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UNGUARDED RECURSION ON COINDUCTIVE RESUMPTIONS *

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Abstract. We study a model of side-effecting processes obtained by starting from a monad modelling base effects and adjoining free operations using a cofree coalgebra construction; one thus arrives at what one may think of as types of non-wellfounded side-effecting trees, generalizing the infinite resumption monad. Correspondingly, the arising monad transformer has been termed the coinductive generalized resumption transformer. Monads of this kind have received some attention in the recent literature; in particular, it has been shown that they admit guarded iteration. Here, we show that they also admit unguarded iteration, i.e. form complete Elgot monads, provided that the underlying base effect supports unguarded iteration. Moreover, we provide a universal characterization of the coinductive resumption monad transformer in terms of coproducts of complete Elgot monads.

1. Introduction

Subsequent to seminal work by Moggi [29], monads are widely used to represent computational effects in program semantics, and in fact in actual programming languages [41]. Their main attraction lies in the fact that they provide an interface to a generic notion of side-effect at the right level of abstraction: they subsume a wide variety of side-effects such as state, nondeterminism, random, and I/O, and at the same time retain enough internal structure to support a substantial amount of generic meta-theory and programming, the latter witnessed, for example, by the monad class implemented in the Haskell basic libraries [31].

In the current work, we study a particular construction on monads motivated partly by the goal of modelling generic side-effects in the semantics of reactive processes. Specifically, given a base monad T and a strong functor Σ, we have final coalgebras

$$T_\Sigma X = \nu\gamma.\, T(X + \Sigma\gamma)$$

for each object X, assuming enough structure on T, Σ, and the base category. Inhabitants of $T_\Sigma X$ are understood as (possibly) nonterminating processes that proceed in steps, where each step produces side-effects specified by T (e.g. writing to shared global memory, nondeterminism) and performs communication actions specified by Σ. E.g. in the simplest case, Σ is of the form $a \times (-)^b$, which may be understood as reading inputs of type b and writing outputs of type a.

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The construction of $T_\Sigma X$ from T is an infinite version of the generalized resumption transformer introduced by Cienciarelli and Moggi [14]. It has been termed the *coinductive generalized resumption* transformer by Piróg and Gibbons [32, 33], who show that on the Kleisli category of T, T_Σ is the free completely iterative monad generated by $T\Sigma$.

The result that T_Σ is a completely iterative monad brings us to the contribution of the current paper. Recall that complete iterativity of T_Σ means that for every morphism

$$e : X \to T_\Sigma(Y + X),$$

read as an equation defining the inhabitants of X, thought of as variables, as terms over the defined variables (from X) and parameters from Y, has a unique *solution*

$$e^\dagger : X \to T_\Sigma Y$$

in the evident sense, *provided* that e is *guarded*. The latter concept is defined in terms of additional structure of T_Σ as an *idealized monad*, which essentially allows distinguishing terms beginning with an operation from mere variables. Guardedness of e then means that recursive calls can happen only under a free operation. Similar results on guarded recursion abound in the literature; for example, the fact that T_Σ admits guarded recursive definitions can also be deduced from more general results by Uustalu on parametrized monads [40].

The central result of the current paper is to remove the guardedness restriction in the above setup. That is, we show that a solution $e^\dagger : X \to T_\Sigma Y$ exists for *every* morphism $e : X \to T_\Sigma(X + Y)$. Of course, the solution is then no longer unique (for example, we admit definitions of the form $x = x$); moreover, we clearly need to make additional assumptions about T. Our result states, more precisely, that T_Σ allows for a principled *choice* of solutions $e^\dagger$ satisfying standard equational laws for recursion [38], thus making T_Σ into a *complete Elgot monad* [4][1]. The assumption on T that we need to enable this result is that T itself is a complete Elgot monad (e.g. partiality, nondeterminism, or combinations of these with state), i.e. we show that *the class of complete Elgot monads is stable under the coinductive generalized resumption transformer*. We show moreover that the structure of T_Σ as a complete Elgot monad is uniquely determined as extending that of T.

The motivation for these results is, well, to free non-wellfounded recursive definitions from the standard guardedness constraint. Note for example that in [32], it was necessary to assume guards in all loop iterations when interpreting a while-language with actions originally proposed by Rutten [37] over a completely iterative monad. Contrastingly, given that T_Σ is a (complete) Elgot monad, one can now just write unrestricted while loops. We elaborate this example in Section 2, and recall a standard example of unguarded recursion in process algebra in Section 3.

An earlier version of this work has appeared as [19]; the present version not only has full proofs, but also works in a generalized setup with an arbitrary strong functor Σ (admitting the requisite final coalgebras) instead of just functors of the form $a \times (-)^b$.

The material is organized as follows. We present the mentioned examples involving unguarded iteration in Sections 2 and 3. In Section 4, we collect preliminaries on (strong) monads and their Kleisli categories. We discuss the concept of complete Elgot monad in

[1] We modify the original definition of Elgot monad, which requires the object X of variables to be a finitely presentable object in an lfp category, by admitting unrestricted objects of variables. This change is owed mostly to the fact that we do not assume the base category to be lfp, and in our own estimate appears to be technically inessential, although we have not checked details for the obvious variants of our results that arise by replacing complete Elgot monads with Elgot monads.

Section 5, and recall the coinductive generalized resumption transformer in Section 6, showing in particular that it preserves strength. Sections 7 and 8 contain our main results, showing that the coinductive generalized resumption transformer preserves complete Elgotness and can be seen as freely extending complete Elgot monads with communication actions. We discuss related work in Section 9, and conclude in Section 10.

2. Example: Unrestricted While Loops

We proceed to discuss examples, aimed mainly at illustrating the benefits of not being restricted to guarded equations in recursive definitions thanks to complete Elgotness of coinductive resumption monads (Theorem 7.1). We work with the intuitive understanding of monads, T_Σ, guardedness, and complete iterativity provided in the introduction, and briefly explain the requisite categorical notation regarding strong monads and distributive categories along the way, deferring a more formal treatment to Sections 4 and 6.

Our first example is a simple while-language with actions proposed by Rutten, given by the grammar

$$P, Q ::= A \mid P; Q \mid \mathsf{if}\ b\ \mathsf{then}\ P\ \mathsf{else}\ Q \mid \mathsf{while}\ b\ \mathsf{do}\ P$$

and, following Piróg and Gibbons [32], interpreted in the Kleisli category of a monad $\mathbb{M}$. Here, A ranges over atomic actions interpreted as Kleisli morphisms $[\![A]\!] : n \to Mn$ for some fixed object n, and b over atomic predicates, interpreted as Kleisli morphisms $[\![b]\!] : n \to M(1+1)$ (where we read the left-hand summand as 'false' and the right-hand one as 'true', and 1 denotes the terminal object). We say that A is of *output type* if $[\![A]\!] : n \to Mn$ has the form $[\![A]\!] = (M\,\mathsf{fst})\,\tau\langle \mathsf{id}_n, p\rangle$ for some $p : n \to M1$, where fst denotes first projection and $\tau : n \times M1 \to M(n \times 1)$ is the strength of M. Moreover, A is of *input type* if $[\![A]\!] : n \to Mn$ factors through the unique morphism $! : n \to 1$. Sequential composition $P; Q$ is interpreted as Kleisli composition $[\![Q]\!]^\star [\![P]\!]$, and

$$[\![\mathsf{if}\ b\ \mathsf{then}\ P\ \mathsf{else}\ Q]\!] = [[\![Q]\!]\,\mathsf{fst}, [\![P]\!]\,\mathsf{fst}]^\star (M\,\mathsf{dist})\,\tau\langle \mathsf{id}_n, [\![b]\!]\rangle$$

where $\mathsf{dist} : n \times (1+1) \to (n \times 1) + (n \times 1)$ is a distributivity isomorphism that we postulate in our general setup (Section 4). The key point, of course, is the interpretation of the while loop, given in the presence of iteration $(–)^\dagger$ by

$$[\![\mathsf{while}\ b\ \mathsf{do}\ P]\!] = \big([(M\,\mathsf{inl})\,\eta\,\mathsf{fst}, (M\,\mathsf{inr})[\![P]\!]\,\mathsf{fst}]^\star (M\,\mathsf{dist})\,\tau\langle \mathsf{id}, [\![b]\!]\rangle\big)^\dagger \tag{2.1}$$

where the typing of the expression under the iteration operator $(–)^\dagger$ is visualized as

$$\begin{aligned} n &\xrightarrow{\langle \mathsf{id}, [\![b]\!]\rangle} n \times M(1+1) \\ &\xrightarrow{\tau} M(n \times (1+1)) \\ &\xrightarrow{M\,\mathsf{dist}} M(n \times 1 + n \times 1) \\ &\xrightarrow{[(M\,\mathsf{inl})\,\eta\,\mathsf{fst}, (M\,\mathsf{inr})[\![P]\!]\,\mathsf{fst}]^\star} M(n+n). \end{aligned}$$

It has been observed by Piróg and Gibbons that if one instantiates $\mathbb{M}$ with a completely iterative monad, one needs to guard every iteration of the while loop, i.e. change the semantics of while to be

$$[\![\mathsf{while}\ b\ \mathsf{do}\ P]\!] = \big([(M\,\mathsf{inl})\,\eta\,\mathsf{fst}, (M\,\mathsf{inr})[\![P]\!]\,\mathsf{fst}]^\star (M\,\mathsf{dist})\,\tau\langle \mathsf{id}, [\![b]\!]\rangle\,\gamma\big)^\dagger$$

where $\gamma : n \to Mn$ is guarded, as otherwise the iteration may fail to be defined (recall from the introduction that over completely iterative monads, definedness of iteration depends on guardedness). If we instantiate $\mathbb{M}$ with a complete Elgot monad, such as $\mathbb{T}_\Sigma$ for a complete Elgot monad $\mathbb{T}$ (by Theorem 7.1), then the guard is unnecessary, i.e. we can stick to the original semantics (2.1). As an example, consider a simple-minded form of processes that input and output symbols from n and have side effects specified by $\mathbb{T}$; i.e. we work in $\mathbb{M} = \mathbb{T}_\Sigma$ for $\Sigma X = n \times X + X^n$ where we think of Σ as being generated by an output operation $1 \to n$ and an input operation $n \to 1$. We correspondingly assume an atomic action *write* that outputs a symbol from n, and an atomic action *read* that inputs a symbol. We interpret *write* as being of output type, i.e. by $[\![\mathit{write}]\!] = (M\,\mathsf{fst})\,\tau\langle \mathsf{id}_n, w\rangle$ where $w : n \to M1$ is obtained from a canonical transformation $\iota^{\mathbb{T}} : \Sigma \to T_\Sigma = M$ that will be introduced in Section 8; intuitively, $\iota^{\mathbb{T}}$ converts actions into single-step processes without side effects. Explicitly, w is the composite

$$n \xrightarrow{\langle \mathsf{id}_n, !_n\rangle} n \times 1 \xrightarrow{\mathsf{inl}} \Sigma 1 \xrightarrow{\iota^{\mathbb{T}}} M1.$$

Moreover, we interpret *read* as being of input type, i.e. $[\![\mathit{read}]\!] = r\ !_n$ where $r : 1 \to Mn$ is obtained analogously, i.e. r is the composite

$$1 \xrightarrow{r_0} n^n \xrightarrow{\mathsf{inr}} \Sigma n \xrightarrow{\iota^{\mathbb{T}}} Mn$$

where $r_0 : 1 \to n^n$ arises by currying $\mathsf{snd} : 1 \times n \to n$. Moreover, assume a basic predicate b whose interpretation is largely irrelevant to the example as long as it may take both truth values; for example, b might just pick a truth value nondeterministically or at random, depending on the nature of the base monad $\mathbb{T}$. Consider the program

$$\mathit{read};\, \mathsf{while}\ \mathit{true}\ \mathsf{do}\ \ \mathsf{if}\ b\ \mathsf{then}\ \mathit{skip}\ \mathsf{else}\ \mathit{write}$$

where *skip* is an atomic action interpreted as the unit of M, a process that does nothing and terminates immediately. It is possible for the loop to not perform any write operations, as b might happen to always pick the left-hand branch; that is, the loop body fails to be guarded. Since M is a complete Elgot monad and not just completely iterative, the semantics of the loop is defined (by (2.1)) nonetheless.

3. Example: Simple Process Algebra

Baeten et al. [7] introduce a simple process algebra BSP (*Basic Sequential Processes*) featuring finite choice and action prefixing, and show that it can express all countable transition systems if unguarded recursion is allowed [7, Theorem 5.7.3]. The idea of the proof is to introduce variables X_{ik} for $i, k \in \mathbb{N}$ representing the k-th transition of the i-th state, with X_{i0} representing the i-th state itself, and (unguarded) recursive equations

$$X_{ik} = b_{ik}.X_{j(i,k),0} + X_{i,k+1} \tag{3.1}$$

where the k-th transition of the i-th state performs action b_{ik} and reaches the $j(i,k)$-th state. (The use of unguarded recursion is essential here, as guarded recursive definitions in BSP will clearly produce only finitely branching systems.) To model this phenomenon using the coinductive generalized resumption transformer, we take $\mathbb{T} = \mathcal{P}_{\omega_1}$, the countable powerset monad on **Set** (details are in Example 6.6), and the functor Σ generated by a-many unary operations where a is the set of actions; that is, $\Sigma X = a \times X$. We thus regard countable nondeterminism as the base effect, and add action prefixing via coinductive generalized resumptions. Representing variables $X_{i,k}$ by their indices (i,k), we then cast the definition (3.1) as an equation morphism

$$e : \mathbb{N} \times \mathbb{N} \to T_\Sigma(\mathbb{N} \times \mathbb{N}) \cong T_\Sigma(0 + \mathbb{N} \times \mathbb{N})$$

as follows. Eliding isomorphic conversions, we write elements of $T_\Sigma(\mathbb{N} \times \mathbb{N})$ as subsets of $(\mathbb{N} \times \mathbb{N}) + a \times T_\Sigma(\mathbb{N} \times \mathbb{N})$; in this notation,

$$e(i,k) = \{\mathsf{inr}(b_{ik}, \{\mathsf{inl}(j(i,k), 0)\}), \mathsf{inl}(i, k+1)\}.$$

Again, our result that $\mathbb{T}_\Sigma$ is a complete Elgot monad (Theorem 7.1) guarantees that this equation has a solution $e^\dagger$, and moreover that the choice $(-)^\dagger$ of solutions in $\mathbb{T}_\Sigma$ is uniquely determined as forming a complete Elgot monad and extending the usual structure of $\mathbb{T} = \mathcal{P}_{\omega_1}$ as a complete Elgot monad, which takes least fixed points. We emphasize that solutions in $\mathbb{T}_\Sigma$ do not arise as least fixed points; in particular, recall that simulation is only a preorder on $T_\Sigma X$.

4. Preliminaries

According to Moggi [28], a notion of computation can be formalized as a strong monad $\mathbb{T}$ over a Cartesian category (i.e. a category with finite products). In order to support the constructions occurring in the main object of study, we work in a *distributive category* $\mathbf{C}$, i.e. a category with finite products and coproducts (including a final and an initial object) such that the natural transformation

$$X \times Y + X \times Z \xrightarrow{[\mathsf{id}\times\mathsf{inl}, \mathsf{id}\times\mathsf{inr}]} X \times (Y + Z)$$

is an isomorphism [15], whose inverse we denote by $\mathsf{dist}_{X,Y,Z}$. Here we denote injections into binary coproducts by $\mathsf{inl} : X \to X + Y$, $\mathsf{inr} : Y \to X + Y$, while $\mathsf{fst} : X \times Y \to X$, $\mathsf{snd} : X \times Y \to Y$ denote projections from binary products; pairing is denoted by $\langle -, - \rangle$, and copairing of $f : X \to Z$, $g : Y \to Z$ by $[f, g] : X + Y \to Z$. Unique morphisms $A \to 1$ into the terminal object are written $!_X$, or just $!$. We write $|\mathbf{C}|$ for the class of objects of $\mathbf{C}$. Distributivity essentially allows using context variables in case expressions, i.e. in copairing. We omit indices on natural transformations where this is unlikely to cause confusion.

A *strong* functor on $\mathbf{C}$ is a functor $F : \mathbf{C} \to \mathbf{C}$ equipped with a natural transformation

$$\rho_{X,Y} : X \times FY \to F(X \times Y)$$

called *strength*, subject to the equations

$$\mathsf{snd} = (F\,\mathsf{snd})\,\rho \tag{STR$_1$}$$

$$(F\,\mathsf{assoc})\,\rho = \tau(\mathsf{id} \times \rho)\,\mathsf{assoc} \tag{STR$_2$}$$

where $\mathsf{assoc} : (X \times Y) \times Z \to X \times (Y \times Z)$ is the associativity isomorphism of products, explicitly, $\mathsf{assoc} = \langle \mathsf{fst}\,\mathsf{fst}, \langle \mathsf{snd}\,\mathsf{fst}, \mathsf{snd} \rangle\rangle$. A natural transformation $\alpha : F \to G$ between strong functors F, G (with the strength denoted ρ in both cases) is *strong* if it commutes with strength:

$$\begin{tikzcd}
X \times FY \arrow[r, "\mathsf{id}_X \times \alpha_Y"] \arrow[d, "\rho_{X,Y}"'] & X \times GY \arrow[d, "\rho_{X,Y}"] \\
F(X \times Y) \arrow[r, "\alpha_{X \times Y}"'] & G(X \times Y)
\end{tikzcd}$$

Recall that a monad $\mathbb{T}$ over $\mathbf{C}$ can be given by a *Kleisli triple* $(T, \eta, \text{-}^\star)$ where T is an endomap of $|\mathbf{C}|$ (in the following, we always denote monads and their functor parts by the same letter, with the former in blackboard bold), the *unit* η is a family of morphisms $\eta_X : X \to TX$, and the *Kleisli lifting* $(\text{-})^\star$ maps $f : X \to TY$ to $f^\star : TX \to TY$, subject to the equations

$$\eta^\star = \mathsf{id} \qquad\qquad f^\star\,\eta = f \qquad\qquad (f^\star\, g)^\star = f^\star\, g^\star.$$

This is equivalent to the presentation in terms of an endofunctor T with natural transformations unit and multiplication.

A *strong monad* is a monad whose underlying endofunctor is strong and the corresponding strength τ additionally satisfies the following additional coherence conditions [28] (with modifications reflecting the switch from monad multiplication to Kleisli lifting):

$$\tau(\mathsf{id} \times \eta) = \eta \tag{STR$_3$}$$

$$(\tau(\mathsf{id} \times f))^\star\, \tau = \tau(\mathsf{id} \times f^\star) \tag{STR$_4$}$$

The typing of the law (STR$_4$) capturing compatibility of the strength with Kleisli lifting is shown in the diagram

$$\begin{tikzcd}
X \times TY \arrow[r, "\mathsf{id}\times f^\star"] \arrow[d, "\tau"'] & X \times TZ \arrow[d, "\tau"] \\
T(X \times Y) \arrow[r, "(\tau(\mathsf{id}\times f))^\star"'] & T(X \times Z)
\end{tikzcd}$$

$$X \times Y \xrightarrow{\mathsf{id}\times f} X \times TZ \xrightarrow{\tau} T(X \times Z)$$

(For distinction, we denote strengths of monads by τ and strengths of functors by ρ throughout.) Strength enables interpreting programs over more than one variable, and allows for internalization of the Kleisli lifting, thus legitimating expressions like $\lambda x.\,(f(x))^\star : X \to (TY \to TZ)$ for $f : X \to (Y \to TZ)$, which encodes $\mathsf{curry}(\mathsf{uncurry}(f)^\star\,\tau)$. Strength is equivalent to the monad being enriched over $\mathbf{C}$ [25]; in particular, every monad on $\mathbf{Set}$ is strong. Henceforth we shall use the term 'monad' to mean 'strong monad' unless explicitly stated otherwise. We emphasize however that all our results remain valid under the removal of all strength assumptions and claims (that is, replacing the terms strong monad, strong functor, and strong natural transformation with monad, functor, and natural transformation, respectively, throughout).

The standard intuition for a monad $\mathbb{T}$ is to think of TX as the set of terms in some algebraic theory, with variables taken from X. In this view, the unit converts variables into terms, and a Kleisli lifting $f^\star$ applies a substitution $f : X \to TY$ to terms over X. In our setting, the 'terms' featuring here are often infinite; nevertheless, we sometimes call them *algebraic terms* for emphasis.

The *Kleisli category* $\mathbf{C}_\mathbb{T}$ of a monad $\mathbb{T}$ has the same objects as $\mathbf{C}$, and $\mathbf{C}$-morphisms $X \to TY$ as morphisms $X \to Y$. The identity on X in $\mathbf{C}_\mathbb{T}$ is η_X; and the *Kleisli composite* of $f : X \to TY$ and $g : Y \to TZ$ is $g^\star f$. A monad $\mathbb{T}$ *has rank* κ for a regular cardinal κ if T preserves κ-filtered colimits. On $\mathbf{Set}$, this condition means that T is determined by its values on sets of cardinality less than κ, in the sense that every element of TX comes from an element of TY for some subset $Y \subseteq X$ with $|Y| < \kappa$; intuitively, all operations of T have arity less than κ. A monad is *ranked* if it has some rank κ.

Example 4.1. As indicated in the introduction, in the main motivating examples the strong functor Σ plays the role of a signature of communication actions. Technical details are as follows. Assume that $\mathbf{C}$ has *exponentials* of the form X^b (for b ranging over a subset of $|\mathbf{C}|$), i.e. objects adjoint to Cartesian products $X \times b$, which means that for any X and Y, there is an isomorphism

$$\mathsf{curry}_{X,Y} : \mathsf{Hom}_{\mathbf{C}}(X \times b, Y) \cong \mathsf{Hom}_{\mathbf{C}}(X, Y^b),$$

natural in X and Y. We write $\mathsf{uncurry}_{X,Y}$ for the inverse map $\mathsf{curry}^{-1}_{X,Y}$. The *evaluation morphism* $\mathsf{ev}_X : X^b \times b \to X$ (natural in X) is obtained as $\mathsf{uncurry}_{X^b,X}(\mathsf{id}_{X^b})$.

It is easy to see that the functors $X \mapsto a \times X$ and $X \mapsto X^b$ are strong and that composites and coproducts of strong functors (as plain functors) are again strong functors. Hence, the functor

$$\Sigma X = \sum\nolimits_i a_i \times X^{b_i}$$

is strong. Intuitively (and formally correctly on $\mathbf{Set}$), ΣX can be seen as the set of flat terms over variables from X in the signature Σ, i.e. the elements of ΣX are of the form $f_i(c; x_1, \dots, x_{n_i})$ where f is a parametrized operation from the signature, c is a *parameter* from a_i, and $x_1, \dots, x_{n_i}$ are elements of X. The computational meaning of exponents in X^b is thus to capture a notion of arity of algebraic operations generating effects, e.g. $b = 2$ would correspond to binary operations such as nondeterministic choice. An example of an operation taking a parameter would be the operation of writing a value *val* to position *ind* of an array, $\mathit{update}(\langle \mathit{ind}, \mathit{val}\rangle; x)$ (see [35] for details).

A more general setup involves categories enriched over a symmetric monoidal closed category $\mathbf{V}$ whose objects are then treated as arities (and *coarities*, i.e. objects used for indexing families of operations) [23, 22]. One then replaces products with *tensors* and exponentials with *cotensors*.

Another example are functors on the topos of nominal sets and equivariant maps $\mathbf{Nom}$ built using constant functors, identity, coproducts, finite products, and the so-called *abstraction functor* $[\mathbb{A}](-)$, where $\mathbb{A}$ is a set of *names* and $[\mathbb{A}]X$ consists of pairs $(a, x) \in \mathbb{A} \times X$ modulo a natural notion of α-equivalence [34]. Such functors represent so-called *binding signatures*, whose operations may bind names, such as λ-abstraction or π-calculus-style fresh name binders ν; terms are then taken modulo α-equivalence. E.g. the λ-calculus syntax is rendered as the initial algebra of the functor $LX = \mathbb{A} + X \times X + [\mathbb{A}]X$ (see [17]).

5. Complete Elgot Monads

As indicated in the introduction, we will be interested in recursive definitions over a monad $\mathbb{T}$; abstractly, these are morphisms

$$f : X \to T(Y + X)$$

thought of as associating to each variable $x : X$ a definition $f(x)$ in the shape of an algebraic term from $T(Y + X)$, which thus employs parameters from Y as well as the defined variables from X. The latter amount to recursive calls of the definition. This notion is agnostic to what happens in the case of non-terminating recursion. For example, T might identify all non-terminating sequences of recursive calls into a single value $\bot$ signifying non-termination; at the other extreme, T might be a type of infinite trees that just records the tree of recursive calls explicitly.

To a recursive definition f as above, we wish to associate a solution

$$f^\dagger : X \to TY,$$

which amounts to a non-recursive definition of the elements of X as terms over Y only. As we do not assume any form of guardedness, this solution will in general fail to be unique. We thus require a coherent selection of solutions $f^\dagger$ for all equations f, where by coherent we mean that the selection satisfies a collection of well-established (quasi-)equational properties. Formally:

Definition 5.1. (Complete Elgot monads) A *complete Elgot monad* is a monad $\mathbb{T}$ equipped with an operator $\text{–}^\dagger$, called *iteration*, that assigns to each morphism $f : X \to T(Y + X)$ a morphism $f^\dagger : X \to TY$ such that the following laws hold:

- *fixpoint:* $[\eta, f^\dagger]^\star f = f^\dagger$;
- *naturality:* $g^\star f^\dagger = ([T\,\mathsf{inl}\; g, \eta\;\mathsf{inr}]^\star f)^\dagger$ for $g : Y \to TZ$;
- *codiagonal:* $(T[\mathsf{id}, \mathsf{inr}]\; g)^\dagger = g^{\dagger\dagger}$ for $g : X \to T((Y + X) + X)$;
- *uniformity:* $f\,h = T(\mathsf{id} + h)\,g$ implies $f^\dagger\,h = g^\dagger$ for $g : Z \to T(Y + Z)$ and $h : Z \to X$.

Additionally, iteration must be *compatible with strength* in the sense that

$$\tau(\mathsf{id} \times f^\dagger) = (T\,\mathsf{dist}\;\tau(\mathsf{id} \times f))^\dagger$$

for $f : X \to T(Y + X)$.

It has recently been shown [16, 18] that *dinaturality*, previously standardly included in axiomatizations of iteration [12], is in fact derivable from the other axioms in Definition 5.1. We record this for future reference:

Lemma 5.2 (Dinaturality)**.** *Every complete Elgot monad satisfies* dinaturality*:*

$$([\eta\,\mathsf{inl}, h]^\star\, g)^\dagger = [\eta, ([\eta\,\mathsf{inl}, g]^\star\, h)^\dagger]^\star\, g \text{ for } g : X \to T(Y + Z) \text{ and } h : Z \to T(Y + X).$$

Remark 5.3. The above definition is inspired by the axioms of parametrized uniform iterativity [38], which go back to Bloom and Ésik [12]. Adámek et al. [4] define *Elgot monads* by means of a slightly different system of axioms: the codiagonal (and dinaturality) laws are replaced with the *Bekić identity*. Both axiomatizations are however equivalent, which is essentially a result about iteration theories [12, Section 6.8]; we record a self-contained proof of this equivalence in Proposition 5.4 below. Moreover, the iteration operator in [4] is defined only for $f : X \to T(Y + X)$ with finitely presentable X, under the assumption that $\mathbf{C}$ is *locally finitely presentable*; hence our use of the term 'complete Elgot monad' instead of 'Elgot monad'. We have the impression that this difference is not technically essential but have not checked details for the finitary variant of our results.

Proposition 5.4 (Bekić identity)**.** *A complete Elgot monad* $\mathbb{T}$ *is equivalently a monad satisfying* fixpoint*,* naturality*,* uniformity *(as in Definition 5.1), and the* Bekić identity

$$((T\alpha)[f, g])^\dagger = [\eta, h^\dagger]^\star[\eta\,\mathsf{inr}, g^\dagger] \tag{Bekić}$$

where $g : X \to T((Z + Y) + X)$*,* $f : Y \to T((Z + Y) + X)$*,* $h = [\eta, g^\dagger]^\star f : Y \to T(Z + Y)$*, with* $\alpha : (A + B) + C \to A + (B + C)$ *being the obvious coproduct associativity morphism.*

Proof. Let us show that complete Elgot monads validate the Bekić identity. Let

$$u = T((\mathsf{id}+\mathsf{inl})+\mathsf{inr})[f,g] : Y+X \to T((Z+(Y+X))+(Y+X)).$$

By *codiagonal*,

$$(T[\mathsf{id},\mathsf{inr}]\,u)^\dagger = (u^\dagger)^\dagger. \tag{5.1}$$

Now the left-hand side of (5.1) simplifies to

$$\begin{aligned} &(T[\mathsf{id},\mathsf{inr}]\,T((\mathsf{id}+\mathsf{inl})+\mathsf{inr})[f,g])^\dagger \\ &= (T[\mathsf{id}+\mathsf{inl},\mathsf{inr}\,\mathsf{inr}][f,g])^\dagger \\ &= ((T\alpha)[f,g])^\dagger, \end{aligned}$$

i.e. to the left-hand side of the Bekić identity. Now observe that, by *uniformity* and *naturality*,

$$u^\dagger\,\mathsf{inr} = (T(\mathsf{id}+\mathsf{inl})+\mathsf{id})\,g)^\dagger = T(\mathsf{id}+\mathsf{inl})\,g^\dagger. \tag{5.2}$$

Therefore, the right-hand side of (5.1) can be rewritten in the form

$$\begin{aligned} (u^\dagger)^\dagger &= ([\eta,u^\dagger]^\star\,u)^\dagger && \text{// fixpoint}\\ &= ([\eta(\mathsf{id}+\mathsf{inl}),u^\dagger\,\mathsf{inr}][f,g])^\dagger \\ &= ([T(\mathsf{id}+\mathsf{inl})\,\eta, T(\mathsf{id}+\mathsf{inl})\,g^\dagger]^\star[f,g])^\dagger && \text{// 5.2}\\ &= (T(\mathsf{id}+\mathsf{inl})[\eta,g^\dagger]^\star[f,g])^\dagger \\ &= (T(\mathsf{id}+\mathsf{inl})[[\eta,g^\dagger]^\star\,f,g^\dagger])^\dagger && \text{// fixpoint}\\ &= ([\eta\,\mathsf{inl},\eta\,\mathsf{inr}\,\mathsf{inl}]^\star[[\eta,g^\dagger]^\star\,f,g^\dagger])^\dagger \\ &= [\eta,([\eta\,\mathsf{inl},[[\eta,g^\dagger]^\star\,f,g^\dagger]]^\star\,\eta\,\mathsf{inr}\,\mathsf{inl})^\dagger]^\star \\ &\quad [[\eta,g^\dagger]^\star\,f,g^\dagger] && \text{// dinaturality, Lemma 5.2}\\ &= [\eta,([\eta,g^\dagger]^\star\,f)^\dagger]^\star[[\eta,g^\dagger]^\star\,f,g^\dagger] \\ &= [([\eta,g^\dagger]^\star\,f)^\dagger,[\eta,([\eta,g^\dagger]^\star\,f)^\dagger]^\star\,g^\dagger] && \text{// fixpoint}\\ &= [h^\dagger,[\eta,h^\dagger]^\star\,g^\dagger] \\ &= [\eta,h^\dagger]^\star[\eta\,\mathsf{inr},g^\dagger], \end{aligned}$$

i.e. equals the right-hand side of the Bekić identity.

For the opposite direction, we need to show that the Bekić identity implies *codiagonal*. So let $k : X \to T((Y+X)+X)$. By the Bekić identity,

$$\begin{aligned} ((T\alpha)[k,k])^\dagger &= [\eta,([\eta,k^\dagger]^\star\,k)^\dagger]^\star[\eta\,\mathsf{inr},k^\dagger] \\ &= [\eta,k^{\dagger\dagger}]^\star[\eta\,\mathsf{inr},k^\dagger]. \end{aligned}$$

Thus, $((T\alpha)[k,k])^\dagger\,\mathsf{inl} = ((T\alpha)[k,k])^\dagger\,\mathsf{inr} = k^{\dagger\dagger}$. On the other hand, by *uniformity*,

$$((T\alpha)[k,k])^\dagger = (T[\mathsf{id},\mathsf{inr}]\,k)^\dagger[\mathsf{id},\mathsf{id}]$$

and therefore

$$k^{\dagger\dagger} = ((T\alpha)[k,k])^\dagger\,\mathsf{inr} = (T[\mathsf{id},\mathsf{inr}]\,k)^\dagger$$

as required. ∎

Given a complete Elgot monad $\mathbb{T}$, we can parametrize the iteration operator $–^{\dagger}$ with an additional argument to be carried over the recursion loop, i.e. we derive an operator $–^{\ddagger}$ sending $f : Z \times X \to T(Y + X)$ to $f^{\ddagger} : Z \times X \to TY$ by

$$f^{\ddagger} = \Big(T(\mathsf{snd} + \mathsf{id})(T\,\mathsf{dist})\,\tau_{Z,Y+X}\langle \mathsf{fst}, f\rangle\Big)^{\dagger}. \tag{5.3}$$

We call the derived operator $–^{\ddagger}$ *strong iteration*.

The key examples of complete Elgot monads are, one the one hand, so-called ω-continuous monads (Definition 5.5), and, on the other hand, extensions of complete Elgot monads, e.g. of ω-continuous monads, with free operations. The latter arise by application of the coinductive generalized resumption transformer as introduced in Section 6. We proceed to discuss ω-continuous monads, which are defined as having a suitable order-enrichment of their Kleisli category. Recall here that a category $\mathbf{D}$ is *enriched* over a category $\mathbf{V}$ [24] (in our application, $\mathbf{V}$ is Cartesian; in general, $\mathbf{V}$ only needs to be monoidal) if $\mathbf{D}$ has hom-objects from $\mathbf{V}$ in place of hom-sets, and both composition and selection of identities are morphisms in $\mathbf{V}$, with the usual equational laws of categories expressed as commuting diagrams in $\mathbf{V}$.

Definition 5.5. **(ω-continuous monad)** An *ω-continuous* monad consists of a monad $\mathbb{T}$ and an enrichment of the Kleisli category $\mathbf{C}_{\mathbb{T}}$ of $\mathbb{T}$ over the category $\mathbf{Cppo}$ of ω-complete partial orders with bottom and (nonstrict) continuous maps, satisfying the following conditions:

- strength is ω-continuous: $\tau(\mathsf{id} \times \bigsqcup_i f_i) = \bigsqcup_i(\tau(\mathsf{id} \times f_i))$;
- copairing in $\mathbf{C}_{\mathbb{T}}$ is ω-continuous in both arguments: $[\bigsqcup_i f_i, \bigsqcup_i g_i] = \bigsqcup_i [f_i, g_i]$;
- bottom elements are preserved by strength and by postcomposition in $\mathbf{C}_{\mathbb{T}}$: $\tau\,(\mathsf{id} \times \bot) = \bot$, $f^{\star}\,\bot = \bot$.

Example 5.6. Many of the standard computational monads on $\mathbf{Set}$ [28] are ω-continuous, including nontermination ($TX = X + 1$), nondeterminism ($TX = \mathcal{P}(X)$), and the nondeterministic state monad ($TX = \mathcal{P}(X \times S)^S$ for a set S of states). On $\mathbf{Cppo}$, lifting ($TX = X_{\bot}$) and the various power domain monads are ω-continuous.

Remark 5.7. As observed by Kock [25], monad strength is equivalent to enrichment over the base category. One consequence of this fundamental fact is that if $\mathbf{C}$ is enriched over the category $\mathbf{Cpo}$ of bottomless ω-complete partial orders and ω-continuous maps (i.e. $\mathbf{C}$ is an $\mathbf{O}$-category in the sense of Wand [42] and of Smyth and Plotkin [39]), with the bi-Cartesian closed structure enriched in the obvious sense, then $\mathbf{C}_{\mathbb{T}}$ is also enriched over $\mathbf{Cpo}$, since T, being a strong functor, is an $\mathbf{Cpo}$-functor (aka *locally continuous* functor [39]). Then $\mathbb{T}$ is ω-continuous in the sense of Definition 5.5 iff each $\mathsf{Hom}(X, TY)$ has a bottom element preserved by strength and postcomposition in $\mathbf{C}_{\mathbb{T}}$. This allows for incorporating numerous domain-theoretic examples by taking $\mathbf{C}$ to be a suitable category of predomains, and $\mathbb{T}$, in the simplest case, the *lifting monad* $TX = X_{\bot}$.

If $\mathbb{T}$ is an ω-continuous monad, then the endomap

$$h \mapsto [\eta, h]^{\star}\, f \tag{5.4}$$

on the hom-set $\mathsf{Hom}_{\mathbf{C}}(A, TB)$ is continuous because copairing and Kleisli composition in T are continuous, and hence has a least fixpoint by Kleene's fixpoint theorem. We can define an iteration operator by taking $f^{\dagger}$ to be this fixpoint; in other words, $f^{\dagger}$ is defined to be the least solution of the fixpoint law as per Definition 5.1. This yields

Theorem 5.8. *On every ω-continuous monad, defining iteration by taking least fixpoints determines a complete Elgot monad structure.*

This result is to be expected in the light of analogous facts known for Bloom and Ésik's *ω-continuous theories* [12, Theorem 8.2.15, Exercise 8.2.17].

Proof. Let $\mathbb{T}$ be an ω-continuous monad, and let $f^\dagger$ be the least fixpoint of (5.4).

Let us verify the axioms of complete Elgot monads one by one. To that end we employ the following *uniformity rule* for least fixpoints of continuous functionals [38]:

$$\frac{UF = GU \qquad U(\bot) = \bot}{U(\mu F) = \mu G} \tag{5.5}$$

Moreover, in several places below we use *fixpoint induction* to show that $f^\dagger \sqsubseteq g$ for given $f : A \to T(B + A)$ and $g : A \to TB$: Since $f^\dagger$ is a supremum of the chain $(F^i(\bot))_{i\in\mathbb{N}}$ where $F : (A \to TB) \to (A \to TB)$ is the functional defined by

$$F(h) = [\eta, h]^\star f^\dagger,$$

$f^\dagger \sqsubseteq g$ follows as soon as we prove $F^i(\bot) \sqsubseteq g$ for all $i \in \mathbb{N}$, a claim that we typically prove by induction on i. The induction base $i = 0$ is always trivial, so we consistently do only the inductive step. More generally, we can apply the same principle to conclude $\alpha(f^\dagger) \sqsubseteq r$, for given $r : C \to TD$ and a function $\alpha : (A \to TB) \to (C \to TD)$, from $\alpha(F^i(\bot)) \sqsubseteq r$ for all i, provided that α is ω-continuous, a condition that will always be immediate from our assumptions. In the more general case, we need to pay attention to the base case, typically be ensuring that α preserves $\bot$.

- *Fixpoint.* This holds by definition.
- *Naturality.* In (5.5) take $F(u) = [\eta, u]^\star f$, $G(u) = [\eta, u]^\star[(T\,\mathsf{inl})g, \eta\,\mathsf{inr}]^\star f$ and $U(u) = g^\star u$. By definition, $U(\bot) = \bot$, $\mu F = f^\dagger$, $\mu G = ([(T\,\mathsf{inl})g, \eta\,\mathsf{inr}]^\star f)^\dagger$. Then we have

$$U(F(u)) = g^\star[\eta, u]^\star f = [\eta, g^\star u]^\star[(T\,\mathsf{inl})g, \eta\,\mathsf{inr}]^\star f = G(U(u)).$$

Therefore, by (5.5), $g^\star f^\dagger = U(\mu F) = \mu G = ([(T\,\mathsf{inl})g, \eta\,\mathsf{inr}]^\star f)^\dagger$.

- *Codiagonal.* Recall that we are claiming that

$$(T[\mathsf{id}, \mathsf{inr}]g)^\dagger = (g^\dagger)^\dagger$$

with $g : A \to T((B + A) + A)$. We first show that $g^{\dagger\dagger}$ is a fixpoint of the functional defining the left-hand side as a least fixpoint, thus proving $\sqsubseteq$. That is, we have to show that

$$g^{\dagger\dagger} = [\eta, g^{\dagger\dagger}]^\star T[\mathsf{id}, \mathsf{inr}]g. \tag{5.6}$$

We proceed as follows:

$$\begin{aligned}
g^{\dagger\dagger} &= [\eta, g^{\dagger\dagger}]^\star g^\dagger && \text{// fixpoint}\\
&= [\eta, g^{\dagger\dagger}]^\star[\eta, g^\dagger]^\star g && \text{// fixpoint}\\
&= [[\eta, g^{\dagger\dagger}], [\eta, g^{\dagger\dagger}]^\star g^\dagger]^\star g\\
&= [[\eta, g^{\dagger\dagger}], g^{\dagger\dagger}]^\star g && \text{// fixpoint}\\
&= [[\eta, g^{\dagger\dagger}], [\eta, g^{\dagger\dagger}]\,\mathsf{inr}]^\star g\\
&= [\eta, g^{\dagger\dagger}]^\star\, T[\mathsf{id}, \mathsf{inr}]g.
\end{aligned}$$

For the converse inequality, we use fixpoint induction. So let $f \sqsubseteq (T[\mathsf{id},\mathsf{inr}]\ g)^\dagger$. We have to show

$$[\eta, f]^\star g^\dagger \sqsubseteq (T[\mathsf{id},\mathsf{inr}]g)^\dagger.$$

We establish this by a second fixpoint induction on the occurrence of $g^\dagger$ on the left hand side. For the base case, just recall that Kleisli composition from the left preserves $\bot$. For the inductive step, assume that $[\eta, f]^\star h \sqsubseteq (T[\mathsf{id},\mathsf{inr}]g)^\dagger$, with $h : A \to T(B + A)$; we have to show that

$$[\eta, f]^\star[\eta, h]^\star g \sqsubseteq (T[\mathsf{id},\mathsf{inr}]g)^\dagger.$$

We calculate as follows:

$$\begin{aligned}
[\eta, f]^\star[\eta, h]^\star g &= [[\eta, f], [\eta, f]^\star h]^\star g \\
&\sqsubseteq [[\eta, f], (T[\mathsf{id},\mathsf{inr}]g)^\dagger]^\star g && \text{// inner IH} \\
&\sqsubseteq [[\eta, (T[\mathsf{id},\mathsf{inr}]g)^\dagger], (T[\mathsf{id},\mathsf{inr}]g)^\dagger]^\star g && \text{// outer IH} \\
&= [\eta, (T[\mathsf{id},\mathsf{inr}]g)^\dagger] T[\mathsf{id},\mathsf{inr}]g \\
&= (T[\mathsf{id},\mathsf{inr}]g)^\dagger. && \text{// fixpoint}
\end{aligned}$$

- *Uniformity.* Let $f : A \to T(X + A)$, $g : B \to T(X + B)$, $h : B \to A$ and assume that $f\,h = T(\mathsf{id} + h)\,g$. Let us define $G(u) = [\eta, u]^\star g$, $F(u) = [\eta, u]^\star f$, and $U(u) = u\,h$. Then $U(\bot) = \bot$ and

$$\begin{aligned}
UF(u) &= [\eta, u]^\star f\,h \\
&= [\eta, u]^\star T(\mathsf{id} + h)\,g \\
&= [\eta, u\,h]^\star\,g \\
&= GU(u).
\end{aligned}$$

Therefore by (5.5), $f^\dagger\,h = U(\mu F) = \mu G = g^\dagger$.

To prove compatibility of strength and iteration, we proceed by first showing

$$((T\,\mathsf{dist})\tau(\mathsf{id} \times f))^\dagger \sqsubseteq \tau(\mathsf{id} \times f^\dagger).$$

First observe that, for any $g : A \to TB$,

$$\begin{tikzcd}
C \times (B + A) \arrow[r, shift left, "\mathsf{dist}"] \arrow[d, "\mathsf{id}\times[\eta,g]"'] & C \times B + C \times A \arrow[l, shift left, "\mathsf{dist}^{-1}"] \arrow[d, "{[\eta,\tau(\mathsf{id}\times g)]}"] \\
C \times TB \arrow[r, "\tau"] & T(C \times B).
\end{tikzcd} \tag{5.7}$$

This is easily checked componentwise starting from $C \times B + C \times A$ and using the fact that by definition $\mathsf{dist}^{-1} = [\mathsf{id} \times \mathsf{inl}, \mathsf{id} \times \mathsf{inr}]$. Then we have

$$\begin{aligned}
&\tau(\mathsf{id} \times f^\dagger) \\
&= \tau(\mathsf{id} \times [\eta, f^\dagger]^\star f) \\
&= \tau(\mathsf{id} \times [\eta, f^\dagger]^\star)(\mathsf{id} \times f) \\
&= (\tau(\mathsf{id} \times [\eta, f^\dagger]))^\star \tau(\mathsf{id} \times f) && \text{// STR}_4 \\
&= ([\eta, \tau(\mathsf{id} \times f^\dagger)]\,\mathsf{dist})^\star \tau(\mathsf{id} \times f) && \text{// 5.7}
\end{aligned}$$

$$= [\eta, \tau(\mathsf{id} \times f^\dagger)]^\star (T\,\mathsf{dist})\tau(\mathsf{id} \times f).$$

Therefore, $\tau(\mathsf{id} \times f^\dagger)$ is a fixed point of the functional defining $((T\,\mathsf{dist})\tau(\mathsf{id} \times f))^\dagger$ as a least fixpoint and the inequality above holds. The converse inequality,

$$\tau(\mathsf{id} \times f^\dagger) \sqsubseteq ((T\,\mathsf{dist}\,\tau)(\mathsf{id} \times f))^\dagger,$$

is shown by fixpoint induction. For the base case, we calculate the left hand side:

$$\tau(\mathsf{id} \times \bot) = \tau\langle \mathsf{fst}, \bot\,\mathsf{snd}\rangle = \tau\langle \mathsf{fst}, \bot\,\mathsf{fst}\rangle = \tau\langle \mathsf{id}, \bot\rangle\,\mathsf{fst} = \bot\,\mathsf{fst} = \bot.$$

For the inductive step, assume that $\tau(\mathsf{id} \times g) \sqsubseteq (T\,\mathsf{dist}\,\tau(\mathsf{id} \times f))^\dagger$. We can then calculate

$$\begin{aligned}
&\tau(\mathsf{id} \times [\eta, g]^\star f) \\
&= \tau(\mathsf{id} \times [\eta, g]^\star)(\mathsf{id} \times f) \\
&= (\tau(\mathsf{id} \times [\eta, g]))^\star \tau(\mathsf{id} \times f) && \text{// STR}_4 \\
&= ([\eta, \tau(\mathsf{id} \times g)]\,\mathsf{dist})^\star \tau(\mathsf{id} \times f) && \text{// 5.7} \\
&\sqsubseteq ([\eta, (T\,\mathsf{dist}\,\tau(\mathsf{id} \times f))^\dagger]\,\mathsf{dist})^\star \tau(\mathsf{id} \times f) \\
&= [\eta, (T\,\mathsf{dist}\,\tau(\mathsf{id} \times f))^\dagger]^\star T\,\mathsf{dist}\,\tau(\mathsf{id} \times f) \\
&= (T\,\mathsf{dist}\,\tau(\mathsf{id} \times f))^\dagger
\end{aligned}$$

which completes the proof. □

Every complete Elgot monad $\mathbb{T}$ can express *unproductive divergence* as the generic effect

$$\bot_{X,Y} = \left(X \xrightarrow{\eta\,\mathsf{inr}} T(Y+X)\right)^\dagger : X \to TY.$$

This computation never produces any effects, i.e. behaves like a deadlock. If $\mathbb{T}$ is ω-continuous, then unproductive divergence coincides with the least element of $\mathsf{Hom}(X, TY)$, for which reason we use the same symbol $\bot_X$, but in general, there is no ordering in which unproductive divergence could be a least element.

Lemma 5.9. *Unproductive divergence is* constant*, i.e. for $f : Z \to X$, we have $\bot_{X,Y} f = \bot_{Z,Y}$, and* coconstant*, i.e. for $h : Y \to TW$ we have $h^\star \bot_{X,Y} = \bot_{X,W}$.*

Proof. Constancy: We have to show $(\eta_{Y+X}\,\mathsf{inr})^\dagger f = (\eta_{Y+Z}\,\mathsf{inr})^\dagger$. By uniformity, it suffices to show that $\eta_{Y+X}\,\mathsf{inr}\, f = T(\mathsf{id} + f)\,\eta_{Y+Z}\,\mathsf{inr}$. We calculate the right-hand side:

$$\begin{aligned}
T(\mathsf{id} + f)\eta_{Y+Z}\,\mathsf{inr} &= \eta_{Y+X}(\mathsf{id} + f)\,\mathsf{inr} && \text{// naturality of } \eta \\
&= \eta_{Y+X}\,\mathsf{inr}\, f.
\end{aligned}$$

Coconstancy: We have

$$\begin{aligned}
h^\star \bot_{X,Y} &= h^\star(\eta\,\mathsf{inr}_{Y+X})^\dagger \\
&= ([T\,\mathsf{inl}\,h, \eta_{W+X}\,\mathsf{inr}]^\star \eta_{Y+X}\,\mathsf{inr})^\dagger && \text{// naturality of } \dagger \\
&= (\eta_{W+X}\,\mathsf{inr})^\dagger = \bot_{X,W}.
\end{aligned}$$
□

The following lemma shows that there can be only one unproductive divergence:

Lemma 5.10. *Let $e : X \to T(Y+X)$ have the form $e = \eta\,\mathsf{inr}\,u$ for $u : X \to X$. Then $e^\dagger = \bot_{X,Y}$.*

Proof. By constancy, $\bot_{X,Y} = \bot_{1,Y}\,!_X$, so we are to show $(\eta_{Y+1}\,\mathsf{inr})^\dagger\,!_X = e^\dagger$. By uniformity, it suffices to show $\eta_{Y+1}\,\mathsf{inr}\,!_X = T(\mathsf{id} + !_X)\eta\,\mathsf{inr}\,u$, which is immediate by naturality of η. □

6. The Coinductive Generalized Resumption Transformer

We proceed to recall the definition of the coinductive generalized resumption transformer [32]. One of our main results will be stability of the class of complete Elgot monads under this construction (Theorem 7.1). In the remainder of the paper, we work with the following set of standing assumptions.

Assumption 6.1. We fix
- a distributive category $\mathbf{C}$;
- a strong functor $\Sigma : \mathbf{C} \to \mathbf{C}$ with strength ρ;
- a strong monad $\mathbb{T}$ on $\mathbf{C}$ with strength τ;

and assume that *the final coalgebra $\nu\gamma.\, T(X + \Sigma\gamma)$ of $T(X + \Sigma)$ exists for all $X \in |\mathbf{C}|$.*

As indicated in the introduction, we think of Σ as specifying a signature of communication actions, and of T as encapsulating a notion of side-effect.

We can then define a functor T_Σ whose action on objects is given by

$$T_\Sigma X = \nu\gamma.\, T(X + \Sigma\gamma).$$

Intuitively, $T_\Sigma X$ is a type of possibly non-terminating computation trees, in which each step triggers a computational effect specified by T, and then either terminates with a result in X or branches according to an operation from the signature represented by Σ, with arguments being again computation trees.

Remark 6.2. There are two broad classes of models satisfying Assumption 6.1:
- $\mathbf{C}$ is a locally presentable category and $\mathbb{T}$ is ranked; or
- $\mathbf{C}$ is $\mathbf{Cpo}$-enriched and has colimits of ω-chains, and T is ω-continuous (Remark 5.7).

Satisfaction of Assumption 6.1 in the first case follows from the fact that categories of coalgebras for accessible functors over locally presentable categories are again locally presentable, in particular complete [6, Exercise 2.j, Chapter 2]. This covers most of the interesting choices of base categories, such as **Set**, **Cpo**, various categories of predomains, and presheaf categories, as well as almost all computationally relevant monads [28, 35]. The fact that Assumption 6.1 is satisfied in the second case follows from Barr's work on algebraically compact functors [8, Theorem 5.4], which also implies that the greatest fixed points of interest coincide with least fixed points. One example covered by the second clause but not by the first one is the continuation monad $TX = (X \to R) \to R$ on $\mathbf{Cpo}$, provided that R has a least element.

Let

$$\mathsf{out}_X : T_\Sigma X \to T(X + \Sigma T_\Sigma X)$$

be the final coalgebra structure, and let $\mathsf{coit}(g) : Y \to T_\Sigma X$ denote the final morphism induced by a coalgebra $g : Y \to T(X + \Sigma Y)$:

$$\begin{tikzcd}
Y \arrow[r, "\mathsf{coit}(g)"] \arrow[d, "g"'] & T_\Sigma X \arrow[d, "\mathsf{out}_X"] \\
T(X + \Sigma Y) \arrow[r, "T(X+\Sigma\,\mathsf{coit}(g))"] & T(X + \Sigma T_\Sigma X).
\end{tikzcd}$$

Intuitively, $\mathsf{coit}(g)$ encapsulates (in $T_\Sigma X$) a computation tree that begins by executing g, terminates in a leaf of type X if g does, and otherwise (co-)recursively continues to execute g, forming a new tree node for each recursive call. By Lambek's lemma, out_X is an isomorphism. As we see below, it is also natural in X. Thus, T maps into T_Σ via

$$\mathsf{ext} = \Big(T \xrightarrow{T\,\mathsf{inl}} T(\mathrm{Id} + \Sigma T_\Sigma) \xrightarrow{\mathsf{out}^{-1}} T_\Sigma\Big). \tag{6.1}$$

We record explicitly that T_Σ is a strong monad:

Theorem 6.3. *Given a monad $\mathbb{T}$, T_Σ is the functorial part of a monad $\mathbb{T}_\Sigma$, with the strong monad structure denoted τ^ν, η^ν, and $(-)^{\maltese}$ (for Kleisli star) and characterized by the following properties.*

(1) *The unit $\eta^\nu : X \to T_\Sigma X$ is defined by $\mathsf{out}\,\eta^\nu = \eta\,\mathsf{inl}$ (i.e. $\eta^\nu = \mathsf{out}^{-1}\,\eta\,\mathsf{inl}$).*
(2) *Given $f : X \to T_\Sigma Y$, the Kleisli lifting $f^{\maltese} : T_\Sigma X \to T_\Sigma Y$ is the unique solution of the equation*
$$\mathsf{out}\, f^{\maltese} = [\mathsf{out}\, f, \eta\,\mathsf{inr}\,\Sigma f^{\maltese}]^\star\,\mathsf{out}\,. \tag{6.2}$$
(3) *Given $f : X \to T_\Sigma Y$, let $g = [f, \eta^\nu] : X + Y \to T_\Sigma Y$; then $g^{\maltese}$ is a final morphism from $(T_\Sigma(X+Y), [T(\mathsf{id} + \Sigma T_\Sigma\,\mathsf{inr})\,\mathsf{out}\,g, \eta\,\mathsf{inr}]^\star\,\mathsf{out} : T_\Sigma(X+Y) \to T(Y + \Sigma T_\Sigma(X+Y)))$ to $(T_\Sigma Y, \mathsf{out}_Y)$, i.e.*
$$g^{\maltese} = \mathsf{coit}\Big([T(\mathsf{id} + \Sigma T_\Sigma\,\mathsf{inr})\,\mathsf{out}\,g, \eta\,\mathsf{inr}]^\star\,\mathsf{out}\Big). \tag{6.3}$$
(4) *The strength $\tau^\nu : X \times T_\Sigma Y \to T_\Sigma(X \times Y)$ is the unique solution of*
$$\mathsf{out}\,\tau^\nu = T(\mathsf{id} + \Sigma\tau^\nu)(T\delta)\,\tau(\mathsf{id} \times \mathsf{out}) \tag{6.4}$$
with $\delta : X \times (Y + \Sigma Z) \to X \times Y + \Sigma(X \times Z)$ being the transformation $\delta = (\mathsf{id} + \rho)\,\mathsf{dist}$ where $\rho_{X,Y} : X \times \Sigma Y \to \Sigma(X \times Y)$ is the strength of Σ.

This justifies calling $\mathbb{T}_\Sigma$ the *coinductive generalized resumption monad* (over $\mathbb{T}$). The proof of Theorem 6.3 is facilitated by the fact that $T(X + \Sigma)$ can be shown to be a *parametrized monad*, which implies that $\mathbb{T}_\Sigma$ is a monad [40, Theorems 3.7 and 3.9]. Alternatively, the fact that $\mathbb{T}_\Sigma$ is a monad can be read off directly from the results of [32]. What is new here is that we show that $\mathbb{T}_\Sigma$ is, in fact, strong, and hence supports an interpretation of Moggi's computational metalanguage [28]. This amounts to showing that the strength defined in the last item satisfies the requisite laws in p. 6. One preliminary fact of potentially independent interest used in the proof of these laws is

Lemma 6.4. *The object assignment $X \mapsto T_\Sigma X$ extends to a functor T_Σ, and $\mathsf{out} : T_\Sigma \to T(\mathrm{Id} + \Sigma T_\Sigma)$ then becomes a natural transformation. For any functor $G : \mathbf{B} \to \mathbf{C}$, $\mathsf{out}_G : T_\Sigma G \to T(G + \Sigma T_\Sigma G)$ is a final $T(G + \Sigma(-))$-coalgebra in $[\mathbf{B}, \mathbf{C}]$.*

Proof. Functoriality follows from the fact that, as stated in Theorem 6.3 and proved independently from this lemma in the proof of the theorem, T_Σ carries a monad structure. That is, $T_\Sigma f = (\eta^\nu f)^{\maltese}$, so by the description of $\maltese$ we have

$$\begin{aligned}
\mathsf{out}\,T_\Sigma f &= [\mathsf{out}\,\eta^\nu f, \eta\,\mathsf{inr}\,\Sigma T_\Sigma f]^\star\,\mathsf{out} \\
&= [\eta\,\mathsf{inl}\,f, \eta\,\mathsf{inr}\,\Sigma T_\Sigma f]^\star\,\mathsf{out} \\
&= T[\mathsf{inl}\,f, \mathsf{inr}\,\Sigma T_\Sigma f]\,\mathsf{out} \\
&= T(f + \Sigma T_\Sigma f)\,\mathsf{out},
\end{aligned}$$

i.e. out is natural.

To show finality, let $\beta : F \to T(G + \Sigma F)$ be a natural transformation. We define the universal arrow $f : F \to T_\Sigma G$ componentwise by the equation

$$\mathsf{out}\, f_X = T(\mathsf{id} + \Sigma f_X)\beta_X$$

using finality of the components $\mathsf{out}_{GX} : T_\Sigma GX \to T(GX + \Sigma T_\Sigma GX)$. We have to show that f is natural (uniqueness is clear). So let $g : X \to Y$; we have to show $f_Y Fg = (T_\Sigma Gg) f_X$. Note that we have a $T(GY + F)$-coalgebra

$$FX \xrightarrow{\beta_X} T(GX + \Sigma FX) \xrightarrow{T(Gg+\mathsf{id})} T(GY + \Sigma FX);$$

we show that both $f_Y Fg$ and $(T_\Sigma Gg) f_X$ are coalgebra morphisms into $T_\Sigma GY$ for $T(Gg + \mathsf{id})\beta_X$. On the one hand, we have

$$\begin{aligned} \mathsf{out}\, f_Y Fg &= T(\mathsf{id} + \Sigma f_Y)\beta_Y Fg && \text{// definition of } f_Y \\ &= T(\mathsf{id} + \Sigma f_Y)T(Gg + \Sigma Fg)\beta_X && \text{// naturality of } \beta \\ &= T(Gg + \Sigma(f_Y Fg))\beta_X. \end{aligned}$$

On the other hand,

$$\begin{aligned} \mathsf{out}(T_\Sigma Gg) f_X &= T(Gg + \Sigma T_\Sigma Gg)\,\mathsf{out}\, f_X && \text{// naturality of } \mathsf{out} \\ &= T(Gg + \Sigma T_\Sigma Gg)T(\mathsf{id} + \Sigma f_X)\beta_X && \text{// definition of } f_X \\ &= T(Gg + \Sigma((T_\Sigma Gg) f_X))\beta_X. \end{aligned}$$

Using the fact that there is unique morphism from a given coalgebra to the final one, we conclude that indeed $f_Y Fg = (T_\Sigma Gg) f_X$. □

Proof of Theorem 6.3. Since $T(X + \Sigma)$ extends to a *parametrized monad*, as shown in [40, Theorems 3.7 and 3.9], $\mathbb{T}_\Sigma$ is a monad whose Kleisli lifting is uniquely characterized by (6.2). What is missing is to show that $\mathbb{T}_\Sigma$ is a strong monad, as we need here. Let g be defined as in clause (3) of the theorem, and let us first show (6.3). By definition, $g^{\maltese}$ is the unique morphism making the following diagram commute:

$$\begin{tikzcd} T_\Sigma(X+Y) \arrow[r, "g^{\maltese}"] \arrow[d, "\mathsf{out}"'] & T_\Sigma Y \arrow[d, "\mathsf{out}"] \\ T(X+Y+\Sigma T_\Sigma(X+Y)) \arrow[r, "{[\mathsf{out}\, g, \eta\, \mathsf{inr}\, \Sigma g^{\maltese}]^\star}"] & T(Y + \Sigma T_\Sigma Y) \end{tikzcd}$$

We then have on the one hand,

$$\begin{aligned} \mathsf{out}\, g^{\maltese} &= [\mathsf{out}[f, \eta^\nu],\, \eta\, \mathsf{inr}\, \Sigma g^{\maltese}]^\star\, \mathsf{out} && \text{// definition of } \text{–}^{\maltese} \\ &= [[\mathsf{out}\, f, \mathsf{out}\, \eta^\nu], \eta\, \mathsf{inr}\, \Sigma g^{\maltese}]^\star\, \mathsf{out} \\ &= [[\mathsf{out}\, f, \eta\, \mathsf{inl}], \eta\, \mathsf{inr}\, \Sigma g^{\maltese}]^\star\, \mathsf{out} && \text{// definition of } \eta^\nu \end{aligned}$$

and also on the other hand,

$$\begin{aligned} & T(\mathsf{id} + \Sigma g^{\maltese})\ [T(\mathsf{id} + \Sigma T_\Sigma\, \mathsf{inr})\,\mathsf{out}\, g, \eta\, \mathsf{inr}]^\star\, \mathsf{out} \\ &\quad = [T(\mathsf{id} + \Sigma(g^{\maltese} T_\Sigma\, \mathsf{inr}))[\mathsf{out}\, f, \mathsf{out}\, \eta^\nu], \eta\, \mathsf{inr}\, \Sigma g^{\maltese}]^\star\, \mathsf{out} \\ &\quad = [[\mathsf{out}\, f, \eta\, \mathsf{inl}], \eta\, \mathsf{inr}\, \Sigma g^{\maltese}]^\star\, \mathsf{out}, \end{aligned}$$

i.e. indeed $g^{\maltese}$ satisfies the characteristic property of the final morphism (6.3).

We proceed to prove that $\mathbb{T}_\Sigma$ is strong. We define the strength τ^ν as the unique final coalgebra morphism shown in the following diagram:

$$\begin{tikzcd}
X \times T_\Sigma Y \arrow[r, "(T\delta)\tau(\mathsf{id}\times\mathsf{out})"] \arrow[d, "\tau^\nu"'] & T(X \times Y + \Sigma(X \times T_\Sigma Y)) \arrow[d, "T(\mathsf{id}+\Sigma\tau^\nu)"] \\
T_\Sigma(X \times Y) \arrow[r, "\mathsf{out}"] & T(X \times Y + \Sigma T_\Sigma(X \times Y))
\end{tikzcd}$$

That is, τ^ν is the unique solution of equation $\mathsf{out}\,\tau^\nu = T(\mathsf{id} + \Sigma\tau^\nu)(T\delta)\tau(\mathsf{id} \times \mathsf{out})$. By Lemma 6.4, τ^ν is a composite of natural transformations and hence itself natural. Let us check the axioms of strength from p. 6.

- (STR_1) The identity $\mathsf{snd} = (T_\Sigma\,\mathsf{snd})\tau^\nu$ follows from $T_\Sigma\langle !, \mathsf{id}\rangle\,\mathsf{snd} = \tau^\nu$ where ! is a suitable terminal morphism $X \to 1$, since obviously $\mathsf{snd} = (T_\Sigma\,\mathsf{snd})T_\Sigma\langle !, \mathsf{id}\rangle\,\mathsf{snd}$. Since τ^ν is uniquely defined by the corresponding characteristic identity (6.4), it suffices to show that $T_\Sigma\langle !, \mathsf{id}\rangle\,\mathsf{snd}$ satisfies the same identity. Indeed,

$$\begin{aligned}
&T(\mathsf{id}+ \Sigma(T_\Sigma\langle !, \mathsf{id}\rangle\,\mathsf{snd}))(T\delta)\tau(\mathsf{id} \times \mathsf{out}) \\
&\quad = T(\langle !, \mathsf{id}\rangle\,\mathsf{snd} + \Sigma(T_\Sigma\langle !, \mathsf{id}\rangle\,\mathsf{snd}))(T\delta)\tau(\mathsf{id} \times \mathsf{out}) \\
&\quad = T(\langle !, \mathsf{id}\rangle + \Sigma T_\Sigma\langle !, \mathsf{id}\rangle)T(\mathsf{snd} + \Sigma\,\mathsf{snd})(T\delta)\tau(\mathsf{id} \times \mathsf{out}) \\
&\quad = T(\langle !, \mathsf{id}\rangle + \Sigma T_\Sigma\langle !, \mathsf{id}\rangle)T(\mathsf{snd} + \mathsf{snd})(T\,\mathsf{dist})\tau(\mathsf{id} \times \mathsf{out}) && /\!/\ \mathrm{STR}_1 \text{ for } \rho \\
&\quad = T(\langle !, \mathsf{id}\rangle + \Sigma T_\Sigma\langle !, \mathsf{id}\rangle)(T\,\mathsf{snd})\tau(\mathsf{id} \times \mathsf{out}) && /\!/\ \text{defintion of } \mathsf{dist} \\
&\quad = T(\langle !, \mathsf{id}\rangle + \Sigma T_\Sigma\langle !, \mathsf{id}\rangle)\,\mathsf{out}\,\mathsf{snd} && /\!/\ \mathrm{STR}_1 \text{ for } \tau \\
&\quad = \mathsf{out}(T_\Sigma\langle !, \mathsf{id}\rangle)\,\mathsf{snd}\,. && /\!/\ \text{naturality of } \tau
\end{aligned}$$

- (STR_2) In order to prove that $(T_\Sigma\,\mathsf{assoc})\tau^\nu = \tau^\nu(\mathsf{id} \times \tau^\nu)\,\mathsf{assoc} : (X \times Y) \times T_\Sigma Z \to T_\Sigma((X\times Y)\times Z)$, it suffices to show that $(T_\Sigma\,\mathsf{assoc}^{-1})\tau^\nu(\mathsf{id}\times\tau^\nu)\,\mathsf{assoc}$ satisfies the characteristic identity (6.4) for τ^ν, i.e.

$$\mathsf{out}(T_\Sigma\,\mathsf{assoc}^{-1})\tau^\nu(\mathsf{id}\times\tau^\nu)\,\mathsf{assoc} = T(\mathsf{id}+\Sigma((T_\Sigma\,\mathsf{assoc}^{-1})\tau^\nu(\mathsf{id}\times\tau^\nu)\,\mathsf{assoc}))(T\delta)\tau(\mathsf{id}\times\mathsf{out}).$$

We calculate, transforming the left hand side,

$$\begin{aligned}
&\mathsf{out}(T_\Sigma\,\mathsf{assoc}^{-1})\tau^\nu(\mathsf{id} \times \tau^\nu)\,\mathsf{assoc} \\
&\quad = T(\mathsf{assoc}^{-1} + \Sigma T_\Sigma\,\mathsf{assoc}^{-1})\,\mathsf{out}\,\tau^\nu(\mathsf{id} \times \tau^\nu)\,\mathsf{assoc} && /\!/\ \text{naturality of } \mathsf{out} \\
&\quad = T(\mathsf{assoc}^{-1} + \Sigma T_\Sigma\,\mathsf{assoc}^{-1}) \\
&\qquad\quad T(\mathsf{id} + \Sigma\tau^\nu)(T\delta)\tau(\mathsf{id} \times \mathsf{out})(\mathsf{id} \times \tau^\nu)\,\mathsf{assoc} && /\!/\ \text{definition of } \tau^\nu \\
&\quad = T(\mathsf{assoc}^{-1} + \Sigma((T_\Sigma\,\mathsf{assoc}^{-1})\tau^\nu)) \\
&\qquad\quad (T\delta)\tau(\mathsf{id} \times T(\mathsf{id} + \Sigma\tau^\nu)(T\delta)\tau(\mathsf{id} \times \mathsf{out}))\,\mathsf{assoc} && /\!/\ \text{definition of } \tau^\nu \\
&\quad = T(\mathsf{assoc}^{-1} + \Sigma((T_\Sigma\,\mathsf{assoc}^{-1})\tau^\nu)) \\
&\qquad\quad (T\delta)T(\mathsf{id} \times (\mathsf{id} + \Sigma\tau^\nu)\delta)\tau(\mathsf{id} \times \tau(\mathsf{id} \times \mathsf{out}))\,\mathsf{assoc} && /\!/\ \text{naturality of } \tau.
\end{aligned}$$

and then continue to transform the last part of the term:

$$\begin{aligned}
&\tau(\mathsf{id} \times\ \tau(\mathsf{id} \times \mathsf{out}))\,\mathsf{assoc} \\
&\quad = \tau(\mathsf{id} \times \tau)(\mathsf{id} \times (\mathsf{id} \times \mathsf{out}))\,\mathsf{assoc}
\end{aligned}$$

$$
\begin{aligned}
&= \tau(\mathsf{id} \times \tau)\,\mathsf{assoc}((\mathsf{id} \times \mathsf{id}) \times \mathsf{out}) && \text{// naturality of } \mathsf{assoc}\\
&= (T\,\mathsf{assoc})\tau(\mathsf{id} \times \mathsf{out}) && \text{// } \textsc{str}_2 \text{ for } \tau
\end{aligned}
$$

(contracting a product of identities into an identity in the last step). Summing up, it remains to show that

$$
\begin{aligned}
&T(\mathsf{assoc}^{-1} + \Sigma((T_\Sigma\,\mathsf{assoc}^{-1})\tau^\nu))(T\delta)T(\mathsf{id} \times (\mathsf{id} + \Sigma\tau^\nu)\delta)(T\,\mathsf{assoc})\tau(\mathsf{id} \times \mathsf{out})\\
&= T(\mathsf{id} + \Sigma((T_\Sigma\,\mathsf{assoc}^{-1})\tau^\nu(\mathsf{id} \times \tau^\nu)\,\mathsf{assoc}))(T\delta)\tau(\mathsf{id} \times \mathsf{out}),
\end{aligned}
$$

which we reduce, removing $\tau(\mathsf{id}\times\mathsf{out})$ and T, multiplying from the left with $\mathsf{assoc} + \Sigma T_\Sigma\,\mathsf{assoc}$, and removing τ^ν on the left, to

$$\delta(\mathsf{id} \times (\mathsf{id} + \Sigma\tau^\nu)\delta)\,\mathsf{assoc} = (\mathsf{assoc} + (\Sigma(\mathsf{id} \times \tau^\nu)\,\mathsf{assoc}))\delta.$$

For the latter we calculate

$$
\begin{aligned}
&\delta(\mathsf{id} \times (\mathsf{id} + \Sigma\tau^\nu)\delta)\,\mathsf{assoc}\\
&\quad= \delta(\mathsf{id} \times (\mathsf{id} + \Sigma\tau^\nu))(\mathsf{id} \times \delta)\,\mathsf{assoc}\\
&\quad= (\mathsf{id} \times \mathsf{id} + \Sigma(\mathsf{id} \times \tau^\nu))\delta(\mathsf{id} \times \delta)\,\mathsf{assoc} && \text{// naturality of } \delta\\
&\quad= (\mathsf{id} + \Sigma(\mathsf{id} \times \tau^\nu))(\mathsf{id} + \rho)\,\mathsf{dist}(\mathsf{id} \times (\mathsf{id} + \rho)\,\mathsf{dist})\,\mathsf{assoc} && \text{// definition of } \delta\\
&\quad= (\mathsf{id} + \Sigma(\mathsf{id} \times \tau^\nu))(\mathsf{id} + \rho(\mathsf{id} \times \rho))\,\mathsf{dist}(\mathsf{id} \times \mathsf{dist})\,\mathsf{assoc} && \text{// naturality of } \mathsf{dist}\\
&\quad= (\mathsf{id} + \Sigma(\mathsf{id} \times \tau^\nu))(\mathsf{id} + \rho(\mathsf{id} \times \rho))(\mathsf{assoc} + \mathsf{assoc})\,\mathsf{dist} && \text{// 6.5}\\
&\quad= (\mathsf{id} + \Sigma(\mathsf{id} \times \tau^\nu))(\mathsf{assoc} + \Sigma\,\mathsf{assoc}\,\rho)\,\mathsf{dist} && \text{// } \textsc{str}_2 \text{ for } \rho\\
&\quad= (\mathsf{id} + \Sigma(\mathsf{id} \times \tau^\nu))(\mathsf{assoc} + \Sigma\,\mathsf{assoc})\delta && \text{// definition of } \delta\\
&\quad= (\mathsf{assoc} + \Sigma((\mathsf{id} \times \tau^\nu)\,\mathsf{assoc}))\delta.
\end{aligned}
$$

Here, we use the obvious coherence property

$$\mathsf{dist}(\mathsf{id} \times \mathsf{dist})\,\mathsf{assoc} = (\mathsf{assoc} + \mathsf{assoc})\,\mathsf{dist}\,. \tag{6.5}$$

• ($\textsc{str}_3$) In order to obtain the identity $\tau^\nu(\mathsf{id} \times \eta^\nu) = \eta^\nu$, we show that the left hand side satisfies the characteristic equation for η^ν, i.e. $\mathsf{out}\,\tau^\nu(\mathsf{id} \times \eta^\nu) = \eta\,\mathsf{inl}$. Indeed,

$$
\begin{aligned}
\mathsf{out}\,\tau^\nu(\mathsf{id} \times \eta^\nu) &= T(\mathsf{id} + \Sigma\tau^\nu)(T\delta)\tau(\mathsf{id} \times \mathsf{out})(\mathsf{id} \times \eta^\nu) && \text{// definition of } \tau^\nu\\
&= T(\mathsf{id} + \Sigma\tau^\nu)(T\delta)\tau(\mathsf{id} \times \eta)(\mathsf{id} \times \mathsf{inl}) && \text{// definition of } \eta^\nu\\
&= T(\mathsf{id} + \Sigma\tau^\nu)(T\delta)\eta(\mathsf{id} \times \mathsf{inl}) && \text{// } \textsc{str}_3 \text{ for } \tau\\
&= T(\mathsf{id} + \Sigma\tau^\nu)(T\delta)T(\mathsf{id} \times \mathsf{inl})\eta && \text{// naturality of } \eta\\
&= T(\mathsf{id} + \Sigma\tau^\nu)(T\,\mathsf{inl})\eta\\
&= \eta\,\mathsf{inl}\,.
\end{aligned}
$$

• ($\textsc{str}_4$) Given $f : X \to T_\Sigma Z$, we show that $(\tau^\nu(\mathsf{id} \times f))^\star\tau^\nu = \tau^\nu(\mathsf{id} \times f^\star)$. Let $g = [f, \eta^\nu]$ and let us show first that $(\tau^\nu(\mathsf{id} \times g))^\star\tau^\nu = \tau^\nu(\mathsf{id} \times g^\star)$. This implies the identity for f as follows:

$$
\begin{aligned}
(\tau^\nu(\mathsf{id} \times f))^\star\tau^\nu &= (\tau^\nu(\mathsf{id} \times g)(\mathsf{id} \times \mathsf{inl}))^\star\tau^\nu\\
&= (\tau^\nu(\mathsf{id} \times g))^\star T_\Sigma(\mathsf{id} \times \mathsf{inl})\tau^\nu\\
&= (\tau^\nu(\mathsf{id} \times g))^\star\tau^\nu(\mathsf{id} \times T_\Sigma\,\mathsf{inl}) && \text{// naturality of } \tau^\nu\\
&= \tau^\nu(\mathsf{id} \times g^\star)(\mathsf{id} \times T_\Sigma\,\mathsf{inl}) && \text{// } \textsc{str}_4 \text{ for } g \text{ and } \tau^\nu\\
&= \tau^\nu(\mathsf{id} \times g^\star T_\Sigma\,\mathsf{inl})
\end{aligned}
$$

$$= \tau^\nu(\mathsf{id} \times f^{\ast}).$$

As we have shown above both $g^{\ast}$ and τ^ν are final morphisms from suitable coalgebras. By composing the corresponding commutative squares we obtain the following diagram:

$$\begin{tikzcd}
X \times T_\Sigma Y \arrow[r, "\mathsf{id}\times[T(\mathsf{id}+\Sigma T_\Sigma\,\mathsf{inr})\,\mathsf{out}\,g,\eta\,\mathsf{inr}]^\star\,\mathsf{out}"] \arrow[dd, "\mathsf{id}\times g^{\ast}"'] & X \times T(Z + \Sigma T_\Sigma Y) \arrow[r, "(T\delta)\tau"] \arrow[d, "\mathsf{id}\times T(\mathsf{id}+\Sigma g^{\ast})"] & T(X \times Z + \Sigma(X \times T_\Sigma Y)) \arrow[dddl, bend left, "T(\mathsf{id}+\Sigma(\tau^\nu(\mathsf{id}\times g^{\ast})))"] \\
 & X \times T(Z + T_\Sigma Z) \arrow[d, "(T\delta)\tau"] & \\
X \times T_\Sigma Z \arrow[ur, "\mathsf{id}\times\mathsf{out}"] \arrow[r, "(T\delta)\tau(\mathsf{id}\times\mathsf{out})"'] \arrow[d, "\tau^\nu"'] & T(X \times Z + \Sigma(X \times T_\Sigma Z)) \arrow[d, "T(\mathsf{id}+\Sigma\tau^\nu)"] & \\
T_\Sigma(X \times Z) \arrow[r, "\mathsf{out}"'] & T_\Sigma(X \times Z + \Sigma T_\Sigma(X \times Z)) &
\end{tikzcd}$$

from which we conclude that

$$\tau^\nu(\mathsf{id} \times g^{\ast}) = \mathsf{coit}\Big((T\delta)\tau(\mathsf{id} \times [h, \eta\,\mathsf{inr}]^\star\,\mathsf{out})\Big)$$

where h denotes $T(\mathsf{id} + \Sigma T_\Sigma\,\mathsf{inr})\,\mathsf{out}\,g$.

We will be done once we show that also $(\tau^\nu(\mathsf{id} \times g))^{\ast}\tau^\nu$ is a morphism from the same coalgebra to the final one, i.e. the identity

$$T(\mathsf{id} + \Sigma((\tau^\nu(\mathsf{id} \times g))^{\ast}\tau^\nu))(T\delta)\tau(\mathsf{id} \times [h, \eta\,\mathsf{inr}]^\star\,\mathsf{out}) = \mathsf{out}(\tau^\nu(\mathsf{id} \times g))^{\ast}\tau^\nu. \tag{6.6}$$

Let us show that the following diagram commutes:

$$\begin{tikzcd}
T(X \times (Y + \Sigma(T_\Sigma Y))) \arrow[r, "T\delta"] \arrow[d, "(\tau(\mathsf{id}\times[h,\eta\,\mathsf{inr}]))^\star"'] & T(X \times Y + \Sigma(X \times T_\Sigma Y)) \arrow[d, "{[(T\delta)\tau(\mathsf{id}\times h),\eta\,\mathsf{inr}]^\star}"] \\
T(X \times (Z + \Sigma(T_\Sigma Y))) \arrow[r, "T\delta"] & T(X \times Z + \Sigma(X \times T_\Sigma Y))
\end{tikzcd} \tag{6.7}$$

Indeed,

$$\begin{aligned}
&[(T\delta)\tau(\mathsf{id} \times h), \eta\,\mathsf{inr}]^\star(T\delta) \\
&\quad = ([(T\delta)\tau(\mathsf{id} \times h), \eta\,\mathsf{inr}]\delta)^\star \\
&\quad = ([(T\delta)\tau(\mathsf{id} \times h), \eta\,\mathsf{inr}](\mathsf{id} + \rho)\,\mathsf{dist})^\star && \text{// definiton of } \delta \\
&\quad = ([(T\delta)\tau(\mathsf{id} \times h), \eta\,\mathsf{inr}\,\rho]\,\mathsf{dist})^\star \\
&\quad = ([(T\delta)\tau(\mathsf{id} \times h), \eta\,\delta(\mathsf{id} \times \mathsf{inr})]\,\mathsf{dist})^\star && \text{// definiton of } \delta \\
&\quad = ([(T\delta)\tau(\mathsf{id} \times h), (T\delta)\tau(\mathsf{id} \times \eta)(\mathsf{id} \times \mathsf{inr})]\,\mathsf{dist})^\star && \text{// } \textsc{str}_3 \text{ for } \tau \\
&\quad = ((T\delta)\tau[\mathsf{id} \times h, \mathsf{id} \times \eta\,\mathsf{inr}]\,\mathsf{dist})^\star \\
&\quad = (T\delta)(\tau(\mathsf{id} \times [h, \eta\,\mathsf{inr}]))^\star
\end{aligned}$$

where the last step is due to the obvious identity $[u \times v, u \times w] = (u \times [v, w])\,\mathsf{dist}^{-1}$. Finally, we obtain (6.6) as follows:

$$
\begin{aligned}
&T(\mathsf{id} + \Sigma((\tau^\nu(\mathsf{id} \times g))^{\bigstar}\tau^\nu))\\
&\qquad (T\delta)\tau(\mathsf{id} \times [h, \eta\,\mathsf{inr}]^\star\,\mathsf{out})\\
&\quad = T(\mathsf{id} + \Sigma((\tau^\nu(\mathsf{id} \times g))^{\bigstar}\tau^\nu))\\
&\qquad (T\delta)(\tau(\mathsf{id} \times [h, \eta\,\mathsf{inr}]))^\star\tau(\mathsf{id} \times \mathsf{out}) && /\!/ \ \textsc{str}_4 \text{ for } \tau\\
&\quad = T(\mathsf{id} + \Sigma((\tau^\nu(\mathsf{id} \times g))^{\bigstar}\tau^\nu))\\
&\qquad [(T\delta)\tau(\mathsf{id} \times h), \eta\,\mathsf{inr}]^\star(T\delta)\tau(\mathsf{id} \times \mathsf{out}) && /\!/ \ 6.7\\
&\quad = T(\mathsf{id} + \Sigma((\tau^\nu(\mathsf{id} \times g))^{\bigstar}\tau^\nu))\\
&\qquad [(T\delta)\tau(\mathsf{id} \times T(\mathsf{id} + \Sigma T_\Sigma\,\mathsf{inr}))(\mathsf{id} \times \mathsf{out}\,g), \eta\,\mathsf{inr}]^\star\\
&\qquad (T\delta)\tau(\mathsf{id} \times \mathsf{out}) && /\!/ \ \text{definition of } h\\
&\quad = T(\mathsf{id} + \Sigma((\tau^\nu(\mathsf{id} \times g))^{\bigstar}\tau^\nu))\\
&\qquad [(T\delta)T(\mathsf{id} \times (\mathsf{id} + \Sigma T_\Sigma\,\mathsf{inr}))\tau(\mathsf{id} \times \mathsf{out}\,g), \eta\,\mathsf{inr}]^\star\\
&\qquad (T\delta)\tau(\mathsf{id} \times \mathsf{out}) && /\!/ \ \text{naturality of } \tau\\
&\quad = T(\mathsf{id} + \Sigma((\tau^\nu(\mathsf{id} \times g))^{\bigstar}\tau^\nu))\\
&\qquad [T(\mathsf{id} + \Sigma(\mathsf{id} \times T_\Sigma\,\mathsf{inr}))(T\delta)\tau(\mathsf{id} \times \mathsf{out}\,g), \eta\,\mathsf{inr}]^\star\\
&\qquad (T\delta)\tau(\mathsf{id} \times \mathsf{out}) && /\!/ \ \text{naturality of } \delta\\
&\quad = [T(\mathsf{id} + \Sigma((\tau^\nu(\mathsf{id} \times g))^{\bigstar}\tau^\nu(\mathsf{id} \times T_\Sigma\,\mathsf{inr})))(T\delta)\tau(\mathsf{id} \times \mathsf{out}\,g),\\
&\qquad \eta\,\mathsf{inr}\,\Sigma((\tau^\nu(\mathsf{id} \times g))^{\bigstar}\tau^\nu)]^\star(T\delta)\tau(\mathsf{id} \times \mathsf{out}) && /\!/ \ \text{naturality of } \eta\\
&\quad = [T(\mathsf{id} + \Sigma((\tau^\nu(\mathsf{id} \times g\,\mathsf{inr}))^{\bigstar}\tau^\nu))(T\delta)\tau(\mathsf{id} \times \mathsf{out}\,g),\\
&\qquad \eta\,\mathsf{inr}\,\Sigma((\tau^\nu(\mathsf{id} \times g))^{\bigstar}\tau^\nu))]^\star(T\delta)\tau(\mathsf{id} \times \mathsf{out}) && /\!/ \ \text{naturality of } \tau^\nu\\
&\quad = [T(\mathsf{id} + \Sigma((\tau^\nu(\mathsf{id} \times \eta^\nu))^{\bigstar}\tau^\nu))(T\delta)\tau(\mathsf{id} \times \mathsf{out}\,g),\\
&\qquad \eta\,\mathsf{inr}\,\Sigma((\tau^\nu(\mathsf{id} \times g))^{\bigstar}\tau^\nu))]^\star(T\delta)\tau(\mathsf{id} \times \mathsf{out}) && /\!/ \ \text{since } g = [f, \eta^\nu]\\
&\quad = [T(\mathsf{id} + \Sigma\tau^\nu)(T\delta)\tau(\mathsf{id} \times \mathsf{out}\,g),\\
&\qquad \eta\,\mathsf{inr}\,\Sigma((\tau^\nu(\mathsf{id} \times g))^{\bigstar}\tau^\nu))]^\star(T\delta)\tau(\mathsf{id} \times \mathsf{out}) && /\!/ \ \textsc{str}_3 \text{ for } \tau^\nu\\
&\quad = [T(\mathsf{id} + \Sigma\tau^\nu)(T\delta)\tau(\mathsf{id} \times \mathsf{out})(\mathsf{id} \times g),\\
&\qquad \eta\,\mathsf{inr}\,\Sigma((\tau^\nu(\mathsf{id} \times g))^{\bigstar}\tau^\nu)]^\star(T\delta)\tau(\mathsf{id} \times \mathsf{out})\\
&\quad = [\mathsf{out}\,\tau^\nu(\mathsf{id} \times g), \eta\,\mathsf{inr}\,\Sigma((\tau^\nu(\mathsf{id} \times g))^{\bigstar}\tau^\nu)]^\star(T\delta)\tau(\mathsf{id} \times \mathsf{out}) && /\!/ \ \text{definition of } \tau^\nu\\
&\quad = [\mathsf{out}\,\tau^\nu(\mathsf{id} \times g), \eta\,\mathsf{inr}\,\Sigma(\tau^\nu(\mathsf{id} \times g))^{\bigstar})]^\star\\
&\qquad T(\mathsf{id} + \Sigma\tau^\nu)(T\delta)\tau(\mathsf{id} \times \mathsf{out})\\
&\quad = [\mathsf{out}\,\tau^\nu(\mathsf{id} \times g), \eta\,\mathsf{inr}\,\Sigma(\tau^\nu(\mathsf{id} \times g))^{\bigstar}]^\star\,\mathsf{out}\,\tau^\nu && /\!/ \ \text{definition of } \tau^\nu\\
&\quad = \mathsf{out}(\tau^\nu(\mathsf{id} \times g))^{\bigstar}\tau^\nu. && /\!/ \ \text{definition of } \mathsf{-}^{\bigstar}
\end{aligned}
$$

We have thus shown all properties ($\textsc{str}_1$)–($\textsc{str}_4$) and the proof is completed. □

Following Uustalu [40] (and other work [32, 1]), we next introduce a notion of guardedness.

Definition 6.5. (Guardedness) A morphism $f : X \to T_\Sigma(Y+Z)$ is *guarded* if there is $u : X \to T(Y + \Sigma T_\Sigma(Y+Z))$ such that $\mathsf{out}\, f = T(\mathsf{inl} + \mathsf{id})\, u$:

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[d, "u"'] & & T_\Sigma(Y+Z) \arrow[d, "\mathsf{out}"] \\ T(Y + \Sigma T_\Sigma(Y+Z)) \arrow[rr, "T(\mathsf{inl} + \mathsf{id})"] & & T((Y+Z) + \Sigma T_\Sigma(Y+Z)). \end{tikzcd}$$

Guardedness of $f : X \to T_\Sigma(Y+Z)$ intuitively means that any call to a computation of type Z in f occurs only under a free operation, i.e. via the right hand summand in $T((Y+Z) + \Sigma T_\Sigma(Y+Z))$. A familiar instance of this notion occurs in process algebra [10], illustrated in simplified form as follows.

Example 6.6. Let $\mathbb{T}$ be the countable powerset monad over a suitable category, i.e. $TX = \mathcal{P}_{\omega_1} X = \{Y \subseteq X \mid |Y| \leq \omega\}$. Take $\Sigma = A \times (-)$; then the object $T_\Sigma X = \nu\gamma.\, \mathcal{P}_{\omega_1}(X + A \times \gamma)$ can be considered as the domain of possibly infinite countably nondeterministic processes over actions from A with final results in X. A morphism $n \to T_\Sigma(X+n)$ can be seen as a system of n mutually recursive process definitions; the latter is guarded in the sense of Definition 6.5 iff every recursive call of a process is preceded by an action, which coincides with the standard notion of guardedness from process algebra. We recall an example of an *unguarded* definition in this setting in Section 3.

7. Iteration on Coinductive Resumptions

We next establish one of the main technical contributions of the paper by proving that iteration operators, i.e. Elgot monad structures, propagate uniquely along extensions $\mathbb{T} \to \mathbb{T}_\Sigma$, implying that Elgot monads are closed under the coinductive generalized resumption transformer.

Theorem 7.1. *Let $\mathbb{T}$ be a complete Elgot monad and let $\mathbb{T}_\Sigma$ be the monad identified in Theorem 6.3, i.e. the coinductive generalized resumption monad over $\mathbb{T}$.*

(1) *There is a unique iteration operator making $\mathbb{T}_\Sigma$ a complete Elgot monad that extends iteration of $\mathbb{T}$ in the sense that for $f : X \to T_\Sigma(Y+X)$ and $g : X \to T(Y+X)$, if*
$$\mathsf{out}\, f = (T\, \mathsf{inl})\, g \tag{7.1}$$
(i.e. $f = \mathsf{out}^{-1}(T\,\mathsf{inl})\, g$) then
$$\mathsf{out}\, f^\dagger = (T\, \mathsf{inl})\, g^\dagger. \tag{7.2}$$

(2) *For any guarded morphism $f : X \to T_\Sigma(Y+X)$, $f^\dagger$ is the* unique *morphism satisfying the fixpoint law $[\eta^\nu, f^\dagger]^\star f = f^\dagger$.*

The proof of Theorem 7.1 relies on a fairly complicated chain of calculations and will, to aid readability, be partitioned into separate lemmas. Before we dive into these details, let us outline the general idea.

Uustalu already proves that guarded morphisms f have unique iterates $f^\dagger$ satisfying the fixpoint law [40, Theorem 3.11], which readily implies the second clause. In showing the first clause of Theorem 7.1, the key step is then to define $f^\dagger$ for unrestricted f in a consistent manner. For $f : X \to T_\Sigma(Y + X)$, let $\Diamond f : X \to T_\Sigma(Y + X)$ be the composite

$$\begin{aligned} X &\xrightarrow{w^\dagger} T(Y + \Sigma T_\Sigma(Y + X)) \\ &\xrightarrow{T(\mathsf{inl} + \mathsf{id})} T((Y + X) + \Sigma T_\Sigma(Y + X)) \\ &\xrightarrow{\mathsf{out}^{-1}} T_\Sigma(Y + X) \end{aligned}$$

(guarded by definition), where w is the composite

$$\begin{aligned} X &\xrightarrow{f} T_\Sigma(Y + X) \\ &\xrightarrow{\mathsf{out}} T((Y + X) + \Sigma T_\Sigma(Y + X)) \\ &\xrightarrow{T\pi} T((Y + \Sigma T_\Sigma(Y + X)) + X) \end{aligned}$$

with $\pi = [\mathsf{inl} + \mathsf{id}, \mathsf{inl}\,\mathsf{inr}]$. That is, $\Diamond f$ makes f guarded by iterating

$$\mathsf{out}\, f : X \to T((Y + X) + \Sigma T_\Sigma(Y + X))$$

(in the complete Elgot monad $\mathbb{T}$) over the middle summand X. It is easy to check that $\Diamond f = f$ when f is guarded. We hence can consistently define

$$f^\dagger = (\Diamond f)^\dagger \tag{7.3}$$

(in $\mathbb{T}_\Sigma$). The remaining technical challenge is now to prove that this definition indeed satisfies the axioms of complete Elgot monads and that it is the unique such iteration operator on $\mathbb{T}_\Sigma$ extending the given iteration operator on $\mathbb{T}$.

Lemma 7.2. *Given $f : X \to T_\Sigma(Y + X)$, $f^\dagger : f : X \to T_\Sigma Y$ defined by* (7.3) *satisfies* fixpoint*,* naturality*, and* uniformity*.*

Proof. To make sure that definition (7.3) introduces the iteration consistently with the iteration for guarded morphisms we check that $\Diamond f = f$ whenever f is guarded. Suppose that $\mathsf{out}\, f = T(\mathsf{inl} + \mathsf{id})u$. Then $f = \mathsf{out}^{-1}\, T(\mathsf{inl} + \mathsf{id})u$ and therefore

$$\begin{aligned} \Diamond f &= \mathsf{out}^{-1}\, T(\mathsf{inl} + \mathsf{id})((T\pi)\,\mathsf{out}\, f)^\dagger \\ &= \mathsf{out}^{-1}\, T(\mathsf{inl} + \mathsf{id})((T\pi)\,\mathsf{out}\,\mathsf{out}^{-1}\, T(\mathsf{inl} + \mathsf{id})u)^\dagger \\ &= \mathsf{out}^{-1}\, T(\mathsf{inl} + \mathsf{id})((T\pi)T(\mathsf{inl} + \mathsf{id})u)^\dagger \\ &= \mathsf{out}^{-1}\, T(\mathsf{inl} + \mathsf{id})(T\,\mathsf{inl}\, u)^\dagger \\ &= \mathsf{out}^{-1}\, T(\mathsf{inl} + \mathsf{id})u \\ &= f. \end{aligned}$$

Let us check *fixpoint*, *naturality*, and *uniformity* (Definition 5.1) in order.

- *Fixpoint.* For any $f : X \to T_\Sigma(Y + X)$ we have

$$\begin{aligned} f^\dagger &= [\eta^\nu, f^\dagger]^{\maltese} \Diamond f && \text{// definition of } -^\dagger \\ &= [\eta^\nu, f^\dagger]^{\maltese}\, \mathsf{out}^{-1}\, T(\mathsf{inl} + \mathsf{id})((T\pi)\,\mathsf{out}\, f)^\dagger && \text{// definition of } \Diamond \\ &= \mathsf{out}^{-1}\Big[[\eta\,\mathsf{inl}, \mathsf{out}\, f^\dagger], \eta\,\mathsf{inr}\,\Sigma[\eta^\nu, f^\dagger]^{\maltese}\Big]^\star \\ &\quad\; T(\mathsf{inl} + \mathsf{id})((T\pi)\,\mathsf{out}\, f)^\dagger && \text{// definition of } -^{\maltese} \end{aligned}$$

$$
\begin{aligned}
&= \mathsf{out}^{\text{-}1}\Big[\eta\,\mathsf{inl}, \eta\,\mathsf{inr}\,\Sigma[\eta^\nu, f^\dagger]^{\maltese}\Big]^\star((T\pi)\,\mathsf{out}\,f)^\dagger \\
&= \mathsf{out}^{\text{-}1}\,T(\mathsf{id} + \Sigma[\eta^\nu, f^\dagger]^{\maltese})((T\pi)\,\mathsf{out}\,f)^\dagger
\end{aligned}
$$

and thus we obtain the following intermediate equation:

$$
\mathsf{out}\,f^\dagger = T\Big(\mathsf{id} + \Sigma[\eta^\nu, f^\dagger]^{\maltese}\Big)((T\pi)\,\mathsf{out}\,f)^\dagger. \tag{7.4}
$$

Now, continuing the above calculation we obtain

$$
\begin{aligned}
f^\dagger &= \mathsf{out}^{\text{-}1}\,T(\mathsf{id} + \Sigma[\eta^\nu, f^\dagger]^{\maltese})((T\pi)\,\mathsf{out}\,f)^\dagger \\
&= \mathsf{out}^{\text{-}1}\,T(\mathsf{id} + \Sigma[\eta^\nu, f^\dagger]^{\maltese})[\eta, ((T\pi)\,\mathsf{out}\,f)^\dagger]^\star(T\pi)\,\mathsf{out}\,f && \text{// fixpoint} \\
&= \mathsf{out}^{\text{-}1}[T(\mathsf{id} + \Sigma[\eta^\nu, f^\dagger]^{\maltese})\eta, \mathsf{out}\,f^\dagger]^\star(T\pi)\,\mathsf{out}\,f && \text{// 7.4} \\
&= \mathsf{out}^{\text{-}1}[\eta(\mathsf{id} + \Sigma[\eta^\nu, f^\dagger]^{\maltese}), \mathsf{out}\,f^\dagger]^\star(T\pi)\,\mathsf{out}\,f && \text{// naturality of } \eta \\
&= \mathsf{out}^{\text{-}1}\Big[[\eta\,\mathsf{inl}, \mathsf{out}\,f^\dagger], \eta\,\mathsf{inr}\,\Sigma[\eta^\nu, f^\dagger]^{\maltese}\Big]^\star\,\mathsf{out}\,f \\
&= \mathsf{out}^{\text{-}1}\Big[\mathsf{out}[\eta^\nu, f^\dagger], \eta\,\mathsf{inr}\,\Sigma[\eta^\nu, f^\dagger]^{\maltese}\Big]^\star\,\mathsf{out}\,f && \text{// definition of } \eta^\nu \\
&= [\eta^\nu, f^\dagger]^{\maltese}\,\mathsf{out}^{\text{-}1}\,\mathsf{out}\,f && \text{// naturality of } -^{\maltese} \\
&= [\eta^\nu, f^\dagger]^{\maltese} f.
\end{aligned}
$$

• *Naturality.* Assume that $h : X \to T_\Sigma(Y + X)$ is guarded and show that so is $[(T_\Sigma\,\mathsf{inl})g, \eta^\nu\,\mathsf{inr}]^{\maltese}h$ for any $g : Y \to Z$. Let u be such that $\mathsf{out}\,h = T(\mathsf{inl} + \mathsf{id})u$ and let $w = [(T_\Sigma\,\mathsf{inl})g, \eta^\nu\,\mathsf{inr}]$. Then

$$
\begin{aligned}
&\mathsf{out}[\,(T_\Sigma\,\mathsf{inl})g, \eta^\nu\,\mathsf{inr}]^{\maltese}h \\
&\quad= [\mathsf{out}\,w, \eta\,\mathsf{inr}\,\Sigma w^{\maltese}]^\star\,\mathsf{out}\,h \\
&\quad= [\mathsf{out}\,w, \eta\,\mathsf{inr}\,\Sigma w^{\maltese}]^\star T(\mathsf{inl} + \mathsf{id})u \\
&\quad= [\mathsf{out}\,w\,\mathsf{inl}, \eta\,\mathsf{inr}\,\Sigma w^{\maltese}]^\star u \\
&\quad= [\mathsf{out}(T_\Sigma\,\mathsf{inl})g, \eta\,\mathsf{inr}\,\Sigma w^{\maltese}]^\star u \\
&\quad= [T(\mathsf{inl} + \Sigma T_\Sigma\,\mathsf{inl})\,\mathsf{out}\,g, \eta\,\mathsf{inr}\,\Sigma w^{\maltese}]^\star u \\
&\quad= T(\mathsf{inl} + \mathsf{id})\Big[T(\mathsf{id} + \Sigma T_\Sigma\,\mathsf{inl})\,\mathsf{out}\,g, \eta\,\mathsf{inr}\,\Sigma w^{\maltese}\Big]^\star u.
\end{aligned}
$$

Now, since $t = [(T_\Sigma\,\mathsf{inl})g, \eta^\nu\,\mathsf{inr}]^{\maltese}\Diamond f$ is guarded, it is the unique fixpoint of the map

$$
t \mapsto [\eta^\nu, t]^{\maltese}[(T_\Sigma\,\mathsf{inl})g, \eta^\nu\,\mathsf{inr}]^{\maltese}\Diamond f.
$$

However, on the other hand,

$$
\begin{aligned}
&[\eta^\nu, g^{\maltese} f^\dagger]^{\maltese}[(T_\Sigma\,\mathsf{inl})g, \eta^\nu\,\mathsf{inr}]^{\maltese}\Diamond f \\
&\qquad= [g, g^{\maltese} f^\dagger]^{\maltese}\Diamond f \\
&\qquad= [g, g^{\maltese}(\Diamond f)^\dagger]^{\maltese}\Diamond f \\
&\qquad= g^{\maltese}[\eta^\nu, (\Diamond f)^\dagger]^{\maltese}\Diamond f \\
&\qquad= g^{\maltese} f^\dagger
\end{aligned}
$$

and therefore $t^\dagger = g^{\maltese} f^\dagger$. It remains to show that

$$
[(T_\Sigma\,\mathsf{inl})g, \eta^\nu\,\mathsf{inr}]^{\maltese}\Diamond f = \Diamond[(T_\Sigma\,\mathsf{inl})g, \eta^\nu\,\mathsf{inr}]^{\maltese} f.
$$

Since $\diamondsuit f$ is guarded by definition, we know by the calculation above that $[(T_\Sigma\,\mathsf{inl})g, \eta^\nu\,\mathsf{inr}]^{\bigstar}\diamondsuit f$ is guarded and therefore

$$[(T_\Sigma\,\mathsf{inl})g, \eta\,\mathsf{inr}]^{\bigstar}\diamondsuit f = \diamondsuit[(T_\Sigma\,\mathsf{inl})g, \eta\,\mathsf{inr}]^{\bigstar}\diamondsuit f.$$

To finish the proof, we calculate

$$\diamondsuit[(T_\Sigma\,\mathsf{inl})g, \eta^\nu\,\mathsf{inr}]^{\bigstar}\diamondsuit f = \mathsf{out}^{-1}\,T(\mathsf{inl}+\mathsf{id})((T\pi)\,\mathsf{out}[(T_\Sigma\,\mathsf{inl})g, \eta^\nu\,\mathsf{inr}]^{\bigstar}\diamondsuit f)^\dagger.$$

Further transforming the dagger expression in the previous term yields

$$\begin{aligned}
&((T\pi)\ \mathsf{out}[(T_\Sigma\,\mathsf{inl})g, \eta\,\mathsf{inr}]^{\bigstar}\diamondsuit f)^\dagger\\
&\quad= ((T\pi)[\mathsf{out}\,w, \eta\,\mathsf{inr}\,\Sigma w^{\bigstar}]^\star T(\mathsf{inl}+\mathsf{id})((T\pi)\,\mathsf{out}\,f)^\dagger)^\dagger\\
&\quad= ((T\pi)[\mathsf{out}(T_\Sigma\,\mathsf{inl})g, \eta\,\mathsf{inr}\,\Sigma w^{\bigstar}]^\star((T\pi)\,\mathsf{out}\,f)^\dagger)^\dagger\\
&\quad= ([(T\,\mathsf{inl})(T\pi)[\mathsf{out}(T_\Sigma\,\mathsf{inl})g, \eta\,\mathsf{inr}\,\Sigma w^{\bigstar}], \eta\,\mathsf{inr}]^\star(T\pi)\,\mathsf{out}\,f)^{\dagger\dagger} && \text{// nat. for } \mathbb{T}\\
&\quad= (T[\mathsf{id},\mathsf{inr}][(T\,\mathsf{inl})(T\pi)[\mathsf{out}(T_\Sigma\,\mathsf{inl})g, \eta\,\mathsf{inr}\,\Sigma w^{\bigstar}], \eta\,\mathsf{inr}]^\star(T\pi)\,\mathsf{out}\,f)^\dagger && \text{// codiag. for } \mathbb{T}\\
&\quad= ([[(T\pi)\,\mathsf{out}(T_\Sigma\,\mathsf{inl})g, \eta\,\mathsf{inr}], (T\pi)\eta\,\mathsf{inr}\,\Sigma w^{\bigstar}]^\star\,\mathsf{out}\,f)^\dagger\\
&\quad= ((T\pi)[[\mathsf{out}(T_\Sigma\,\mathsf{inl})g, \eta\,\mathsf{inl}\,\mathsf{inr}], \eta\,\mathsf{inr}\,\Sigma w^{\bigstar}]^\star\,\mathsf{out}\,f)^\dagger\\
&\quad= ((T\pi)[\mathsf{out}[(T_\Sigma\,\mathsf{inl})g, \eta^\nu\,\mathsf{inr}], \eta\,\mathsf{inr}\,\Sigma w^{\bigstar}]^\star\,\mathsf{out}\,f)^\dagger\\
&\quad= ((T\pi)\,\mathsf{out}[(T_\Sigma\,\mathsf{inl})g, \eta^\nu\,\mathsf{inr}]^{\bigstar} f)^\dagger
\end{aligned}$$

and therefore

$$\diamondsuit[(T_\Sigma\,\mathsf{inl})g, \eta\,\mathsf{inr}]^{\bigstar}\diamondsuit f = \diamondsuit[(T_\Sigma\,\mathsf{inl})g, \eta\,\mathsf{inr}]^{\bigstar} f.$$

- *Uniformity.* First, we show uniformity under the assumption that g is guarded. Suppose that $fh = T_\Sigma(\mathsf{id}+h)g$. It is then sufficient to verify that $f^\dagger h$ satisfies the fixpoint law for g. Indeed,

$$\begin{aligned}
f^\dagger h &= [\eta^\nu, f^\dagger]^{\bigstar} fh\\
&= [\eta^\nu, f^\dagger]^{\bigstar} T_\Sigma(\mathsf{id}+h)g\\
&= [\eta^\nu, f^\dagger h]^{\bigstar} g.
\end{aligned}$$

Now consider the general case. Suppose that again we have $fh = T_\Sigma(\mathsf{id}+h)g$. We prove the following auxiliary identity:

$$((T\pi)\,\mathsf{out}\,f)^\dagger h = T\big(\mathsf{id} + \Sigma T_\Sigma(\mathsf{id}+h)\big)((T\pi)\,\mathsf{out}\,g)^\dagger. \tag{7.5}$$

Observe that

$$\begin{aligned}
((T\pi)\,\mathsf{out}\,f)h &= (T\pi)\,\mathsf{out}\,T_\Sigma(\mathsf{id}+h)g\\
&= (T\pi)T(\mathsf{id}+h+\Sigma T_\Sigma(\mathsf{id}+h))\,\mathsf{out}\,g\\
&= T(\mathsf{id}+h)T((\mathsf{id}+\Sigma T_\Sigma(\mathsf{id}+h))+\mathsf{id})(T\pi)\,\mathsf{out}\,g,
\end{aligned}$$

from which by uniformity of the iteration operator of $\mathbb{T}$, we obtain

$$((T\pi)\,\mathsf{out}\,f)^\dagger h = \Big(T((\mathsf{id}+\Sigma T_\Sigma(\mathsf{id}+h))+\mathsf{id})(T\pi)\,\mathsf{out}\,g\Big)^\dagger.$$

After transforming the right hand side by naturality of the iteration operator of $\mathbb{T}$ we arrive at (7.5).

Next we prove that $(\diamondsuit f)h = T_\Sigma(\mathsf{id}+h)\diamondsuit g$:

$$\begin{aligned}
(\diamondsuit f)h &= \mathsf{out}^{\text{-}1}\, T(\mathsf{inl}+\mathsf{id})((T\pi)\,\mathsf{out}\, f)^\dagger h && \text{// definition of } \diamondsuit\\
&= \mathsf{out}^{\text{-}1}\, T(\mathsf{inl}+\mathsf{id})T(\mathsf{id}+\Sigma T_\Sigma(\mathsf{id}+h))((T\pi)\,\mathsf{out}\, g)^\dagger && \text{// 7.5}\\
&= \mathsf{out}^{\text{-}1}\, T((\mathsf{id}+h)+\Sigma T_\Sigma(\mathsf{id}+h))T(\mathsf{inl}+\mathsf{id})((T\pi)\,\mathsf{out}\, g)^\dagger\\
&= T_\Sigma(\mathsf{id}+h)\,\mathsf{out}^{\text{-}1}\, T(\mathsf{inl}+\mathsf{id})((T\pi)\,\mathsf{out}\, g)^\dagger && \text{// Lemma 6.4}\\
&= T_\Sigma(\mathsf{id}+h)\diamondsuit g. && \text{// definition of } \diamondsuit
\end{aligned}$$

We have shown before that for guarded g uniformity holds, and therefore $f^\dagger h = (\diamondsuit f)^\dagger h = (\diamondsuit g)^\dagger = g^\dagger$. ∎

We now deal with the last axiom, *codiagonal*, whose proof is more involved that that of the other properties and therefore handled in a separate lemma:

Lemma 7.3. *The assignment of $f^\dagger : X \to T_\Sigma Y$ to $f : X \to T_\Sigma(Y+X)$ defined by* (7.3) *satisfies the* codiagonal *law.*

Proof. Let $g : X \to T_\Sigma((Y+X)+X)$. We shall show below that

$$\diamondsuit(T_\Sigma[\mathsf{id},\mathsf{inr}]\diamondsuit g) = \diamondsuit(T_\Sigma[\mathsf{id},\mathsf{inr}]g). \tag{7.6}$$

Since $T_\Sigma[\mathsf{id},\mathsf{inr}]^\dagger g$ is the unique fixpoint of the map

$$\gamma \mapsto [\eta^\nu,\gamma]^\star \diamondsuit(T_\Sigma[\mathsf{id},\mathsf{inr}]g)$$

we will be done once we show that $g^{\dagger\dagger}$ is also a fixpoint of the same map, i.e.

$$g^{\dagger\dagger} = [\eta^\nu, g^{\dagger\dagger}]^\star \diamondsuit(T_\Sigma[\mathsf{id},\mathsf{inr}]g). \tag{7.7}$$

Let us again denote by $\pi : (Y+X)+X \to (Y+X)+X$ the morphism swapping the last two components of the coproduct. We consider the following three cases.

1. $T_\Sigma[\mathsf{id},\mathsf{inr}]g$ *is guarded.* Then we obtain (7.7) directly as follows

$$\begin{aligned}
g^{\dagger\dagger} &= [\eta^\nu, g^{\dagger\dagger}]^\star g^\dagger && \text{// fixpoint}\\
&= [\eta^\nu, g^{\dagger\dagger}]^\star [\eta^\nu, g^\dagger]^\star g && \text{// fixpoint}\\
&= \big[[\eta^\nu, g^{\dagger\dagger}], [\eta^\nu, g^{\dagger\dagger}]^\star g^\dagger\big]^\star g\\
&= \big[[\eta^\nu, g^{\dagger\dagger}], g^{\dagger\dagger}\big]^\star g && \text{// fixpoint}\\
&= [\eta^\nu, g^{\dagger\dagger}]^\star T_\Sigma[\mathsf{id},\mathsf{inr}]g\\
&= [\eta^\nu, g^{\dagger\dagger}]^\star \diamondsuit(T_\Sigma[\mathsf{id},\mathsf{inr}]g).
\end{aligned}$$

2. $(T_\Sigma\pi)g$ *is guarded.* E.g. let $(T_\Sigma\pi)g = \mathsf{out}^{\text{-}1}\, T(\mathsf{inl}+\mathsf{id})u$. Then $T_\Sigma[\mathsf{id},\mathsf{inr}]\diamondsuit g$ is also guarded, which is certified by the following calculation, involving the definitions of g, $\diamondsuit$ and the naturality law for $-^\dagger$:

$$\begin{aligned}
&T_\Sigma[\mathsf{id},\mathsf{inr}]\diamondsuit g\\
&\quad= T_\Sigma[\mathsf{id},\mathsf{inr}]\diamondsuit((T_\Sigma\pi)\,\mathsf{out}^{\text{-}1}\, T(\mathsf{inl}+\mathsf{id})u)\\
&\quad= T_\Sigma[\mathsf{id},\mathsf{inr}]\,\mathsf{out}^{\text{-}1}\, T(\mathsf{inl}+\mathsf{id})\big((T\pi)\,\mathsf{out}(T_\Sigma\pi)\,\mathsf{out}^{\text{-}1}\, T(\mathsf{inl}+\mathsf{id})u\big)^\dagger\\
&\quad= T_\Sigma[\mathsf{id},\mathsf{inr}]\,\mathsf{out}^{\text{-}1}\, T(\mathsf{inl}+\mathsf{id})\big((T\pi)T(\pi+\Sigma T_\Sigma\pi)T(\mathsf{inl}+\mathsf{id})u\big)^\dagger
\end{aligned}$$

$$
\begin{aligned}
&= T_\Sigma[\mathsf{id},\mathsf{inr}]\,\mathsf{out}^{-1}\,T(\mathsf{inl}+\mathsf{id})\Big(T((\mathsf{inl}+\mathsf{id})+\mathsf{id})(T\pi)T(\mathsf{id}+\Sigma T_\Sigma\pi)u\Big)^\dagger\\
&= T_\Sigma[\mathsf{id},\mathsf{inr}]\,\mathsf{out}^{-1}\,T(\mathsf{inl}+\mathsf{id})T(\mathsf{inl}+\mathsf{id})\Big((T\pi)T(\mathsf{id}+\Sigma T_\Sigma\pi)u\Big)^\dagger\\
&= T_\Sigma[\mathsf{id},\mathsf{inr}]\,\mathsf{out}^{-1}\,T(\mathsf{inl}\,\mathsf{inl}+\mathsf{id})\Big((T\pi)T(\mathsf{id}+\Sigma T_\Sigma\pi)u\Big)^\dagger\\
&= \mathsf{out}^{-1}\,T([\mathsf{id},\mathsf{inr}]+\Sigma T_\Sigma[\mathsf{id},\mathsf{inr}])T(\mathsf{inl}\,\mathsf{inl}+\mathsf{id})\Big((T\pi)T(\mathsf{id}+\Sigma T_\Sigma\pi)u\Big)^\dagger\\
&= \mathsf{out}^{-1}\,T(\mathsf{inl}+\Sigma T_\Sigma[\mathsf{id},\mathsf{inr}])\Big((T\pi)T(\mathsf{id}+\Sigma T_\Sigma\pi)u\Big)^\dagger\\
&= \mathsf{out}^{-1}\,T(\mathsf{inl}+\mathsf{id})T(\mathsf{id}+\Sigma T_\Sigma[\mathsf{id},\mathsf{inr}])\Big((T\pi)T(\mathsf{id}+\Sigma T_\Sigma\pi)u\Big)^\dagger.
\end{aligned}
$$

The proof of (7.7) now can be completed as follows:

$$
\begin{aligned}
g^{\dagger\dagger} &= (\Diamond g)^{\dagger\dagger} && \text{// definition of } -^\dagger\\
&= [\eta^\nu,(\Diamond g)^{\dagger\dagger}]^{\bigstar}\Diamond(T_\Sigma[\mathsf{id},\mathsf{inr}]\Diamond g) && \text{// Clause (1)}\\
&= [\eta^\nu,g^{\dagger\dagger}]^{\bigstar}\Diamond(T_\Sigma[\mathsf{id},\mathsf{inr}]g). && \text{// 7.6}
\end{aligned}
$$

3. *g is guarded.* Let $h=(T_\Sigma\pi)\Diamond(T_\Sigma\pi)g$. It is easy to see that h is guarded. We use the following identity

$$\Diamond g^\dagger = [\eta^\nu, g^\dagger]^{\bigstar} h \tag{7.8}$$

whose proof runs as follows. Let $g=\mathsf{out}^{-1}\,T(\mathsf{inl}+\mathsf{id})u$ for some u and observe that $\pi\,\mathsf{inl} = (\mathsf{inl}+\mathsf{id})$. We apply out to the right-hand side of the equation,

$$
\begin{aligned}
&\mathsf{out}[\eta^\nu,g^\dagger]^{\bigstar}(T_\Sigma\pi)\Diamond(T_\Sigma\pi)g\\
&\quad= [\mathsf{out}[\eta^\nu,g^\dagger],\eta\,\mathsf{inr}\,\Sigma[\eta^\nu,g^\dagger]^{\bigstar}]^\star\\
&\qquad\quad \mathsf{out}(T_\Sigma\pi)\,\mathsf{out}^{-1}\,T(\mathsf{inl}+\mathsf{id})((T\pi)\,\mathsf{out}(T_\Sigma\pi)g)^\dagger && \text{// defn. of } -^\dagger,\ \Diamond\\
&\quad= [\mathsf{out}[\eta^\nu,g^\dagger]\pi\,\mathsf{inl},\eta\,\mathsf{inr}\,\Sigma([\eta^\nu,g^\dagger]^{\bigstar}T_\Sigma\pi)]^\star((T\pi)\,\mathsf{out}(T_\Sigma\pi)g)^\dagger\\
&\quad= [\mathsf{out}[\eta^\nu\,\mathsf{inl},g^\dagger],\eta\,\mathsf{inr}\,\Sigma([\eta^\nu,g^\dagger]^{\bigstar}T_\Sigma\pi)]^\star\\
&\qquad\quad ((T\pi)T(\pi+\Sigma T_\Sigma\pi)\,\mathsf{out}\,g)^\dagger\\
&\quad= ([(T\,\mathsf{inl})[\mathsf{out}[\eta^\nu\,\mathsf{inl},g^\dagger],\eta\,\mathsf{inr}([\eta^\nu,g^\dagger]^{\bigstar}\Sigma T_\Sigma\pi)],\eta\,\mathsf{inr}]^\star\\
&\qquad\quad (T\pi)T(\pi+\Sigma T_\Sigma\pi)\,\mathsf{out}\,g)^\dagger && \text{// naturality}\\
&\quad= ([[(T\,\mathsf{inl})\,\mathsf{out}[\eta^\nu\,\mathsf{inl},g^\dagger],\eta\,\mathsf{inl}\,\mathsf{inr}\,\Sigma([\eta^\nu,g^\dagger]^{\bigstar}T_\Sigma\pi)],\eta\,\mathsf{inr}]^\star\\
&\qquad (T\pi)T(\pi+\Sigma T_\Sigma\pi)\,\mathsf{out}\,g)^\dagger\\
&\quad= ([[(T\,\mathsf{inl})\,\mathsf{out}[\eta^\nu\,\mathsf{inl},g^\dagger],\eta\,\mathsf{inr}],\eta\,\mathsf{inl}\,\mathsf{inr}\,\Sigma([\eta^\nu,g^\dagger]^{\bigstar}T_\Sigma\pi)]^\star\\
&\qquad\quad T(\pi+\Sigma T_\Sigma\pi)\,\mathsf{out}\,g)^\dagger && \text{// defn. } T\pi\\
&\quad= ([[(T\,\mathsf{inl})\,\mathsf{out}[\eta^\nu\,\mathsf{inl},g^\dagger],\eta\,\mathsf{inr}]\pi,\\
&\qquad\quad \eta\,\mathsf{inl}\,\mathsf{inr}\,\Sigma([\eta^\nu,g^\dagger]^{\bigstar}(T_\Sigma\pi)(T_\Sigma\pi))]^\star T(\mathsf{inl}+\mathsf{id})u)^\dagger && \text{// } g \text{ guarded}\\
&\quad= ([[(T\,\mathsf{inl})\,\mathsf{out}[\eta^\nu\,\mathsf{inl},g^\dagger],\eta\,\mathsf{inr}]\pi\,\mathsf{inl},\eta\,\mathsf{inl}\,\mathsf{inr}\,\Sigma[\eta^\nu,g^\dagger]^{\bigstar}]^\star u)^\dagger\\
&\quad= ([[\eta\,\mathsf{inl}\,\mathsf{inl}\,\mathsf{inl},\eta\,\mathsf{inr}],\eta\,\mathsf{inl}\,\mathsf{inr}\,\Sigma[\eta^\nu,g^\dagger]^{\bigstar}]^\star u)^\dagger\\
&\quad= ([\eta(\mathsf{inl}\,\mathsf{inl}+\mathsf{id}),\eta\,\mathsf{inl}\,\mathsf{inr}\,\Sigma[\eta^\nu,g^\dagger]^{\bigstar}]^\star u)^\dagger.
\end{aligned}
$$

On the other hand, applying out to the left-hand side yields the same result:

$$\begin{aligned}
&\mathsf{out}\,\diamondsuit(g^\dagger)\\
&= T(\mathsf{inl}+\mathsf{id})((T\pi)\,\mathsf{out}(g^\dagger))^\dagger\\
&= ([(T\,\mathsf{inl})\eta(\mathsf{inl}+\mathsf{id}),\eta\,\mathsf{inr}]^\star(T\pi)\,\mathsf{out}(g^\dagger))^\dagger && \text{// naturality}\\
&= ([[\eta\,\mathsf{inl}\,\mathsf{inl}\,\mathsf{inl},\mathsf{inl}\,\mathsf{inr}],\eta\,\mathsf{inr}]^\star(T\pi)\,\mathsf{out}[\eta^\nu,g^\dagger]^{\bigstar}g)^\dagger\\
&= ([\eta(\mathsf{inl}\,\mathsf{inl}+\mathsf{id}),\eta\,\mathsf{inl}\,\mathsf{inr}]^\star[\mathsf{out}[\eta^\nu,g^\dagger],\eta\,\mathsf{inr}\,\Sigma[\eta^\nu,g^\dagger]^{\bigstar}]^\star\,\mathsf{out}\,g)^\dagger && \text{// defn. } T\pi\\
&= ([\eta(\mathsf{inl}\,\mathsf{inl}+\mathsf{id}),\eta\,\mathsf{inl}\,\mathsf{inr}]^\star[[\eta\,\mathsf{inl},\mathsf{out}\,g^\dagger],\eta\,\mathsf{inr}\,\Sigma[\eta^\nu,g^\dagger]^{\bigstar}]^\star\\
&\qquad T(\mathsf{inl}+\mathsf{id})u)^\dagger && \text{// } g \text{ guarded}\\
&= ([\eta(\mathsf{inl}\,\mathsf{inl}+\mathsf{id}),\eta\,\mathsf{inl}\,\mathsf{inr}]^\star[\eta\,\mathsf{inl},\eta\,\mathsf{inr}\,\Sigma[\eta^\nu,g^\dagger]^{\bigstar}]^\star u)^\dagger\\
&= ([\eta(\mathsf{inl}\,\mathsf{inl}+\mathsf{id}),\eta\,\mathsf{inl}\,\mathsf{inr}\,\Sigma[\eta^\nu,g^\dagger]^{\bigstar}]^\star u)^\dagger.
\end{aligned}$$

Then the goal can be obtained as follows. First, observe the following:

$$\begin{aligned}
g^{\dagger\dagger} &= ([\eta^\nu,g^\dagger]^{\bigstar}h)^\dagger && \text{// 7.8}\\
&= ([[\eta^\nu\,\mathsf{inl},\eta^\nu\,\mathsf{inr}],g^\dagger]^{\bigstar}h)^\dagger\\
&= ([[\eta^\nu\,\mathsf{inl},g^\dagger],\eta\,\mathsf{inr}]^{\bigstar}\diamondsuit(T_\Sigma\pi)\,g)^\dagger && \text{// defn. of } \pi\\
&= ((T_\Sigma[\mathsf{id},\mathsf{inr}])[[\eta^\nu\,\mathsf{inl}\,\mathsf{inl},T_\Sigma\,\mathsf{inl}\,g^\dagger],\eta\,\mathsf{inr}]^{\bigstar}\diamondsuit(T_\Sigma\pi)\,g)^\dagger\\
&= ([[\eta^\nu\,\mathsf{inl}\,\mathsf{inl},T_\Sigma\,\mathsf{inl}\,g^\dagger],\eta\,\mathsf{inr}]^{\bigstar}\diamondsuit(T_\Sigma\pi)\,g)^{\dagger\dagger} && \text{// Clause (2)}\\
&= ([T_\Sigma\,\mathsf{inl}[\eta^\nu\,\mathsf{inl},g^\dagger],\eta\,\mathsf{inr}]^{\bigstar}\diamondsuit(T_\Sigma\pi)\,g)^{\dagger\dagger}\\
&= ([\eta^\nu\,\mathsf{inl},g^\dagger]^{\bigstar}(\diamondsuit(T_\Sigma\pi)\,g)^\dagger)^\dagger && \text{// naturality}\\
&= ([\eta^\nu\,\mathsf{inl},g^\dagger]^{\bigstar}((T_\Sigma\pi)\,g)^\dagger)^\dagger. && \text{// defn. of } \text{–}^\dagger\\
&= [\eta^\nu,([\eta^\nu\,\mathsf{inl},g^\dagger]^{\bigstar}((T_\Sigma\pi)\,g)^\dagger)^\dagger]^{\bigstar}[\eta^\nu\,\mathsf{inl},g^\dagger]^{\bigstar}((T_\Sigma\pi)\,g)^\dagger && \text{// fixpoint}\\
&= [\eta^\nu,g^{\dagger\dagger}]^{\bigstar}[\eta^\nu\,\mathsf{inl},g^\dagger]^{\bigstar}((T_\Sigma\pi)\,g)^\dagger\\
&= [\eta^\nu,[\eta^\nu,g^{\dagger\dagger}]^{\bigstar}g^\dagger]^{\bigstar}((T_\Sigma\pi)\,g)^\dagger\\
&= [\eta^\nu,g^{\dagger\dagger}]^{\bigstar}((T_\Sigma\pi)\,g)^\dagger. && \text{// fixpoint}
\end{aligned}$$

It is easy to see that $((T_\Sigma\pi)\,g)^\dagger$ is guarded, and hence, by the previous calculation, $g^{\dagger\dagger} = ((T_\Sigma\pi)\,g)^{\dagger\dagger}$. Finally, by Clause (2), $((T_\Sigma\pi)\,g)^{\dagger\dagger} = ((T_\Sigma[\mathsf{id},\mathsf{inr}])(T_\Sigma\pi)\,g)^\dagger = (T_\Sigma[\mathsf{id},\mathsf{inr}]g)^\dagger$.

4. *g is unrestricted.* Then,

$$\begin{aligned}
g^{\dagger\dagger} &= (\diamondsuit g)^{\dagger\dagger}\\
&= ((T_\Sigma[\mathsf{id},\mathsf{inr}])\diamondsuit g)^\dagger && \text{// Clause (3)}\\
&= (\diamondsuit(T_\Sigma[\mathsf{id},\mathsf{inr}])\diamondsuit(T_\Sigma\pi)g)^\dagger\\
&= (\diamondsuit(T_\Sigma[\mathsf{id},\mathsf{inr}])(T_\Sigma\pi)g)^\dagger && \text{// 7.6}\\
&= (T_\Sigma[\mathsf{id},\mathsf{inr}]g)^\dagger
\end{aligned}$$

and we are done. It remains to prove (7.6). Observe that by definiton,

$$\diamondsuit(T_\Sigma[\mathsf{id},\mathsf{inr}])\diamondsuit(T_\Sigma\pi)g$$

$$
\begin{aligned}
&= \diamondsuit T_\Sigma[\mathsf{id},\mathsf{inr}]\,\mathsf{out}^{\text{-}1}\,T(\mathsf{inl}+\mathsf{id})((T\pi)\,\mathsf{out}\,g)^\dagger \\
&= \mathsf{out}^{\text{-}1}\,T(\mathsf{inl}+\mathsf{id})((T\pi)\,\mathsf{out}\,T_\Sigma[\mathsf{id},\mathsf{inr}]\,\mathsf{out}^{\text{-}1}\,T(\mathsf{inl}+\mathsf{id})((T\pi)\,\mathsf{out}\,g)^\dagger)^\dagger
\end{aligned}
$$

Let us further transform the expression after $\mathsf{out}^{\text{-}1}\,T(\mathsf{inl}+\mathsf{id})$:

$$
\begin{aligned}
&((T\pi)T([\mathsf{id},\mathsf{inr}]+\Sigma T_\Sigma[\mathsf{id},\mathsf{inr}])T(\mathsf{inl}+\mathsf{id})((T\pi)\,\mathsf{out}\,g)^\dagger)^\dagger \\
&= ((T\pi)T(\mathsf{id}+\Sigma T_\Sigma[\mathsf{id},\mathsf{inr}])((T\pi)\,\mathsf{out}\,g)^\dagger)^\dagger \\
&= (([T\,\mathsf{inl}\,\pi\eta(\mathsf{id}+\Sigma T_\Sigma[\mathsf{id},\mathsf{inr}]),\eta\,\mathsf{inr}]^\star(T\pi)\,\mathsf{out}\,g)^\dagger)^\dagger && \text{// naturality} \\
&= ([(T\pi)\eta(\mathsf{id}+\Sigma T_\Sigma[\mathsf{id},\mathsf{inr}]),(T\pi)\eta\,\mathsf{inl}\,\mathsf{inr}]^\star(T\pi)\,\mathsf{out}\,g)^\dagger && \text{// codiagonal} \\
&= ((T\pi)[[\eta\,\mathsf{inl},\eta\,\mathsf{inl}\,\mathsf{inr}],\eta\,\mathsf{inr}\,\Sigma T_\Sigma[\mathsf{id},\mathsf{inr}]]^\star\,\mathsf{out}\,g)^\dagger \\
&= ((T\pi)[\mathsf{out}(\eta^\nu[\mathsf{id},\mathsf{inr}]),\eta\,\mathsf{inr}\,\Sigma T_\Sigma[\mathsf{id},\mathsf{inr}]]^\star\,\mathsf{out}\,g)^\dagger \\
&= ((T\pi)\,\mathsf{out}(T_\Sigma[\mathsf{id},\mathsf{inr}])g)^\dagger.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&\diamondsuit(T_\Sigma[\mathsf{id},\mathsf{inr}])\diamondsuit(T_\Sigma\pi)g \\
&= \mathsf{out}^{\text{-}1}\,T(\mathsf{inl}+\mathsf{id})((T\pi)\,\mathsf{out}(T_\Sigma[\mathsf{id},\mathsf{inr}])g)^\dagger \\
&= \diamondsuit T_\Sigma[\mathsf{id},\mathsf{inr}]g
\end{aligned}
$$

and we are done. ∎

Lemma 7.4. *The assignment of $f^\dagger : X \to T_\Sigma Y$ to $f : X \to T_\Sigma(Y+X)$ defined by (7.3) is compatible with strength, i.e.*

$$\tau^\nu(\mathsf{id}\times f^\dagger) = ((T_\Sigma\,\mathsf{dist})\tau^\nu(\mathsf{id}\times f))^\dagger.$$

Proof. Let f be guarded with $\mathsf{out}\,f = T(\mathsf{inl}+\mathsf{id})u$. Then, $f' = (T_\Sigma\,\mathsf{dist})\tau^\nu(\mathsf{id}\times f)$ is also guarded with $\mathsf{out}\,f' = T(\mathsf{inl}+\mathsf{id})T(\mathsf{id}+\Sigma((T_\Sigma\,\mathsf{dist})\tau^\nu))(T\delta)\tau(\mathsf{id}\times u)$ where δ is as in Theorem 6.3 (besides guardedness of f, the proof of this equation uses naturality of out and the definitions of τ and dist). The following calculation shows that $\tau^\nu(\mathsf{id}\times f^\dagger)$ satisfies the fixpoint law for $((T_\Sigma\,\mathsf{dist})\tau^\nu(\mathsf{id}\times f))^\dagger$:

$$
\begin{aligned}
&\tau^\nu(\mathsf{id}\times f^\dagger) \\
&= \tau^\nu(\mathsf{id}\times[\eta^\nu,f^\dagger]^{\maltese}f) \\
&= \tau^\nu(\mathsf{id}\times[\eta^\nu,f^\dagger]^{\maltese})(\mathsf{id}\times f) \\
&= (\tau^\nu(\mathsf{id}\times[\eta^\nu,f^\dagger]))^{\maltese}\tau^\nu(\mathsf{id}\times f) && \text{// STR}_4 \\
&= ([\eta^\nu,\tau^\nu(\mathsf{id}\times f^\dagger)]\,\mathsf{dist})^{\maltese}\tau^\nu(\mathsf{id}\times f) && \text{// 5.7} \\
&= [\eta^\nu,\tau^\nu(\mathsf{id}\times f^\dagger)]^{\maltese}(T_\Sigma\,\mathsf{dist})\tau^\nu(\mathsf{id}\times f) \\
&= [\eta^\nu,\tau^\nu(\mathsf{id}\times f^\dagger)]^{\maltese}\diamondsuit(T_\Sigma\,\mathsf{dist})\tau^\nu(\mathsf{id}\times f),
\end{aligned}
$$

and hence $\tau^\nu(\mathsf{id}\times f^\dagger)$ and $((T_\Sigma\,\mathsf{dist})\tau^\nu(\mathsf{id}\times f))^\dagger$ are equal.

The general case reduces to the guarded case by means of the equation

$$(T_\Sigma\,\mathsf{dist})\tau^\nu(\mathsf{id}\times\diamondsuit f) = \diamondsuit((T_\Sigma\,\mathsf{dist})\tau^\nu(\mathsf{id}\times f)), \tag{7.9}$$

as follows:

$$\begin{aligned}
\tau^\nu(\mathsf{id}\times f^\dagger) &= \tau^\nu(\mathsf{id}\times(\diamondsuit f)^\dagger) && \text{// definition of } \text{–}^\dagger\\
&= ((T\,\mathsf{dist})\tau^\nu(\mathsf{id}\times\diamondsuit f))^\dagger\\
&= (\diamondsuit(T_\Sigma\,\mathsf{dist}\,\tau^\nu(\mathsf{id}\times f)))^\dagger && \text{// 7.9}\\
&= (T_\Sigma\,\mathsf{dist}\,\tau^\nu(\mathsf{id}\times f))^\dagger. && \text{// definition of } \text{–}^\dagger
\end{aligned}$$

We show (7.9) by establishing commutativity of the following diagram where $Q = C\times B + C\times A$ (the identity in question is read from the border):

$$\begin{tikzcd}
 & C\times A \arrow[dl, bend right, "\mathsf{id}\times((T\pi)\,\mathsf{out}\,f)^\dagger"'] \arrow[dr, bend left, "((T\pi)\,\mathsf{out}(T_\Sigma\,\mathsf{dist})\tau^\nu(\mathsf{id}\times f))^\dagger"] & \\
C\times T(B+\Sigma T_\Sigma(B+A)) \arrow[rr, "T(\mathsf{id}+\Sigma T_\Sigma\,\mathsf{dist})T(\mathsf{id}+\Sigma\tau^\nu)(T\delta)\tau"] \arrow[d, "\mathsf{id}\times T(\mathsf{inl}+\mathsf{id})"'] & & T(C\times B+\Sigma T_\Sigma Q) \arrow[d, "T(\mathsf{inl}+\mathsf{id})"] \\
C\times T((B+A)+\Sigma T_\Sigma(B+A)) \arrow[rr, "T(\mathsf{dist}+\Sigma T_\Sigma\,\mathsf{dist})T(\mathsf{id}+\Sigma\tau^\nu)(T\delta)\tau"] \arrow[dr, bend right, "(T_\Sigma\,\mathsf{dist})\tau^\nu(\mathsf{id}\times\mathsf{out}^{-1})"'] & & T(Q+\Sigma T_\Sigma Q) \arrow[dl, bend left, "\mathsf{out}^{-1}"] \\
 & T_\Sigma Q &
\end{tikzcd}$$

The bottom triangle commutes as follows:

$$\begin{aligned}
&(T_\Sigma\,\mathsf{dist})\tau^\nu(\mathsf{id}\times\mathsf{out}^{-1})\\
&= (T_\Sigma\,\mathsf{dist})\,\mathsf{out}^{-1}\,T(\mathsf{id}+\Sigma\tau^\nu)(T\delta)\tau(\mathsf{id}\times\mathsf{out})(\mathsf{id}\times\mathsf{out}^{-1})\\
&= \mathsf{out}^{-1}\,\mathsf{out}(T_\Sigma\,\mathsf{dist})\,\mathsf{out}^{-1}\,T(\mathsf{id}+\Sigma\tau^\nu)(T\delta)\tau\\
&= \mathsf{out}^{-1}\,T(\mathsf{dist}+\Sigma T_\Sigma\,\mathsf{dist})T(\mathsf{id}+\Sigma\tau^\nu)(T\delta)\tau.
\end{aligned}$$

The middle square commutes by properties of τ, dist and δ:

$$\begin{aligned}
&T(\mathsf{dist}+\Sigma(T_\Sigma\,\mathsf{dist}\,\tau^\nu))(T\delta)\tau(\mathsf{id}\times T(\mathsf{inl}+\mathsf{id}))\\
&= T(\mathsf{dist}+\Sigma(T_\Sigma\,\mathsf{dist}\,\tau^\nu))(T\delta)T(\mathsf{id}\times(\mathsf{inl}+\mathsf{id}))\tau && \text{// naturality of } \tau\\
&= T(\mathsf{dist}+\Sigma(T_\Sigma\,\mathsf{dist}\,\tau^\nu))T((\mathsf{id}\times\mathsf{inl})+\mathsf{id})(T\delta)\tau && \text{// naturality of } \delta\\
&= T(\mathsf{dist}(\mathsf{id}\times\mathsf{inl})+\mathsf{id})T(\mathsf{id}+\Sigma(T_\Sigma\,\mathsf{dist}\,\tau^\nu))(T\delta)\tau\\
&= T(\mathsf{inl}+\mathsf{id})T(\mathsf{id}+\Sigma(T_\Sigma\,\mathsf{dist}\,\tau^\nu))(T\delta)\tau.
\end{aligned}$$

This leaves us with the top triangle. Let $\alpha = (\mathsf{id}+\Sigma(T_\Sigma\,\mathsf{dist}\,\tau^\nu))\delta$. We apply the assumption that τ is compatible with iteration to $(T\alpha)\tau(\mathsf{id}\times((T\pi)\,\mathsf{out}\,f)^\dagger)$ and further calculate as follows:

$$\begin{aligned}
&(T\alpha)\tau(\mathsf{id}\times((T\pi)\,\mathsf{out}\,f)^\dagger)\\
&= (T\alpha)(T\,\mathsf{dist}\,\tau(\mathsf{id}\times(T\pi)\,\mathsf{out}\,f))^\dagger\\
&= (T(\alpha+\mathsf{id})(T\,\mathsf{dist})\tau(\mathsf{id}\times(T\pi)\,\mathsf{out}\,f))^\dagger && \text{(naturality)}
\end{aligned}$$

$$
\begin{aligned}
&= (T(\alpha + \mathsf{id})(T\,\mathsf{dist})T(\mathsf{id} \times \pi)\tau(\mathsf{id} \times \mathsf{out})(\mathsf{id} \times f))^\dagger && \text{// naturality of } \tau \\
&= (T((\mathsf{id} + \Sigma(T_\Sigma\,\mathsf{dist}\,\tau^\nu) + \mathsf{id}) \\
&\qquad T(\delta + \mathsf{id})(T\,\mathsf{dist})T(\mathsf{id} \times \pi)\tau(\mathsf{id} \times \mathsf{out})(\mathsf{id} \times f))^\dagger && \text{// definition of } \alpha \\
&= (T((\mathsf{id} + \Sigma(T_\Sigma\,\mathsf{dist}\,\tau^\nu) + \mathsf{id}) \\
&\qquad T((\mathsf{id} + \rho) + \mathsf{id})T(\mathsf{dist} + \mathsf{id})(T\,\mathsf{dist})T(\mathsf{id} \times \pi) \\
&\qquad \tau(\mathsf{id} \times \mathsf{out})(\mathsf{id} \times f))^\dagger. && \text{// definition of } \delta
\end{aligned}
$$

At this position we apply the obvious identity

$$T(\mathsf{dist} + \mathsf{id})(T\,\mathsf{dist})T(\mathsf{id} \times \pi) = (T\pi)T(\mathsf{dist} + \mathsf{id})(T\,\mathsf{dist})$$

and then proceed as follows:

$$
\begin{aligned}
&(T((\mathsf{id} + \Sigma(T_\Sigma\,\mathsf{dist}\,\tau^\nu) + \mathsf{id}) \\
&\quad T((\mathsf{id} + \rho) + \mathsf{id})(T\pi)T(\mathsf{dist} + \mathsf{id})(T\,\mathsf{dist}) \\
&\quad \tau(\mathsf{id} \times \mathsf{out})(\mathsf{id} \times f))^\dagger \\
&= (T((\mathsf{id} + \Sigma(T_\Sigma\,\mathsf{dist}\,\tau^\nu) + \mathsf{id}) \\
&\quad (T\pi)T(\mathsf{dist} + \rho)(T\,\mathsf{dist})\tau(\mathsf{id} \times \mathsf{out})(\mathsf{id} \times f))^\dagger \\
&= (T((\mathsf{id} + \Sigma(T_\Sigma\,\mathsf{dist}\,\tau^\nu)) + \mathsf{id}) \\
&\quad (T\pi)T(\mathsf{dist} + \mathsf{id})(T\delta)\tau(\mathsf{id} \times \mathsf{out})(\mathsf{id} \times f))^\dagger && \text{// definition of } \delta \\
&= ((T\pi)T(\mathsf{dist} + \Sigma T_\Sigma\,\mathsf{dist}) \\
&\quad T(\mathsf{id} + \Sigma\tau^\nu)(T\delta)\tau(\mathsf{id} \times \mathsf{out})(\mathsf{id} \times f))^\dagger \\
&= ((T\pi)T(\mathsf{dist} + \Sigma T_\Sigma\,\mathsf{dist})\,\mathsf{out}\,\tau^\nu(\mathsf{id} \times f))^\dagger && \text{// definition of } \tau^\nu \\
&= ((T\pi)\,\mathsf{out}(T_\Sigma\,\mathsf{dist})\tau^\nu(\mathsf{id} \times f))^\dagger.
\end{aligned}
$$

This yields the proof of the top triangle of the diagram and therefore completes the proof of the lemma. □

Finally, we can return to the proof of Theorem 7.1.

Proof of Theorem 7.1. As we indicated above, the second clause is already proved by Uustalu [40]. To show the existence part of the first clause we call on the above Lemmas 7.2, 7.3 and 7.4 and additionally prove that iteration on $\mathbb{T}_\Sigma$ extends iteration on $\mathbb{T}$, i.e. that (7.1) implies (7.2). Let us call morphisms f for which there is g satisfying (7.1) *completely unguarded*. Suppose that (7.1) holds. Then the proof of (7.2) runs as follows:

$$
\begin{aligned}
\mathsf{out}\, f^\dagger &= (\Diamond f)^\dagger \\
&= \mathsf{out}(\mathsf{out}^{-1}\, T(\mathsf{inl} + \mathsf{id})((T\pi)\,\mathsf{out}\, f)^\dagger)^\dagger \\
&= \mathsf{out}(\mathsf{out}^{-1}\, T(\mathsf{inl} + \mathsf{id})((T\pi)(T\,\mathsf{inl})g)^\dagger)^\dagger && (7.1) \\
&= \mathsf{out}(\mathsf{out}^{-1}\, T(\mathsf{inl} + \mathsf{id})((T\,\mathsf{inl} + \mathsf{id})g)^\dagger)^\dagger \\
&= \mathsf{out}(\mathsf{out}^{-1}\, T(\mathsf{inl} + \mathsf{id})(T\,\mathsf{inl})g^\dagger)^\dagger && \text{// naturality} \\
&= \mathsf{out}(\mathsf{out}^{-1}\, T(\mathsf{inl}\,\mathsf{inl})g^\dagger)^\dagger \\
&= \mathsf{out}[\eta^\nu, f^\dagger]^{\star}\,\mathsf{out}^{-1}\, T(\mathsf{inl}\,\mathsf{inl})g^\dagger && \text{// fixpoint}
\end{aligned}
$$

$$
\begin{aligned}
&= \mathsf{out}[\eta^\nu, f^\dagger]^{\maltese}\, \mathsf{out}^{\text{-}1}\, T(\mathsf{inl}\,\mathsf{inl})g^\dagger \\
&= [\mathsf{out}[\eta^\nu, f^\dagger], \eta\, \mathsf{inr}\, \Sigma[\eta^\nu, f^\dagger]^{\maltese}]^\star T(\mathsf{inl}\,\mathsf{inl})g^\dagger && (6.2)\\
&= [\mathsf{out}\,\eta^\nu]^\star g^\dagger \\
&= [\eta\,\mathsf{inl}]^\star g^\dagger \\
&= (T\,\mathsf{inl})\, g^\dagger.
\end{aligned}
$$

It remains to show the uniqueness part of the first clause. To that end we first show that any morphism $f : X \to T_\Sigma(Y + X)$ can be decomposed by means of morphisms $g : X \to T_\Sigma(Z + X)$ and $h : Z \to T_\Sigma(Y + X)$, where $Z = Y + \Sigma T_\Sigma(Y + X)$, as

$$f = [h, \eta^\nu\, \mathsf{inr}]^{\maltese} g \tag{7.10}$$

with completely unguarded g. Next we show that

$$f^\dagger = (h^{\maltese} g^\dagger)^\dagger \tag{7.11}$$

and that

$$h^{\maltese} g^\dagger = \Diamond f. \tag{7.12}$$

In summary, we obtain that $f^\dagger = (h^{\maltese} g^\dagger)^\dagger = (\Diamond f)^\dagger$. The following proofs of (7.11) and (7.12) do not depend on the concrete definition of $\text{–}^\dagger$ on $\mathbb{T}_\Sigma$ but only use its abstract properties as an iteration operator of a complete Elgot monad and compatibility with the underlying iteration operator for $\mathbb{T}$. Hence, the identity $f^\dagger = (\Diamond f)^\dagger$ would be valid for any other such operator, but since $(\Diamond f)^\dagger$ is uniquely defined all of them must agree.

Let $g = \mathsf{out}^{\text{-}1}\, T(\mathsf{inl}\,\pi)\, \mathsf{out}\, f$ (recall that $\pi = [\mathsf{inl} + \mathsf{id}, \mathsf{inl}\,\mathsf{inr}]$), which is, by definition, completely unguarded, and let $h = \mathsf{out}^{\text{-}1}\, \eta(\mathsf{inl} + \mathsf{id})$.

Then the proof of (7.10) runs as follows:

$$
\begin{aligned}
&[h, \eta^\nu\, \mathsf{inr}]^{\maltese} g \\
&\quad= [\mathsf{out}^{\text{-}1}\, \eta(\mathsf{inl} + \mathsf{id}), \eta^\nu\, \mathsf{inr}]^{\maltese} g \\
&\quad= [\mathsf{out}^{\text{-}1}\, \eta(\mathsf{inl} + \mathsf{id}), \mathsf{out}^{\text{-}1}\, \eta\, \mathsf{inl}\,\mathsf{inr}]^{\maltese} g && \text{// Theorem 6.3}\\
&\quad= (\mathsf{out}^{\text{-}1}\, \eta[\mathsf{inl} + \mathsf{id}, \mathsf{inl}\,\mathsf{inr}])^{\maltese} g \\
&\quad= (\mathsf{out}^{\text{-}1}\, \eta\, \pi)^{\maltese} g \\
&\quad= \mathsf{out}^{\text{-}1}\, [\eta\, \pi, \eta\, \mathsf{inr}\, \Sigma(\mathsf{out}^{\text{-}1}\, \eta\, \pi)^{\maltese}]^\star\, \mathsf{out}\, g && \text{// Theorem 6.3}\\
&\quad= \mathsf{out}^{\text{-}1}\, (\eta\, [\pi, \mathsf{inr}\, \Sigma(\mathsf{out}^{\text{-}1}\, \eta\, \pi)^\star])^\star T(\mathsf{inl}\,\pi)\, \mathsf{out}\, f \\
&\quad= \mathsf{out}^{\text{-}1}\, (\eta\, \pi)^\star (T\pi)\, \mathsf{out}\, f \\
&\quad= \mathsf{out}^{\text{-}1}\, (T\pi)\, (T\pi)\, \mathsf{out}\, f \\
&\quad= f.
\end{aligned}
$$

Next, we show (7.11):

$$
\begin{aligned}
(h^{\maltese} g^\dagger)^\dagger &= ((T_\Sigma \mathsf{inl}\ h, \eta^\nu\, \mathsf{inr}]^{\maltese} g)^\dagger)^\dagger && \text{// naturality}\\
&= (T_\Sigma[\mathsf{id}, \mathsf{inr}][(T_\Sigma \mathsf{inl})h, \eta^\nu\, \mathsf{inr}]^{\maltese} g)^\dagger && \text{// codiagonal}\\
&= ([h, T_\Sigma[\mathsf{id}, \mathsf{inr}]\eta^\nu\, \mathsf{inr}]^{\maltese} g)^\dagger \\
&= ([h, \eta^\nu\, \mathsf{inr}]^{\maltese} g)^\dagger
\end{aligned}
$$

$$= f^\dagger.$$

Finally, we prove (7.12):

$$\begin{aligned}
h^{\bigstar} g^\dagger &= (\mathsf{out}^{\text{-}1}\,\eta(\mathsf{inl} + \mathsf{id}))^{\bigstar} g^\dagger && \text{// definition of } \mathsf{-}^{\bigstar}\\
&= \mathsf{out}^{\text{-}1}\,[\eta(\mathsf{inl} + \mathsf{id}), \eta\,\mathsf{inr}\,\Sigma h^{\bigstar}]^\star\,\mathsf{out}\,g^\dagger\\
&= \mathsf{out}^{\text{-}1}\,[\eta(\mathsf{inl} + \mathsf{id}), \eta\,\mathsf{inr}\,\Sigma h^{\bigstar}]^\star (T\mathsf{inl})((T\pi)\,\mathsf{out}\,f)^\dagger && \text{// } g \text{ compl. ung.}\\
&= \mathsf{out}^{\text{-}1}\,T(\mathsf{inl} + \mathsf{id})((T\pi)\,\mathsf{out}\,f)^\dagger\\
&= \Diamond f.
\end{aligned}$$

This finishes the proof. □

8. A Coproduct Characterization of Coinductive Resumptions

Our second main result is a universal characterization of the coinductive resumption monad transformer. Essentially, we show that $\mathbb{T}_\Sigma$ arises as the coproduct of $\mathbb{T}$ with the free complete Elgot monad over Σ (modulo existence of the latter) in the category of complete Elgot monads on $\mathbf{C}$ (see Section 9 for discussion of a similar result on *completely iterative* monads). In other words, $\mathbb{T}_\Sigma$ really does freely extend $\mathbb{T}$ by Σ in a fully formal sense. We begin by recording the relevant notion of morphism of complete Elgot monads:

Definition 8.1. A *complete Elgot monad morphism* $\xi : \mathbb{R} \to \mathbb{S}$ between complete Elgot monads $\mathbb{R}$, $\mathbb{S}$ is a morphism ξ between the underlying strong monads (i.e. $\xi\,\eta = \eta$, $\xi\,f^\star = (\xi\,f)^\star\,\xi$ for $f : X \to RY$, and $\xi\,\tau = \tau(\mathsf{id} \times \xi)$, see [27]) additionally satisfying

$$(\xi\,g)^\dagger = \xi\,g^\dagger$$

for $g : X \to R(Y + X)$. Complete Elgot monads over $\mathbf{C}$ and their morphisms form an (overlarge) category $\mathbf{CElg}(\mathbf{C})$. We have a forgetful functor from $\mathbf{CElg}(\mathbf{C})$ to the category of strong functors and strong natural transformations; mention of *free* complete Elgot monads refers to this forgetful functor.

Note next that the coinductive resumption monad T_Σ implements, by construction, all the operations of Σ, that is, we have a canonical strong natural transformation $\iota^{\mathbb{T}} : \Sigma \to T_\Sigma$, given by

$$\iota^{\mathbb{T}}_X = \mathsf{out}^{\text{-}1}\,\eta\,\mathsf{inr}\,\Sigma\eta^\nu$$

where the typing of the composite is shown in

$$\Sigma X \xrightarrow{\Sigma\eta^\nu} ST_\Sigma X \xrightarrow{\mathsf{inr}} X + ST_\Sigma X \xrightarrow{\eta} T(X + ST_\Sigma X) \xrightarrow{\mathsf{out}^{\text{-}1}} T_\Sigma X.$$

Moreover, recall that T maps into T_Σ via a natural transformation $\mathsf{ext} : T \to T_\Sigma$ defined in Equation (6.1). We have

Lemma 8.2. *The natural transformation* $\mathsf{ext} : \mathbb{T} \to \mathbb{T}_\Sigma$ *is a complete Elgot monad morphism.*

Proof. Let us verify the identities

$$\xi\eta = \eta \qquad \xi f^\star = (\xi f)^\star\xi \qquad \xi\tau = \tau(\mathsf{id} \times \xi) \qquad (\xi g)^\dagger = \xi g^\dagger \tag{8.1}$$

with $f : X \to TY$ and $g : X \to T(Y + X)$ from left to right.

• Compatibility of ext with unit is a straightforward consequence of Theorem 6.3: $\mathsf{ext}\,\eta = \mathsf{out}^{\text{-}1}\,(T\mathsf{inl})\eta = \mathsf{out}^{\text{-}1}\,\eta\,\mathsf{inl} = \eta^\nu$.

• In order to show compatibility of ext with Kleisli star we call the definition of the latter from Theorem 6.3:

$$\begin{aligned}(\mathsf{ext}\,g)^{\bigstar}\,\mathsf{ext} &= (\mathsf{out}^{\text{-}1}\,(T\mathsf{inl})g)^{\bigstar}\,\mathsf{out}^{\text{-}1}\,(T\mathsf{inl})\\ &= \mathsf{out}^{\text{-}1}\,[\mathsf{out}\,\mathsf{out}^{\text{-}1}\,(T\mathsf{inl})g, \eta\,\mathsf{inr}\,\Sigma(\mathsf{ext}\,g)^{\bigstar}]^{\star}(T\mathsf{inl})\\ &= \mathsf{out}^{\text{-}1}\,((T\mathsf{inl})g)^{\star}\\ &= \mathsf{out}^{\text{-}1}\,(T\mathsf{inl})g^{\star}\\ &= \mathsf{ext}\,g^{\star}.\end{aligned}$$

• Recall the distributivity transformation $\delta : A \times (B + \Sigma C) \to A \times B + \Sigma(A \times C)$ from Theorem 6.3. Then by the corresponding definition of τ^{ν},

$$\begin{aligned}\tau^{\nu}(\mathsf{id}\times\mathsf{ext}) &= \mathsf{out}^{\text{-}1}\,T(\mathsf{id}+\Sigma\tau^{\nu})(T\delta)\tau(\mathsf{id}\times\mathsf{out}\,\mathsf{ext})\\ &= \mathsf{out}^{\text{-}1}\,T(\mathsf{id}+\Sigma\tau^{\nu})(T\delta)\tau(\mathsf{id}\times T\mathsf{inl})\\ &= \mathsf{out}^{\text{-}1}\,T(\mathsf{id}+\Sigma\tau^{\nu})T(\mathsf{id}+\rho)(T\,\mathsf{dist})T(\mathsf{id}+\mathsf{inl})\tau\\ &= \mathsf{out}^{\text{-}1}\,T(\mathsf{id}+\Sigma\tau^{\nu})T(\mathsf{id}+\rho)(T\,\mathsf{inl})\tau\\ &= \mathsf{out}^{\text{-}1}\,T(\mathsf{id}+\Sigma\tau^{\nu})(T\mathsf{inl})\tau\\ &= \mathsf{out}^{\text{-}1}\,(T\mathsf{inl})\tau\\ &= \mathsf{ext}\,\tau.\end{aligned}$$

• Since $\mathsf{out}\ \mathsf{ext}\,g = (T_\Sigma\mathsf{inl})g$, then by Theorem 7.1, $\mathsf{out}(\mathsf{ext}\,g)^{\dagger} = (T\,\mathsf{inl})g^{\dagger}$, from which the last identity in (8.1) follows by composition with $\mathsf{out}^{\text{-}1}$ on the left. □

Summing up, we have, slightly abusing notation, a cospan of strong natural transformations

$$\mathbb{T} \xrightarrow{\mathsf{ext}} \mathbb{T}_\Sigma \xleftarrow{\iota^{\mathbb{T}}} \Sigma$$

with the left arrow being a complete Elgot monad morphism. It turns out that this gives a universal characterization of $\mathbb{T}_\Sigma$ in terms of being composed of $\mathbb{T}$ and Σ:

Theorem 8.3. *The cospan $\mathbb{T} \xrightarrow{\mathsf{ext}} \mathbb{T}_\Sigma \xleftarrow{\iota^{\mathbb{T}}} \Sigma$ is universal. Explicitly: Given a complete Elgot monad $\mathbb{S}$, a strong natural transformation $\upsilon : \Sigma \to S$, and a complete Elgot monad morphism $\sigma : \mathbb{T} \to \mathbb{S}$, there exists a unique complete Elgot monad morphism $\xi : \mathbb{T}_\Sigma \to \mathbb{S}$ such that $\xi\,\mathsf{ext} = \sigma$ and $\xi\iota^{\mathbb{T}} = \upsilon$:*

$$\begin{tikzcd} \mathbb{T} \arrow[r, "\mathsf{ext}"] \arrow[dr, "\sigma"'] & \mathbb{T}_\Sigma \arrow[d, dotted, "\xi"] & \Sigma \arrow[l, "\iota^{\mathbb{T}}"'] \arrow[dl, "\upsilon"] \\ & \mathbb{S}. & \end{tikzcd} \tag{8.2}$$

Specifically, ξ is given as $\xi = \zeta^{\dagger}$ with ζ defined componentwise by

$$T_\Sigma X \xrightarrow{\mathsf{out}} T(X + \Sigma T_\Sigma X) \xrightarrow{\sigma} S(X + \Sigma T_\Sigma X) \xrightarrow{[\eta\,\mathsf{inl}, (S\,\mathsf{inr})\upsilon]^{\star}} S(X + T_\Sigma X).$$

In other words, $\mathbb{T}_\Sigma$ is free as a complete Elgot monad over Σ that extends $\mathbb{T}$.

Example 8.4. Let us spell out what a strong natural transformation $\upsilon : \Sigma \to S$ amounts to in the case $\Sigma X = \sum_i a_i \times X^{b_i}$ (Example 4.1). A natural transformation $\upsilon : \Sigma \to S$ is equivalent to a family of natural transformations $a_i \times (-)^{b_i} \to S$, equivalently $(-)^{b_i} \to S^{a_i}$, each of which is, by the enriched Yoneda lemma, equivalent to an element of $(Sb_i)^{a_i}$, i.e. a morphism $u_i : a_i \to Sb_i$. Concretely, υ is assembled from the u_i as follows:

$$\upsilon_X = [\lambda\langle x, f\rangle.\,(Sf)(u_1(x)), \dots, \lambda\langle x, f\rangle.\,(Sf)(u_n(x))].$$

Note that the above generic argument makes use of the assumption that $\mathbf{C}$ is Cartesian closed. In fact it suffices to assume that only the exponentials $(-)^{b_i}$ exist (in particular we do not actually need the exponentials $(-)^{a_i}$ mentioned in between). The expressions $\lambda\langle x, f\rangle.\,(Sf)(u_i(x))$ above then have to be read as

$$a_i \times X^{b_i} \xrightarrow{\mathsf{swap}} X^{b_i} \times a_i \xrightarrow{\mathsf{id}\times u_i} X^{b_i} \times Sb_i \xrightarrow{\tau} S(X^{b_i} \times b_i) \xrightarrow{S\,\mathsf{ev}} SX.$$

where swap and ev are the obvious swapping and evaluation transformations respectively.

If $\mathbf{CElg}(\mathbf{C})$ has an initial object, then the statement of Theorem 8.3 can be phrased slightly more concisely. We later give a sufficient criterion on $\mathbf{C}$ that ensures this (Theorem 8.10).

Corollary 8.5. *Suppose that $\mathbf{CElg}(\mathbf{C})$ has an initial object $\mathbb{L}$. Then*

(1) *$\mathbb{L}_\Sigma$ is the free complete Elgot monad over the strong functor $\Sigma : \mathbf{C} \to \mathbf{C}$, with universal arrow*
$$\iota^{\mathbb{L}} : \Sigma \to L_\Sigma.$$
(2) *For any complete Elgot monad $\mathbb{T}$, the coinductive generalized resumption monad $\mathbb{T}_\Sigma$ is the coproduct of $\mathbb{T}$ and $\mathbb{L}_\Sigma$ in $\mathbf{CElg}(\mathbf{C})$, with left injection $\mathsf{ext} : \mathbb{T} \to \mathbb{T}_\Sigma$ and with the right injection being the free extension of $\iota^{\mathbb{T}} : \Sigma \to T_\Sigma$ to $\mathbb{L}_\Sigma$.*

Proof. Claim (1) is proved by taking $\mathbb{T} = \mathbb{L}$ in Theorem 8.3. Claim (2) is then immediate. □

We assemble some auxiliary results before embarking on the proof of Theorem 8.3.

Lemma 8.6. *The Kleisli composition of a complete Elgot monad $\mathbb{T}$ can be characterized in terms of iteration as follows:*

$$g^\star f = [T(\mathsf{inr}\,\mathsf{inr})f, (T\,\mathsf{inl})\;g]^\dagger\,\mathsf{inl} \tag{8.3}$$

Proof. By straightforward calculation:

$$\begin{aligned}
&[T(\mathsf{inr}\,\mathsf{inr})f, (T\,\mathsf{inl})g]^\dagger\,\mathsf{inl}\\
&= [\eta, [T(\mathsf{inr}\,\mathsf{inr})f, (T\,\mathsf{inl})g]^\dagger]^\star T(\mathsf{inr}\,\mathsf{inr})f && \text{// fixpoint}\\
&= ([\eta, [T(\mathsf{inr}\,\mathsf{inr})f, (T\,\mathsf{inl})g]^\dagger]\,\mathsf{inr}\,\mathsf{inr})^\star f\\
&= ([T(\mathsf{inr}\,\mathsf{inr})f, (T\,\mathsf{inl})g]^\dagger\,\mathsf{inr})^\star f\\
&= ([\eta, [T(\mathsf{inr}\,\mathsf{inr})f, (T\,\mathsf{inl})\,g]^\dagger]^\star (T\,\mathsf{inl})\;g)^\star f && \text{// fixpoint}\\
&= g^\star f.
\end{aligned}$$
□

Lemma 8.7. *Let $f : X \to T(Y + X)$. Then $[\eta, f^\dagger]^\star = (T(\mathsf{id} + f))^\dagger$.*

Proof. Consider the following trivially commuting diagram

$$\begin{tikzcd}
X \arrow[rr, "f"] \arrow[d, "f"] && T(Y+X) \arrow[d, "T(\mathsf{id}+f)"]\\
T(Y+X) \arrow[rr, "T(\mathsf{id}+f)"] && T(Y+T(Y+X))
\end{tikzcd}$$

By uniformity, this implies $f^\dagger = (T(\mathsf{id} + f))^\dagger f$. Therefore $[\eta, f^\dagger]^\star = [\eta, (T(\mathsf{id} + f))^\dagger f]^\star = [\eta, (T(\mathsf{id} + f))^\dagger]^\star T(\mathsf{id} + f) = (T(\mathsf{id} + f))^\dagger$ and we are done. □

We now proceed with the proof of the universal property:

Proof of Theorem 8.3. We first show that ξ has the requisite properties, and then prove uniqueness.

Commutation of Diagram 8.2. We need to show that $\xi\iota^{\mathbb{T}} = \upsilon$ and $\xi\,\mathsf{ext} = \sigma$. Put $w = [\eta\,\mathsf{inl}, (S\,\mathsf{inr})\upsilon]^\star\sigma$. Then we have

$$
\begin{aligned}
\xi\iota^{\mathbb{T}} &= \xi\,\mathsf{out}^{-1}\,\eta\,\mathsf{inr}\,\Sigma\eta^\nu \\
&= (w\,\mathsf{out})^\dagger\,\mathsf{out}^{-1}\,\eta\,\mathsf{inr}\,\Sigma\eta^\nu \\
&= [\eta, (w\,\mathsf{out})^\dagger]^\star w\,\mathsf{out}\,\mathsf{out}^{-1}\,\eta\,\mathsf{inr}\,\Sigma\eta^\nu \\
&= [\eta, (w\,\mathsf{out})^\dagger]^\star[\eta\,\mathsf{inl}, (S\,\mathsf{inr})\upsilon]^\star\sigma\eta\,\mathsf{inr}\,\Sigma\eta^\nu \\
&= [\eta, (w\,\mathsf{out})^\dagger]^\star[\eta\,\mathsf{inl}, (S\,\mathsf{inr})\upsilon]\,\mathsf{inr}\,\Sigma\eta^\nu \\
&= [\eta, (w\,\mathsf{out})^\dagger]^\star S(\mathsf{inr})\upsilon\Sigma\eta^\nu \\
&= [\eta, \xi]^\star S(\mathsf{inr}\,\eta^\nu)\upsilon \\
&= (\xi\eta^\nu)^\star\upsilon \\
&= \eta^\star\upsilon \\
&= \upsilon,
\end{aligned}
$$

and

$$
\begin{aligned}
\xi\,\mathsf{ext} &= \Big([\eta\,\mathsf{inl}, (S\,\mathsf{inr})\upsilon]^\star\sigma\,\mathsf{out}\Big)^\dagger\,\mathsf{ext} \\
&= [\eta, ([\eta\,\mathsf{inl}, \sigma\,\mathsf{out}]^\star(S\,\mathsf{inr})\upsilon)^\dagger]^\star\sigma\,\mathsf{out}\,\mathsf{ext} && \text{// dinaturality, Lemma 5.2} \\
&= [\eta, ([\eta\,\mathsf{inl}, \sigma\,\mathsf{out}]^\star(S\,\mathsf{inr})\upsilon)^\dagger]^\star\sigma\,\mathsf{out}\,\mathsf{out}^{-1}\,T\,\mathsf{inl} \\
&= [\eta, ([\eta\,\mathsf{inl}, \sigma\,\mathsf{out}]^\star(S\,\mathsf{inr})\upsilon)^\dagger]^\star\sigma\,T\,\mathsf{inl} \\
&= [\eta, ([\eta\,\mathsf{inl}, \sigma\,\mathsf{out}]^\star(S\,\mathsf{inr})\upsilon)^\dagger]^\star(S\,\mathsf{inl})\sigma \\
&= \sigma
\end{aligned}
$$

ξ is a complete Elgot monad morphism: We have to show that $\xi_X = \zeta_X^\dagger : T_\Sigma X \to SX$ is natural in X and satisfies the identities (8.1). We successively reduce verification of these properties to the last identity in (8.1), whose proof is the major challenge in establishing the claim.

Note that ζ is a natural transformation (being a composite of natural transformations), and hence $\zeta\,T_\Sigma f = S(f + T_\Sigma f)\zeta = S(\mathsf{id} + T_\Sigma f)S(f + \mathsf{id})\zeta$ for any f. Therefore, by the uniformity and naturality laws we obtain

$$\xi\,T_\Sigma f = \zeta^\dagger T_\Sigma f = (S(f + \mathsf{id})\zeta)^\dagger = (Sf)\zeta^\dagger = (Sf)\xi,$$

i.e. ξ is natural. The equation $\xi\eta = \eta$ is shown as as follows:

$$\begin{aligned}\xi\eta &= [\eta,\xi]^\star[\eta\,\mathsf{inl},(S\,\mathsf{inr})\upsilon]^\star\sigma\,\mathsf{out}\,\eta\\ &= [\eta,\xi]^\star[\eta\,\mathsf{inl},S(\mathsf{inr}\,f)\upsilon]^\star\sigma\,\mathsf{out}\,\mathsf{out}^{\text{-}1}\,\eta\,\mathsf{inl}\\ &= [\eta,\xi]^\star\eta\,\mathsf{inl}\\ &= \eta.\end{aligned}$$

Compatibility of ξ with Kleisli lifting follows from Lemma 8.6 and compatibility of ξ with iteration, which we argue later:

$$\begin{aligned}\xi\, g^\star f &= \xi[T_\Sigma(\mathsf{inr}\,\mathsf{inr})f,(T_\Sigma\,\mathsf{inl})g]^\dagger\,\mathsf{inl}\\ &= [S(\mathsf{inr}\,\mathsf{inr})\xi f,(S\,\mathsf{inl})\xi g]^\dagger\,\mathsf{inl}\\ &= (\xi g)^\star(\xi f).\end{aligned}$$

We now show that $\xi=\zeta^\dagger$ is compatible with strength, i.e. $\xi\tau^\nu=\tau(\mathsf{id}\times\xi)$. With a view to applying uniformity, we calculate $\zeta\tau^\nu$:

$$\begin{aligned}\zeta\tau^\nu &= [\eta\,\mathsf{inl},(S\,\mathsf{inr})\upsilon]^\star\sigma\,\mathsf{out}\,\tau^\nu\\ &= [\eta\,\mathsf{inl},(S\,\mathsf{inr})\upsilon]^\star\sigma T(\mathsf{id}+\Sigma\tau^\nu)(T\delta)\tau(\mathsf{id}\times\mathsf{out}) && (6.4)\\ &= [\eta\,\mathsf{inl},(S\,\mathsf{inr})\upsilon]^\star S(\mathsf{id}+\Sigma\tau^\nu)(S\delta)\tau(\mathsf{id}\times\sigma\,\mathsf{out}) && /\!/\ \sigma\text{ monad morph.}\\ &= [\eta\,\mathsf{inl},(S\,\mathsf{inr})\upsilon(\Sigma\tau^\nu)]^\star(S\delta)\tau(\mathsf{id}\times\sigma\,\mathsf{out})\\ &= [\eta\,\mathsf{inl},(S\,\mathsf{inr})(S\tau^\nu)\upsilon]^\star(S\delta)\tau(\mathsf{id}\times\sigma\,\mathsf{out}) && /\!/\text{ naturality of }\upsilon\\ &= [\eta\,\mathsf{inl},S(\mathsf{id}+\tau^\nu)(S\,\mathsf{inr})\upsilon]^\star(S\delta)\tau(\mathsf{id}\times\sigma\,\mathsf{out})\\ &= S(\mathsf{id}+\tau^\nu)[\eta\,\mathsf{inl},(S\,\mathsf{inr})\upsilon]^\star(S\delta)\tau(\mathsf{id}\times\sigma\,\mathsf{out})\\ &= S(\mathsf{id}+\tau^\nu)[\eta\,\mathsf{inl},(S\,\mathsf{inr})\upsilon]^\star S(\mathsf{id}+\rho)(S\,\mathsf{dist})\tau(\mathsf{id}\times\sigma\,\mathsf{out}) && /\!/\text{ def. of }\delta\\ &= S(\mathsf{id}+\tau^\nu)([\eta\,\mathsf{inl},(S\,\mathsf{inr})\upsilon\rho]\,\mathsf{dist})^\star\tau(\mathsf{id}\times\sigma\,\mathsf{out})\\ &= S(\mathsf{id}+\tau^\nu)([\eta\,\mathsf{inl},(S\,\mathsf{inr})\tau(\mathsf{id}\times\upsilon)]\,\mathsf{dist})^\star\tau(\mathsf{id}\times\sigma\,\mathsf{out}). && /\!/\text{ strong nat. of }\upsilon\end{aligned}$$

Furthermore we simplify the tail of the latter expression:

$$\begin{aligned}&([\eta\,\mathsf{inl},(S\,\mathsf{inr})\tau(\mathsf{id}\times\upsilon)]\,\mathsf{dist})^\star\tau(\mathsf{id}\times\sigma\,\mathsf{out})\\ &= (S\,\mathsf{dist})(S\,\mathsf{dist}^{\text{-}1})([\eta\,\mathsf{inl},(S\,\mathsf{inr})\tau(\mathsf{id}\times\upsilon)]\,\mathsf{dist})^\star\tau(\mathsf{id}\times\sigma\,\mathsf{out})\\ &= (S\,\mathsf{dist})([\eta(\mathsf{id}\times\mathsf{inl}),S(\mathsf{id}\times\mathsf{inr})\tau(\mathsf{id}\times\upsilon)]\,\mathsf{dist})^\star\tau(\mathsf{id}\times\sigma\,\mathsf{out}) && /\!/\text{ def. of }\mathsf{dist}^{\text{-}1}\\ &= (S\,\mathsf{dist})([\tau(\mathsf{id}\times\eta\,\mathsf{inl}),\tau(\mathsf{id}\times(S\,\mathsf{inr})\upsilon)]\,\mathsf{dist})^\star\tau(\mathsf{id}\times\sigma\,\mathsf{out}) && /\!/\ \textsc{str}_3\\ &= (S\,\mathsf{dist})(\tau(\mathsf{id}\times[\eta\,\mathsf{inl},(S\,\mathsf{inr})\upsilon]))^\star\tau(\mathsf{id}\times\sigma\,\mathsf{out})\\ &= (S\,\mathsf{dist})\tau(\mathsf{id}\times[\eta\,\mathsf{inl},(S\,\mathsf{inr})\upsilon]^\star)(\mathsf{id}\times\sigma\,\mathsf{out})\\ &= (S\,\mathsf{dist})\tau(\mathsf{id}\times\zeta).\end{aligned}$$

We have obtained in summary that $\zeta\tau^\nu=S(\mathsf{id}+\tau^\nu)(S\,\mathsf{dist})\tau(\mathsf{id}\times\zeta)$. Therefore, by uniformity and compatibility of strength and iteration we obtain the desired identity:

$$\xi\tau^\nu=\zeta^\dagger\tau^\nu=((S\,\mathsf{dist})\tau(\mathsf{id}\times\zeta))^\dagger=\tau(\mathsf{id}\times\zeta^\dagger)=\tau(\mathsf{id}\times\xi).$$

Finally, we are left to show that

$$\xi f^\dagger=(\xi f)^\dagger \tag{8.4}$$

for any $f : X \to T_\Sigma(Y+X)$ where $\xi = ([\eta\,\mathsf{inl}, (S\,\mathsf{inr})\upsilon]^\star\sigma\,\mathsf{out})^\dagger$. We proceed by successive reduction of the unrestricted identity (8.4) to the partial cases when f is guarded, and when f is *strongly guarded*. The latter auxiliary notion is defined as follows. Recall that guardedness of f means that $\mathsf{out}\,f$ factors through some $g : X \to T(Y+\Sigma T_\Sigma(Y+X))$. We us call f *strongly guarded* if moreover there is $g' : X \to T(Y+\Sigma X)$ such that $\mathsf{out}\,f = T(\mathsf{inl} + \Sigma(\eta^\nu\,\mathsf{inr}))g'$.

- *Reduction from unrestricted f to guarded f.* Assuming that (8.4) holds for guarded f, we obtain that for any f, $\xi f^\dagger = \xi(\Diamond f)^\dagger = (\xi\Diamond f)^\dagger$. We are left to show that $(\xi\Diamond f)^\dagger = (\xi f)^\dagger$. To that end, consider the morphism w given by the composition

$$X \xrightarrow{\mathsf{out}\,f} T((Y+X)+\Sigma T_\Sigma(Y+X)) \xrightarrow{[\eta(\mathsf{inl}+\mathsf{id}),(S\,\mathsf{inl})\xi^\star\upsilon]^\star\sigma} S((Y+X)+X).$$

Now, on the one hand

$$\begin{aligned}(S[\mathsf{id},\mathsf{inr}]w)^\dagger &= ([\eta[\mathsf{inl},\mathsf{inr}],\xi^\star\upsilon]^\star\sigma\,\mathsf{out}\,f)^\dagger\\ &= ([\eta,\xi^\star\upsilon]^\star\sigma\,\mathsf{out}\,f)^\dagger\\ &= ([\eta,\xi]^\star[\eta\,\mathsf{inl},(S\,\mathsf{inr})\upsilon]^\star\sigma\,\mathsf{out}\,f)^\dagger\\ &= (\xi f)^\dagger\end{aligned}$$

and on the other hand, by naturality of $-^\dagger$,

$$\begin{aligned}w^{\dagger\dagger} &= \big(([\eta(\mathsf{inl}+\mathsf{id}),(S\,\mathsf{inl})\xi^\star\upsilon]^\star\sigma\,\mathsf{out}\,f)^\dagger\big)^\dagger\\ &= \big(([(S\,\mathsf{inl})[\eta\,\mathsf{inl},\xi^\star\upsilon],\eta\,\mathsf{inr}]^\star S[\mathsf{inl}+\mathsf{id},\mathsf{inl}\,\mathsf{inr}]\sigma\,\mathsf{out}\,f)^\dagger\big)^\dagger\\ &= \big(([(S\,\mathsf{inl})[\eta\,\mathsf{inl},\xi^\star\upsilon],\eta\,\mathsf{inr}]^\star\sigma(T\pi)\,\mathsf{out}\,f)^\dagger\big)^\dagger\\ &= \big([\eta\,\mathsf{inl},\xi^\star\upsilon]^\star(\sigma(T\pi)\,\mathsf{out}\,f)^\dagger\big)^\dagger && \text{// naturality}\\ &= \big([\eta,\xi^\star\upsilon]^\star S(\mathsf{inl}+\mathsf{id})(\sigma(T\pi)\,\mathsf{out}\,f)^\dagger\big)^\dagger\\ &= \big([\eta,\xi^\star\upsilon]^\star\sigma T(\mathsf{inl}+\mathsf{id})((T\pi)\,\mathsf{out}\,f)^\dagger\big)^\dagger\\ &= ([\eta,\xi^\star\upsilon]^\star\sigma\,\mathsf{out}\,\Diamond f)^\dagger && \text{// definition of }\Diamond\\ &= ([\eta,\xi]^\star[\eta\,\mathsf{inl},(S\,\mathsf{inr})\upsilon]^\star\sigma\,\mathsf{out}\,\Diamond f)^\dagger\\ &= (\xi\Diamond f)^\dagger. && \text{// fixpoint for }\xi\end{aligned}$$

We thus obtain by the codiagonal law that

$$(\xi f)^\dagger = (S[\mathsf{id},\mathsf{inr}]w)^\dagger = w^{\dagger\dagger} = (\xi\Diamond f)^\dagger.$$

- *Reduction from guarded f to strongly guarded f.* We proceed under the assumption that f is guarded, i.e. $\mathsf{out}\,f = T(\mathsf{inl}+\mathsf{id})g$ for some $g : X \to T(Y+\Sigma T_\Sigma(Y+X))$. Let w be the following morphism:

$$\begin{aligned}&T_\Sigma(Y+X) \xrightarrow{\mathsf{out}} T((Y+X)+\Sigma T_\Sigma(Y+X))\\ &\xrightarrow{[[\eta\,\mathsf{inl}\,\mathsf{inl},[\eta\,\mathsf{inl}\,\mathsf{inl},(S\,\mathsf{inl}\,\mathsf{inr})\upsilon]^\star\sigma g],(S\,\mathsf{inr})\upsilon]^\star\sigma} S((Y+T_\Sigma(Y+X))+T_\Sigma(Y+X)).\end{aligned}$$

Then, on the one hand, using dinaturality (Lemma 5.2),

$$w^{\dagger\dagger} = \big(([[\eta\,\mathsf{inl}\,\mathsf{inl},[\eta\,\mathsf{inl}\,\mathsf{inl},(S\,\mathsf{inl}\,\mathsf{inr})\upsilon]^\star\sigma g],(S\,\mathsf{inr})\upsilon]^\star\sigma\,\mathsf{out})^\dagger\big)^\dagger$$

$$
\begin{aligned}
&= \big([\eta\,\mathsf{inl}, [\eta\,\mathsf{inl}, (S\,\mathsf{inr})\upsilon]^\star\sigma g]^\star([\eta\,\mathsf{inl}, (S\,\mathsf{inr})\upsilon]^\star\sigma\,\mathsf{out})^\dagger\big)^\dagger && \text{// naturality}\\
&= ([\eta\,\mathsf{inl}, [\eta\,\mathsf{inl}, (S\,\mathsf{inr})\upsilon]^\star\sigma g]^\star\xi)^\dagger && \text{// definition of } \xi\\
&= [\eta, ([\eta\,\mathsf{inl}, \xi]^\star[\eta\,\mathsf{inl}, (S\,\mathsf{inr})\upsilon]^\star\sigma g)^\dagger]^\star\xi && \text{// dinaturality}\\
&= [\eta, ([\eta, \xi]^\star[\eta\,\mathsf{inl}\,\mathsf{inl}, (S\,\mathsf{inr})\upsilon]^\star\sigma g)^\dagger]^\star\xi\\
&= [\eta, ([\eta, \xi]^\star[\eta\,\mathsf{inl}, (S\,\mathsf{inr})\upsilon]^\star\sigma\,\mathsf{out}\,f)^\dagger]^\star\xi && \text{// definition of } g\\
&= [\eta, ([\eta, \xi]^\star\xi f)^\dagger]^\star\xi && \text{// definition of } \xi\\
&= [\eta, (\xi f)^\dagger]^\star\xi. && \text{// fixpoint}
\end{aligned}
$$

and hence $w^{\dagger\dagger}\eta^\nu\,\mathsf{inr} = [\eta, (\xi f)^\dagger]^\star\xi\eta^\nu\,\mathsf{inr} = [\eta, (\xi f)^\dagger]^\star\eta^\nu\,\mathsf{inr} = (\xi f)^\dagger$. Next we introduce the following morphism t:

$$
\begin{aligned}
T_\Sigma(Y+X) &\xrightarrow{\mathsf{out}} T((Y+X) + \Sigma T_\Sigma(Y+X))\\
&\xrightarrow{[[\eta\,\mathsf{inl}, g], \eta\,\mathsf{inr}]^\star} T(Y + \Sigma T_\Sigma(Y+X))\\
&\xrightarrow{T(\mathsf{inl} + \Sigma(\eta^\nu\,\mathsf{inr}))} T((Y + T_\Sigma(Y+X)) + \Sigma T_\Sigma(Y + T_\Sigma(Y+X)))\\
&\xrightarrow{\mathsf{out}^{-1}} T_\Sigma(Y + T_\Sigma(Y+X)).
\end{aligned}
$$

By definition, t is strongly guarded, hence $\xi t^\dagger = (\xi t)^\dagger$.

$$
\begin{aligned}
\xi t &= [\eta, \xi]^\star\xi t\\
&= [\eta, \xi]^\star[\eta\,\mathsf{inl}, (S\,\mathsf{inr})\upsilon]^\star\sigma\,\mathsf{out}\,t\\
&= [\eta, \xi^\star\upsilon]^\star\sigma\,\mathsf{out}\,t\\
&= [\eta, \xi^\star\upsilon]^\star S(\mathsf{inl} + \Sigma(\eta^\nu\,\mathsf{inr}))[[\eta\,\mathsf{inl}, \sigma g], \eta\,\mathsf{inr}]^\star\sigma\,\mathsf{out}\\
&= [\eta\,\mathsf{inl}, \xi^\star S(\eta^\nu\,\mathsf{inr})\upsilon]^\star[[\eta\,\mathsf{inl}, \sigma g], \eta\,\mathsf{inr}]^\star\sigma\,\mathsf{out}\\
&= [\eta\,\mathsf{inl}, (S\,\mathsf{inr})\upsilon]^\star[[\eta\,\mathsf{inl}, \sigma g], \eta\,\mathsf{inr}]^\star\sigma\,\mathsf{out}\\
&= [[\eta\,\mathsf{inl}, [\eta\,\mathsf{inl}, (S\,\mathsf{inr})\upsilon]^\star\sigma g], (S\,\mathsf{inr})\upsilon]^\star\sigma\,\mathsf{out}\\
&= S[\mathsf{id}, \mathsf{inr}]w
\end{aligned}
$$

Using the identities derived above and the codiagonal law, we obtain that

$$
(\xi f)^\dagger = w^{\dagger\dagger}\eta^\nu\,\mathsf{inr} = (S[\mathsf{id}, \mathsf{inr}]w)^\dagger\eta^\nu\,\mathsf{inr} = (\xi t)^\dagger\eta^\nu\,\mathsf{inr} = \xi t^\dagger\eta^\nu\,\mathsf{inr}\,.
$$

We are left to show that $\xi t^\dagger\eta^\nu\,\mathsf{inr} = \xi f^\dagger$. We strengthen the latter to $t^\dagger = [\eta^\nu, f^\dagger]^{\bigstar}$, which would imply it as follows: $\xi t^\dagger\eta^\nu\,\mathsf{inr} = \xi[\eta^\nu, f^\dagger]^{\bigstar}\eta^\nu\,\mathsf{inr} = \xi f^\dagger$.

Since t is guarded, we will be done once we show that $[\eta^\nu, f^\dagger]^{\bigstar}$ satisfies the fixpoint law for $t^\dagger$. It is easy to verify that $\mathsf{out}\,f^\dagger = T(\mathsf{id} + \Sigma[\eta^\nu, f^\dagger]^{\bigstar})g$. Then we have

$$
\begin{aligned}
\mathsf{out}[\eta^\nu, f^\dagger]^{\bigstar} &= [\mathsf{out}[\eta^\nu, f^\dagger], \eta\,\mathsf{inr}\,\Sigma[\eta^\nu, f^\dagger]^{\bigstar}]^\star\,\mathsf{out}\\
&= [[\eta\,\mathsf{inl}, \mathsf{out}\,f^\dagger], \eta\,\mathsf{inr}\,\Sigma[\eta^\nu, f^\dagger]^{\bigstar}]^\star\,\mathsf{out}\\
&= [[\eta\,\mathsf{inl}, T(\mathsf{id} + \Sigma[\eta^\nu, f^\dagger]^{\bigstar})g], \eta\,\mathsf{inr}\,\Sigma[\eta^\nu, f^\dagger]^{\bigstar}]^\star\,\mathsf{out}
\end{aligned}
$$

while, on the other hand,

$$
\begin{aligned}
\mathsf{out}\,t^\dagger &= \mathsf{out}[\eta^\nu, t^\dagger]^{\bigstar}\,\mathsf{out}^{-1}\,T(\mathsf{inl} + \Sigma(\eta^\nu\,\mathsf{inr}))[[\eta\,\mathsf{inl}, g], \eta\,\mathsf{inr}]^\star\,\mathsf{out}\\
&= [\mathsf{out}[\eta^\nu, t^\dagger], \eta\,\mathsf{inr}\,\Sigma[\eta^\nu, t^\dagger]^{\bigstar}]^\star T(\mathsf{inl} + \Sigma(\eta^\nu\,\mathsf{inr}))[[\eta\,\mathsf{inl}, g], \eta\,\mathsf{inr}]^\star\,\mathsf{out}
\end{aligned}
$$

$$
\begin{aligned}
&= [\mathsf{out}\,\eta^\nu, \eta\,\mathsf{inr}\,\Sigma t^\dagger]^\star[[\eta\,\mathsf{inl}, g], \eta\,\mathsf{inr}]^\star\,\mathsf{out}\\
&= [[\eta\,\mathsf{inl}, T(\mathsf{id} + \Sigma t^\dagger)g], \eta\,\mathsf{inr}\,\Sigma t^\dagger]^\star\,\mathsf{out}\,.
\end{aligned}
$$

Hence, indeed, $[\eta^\nu, f^\dagger]^{\bigstar} = t^\dagger$.

- *Strongly guarded f.* Finally, let us show (8.4) with strongly guarded f. Suppose that h is such that $\mathsf{out}\, f = T(\mathsf{inl} + \Sigma(\eta^\nu\,\mathsf{inr}))h$. Recall that $\xi = ([\eta\,\mathsf{inl}, (S\,\mathsf{inr})\upsilon]^\star\sigma\,\mathsf{out})^\dagger$. By uniformity, it suffices to show that

$$[\eta\,\mathsf{inl}, (S\,\mathsf{inr})\upsilon]^\star\sigma\,\mathsf{out}\, f^\dagger = S(\mathsf{id} + f^\dagger)\xi f.$$

On the one hand,

$$
\begin{aligned}
&[\eta\,\mathsf{inl}, (S\,\mathsf{inr})\upsilon]^\star\sigma\,\mathsf{out}\, f^\dagger\\
&\quad= [\eta\,\mathsf{inl}, (S\,\mathsf{inr})\upsilon]^\star\sigma\,\mathsf{out}[\eta^\nu, f^\dagger]^{\bigstar} f\\
&\quad= [\eta\,\mathsf{inl}, (S\,\mathsf{inr})\upsilon]^\star[\mathsf{out}[\eta^\nu, f^\dagger], \eta\,\mathsf{inr}\,\Sigma[\eta^\nu, f^\dagger]^{\bigstar}]^\star\sigma\,\mathsf{out}\, f\\
&\quad= [\eta\,\mathsf{inl}, (S\,\mathsf{inr})\upsilon]^\star[\mathsf{out}[\eta^\nu, f^\dagger], \eta\,\mathsf{inr}\,\Sigma[\eta^\nu, f^\dagger]^{\bigstar}]^\star S(\mathsf{inl} + \Sigma(\eta^\nu\,\mathsf{inr}))\sigma h\\
&\quad= [\eta\,\mathsf{inl}, (S\,\mathsf{inr})\upsilon]^\star[\eta\,\mathsf{inl}, \eta\,\mathsf{inr}\,\Sigma f^\dagger]^\star\sigma h\\
&\quad= [\eta\,\mathsf{inl}, S(\mathsf{inr}\, f^\dagger)\upsilon]^\star\sigma h.
\end{aligned}
$$

and on the other hand,

$$
\begin{aligned}
S(\mathsf{id} + f^\dagger)\xi f &= S(\mathsf{id} + f^\dagger)[\eta, \xi]^\star[\eta\,\mathsf{inl}, (S\,\mathsf{inr})\upsilon]^\star\sigma\,\mathsf{out}\, f\\
&= S(\mathsf{id} + f^\dagger)[\eta, \xi]^\star[\eta\,\mathsf{inl}, (S\,\mathsf{inr})\upsilon]^\star\sigma T(\mathsf{inl} + \Sigma(\eta^\nu\,\mathsf{inr}))h\\
&= S(\mathsf{id} + f^\dagger)[\eta, \xi]^\star[\eta\,\mathsf{inl}, (S\,\mathsf{inr})\upsilon]^\star S(\mathsf{inl} + \Sigma(\eta^\nu\,\mathsf{inr}))\sigma h\\
&= S(\mathsf{id} + f^\dagger)[\eta, \xi]^\star[\eta\,\mathsf{inl}\,\mathsf{inl}, S(\mathsf{inr}\,\eta^\nu\,\mathsf{inr})\upsilon]^\star\sigma h\\
&= S(\mathsf{id} + f^\dagger)[\eta\,\mathsf{inl}, \xi^\star S(\eta^\nu\,\mathsf{inr})\upsilon]^\star\sigma h\\
&= S(\mathsf{id} + f^\dagger)[\eta\,\mathsf{inl}, (S\,\mathsf{inr})\upsilon]^\star\sigma h\\
&= [\eta\,\mathsf{inl}, S(\mathsf{inr}\, f^\dagger)\upsilon]^\star\sigma h.
\end{aligned}
$$

This finishes the proof that ξ is a complete Elgot monad morphism.

Uniqueness. Let $\rho : \mathbb{T}_\Sigma \to \mathbb{S}$ be a complete Elgot monad morphism such that $\sigma = \rho\,\mathsf{ext}$ and $\upsilon = \rho\,\mathsf{out}^{-1}\,\eta\,\mathsf{inr}\,\Sigma\eta^\nu$. We have to show that $\rho = \xi = ([\eta\,\mathsf{inl}, (S\,\mathsf{inr})\upsilon]^\star\sigma\,\mathsf{out})^\dagger$. We rewrite the last term as follows:

$$
\begin{aligned}
&\big([\eta\,\mathsf{inl}, (S\,\mathsf{inr})\upsilon]^\star\sigma\,\mathsf{out}\big)^\dagger\\
&= ([\eta\,\mathsf{inl}, (S\,\mathsf{inr})\rho\,\mathsf{out}^{-1}\,\eta\,\mathsf{inr}\,\Sigma\eta^\nu]^\star\sigma\,\mathsf{out})^\dagger\\
&= ([\eta\,\mathsf{inl}, \rho\,(T_\Sigma\,\mathsf{inr})\,\mathsf{out}^{-1}\,\eta\,\mathsf{inr}\,\Sigma\eta^\nu]^\star\sigma\,\mathsf{out})^\dagger\\
&= ([\eta\,\mathsf{inl}, \rho\,\mathsf{out}^{-1}\,T(\mathsf{inr} + (\Sigma T_\Sigma\,\mathsf{inr}))\eta\,\mathsf{inr}\,\Sigma\eta^\nu]^\star\sigma\,\mathsf{out})^\dagger\\
&= ([\eta\,\mathsf{inl}, \rho\,\mathsf{out}^{-1}\,\eta\,\mathsf{inr}\,\Sigma(T_\Sigma\,\mathsf{inr})\Sigma\eta^\nu]^\star\sigma\,\mathsf{out})^\dagger\\
&= ([\eta\,\mathsf{inl}, \rho\,\mathsf{out}^{-1}\,\eta\,\mathsf{inr}\,\Sigma(\eta^\nu\,\mathsf{inr})]^\star\sigma\,\mathsf{out})^\dagger\\
&= ((\rho[\eta\,\mathsf{inl}, \mathsf{out}^{-1}\,\eta\,\mathsf{inr}\,\Sigma(\eta^\nu\,\mathsf{inr})])^\star\rho\,\mathsf{ext}\,\mathsf{out})^\dagger\\
&= (\rho[\eta\,\mathsf{inl}, \mathsf{out}^{-1}\,\eta\,\mathsf{inr}\,\Sigma(\eta^\nu\,\mathsf{inr})]^{\bigstar}\,\mathsf{ext}\,\mathsf{out})^\dagger
\end{aligned}
$$

$$= \rho([\eta\,\mathsf{inl}, \mathsf{out}^{-1}\,\eta\,\mathsf{inr}\,\Sigma(\eta^{\nu}\,\mathsf{inr})]^{\maltese}\,\mathsf{ext}\,\mathsf{out})^{\dagger}.$$

To finish the calculation we have to verify that the term after ρ vanishes. Note that the term under the iteration operator is guarded. Hence, it suffices to show that id satisfies the corresponding characteristic equation for iteration, i.e. that

$$[\eta, \mathsf{id}]^{\maltese}[\eta\,\mathsf{inl}, \mathsf{out}^{-1}\,\eta\,\mathsf{inr}\,\Sigma(\eta^{\nu}\,\mathsf{inr})]^{\maltese}\,\mathsf{ext}\,\mathsf{out} = \mathsf{id}.$$

We reduce the left hand side to id as follows:

$$\begin{aligned}
&[\eta, \mathsf{id}]^{\maltese}[\eta\,\mathsf{inl}, \mathsf{out}^{-1}\,\eta\,\mathsf{inr}\,\Sigma(\eta^{\nu}\,\mathsf{inr})]^{\maltese}\,\mathsf{ext}\,\mathsf{out}\\
&= [\eta, [\eta, \mathsf{id}]^{\maltese}\,\mathsf{out}^{-1}\,\eta\,\mathsf{inr}\,\Sigma(\eta^{\nu}\,\mathsf{inr})]^{\maltese}\,\mathsf{ext}\,\mathsf{out}\\
&= [\eta, \mathsf{out}^{-1}[\mathsf{out}[\eta, \mathsf{id}], \eta\,\mathsf{inr}\,\Sigma[\eta, \mathsf{id}]^{\maltese}]^{\star}\eta\,\mathsf{inr}\,\Sigma(\eta^{\nu}\,\mathsf{inr})]^{\maltese}\,\mathsf{ext}\,\mathsf{out}\\
&= [\eta, \mathsf{out}^{-1}\,\eta\,\mathsf{inr}]^{\maltese}\,\mathsf{ext}\,\mathsf{out}\\
&= [\eta, \mathsf{out}^{-1}\,\eta\,\mathsf{inr}]^{\maltese}\,\mathsf{out}^{-1}(T\,\mathsf{inl})\,\mathsf{out}\\
&= \mathsf{out}^{-1}[\mathsf{out}[\eta, \mathsf{out}^{-1}\,\eta\,\mathsf{inr}], \eta\,\mathsf{inr}\,\Sigma[\eta, \mathsf{out}^{-1}\,\eta\,\mathsf{inr}]^{\maltese}]^{\star}(T\,\mathsf{inl})\,\mathsf{out} && \text{// Theorem 6.3}\\
&= \mathsf{out}^{-1}(\mathsf{out}[\eta, \mathsf{out}^{-1}\,\eta\,\mathsf{inr}])^{\star}\,\mathsf{out}\\
&= \mathsf{out}^{-1}[\mathsf{out}\,\eta, \eta\,\mathsf{inr}]^{\star}\,\mathsf{out}\\
&= \mathsf{out}^{-1}[\eta\,\mathsf{inl}, \eta\,\mathsf{inr}]^{\star}\,\mathsf{out}\\
&= \mathsf{out}^{-1}\,\mathsf{out}\\
&= \mathsf{id}.
\end{aligned}$$

This finishes the proof. □

The existence and the exact shape of the initial complete Elgot monad $\mathbb{L}$ mentioned in Corollary 8.5 depend on the properties of the base category. We recall the key definition of a hyperextensive category [2]:

Definition 8.8. A category $\mathbf{C}$ is *hyperextensive* if

(1) $\mathbf{C}$ has countable coproducts that are *disjoint*, i.e. the pullback of any two distinct injections is an initial object, and *universal*, i.e. stable under pullbacks; and
(2) in $\mathbf{C}$, subobjects that are coproduct injections are closed under countable disjoint unions; that is, given countably many pairwise disjoint subobjects $A_i \to B$ that are coproduct injections, their union $\sum_i A_i \to B$ is again a coproduct injection.

Examples of hyperextensive categories include **Set**, **Cpo**, and bounded complete metric spaces as well as all presheaf categories [2]. We refer to subobjects whose inclusion morphisms are (binary) coproduct injections as *summands*, and given a summand, we refer to the partner injection of the corresponding binary coproduct as its *coproduct complement* (we will not need uniqueness of complements). In this terminology, summands are closed under pullbacks (i.e. under preimages) and under countable disjoint unions in hyperextensive categories. From countable disjoint unions we obtain unions of chains:

Lemma 8.9. *Let* $\mathbf{C}$ *be hyperextensive. Then* $\mathbf{C}$ *has unions of* ω*-chains of summands; such unions are again summands, and are* universal*, i.e. stable under pullbacks (and, hence, under products).*

Proof. Any ascending chain of summands can be transformed into a disjoint union of summands: if A_1 and A_2 are summands of X and A_1 is contained in A_2, then by universality of coproducts, A_1 is also a summand of A_2 so we can replace A_2 with the coproduct complement of A_1 in A_2, preserving the union. Universality of unions of ascending chains of summands is then inherited from countable disjoint unions. □

Theorem 8.10. *Let* $\mathbf{C}$ *be hyperextensive and have binary coproducts. Then the monad* $\mathbb{L}$ *given by* $LX = X + 1$ *is* ω*-continuous. Equipped with the arising complete Elgot monad structure according to Theorem 5.8,* $\mathbb{L}$ *is the initial complete Elgot monad over* $\mathbf{C}$*.*

(The conditions of the theorem imply our running assumption that $\mathbf{C}$ is distributive [13].)

Remark 8.11. Let us spell out the definition of the iteration operator figuring in the statement of Theorem 8.10 explicitly. Suppose that $e : X \to L(Y + X)$. Let X_1 be the preimage of Y under e and $e_1 : X_1 \to Y$ the arising restriction of e; for $i \geq 1$ let X_{i+1} be the preimage of X_i under e, and let $e_{i+1} : X_{i+1} \to X_i$ be the arising restriction of e. By universality of finite coproducts, the X_i are pairwise disjoint summands. By stability of summands under countable disjoint unions, $\sum_i X_i$ is a summand of X, whose complement we denote X_∞. We obtain the presentation $X = \sum_i X_i + X_\infty$. Now $e^\dagger : X \to LY$ is the universal map induced by the $\eta\, e_1 \dots e_i : X_i \to LY$ and $\perp : X_\infty \to LY$.

Now L clearly admits only very simple recursive definitions: an equation morphism $e : X \to L(Y + X) = (Y + X) + 1$ essentially defines each variable in X either as a result from Y or as another variable from X or as divergence. In preparation of the proof of Theorem 8.10, the following lemma shows that the solution of all possible such definitions of this shape is, in any complete Elgot monad, uniquely determined by the complete Elgot monad laws.

Lemma 8.12. *Let* $\mathbb{T}$ *be a complete Elgot monad over a hyperextensive category* $\mathbf{C}$*, let* $e : X \to T(Y + X)$*, and let* $m : Z \to X$*. Then the following holds.*

(1) *If* $e\, m = \eta\, \mathsf{inl}\, u$ *for some* $u : Z \to Y$ *then* $e^\dagger\, m = \eta\, u$*.*
(2) *If* $e\, m = \perp_{Z,Y+X}$ *then* $e^\dagger\, m = \perp_{Z,X}$*.*
(3) *If* $e\, m = \eta\, \mathsf{inr}\, u$ *for some* $u : Z \to X$ *then* $e^\dagger\, m = e^\dagger\, u$*.*
(4) *If* $e\, m = \eta\, \mathsf{inr}\, m\, u$ *for some* $u : Z \to Z$ *then* $e^\dagger\, m = \perp_{Z,Y}$*.*

That is: If a variable is defined as a result value, then the solution of the recursive definition for that variable is that result value; if a variable is defined as $\perp$, then the solution is $\perp$; if a variable is defined as another variable, then its solution is that of the other variable; and if a set of variables is defined by mutual recursion without any base case and without use of the algebraic operations of the monad, then the solution for all these variables is $\perp$.

Proof. The first three claims are immediate from the fixpoint law and coconstancy of $\perp$ (Lemma 5.9). We show the last claim. We have

$$e\, m = \eta\, \mathsf{inr}\, m\, u = \eta(\mathsf{id} + m)\, \mathsf{inr}\, u = T(\mathsf{id} + m)\, \eta\, \mathsf{inr}\, u,$$

which by uniformity implies $e^\dagger m = (\eta\, \mathsf{inr}\, u)^\dagger$. The claim then follows by Lemma 5.10. □

Proof of Theorem 8.10. The base category $\mathbf{C}$ is, a fortiori, extensive. In any extensive category, $\mathbb{L}$ is the partial map classifier for partial morphisms whose domains are summands; we will call such partial morphisms *summand-partial.* Explicitly, a summand-partial morphism f from X to Y is thus a span $X \xleftarrow{m} D \xrightarrow{f} Y$ where m is a summand; the *domain* of f is m or, by abuse of notation, D. By *preimages under* f we mean pullbacks along the map $f : D \to Y$ in this span.

Thus, the Kleisli category of $\mathbb{L}$ inherits orderings on its hom-sets from the extension ordering on partial functions. The fact that $\mathbf{C}$ has unions of ω-chains of summands which are again summands (Lemma 8.9) then guarantees that these orderings are ω-complete, and since 0 is a summand, they have bottoms $\bullet \leftarrow 0 \rightarrow \bullet$. We have to verify that Kleisli composition for $\mathbb{L}$ is continuous on both sides and that the remaining conditions of Definition 5.5 are satisfied. We will phrase all arguments in terms of summand-partial morphisms.

Continuity of left Kleisli composition: Let g be a summand-partial morphism from Y to Z, and let $(f_i)_{i\in\mathbb{N}}$ be an ascending chain of summand-partial morphisms from X to Y, with domains D_i. Denoting unions and joins of ascending chains by $\bigsqcup$ and composition of partial morphisms simply by juxtaposition, we have to show that $(\bigsqcup_i f_i)g = \bigsqcup_i f_i g$. The only problem here is to show that the domains of the two sides agree. The domain of $f_i g$ is the preimage E_i of D_i under g; the domain of $\bigsqcup_i f_i g$ is the union $\bigsqcup_i E_i$ of the ascending chain $(E_i)_i$; the domain of $\bigsqcup_i f_i$ is the union $D = \bigsqcup_i D_i$; and the domain of $(\bigsqcup_i f_i)g$ is the preimage E of D under g. By universality of unions of ascending chains, $E = \bigsqcup_i E_i$.

Continuity of right Kleisli composition: Let g be a summand-partial morphism from X to Y with domain C, and let $(f_i)_{i\in\mathbb{N}}$ be an ascending chain of summand-partial morphisms from Y to Z, with domains D_i and supremum f. We have to show $g(\bigsqcup_i f_i) = \bigsqcup_i g\, f_i$; again, we focus only on the domains. The domain of gf_i is the preimage E_i of C under f_i; the domain of $\bigsqcup_i f_i g$ is the union $\bigsqcup_i E_i$; the domain of f is the union $D = \bigsqcup_i D_i$; and the domain of gf is the preimage E of C under f. By construction, E_i is contained in D_i, and E is contained in D. Moreover, since f_i maps E_i into C, so does f, and hence E_i is also contained in E (by the universal property of E as a pullback). Denoting the restriction of $f : D \to Y$ to $E \to C$ by f', we thus have the diagram

$$\begin{tikzcd}
E_i \arrow[r, tail] \arrow[d, tail] & E \arrow[r, "f'"] \arrow[d, tail] & C \arrow[d, tail] \\
D_i \arrow[r, tail] & \bigsqcup D_i \arrow[r, "f"'] & Y
\end{tikzcd}$$

where the outer rectangle and the right hand square are pullbacks by construction. By the pullback lemma, it follows that the left hand square is also a pullback. By universality of unions of ascending chains of summands, it now follows that $E = \bigsqcup_i E_i$, as required.

Continuity of the strength: If the Kleisli morphism $f : Y \to Z + 1$ corresponds to a summand-partial map with domain D, the Kleisli morphism $\tau(\mathsf{id} \times f) : X \times Y \to Z + 1$ corresponds to a summand-partial map with domain $X \times D$. Continuity of $\tau(\mathsf{id} \times (-))$ is then immediate from stability of unions of ascending chains of summands under products.

Continuity of copairing: Immediate from the fact that generally, $\bigsqcup_i (D_i + E_i) = \bigsqcup_i E_i + \bigsqcup_i D_i$ because unions of ascending chains of summands are defined via coproducts.

Preservation of $\bot$ by left Kleisli composition: The bottom element of the Kleisli hom-set from X to Y is the unique (summand-)partial morphism with domain 0. Left Kleisli composites of this morphism have domains that are pullbacks of 0, which in extensive categories are again 0.

Preservation of $\bot$ by the strength: The domain of the partial morphism corresponding to $\tau(\mathsf{id} \times \bot) : X \to Y + 1$ is $X \times 0$, which by extensivity (in fact already by distributivity) is 0.

This establishes that $\mathbb{L}$ is ω-continuous, and hence a complete Elgot monad; by the standard construction of least fixpoints in ω-cpos, the iteration operator of $\mathbb{L}$ then has the form described in Remark 8.11. To see initiality of $\mathbb{L}$, let $\mathbb{S}$ be a complete Elgot monad on $\mathbf{C}$. For clarity, we denote the unit of $\mathbb{L}$ by $\eta^{\mathbb{L}}$ and that of $\mathbb{S}$ by $\eta^{\mathbb{S}}$. We need to show existence of a unique complete Elgot monad morphism $\xi : \mathbb{L} \to \mathbb{S}$. Since ξ must preserve the unit and unproductive divergence $\bot$ (the latter by preservation of iteration), the only candidate is $\xi = [\eta^{\mathbb{S}}, \bot]$. It remains to show that ξ is a complete Elgot monad morphism. Thanks to the simplicity of the monad structure of $\mathbb{L}$, it is clear that ξ is a strong monad morphism. The main task is to prove preservation of iteration. So let $e : X \to L(Y + X) = (Y + X) + 1$. We inductively construct infinite sequences $X_1, X_2, \dots$ and $D_1, D_2, \dots$ of summands of X as follows: Like in Remark 8.11, we take X_1 to be the preimage of Y under e, and for $i > 1$ we take X_i to be the preimage of X_{i-1} under e; similarly, we take D_1 to be the preimage of 1 under e, and for $i > 1$ we take D_i to be the preimage of D_{i-1} under e. By universality of coproducts, the X_i and D_i are pairwise disjoint summands (that is, the X_i are pairwise disjoint, the D_i are pairwise disjoint, and every X_i is disjoint with every D_j).

Let $X' = \sum_i X_i$, $D' = \sum_i D_i$ and let Z be the complement of $X' + D'$ in X. For the remainder we regard X as being decomposed into the coproduct $X = X' + X_\infty$ where $X_\infty = D' + Z$.

By definition, there are $e_1 : X_1 \to Y$ and $e_i : X_i \to X_{i-1}$ $(i > 1)$ such that

$$e \,\mathsf{inl}\,\mathsf{in}_1 = \eta^{\mathbb{L}}\,\mathsf{inl}\,e_1 \qquad\qquad e\,\mathsf{inl}\,\mathsf{in}_i = \eta^{\mathbb{L}}\,\mathsf{inr}\,\mathsf{inl}\,\mathsf{in}_{i-1}\,e_i$$

where in_i denotes the i-th coproduct injection into a countable coproduct.

By applying the fixpoint law i times we obtain $(\xi e)^\dagger\,\mathsf{inl}\,\mathsf{in}_i = \eta^{\mathbb{S}}\,e_1 \dots e_i$ and analogously, $\xi\,e^\dagger\,\mathsf{inl}\,\mathsf{in}_i = \xi\,\eta^{\mathbb{L}}\,e_1 \dots e_i = \eta^{\mathbb{S}}\,e_1 \dots e_i$. We are left to show that $(\xi e)^\dagger\,\mathsf{inr} = \xi\,e^\dagger\,\mathsf{inr}$. Noting that by definition, $e^\dagger\,\mathsf{inr} = \bot$ and ξ preserves $\bot$, this amounts to showing that $(\xi e)^\dagger\,\mathsf{inr} = \bot$. By construction of the D_i, for every $i > 1$ there is $d_i : D_i \to D_{i-1}$ such that

$$e\,\mathsf{inr}\,\mathsf{inl}\,\mathsf{in}_1 = \bot \qquad\qquad e\,\mathsf{inr}\,\mathsf{inl}\,\mathsf{in}_i = \eta^{\mathbb{L}}\,\mathsf{inr}\,\mathsf{inr}\,\mathsf{inl}\,\mathsf{in}_{i-1}\,d_i,$$

hence, by applying the fixpoint law i times we obtain that $(\xi e)^\dagger\,\mathsf{inr}\,\mathsf{inl}\,\mathsf{in}_i = \bot$, which implies $(\xi e)^\dagger\,\mathsf{inr}\,\mathsf{inl} = \bot$ and hence we are left to show $(\xi e)^\dagger\,\mathsf{inr}\,\mathsf{inr} = \bot$.

Notice that the preimages of Y and 1 under $e\,\mathsf{inr}\,\mathsf{inr}$ must be 0 and therefore there is $m_1 : Z \to X$ such that $e\,\mathsf{inr}\,\mathsf{inr} = \eta^{\mathbb{L}}\,\mathsf{inr}\,m_1$. Analogously for every $i > 1$ we construct $m_i : Z \to X$ such that $e\,m_{i-1} = \eta^{\mathbb{L}}\,\mathsf{inr}\,m_i$. Let us denote by $\widehat{Z}$ the sum of ω copies of Z and by $\hat{m} : \widehat{Z} \to X$ the cotuple formed by the morphisms m_i with $i > 0$. Now,

$$\begin{aligned}
\xi\,e[\mathsf{inr}\,\mathsf{inr}, \hat{m}] &= \xi\,\eta^{\mathbb{L}}\,\mathsf{inr}\,\hat{m}\,w \\
&= \eta^{\mathbb{S}}\,\mathsf{inr}[\mathsf{inr}\,\mathsf{inr}, \hat{m}]\,\mathsf{inr}\,w \\
&= \eta^{\mathbb{S}}(\mathsf{id} + [\mathsf{inr}\,\mathsf{inr}, \hat{m}])\,\mathsf{inr}\,\mathsf{inr}\,w \\
&= S(\mathsf{id} + [\mathsf{inr}\,\mathsf{inr}, \hat{m}])\,\eta^{\mathbb{S}}\,\mathsf{inr}\,\mathsf{inr}\,w
\end{aligned}$$

where $w : Z + \widehat{Z} \to \widehat{Z}$ is the obvious canonical isomorphism. By uniformity, and by Lemma 5.10, this implies $(\xi\,e)^\dagger[\mathsf{inr}\,\mathsf{inr}, \hat{m}] = (\eta^{\mathbb{S}}\,\mathsf{inr}\,\mathsf{inr}\,w)^\dagger = \bot$ and therefore $(\xi\,e)^\dagger\,\mathsf{inr}\,\mathsf{inr} = (\xi\,e)^\dagger[\mathsf{inr}\,\mathsf{inr}, \hat{m}]\,\mathsf{inl} = \bot$ as required. □

Remark 8.13. The above proof of Theorem 8.10 uses the full power of the definition of hyperextensive categories, including universality of countable coproducts. It has been shown previously [11] that assuming only universality of finite coproducts and stability of summands under countable disjoint unions, one can still define the iteration operator and prove the fixpoint law. However, we do not see how to show the uniformity law in this weaker setting.

At the same time, we have the impression that the uniformity law is the only place where universality of countable coproducts is needed.

9. Related Work

The above results benefit from extensive previous work on monad-based axiomatic iteration. In particular we draw on the concept of *complete Elgot monad* studied by Adámek et al. [4]; the construction of the free complete Elgot monad over a functor [5] is strongly related to Corollary 8.5.(1), and we do not claim Part (1) of Corollary 8.5 as a contribution of this paper. There is extensive literature on solutions of (co)recursive program schemes [9, 1, 26, 20, 32, 33], from which our present work differs primarily in that we do not restrict to guarded systems of equations. In particular, as mentioned in the introduction, Piróg and Gibbons [32] actually work with the same monad transformer, the coinductive generalized resumption transformer. The same authors [33, Corollary 4.6] prove a coproduct characterization of the coinductive generalized resumption transformer that is similar to our Theorem 8.3; but again, this takes place in a different category, that is, in completely iterative monads (admitting guarded recursive definitions) rather than complete Elgot monads (admitting unrestricted recursive definitions). Technically, results on T_Σ being a completely iterative monad are incomparable to our result on T_Σ being a complete Elgot monad – we prove a stronger recursion scheme for T_Σ but need to assume that T is a complete Elgot monad, while T_Σ is completely iterative without any assumptions on T.

Moss [30] proves that given a **Set**-endofunctor F and a distinguished point $\bot : 1 \to \nu F$ of the final F-coalgebra, the monad M given on objects by $M_F X = X + \nu\gamma.\, F(X + \gamma) \cong \nu\gamma.\, X + F\gamma$ is completely Elgot, with unproductive divergence induced by $\bot$ (Moss in fact establishes a completeness result over such monads). This result does not appear to be an immediate application of our Theorem 7.1, as there is no implicit complete Elgot base monad in M_F.

We construct solutions of unguarded recursive equations from solutions of guarded recursive equations, for the latter relying crucially on results by Uustalu on guarded recursion over parametrized monads [40], which in particular has allowed us to make do without idealized monads.

The axiomatic treatment of iteration via complete Elgot monads is essentially dual to the axiomatic treatment of recursion by Simpson and Plotkin [38], who work in a category $\mathbf{D}$ with a *parametrized uniform recursion operator* $\mathsf{Hom}_{\mathbf{D}}(Y \times X, X) \to \mathsf{Hom}_{\mathbf{D}}(Y, X)$ and a subcategory $\mathbf{S}$ of *strict* functions in $\mathbf{D}$. Given a distributive category $\mathbf{C}$ equipped with a complete Elgot monad, we can take $\mathbf{S} = \mathbf{C}^{op}$ and $\mathbf{D} = (\mathbf{C}_{\mathbb{T}})^{op}$. Then the iteration operator over $\mathbf{C}_{\mathbb{T}}$ sending $f : X \to T(Y + X)$ to $f^\dagger : X \to TY$ induces precisely a parametrized uniform recursion operator for the pair $(\mathbf{D}, \mathbf{S})$ in the sense of Simpson and Plotkin.

The proof of Theorem 7.1 can be embedded into a generic framework connecting guarded and unguarded iteration that we have developed in further work [21].

10. Conclusions and Future Work

We have developed semantic foundations for non-wellfounded side-effecting recursive definitions, specifically for recursive definitions over the so-called *coinductive generalized resumption transformer* that extends a base monad $\mathbb{T}$ with operations represented by a functor Σ to obtain a monad $\mathbb{T}_\Sigma$ defined by taking final coalgebras, i.e. consisting of non-wellfounded trees.

While previous work on the same monad transformer was focussed on guarded recursive definitions, in the framework of completely iterative monads, we work in the setting of (complete) Elgot monads, which admit unrestricted recursive definitions. Our main results state that

- $\mathbb{T}_\Sigma$ is a complete Elgot monad if $\mathbb{T}$ is a complete Elgot monad (Theorem 7.1);
- the structure of $\mathbb{T}_\Sigma$ as a complete Elgot monad is uniquely determined as extending that of $\mathbb{T}$ (Theorem 7.1);
- if the underlying category $\mathbf{C}$ admits an initial complete Elgot monad $\mathbb{L}$ (often $L = (\text{-}) + 1$), then $\mathbb{T}_\Sigma \cong \mathbb{T} + \mathbb{L}_\Sigma$ in the category of complete Elgot monads on $\mathbf{C}$ (Theorem 8.3/Corollary 8.5).

In particular this requires proving the equational laws of complete Elgot monads for the solution operator that we construct on T_Σ. We have implemented a formal verification of our results, which are technically quite involved, in the Coq proof assistant, see `https://git8.cs.fau.de/redmine/projects/corque`.

Besides the fact that applying the coinductive resumption monad transformer to a complete Elgot monad $\mathbb{T}$ again yields a complete Elgot monad $\mathbb{T}_\Sigma$, the resulting object obviously has a richer structure provided by the adjoined free operations. One topic for further investigation is to identify (and possibly axiomatize) this structure. We aim to use this structure to program definitions of free operations as morphisms $T_\Sigma X \to TX$ in a similar spirit as in the paradigm of *handling algebraic effects* [36]. In conjunction with iteration this actually produces a recursion operator that is more expressive than iteration. This however requires going beyond the first-order setting of this paper (which was sufficient for iteration), as call-by-value recursion is known to be an inherently higher-order concept. There is an concept of complete *Elgot algebra* [3] complementing complete Elgot monads. It has been shown that the algebras of complete Elgot monads are complete Elgot algebras satisfying additional conditions [18]; the precise relationship between complete Elgot monads and complete Elgot algebras remains to be determined, possibly using our results on iteration-congruent retracts of monads with iteration [21].

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