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PROPER FUNCTORS AND FIXED POINTS FOR FINITE BEHAVIOUR

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Abstract. The rational fixed point of a set functor is well-known to capture the behaviour of finite coalgebras. In this paper we consider functors on algebraic categories. For them the rational fixed point may no longer be fully abstract, i.e. a subcoalgebra of the final coalgebra. Inspired by Ésik and Maletti's notion of a proper semiring, we introduce the notion of a proper functor. We show that for proper functors the rational fixed point is determined as the colimit of all coalgebras with a free finitely generated algebra as carrier and it is a subcoalgebra of the final coalgebra. Moreover, we prove that a functor is proper if and only if that colimit is a subcoalgebra of the final coalgebra. These results serve as technical tools for soundness and completeness proofs for coalgebraic regular expression calculi, e.g. for weighted automata.

1. Introduction

Coalgebras allow to model many types of systems within a uniform and conceptually clear mathematical framework [33]. One of the key features of this framework is *final semantics*; the final coalgebra provides a fully abstract domain of system behaviour (i.e. it identifies precisely the behaviourally equivalent states). For example, the standard coalgebraic modelling of deterministic automata (without restricting to finite state sets) yields the set of formal languages as final coalgebra. Restricting to finite automata, one obtains precisely the regular languages [32]. It is well-known that this correspondence can be generalized to locally finitely presentable (lfp) categories [8], where *finitely presentable* objects play the role of finite sets. For a finitary functor F (modelling a coalgebraic system type) one then obtains the *rational fixed point* ϱF, which provides final semantics to all coalgebras with a finitely presentable carrier [25]. Moreover, the rational fixed point is *fully abstract*, i.e. ϱF is a subcoalgebra of the final one νF, whenever the classes of finitely presentable and finitely generated objects agree in the base category and F preserves non-empty monomorphisms [27, Section 5]. While the latter assumption on F is very mild, the former one on the base category is more restrictive. However, it is still true for many categories used in the construction of coalgebraic system models (e.g. sets, posets, graphs, vector spaces, commutative monoids, nominal sets and

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positively convex algebras). Hence, in these cases the rational fixed point ϱF is the canonical domain of *regular* behaviour, i.e. the behaviour of 'finite' systems of type F.

In this paper we will consider rational fixed points in algebraic categories (a.k.a. finitary varieties), i.e. categories of algebras specified by a signature of operation symbols with finite arity and a set of equations (equivalently, these are precisely the Eilenberg-Moore categories for finitary monads on sets). Being the target of generalized determinization [36], these categories provide a paradigmatic setting for coalgebraic modelling beyond sets. For example, non-deterministic automata, weighted or probabilistic automata [23], or context-free grammars [43] are coalgebraically modelled over the categories of join-semilattices, modules for a semiring, positively convex algebras, and idempotent semirings, respectively. In algebraic categories one would like that the rational fixed point, in addition to being fully abstract, is determined already by those coalgebras carried by free finitely generated algebras, i.e. precisely those coalgebras arising by generalized determinization. In particular, this feature is used in completeness proofs for generalized regular expressions calculi [12, 36, 37]; there one proves that the quotient of syntactic expressions modulo axioms of the calculus is (isomorphic to) the rational fixed point by establishing its universal property as a final object for that quotient. A key feature of the settings in loc. cit. is that it suffices to verify the finality only w.r.t. coalgebras with a free finitely generated carrier.

The purpose of the present paper is to provide sufficient conditions on the algebraic base category and coalgebraic type functor that ensure that the rational fixed point is fully abstract and that such finality proofs are sound. To this end we form a coalgebra that serves as the semantics domain of all behaviours of target coalgebras of generalized determinization (modulo bisimilarity on the level of these coalgebra). More precisely, let $T : \mathsf{Set} \to \mathsf{Set}$ be a finitary monad on sets and $F : \mathsf{Set}^T \to \mathsf{Set}^T$ be a finitary endofunctor preserving surjective T-algebra morphisms (note that the last assumption always holds if F is lifted from some endofunctor on Set). Now form the colimit φF of the inclusion functor of the full subcategory $\mathsf{Coalg}_{\mathsf{ffg}}\, F$ formed by all F-coalgebras of the form $TX \to FTX$, where X is a finite set. Urbat [41] has shown that φF is a fixed point of F. We first first provide a characterization of φF that uniquely determines it up to isomorphism: based on Adámek et al.'s notion of a Bloom algebra [2], we introduce the new notion of an *ffg-Bloom algebra*, and we prove that, considered as an algebra for F, φF is the initial ffg-Bloom algebra (Theorem 4.4).

Then we turn to the full abstractness of the rational fixed point ϱF and the soundness of the above mentioned finality proofs. Inspired by Ésik and Maletti's notion of a proper semiring (which is in fact a notion concerning weighted automata), we introduce *proper functors* (Definition 5.2), and we prove that for a proper functor on an algebraic category the rational fixed point is determined by the coalgebras with a free finitely generated carrier. More precisely, if F is proper, then the rational fixed point ϱF is (isomorphic to) initial Bloom algebra φF. Moreover, we show that a functor F is proper if and only if φF is a subcoalgebra of the final coalgebra νF (Theorem 5.6). As a consequence we also obtain the desired result that for a proper functor F the finality property of ϱF can be established by only verifying that property for all coalgebras from $\mathsf{Coalg}_{\mathsf{ffg}}\, F$ (Corollary 5.9).

In addition, we provide more easily established sufficient conditions on Set^T and F that ensure properness: F is proper if finitely generated algebras of Set^T are closed under kernel pairs and F maps kernel pairs to weak pullbacks in Set. For a lifting F this holds whenever the lifted functor on sets preserves weak pullbacks; in fact, in this case the above conditions were shown to entail Corollary 5.9 in previous work [12, Corollary 3.36]. However, the type functor (on the category of commutative monoids) of weighted automata with weights drawn from the semiring of natural numbers provides an example of a proper functor for which the above condition on Set^T fails.

Another recent related work concerns the so-called *locally finite fixed point* ϑF [27]; this provides a fully abstract behavioural domain whenever F is a finitary endofunctor on an lfp category preserving non-empty monomorphisms. In loc. cit. it was shown that ϑF captures a number of instances that cannot be captured by the rational fixed point, e.g. context free languages [43], constructively algebraic formal power-series [30, 44], Courcelle's algebraic trees [6, 13] and the behaviour of stack machines [22]. However, as far as we know, ϑF is not amenable to the simplified finality check mentioned above unless F is proper.

Putting everything together, in an algebraic category we obtain the following picture of fixed points of F (where $\twoheadrightarrow$ denotes quotient coalgebras and $\rightarrowtail$ a subcoalgebra):

$$\varphi F \twoheadrightarrow \varrho F \twoheadrightarrow \vartheta F \rightarrowtail \nu F. \tag{1.1}$$

We exhibit an example, where all four fixed points are different. However, if F is proper and preserves monomorphisms, then φF, ϱF and ϑF are isomorphic and fully abstract, i.e. they collapse to a subcoalgebra of the final one: $\varphi F \cong \varrho F \cong \vartheta F \rightarrowtail \nu F$.

At this point, note that Urbat's above mentioned recent work [41] also provides a framework which covers the four above fixed points as four instances of one theory. This provides, for example, a uniform proof of the fact that they are fixed points and their universal properties (in the case of ϱF, ϑF and φF). However, Urbat's paper does not study the relationship between the four fixed points.

The rest of the paper is structured as follows: in Section 2 we collect some technical preliminaries and recall the rational and locally finite fixed points more in detail. Section 3 introduces the new fixed point φF and establishes the picture in (1.1). Next, Section 4 provides the characterization of φF as the initial ffg-Bloom algebra for F. Section 5 introduces proper functors and presents our main results, while in Section 6 we present the proof of Theorem 5.6. Finally, Section 7 concludes the paper.

This paper is a reworked full version of the conference paper [26]. We have included detailed proofs, and in addition, we have added the new results in Section 4.

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2. Preliminaries

In this section we recall a few preliminaries needed for the subsequent development. We assume that readers are familiar with basic concepts of category theory.

We denote the coproduct of two objects X and Y of a category $\mathscr{A}$ by $X + Y$ with injections $\mathsf{inl} : X \to X + Y$ and $\mathsf{inr} : Y \to X + Y$.

Remark 2.1. Recall that a *strong epimorphism* in a category $\mathscr{A}$ is an epimorphism $e : A \twoheadrightarrow B$ of $\mathscr{A}$ that has the *unique diagonal property* w.r.t. any monomorphism. More precisely, whenever the outside of the following square

$$\begin{array}{ccc} A & \xrightarrow{e} & B \\ {\scriptstyle f}\downarrow & \swarrow{\scriptstyle d} & \downarrow{\scriptstyle g} \\ C & \xrightarrow[m]{} & D \end{array}$$

commutes, where $m : C \rightarrowtail D$ is a monomorphism, then there exists a unique morphism $d : B \to C$ with $d \cdot e = f$ and $m \cdot d = g$.

Similarly, a jointly epimorphic family $e_i : A_i \to B$, $i \in I$, is *strong* if it has the following similar unique diagonal property: for every monomorphism $m : C \rightarrowtail D$ and morphisms $g : B \to D$ and $f_i : A_i \to C$, $i \in I$, such that $m \cdot f_i = g \cdot e_i$ holds for all $i \in I$, there exists a unique $d : C \to D$ such that $m \cdot d = g$ and $d \cdot e_i = f_i$ for all $i \in I$.

On several occasions we will make use of the following fact.

Lemma 2.2. *Let $D : \mathscr{D} \to \mathscr{C}$ be a diagram with a colimit cocone $\mathsf{in}_d : Dd \to C$. Then the colimit injections in_d form a strongly epimorphic family.*

Proof. First, it is easy to see that the in_d form a jointly epimorphic family. To see that it is strong, suppose we have a monomorphism $m : M \rightarrowtail N$ and morphisms $g : C \to N$ and $f_d : Dd \to M$ for every object d in $\mathscr{D}$ such that $m \cdot f_d = g \cdot \mathsf{in}_d$. Then the $f_d : Dd \to M$ form a cocone of D. Indeed, for every morphism $h : d \to d'$ of $\mathscr{D}$ we have

$$m \cdot f_{d'} \cdot Dh = g \cdot \mathsf{in}_{d'} \cdot Dh = g \cdot \mathsf{in}_d = m \cdot f_d,$$

which implies that $f_{d'} \cdot Dh = f_d$ since m is a monomorphism. Therefore there exists a unique $i : C \to M$ such that $f_d = i \cdot \mathsf{in}_d$ for every d in $\mathscr{D}$. It follows that also $m \cdot i = g$ since this equation holds when extended by every in_d; then use that the in_d form an epimorphic family. □

2.1. **Algebras and Coalgebras.** We also assume that readers are familiar with algebras and coalgebras for an endofunctor. Given an endofunctor F on some category $\mathscr{A}$ we write $(\nu F, t)$ for the final F-coalgebra (if it exists). Recall, that the final F-coalgebra exists under mild assumptions on $\mathscr{A}$ and F, e.g. whenever $\mathscr{A}$ is locally presentable and F an accessible functor (see [8]). For any coalgebra $c : C \to FC$ we will write $\dagger c : C \to \nu F$ for the unique coalgebra morphism. We write

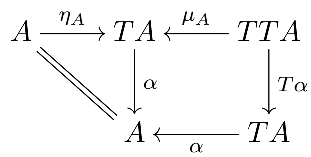
$$\mathsf{Coalg}\, F$$

for the category of F-coalgebras and their morphisms. Recall that all colimits in $\mathsf{Coalg}\, F$ are formed on the level of $\mathscr{A}$, i.e. the canonical forgetful functor $\mathsf{Coalg}\, F \to \mathscr{A}$ creates all colimits (see e.g. [1, Prop. 4.3]).

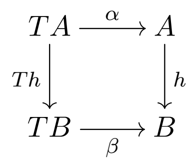
If $\mathscr{A}$ is a concrete category, i.e. equipped with a faithful functor $|\cdot| : \mathscr{A} \to \mathsf{Set}$, one defines *behavioural equivalence* as the following relation $\sim$: given two F-coalgebras (X, c) and (Y, d) then $x \sim y$ holds for $x \in |X|$ and $y \in |Y|$ if there is another F-coalgebra (Z, e) and F-coalgebra morphisms $f : X \to Z$ and $g : Y \to Z$ with $|f|(x) = |g|(y)$.

The base categories $\mathscr{A}$ of interest in this paper are the *algebraic categories*, i.e. categories of Eilenberg-Moore algebras (or T-algebras, for short) for a finitary monad T on Set. Recall that, for a monad T on Set with unit η and multiplication μ, a T-algebra is a pair (A, α)

where $\alpha : TA \to A$, called the *algebra structure*, is a map such that the diagram below commutes:

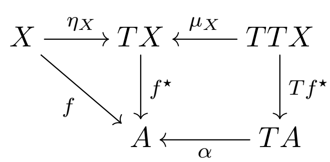


Morphisms of T-algebras are just the usual morphisms of functor algebra, i.e. a T-algebra morphism $h : (A, \alpha) \to (B, \beta)$ is a map $h : A \to B$ such that the square below commutes:



The category of T-algebras and their morphisms is denoted by Set^T as usual. Equivalently, those categories are precisely the finitary varieties, i.e. category of Σ-algebras for a signature Σ, whose operation symbols have finite arity, satisfying a set of equations (e.g. the categories of monoids, groups, vector spaces, or join-semilattices).

We will frequently make use of the fact that (TX, μ_X) is the *free* T-algebra on the set X (of generators). This means that for every T-algebra (A, α) and every map $f : X \to A$ there exists a unique extension of f to a T-algebra morphisms, i.e., there exists a unique T-algebra morphism $f^\star$ such that $f^\star \cdot \eta_X = f$:



Moreover, it is easy to verify that $f^\star = \mu_A \cdot Tf$ holds.

A free T-algebra (TX, μ_X) where X is a finite set is called *free finitely generated.*

In the following we will often drop algebra structures when we discuss a T-algebra (A, α) and simply speak of the algebra A.

Example 2.3. (1) The leading example in this paper are weighted automata considered as coalgebras. Let $(\mathbb{S}, +, \cdot, 0, 1)$ be a semiring, i.e. $(\mathbb{S}, +, 0)$ is a commutative monoid, $(\mathbb{S}, \cdot, 1)$ a monoid and the usual distributive laws hold: $r \cdot 0 = 0 = 0 \cdot r$, $r \cdot (s + t) = r \cdot s + r \cdot t$ and $(r + s) \cdot t = r \cdot t + s \cdot t$. We just write $\mathbb{S}$ to denote a semiring. As base category $\mathscr{A}$ we consider the category $\mathbb{S}$-Mod of $\mathbb{S}$-semimodules; recall that a (left) $\mathbb{S}$-*semimodule* is a commutative monoid $(M, +, 0)$ together with an action $\mathbb{S} \times M \to M$, written as juxtaposition sm for $r \in \mathbb{S}$ and $m \in M$, such that for every $r, s \in \mathbb{S}$ and every $m, n \in M$ the following laws hold:

$$\begin{aligned} (r+s)m &= rm + sm \qquad & 0m &= 0 \qquad & 1m &= m \\ r(m+n) &= rm + rn \qquad & r0 &= 0 \qquad & r(sm) &= (r \cdot s)m \end{aligned}$$

An $\mathbb{S}$-semimodule morphism is a monoid homomorphism $h\colon M_1 \to M_2$ such that $h(rm) = rh(m)$ for each $r \in \mathbb{S}$ and $m \in M_1$.

An $\mathbb{S}$-weighted automaton over the fixed input alphabet Σ is a triple $(i, (M^a)_{a \in \Sigma}, o)$, where i and o are a row and a column vector in $\mathbb{S}^n$, respectively, of input and output weights, respectively, and M_a is an $n \times n$-matrix over $\mathbb{S}$, for some natural number n.

This number n is the number of states of the weighted automaton and the matrices M^a represent $\mathbb{S}$-weighted transitions; in fact, $M^a_{i,j}$ is the weight of the a-transition from state i to state j (with a weight of 0 meaning that there is no a-transition). Every weighted automaton accepts a *formal power series* (or *weighted language*) $L : \Sigma^* \to \mathbb{S}$ defined in the following way: $L(w) = i \cdot M^w \cdot o$ where M^w is the obvious inductive extension of $a \mapsto M^a$ to words in Σ^*: M^ε is the identity matrix and $M^{av} = M^a \cdot M^v$ for every $a \in \Sigma$ and $v \in \Sigma^*$.

Now consider the functor $FX = \mathbb{S} \times X^\Sigma$ on $\mathbb{S}$-Mod. Clearly, a weighted automaton (without its initial vector) on n states is equivalently an F-coalgebra on $\mathbb{S}^n$; in fact, to give a coalgebra structure $\mathbb{S}^n \to \mathbb{S} \times (\mathbb{S}^n)^\Sigma$ amount to specifying two $\mathbb{S}$-semimodule morphisms $o : \mathbb{S}^n \to \mathbb{S}$ (equivalently, a column vector over in $\mathbb{S}$) and $t : \mathbb{S}^n \to (\mathbb{S}^n)^\Sigma$ (equivalently, an Σ-indexed family of $\mathbb{S}$-semimodule morphisms on $\mathbb{S}^n$ each of which can be represented by an $n \times n$-matrix).

The final F-coalgebra is carried by the set $\mathbb{S}^{\Sigma^*}$ of all weighted languages over Σ with the obvious (coordinatewise) $\mathbb{S}$-semimodule structure and with the F-coalgebra structure given by $\langle o, t\rangle : \mathbb{S}^{\Sigma^*} \to \mathbb{S} \times (\mathbb{S}^{\Sigma^*})^\Sigma$ with $o(L) = L(\varepsilon)$ and $t(L)(a) = \lambda w.L(aw)$; it is straightforward to verify that o and t are $\mathbb{S}$-semimodule morphisms and form a final coalgebra. Moreover, for every F-coalgebra on $\mathbb{S}^n$ the unique coalgebra morphism $\mathbb{S}^n \to \mathbb{S}^{\Sigma^*}$ assigns to every element i of $\mathbb{S}^n$ (perceived as the row input vector of the weighted automaton associated to the given coalgebra) the weighted language accepted by that automaton.

(2) An important special case of $\mathbb{S}$-weighted automata are ordinary non-deterministic automata. One takes $\mathbb{S} = \{0, 1\}$ the Boolean semiring for which the category of $\mathbb{S}$-semimodules is (isomorphic to) the category of join-semilattices. Then $FX = \{0, 1\} \times X^\Sigma$ is the coalgebraic type functor of deterministic automata with input alphabet Σ, and there is a bijective correspondence between an F-coalgebra on a free join-semilattice and non-deterministic automata. In fact in one direction one restricts $\mathcal{P}_\mathsf{f} X \to \{0, 1\} \times (\mathcal{P}_\mathsf{f} X)^\Sigma$ to the set X of generators, and in the other direction one performs the well-known subset construction. The final coalgebra is carried by the set of all formal languages on Σ in this case.

(3) Another special case is where $\mathbb{S}$ is a field. In this case, $\mathbb{S}$-semimodules are precisely the vector spaces over the field $\mathbb{S}$. Moreover, since every field is freely generated by its basis, it follows that the $\mathbb{S}$-weighted automata are precisely those F-coalgebras whose carrier is a finite dimensional vector space over $\mathbb{S}$.

We will now recall a few properties of algebraic categories Set^T, where T is a finitary set monad, needed for our proofs.

Remark 2.4. (1) Recall that every strong epimorphism e in Set^T is *regular*, i.e. e is the coequalizer of some pair of T-algebra morphisms. It follows that the classes of strong and regular epimorphisms coincide, and these are precisely the surjective T-algebra morphisms. Similarly, jointly strongly epimorphic families of morphisms are precisely the jointly surjective families. Finally, monomorphisms in Set^T are precisely the injective T-algebra morphisms since the canonical forgetful functor $\mathsf{Set}^T \to \mathsf{Set}$ creates all limits (and pullbacks in particular).

(2) Every free T-algebra TX is *(regular) projective*, i.e. given any surjective T-algebra morphism $q : A \twoheadrightarrow B$ then for every T-algebra morphism $h : TX \to B$ there exists a

T-algebra morphism $g : TX \to A$ such that $q \cdot g = h$:

$$\begin{array}{ccc} & & A \\ & {\scriptstyle g} \nearrow & \big\downarrow {\scriptstyle q} \\ TX & \xrightarrow[h]{} & B \end{array}$$

(3) Furthermore, note that every finitely presentable T-algebra A is a regular (= strong) quotient of a free T-algebra TX with a finite set X of generators. Indeed, A is presented by finitely many generators and relations. So by taking X as a finite set of generators of A, the unique extension of the embedding $X \hookrightarrow A$ yields a surjective T-algebra morphism $TX \twoheadrightarrow A$.

2.2. **The Rational Fixed Point.** As we mentioned in the introduction the canonical domain of behaviour of 'finite' coalgebras is the rational fixed point of an endofunctor. Its theory can be developed for every finitary endofunctor on a locally finitely presentable category. We will now recall the necessary background material.

A *filtered colimit* is the colimit of a diagram $\mathscr{D} \to \mathscr{C}$ where $\mathscr{D}$ is a filtered category (i.e. every finite subcategory $\mathscr{D}_0 \hookrightarrow \mathscr{D}$ has a cocone in $\mathscr{D}$), and a *directed colimit* is a colimit whose diagram scheme $\mathscr{D}$ is a directed poset. A functor is called *finitary* if it preserves filtered (equivalently directed) colimits. An object C is called *finitely presentable* (fp) if the hom-functor $\mathscr{C}(C, -)$ preserves filtered (equivalently directed) colimits, and *finitely generated* (fg) if $\mathscr{C}(C, -)$ preserves directed colimits of monos (i.e. colimits of directed diagrams $D : \mathscr{D} \to \mathscr{C}$ where all connecting morphisms Df are monic in $\mathscr{C}$). Clearly, every fp object is fg, but the converse fails in general. In addition, fg objects are closed under strong epis (quotients), which fails for fp objects in general.

A cocomplete category $\mathscr{C}$ is called *locally finitely presentable* (lfp) if there is a set of finitely presentable objects in $\mathscr{C}$ such that every object of $\mathscr{C}$ is a filtered colimit of objects from that set. We refer to [8] for further details.

Example 2.5. Examples of lfp categories are the categories of sets, posets and graphs, with finitely presentable objects precisely the finite sets, posets, and graphs, respectively. The category of vector spaces over the field k is lfp with finite-dimensional spaces being the fp-objects. Every algebraic category is lfp. The finitely generated objects are precisely the finitely generated algebras (in the sense of general algebra), and finitely presentable objects are precisely those algebras specified by finitely many generators and finitely many relations.

Example 2.6. Finitary functors abound. We just mention a few examples of finitary functors on Set.

Constant functors and the identity functor are, of course, finitary. For every finitary signature Σ, i.e. $\Sigma = (\Sigma_n)_{n<\omega}$ is a sequence of sets with Σ_n containing operation symbols of the finite arity n, the associated polynomial functor given by

$$F_\Sigma X = \coprod_{n<\omega} \Sigma_n \times X^n$$

is finitary. The finite power-set functor given by $\mathcal{P}_\mathsf{f} X = \{Y \mid Y \subseteq X, \text{Y finite}\}$ is finitary (while the full power-set functor is not) and so is the bag functor mapping a set X to the set of finite multisets on X. The class of finitary functors enjoys good closure properties: it is closed under composition, finite products, arbitrary coproducts, and, in fact, arbitrary colimits.

As we have mentioned already, the finitary monads (i.e. whose functor part is finitary) on Set are precisely those monads whose Eilenberg-Moore category Set^T is (isomorphic to) a finitary variety of algebras.

Assumptions 2.7. *For the rest of this section we assume that F denotes a finitary endofunctor on the lfp category $\mathscr{A}$.*

Remark 2.8. Recall that an lfp category, besides being cocomplete, is complete and has (strong epi, mono)-factorizations of morphisms [8], i.e. every morphism $f : X \to Y$ can be decomposed as $f = m \cdot e$ where $e : X \twoheadrightarrow I$ is a strong epi and $m : I \rightarrowtail Y$ a mono. One should think of I as the image of X in Y under f.

The rational fixed point is a fully abstract model of behaviour for all F-coalgebras whose carrier is an fp-object. We now recall its construction [5].

Notation 2.9. Denote by $\mathsf{Coalg}_{\mathsf{fp}}\, F$ the full subcategory of all F-coalgebras on fp carriers, and let $(\varrho F, r)$ be the colimit of the inclusion functor of $\mathsf{Coalg}_{\mathsf{fp}}\, F$ into $\mathsf{Coalg}\, F$:

$$(\varrho F, r) = \operatorname{colim}(\mathsf{Coalg}_{\mathsf{fp}}\, F \hookrightarrow \mathsf{Coalg}\, F)$$

with the colimit injections $a^\sharp : A \to \varrho F$ for every coalgebra $a : A \to FA$ in $\mathsf{Coalg}_{\mathsf{fp}}\, F$.

We call $(\varrho F, r)$ the *rational fixed point* of F; indeed, it is a fixed point:

Proposition 2.10 ([5])**.** *The coalgebra structure $r : \varrho F \to F(\varrho F)$ is an isomorphism.*

The rational fixed point can be characterized by a universal property both as a coalgebra and as an algebra for F: as a coalgebra ϱF is the *final locally finitely presentable coalgebra* [25], and as an algebra it is the *initial iterative algebra* [5]. We will not recall the latter notion as it is not needed for the technical development in this paper. Locally finitely presentable (locally fp, for short) coalgebras for F can be characterized as precisely those F-coalgebra obtained as a filtered colimit of a diagram of coalgebras from $\mathsf{Coalg}_{\mathsf{fp}}\, F$:

Proposition 2.11 ([25], Corollary III.13)**.** *An F-coalgebra is locally fp if and only if it is a colimit of some filtered diagram $\mathscr{D} \to \mathsf{Coalg}_{\mathsf{fp}}\, F \hookrightarrow \mathsf{Coalg}\, F$.*

For $\mathscr{A} = \mathsf{Set}$ an F-coalgebra (X, c) is locally fp iff it is *locally finite*, i.e. every element of X is contained in a finite subcoalgebra. Analogously, for $\mathscr{A}$ the category of vector spaces over the field k an F-coalgebra (X, c) is locally fp iff it is *locally finite dimensional*, i.e. every element of X is contained in a finite dimensional subcoalgebra.

Of course, there is a unique coalgebra morphism $\varrho F \to \nu F$. Moreover, in many cases ϱF is *fully abstract* for locally fp coalgebras, i.e. besides being the final locally fp coalgebra the above coalgebra morphism is monic; more precisely, if the classes of fp- and fg-objects coincide and F preserves non-empty monos, then ϱF is fully abstract (cf. Theorem 2.17 below). The assumption that the two object classes coincide is often true:

Example 2.12. (1) In the category of sets, posets, and graphs, fg-objects are fp and those are precisely the finite sets, posets, and graphs, respectively.

(2) A *locally finite variety* is a variety of algebras, where every free algebra on a finite set of generators is finite. It follows that fp- and fg-objects coincide and are precisely the finite algebras. Concrete examples are the categories of Boolean algebras, distributive lattices and join-semilattices.

(3) In the category of $\mathbb{S}$-semimodules for a semiring $\mathbb{S}$ the fp- and fg-objects need not coincide in general. However, if the semiring $\mathbb{S}$ is *Noetherian* in the sense of Ésik and Maletti [18], i.e. every subsemimodule of a finitely generated $\mathbb{S}$-semimodule is itself finitely generated, then fg- and fp-semimodules coincide. Examples of Noetherian semirings are: every finite semiring, every field, every principal ideal domain such as the ring of integers and therefore every finitely generated commutative ring by Hilbert's Basis Theorem. The tropical semiring $(\mathbb{N} \cup \{\infty\}, \min, +, \infty, 0)$ is not Noetherian [17]. The usual semiring of natural numbers is also not Noetherian: the $\mathbb{N}$-semimodule $\mathbb{N} \times \mathbb{N}$ is finitely generated but its subsemimodule generated by the infinite set $\{(n, n+1) \mid n \geqslant 1\}$ is not. However, $\mathbb{N}$-semimodules are precisely the commutative monoids, and for them fg- and fp-objects coincide (this is known as Redei's theorem [31]; see Freyd [20] for a very short proof).

(4) The category PCA of *positively convex algebras* [14, 15] is the Eilenberg-Moore category for the monad $\mathcal{D}$ of finitely supported subprobability distributions on sets. This monad maps a set X to

$$\mathcal{D}X = \{d : X \to [0,1] \mid \mathsf{supp}\, d \text{ is finite and } \sum_{x \in X} d(x) \leqslant 1\},$$

where $\mathsf{supp}\, d = \{x \in X \mid d(x) \neq 0\}$, and a function $f : X \to Y$ to $\mathcal{D}f : \mathcal{D}X \to \mathcal{D}Y$ with

$$\mathcal{D}f(d) = \lambda y. \sum_{fx = y} d(x).$$

More concretely, a positively convex algebra is a set X equipped with finite convex sum operations: for every n and $p_1, \ldots, p_n \in [0,1]$ with $\sum_{i=1}^{n} p_i \leqslant 1$ we have an n-ary operation assigning to $x_1, \ldots, x_n \in X$ an element $\boxplus_{i=1}^{n} p_i x_i$ subject to the following axioms:

(a) $\boxplus_{i=1}^{n} p_i^k x_i = x_k$ whenever $p_k^k = 1$ and $p_i^k = 0$ for $i \neq k$, and

(b) $\boxplus_{i=1}^{n} p_i \left(\boxplus_{j=1}^{k} q_{i,j} x_j \right) = \boxplus_{j=1}^{k} \left(\sum_{i=1}^{n} p_i q_{i,j} \right) x_j.$

For $n = 1$ we write the convex sum operation for $p \in [0,1]$ simply as px. The morphisms of PCA are maps preserving finite convex sums in the obvious sense.

The point of mentioning this example at length is that PCA is used for the coalgebraic modelling of the trace semantics of probabilistic systems (see e.g. [38]), and recently, it was established by Sokolova and Woracek [39] that in PCA, the classes of fp- and fg-objects coincide. We shall come back to this example in Section 5 when we introduce and discuss proper functors.

Example 2.13. We list a number of examples of rational fixed points for cases where they do form subcoalgebras of the final coalgebra.

(1) For the functor $FX = \{0,1\} \times X^A$ on Set the finite coalgebras are deterministic automata, and the rational fixed point is carried by the set of regular languages on the alphabet A.

(2) For every finitary signature Σ, the final coalgebra for the associated polynomial functor F_Σ (see Example 2.6) is carried by the set of all (finite and infinite) Σ-trees, i.e. rooted and ordered trees where each node with n-children is labelled by an n-ary operation symbol. The rational fixed point is the subcoalgebra given by *rational* (or *regular* [13]) Σ-trees, i.e. those Σ-trees that have only finitely many different subtrees (up to isomorphism) –

this characterization is due to Ginali [21]. For example, for the signature Σ with a binary operation symbol $*$ and a constant c the following infinite Σ-tree (here written as an infinite term) is rational:

$$c * (c * (c * \cdots)));$$

in fact, its only subtrees are the whole tree and the single node tree labelled by c.

(3) For the functor $FX = \mathbb{R} \times X$ on Set the final coalgebra is carried by the set $\mathbb{R}^\omega$ of real streams, and the rational fixed point is carried by its subset of eventually periodic streams (or lassos). Considered as a functor on the category of vector spaces over $\mathbb{R}$, the final coalgebra νF remains the same, but the rational fixed point ϱF consists of all rational streams [34].

(4) For the functor $FX = \mathbb{S} \times X^A$ on the category $\mathbb{S}$-Mod of $\mathbb{S}$-semimodules for the semiring $\mathbb{S}$ we already mentioned that $\nu F = \mathbb{S}^{A^*}$ consists of all formal power-series. Whenever the classes of fg- and fp-semimodules coincide, e.g. for every Noetherian semiring $\mathbb{S}$ or the semiring of natural numbers, then ϱF is formed by the *recognizable* formal power-series; from the Kleene-Schützenberger theorem [35] (see also [11]) it follows that these are, equivalently, the *rational* formal power-series.

(5) On the category of presheaves $\mathsf{Set}^{\mathcal{F}}$, where $\mathcal{F}$ is the category of all finite sets and maps between them, consider the functor $FX = V + X \times X + \delta(X)$, where $V : \mathcal{F} \hookrightarrow \mathsf{Set}$ is the embedding and $\delta(X)(n) = X(n+1)$. This is a paradigmatic example of a functor arising from a *binding signature* for which initial semantics was studied by Fiore et al. [19].

The final coalgebra νF is carried by the presheaf of all λ-trees modulo α-equivalence: $\nu F(n)$ is the set of (finite and infinite) λ-trees in n free variables (note that such a tree may have infinitely many bound variables). And ϱF is carried by the rational λ-trees, where an α-equivalence class is called *rational* if it contains at least one λ-tree which has (up to isomorphism) only finitely many different subtrees (see [7] for details). Rational λ-trees also appear as the rational fixed point of a very similar functor on the category of nominal sets [29]. An analogous characterization can be given for every functor on nominal sets arising from a binding signature [28].

As we mentioned previously, whether fg- and fp-objects coincide is currently unknown in some base categories used in the coalgebraic modelling of systems, for example, in idempotent semirings (used in the treatment of context-free grammars [43]), in algebras for the stack monad (used for modelling configurations of stack machines [22]); or it even fails, for example in the category of finitary monads on sets (used in the categorical study of algebraic trees [6]) or in Eilenberg-Moore categories for a monad in general (the target categories of generalized determinization [36]).

As a remedy, in recent joint work with Pattinson and Wißmann [27], we have introduced the *locally finite fixed point* which provides a fully abstract model of finitely generated behaviour. Its construction is very similar to that of the rational fixed point but based on fg- in lieu of fp-objects. In more detail, one considers the full subcategory $\mathsf{Coalg}_{\mathsf{fg}}\, F$ of all F-coalgebras carried by an fg-object and takes the colimit of its inclusion functor:

$$(\vartheta F, \ell) = \mathrm{colim}(\mathsf{Coalg}_{\mathsf{fg}}\, F \hookrightarrow \mathsf{Coalg}\, F).$$

Theorem 2.14 ([27], Theorems 3.10 and 3.12)**.** *Suppose that the finitary functor $F : \mathscr{A} \to \mathscr{A}$ preserves non-empty monos. Then $(\vartheta F, \ell)$ is a fixed point for F, and it is a subcoalgebra of νF.*

Remark 2.15. (1) Note that for an arbitrary (not necessarily concrete) lfp category $\mathscr{A}$ the notion of a non-empty monomorphisms needs explanation: a monomorphism $m : X \rightarrowtail Y$ is said to be *empty* if its domain X is a strict initial object of $\mathscr{A}$, where recall that the initial object 0 of $\mathscr{A}$ is *strict* provided that every morphism $A \to 0$ is an isomorphism. In particular, if the initial object of $\mathscr{A}$ is not strict, then all monomorphisms are non-empty.

(2) For a functor $F : \mathscr{A} \to \mathscr{A}$ preserving non-empty monos the category $\mathsf{Coalg}\, F$ of all F-coalgebras inherits the (strong epi, mono)-factorization system from $\mathscr{A}$ (see Remark 2.8) in the following sense: every coalgebra morphism $f : (X, c) \to (Y, d)$ can be factorized into coalgebra morphisms e and m carried by a strong epi and a mono in $\mathscr{A}$, respectively. In fact, one (strong epi, mono)-factorizes $f = m \cdot e$ in $\mathscr{A}$ and obtains a unique coalgebra structure on the 'image' I such that e and m are coalgebra morphisms:

$$\begin{array}{ccc} X & \xrightarrow{c} & FX \\ {\scriptstyle e}\downarrow & & \downarrow{\scriptstyle Fe} \\ I & \dashrightarrow & FI \\ {\scriptstyle m}\downarrow & & \downarrow{\scriptstyle Fm} \\ Y & \xrightarrow[d]{} & FY \end{array}$$

Indeed, if m is a non-empty mono, we know that Fm is monic by assumption and we use the unique diagonal property. Otherwise, m is an empty mono, which implies that $e : X \twoheadrightarrow I$ is an isomorphism since I is a strict initial object. Then $Fe \cdot c \cdot e^{-1}$ is the desired coalgebra structure on I.

Furthermore, like its brother, the rational fixed point, ϑF is characterized by a universal property both as a coalgebra and as an algebra: it is the final locally finitely generated coalgebra and the initial fg-iterative algebra [27, Theorem 3.8 and Corollary 4.7].

Under additional assumptions, which all hold in every algebraic category, we have a close relation between ϱF and ϑF; in fact, the following is a consequence of [27, Theorem 5.4]:

Theorem 2.16. *Suppose that $\mathscr{A}$ is an lfp category such that every fp-object is a strong quotient of a strong epi projective fp-object, and let $F : \mathscr{A} \to \mathscr{A}$ be finitary and preserving non-empty monos. Then ϑF is the image of ϱF in the final coalgebra.*

More precisely, taking the (strong epi, mono)-factorization of the unique F-coalgebra morphism $\varrho F \to \nu F$ yields ϑF, i.e. for F preserving monos on an algebraic category we have the following picture:

$$\varrho F \twoheadrightarrow \vartheta F \rightarrowtail \nu F.$$

A sufficient condition under which ϱF and ϑF coincide is the following (cf. [27, Section 5]):

Theorem 2.17. *Suppose that in addition to the assumption in Theorem 2.16 the classes of fg- and fp-objects coincide in $\mathscr{A}$. Then $\varrho F \cong \vartheta F$, i.e. the left-hand morphism above is an isomorphism.*

In the introduction we briefly mentioned a number of interesting instances of ϑF that are not (known to be) instances of the rational fixed point; see [27] for details. A concrete example, where ϱF is not a subcoalgebra of νF (and hence not isomorphic to ϑF) was given in [12, Example 3.15]. We present a new, simpler example based on similar ideas:

Example 2.18. (1) Let $\mathscr{A}$ be the category of algebras for the signature Σ with two unary operation symbols u and v. The natural numbers $\mathbb{N}$ with the successor function as both operations $u^{\mathbb{N}}$ and $v^{\mathbb{N}}$ form an object of $\mathscr{A}$. We consider the functor $FX = \mathbb{N} \times X$ on $\mathscr{A}$. Coalgebras for F are automata carried by an algebra A in $\mathscr{A}$ equipped with two Σ-algebra morphisms: an output morphism $A \to \mathbb{N}$ and a next state morphism $A \to A$. The final coalgebra is carried by the set $\mathbb{N}^\omega$ of streams of natural numbers with the coordinatewise algebra operations and with the coalgebra structure given by the usual head and tail functions.

Note that the free Σ-algebra on a set X of generators is $TX \cong \{u, v\}^* \times X$; we denote its elements by $w(x)$ for $w \in \{u, v\}^*$ and $x \in X$. The operations are given by prefixing words by the letters u and v, respectively: $s^{TX} : w(x) \mapsto sw(x)$ for $s = u$ or v.

Now one considers the F-coalgebra $a : A \to FA$, where $A = T\{x\}$ is free Σ-algebra on one generator x and a is determined by $a(x) = (0, u(x))$. Recall our notation $\dagger a : A \to \nu F$ for the unique coalgebra morphism. Clearly, $\dagger a(x)$ is the stream $(0, 1, 2, 3, \cdots)$ of all natural numbers, and since $\dagger a$ is a Σ-algebra morphism we have

$$\dagger a(u(x)) = \dagger a(v(x)) = (1, 2, 3, 4, \cdots).$$

Since A is (free) finitely generated, it is of course, finitely presentable as well. Thus, (A, a) is a coalgebra in $\mathsf{Coalg}_{\mathsf{fp}} F$. However, we shall now prove that the (unique) F-coalgebra morphism $a^\sharp : A \to \varrho F$ maps $u(x)$ and $v(x)$ to two distinct elements of ϱF.

We prove this by contradiction. So suppose that $a^\sharp(u(x)) = a^\sharp(v(x))$. By the construction of ϱF as a filtered colimit (see Notation 2.9) we know that there exists a coalgebra $b : B \to FB$ in $\mathsf{Coalg}_{\mathsf{fp}} F$ and an F-coalgebra morphism $h : A \to B$ with

$$h(u(x)) = h(v(x)). \tag{2.1}$$

Since B is a finitely presented Σ-algebra it is the quotient in $\mathscr{A}$ of a free algebra A' via some surjective Σ-algebra morphism $q : A' \twoheadrightarrow B$, say. Next observe, that there is a coalgebra structure $a' : A' \to FA'$ such that q is an F-coalgebra morphism from (A', a') to (B, b): for Fq is a surjective Σ-algebra morphism and so we obtain q' by using projectivity of A' w.r.t. $b \cdot q : A' \to FB$ (cf. Remark 2.4(2)):

$$\begin{array}{ccc} A' & \overset{a'}{\dashrightarrow} & FA' \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle Fq} \\ B & \underset{b}{\longrightarrow} & FB \end{array}$$

Now choose a term t_x in A' with $q(t_x) = h(x)$. Using that q and h are Σ-algebra morphisms we see that $q(u(t_x)) = q(v(t_x))$ as follows:

$$q(u(t_x)) = u^B(q(t_x)) = u^B(h(x)) = h(u(x)) = v^B(h(x)) = v^B(q(t_x)) = q(v(t_x)). \tag{2.2}$$

Since h is an F-coalgebra morphism, we obtain from (2.1) that h merges the right-hand components of $a(u(x))$ and $a(v(x))$, in symbols: $h(uu(x)) = h(vu(x))$. It follows that q satisfies $q(uu(t_x)) = q(vu(t_x))$ using a similar argument as in (2.2) above.

Continuing to use that h and q are F-coalgebra morphisms, we obtain the following infinite list of elements (terms) of A' that are merged by q (we write these pairs as equations):

$$q(u^{n+1}(t_x)) = q(vu^n(t_x)) \qquad \text{for } n \in \mathbb{N}. \tag{2.3}$$

We need to prove that there exists no finite set of relations $E \subseteq A' \times A'$ generating the above congruence on A' given by $q : A' \twoheadrightarrow B$. So suppose the contrary, and let A'_0 be the Σ-subalgebra of A' generated by $\{t_x\}$, i.e. $A'_0 \cong \{u, v\}^* \times \{t_x\}$. Since $q(t_x) = h(x)$ and q and h are both coalgebra morphisms we know that $\dagger a' = \dagger b \cdot q$ and $\dagger b \cdot h = \dagger a$ and therefore

$$\dagger a'(t_x) = \dagger b(q(t_x)) = \dagger b(h(x)) = \dagger a(x) = (0, 1, 2, 3, \cdots).$$

Since $\dagger a'$ is a Σ-algebra morphism it follows that for a word $w \in \{u, v\}^*$ of length n we have

$$\dagger a'(w(t_x)) = (n, n+1, n+2, n+3, \cdots). \tag{2.4}$$

Thus, when $w, w' \in \{u, v\}^*$ are of different length, then the pair $(w(t_x), w'(t_x))$ cannot be in the congruence generated by E; otherwise we would have $q(w(t_x)) = q(w'(t_x))$ which implies $\dagger a'(w(t_x)) = \dagger a'(w'(t_x))$ contradicting (2.2).

Now let ℓ be the maximum length of words from $\{u, v\}^*$ occurring in any pair contained in the finite set E. Then the pair $(u^{\ell+2}(t_x), vu^{\ell+1}(t_x))$ obtained from the $\ell+1$-st equation in (2.3) is not in the congruence generated by E; for if any pair of terms of height greater then ℓ are related by that congruence, these two terms must have the same head symbol. Thus we arrive at a contradiction as desired.

(2) In this example we also have that ϑF and νF do not coincide. To see this we use that ϑF is the union of images of all $\dagger c : TX \to \nu F$ where (TX, c) ranges over those F-coalgebras whose carrier TX is free finitely generated (i.e. $TX \cong \{u, v\}^* \times X$ for some finite set X) [27, Theorem 6.5]. Hence, each such algebra TX is countable, and there exist only countably many of them, up to isomorphism. Furthermore, note that on every free finitely generated algebra TX there exist only countably many coalgebra structures $c : TX \to FTX$, since $FTX = \mathbb{N} \times TX$ is countable and c, being a Σ-algebra morphism, is determined by its action on the finitely many generators. Thus, ϑF is countable because it is the above union of countably many countable coalgebras. However, νF being carried by the set $\mathbb{N}^\omega$ of all streams over $\mathbb{N}$ is uncountable.

3. A Fixed Point Based on Coalgebras Carried by Free Algebras

In this section we study coalgebras for a functor F on an algebraic category Set^T whose carrier is a free finitely generated algebra. These coalgebras are of interest because they are precisely those coalgebras arising as the results of the generalized determinization [36].

We shall see that their colimit yields yet another fixed point of F (besides the rational fixed point and the locally finite one). Moreover, in the next section we show that this fixed point is characterized by a universal property as an algebra.

Assumptions 3.1. *Throughout the rest of the paper we assume that $\mathscr{A}$ is an* algebraic category*, i.e. $\mathscr{A}$ is (equivalent to) the Eilenberg-Moore category Set^T for a finitary monad T on* Set. *Furthermore, we assume that $F : \mathscr{A} \to \mathscr{A}$ is a finitary endofunctor preserving surjective T-algebra morphisms.*

Remark 3.2. (1) Note that we do not assume here that F preserves non-empty monomorphisms (cf. Theorems 2.14 and 2.16) as this assumption is not needed for our main result Theorem 5.6. However, we will make this assumption at the end, in order to obtain the picture in (1.1) (see Corollary 5.8).

(2) The most common instance of a functor F on an algebraic category $\mathscr{A}$ is a lifting of an endofunctor $F_0 : \mathsf{Set} \to \mathsf{Set}$, i.e. we have a commutative square

$$\begin{array}{ccc} \mathsf{Set}^T & \xrightarrow{F} & \mathsf{Set}^T \\ {\scriptstyle U}\downarrow & & \downarrow{\scriptstyle U} \\ \mathsf{Set} & \xrightarrow[F_0]{} & \mathsf{Set} \end{array}$$

where $U : \mathscr{A} \to \mathsf{Set}$ is the forgetful functor. Recall that monomorphisms in Set^T are precisely the injective T-algebra morphisms (see Remark 2.4(1)). Hence, a lifting F preserves all non-empty monos since the lifted set functor F_0 does so. Similarly, F preserves surjective T-algebra morphisms since F_0 preserves surjections (which are split epis in Set). Finally, F is finitary whenever F_0 is so because filtered colimits in Set^T are created by U.

(3) It is well known that liftings $F : \mathsf{Set}^T \to \mathsf{Set}^T$ are in bijective correspondence with distributive laws of the monad T over the functor F_0, i.e. natural transformations $\lambda : TF_0 \to F_0T$ satisfying two obvious axioms w.r.t. the unit and multiplication of T (see e.g. Johnstone [24]):

$$\begin{array}{ccc} F_0 & \xrightarrow{F_0\eta} & TF_0 \\ & {\scriptstyle \eta F_0}\searrow & \downarrow{\scriptstyle \lambda} \\ & & F_0T \end{array} \qquad \begin{array}{ccccc} TTF_0 & \xrightarrow{T\lambda} & TF_0T & \xrightarrow{\lambda T} & F_0TT \\ {\scriptstyle \mu F_0}\downarrow & & & & \downarrow{\scriptstyle F_0\mu} \\ TF_0 & & \xrightarrow[\lambda]{} & & F_0T \end{array}$$

Moreover, coalgebras for the lifting F are precisely the λ-*bialgebras*, i.e. sets X equipped with an Eilenberg-Moore algebra structure $\alpha : TX \to X$ and a coalgebra structure $c : X \to F_0X$ subject to the following commutativity condition

$$\begin{array}{ccccc} TX & \xrightarrow{Tc} & TF_0X & \xrightarrow{\lambda_X} & F_0TX \\ {\scriptstyle \alpha}\downarrow & & & & \downarrow{\scriptstyle F_0\alpha} \\ X & & \xrightarrow[c]{} & & F_0X \end{array}$$

which states that c is a T-algebra morphism from (X, α) to $F(X, \alpha)$.

(4) Let $F_0 : \mathsf{Set} \to \mathsf{Set}$ have a lifting to Set^T. *Generalized determinization* [36] is the process of turning a given coalgebra $c : X \to F_0TX$ in Set into the coalgebra $c^\star : TX \to FTX$ in Set^T. For example, for the functor $F_0X = \{0, 1\} \times X^\Sigma$ on Set and the finite power-set monad $T = \mathcal{P}_\mathsf{f}$, F_0T-coalgebras are precisely non-deterministic automata and generalized determinization is the construction of a deterministic automaton by the well-known subset construction. The unique F-coalgebra morphism $\dagger(c^\star)$ assigns to each state $x \in X$ the language accepted by x in the given non-deterministic automaton (whereas the final semantics for F_0T on Set provides a kind of process semantics taking the non-deterministic branching into account).

Thus studying the behaviour of F-coalgebras whose carrier is a free finitely generated T-algebra TX is precisely the study of a *coalgebraic language semantics* of finite F_0T-coalgebras.

Notation 3.3. We denote by

$$\mathsf{Coalg}_{\mathsf{ffg}}\, F$$

the full subcategory of $\mathsf{Coalg}\, F$ given by all coalgebras $c : TX \to FTX$ whose carrier is a free finitely generated T-algebra, i.e. where X is a finite set X.

The colimit of the inclusion functor of $\mathsf{Coalg}_{\mathsf{ffg}}\, F$ into the category of all F-coalgebras is denoted by

$$(\varphi F, \zeta) = \operatorname{colim}(\mathsf{Coalg}_{\mathsf{ffg}}\, F \hookrightarrow \mathsf{Coalg}\, F)$$

with the colimit injections $\mathsf{in}_c : TX \to \varphi F$ for every $c : TX \to FTX$ in $\mathsf{Coalg}_{\mathsf{ffg}}\, F$.

Notation 3.4. Since every free finitely generated algebra TX is clearly fp (being presented by the finite set X of generators and no relations), $\mathsf{Coalg}_{\mathsf{ffg}}\, F$ is a full subcategory of $\mathsf{Coalg}_{\mathsf{fp}}\, F$. Therefore, the universal property of the colimit φF induces a coalgebra morphism denoted by $h : \varphi F \to \varrho F$. Furthermore we write $m : \varphi F \to \nu F$ for the unique F-coalgebra morphisms into the final coalgebra, respectively.

We shall show in Proposition 3.9 that h is a strong epimorphism. Thus, whenever F preserves non-empty monos, we have the picture (1.1) from the introduction.

Remark 3.5. We will also use that the colimit φF is a sifted colimit.

(1) Recall that a small category $\mathscr{D}$ is called *sifted* [9] if finite products commute with colimits over $\mathscr{D}$ in Set. More precisely, $\mathscr{D}$ is sifted iff given any diagram $D : \mathscr{D} \times \mathscr{J} \to \mathsf{Set}$, where $\mathscr{J}$ is a finite discrete category, the canonical map

$$\operatorname*{colim}_{d \in \mathscr{D}} \Big(\prod_{j \in \mathscr{J}} D(d, j) \Big) \to \prod_{j \in \mathscr{J}} (\operatorname*{colim}_{d \in \mathscr{D}} D(d, j))$$

is an isomorphism. A *sifted colimit* is a colimit of a diagram with a sifted diagram scheme.

(2) It is well-known that the forgetful functor $\mathsf{Set}^T \to \mathsf{Set}$ preserves and reflects sifted colimits; this follows from [9, Corollary 11.9].

(3) Further recall [9, Example 2.16] that every small category $\mathscr{D}$ with finite coproducts is sifted. Thus, from Lemma 3.6 below it follows that $\mathscr{D} = \mathsf{Coalg}_{\mathsf{ffg}}\, F$ is sifted, and therefore φF is a sifted colimit.

Lemma 3.6. *The category $\mathsf{Coalg}_{\mathsf{ffg}}\, F$ is closed under finite coproducts in $\mathsf{Coalg}\, F$.*

Proof. The empty map $0 \to FT0$ extends uniquely to a T-algebra morphism $T0 \to FT0$, i.e. an F-coalgebra, and this coalgebra is the initial object of $\mathsf{Coalg}_{\mathsf{ffg}}\, F$.

Given coalgebras $c : TX \to FTX$ and $d : TY \to FTY$ one uses that $T(X + Y)$ together with the injections $T\mathsf{inl} : TX \to T(X + Y)$ and $T\mathsf{inr} : TY \to T(X + Y)$ form a coproduct in Set^T. This implies that forming the coproduct of (TX, c) and (TY, d) in $\mathsf{Coalg}\, F$ we obtain an F-coalgebra on $T(X + Y)$, and this is an object of $\mathsf{Coalg}_{\mathsf{ffg}}\, F$ since $X + Y$ is finite. □

Theorem 3.7 (Urbat [41], Lemma 4.5)**.** *If F preserves sifted colimits, then φF is a fixed point of F, i.e. $\zeta : \varphi F \to F(\varphi F)$ is an isomorphism.*

Recall that every finitary endofunctor on Set preserves sifted colimits (this follows from [9, Corollary 6.30]). Thus, so does every lifting $F : \mathsf{Set}^T \to \mathsf{Set}^T$ of a finitary endofunctor on Set, using Remark 3.5(2). In general, finitary functors need not preserve sifted colimits [9, Example 7.11].

One might now expect that φF is characterized as a coalgebra by a universal property similar to finality properties that characterize ϱF and ϑF. However, Urbat [41] shows that this is not the case. In fact, he provides the following example of a coalgebra $c : TX \to FTX$ where $\mathsf{in}_c : TX \to \varphi F$ is not the only F-coalgebra morphism:

Example 3.8. (1) Let $\mathscr{A}$ be the category of algebras for the signature with one unary operation symbol u (and no equations), and let $F = \mathsf{Id}$ be the identity functor on $\mathscr{A}$. Let A be the free (term) algebra on one generator x, and let B be the free algebra on one generator y (i.e. both A and B are isomorphic to $\mathbb{N}$). We equip A and B with the F-coalgebra structures $a = \mathsf{id} : A \to A$ and $b : B \to B$ given by $b(y) = u(y)$. Then the mapping $t \mapsto u(t)$ clearly is an F-coalgebra morphism from B to itself, i.e. a morphism in $\mathsf{Coalg}_{\mathsf{ffg}}\, F$. Therefore we have $\mathsf{in}_b(y) = \mathsf{in}_b(u(y))$.

Now define a morphism $g : A \to \varphi F$ in $\mathscr{A}$ by $g(x) = \mathsf{in}_b(y)$. Then g is an F-coalgebra morphism since

$$g \cdot a(x) = g(x) = \mathsf{in}_b(y) = \mathsf{in}_b(u(y)) = \mathsf{in}_b(b(y)) = \zeta(\mathsf{in}_b(y)) = \zeta(g(x)),$$

where $\zeta : \varphi F \to F(\varphi F)$ is the coalgebra structure on φF.

We prove the following property: for every morphism f in $\mathsf{Coalg}_{\mathsf{ffg}}\, F$ from $\alpha : TX \to TX$ to $\beta : TY \to TY$, any $t \in TX$ reaches finitely many states iff $f(t)$ does so, more precisely:

$$\{\alpha^n(t) \mid n \in \mathbb{N}\} \text{ is finite} \qquad \iff \qquad \{\beta^n(f(t)) \mid n \in \mathbb{N}\} \text{ is finite.}$$

Indeed, if t reaches finitely many states, then the $f(\alpha^n(t))$, for $n \in \mathbb{N}$, form a finite set, and $\beta^n(f(t))$, $n \in \mathbb{N}$ is the same set since f is a coalgebra morphism.

Conversely, suppose that t reaches infinitely many states. Since f is a morphism in $\mathscr{A}$, we know that if $\alpha^n(t)$ is $u^k(x)$ for some $x \in X$ then $f(\alpha^n(t)) = \beta^n(f(t))$ must be $u^l(y)$ with $l \geqslant k$ for some $y \in Y$. Thus, $f(t)$ must also reach infinitely many states.

We can now conclude that $g, \mathsf{in}_a : A \to \varphi F$ are different coalgebra morphisms. Indeed, $\mathsf{in}_a(x)$ reaches only itself since x does so, but $g(x) = \mathsf{in}_b(y)$ reaches infinitely many states since y does so. Thus, $g(x) \neq \mathsf{in}_a(x)$.

It follows that $|\varphi F| \geqslant 2$, while $\varrho F = \vartheta F = \nu F = 1$; to see the latter equation use that $\mathsf{id} : 1 \to 1$ is a coalgebra in $\mathsf{Coalg}_{\mathsf{fp}}\, F$ since 1 is the object of $\mathscr{A}$ presented by one generator z and one relation $z = u(z)$.

(2) Using similar ideas as in the previous point one can show that, for the category $\mathscr{A}$ and $FX = \mathbb{N} \times X$ from Example 2.18, φF and ϱF do not coincide. Consequently, in this example, none of the arrows in (1.1) is an isomorphism.

In order to see that φF and ϱF do not coincide, consider the two coalgebras $a : A \to FA$ and $b : B \to FB$ with $A = T\{x\}$ and $B = T\{y\}$ and with the coalgebra structure given by $a(x) = (0, u(x))$ and $b(y) = (0, v(y))$. These coalgebras both lie in $\mathsf{Coalg}_{\mathsf{ffg}}\, F$. Consider also the coalgebra $p : P \to FP$ where P is presented by one generator z and one relation $u(z) = v(z)$, i.e. $P = T\{z\}/{\sim}$, where $\sim$ is the smallest congruence with $u(z) \sim v(z)$. Hence, $w(z) \sim w'(z)$ for $w, w' \in \{u, v\}^*$ iff w and w' have the same length. The coalgebra structure is defined by $p([w(x)]) = (0, [uw(x)])$. The coalgebra (P, p) lies in $\mathsf{Coalg}_{\mathsf{fp}}\, F$. Now $f : A \to P$ and $g : B \to P$ determined by $f(x) = z = g(y)$ are easily seen to be F-coalgebra morphisms, and therefore $a^\sharp = p^\sharp \cdot f$ and $b^\sharp = p^\sharp \cdot g$. Therefore

$$a^\sharp(x) = p^\sharp(f(x)) = p^\sharp(z) = p^\sharp(g(y)) = b^\sharp(y).$$

However, we will prove that $\mathsf{in}_a(x) \neq \mathsf{in}_b(x)$. For any (TX, c) in $\mathsf{Coalg}_{\mathsf{ffg}}\, F$ and $t \in TX$, we say that *t-reachable states are u-bounded* if there exists a natural number k such that, for any state $s = w(x)$ reachable from t via the next state function, the number $|w|_u$ of u's in w is at most k. Now we prove for any morphism $f : (TX, c) \to (TY, d)$ in

$\mathsf{Coalg}_{\mathsf{ffg}}\, F$ and any $t \in TX$ the following claim:

t-reachable states are u-bounded iff $f(t)$-reachable states are u-bounded.

Indeed, a state $s = w(x)$ is reachable from t iff $f(s) = wf(x)$ is reachable from $f(t)$. Then the 'only if' direction is clear: if t-reachable states are not u-bounded, then neither are $f(t)$-reachable states. For the 'if' direction suppose t-reachable states are u-bounded by k. Then $f(t)$-reachable states are bounded by $k + \max\{|f(x)|_u \mid x \in X\}$.

Coming back to the discussion of properties that φF does have, the following proposition shows that ϱF is always a strong quotient of φF. Recall from Notation 3.4 the canonical coalgebra morphism h from φF to ϱF:

Proposition 3.9. *The morphism $h : \varphi F \twoheadrightarrow \varrho F$ is a strong epimorphism in $\mathscr{A}$.*

The following proof is set theoretic and makes explicit use of the fact that $\mathscr{A}$ is algebraic over Set, i.e. we use that strong epimorphisms in $\mathscr{A}$ are precisely surjective T-algebra morphisms. In the appendix we provide a purely category theoretic proof, which is somewhat longer, however. That proof shows that the above result holds for more general base categories than sets.

Proof. We first prove the following fact:

every coalgebra in $\mathsf{Coalg}_{\mathsf{fp}}\, F$ is a regular quotient of some coalgebra in $\mathsf{Coalg}_{\mathsf{ffg}}\, F$.

Indeed, given any $a : A \to FA$ in $\mathsf{Coalg}_{\mathsf{fp}}\, F$ we know that its carrier is a regular quotient of some free T-algebra TX with X finite, via $q : TX \twoheadrightarrow A$, say (see Remark 2.4.3). Since F preserves regular epis (= surjections) we can use projectivity of TX (see Remark 2.4.2) to obtain a coalgebra structure c on TX making q an F-coalgebra morphism:

$$\begin{array}{ccc} TX & \overset{c}{\dashrightarrow} & FTX \\ {\scriptstyle q}\big\downarrow & & \big\downarrow{\scriptstyle Fq} \\ A & \underset{a}{\longrightarrow} & FA \end{array}$$

This implies that we have $c^\sharp = a^\sharp \cdot q$.

Now let $p \in \varrho F$. Since ϱF is the colimit of all coalgebras in $\mathsf{Coalg}_{\mathsf{fp}}\, F$, we know from Lemma 2.2 that there exists some coalgebra $a : A \to FA$ in $\mathsf{Coalg}_{\mathsf{fp}}\, F$ and $r \in A$ such that $a^\sharp(r) = p$. By the above fact, we have (TX, c) in $\mathsf{Coalg}_{\mathsf{ffg}}\, F$ and the surjective coalgebra morphism $q : TX \twoheadrightarrow A$. Hence there exists some $s \in TX$ with $q(s) = r$. By the finality of ϱF we have the commuting square below:

$$\begin{array}{ccc} TX & \overset{q}{\twoheadrightarrow} & A \\ {\scriptstyle \mathsf{in}_c}\big\downarrow & & \big\downarrow{\scriptstyle a^\sharp} \\ \varphi F & \underset{h}{\twoheadrightarrow} & \varrho F \end{array}$$

Thus we have $p = a^\sharp(q(s)) = h(\mathsf{in}_c(s))$, which shows that h is surjective as desired. □

Corollary 3.10. *If F preserves non-empty monomorphisms, then we obtain the situation displayed in* (1.1)*:*

$$\varphi F \twoheadrightarrow \varrho F \twoheadrightarrow \vartheta F \rightarrowtail \nu F.$$

Indeed, this follows from Proposition 3.9 and Theorem 2.16.

4. A Universal Property of φF

We have seen in Example 3.8(1) that φF, unlike ϱF and ϑF, does not enjoy a finality property as a coalgebra. In this section we will prove that, as an algebra for F, φF is characterized by a universal property. This property then determines φF uniquely up to isomorphism. To this end we make the

Assumption 4.1. *In addition to Assumptions 3.1 we assume in this section that F preserves sifted colimits (cf. Remark 3.5).*

By Theorem 3.7, we know that φF is then a fixed point of F so that by inverting its coalgebra structure we may regard it as the F-algebra $\zeta^{-1} : F(\varphi F) \to \varphi F$.

We have already mentioned that both ϱF and ϑF are characterized by universal properties as F-algebras: they are the initial iterative and initial fg-iterative algebras, respectively. However, those properties entail that there exists a unique F-coalgebra morphism from every coalgebra in $\mathsf{Coalg}_{\mathsf{fp}}\, F$ to ϱF, and from every coalgebra in $\mathsf{Coalg}_{\mathsf{fg}}\, F$ to ϑF, respectively. That means that simply adjusting the definition of the notion of iterative algebra does not yield the desired universal property of φF, again due to Example 3.8(1).

The key to establishing a universal property of φF is to consider algebras which admit canonical (rather than unique) coalgebra-to-algebra homomorphisms. The following notion is inspired by the Bloom algebras introduced by Adámek et al. [2].

Definition 4.2. An *ffg-Bloom algebra* for the functor F is a triple $(A, a, \dagger)$ where $a : FA \to A$ is an F-algebra and $\dagger$ is an operation

$$\frac{TX \xrightarrow{c} FTX,\, X \text{ finite}}{TX \xrightarrow{c^\dagger} A}$$

subject to the following axioms:

(1) solution: $c^\dagger$ is a coalgebra-to-algebra morphism, i.e. the diagram below commutes:

$$\begin{array}{ccc} TX & \xrightarrow{c^\dagger} & A \\ {\scriptstyle c}\downarrow & & \uparrow{\scriptstyle a} \\ FTX & \xrightarrow[Fc^\dagger]{} & FA \end{array}$$

(2) functoriality: for every coalgebra morphism $m : (TX, c) \to (TY, d)$ in $\mathsf{Coalg}_{\mathsf{ffg}}\, F$ we have $c^\dagger = d^\dagger \cdot m$:

$$\begin{array}{ccc} TX & \xrightarrow{c} & FTX \\ {\scriptstyle m}\downarrow & & \downarrow{\scriptstyle Fm} \\ TY & \xrightarrow[d]{} & FY \end{array} \quad\Longrightarrow\quad \begin{array}{ccc} TX & & \\ {\scriptstyle m}\downarrow & \searrow^{c^\dagger} & A \\ TY & \nearrow_{d^\dagger} & \end{array}$$

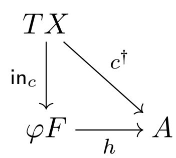
A *morphism of ffg-Bloom algebras* from $(A, a, \dagger)$ to $(B, b, \ddagger)$ is an F-algebra morphism preserving solutions, i.e. an F-algebra morphism $h : (A, a) \to (B, b)$ such that for every $c : TX \to FTX$ in $\mathsf{Coalg}_{\mathsf{ffg}}\, F$ we have

$$c^\ddagger = (TX \xrightarrow{c^\dagger} A \xrightarrow{h} B).$$

Observation 4.3. The algebra $\zeta^{-1} : F(\varphi F) \to \varphi F$ together with the operation $\ddagger$ given by the colimit injections, i.e. $c^\ddagger = \mathsf{in}_c : TX \to \varphi F$ for every $c : TX \to FTX$ in $\mathsf{Coalg}_{\mathsf{ffg}}\, F$, clearly is an ffg-Bloom algebra. Indeed, the solution axiom holds since in_c is a coalgebra morphisms from (TX, c) to $(\varphi F, \zeta)$ and functoriality holds since the in_c form a compatible cocone of the diagram $D : \mathsf{Coalg}_{\mathsf{ffg}} F \hookrightarrow \mathsf{Coalg}\, F$.

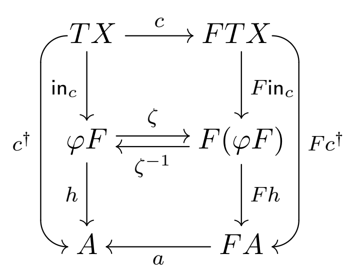
Theorem 4.4. *The above Bloom algebra on φF is the initial ffg-Bloom algebra.*

Proof. It remains to prove the universal property. Let $(A, a, \dagger)$ be any ffg-Bloom algebra. Then the morphisms $c^\dagger : TX \to A$, for $c : TX \to FTX$ ranging over $\mathsf{Coalg}_{\mathsf{ffg}} F$, form a compatible cocone on the diagram D by functoriality. Therefore we have a unique morphism $h : \varphi F \to A$ such that the triangles below commute



for every $c : TX \to FTX$ in $\mathsf{Coalg}_{\mathsf{ffg}} F$.

In order to see that h is an F-algebra morphism consider the diagram below:



(4.1)

Its outside commutes, for every $c : TX \to FTX$ in $\mathsf{Coalg}_{\mathsf{ffg}} F$, by the solution axiom for A, and the left-hand and right-hand parts by the definition of h. The upper square commutes by the solution axiom for φF. Therefore, for every $c : TX \to FTX$ in $\mathsf{Coalg}_{\mathsf{ffg}} F$ we have

$$h \cdot \mathsf{in}_c = a \cdot Fh \cdot \zeta \cdot \mathsf{in}_c.$$

Use that the colimit injections in_c form an epimorphic family to conclude that h is an F-algebra morphism, i.e. $h \cdot \zeta^{-1} = a \cdot Fh$. This proves existence of a morphism of ffg-Bloom algebras from φF to A.

For the uniqueness suppose that $g : \varphi F \to A$ is any morphism of ffg-Bloom algebras. Then

$$g \cdot \mathsf{in}_c = g \cdot c^\ddagger = c^\dagger$$

holds for every $c : TX \to FTX$ in $\mathsf{Coalg}_{\mathsf{ffg}} F$. Thus, $g = h$ by the universal property of the colimit φF. □

The following result provides a simple alternative characterization of the category of ffg-Bloom algebras for F without mentioning $\dagger$ and its axioms. This result is similar to [2, Prop. 3.4] for ordinary Bloom algebras. Here $\mathsf{Alg}\, F$ denotes the category of all F-algebras.

Proposition 4.5. *The category of ffg-Bloom algebras is isomorphic to the slice category $(\varphi F, \zeta^{-1})/\mathsf{Alg}\, F$.*

Proof. (1) Given an ffg-Bloom algebra $(A, a, \dagger)$, initiality of φF provides an F-algebra morphism $h : \varphi F \to A$, i.e. an object of the slice category. Moreover, this object assignment clearly gives rise to a functor using the initiality of φF.

(2) In the reverse direction, suppose we are given any F-algebra (A, a) and F-algebra morphism $h : (\varphi F, \zeta^{-1}) \to (A, a)$. Then we define for every $c : TX \to FTX$ in $\mathsf{Coalg}_{\mathsf{ffg}}\, F$,

$$c^\dagger = (TX \xrightarrow{\mathsf{in}_c} \varphi F \xrightarrow{h} A).$$

Then using diagram (4.1) we see that $c^\dagger$ satisfies the solution axiom: indeed, the outside of the diagram commutes since all its inner parts do. Moreover, functoriality of $\dagger$ follows from that of $\ddagger$: given any $m : (TX, c) \to (TY, d)$ in $\mathsf{Coalg}_{\mathsf{ffg}}\, F$ we have

$$d^\dagger \cdot m = h \cdot \mathsf{in}_d \cdot m = h \cdot \mathsf{in}_c = c^\dagger.$$

Furthermore, given a morphism in the slice category, i.e. we have $h : (\varphi F, \zeta^{-1}) \to (A, a)$, $g : (\varphi F, \zeta^{-1}) \to (B, b)$ and $m : (A, a) \to (B, b)$ such that $m \cdot h = g$, we see that m is a morphism of ffg-Bloom algebras from $(A, a, \dagger)$ to $(B, b, \ddagger)$, where $c^\ddagger : TX \to B$ is defined as $g \cdot \mathsf{in}_c$: indeed, m is an F-algebra morphism and we have

$$m \cdot c^\dagger = m \cdot h \cdot \mathsf{in}_c = g \cdot \mathsf{in}_c = c^\ddagger.$$

That this gives a functor from the slice category to the category of ffg-Bloom algebras is again straightforward.

(3) We have defined two identity-on-morphisms functors and it remains to show that they are mutually inverse on objects.

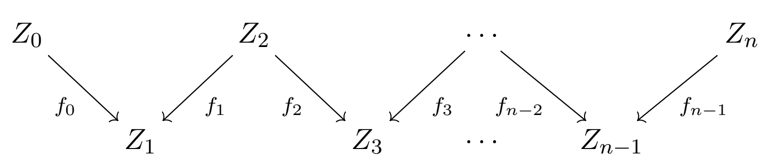
From ffg-Bloom algebras to the slice category and back we form for the given ffg-Bloom algebra $(A, a, \dagger)$ the ffg-Bloom algebra $(A, a, \ddagger)$ where $c^\ddagger = h \cdot \mathsf{in}_c$ for the unique morphism $h : \varphi F \to A$ of ffg-Bloom algebras. Hence, since h preserves solutions we thus have $c^\ddagger = h \cdot \mathsf{in}_c = c^\dagger$ for every $c : TX \to FTX$ in $\mathsf{Coalg}_{\mathsf{ffg}}\, F$.

From the slice category to ffg-Bloom algebras and back we take for a given F-algebra morphism $h : (\varphi F, \zeta^{-1}) \to (A, a)$ the Bloom algebra $(A, a, \dagger)$ with $c^\dagger = h \cdot \mathsf{in}_c$, which shows that h is a morphism of ffg-Bloom algebras. Thus, going back to the slice category we get back to h. □

5. Proper Functors and Full Abstractness of φF

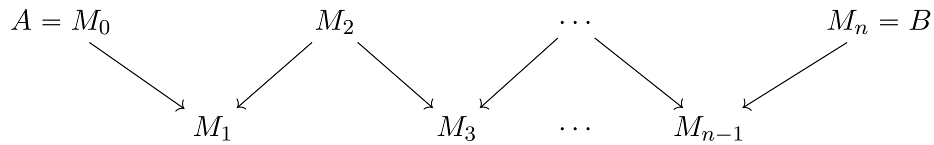
In this section we are going to investigate when the three left-hand fixed points in (1.1) collapse to one, i.e. $\varphi F \cong \varrho F \cong \vartheta F$. We introduce proper functors and show that a functor is proper if and only if φF is fully abstract, i.e. a subcoalgebra of the final one. This also entails that the rational fixed point ϱF is fully abstract and at the same time it is determined by the coalgebras with free finitely generated carrier. More precisely, the finality of a given locally fp coalgebra for F can be established by checking the universal property only for the coalgebras in $\mathsf{Coalg}_{\mathsf{ffg}}\, F$ (Corollary 5.9). Here we continue to work under Assumptions 3.1.

Remark 5.1. (1) Recall that a *zig-zag* in a category $\mathscr{A}$ is a diagram of the form



For $\mathscr{A} = \mathsf{Set}^T$, we say that the zig-zag *relates* $z_0 \in Z_0$ and $z_n \in Z_n$ if there exist $z_i \in Z_i$, $i = 1, \ldots, n-1$ such that $f_i(z_i) = z_{i+1}$ for i even and $f_i(z_{i+1}) = z_i$ for i odd.

(2) Ésik and Maletti [17] introduced the notion of a proper semiring in order to obtain the decidability of the (language) equivalence of weighted automata. A semiring $\mathbb{S}$ is called *proper* provided that for every two $\mathbb{S}$-weighted automata A and B whose initial states x and y, respectively, accept the same weighted language there exists a zig-zag



of simulations that *relates* x and y. Recall here that a *simulation* from a weighted automaton $(i, (M^a)_a a \in A, o)$ with n states to another one $(j, (N^a)_{a\in A}, p)$ with m states is an $\mathbb{S}$-semimodule morphism represented by an $n \times m$ matrix H over $\mathbb{S}$ such that $i \cdot H = j$, $o \cdot H = p$ and $M_a \cdot H = H \cdot N_a$.

Ésik and Maletti show that every Noetherian semiring is proper as well as the semiring $\mathbb{N}$ of natural numbers, which is not Noetherian. However, the tropical semiring $(\mathbb{N} \cup \{\infty\}, \min, +, \infty, 0)$ is not proper.

Recall from Example 2.3 that $\mathbb{S}$-weighted automata with input alphabet Σ are equivalently coalgebras with carrier $\mathbb{S}^n$, where $n \geqslant 1$ is the number of states, for the functor $FX = \mathbb{S} \times X^\Sigma$ on the category $\mathbb{S}$-Mod. Note that the $\mathbb{S}^n$ are precisely the free finitely generated $\mathbb{S}$-semimodules, whence $\mathbb{S}$-weighted automata are precisely the coalgebras in $\mathsf{Coalg}_{\mathsf{ffg}}\, F$, which explains why we are interested in collecting precisely their behaviour in the form of the fixed point φF. Moreover, since simulations of $\mathbb{S}$-weighted automata are clearly in one to one correspondence with F-coalgebra morphisms, one easily generalizes the notion of a proper semiring as follows. Recall that $\eta_X : X \to TX$ denotes the unit of the monad T.

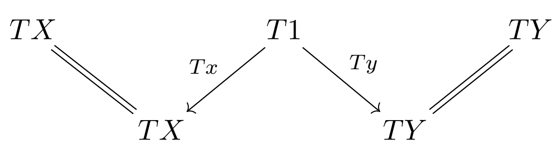
Definition 5.2. We call the functor $F : \mathscr{A} \to \mathscr{A}$ *proper* whenever for every pair of coalgebras $c : TX \to FTX$ and $d : TY \to FTY$ in $\mathsf{Coalg}_{\mathsf{ffg}}\, F$ and every $x \in X$ and $y \in Y$ such that $\eta_X(x) \sim \eta_Y(y)$ are behaviourally equivalent there exists a zig-zag in $\mathsf{Coalg}_{\mathsf{ffg}}\, F$ relating $\eta_X(x)$ and $\eta_Y(y)$.

Example 5.3. A semiring $\mathbb{S}$ is proper iff the functor $FX = \mathbb{S} \times X^\Sigma$ on $\mathbb{S}$-Mod is proper for every input alphabet Σ. We know that Noetherian semirings are proper (cf. Example 2.12.3), and the semiring $\mathbb{N}$ of natural numbers is proper. Recently, Sokolova and Woracek [40] have shown that the non-negative rationals $\mathbb{Q}_+$ and non-negative reals $\mathbb{R}_+$ form proper semirings.

Example 5.4. Constant functors are always proper. Indeed, suppose that F is the constant functor on some algebra A. Then we have $\nu F = A$, and for any F-coalgebra B its coalgebra structure $c : B \to FB = A$ is also the unique F-coalgebra morphism from B to $\nu F = A$.

Now given any $c : TX \to FTX = A$ and $d : TY \to FTY = A$ and $x \in TX$, $y \in TY$ as in Definition 5.2. Then $\eta_X(x) \sim \eta_Y(y)$ is equivalent to $c(\eta_X(x)) = d(\eta_Y(y))$. Let a be this element of A, and extend $x : 1 \to X$, $y : 1 \to Y$ and $a : 1 \to A$ to T-algebra morphisms $Tx : T1 \to TX$, $Ty : T1 \to TY$ and

$a^\star : T1 \to A = FT1$ (the latter yielding an F-coalgebra). Then



is the required zig-zag in $\mathsf{Coalg}_{\mathsf{ffg}}\, F$ relating $\eta_X(x)$ and $\eta_Y(y)$.

Example 5.5. Sokolova and Woracek [40] have recently proved that the functor $FX = [0,1] \times X^\Sigma$ on the category PCA of positively convex algebras (see Example 2.12.4) is proper. In addition, its subfunctor $\hat{F}$ given by

$$\hat{F}X = \{(o, f) \in [0,1] \times X^\Sigma \mid \forall s \in \Sigma : \exists p_s \in [0,1], x_s \in X : \\ o + \sum_{s\in\Sigma} p_s \leqslant 1, f(s) = p_s x_s\}$$

is proper. The latter functor was used as coalgebraic type functor for the axiomatization of probabilistic systems in [38]. In fact, the completeness proof of the expression calculus in loc. cit. makes use of our Corollary 5.9 below.

In general, it seems to be non-trivial to establish that a given functor is proper (even for the identity functor this may fail; in the light of Theorem 5.6 below this follows from Example 3.8(1)). However, we will provide in Proposition 5.10 sufficient conditions on $\mathscr{A}$ and F the entail properness using our main result:

Theorem 5.6. *The functor F is proper iff the coalgebra φF is a subcoalgebra of νF.*

The latter condition states that the unique coalgebra morphism $m : \varphi F \to \nu F$ is a monomorphism in $\mathscr{A}$.

We present the proof of this theorem in Section 6. Here we continue with a discussion of the consequences of this result.

Corollary 5.7. *If F is proper, then φF is the rational fixed point of F.*

Proof. Let $u : \varrho F \to \nu F$ be the unique F-coalgebra morphism. Then we have a commutative triangle of F-coalgebra morphisms due to finality of νF:

$$\varphi F \xrightarrow{\;h\;} \!\!\!\!\to \varrho F \xrightarrow{\;u\;} \nu F, \qquad m : \varphi F \to \nu F.$$

Since F is proper, m is a monomorphism in $\mathscr{A}$, hence so is h. Since h is also a strong epimorphism by Proposition 3.9, it is an isomorphism. Thus, $\varphi F \cong \varrho F$ is the rational fixed point of F. □

Corollary 5.8. *Suppose that F preserves non-empty monomorphisms. Then the functor F is proper iff $\varphi F \cong \varrho F \cong \vartheta F \rightarrowtail \nu F$.*

Proof. If the three fixed points are isomorphic, then F is proper by Theorem 5.6.

Conversely, since F preserves non-empty monomorphisms, we have the situation displayed in (1.1) (see Corollary 3.10). Now if F is proper we know from Corollary 5.7 that $\varphi F \cong \varrho F$. Thus, ϱF is a subcoalgebra of νF, i.e. the composition of the last two morphisms in (1.1) is a monomorphism. Thus, so is $\varrho F \twoheadrightarrow \vartheta F$. Since this is also a strong epimorphism, we conclude that $\varrho F \cong \vartheta F$. □

Note that this result also entails full abstractness of $\varphi F \cong \varrho F$.

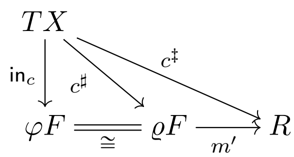
A key result for establishing soundness and completeness of coalgebraic regular expression calculi is the following corollary (cf. [12, Corollary 3.36] and its applications in Sections 4 and 5 of loc. cit.).

Corollary 5.9. *Suppose that F is proper. Then an F-coalgebra (R, r) is a final locally fp coalgebra if and only if (R, r) is locally fp and for every coalgebra (TX, c) in $\mathsf{Coalg}_{\mathsf{ffg}}\, F$ there exists a unique F-coalgebra morphism from TX to R.*

Proof. The implication "⇒" clearly holds

For "⇐" it suffices to prove that for every $a : A \to FA$ in $\mathsf{Coalg}_{\mathsf{fp}}\, F$ there exists a unique F-coalgebra morphism from A to R. In fact, it then follows that R is the final locally fp coalgebra. To see this write an arbitrary locally fp coalgebra A as a filtered colimit of a diagram $D : \mathscr{D} \to \mathsf{Coalg}_{\mathsf{fp}}\, F \hookrightarrow \mathsf{Coalg}\, F$ with colimit injections $h_d : Dd \to A$ (d an object in $\mathscr{D}$). Then the unique F-coalgebra morphisms $u_d : Dd \to R$ form a compatible cocone, and so one obtains a unique $u : A \to R$ such that $u \cdot h_d = u_d$ holds for every object d of $\mathscr{D}$. It is now straightforward to prove that u is a unique F-coalgebra morphism from A to R.

Now let $a : A \to FA$ be a coalgebra in $\mathsf{Coalg}_{\mathsf{fp}}\, F$. For every (TX, c) in $\mathsf{Coalg}_{\mathsf{ffg}}\, F$ denote by $c^\ddagger : TX \to R$ the unique F-coalgebra morphism that exists by assumption. These morphisms $c^\ddagger$ form a compatible cocone of the diagram $\mathsf{Coalg}_{\mathsf{ffg}}\, F \hookrightarrow \mathsf{Coalg}\, F$. Thus, we obtain a unique F-coalgebra morphism $m' : \varphi F \cong \varrho F \to R$ such that the following diagram commutes for every $c : TX \to FTX$ in $\mathsf{Coalg}_{\mathsf{ffg}}\, F$:



Therefore we have an F-coalgebra morphism

$$h = (A \xrightarrow{a^\sharp} \varrho F \xrightarrow{m'} R).$$

To prove it is unique, assume that $g : A \to R$ is any F-coalgebra morphism. As in the proof of Proposition 3.9, we know that A is the quotient of some TX in $\mathsf{Coalg}_{\mathsf{ffg}}\, F$ via $q : TX \twoheadrightarrow A$, say. Then we have

$$m' \cdot a^\sharp \cdot q = g \cdot q$$

because there is only one F-coalgebra morphism from TX to R by hypothesis. It follows that $h = m' \cdot a^\sharp = g$ since q is epimorphic. □

The next result provides sufficient conditions for properness of F. It can be seen as a category-theoretic generalization of Ésik's and Maletti's result [17, Theorem 4.2] that Noetherian semirings are proper.

Proposition 5.10. *Suppose that finitely generated algebras in $\mathscr{A}$ are closed under kernel pairs and that F maps kernel pairs to weak pullbacks in Set. Then F is proper.*

Proof. First, since F maps kernel pairs to weak pullbacks in Set we see that F preserves monomorphisms; indeed, $m : A \rightarrowtail B$ is a mono in $\mathscr{A}$ iff and only if its kernel pair is $\mathsf{id}_A, \mathsf{id}_A$. Thus $F\mathsf{id}_A, F\mathsf{id}_A$ form a weak pullback in Set, which is in fact a pullback, whence Fm is monomorphic.

Now let (TX, c) and (TY, d) be in $\mathsf{Coalg}_{\mathsf{ffg}}\, F$, $x \in X$ and $y \in Y$ such that $\dagger c(\eta_X(x)) = \dagger d(\eta_Y(y))$. It is our task to construct a zig-zag relating $\eta_X(x)$ and $\eta_Y(y)$.

Form $Z = X + Y$ and let $e : TZ \to FTZ$ be the coproduct of the coalgebras (TX, c) and (TY, d) in $\mathsf{Coalg}_{\mathsf{ffg}}\, F$ (see Lemma 3.6). Take the factorization of $\dagger e : TZ \to \nu F$ into a strong epi $q : TZ \twoheadrightarrow A$ followed by a monomorphism $m : A \rightarrowtail \nu F$. Since F preserves non-empty monos, we obtain a unique coalgebra structure $a : A \to FA$ such that q and m are coalgebra morphisms (see Remark 2.15(2)). Now take the kernel pair $f, g : K \rightrightarrows TZ$ of q. Since TZ and its quotient A are finitely generated T-algebras, so is K because finitely generated T-algebras are closed under taking kernel pairs by assumption. Now F maps the kernel pair f, g to a weak pullback Ff, Fg of Fq along itself in Set. Thus, we have a map $k\colon K \to FK$ such that the diagram below commutes:

$$\begin{array}{ccc} K & \overset{k}{\dashrightarrow} & FK \\ {\scriptstyle f}\downdownarrows{\scriptstyle g} & & {\scriptstyle Ff}\downdownarrows{\scriptstyle Fg} \\ TZ & \xrightarrow{e} & FTZ \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle Fq} \\ A & \xrightarrow[a]{} & FA \end{array} \tag{5.1}$$

Notice that we do not claim that k is a T-algebra morphism. However, since K is a finitely generated T-algebra, it is the quotient of some free finitely generated T-algebra TR via $p\colon TR \twoheadrightarrow K$, say. Now we choose some splitting $s\colon K \to TR$ of p in Set, i. e., s is a map such that $p \cdot s = \mathsf{id}$. Next we extend the map $r_0 = Fs \cdot k \cdot p \cdot \eta_R$ to a T-algebra morphism $r\colon TR \to FTR$; it follows that the outside of the diagram below commutes:

$$\begin{array}{ccc} R & & \\ {\scriptstyle \eta_R}\downarrow & \overset{r_0}{\searrow} & \\ TR & \overset{r}{\dashrightarrow} & FTR \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle Fp} \\ K & \xrightarrow[k]{} & FK \end{array} \tag{5.2}$$

(Notice that to obtain r we cannot simply use projectivity of TR since k is not necessarily a T-algebra homomorphism.)

We do not claim that this makes p a coalgebra morphism (i. e., we do not claim the lower square in (5.2) commutes). However, $f \cdot p$ and $g \cdot p$ are coalgebra morphisms from (TR, r) to (TZ, e); in fact, to see that

$$e \cdot (f \cdot p) = F(f \cdot p) \cdot r$$

it suffices that this equation of T-algebra morphisms holds when both sides are precomposed with η_R. To this end we compute

$$\begin{array}{rcll} e \cdot f \cdot p \cdot \eta_R & = & Ff \cdot k \cdot p \cdot \eta_R & \text{see (5.1),} \\ & = & Ff \cdot Fp \cdot r_0 & \text{outside of (5.2),} \\ & = & Ff \cdot Fp \cdot r \cdot \eta_R & \text{definition of } d. \end{array}$$

Similarly, $g \cdot p$ is a coalgebra morphism.

Now consider the following zig-zag in $\mathsf{Coalg}_{\mathsf{ffg}}\, F$ (recall that the algebra TZ is the coproduct of TX and TY with coproduct injections $T\mathsf{inl}$ and $T\mathsf{inr}$):

$$TX \xrightarrow{T\mathsf{inl}} TZ \xleftarrow{f \cdot p} TR \xrightarrow{g \cdot p} TZ \xleftarrow{T\mathsf{inr}} TY$$

We now show that this zig-zag relates $\eta_X(x)$ and $\eta_Y(y)$. Let $x' = T\mathsf{inl}(\eta_X(x))$ and $y' = T\mathsf{inr}(\eta_Y(y))$. Then we have

$$\dagger e(x') = \dagger e \cdot T\mathsf{inl}(\eta_X(x)) = \dagger c(\eta_X(x)) = \dagger d(\eta_Y(y)) = \dagger e \cdot T\mathsf{inr}(\eta_Y(y)) = \dagger e(y').$$

Hence, since $\dagger e = m \cdot q$ and m is monomorphic, we obtain $q(x') = q(y')$. Thus, there exists some $k \in K$ such that $f(k) = x'$ and $g(k) = y'$ by the universal property of the kernel pair. Finally, since $p : TR \twoheadrightarrow K$ is surjective we obtain some $z \in TR$ such that $p(z) = k$ whence $f \cdot p(z) = x'$ and $g \cdot p(z) = y'$. This completes the proof. □

Remark 5.11. (1) Note that closure of finitely generated algebras under kernel pairs can equivalently be stated in general algebra terms as follows: every congruence R of a finitely generated algebra A is finitely generated as a subalgebra $R \hookrightarrow A \times A$ (observe that this is *not* equivalent to stating that R is a finitely generated congruence).

(2) For a lifting F of a set functor F_0, the condition that F maps kernel pairs to weak pullbacks in Set holds whenever F_0 preserves weak pullbacks. Hence, all the functors on algebraic categories mentioned in Example 2.13 satisfy this assumption.

(3) For the special case of a lifting, a variant of the argument in the proof of Proposition 5.10 was used in [12, Proposition 3.34] in order to prove that every coalgebra in $\mathsf{Coalg}_{\mathsf{fp}}\, F$ is a coequalizer of a parallel pair of morphisms in $\mathsf{Coalg}_{\mathsf{ffg}}\, F$. This has inspired Winter [42, Proposition 7] who uses a very similar argument to prove that, for a distributive law λ, λ-bisimulations (see Bartels [10]) are sound and complete for λ-bialgebras (see Remark 3.2). It turns out that, for a lifting F, Proposition 5.10 is a consequence of Winter's result, or, in other words, our result can be understood as a slight generalization of Winter's one.

Examples 5.12. (1) The first condition in Proposition 5.10 is not necessary for properness of F. In fact, it fails in the category of semimodules for $\mathbb{N}$, viz. the category of commutative monoids: in fact, consider the finitely generated commutative monoid $\mathbb{N} \times \mathbb{N}$ and its submonoid infinitely generated by

$$\{(n, n+1) \mid n \in \mathbb{N}\},$$

which is easily seen not be finitely generated. However, as we mentioned in Example 5.3, $FX = \mathbb{N} \times X^\Sigma$ is proper on the category of commutative monoids.

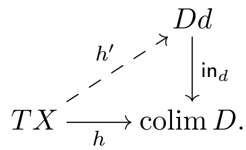
(2) In Example 2.12(4) we mentioned that, in the category PCA of positively convex algebras, fg- and fp-objects coincide. However, fg-objects are not closed under kernel pairs. In fact, the interval $[0, 1]$ is the free positively convex algebra on two generators, but $\{(0, 0), (1, 1)\} \cup (0, 1) \times (0, 1)$ is a congruence on $[0, 1]$ that is not an fg-object (i.e. a polytope) [39, Example 4.13]. Thus, properness of the functors in Example 5.5 does not follow from Proposition 5.10.

6. Proof of Theorem 5.6

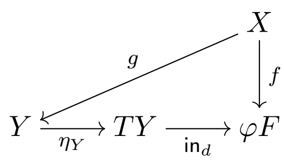
In this section we will present the proof of our main result Theorem 5.6. We start with two technical lemmas.

Remark 6.1. Recall [9, Proposition 11.28.2] that every free T-algebra TX is *perfectly presentable*, i.e. the hom-functor $\mathsf{Set}^T(TX, -)$ preserves sifted colimits (cf. Remark 3.5).

It follows that for every sifted diagram $D : \mathscr{D} \to \mathsf{Set}^T$ and every T-algebra morphism $h : TX \to \operatorname{colim} D$ there exists some $d \in \mathscr{D}$ and $h' : TX \to Dd$ such that



Lemma 6.2. *For every finite set X and map $f : X \to \varphi F$ there exists an object (TY, d) in $\mathsf{Coalg}_{\mathsf{ffg}} F$ and a map $g : X \to Y$ such that the triangle below commutes:*



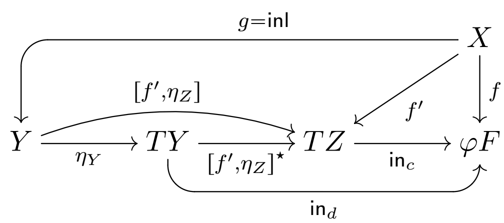
Proof. We begin by extending f to a T-algebra morphism $h = f^\star : TX \to \varphi F$. By Remark 6.1, there exists some $c : TZ \to FTZ$ in $\mathsf{Coalg}_{\mathsf{ffg}} F$ and a T-algebra morphism $h' : TX \to TZ$ such that $h = \mathsf{in}_c \cdot h'$. Let $f' = h' \cdot \eta_X$, let $Y = X + Z$ and consider the T-algebra morphism $[f', \eta_Z]^\star : TY \to TZ$. This is a split epimorphism in Set^T; we have $T\mathsf{inr} : TZ \to TY$ with

$$[f', \eta_Z]^\star \cdot T\mathsf{inr} = {\eta_Z}^\star = \mathsf{id}_{TZ},$$

where the last equation follows from the uniqueness property of ${\eta_Z}^\star$ (see Section 2.1) by the laws of $(-)^\star$. We therefore get a coalgebra structure

$$d = (TY \xrightarrow{[f',\eta_Z]^\star} TZ \xrightarrow{c} FTZ \xrightarrow{T\mathsf{inr}} FTY)$$

such that $[f', \eta_Z]^\star$ is an F-coalgebra morphism from (TY, d) to (TZ, c). Since Y is a finite set, (TY, d) is an F-coalgebra in $\mathsf{Coalg}_{\mathsf{ffg}} F$, and hence $\mathsf{in}_c \cdot [f', \eta_Z]^\star = \mathsf{in}_d$. Thus we see that $g = \mathsf{inl} : X \to Y$ is the desired morphism due to the commutative diagram below:

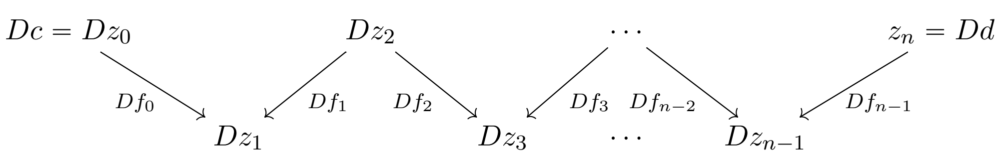


□

Remark 6.3. Recall that a colimit of a diagram $D : \mathscr{D} \to \mathsf{Set}$ is computed as follows:

$$\operatorname{colim} D = \Big(\coprod_{d \in \mathscr{D}} Dd\Big)/\!\sim,$$

where $\sim$ is the least equivalence on the coproduct (i.e. the disjoint union) of all Dd with $x \sim Df(x)$ for every $f : d \to d'$ in $\mathscr{D}$ and every $x \in Dd$. In other words, for every pair of objects c, d of $\mathscr{D}$ and $x \in Dc$, $y \in Dd$ we have $x \sim y$ iff there is a zig-zag in $\mathscr{D}$ whose D-image



relates x and y (cf. Remark 5.1).

Lemma 6.4. *Let* (TX, c) *and* (TY, d) *be coalgebras in* $\mathsf{Coalg}_{\mathsf{ffg}} F$*,* $x \in TX$*, and* $y \in TY$*. Then the following are equivalent:*

(1) $\mathsf{in}_c(x) = \mathsf{in}_d(y) \in \varphi F$*, and*
(2) *there is a zig-zag in* $\mathsf{Coalg}_{\mathsf{ffg}} F$ *relating* x *and* y*.*

Proof. By Remark 3.5(3), φF is a sifted colimit. Hence, the forgetful functor $\mathsf{Coalg}\, F \to \mathsf{Set}^T \to \mathsf{Set}$ preserves this colimit. Thus the colimit φF is formed as recalled in Remark 6.3:

$$\varphi F \cong \big(\coprod_c TX_c\big)/{\sim},$$

where $c : TX_c \to FTX_c$ ranges over the objects of $\mathsf{Coalg}_{\mathsf{ffg}} F$. Therefore, we have the desired equivalence. □

Proof of Theorem 5.6. "⇒" Suppose that for $m : \varphi F \to \nu F$ we have $x, y \in \varphi F$ with $m(x) = m(y)$. We apply Lemma 6.2 to

$$1 \xrightarrow{x} \varphi F \qquad \text{and} \qquad 1 \xrightarrow{y} \varphi F,$$

respectively, to obtain two objects $c : TX \to FTX$ and $d : TY \to FTY$ in $\mathsf{Coalg}_{\mathsf{ffg}} F$ with $x' \in X$ and $y' \in Y$ such that $\mathsf{in}_c(\eta_X(x')) = x$ and $\mathsf{in}_d(\eta_Y(y')) = y$. By the uniqueness of coalgebra morphisms into νF we have

$$\dagger c = m \cdot \mathsf{in}_c \qquad \text{and} \qquad \dagger d = m \cdot \mathsf{in}_d. \tag{6.1}$$

Thus we compute:

$$\dagger c(\eta_X(x')) = m \cdot \mathsf{in}_c \cdot \eta_X(x') = m(x) = m(y) = m \cdot \mathsf{in}_d \cdot \eta_Y(y') = \dagger d(\eta_Y(y')).$$

Since F is proper by assumption, we obtain a zig-zag in $\mathsf{Coalg}_{\mathsf{ffg}} F$ relating $\eta_X(x')$ and $\eta_Y(y')$. By Lemma 6.4, these two elements are merged by the colimit injections, and we have $x = \mathsf{in}_c(\eta_X(x')) = \mathsf{in}_d(\eta_Y(y') = y$. We conclude that m is monomorphic.

"⇐" Suppose that $m : \varphi F \rightarrowtail \nu F$ is a monomorphism. Let $c : TX \to FTX$ and $d : TY \to FTY$ be objects of $\mathsf{Coalg}_{\mathsf{ffg}} F$, and let $x \in X$ and $y \in Y$ be such that $\dagger c(\eta_X(x)) = \dagger d(\eta_Y(y))$. Using (6.1) and the fact that m is monomorphic we get $\mathsf{in}_c(\eta_X(x)) = \mathsf{in}_d(\eta_Y(y))$. By Lemma 6.4, we thus obtain a zig-zag in $\mathsf{Coalg}_{\mathsf{ffg}} F$ relating $\eta_X(x)$ and $\eta_Y(y)$. This proves that F is proper. □

7. Conclusions and Further Work

Inspired by Ésik and Maletti's notion of a proper semiring, we have introduced the notion of a proper functor. We have shown that, for a proper endofunctor F on an algebraic category preserving regular epis and monos, the rational fixed point ϱF is fully abstract and moreover determined by those coalgebras with a free finitely generated carrier (i.e. the target coalgebras of generalized determinization).

Our main result also shows that properness is necessary for this kind of full abstractness. For categories in which fg-objects are closed under kernel pairs we saw that when F maps kernel pairs to weak pullbacks in Set, then it is proper. This provides a number of examples of proper functors. However, in several categories of interest the condition on kernel pairs fails, e.g. in $\mathbb{N}$-semimodules (commutative monoids) and positively convex algebras. There can still be proper functors, e.g. $FX = \mathbb{N} \times X^\Sigma$ on the former and $FX = [0, 1] \times X^\Sigma$ on the latter. But establishing properness of a functor without using Proposition 5.10 seems non-trivial, and we leave the task of finding more examples of proper functors for further work.

One immediate consequence of our results is that the soundness and completeness proof for the expression calculi for weighted automata [12] extends from Noetherian to proper semirings. In fact, Ésik and Kuich [16, Theorems 7.1 and 8.5] already provide sound and complete axiomatizations of weighted language equivalence for (certain subclasses of) proper semirings $\mathbb{S}$ by showing that $\mathbb{S}$-rational weighted languages form certain free algebras.

In the future, when additional proper functors are known, it will be interesting to study regular expression calculi for their coalgebras and use the technical machinery developed in the present paper for soundness and completeness proofs.

Another task for future work is to study the new fixed point φF in its own right. Here we have already proven that φF is characterized uniquely (up to isomorphism) as the initial ffg-Bloom algebra. In the future, it might be interesting to investigate *free* (rather than initial) ffg-Bloom algebras. Moreover, related to ordinary Bloom algebras [2] there is the notion of an Elgot algebra [4]. It is known that for every object Y of an lfp category, the parametric rational fixed point $\varrho(F(-)+Y)$ yields a free Elgot algebra on Y. In addition, the category of algebras for the ensuing monad is isomorphic to the category of Elgot algebras for F. In [3], the new notion of an ffg-Elgot algebra for F is introduced, and it is shown that for free finitely generated algebras Y the parametric fixed point $\varphi(F(-)+Y)$ forms a free ffg-Elgot algebra for F on Y, and furthermore the category of ffg-Elgot algebras for F is monadic over our algebraic base category $\mathscr{A}$. It remains an open question whether ffg-Elgot algebras (or ffg-Bloom algebras) are monadic over Set.

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Appendix: Category Theoretic Proof of Proposition 3.9

Note first that for every $c : TX \to FTX$ in $\mathsf{Coalg}_{\mathsf{fp}}\, F$ we clearly have

$$c^\sharp = (TX \xrightarrow{\mathsf{in}_c} \varphi F \xrightarrow{h} \varrho F)$$

by the finality of ϱF. Recall that for strong epis the same cancellation law as for epis holds: if $e \cdot e'$ is a strong epi, then so is e; a similar law holds for strongly epimorphic families. Hence, we are done if we show that the $c^\sharp$ where $c : TX \to FTX$ ranges over $\mathsf{Coalg}_{\mathsf{ffg}}\, F$ forms a jointly strongly epimorphic family, too. This is done by using that the $a^\sharp$, where $a : A \to FA$ ranges over $\mathsf{Coalg}_{\mathsf{fp}}\, F$, form a strongly epimorphic family (to see this use Lemma 2.2 once again).

The key observation is as follows: given any $a : A \to FA$ in $\mathsf{Coalg}_{\mathsf{fp}}\, F$ we know that its carrier is a regular quotient of some free T-algebra TX with X finite, via $q : TX \twoheadrightarrow A$, say. Since F preserves regular epis (= surjections) we can use projectivity of TX (see Remark 2.4(2)) to obtain a coalgebra structure c on TX making q an F-coalgebra morphism:

$$\begin{array}{ccc} TX & \overset{c}{\dashrightarrow} & FTX \\ {\scriptstyle q}\big\downarrow & & \big\downarrow{\scriptstyle Fq} \\ A & \xrightarrow[a]{} & FA \end{array}$$

This implies that we have $c^\sharp = a^\sharp \cdot q$.

Now suppose that we have two parallel morphisms f, g such that for every $c : TX \to FTX$ in $\mathsf{Coalg}_{\mathsf{ffg}}\, F$ we have $f \cdot c^\sharp = g \cdot c^\sharp$. Then for every $a : A \to FA$ in $\mathsf{Coalg}_{\mathsf{fp}}\, F$ we obtain

$$f \cdot a^\sharp \cdot q = f \cdot c^\sharp = g \cdot c^\sharp = g \cdot a^\sharp \cdot q,$$

which implies that $f \cdot a^\sharp = g \cdot a^\sharp$ since q is epimorphic. Hence $f = g$ since the $a^\sharp$ form a jointly epimorphic family. This proves that the $c^\sharp$ form a jointly epimorphic family.

To see that they form a strongly jointly epimorphic family, assume that we are given a monomorphism $m : M \rightarrowtail N$ and morphisms $g : \varrho F \to N$ and $f_c : TX \to M$ for every $c : TX \to FTX$ in $\mathsf{Coalg}_{\mathsf{ffg}}\, F$ such that $m \cdot f_c = g \cdot c^\sharp$. We extend the family (f_c) to one indexed by all $a : A \to FA$ in $\mathsf{Coalg}_{\mathsf{fp}}\, F$ as follows. We have that any such (A, a) is a quotient coalgebra of some (TX, c) via $q : TX \twoheadrightarrow A$, which is the coequalizer of some parallel pair $k_1, k_2 : K \to TX$ in $\mathscr{A}$. Thus we have

$$\begin{aligned} m \cdot f_c \cdot k_1 &= g \cdot c^\sharp \cdot k_1 \\ &= g \cdot a^\sharp \cdot q \cdot k_1 \\ &= g \cdot a^\sharp \cdot q \cdot k_2 \\ &= g \cdot c^\sharp \cdot k_2 \\ &= m \cdot f_c \cdot k_2, \end{aligned}$$

which implies that $f_c \cdot k_1 = f_c \cdot k_2$ since m is monomorphic. Therefore we obtain a unique $f_a : A \to M$ such that $f_a \cdot q = f_c$ using the universal property of the coequalizer q. Hence we can compute

$$m \cdot f_a \cdot q = m \cdot f_c = g \cdot c^\sharp = g \cdot a^\sharp \cdot q,$$

which implies $m \cdot f_a = g \cdot a^\sharp$ since q is epimorphic. Now we use that the $a^\sharp$ are jointly strongly epimorphic (cf. Lemma 2.2) to obtain a unique morphism $d : \varrho F \to M$ with $d \cdot a^\sharp = f_a$ and

$m \cdot d = g$ for all $a : A \to FA$ in $\mathsf{Coalg}_{\mathsf{fp}}\, F$. In particular, d is the desired diagonal fill-in since $\mathsf{Coalg}_{\mathsf{ffg}}\, F$ is a full subcategory of $\mathsf{Coalg}_{\mathsf{fp}}\, F$. As for the uniqueness of the fill-in d we still need to check that any d with $d \cdot c^\sharp = f_c$ for all $c : TX \to FTX$ in $\mathsf{Coalg}_{\mathsf{ffg}}\, F$ and $m \cdot d = g$ also fulfils $d \cdot a^\sharp = f_a$ for every $a : A \to FA$ in $\mathsf{Coalg}_{\mathsf{fp}}\, F$. Indeed, this follows from

$$d \cdot a^\sharp \cdot q = d \cdot c^\sharp = f_c = f_a \cdot q$$

using that q is epimorphic. □