarrow, \mathrm{rec}'_\rightarrow)$ is a recursive weak indicator of recurrence for ξ. Then there are $x, y \in \mathbb{Z}$ such that $\xi(-\infty,x]^R$ and $\xi[y,\infty)$ are recursive with recursive weak indicators of recurrence. Hence, by Corollary 3.4, these two infinite words have decidable MSO-theories. Since $\xi[y,\infty)$ and $\xi_\rightarrow = \xi[0,\infty)$ only differ in a finite word, also $\mathsf{MTh}(\xi_\rightarrow)$ is decidable (and similarly for $\xi_\leftarrow = \xi(-\infty,-1]$). From $\xi = \xi_\leftarrow\,\xi_\rightarrow$, it follows that also $\mathsf{MTh}(\xi)$ is decidable.

We now consider the case that ξ is recurrent. Then, by Theorem 5.12, there exists a recursive indicator of recurrence $(\mathrm{rec}_\leftarrow, \mathrm{rec}_\rightarrow)$ for ξ. Define the recursive functions $\mathrm{rec}'_\leftarrow, \mathrm{rec}'_\rightarrow \colon \mathsf{Sent} \to \{0,1,\top\}$ as follows:

$$\mathrm{rec}'_\leftarrow(\varphi) = \begin{cases} \top & \text{if } \mathrm{rec}_\leftarrow(\varphi) = \top \\ 0 & \text{otherwise} \end{cases} \qquad \mathrm{rec}'_\rightarrow(\varphi) = \begin{cases} \top & \text{if } \mathrm{rec}_\rightarrow(\varphi) = \top \\ 0 & \text{otherwise} \end{cases}$$

Since ξ is recurrent, $\mathrm{rec}'_\leftarrow$ is the weak indicator of recurrence for $\xi(-\infty,-1]^R$ and $\mathrm{rec}'_\rightarrow$ is the one for $\xi[0,\infty)$.

Conversely, suppose $(\mathrm{rec}'_\leftarrow, \mathrm{rec}'_\rightarrow)$ is a recursive weak indicator of recurrence for ξ. Since ξ is recurrent, it is also an indicator of recurrence for ξ. Hence, by Theorem 5.12, $\mathsf{MTh}(\xi)$ is decidable. □

5.4. **A characterization à la Rabinovich-Thomas I.**

Definition 5.18. Let ξ be a bi-infinite word, $u, v, w \in \{0,1\}^+$, $k \in \mathbb{N}$, and $H_\leftarrow, H_\rightarrow \subseteq \mathbb{Z}$ be infinite.

- The pair $(H_\leftarrow, H_\rightarrow)$ is a *k-homogeneous factorisation of ξ into (u,v,w)* if
  - $\xi[i,j-1] \equiv_k u$ for all $i, j \in H_\leftarrow$ with $i < j$,
  - $\xi[i,j-1] \equiv_k v$ for all $i \in H_\leftarrow$ and $j \in H_\rightarrow$ with $i < j$ and
  - $\xi[i,j-1] \equiv_k w$ for all $i, j \in H_\rightarrow$ with $i < j$.
- The pair $(H_\leftarrow, H_\rightarrow)$ is *k-homogeneous for ξ* if it is a k-homogeneous factorisation of ξ into some finite words (u,v,w).
- Let $H_\leftarrow = \{h_i^- \mid i \in \mathbb{N}\}$ and $H_\rightarrow = \{h_i^+ \mid i \in \mathbb{N}\}$ with $h_0^- > h_1^- > \dots$ and $h_0^+ < h_1^+ < \dots$. The pair $(H_\leftarrow, H_\rightarrow)$ is *uniformly homogeneous for ξ* if, for all $k \in \mathbb{N}$, the pair $(\{h_i^- \mid i \geq k\}, \{h_i^+ \mid i \geq k\})$ is k-homogeneous for ξ.

Let ξ be a bi-infinite word split into an ω^*-word $\xi_\leftarrow$ and an ω-word $\xi_\rightarrow$. As for any ω-word, there exists a uniformly homogeneous set $H_\rightarrow$ for $\xi_\rightarrow$. Symmetrically, there exists a set $H_\leftarrow \subseteq \widetilde{\mathbb{N}}$ that is "uniformly homogeneous" for $\xi_\leftarrow$. Then the pair $(H_\leftarrow, H_\rightarrow)$ is a uniformly homogeneous pair for $\xi = \xi_\leftarrow\xi_\rightarrow$.

We will now see that Theorem 3.6 naturally extends to *recursive* bi-infinite words (Theorem 5.20 below demonstrates that it does not extend to non-recursive bi-infinite words).

Theorem 5.19. *A recursive bi-infinite word ξ has a decidable* MSO*-theory if and only if there exists a recursive uniformly homogeneous pair for ξ.*

Proof. Suppose $\mathsf{MTh}(\xi)$ is decidable. Consider the infinite words $\xi_\leftarrow = \xi(-\infty,-1]$ and $\xi_\rightarrow = \xi[0,\infty)$. By Corollary 5.15, the MSO-theories of $\xi_\leftarrow^R = \xi(-\infty,-1]^R$ and of $\xi_\rightarrow = \xi[0,\infty)$ are both decidable. Consequently, by Theorem 3.6, there are recursive uniformly homogeneous factorisations $H_\leftarrow^R, H_\rightarrow \subseteq \mathbb{N}$ for $\xi_\leftarrow^R$ and $\xi_\rightarrow$ into (x^R, y^R) and (y', z), respectively. Deleting, if necessary, the minimal element from $H_\leftarrow^R$, we can assume $0 \notin H_\leftarrow^R$. We set $H_\leftarrow = \{-n \mid n \in H_\leftarrow^R\} \subseteq \widetilde{\mathbb{N}}$ and show that $(H_\leftarrow, H_\rightarrow)$ is a uniformly homogeneous pair for ξ: Let $H_\leftarrow = \{h_i^- \mid i \in \mathbb{N}\}$ and $H_\rightarrow = \{h_i^+ \mid i \in \mathbb{N}\}$ such that $h_0^- > h_1^- > \dots$ and $h_0^+ < h_1^+ < \dots$.

- Let $j > i \geq k$. Then
$$\begin{aligned}\xi[h_j^-, h_i^- - 1] &= \xi_\leftarrow[h_j^-, h_i^- - 1]\\ &= (\xi_\leftarrow^R[|h_i^-|, |h_j^-| - 1])^R\\ &= (\xi_\leftarrow^R[-h_i^-, -h_j^- - 1])^R\\ &\equiv_k y^R\end{aligned}$$
since $-h_i^-, -h_j^i \in H_\leftarrow^R$ and $k \leq i < j$.
- Let $i, j \geq k$. Then
$$\begin{aligned}\xi[h_i^-, h_j^+ - 1] &= \xi_\leftarrow[h_i^- + 1, 0]\, \xi_\rightarrow[0, h_j^+ - 1]\\ &= (\xi_\leftarrow^R[0, |h_i^- + 1|])^R\, \xi_\rightarrow[0, h_j^+ - 1]\\ &= (\xi_\leftarrow^R[0, -h_i^- - 1])^R\, \xi_\rightarrow[0, h_j^+ - 1]\\ &\equiv_k x^R y'\end{aligned}$$
since $-h_i^- \in H_\leftarrow^R$, $h_j^+ \in H_\rightarrow$, and $i, j \geq k$.
- Let $j > i \geq k$. Then $\xi[h_i^+, h_j^+ - 1] = \xi_\rightarrow[h_i^+, h_j^+ - 1] \equiv_k z$.

Hence the pair $(\{h_i^- \mid i \geq k\}, \{h_i^+ \mid i \geq k\})$ is a k-homogeneous factorisation of ξ into $(y^R, x^R y', z)$. Since k is arbitrary, $(H_\leftarrow, H_\rightarrow)$ is uniformly homogeneous for ξ. Since these two sets are clearly recursive, this proves the first implication.

Conversely, suppose there exists a recursive uniformly homogeneous pair $(H_\leftarrow, H_\rightarrow)$ for ξ. Then the sets $H_\leftarrow^R = \{|n| \mid n \in H_\leftarrow \cap \widetilde{\mathbb{N}}\}$ and $H_\rightarrow \cap \mathbb{N}$ are recursive and uniformly homogeneous for $\xi_\leftarrow^R$ and $\xi_\rightarrow$, resp. Since $\xi_\leftarrow$ and $\xi_\rightarrow$ are both recursive, we can apply Theorem 3.6. Hence the infinite words $\xi_\leftarrow$ and $\xi_\rightarrow$ both have decidable MSO-theories. Since $\xi = \xi_\leftarrow \xi_\rightarrow$, the MSO-theory of ξ is decidable. □

We next show that we cannot hope to extend the characterisation from Theorem 5.19 to non-recursive words. The counterexample we construct is simplest possible (namely, recursively enumerable) and does not even have a uniformly homogeneous pair that is recursively enumerable.

Theorem 5.20. *There exists a recurrent recursively enumerable bi-infinite word ξ with decidable* MSO*-theory such that there is no recursively enumerable uniformly homogeneous pair for ξ.*

Proof. We prove this theorem by constructing a recurrent bi-infinite word ξ such that the set $F(\xi)$ of factors is $\{0,1\}^*$. Hence ξ has decidable MSO-theory by Theorem 5.1.

Let $e, s \in \mathbb{N}$ and define the function $g_{e,s} \colon \mathbb{N} \to \mathbb{N}$ by

$$g_{e,s}(n) = \begin{cases} 1 & \text{if } n \leq s \text{ and the computation of } \Phi_e(n) \text{ halts in } \leq s \text{ steps} \\ 0 & \text{otherwise.} \end{cases}$$

The function $g_{e,s}$ is computable and, even more, from e and s, one can compute an index $f(e,s)$ such that $g_{e,s} = \Phi_{f(e,s)}$. With $W_{e,s} = \{n \in \mathbb{N} \mid \Phi_{f(e,s)}(n) = 1\}$, we get

- $\Phi_{f(e,s)}$ is total,
- $W_{e,s} \subseteq \{0, 1, \ldots, s\}$, and
- $W_e = \bigcup_{s \in \mathbb{N}} W_{e,s}$

Furthermore, we fix some recursive enumeration $u_0, u_1, \ldots$ of the set $\{0,1\}^+$ of non-empty finite words.

Construction. By induction on $s \in \mathbb{N}$, we construct tuples

$$t_s = (w_s, m_{0,s}, m_{1,s}, \ldots, m_{s,s}, P_s) \in \{0,1\}^* \times \mathbb{N}^{s+1} \times 2^{\{0,\ldots,s\}}$$

such that

- $m_{i,s} + |u_i| \leq m_{i+1,s}$ for all $0 \leq i < s$ and $m_{s,s} + |u_s| \leq |w_s|$ (in particular, $|w_s| > s$),
- $w_s[m_{i,s}, m_{i,s} + |u_i| - 1] = u_i$ for all $0 \leq i \leq s$, and
- for all $e \in P_s$, there exist $a, b \in W_e$ with $a < b < |w_s|$ and $w_s[a, b-1] \in 1^*$.

In other words, the finite word w_s contains disjoint occurrences of the factors $u_0, u_1, \ldots, u_s$ at positions $m_{0,s}, m_{1,s}, \ldots, m_{s,s}$ and a factor from 1^* between two positions from W_e (for $e \in P_s$).

At the beginning, set $w_0 = u_0$, $m_{0,0} = 0$, and $P_0 = \emptyset$. Then the inductive invariant holds for the tuple $t_0 = (w_0, m_{0,0}, P_0)$.

Now suppose the tuple t_s has been constructed. Let H_{s+1} denote the set of indices $0 \leq e \leq s+1$ with $e \notin P_s$ such that $W_{e,s}$ contains at least two numbers $b > a \geq m_{e,s}$. In the construction of the tuple t_{s+1}, we distinguish two cases:

- 1st case: $H_{s+1} = \emptyset$. Then set $w_{s+1} = w_s u_{s+1}$, $m_{i,s+1} = m_{i,s}$ for $0 \leq i \leq s$, $m_{s+1,s+1} = |w_s|$, and $P_{s+1} = P_s$. Since the inductive invariant holds for the tuple t_s, it also holds for the newly constructed tuple t_{s+1}.
- 2nd case: $H_{s+1} \neq \emptyset$. Let e_{s+1} be the minimal element of H_{s+1} and let a_{s+1} and b_{s+1} be the minimal elements of $W_{e_{s+1},s}$ satisfying $m_{e,s} \leq a_{s+1} < b_{s+1}$. Then set
  - $w_{s+1} = w_s[0, a_{s+1} - 1]\, 1^{b_{s+1} - a_{s+1}}\, w_s[b_{s+1}, |w_s| - 1]\, u_{e_{s+1}} u_{e_{s+1}+1} \ldots u_{s+1}$ (in other words, the words $u_{e_{s+1}}$ up to u_{s+1} are appended to w_s and the positions between a_{s+1} and $b_{s+1} - 1$ are set to 1).
  - $m_{i,s+1} = \begin{cases} m_{i,s} & \text{if } i < e_{s+1} \\ |w_s u_{e_{s+1}} u_{e_{s+1}+1} \ldots u_{i-1}| & \text{if } e_{s+1} \leq i \leq s+1 \end{cases}$
  - $P_{s+1} = P_s \cup \{e_{s+1}\}$

  The first two conditions of the inductive invariant are obvious. Regarding the last one, let $e \in P_{s+1}$. If $e \neq e_{s+1}$, then $e \in P_s$ and therefore there exist $a, b \in W_e$ with $a < b < |w_s| < |w_{s+1}|$ such that $w_s[a, b-1] \in 1^*$. Note that any position in w_s that carries 1 also carries 1 in w_{s+1}. Hence $w_{s+1}[a, b-1] \in 1^*$ as well. It remains to consider the case $e = e_{s+1}$. But then, by the very construction, $a_{s+1} < b_{s+1}$ belong to $W_{e_{s+1},s} \subseteq W_e$ and satisfy $w_{s+1}[a_{s+1}, b_{s+1} - 1] \in 1^*$.

This finishes the construction of the sequence of tuples t_s.

Let $\xi_\rightarrow$ be the ω-word with $\xi_\rightarrow(i) = 1$ iff there exists $s \in \mathbb{N}$ with $w_s(i) = 1$.

Claim 1. The ω-word $\xi_\rightarrow$ is recursively enumerable.

Proof of Claim 1. Note that the tuple t_{s+1} is computable from the tuple t_s. **q.e.d.**

Claim 2. The ω-word $\xi_\rightarrow$ is rich, i.e., any finite word is a factor of $\xi_\rightarrow$.

Proof of Claim 2. Let $u \in \{0,1\}^+$. Then there exists $e \in \mathbb{N}$ with $u = u_e$. Note that $m_{e,s} \leq m_{e,s+1}$ for all $e, s \in \mathbb{N}$. Furthermore, $m_{e,s} < m_{e,s+1}$ iff $H_{s+1} \neq \emptyset$ and $e_{s+1} \leq e$. Since the numbers $e_{s'+1}$ for $s' \in \mathbb{N}$ (if defined) are mutually distinct, there exists $s \in \mathbb{N}$ such that $e_{t+1} > e$ and therefore $m_{e,s} = m_{e,t}$ for all $t \geq s$.

Consequently, $\xi_\rightarrow[m_{e,s}, m_{e,s} + |u_e| - 1] = w_s[m_{e,s}, m_{e,s} + |u_e| - 1] = u_e = u$. **q.e.d.**

Claim 3. If W_e is infinite, then $e \in \bigcup_{s\in\mathbb{N}} P_s$.

Proof of Claim 3. By contradiction, suppose this is not the case. Let $e \in \mathbb{N}$ be minimal with W_e infinite and $e \notin \bigcup_{s\in\mathbb{N}} P_s$. Since W_e is infinite, we get $e \in H_{s+1}$ for almost all $s \in \mathbb{N}$. Since e was chosen minimal, there exists $s \in \mathbb{N}$ with $e = \min H_{s+1}$. But then $e_{s+1} = e$ and therefore $e \in P_{s+1}$. **q.e.d.**

Claim 4. No recursively enumerable set W is uniformly homogeneous for the ω-word $\xi_\rightarrow$.

Proof of Claim 4. Suppose W is recursively enumerable and uniformly homogeneous for $\xi_\rightarrow$. Then W is infinite and there exists $e \in \mathbb{N}$ with $W = W_e$. By claim 3, there exists $s \in \mathbb{N}$ with $e \in P_s$. Hence there are $a, b \in W_e$ with $w_s[a, b-1] \in 1^*$ and therefore $\xi_\rightarrow[a, b-1] = w_s[a, b-1]$. By claim 2, there are $d > c > b$ in W_e such that $\xi_\rightarrow[c, d-1] \notin 1^*$. But then $\xi_\rightarrow[a, b-1]$ and $\xi_\rightarrow[c, d-1]$ do not have the same 1-type. Hence the set W_e is not 1- and therefore not uniformly homogeneous for $\xi_\rightarrow$. **q.e.d.**

Finally, let $\xi_\leftarrow$ be the reversal of $\xi_\rightarrow$ and consider the bi-infinite word $\xi = \xi_\leftarrow \xi_\rightarrow$. By Theorem 5.12, $\mathsf{MTh}(\xi)$ is decidable since ξ is recurrent and contains every finite word as a factor. It is recursively enumerable by claim 1. Finally, suppose $(H_\leftarrow, H_\rightarrow)$ is uniformly homogeneous for ξ. Then $H_\rightarrow \cap \mathbb{N}$ is uniformly homogeneous for $\xi_\rightarrow$. By claim 4, this set cannot be recursively enumerable. Hence $(H_\leftarrow, H_\rightarrow)$ is not recursively enumerable either. $\square$

5.5. **A characterization à la Rabinovich-Thomas II.** We next extend the 2nd characterisation by Rabinovich and Thomas (Theorem 3.8) to bi-infinite words. Differently from the first characterization, this will also cover non-recursive bi-infinite words.

Definition 5.21. Let ξ be some bi-infinite word and $\mathrm{tp}\colon \mathbb{N} \to \{0,1\}^+ \times \{0,1\}^+ \times \{0,1\}^+$. The function tp is a *type-function for* ξ if, for all $k \in \mathbb{N}$, the bi-infinite word ξ has a k-homogeneous factorisation into $\mathrm{tp}(k)$.

We will show that the MSO-theory of a bi-infinite word is decidable if and only if it has a recursive type-function.

Theorem 5.22. *Let ξ be a bi-infinite word. Then $\mathsf{MTh}(\xi)$ is decidable if and only if ξ has a recursive type-function.*

Proof. First suppose that $\mathsf{MTh}(\xi)$ is decidable. We have to construct a recursive type-function $\mathrm{tp}\colon \mathbb{N} \to (\{0,1\}^+)^3$. To this aim, let $k \in \mathbb{N}$. Then one can compute a finite sequence $\varphi_1, \dots, \varphi_n$ of MSO-sentences of quantifier-rank k such that, for all finite words u and v, we have $u \equiv_k v$ if and only if

$$\forall 1 \le i \le n\colon u \models \varphi_i \iff v \models \varphi_i\,.$$

For finite words u, v, and w, consider the following statement:

$$\begin{array}{lll} \exists H_\leftarrow, H_\rightarrow\colon & & \forall y\colon (\exists x,z\colon x<y<z \wedge H_\leftarrow(x) \wedge H_\rightarrow(z)) \\ & \wedge & \forall x,z\colon (H_\leftarrow(x) \wedge H_\rightarrow(z) \to x<z) \\ & \wedge & \forall x,y\colon (x<y \wedge H_\leftarrow(x) \wedge H_\leftarrow(y) \to \xi[x,y-1] \equiv_k u) \\ & \wedge & \exists x,y\colon (H_\leftarrow(x) \wedge H_\rightarrow(y) \wedge \xi[x,y-1] \equiv_k v) \\ & \wedge & \forall x,y\colon (x<y \wedge H_\rightarrow(x) \wedge H_\rightarrow(y) \to \xi[x,y-1] \equiv_k w) \end{array}$$

This statement holds for a bi-infinite word ξ if and only if ξ has a k-homogeneous factorisation into (u,v,w). Using the MSO-sentences $\varphi_1, \dots, \varphi_n$, the statements $\xi[x,y-1] \equiv_k u$ etc. can be expressed as MSO-formulas with free variables x and y. Since the MSO-theory of ξ is decidable, we can therefore decide (given k, u, v, and w) whether ξ has a k-homogeneous factorisation into (u,v,w). Since some k-homogeneous factorisation always exist, this allows to compute, from k, a tuple $\mathrm{tp}(k)$ such that ξ has a k-homogeneous factorisation into $\mathrm{tp}(k)$. Thus, we obtained a recursive type-function tp.

Conversely suppose that tp is a recursive type-function for ξ. To show that $\mathsf{MTh}(\xi)$ is decidable, let $\varphi \in \mathsf{Sent}$ be any MSO-sentence. Let k denote the quantifier-rank of φ. First, compute $\mathrm{tp}(k) = (u,v,w)$. Then $\xi \models \varphi$ iff $u^{\omega^*} v w^\omega \models \varphi$ which is decidable since this bi-infinite word is ultimately periodic on the left and on the right. □

6. How many MSO-equivalent bi-infinite words are there?

If α and β are ω-words and MSO-equivalent, then $\alpha = \beta$. In this final section we study this question for bi-infinite words. Shift-equivalence and period will be important notions in this context.

To count the number of MSO-equivalent bi-infinite words, we need a characterisation when two bi-infinite words are MSO-equivalent.

Theorem 6.1. [8, Chp. 9, Theorem 6.1] *Two bi-infinite words ξ and ζ are MSO-equivalent if and only if one of the following conditions is satisfied:*

(1) *ξ and ζ are shift-equivalent.*
(2) *ξ and ζ are recurrent and have the same set of factors.*

This characterisation is the central ingredient in the proof of the main result of this final section:

Theorem 6.2. *Let ξ be a bi-infinite word.*

(a) *If ξ is periodic, then the cardinality of the type of ξ is finite and equals the period of w.*
(b) *If ξ is non-recurrent, then the cardinality of the type of ξ is $\aleph_0$.*
(c) *If ξ is recurrent and non-periodic, then the cardinality of the type of ξ is $2^{\aleph_0}$.*

Proof. (a) Let p be the period of ξ. Since p is minimal, there are precisely p distinct bi-infinite words that are shift-equivalent with ξ. Since shift-equivalent words are MSO-equivalent, the type of ξ contains at least p elements. It remains to be shown that no further MSO-equivalent word exists. So let ζ be some MSO-equivalent word. Then ζ is p-periodic since ξ (and therefore ζ) satisfies $\forall x\colon (P(x) \Leftrightarrow P(x+p))$ and does not satisfy $\forall x\colon (P(x) \Leftrightarrow P(x+q))$ for any $1 \leq q < p$. Furthermore $u = \xi[1,p]$ is a factor of ξ and therefore of ζ of length p. Hence $\zeta = u^{\omega^*}u^{\omega}$.

(b) This claim follows immediately from Theorem 6.1.

(c) Above any Turing-degree, there are $2^{\aleph_0}$ Turing-degrees. Hence the claim follows from Theorem 5.8.

$\square$

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