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ON INTERPOLATION AND SYMBOL ELIMINATION IN THEORY EXTENSIONS

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Abstract. In this paper we study possibilities of interpolation and symbol elimination in extensions of a theory $\mathcal{T}_0$ with additional function symbols whose properties are axiomatised using a set of clauses. We analyze situations in which we can perform such tasks in a hierarchical way, relying on existing mechanisms for symbol elimination in $\mathcal{T}_0$. This is for instance possible if the base theory allows quantifier elimination. We analyze possibilities of extending such methods to situations in which the base theory does not allow quantifier elimination but has a model completion which does. We illustrate the method on various examples.

1. Introduction

Many problems in computer science (e.g. in program verification) can be reduced to checking satisfiability of ground formulae w.r.t. a theory which can be a standard theory (for instance linear arithmetic) or a complex theory (typically the extension of a base theory $\mathcal{T}_0$ with additional function symbols axiomatized by a set $\mathcal{K}$ of formulae, or a combination of theories). SMT solvers are tuned for efficiently checking satisfiability of ground formulae in increasingly complex theories; the output can be "satisfiable", "unsatisfiable", or "unknown" (if incomplete methods are used, or else termination cannot be guaranteed).

More interesting is to go beyond yes/no answers, i.e. to consider parametric systems and infer constraints on parameters (which can be values or functions) which guarantee that certain properties are met (e.g. constraints which guarantee the unsatisfiability of ground clauses in suitable theory extensions). In [31, 32] – in a context specially tailored for the parametric verification of safety properties in increasingly more complex systems – we showed that such constraints could be generated in extensions of a theory allowing quantifier elimination.

In this paper, we propose a symbol elimination method in theory extensions and analyze its properties. We also discuss possibilities of applying such methods to extensions of theories which do not allow quantifier elimination provided that they have a model completion which does.

Key words and phrases: Quantifier Elimination, Theory Extensions, SMT, Hierarchical Reasoning, Ground Interpolation.

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Another problem we analyze is interpolation (widely used in program verification [22, 23, 24, 16, 18]). Intuitively, interpolants can be used for describing separations between the sets of "good" and "bad" states; they can help to discover relevant predicates in predicate abstraction with refinement and for over-approximation in model checking. It often is desirable to obtain "ground" interpolants of ground formulae. The first algorithms for interpolant generation in program verification required explicit constructions of proofs [19, 23], which in general is a relatively difficult task. In [18] the existence of ground interpolants for *arbitrary formulae* w.r.t. a theory $\mathcal{T}$ is studied. It is proved that this is the case if and only if $\mathcal{T}$ allows quantifier elimination, which limits the applicability of the results in [18]. Symbol elimination (e.g. using resolution and/or superposition) has been used for interpolant generation in e.g. [11]. In [30] we identify classes of local theory extensions in which interpolants can be computed hierarchically, using a method of computing interpolants in the base theory. [27] proposes an algorithm for the generation of interpolants for linear arithmetic with uninterpreted function symbols which reduces the problem to constraint solving in linear arithmetic. In both cases, when considering theory extensions $\mathcal{T}_0 \subseteq \mathcal{T}_0 \cup \mathcal{K}$ we devise ways of "separating" the instances of axioms in $\mathcal{K}$ and of the congruence axioms.

There also exist results which relate ground interpolation to amalgamation or the injection transfer property [17, 1, 39, 2, 8]. We use such results for obtaining criteria which allow us to recognize theories with ground interpolation. However, in general just knowing that ground interpolants exist is not sufficient: we want to construct the interpolants fast (in a hierarchical or modular way) and characterize situations in which we know which (extension) terms these interpolants contain. For this, [35, 36] introduce the notion of W-separability and study its links to a form of hierarchical interpolation. We here make the results in [35, 36] more precise, and extend them.

The main results of this paper can be summarized as follows:

- We link the existence (and computation) of ground interpolants in a theory $\mathcal{T}$ to their existence (and computation) in a model completion $\mathcal{T}^*$ of $\mathcal{T}$.
- We study possibilities of effective symbol elimination in theory extensions (based on quantifier elimination in the base theory or in a model completion thereof) and analyze the properties of the formulae obtained this way.
- We analyze possibilities of hierarchical interpolation in local theory extensions. Our analysis extends both results in [30] and results in [35] by avoiding the restriction to convex base theories. We explicitly point out all conditions needed for hierarchical interpolation and show how to check them.

This paper is an extended version of [33]; it extends and refines results described there as follows: We include a more comprehensive overview of prior work, as well as full proofs of the main results and detailed examples that explain the different procedures we propose. We expanded the considerations on the link between amalgamation and ground interpolation for theories which are not necessarily universal and, when describing the symbol elimination procedure for theory extensions, we also explicitly consider situations in which instead of a single theory extension we need to consider chains of theory extensions. We analyze the relationship between the partial amalgamation property proposed in [35] and a weaker W-amalgamation property proposed in [33].

The paper is structured as follows. In Section 2 we present the main results on model theory needed in the paper. In Section 3 we present existing results linking (sub-)amalgamation, quantifier elimination and the existence of ground interpolants, which we then combine to obtain efficient ways of proving ground interpolation and computing ground interpolants. Section 4 contains the main definitions and results on local theory extensions; these are used in Section 5 for symbol elimination and in Section 6 for ground interpolation in theory extensions.

2. Preliminaries

In this section we present the main results on model theory needed in the paper.
We consider signatures of the form $\Pi = (\Sigma, \mathsf{Pred})$, where Σ is a family of function symbols and Pred a family of predicate symbols. We assume known standard definitions from first-order logic such as Π-structures, models, homomorphisms, satisfiability, unsatisfiability. We denote "falsum" with $\perp$.
Theories can be defined by specifying a set of axioms, or by specifying a set of structures (the models of the theory). In this paper, (logical) theories are simply sets of sentences.

Definition 2.1 (Entailment)**.** If F, G are formulae and $\mathcal{T}$ is a theory we write:
(1) $F \models G$ to express the fact that every model of F is a model of G;
(2) $F \models_{\mathcal{T}} G$ – also written as $\mathcal{T} \cup F \models G$ – to express the fact that every model of F which is also a model of $\mathcal{T}$ is a model of G.

If $F \models G$ we say that F *entails* G. If $F \models_{\mathcal{T}} G$ we say that F *entails* G *w.r.t.* $\mathcal{T}$.

$F \models \perp$ means that F is unsatisfiable; $F \models_{\mathcal{T}} \perp$ means that there is no model of $\mathcal{T}$ in which F is true. If there is a model of $\mathcal{T}$ which is also a model of F we say that F is $\mathcal{T}$-consistent (or satisfiable w.r.t. $\mathcal{T}$).

If $\mathcal{T}$ is a theory over a signature $\Pi = (\Sigma, \mathsf{Pred})$ we denote by $\mathcal{T}_\forall$ (the universal theory of $\mathcal{T}$) the set of all universal sentences which are entailed by $\mathcal{T}$.

If $\mathcal{A} = (A, \{f_{\mathcal{A}}\}_{f \in \Sigma}, \{P_{\mathcal{A}}\}_{P \in \mathsf{Pred}})$ is a Π-structure, in what follows we will sometimes denote the universe A of the structure $\mathcal{A}$ by $|\mathcal{A}|$.

Definition 2.2 (Embedding)**.** For Π-structures $\mathcal{A}$ and $\mathcal{B}$, a map $\varphi : \mathcal{A} \to \mathcal{B}$ is an embedding if and only if it is an injective homomorphism and has the property that for every $P \in \mathsf{Pred}$ with arity n and all $(a_1, \dots, a_n) \in |\mathcal{A}|^n$, $(a_1, \dots, a_n) \in P_{\mathcal{A}}$ iff $(\varphi(a_1), \dots, \varphi(a_n)) \in P_{\mathcal{B}}$.

In particular, an embedding preserves the truth of all literals.

Definition 2.3 (Elementary Embedding)**.** An elementary embedding between two Π-structures is an embedding that preserves the truth of all first-order formulae over Π.

Definition 2.4 (Elementarily Equivalent Structures)**.** Two Π-structures are elementarily equivalent if they satisfy the same first-order formulae over Π.

Definition 2.5 (Diagram of a Structure)**.** Let $\mathcal{A} = (A, \{f_{\mathcal{A}}\}_{f \in \Sigma}, \{P_{\mathcal{A}}\}_{P \in \mathsf{Pred}})$ be a Π-structure. The *diagram* $\Delta(\mathcal{A})$ of $\mathcal{A}$ is the set of all literals true in the extension $\mathcal{A}^A$ of $\mathcal{A}$ where we have an additional constant for each element of A (which we here denote with the same symbol) with the natural expanded interpretation mapping the constant a to the element a of $|\mathcal{A}|$ (this is a set of sentences over the signature $\Pi^{\bar{a}}$ obtained by expanding Π with a fresh constant a for every element a from $|\mathcal{A}|$).

Note that if $\mathcal{A}$ is a Π-structure and $\mathcal{T}$ a theory and $\Delta(\mathcal{A})$ is $\mathcal{T}$-consistent then there exists a Π-structure $\mathcal{B}$ which is a model of $\mathcal{T}$ and into which $\mathcal{A}$ embeds.

Definition 2.6 (Quantifier Elimination)**.** A theory $\mathcal{T}$ over a signature Π *allows quantifier elimination* if for every formula ϕ over Π there exists a quantifier-free formula ϕ^* over Π which is equivalent to ϕ modulo $\mathcal{T}$.

Quantifier elimination can, in particular, be used for eliminating certain constants from ground formulae:

Theorem 2.7. *Let $\mathcal{T}$ be a theory with signature Π and $A(c_1, \dots, c_n, d_1, \dots, d_m)$ a ground formula over an extension Π^C of Π with additional constants $c_1, \dots, c_n$, $d_1, \dots, d_m$. If $\mathcal{T}$ has quantifier elimination then there exists a ground formula $\Gamma(c_1, \dots, c_n)$ containing only constants $c_1, \dots, c_n$, which is satisfiable w.r.t. $\mathcal{T}$ iff $A(c_1, \dots, c_n, d_1, \dots, d_m)$ is satisfiable w.r.t. $\mathcal{T}$.*

Proof. Assume that $\mathcal{T}$ has quantifier elimination. Let $A(c_1, \dots, c_n, y_1, \dots, y_m)$ be the formula obtained from $A(c_1, \dots, c_n, d_1, \dots, d_m)$ by replacing every occurrence of d_i with the variable y_i for $i = 1, \dots, m$. Let $\Gamma(c_1, \dots, c_n)$ be the formula obtained by eliminating the quantified variables $y_1, \dots y_m$ from the formula $\exists y_1, \dots, y_m A(c_1, \dots, c_n, y_1, \dots, y_m)$. The formula $\Gamma(c_1, \dots, c_n)$ is equivalent with $\exists y_1, \dots, y_m A(c_1, \dots, c_n, y_1, \dots, y_m)$ w.r.t. $\mathcal{T}$, i.e. they are true in the same models of $\mathcal{T}$. The following are equivalent:

- $A(c_1, \dots, c_n, d_1, \dots, d_m)$ is satisfiable w.r.t. $\mathcal{T}$.
- $\exists y_1, \dots, y_m A(c_1, \dots, c_n, y_1, \dots, y_m)$ is satisfiable w.r.t. $\mathcal{T}$.
- $\Gamma(c_1, \dots, c_n)$ is satisfiable w.r.t. $\mathcal{T}$. $\square$

Definition 2.8 (Model Complete Theory)**.** A *model complete* theory has the property that all embeddings between its models are elementary.

Every theory which allows quantifier elimination (QE) is model complete (cf. [12], Theorem 7.3.1).

Example 2.9. The following theories have QE and are therefore model complete.
(1) Presburger arithmetic with congruence mod. n ($\equiv_n$), $n = 2, 3, ...$ ([5], p.197).
(2) Rational linear arithmetic in the signature $\{+, 0, \leq\}$ ([37]).
(3) Real closed ordered fields ([12], 7.4.4), e.g., the real numbers.
(4) Algebraically closed fields ([3], Ex. 3.5.2; Rem. p.204; [12], Ch. 7.4, Ex. 2).
(5) Finite fields ([12], Ch. 7.4, Example 2).
(6) The theory of acyclic lists in the signature $\{\mathrm{car}, \mathrm{cdr}, \mathrm{cons}\}$ ([20, 7]).

A model complete theory can sometimes be regarded as the completion of another theory with the same universal fragment. Two theories $\mathcal{T}_1, \mathcal{T}_2$ are *companions* (or co-theories) if every model of $\mathcal{T}_1$ can be embedded (not necessarily elementarily) into a model of $\mathcal{T}_2$ and vice versa. This is the case iff $\mathcal{T}_1$ and $\mathcal{T}_2$ have the same universal consequences (i.e. iff $\mathcal{T}_{1\forall} = \mathcal{T}_{2\forall}$).

Definition 2.10 (Model Companion)**.** A theory $\mathcal{T}^*$ is called a *model companion* of $\mathcal{T}$ if
(i) $\mathcal{T}$ and $\mathcal{T}^*$ are co-theories,
(ii) $\mathcal{T}^*$ is model complete.

Definition 2.11. A theory $\mathcal{T}$ is called *complete* if it has models and every two models of $\mathcal{T}$ are elementarily equivalent (this is the same as saying that for every formula ϕ in the language of $\mathcal{T}$ exactly one of ϕ, $\neg\phi$ is a consequence of $\mathcal{T}$).

Definition 2.12 (Model Completion)**.** A theory $\mathcal{T}^*$ is called a *model completion* of $\mathcal{T}$ if it is a model companion of $\mathcal{T}$ with the additional property
(iii) for every model $\mathcal{A}$ of $\mathcal{T}$, $\mathcal{T}^* \cup \Delta(\mathcal{A})$ is a complete theory (where $\Delta(\mathcal{A})$ is the diagram of $\mathcal{A}$).

Thus, the model completion $\mathcal{T}^*$ of a theory $\mathcal{T}$ is model complete (because it is a model companion of $\mathcal{T}$). Condition (iii) states that every model of $\mathcal{T}$ is embeddable into a model of $\mathcal{T}^*$ "in a unique way".

A model complete theory is its own model completion. A theory that admits quantifier elimination is the model completion of every one of its companions. A theory $\mathcal{T}$ is the model completion of every one of its companions iff it is the model completion of the weakest of them, $\mathcal{T}_\forall$ (cf. e.g. [25]).

Example 2.13. Below we present some examples of model completions:
(1) The theory of infinite sets is the model completion of the pure theory of equality in the minimum signature containing only the equality predicate (cf. e.g. [7]).
(2) The theory of algebraically closed fields is the model completion of the theory of fields. This was the motivating example for developing the theory of model completions ([3], Examples 3.5.2, 3.5.12; Remark 3.5.6 ff.; [12], 7.3).
(3) The theory of dense total orders without endpoints is the model completion of the theory of total orders (cf. e.g. [7]).
(4) The theory of atomless Boolean algebras is the model completion of the theory of Boolean algebras ([3], Example 3.5.12, cf. also p.196).
(5) Universal Horn theories in finite signatures have a model completion if they are locally finite and have the amalgamation property (e.g., graphs, posets) ([38], cf. also [7]).

The following are easy consequences of the definitions:

Remark 2.14. If $\mathcal{T}$ and $\mathcal{T}'$ are co-theories then $\mathcal{T}_\forall = \mathcal{T}'_\forall$. If $\mathcal{T}^*$ is a model companion (or model completion) of $\mathcal{T}$ then $(\mathcal{T}^*)_\forall = \mathcal{T}_\forall$ and $\mathcal{T}^{**} = \mathcal{T}^*$.

Lemma 2.15. *Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be two co-theories with signature Π, and $A(c_1, \dots, c_n)$ be a ground clause over an extension Π^C of Π with additional constants $c_1, \dots, c_n$. Then A is satisfiable w.r.t. $\mathcal{T}_1$ if and only if it is satisfiable w.r.t. $\mathcal{T}_2$.*

Proof. For $i = 1, 2$, $A(c_1, \dots, c_n)$ is unsatisfiable w.r.t. $\mathcal{T}_i$ if and only if the formula $\exists y_1, \dots, y_n A(y_1, \dots, y_n)$ is false in all models of $\mathcal{T}_i$. This is the case if and only if $\mathcal{T}_i \models \forall y_1, \dots, y_n \neg A(y_1, \dots, y_n)$. As $\mathcal{T}_1$ and $\mathcal{T}_2$ are co-theories, they have the same universal fragment. Thus, $\mathcal{T}_1 \models \forall y_1, \dots, y_n \neg A(y_1, \dots, y_n)$ if and only if $\mathcal{T}_2 \models \forall y_1, \dots, y_n \neg A(y_1, \dots, y_n)$. It follows that $A(c_1, \dots, c_n)$ is satisfiable w.r.t. $\mathcal{T}_1$ if and only if it is satisfiable w.r.t. $\mathcal{T}_2$. $\square$

Notation. We denote with (indexed versions of) x, y, z variables and with (indexed versions of) a, b, c, d constants. As we will often refer to tuples of variables or constants, we will succinctly denote them as follows: $\overline{x}$ will stand for a sequence of variables $x_1, \dots, x_n$, $\overline{x}^i$ for a sequence of variables $x^i_1, \dots, x^i_n$, and $\overline{c}$ for a sequence of constants $c_1, \dots, c_n$.

3. Ground Interpolation

A Π-theory $\mathcal{T}$ has interpolation if, for all Π-formulae ϕ and ψ, if $\phi \models_{\mathcal{T}} \psi$ then there exists a formula I containing only symbols common[1] to ϕ and ψ such that $\phi \models_{\mathcal{T}} I$ and $I \models_{\mathcal{T}} \psi$. The formula I is then called *the interpolant* of ϕ and ψ.
Craig proved that first order logic has interpolation [4] but even if ϕ and ψ are e.g. conjunctions of ground literals the interpolant I may still be an arbitrary formula. It is often important to identify situations in which ground clauses have ground interpolants.

Definition 3.1 (Ground Interpolation)**.** A theory $\mathcal{T}$ has the *ground interpolation property* (for short: $\mathcal{T}$ has *ground interpolation*) if for every pair of ground formulae $A(\overline{c}, \overline{a})$ (containing constants $\overline{c}, \overline{a}$) and $B(\overline{c}, \overline{b})$ (containing constants $\overline{c}, \overline{b}$), if $A(\overline{c}, \overline{a}) \wedge B(\overline{c}, \overline{b}) \models_{\mathcal{T}} \perp$ then there exists a ground formula $I(\overline{c})$, containing only the constants $\overline{c}$ occurring both in A and B, such that $A(\overline{c}, \overline{a}) \models_{\mathcal{T}} I(\overline{c})$ and $B(\overline{c}, \overline{b}) \wedge I(\overline{c}) \models_{\mathcal{T}} \perp$.

Let $\mathcal{T}$ be a theory in a signature Σ and Σ' a signature disjoint from Σ. We denote by $\mathcal{T} \cup \mathsf{UIF}_{\Sigma'}$ the extension of $\mathcal{T}$ with uninterpreted symbols in Σ'.

Definition 3.2 (General Ground Interpolation [2])**.** We say that a theory $\mathcal{T}$ in a signature Σ has the *general ground interpolation property* (or, shorter, that $\mathcal{T}$ has *general ground interpolation*) if for every signature Σ' disjoint from Σ and every pair of ground $\Sigma \cup \Sigma'$-formulae A and B, if $A \wedge B \models_{\mathcal{T} \cup \mathsf{UIF}_{\Sigma'}} \perp$ then there exists a ground formula I such that:

(i) all constants, predicate and function symbols from Σ' occurring in I occur both in A and B, and
(ii) $A \models_{\mathcal{T} \cup \mathsf{UIF}_{\Sigma'}} I$ and $B \wedge I \models_{\mathcal{T} \cup \mathsf{UIF}_{\Sigma'}} \perp$.

Remark 3.3. *When defining ground interpolation, in many papers a difference is made between interpreted and uninterpreted function symbols or constants: The interpolant I of two (ground) formulae A and B is often required to contain only constants and uninterpreted function symbols occurring in both A and B; no restriction is imposed on the interpreted function symbols. We explain how these aspects are addressed in the previously given definitions:*

- *Definition 3.1 assumes that all function and predicate symbols in the signature of $\mathcal{T}$ which are not constant are* interpreted, *i.e. can be contained in the interpolant of two formulae A and B also if they are not common to the two formulae.*
- *In Definition 3.2 (with the notation used there) the function and predicate symbols from the signature Σ of the theory $\mathcal{T}$ are considered to be* interpreted *(thus can be contained in the interpolant of two formulae A and B also if they are not common to the two formulae), whereas the constants and the function and predicate symbols from Σ' are considered to be* uninterpreted *(thus all constants and all predicate and function symbols occurring in the interpolant of two formulae A and B must occur in both A and B).*

[1]For full first-order logic, the symbols common to ϕ and ψ are the function and predicate symbols which occur in both ϕ and ψ. Remark 3.3 discusses which symbols are considered to be common to ϕ and ψ in articles in which interpolation modulo a theory is considered.

3.1. **Amalgamation and Ground Interpolation.** There exist results which relate ground interpolation to amalgamation [17, 1, 39, 2, 8] and thus allow us to recognize many theories with ground interpolation.
For instance, Bacsich [1] shows that every *universal theory* with the amalgamation property has ground interpolation. The terminology is defined below.

Definition 3.4 (Amalgamation Property)**.** A theory $\mathcal{T}$ has the *amalgamation property* iff whenever we are given models M_1 and M_2 of $\mathcal{T}$ with a common substructure A which is a model of $\mathcal{T}$, there exists a further model M of $\mathcal{T}$ endowed with embeddings $\mu_i : M_i \rightarrow M$, $i = 1, 2$ whose restrictions to A coincide.

A theory $\mathcal{T}$ has the *strong amalgamation property* if the preceding embeddings μ_1, μ_2 and the preceding model M can be chosen so as to satisfy the following additional condition: if for some m_1, m_2 we have $\mu_1(m_1) = \mu_2(m_2)$, then there exists an element $a \in A$ such that $m_1 = m_2 = a$.

Theorem 3.5 ([1])**.** *Every universal theory with the amalgamation property has the ground interpolation property.*

Theorem 3.5 can be used to show that equational classes such as (abelian) groups, partially-ordered sets, lattices, semilattices, distributive lattices and Boolean algebras have ground interpolation.

If $\mathcal{T}$ is not a universal theory, the amalgamation property does not necessarily imply the ground interpolation property. We present two possible solutions in this situation:

Solution 1: Apply Theorem 3.5 to the universal fragment $\mathcal{T}_\forall$ to check whether $\mathcal{T}_\forall$ has ground interpolation and note that a theory $\mathcal{T}$ has ground interpolation iff its universal fragment $\mathcal{T}_\forall$ does.

Solution 2: Extend the amalgamation property.

We discuss and compare these two approaches in what follows.

Solution 1: Regard $\mathcal{T}_\forall$ instead of $\mathcal{T}$. We relate existence of ground interpolants in $\mathcal{T}$ and $\mathcal{T}_\forall$, and use Theorem 3.5 to check whether $\mathcal{T}_\forall$ has ground interpolation.

Lemma 3.6. *Let $\mathcal{T}$ be a logical theory. $\mathcal{T}$ has ground interpolation iff $\mathcal{T}_\forall$ has ground interpolation.*

Proof. ($\Rightarrow$) Assume first that $\mathcal{T}$ has ground interpolation. We show that $\mathcal{T}_\forall$ has ground interpolation. Let $A(\overline{a}, \overline{c})$ and $B(\overline{b}, \overline{c})$ be ground formulae in the signature of $\mathcal{T}$ possibly containing new constants $\overline{a}, \overline{b}, \overline{c}$ such that $A(\overline{a}, \overline{c}) \wedge B(\overline{b}, \overline{c}) \models_{\mathcal{T}_\forall} \perp$. As all formulae in $\mathcal{T}_\forall$ are consequences of $\mathcal{T}$, $A(\overline{a}, \overline{c}) \wedge B(\overline{b}, \overline{c}) \models_{\mathcal{T}} \perp$. As $\mathcal{T}$ has ground interpolation, there exists a ground formula $I(\overline{c})$ in the signature of $\mathcal{T}$ containing only additional constants occurring in both A and B, such that $A(\overline{a}, \overline{c}) \models_{\mathcal{T}} I(\overline{c})$ and $I(\overline{c}) \wedge B(\overline{b}, \overline{c}) \models_{\mathcal{T}} \perp$. We argue that in this case $A(\overline{a}, \overline{c}) \models_{\mathcal{T}_\forall} I(\overline{c})$ and $I(\overline{c}) \wedge B(\overline{b}, \overline{c}) \models_{\mathcal{T}_\forall} \perp$, i.e. $I(\overline{c})$ is a ground interpolant of $A(\overline{a}, \overline{c})$ and $B(\overline{b}, \overline{c})$ also w.r.t. $\mathcal{T}_\forall$. Indeed, the following are equivalent:

(1) $A(\overline{a}, \overline{c}) \models_{\mathcal{T}} I(\overline{c})$
(2) $A(\overline{a}, \overline{c}) \wedge \neg I(\overline{c}) \models_{\mathcal{T}} \perp$
(3) $\exists \overline{a} \exists \overline{c}\, (A(\overline{a}, \overline{c}) \wedge \neg I(\overline{c})) \models_{\mathcal{T}} \perp$
(4) $\mathcal{T} \models \forall \overline{a} \forall \overline{c}\, \neg (A(\overline{a}, \overline{c}) \wedge \neg I(\overline{c}))$.

(To simplify notation, we regarded the additional constants in A and I as existentially quantified variables; after negation they became universally quantified.)

Thus, $A(\overline{a}, \overline{c}) \models_{\mathcal{T}} I(\overline{c})$ iff $\forall \overline{a} \forall \overline{c} \neg (A(\overline{a}, \overline{c}) \wedge \neg I(\overline{c})) \in \mathcal{T}_\forall$ iff $\mathcal{T}_\forall \models \forall \overline{a} \forall \overline{c} \neg (A(\overline{a}, \overline{c}) \wedge \neg I(\overline{c}))$. We can now use the chain of equivalences established before to conclude that $A(\overline{a}, \overline{c}) \models_{\mathcal{T}} I(\overline{c})$ iff $A(\overline{a}, \overline{c}) \models_{\mathcal{T}_\forall} I(\overline{c})$. Similarly we can show that $I(\overline{c}) \wedge B(\overline{b}, \overline{c}) \models_{\mathcal{T}} \perp$ iff $I(\overline{c}) \wedge B(\overline{b}, \overline{c}) \models_{\mathcal{T}_\forall} \perp$.

($\Leftarrow$) Assume now that $\mathcal{T}_\forall$ has ground interpolation. Let $A(\overline{a}, \overline{c}), B(\overline{b}, \overline{c})$ be ground formulae in the signature of $\mathcal{T}$ possibly containing new constants $\overline{a}, \overline{b}, \overline{c}$ such that $A(\overline{a}, \overline{c}) \wedge B(\overline{b}, \overline{c}) \models_{\mathcal{T}} \perp$. Then $\exists \overline{a}, \overline{b}, \overline{c}\, A(\overline{a}, \overline{c}) \wedge B(\overline{b}, \overline{c}) \models_{\mathcal{T}} \perp$. (For the sake of simplicity we again regarded the additional constants in A and I as constants, which we quantified existentially since we talk about satisfiability; after negation they became universally quantified.) Hence, $\models_{\mathcal{T}} \forall \overline{a}, \overline{b}, \overline{c} \neg (A(\overline{a}, \overline{c}) \wedge B(\overline{b}, \overline{c}))$, i.e. $\forall \overline{a}, \overline{b}, \overline{c} \neg (A(\overline{a}, \overline{c}) \wedge B(\overline{b}, \overline{c})) \in \mathcal{T}_\forall$. Then $\exists \overline{a}, \overline{b}, \overline{c}\, A(\overline{a}, \overline{c}) \wedge B(\overline{b}, \overline{c}) \models_{\mathcal{T}_\forall} \perp$, so $A(\overline{a}, \overline{c}) \wedge B(\overline{b}, \overline{c}) \models_{\mathcal{T}_\forall} \perp$. From the fact that $\mathcal{T}_\forall$ has the ground interpolation property it follows that there exists a ground interpolant $I(\overline{c})$ such that

$$A(\overline{a}, \overline{c}) \models_{\mathcal{T}_\forall} I(\overline{c}) \text{ and } B(\overline{b}, \overline{c}) \wedge I(\overline{c}) \models_{\mathcal{T}_\forall} \perp .$$

Since $\mathcal{T} \models \mathcal{T}_\forall$ we then know that $A(\overline{a}, \overline{c}) \models_{\mathcal{T}} I(\overline{c})$ and $B(\overline{b}, \overline{c}) \wedge I(\overline{c}) \models_{\mathcal{T}} \perp$, so I is an interpolant of $A \wedge B$ also w.r.t. $\mathcal{T}$. □

Corollary 3.7. *Let $\mathcal{T}$ be a logical theory. Assume that $\mathcal{T}_\forall$ has the amalgamation property. Then both $\mathcal{T}_\forall$ and $\mathcal{T}$ have ground interpolation.*

Proof. Since $\mathcal{T}_\forall$ is a universal theory, by Theorem 3.5 if $\mathcal{T}_\forall$ has the amalgamation property then it has ground interpolation. By Lemma 3.6 it follows that $\mathcal{T}$ has ground interpolation: □

Solution 2: Extend the amalgamation property. In [2] Theorem 3.5 is extended to theories which are not necessarily universal. If a theory $\mathcal{T}$ is not necessarily universal its class of models is not closed under substructures. In order to extend Theorem 3.5 to this case it was necessary to define a variant of the amalgamation property (called the sub-amalgamation property), in which it is not required that the common substructure A of M_1 and M_2 is a model of the theory $\mathcal{T}$.

Definition 3.8 (Sub-Amalgamation Property [2])**.** A theory $\mathcal{T}$ has the *sub-amalgamation property* iff whenever we are given models M_1 and M_2 of $\mathcal{T}$ with a common substructure A, there exists a further model M of $\mathcal{T}$ endowed with embeddings $\mu_i : M_i \to M$, $i = 1, 2$ whose restrictions to A coincide.

A theory $\mathcal{T}$ has the *strong sub-amalgamation property* if the preceding embeddings μ_1, μ_2 and the preceding model M can be chosen so as to satisfy the following additional condition: if for some m_1, m_2 we have $\mu_1(m_1) = \mu_2(m_2)$, then there exists an element $a \in A$ such that $m_1 = m_2 = a$.

Clearly, for universal theories the amalgamation property and the sub-amalgamation property coincide.

Definition 3.9 (Equality Interpolating Theories [2])**.** A theory $\mathcal{T}$ is equality interpolating iff it has the ground interpolation property and has the property that for all tuples $\overline{x} = x_1, \dots, x_n$, $\overline{y}^1 = y^1_1, \dots, y^1_{n_1}$, $\overline{z}^1 = z^1_1, \dots, z^1_{m_1}$, $\overline{y}^2 = y^2_1, \dots, y^2_{n_2}$, $\overline{z}^2 = z^2_1, \dots, z^2_{m_2}$ of constants, and for every pair of ground formulae $A(\overline{x}, \overline{z}^1, \overline{y}^1)$ and $B(\overline{x}, \overline{z}^2, \overline{y}^2)$ such that $A(\overline{x}, \overline{z}^1, \overline{y}^1) \wedge B(\overline{x}, \overline{z}^2, \overline{y}^2) \models_{\mathcal{T}} \bigvee_{i=1}^{n_1} \bigvee_{j=1}^{n_2} y^1_i \approx y^2_j$ there exists a tuple of terms containing only the constants in $\overline{x}$, $v(\overline{x}) = v_1, \dots, v_k$ such that

$$A(\overline{x}, \overline{z}^1, \overline{y}^1) \wedge B(\overline{x}, \overline{z}^2, \overline{y}^2) \models_{\mathcal{T}} \bigvee_{i=1}^{n_1} \bigvee_{u=1}^{k} y_i^1 \approx v_u \vee \bigvee_{j=1}^{n_2} \bigvee_{u=1}^{k} v_u \approx y_j^2.$$

Theorem 3.10 ([2])**.** *The following hold:*
(1) *A theory $\mathcal{T}$ has the sub-amalgamation property iff it has ground interpolation.*
(2) *$\mathcal{T}$ is strongly sub-amalgamable iff it has general ground interpolation.*
(3) *If $\mathcal{T}$ has ground interpolation, then $\mathcal{T}$ is strongly sub-amalgamable iff it is equality interpolating.*
(4) *If $\mathcal{T}$ is universal and has quantifier elimination, $\mathcal{T}$ is equality interpolating.*

Theorem 3.11 ([3])**.** *If $\mathcal{T}^*$ is a model companion of $\mathcal{T}$ the following are equivalent:*
(1) *$\mathcal{T}^*$ is a model completion of $\mathcal{T}$.*
(2) *$\mathcal{T}$ has the amalgamation property.*
If, additionally, $\mathcal{T}$ has universal axiomatization, either of the conditions (1) or (2) above is equivalent to (3) $\mathcal{T}^*$ allows quantifier elimination.

Theorem 3.12 ([12], p.390)**.** *If $\mathcal{T}^*$ is a model companion of $\mathcal{T}$ the following are equivalent:*
(1) *$\mathcal{T}^*$ allows quantifier elimination.*
(2) *$\mathcal{T}_\forall$ has the amalgamation property.*

We now show that for every theory $\mathcal{T}$ which has a model companion $\mathcal{T}^*$ Solutions 1 and 2 are equivalent.

Theorem 3.13. *Let $\mathcal{T}$ be a theory and let $\mathcal{T}^*$ be a model companion of $\mathcal{T}$. Then $\mathcal{T}_\forall$ has the amalgamation property iff $\mathcal{T}$ has the sub-amalgamation property.*

Proof. ($\Rightarrow$) Assume that $\mathcal{T}_\forall$ has the amalgamation property. Then, by Corollary 3.7, $\mathcal{T}$ has ground interpolation, hence, by Theorem 3.10 (1), $\mathcal{T}$ has the sub-amalgamation property.

($\Leftarrow$) Assume now that $\mathcal{T}$ has the sub-amalgamation property. By Theorem 3.10 (1), $\mathcal{T}$ has ground interpolation, so by Theorem 3.6 $\mathcal{T}_\forall$ has ground interpolation. Then, again by Theorem 3.10 (1), $\mathcal{T}_\forall$ has the sub-amalgamation property. Since $\mathcal{T}_\forall$ is a universal theory, it immediately follows that $\mathcal{T}_\forall$ has the amalgamation property. □

3.2. **Quantifier Elimination and Ground Interpolation.** Clearly, if a theory $\mathcal{T}$ allows quantifier elimination then it has ground interpolation: Assume $A \wedge B \models_{\mathcal{T}} \perp$. We can simply use quantifier elimination to eliminate the non-shared constants from A w.r.t. $\mathcal{T}$ and obtain an interpolant. The converse is not true (the theory of uninterpreted function symbols over a signature Σ has ground interpolation but does not allow quantifier elimination).

Theorem 3.14. *If $\mathcal{T}$ is a universal theory which allows quantifier elimination then $\mathcal{T}$ has* general ground interpolation*.*

Proof. Clearly, if $\mathcal{T}$ allows quantifier elimination then it has ground interpolation. By Theorem 3.10(4) we know that if a theory $\mathcal{T}$ is universal and allows quantifier elimination then it is equality interpolating. By Theorem 3.10(3), if a theory $\mathcal{T}$ has ground interpolation and is equality interpolating then it has the strong sub-amalgamation property, hence, by Theorem 3.10(2), it has general ground interpolation. □

Example 3.15. (1) All theories in Example 2.9 allow quantifier elimination, hence have ground interpolation.
(2) The theory of pure equality has the strong (sub-)amalgamation property [2], hence by Theorem 3.10 it allows general ground interpolation.
(3) The theory of absolutely-free data structures [20] is universal and has quantifier elimination, hence by Theorem 3.14 it has general ground interpolation.

3.3. **Model Companions and Ground Interpolation.** In what follows we establish links between ground interpolation resp. quantifier elimination in a theory and in its model companions (if they exist).

Theorem 3.16. *If $\mathcal{T}$ is a universal theory which has ground interpolation, and $\mathcal{T}^*$ is a model companion of $\mathcal{T}$ then $\mathcal{T}^*$ allows quantifier elimination (and it is a model completion of $\mathcal{T}$).*

Proof. Assume that $\mathcal{T}$ is a universal theory which has ground interpolation. Then, by [1], $\mathcal{T}$ has the amalgamation property. By Theorem 3.11, $\mathcal{T}^*$ is a model completion of $\mathcal{T}$ and it allows quantifier elimination. □

We now analyze situations when $\mathcal{T}$ is not necessarily a universal theory.

Theorem 3.17. *Let $\mathcal{T}$ be a theory. Assume that $\mathcal{T}$ has a model companion $\mathcal{T}^*$. If $\mathcal{T}^*$ has ground interpolation then so does $\mathcal{T}$; the ground interpolants computed w.r.t. $\mathcal{T}^*$ are also interpolants w.r.t. $\mathcal{T}$.*

Proof. If $\mathcal{T}^*$ is the model companion of $\mathcal{T}$ they are co-theories, so $\mathcal{T}_\forall = \mathcal{T}^*_\forall$, cf. Remark 2.14. Assume that $\mathcal{T}^*$ has ground interpolation. Let A, B be two sets of ground clauses such that $\mathcal{T} \cup A \cup B \models \perp$. As $\mathcal{T}_\forall = \mathcal{T}^*_\forall$, by Lemma 2.15, $\mathcal{T}^* \cup A \cup B \models \perp$. As $\mathcal{T}^*$ has ground interpolation, there exists a ground formula I containing only constants occurring in both A and B such that $\mathcal{T}^* \cup A \cup \neg I$ and $\mathcal{T}^* \cup B \cup I$ are unsatisfiable. Then, again by Lemma 2.15, $\mathcal{T} \cup A \cup \neg I$ and $\mathcal{T} \cup B \cup I$ are unsatisfiable, i.e. I is an interpolant w.r.t. $\mathcal{T}$. □

Corollary 3.18. *Let $\mathcal{T}$ be a universal theory. Assume that $\mathcal{T}$ has a model companion $\mathcal{T}^*$. Then $\mathcal{T}$ has ground interpolation iff $\mathcal{T}^*$ has ground interpolation.*

Proof. If $\mathcal{T}$ is a universal theory and has ground interpolation then, by Theorem 3.16, $\mathcal{T}^*$ allows quantifier elimination hence has ground interpolation. The converse follows from Theorem 3.17. □

Corollary 3.19. *Let $\mathcal{T}$ be a theory. Assume that $\mathcal{T}$ has a model companion $\mathcal{T}^*$. If $\mathcal{T}^*$ allows quantifier elimination then $\mathcal{T}$ has ground interpolation.*

Example 3.20. The following theories have ground interpolation:
(1) The pure theory of equality (its model completion is the theory of an infinite set, which allows quantifier elimination, cf. e.g. [7]).
(2) The theory of total orderings (its model completion is the theory of dense total orders without endpoints, which allows quantifier elimination, cf. e.g. [7]).
(3) The theory of Boolean algebras (its model completion is the theory of atomless Boolean algebras, which allows quantifier elimination, cf. [3]).

(4) The theory of fields (its model completion is the theory of algebraically closed fields, which allows quantifier elimination, cf. [3]).

For a theory $\mathcal{T}$ which is not universal the amalgamation property does not necessarily imply the ground interpolation property. Instead, we can check whether the universal fragment $\mathcal{T}_\forall$ of $\mathcal{T}$ has the amalgamation property as explained in Corollary 3.7.

Theorem 3.21. *Let $\mathcal{T}$ be a logical theory such that $\mathcal{T}_\forall$ has the amalgamation property. If $\mathcal{T}$ has a model companion $\mathcal{T}^*$ then $\mathcal{T}^*$ allows quantifier elimination (so it is a model completion of $\mathcal{T}$) hence interpolants in $\mathcal{T}$ can be computed by quantifier elimination in $\mathcal{T}^*$.*

Proof. If $\mathcal{T}^*$ is a model companion of $\mathcal{T}$ and $\mathcal{T}_\forall$ has the amalgamation property then by Theorem 3.12 $\mathcal{T}^*$ allows quantifier elimination, so it has ground interpolation and by Theorem 3.17 so does $\mathcal{T}$; the ground interpolants computed w.r.t. $\mathcal{T}^*$ are also interpolants w.r.t. $\mathcal{T}$. □

A summary of the results obtained in this section is given, in succinct form, in Section 7.1.

Until now, we discussed possibilities for symbol elimination and ground interpolation in arbitrary theories. However, often the theories we consider are extensions of a "base" theory with additional function symbols satisfying certain properties axiomatized using clauses; we now analyze such theories. In Section 4 we recall the main definitions and results related to (local) theory extensions. We use these results in Section 5 to study possibilities of symbol elimination in theory extensions and in Section 6 to identify theory extensions with ground interpolation.

4. Local Theory Extensions

Let $\Pi_0{=}(\Sigma_0, \mathsf{Pred})$ be a signature, and $\mathcal{T}_0$ be a "base" theory with signature Π_0. We consider extensions $\mathcal{T} := \mathcal{T}_0 \cup \mathcal{K}$ of $\mathcal{T}_0$ with new function symbols Σ (*extension functions*) whose properties are axiomatized using a set $\mathcal{K}$ of (universally closed) clauses in the extended signature $\Pi = (\Sigma_0 \cup \Sigma, \mathsf{Pred})$, which contain function symbols in Σ. Let C be a fixed countable set of fresh constants. We denote by Π^C the extension of Π with constants in C.

4.1. Locality Conditions.

If G is a finite set of ground Π^C-clauses and $\mathcal{K}$ a set of Π-clauses, we denote by $\mathsf{st}(\mathcal{K}, G)$ the set of all ground terms which occur in G or $\mathcal{K}$. We denote by $\mathsf{est}(\mathcal{K}, G)$ the set of all extension ground terms (i.e. terms starting with a function in Σ) which occur in G or $\mathcal{K}$.

We regard every finite set G of ground clauses as the ground formula $\bigwedge_{C \in G} C$. If T is a set of ground terms in the signature Π^C, we denote by $\mathcal{K}[T]$ the set of all instances of $\mathcal{K}$ in which the terms starting with a function symbol in Σ are in T. Formally:

$$\begin{aligned}\mathcal{K}[T] := \{\varphi\sigma \mid \forall \bar{x}.\, \varphi(\bar{x}) \in \mathcal{K},\ &\text{where (i) if } f \in \Sigma \text{ and } t = f(t_1, ..., t_n) \text{ occurs in } \varphi\sigma \\ &\text{then } t \in T\text{; (ii) if } x \text{ is a variable that does not appear below some} \\ &\Sigma\text{-function in } \varphi \text{ then } \sigma(x) = x\}.\end{aligned}$$

Definition 4.1 ([15])**.** Let Ψ be a map associating with every set T of ground Π^C-terms a set $\Psi(T)$ of ground Π^C-terms. For any set G of ground Π^C-clauses we write $\mathcal{K}[\Psi_\mathcal{K}(G)]$ for $\mathcal{K}[\Psi(\mathsf{est}(\mathcal{K}, G))]$. Let $\mathcal{T}_0 \cup \mathcal{K}$ be an extension of $\mathcal{T}_0$ with clauses in $\mathcal{K}$. We define:

(Loc_f^Ψ) For every finite set G of ground clauses in Π^C it holds that
$\mathcal{T}_0 \cup \mathcal{K} \cup G \models \perp$ if and only if $\mathcal{T}_0 \cup \mathcal{K}[\Psi_{\mathcal{K}}(G)] \cup G$ is unsatisfiable.

Extensions satisfying condition (Loc_f^Ψ) are called *Ψ-local*.

If Ψ is the identity we obtain the notion of *local theory extensions* [28, 29], which generalizes the notion of *local theories* [9, 21, 10, 6].

4.2. **Partial Structures.** In [28] we showed that local theory extensions can be recognized by showing that certain partial models embed into total ones, and in [15] we established similar results for Ψ-local theory extensions and generalizations thereof. We introduce the main definitions here.

Let $\Pi = (\Sigma, \mathsf{Pred})$ be a first-order signature with set of function symbols Σ and set of predicate symbols Pred. A *partial* Π-*structure* is a structure $\mathcal{A} = (A, \{f_{\mathcal{A}}\}_{f\in\Sigma}, \{P_{\mathcal{A}}\}_{P\in\mathsf{Pred}})$, where A is a non-empty set, for every n-ary $f \in \Sigma$, $f_{\mathcal{A}}$ is a partial function from A^n to A, and for every n-ary $P \in \mathsf{Pred}$, $P_{\mathcal{A}} \subseteq A^n$. We consider constants (0-ary functions) to be always defined. $\mathcal{A}$ is called a *total structure* if the functions $f_{\mathcal{A}}$ are all total. Given a (total or partial) Π-structure $\mathcal{A}$ and $\Pi_0 \subseteq \Pi$ we denote the reduct of $\mathcal{A}$ to Π_0 by $\mathcal{A}|_{\Pi_0}$.

The notion of evaluating a term t with variables X w.r.t. an assignment $\beta : X \to A$ for its variables in a partial structure $\mathcal{A}$ is the same as for total algebras, except that the evaluation is undefined if $t = f(t_1, \dots, t_n)$ and at least one of $\beta(t_i)$ is undefined, or else $(\beta(t_1), \dots, \beta(t_n))$ is not in the domain of $f_{\mathcal{A}}$.

Definition 4.2. A *weak* Π-*embedding* between two partial Π-structures $\mathcal{A}$ and $\mathcal{B}$, where $\mathcal{A} = (A, \{f_{\mathcal{A}}\}_{f\in\Sigma}, \{P_{\mathcal{A}}\}_{P\in\mathsf{Pred}})$ and $\mathcal{B} = (B, \{f_{\mathcal{B}}\}_{f\in\Sigma}, \{P_{\mathcal{B}}\}_{P\in\mathsf{Pred}})$ is a total map $\varphi : A \to B$ such that

(i) φ is an embedding w.r.t. $\mathsf{Pred} \cup \{=\}$, i.e. for every $P \in \mathsf{Pred}$ with arity n and every $a_1, \dots, a_n \in \mathcal{A}$, $(a_1, \dots, a_n) \in P_{\mathcal{A}}$ if and only if $(\varphi(a_1), \dots, \varphi(a_n)) \in P_{\mathcal{B}}$.
(ii) whenever $f_{\mathcal{A}}(a_1, \dots, a_n)$ is defined (in $\mathcal{A}$), then $f_{\mathcal{B}}(\varphi(a_1), \dots, \varphi(a_n))$ is defined (in $\mathcal{B}$) and $\varphi(f_{\mathcal{A}}(a_1, \dots, a_n)) = f_{\mathcal{B}}(\varphi(a_1), \dots, \varphi(a_n))$, for all $f \in \Sigma$.

Definition 4.3 (Weak validity)**.** Let $\mathcal{A}$ be a partial Π-algebra and $\beta : X \to A$ a valuation for its variables. $(\mathcal{A}, \beta)$ *weakly satisfies a clause* C (notation: $(\mathcal{A}, \beta) \models_w C$) if either some of the literals in $\beta(C)$ are not defined or otherwise all literals are defined and for at least one literal L in C, L is true in $\mathcal{A}$ w.r.t. β. $\mathcal{A}$ is a *weak partial model* of a set of clauses $\mathcal{K}$ if $(\mathcal{A}, \beta) \models_w C$ for every valuation β and every clause C in $\mathcal{K}$.

4.3. **Recognizing Ψ-Local Theory Extensions.** In [28] we proved that if every weak partial model of an extension $\mathcal{T}_0 \cup \mathcal{K}$ of a base theory $\mathcal{T}_0$ with total base functions can be embedded into a total model of the extension, then the extension is local. In [13] we lifted these results to Ψ-locality.

Let $\mathcal{A} = (A, \{f_{\mathcal{A}}\}_{f\in\Sigma_0\cup\Sigma} \cup C, \{P_{\mathcal{A}}\}_{P\in\mathsf{Pred}})$ be a partial Π^C-structure with total Σ_0-functions. Let Π^A be the extension of the signature Π with constants from A. We denote by $T(\mathcal{A})$ the following set of ground Π^A-terms:

$$T(\mathcal{A}) := \{f(a_1, ..., a_n) \mid f \in \Sigma, a_i \in A, i = 1, \dots, n, f_{\mathcal{A}}(a_1, ..., a_n) \text{ is defined } \}.$$

Let $\mathsf{PMod}^\Psi_{w,f}(\Sigma, \mathcal{T})$ be the class of all weak partial models $\mathcal{A}$ of $\mathcal{T}_0 \cup \mathcal{K}$, such that $\mathcal{A}|_{\Pi_0}$ is a total model of $\mathcal{T}_0$, the Σ-functions are possibly partial, $T(\mathcal{A})$ is finite and all terms in

$\Psi(\mathsf{est}(\mathcal{K}, T(\mathcal{A})))$ are defined (in the extension $\mathcal{A}^A$ with constants from A). We consider the following embeddability property of partial algebras:

$(\mathsf{Emb}^{\Psi}_{w,f})$ Every $\mathcal{A} \in \mathsf{PMod}^{\Psi}_{w,f}(\Sigma, \mathcal{T})$ weakly embeds into a total model of $\mathcal{T}$.

We also consider the properties $(\mathsf{EEmb}^{\Psi}_{w,f})$, which additionally requires the embedding to be *elementary* and (Comp_f) which requires that every structure $\mathcal{A} \in \mathsf{PMod}^{\Psi}_{w,f}(\Sigma, \mathcal{T})$ embeds into a total model of $\mathcal{T}$ *with the same support.*

When establishing links between locality and embeddability we require that the clauses in $\mathcal{K}$ are *flat* and *linear* w.r.t. Σ-functions. When defining these notions we distinguish between ground and non-ground clauses.

Definition 4.4. An *extension clause D is flat* (resp. *quasi-flat*) when all symbols below a Σ-function symbol in D are variables. (resp. variables or ground Π_0-terms). D is *linear* if whenever a variable occurs in two terms of D starting with Σ-functions, the terms are equal, and no term contains two occurrences of a variable.

A *ground clause D is flat* if all symbols below a Σ-function in D are constants. A *ground clause D is linear* if whenever a constant occurs in two terms in D whose root symbol is in Σ, the two terms are identical, and if no term which starts with a Σ-function contains two occurrences of the same constant.

Definition 4.5 ([15])**.** With the above notations, let Ψ be a map associating with $\mathcal{K}$ and a set of Π^C-ground terms T a set $\Psi_{\mathcal{K}}(T)$ of Π^C-ground terms. We call $\Psi_{\mathcal{K}}$ a *term closure operator* if the following holds for all sets of ground terms T, T':
(1) $\mathrm{est}(\mathcal{K}, T) \subseteq \Psi_{\mathcal{K}}(T)$,
(2) $T \subseteq T' \Rightarrow \Psi_{\mathcal{K}}(T) \subseteq \Psi_{\mathcal{K}}(T')$,
(3) $\Psi_{\mathcal{K}}(\Psi_{\mathcal{K}}(T)) \subseteq \Psi_{\mathcal{K}}(T)$,
(4) for any map $h : C \to C$, $\bar{h}(\Psi_{\mathcal{K}}(T)) = \Psi_{\bar{h}\mathcal{K}}(\bar{h}(T))$, where $\bar{h}$ is the canonical extension of h to extension ground terms.

Theorem 4.6 ([13, 15])**.** *Let $\mathcal{T}_0$ be a first-order theory and $\mathcal{K}$ a set of universally closed flat clauses in the signature* Π. *The following hold:*
(1) *If all clauses in $\mathcal{K}$ are linear and Ψ is a term closure operator with the property that for every flat set of ground terms T, $\Psi(T)$ is flat then either of the conditions* $(\mathsf{Emb}^{\Psi}_{w,f})$ *and* $(\mathsf{EEmb}^{\Psi}_{w,f})$ *implies* $(\mathsf{Loc}^{\Psi}_{f})$.
(2) *If the extension $\mathcal{T}_0 \subseteq \mathcal{T}{=}\mathcal{T}_0 \cup \mathcal{K}$ satisfies* $(\mathsf{Loc}^{\Psi}_{f})$ *then* $(\mathsf{Emb}^{\Psi}_{w,f})$ *holds.*

4.4. **Examples of Local Theory Extensions.** Using a variant of Theorem 4.6, in [28] we gave several examples of local theory extensions:
(1) any extension $\mathcal{T}_0 \cup \mathsf{UIF}_{\Sigma}$ of a theory $\mathcal{T}_0$ with free functions in a set Σ;
(2) extensions of a theory $\mathcal{T}_0$ with signature Σ_0 having an injective function (constructor) c with arity n with suitable selector functions $s_1, \dots, s_n$;
(3) extensions of $\mathbb{R}$ with one or several functions satisfying conditions such as boundedness, or boundedness on the slope;

(4) extensions of partially ordered theories – in a class Ord consisting of the theories of posets, (dense) totally-ordered sets, semilattices, (distributive) lattices, Boolean algebras, or $\mathbb{R}$ – with a monotone function f, i.e. satisfying:

$$(\mathsf{Mon}(f)) \qquad \bigwedge_{i=1}^{n} x_i \leq y_i \rightarrow f(x_1, \dots, x_n) \leq f(y_1, \dots, y_n).$$

(5) Generalized monotonicity conditions – combinations of monotonicity in some arguments and antitonicity in other arguments – as well as extensions with functions defined by case distinction (over a disjoint set of conditions) were studied in [34].

We now present some more examples which were studied in [30].

Theorem 4.7 ([30])**.** *We consider the following base theories $\mathcal{T}_0$: (1) $\mathcal{P}$ (posets), (2) TOrd (totally-ordered sets), (3) SLat (semilattices), (4) DLat (distributive lattices), (5) Bool (Boolean algebras), (6) the theory $\mathbb{R}$ of reals resp. $\mathsf{LI}(\mathbb{R})$ (linear arithmetic over $\mathbb{R}$), or the theory $\mathbb{Q}$ of rationals resp. $\mathsf{LI}(\mathbb{Q})$ (linear arithmetic over $\mathbb{Q}$), or (a subtheory of) the theory of integers (e.g. Presburger arithmetic). The following theory extensions are local:*

(a) *Extensions of any theory $\mathcal{T}_0$ for which $\leq$ is reflexive with functions satisfying boundedness $(\mathsf{Bound}^t(f))$ or guarded boundedness $(\mathsf{GBound}^t(f))$ conditions*

$$(\mathsf{Bound}^t(f)) \qquad \forall x_1, \dots, x_n (f(x_1, \dots, x_n) \leq t(x_1, \dots, x_n))$$

$$(\mathsf{GBound}^t(f)) \quad \forall x_1, \dots, x_n (\phi(x_1, \dots, x_n) \rightarrow f(x_1, \dots, x_n) \leq t(x_1, \dots, x_n)),$$

where $t(x_1, \dots, x_n)$ is a term in the base signature Π_0 and $\phi(x_1, \dots, x_n)$ a conjunction of literals in the signature Π_0, whose variables are in $\{x_1, \dots, x_n\}$.

(b) *Extensions of any theory $\mathcal{T}_0$ in (1)–(6) with $\mathsf{Mon}(f) \wedge \mathsf{Bound}^t(f)$, if $t(x_1, \dots, x_n)$ is a term in the base signature Π_0 in the variables $x_1, \dots, x_n$ such that for every model of $\mathcal{T}_0$ the associated function is monotone in the variables $x_1, \dots, x_n$.*

(c) *Extensions of any theory $\mathcal{T}_0$ in (1)–(6) with functions satisfying $\mathsf{Leq}(f, g) \wedge \mathsf{Mon}(f)$.*

$$(\mathsf{Leq}(f, g)) \quad \forall x_1, \dots, x_n (\textstyle\bigwedge_{i=1}^n x_i \leq y_i \rightarrow f(x_1, \dots, x_n) \leq g(y_1, \dots, y_n))$$

(d) *Extensions of any theory $\mathcal{T}_0$ which is one of the totally-ordered theories in (2) or (6) (i.e. the theory TOrd of totally ordered sets or the theory of real numbers) with functions satisfying $\mathsf{SGc}(f, g_1, \dots, g_n) \wedge \mathsf{Mon}(f, g_1, \dots, g_n)$.*

$$(\mathsf{SGc}(f, g_1, \dots, g_n)) \quad \forall x_1, \dots, x_n, x (\textstyle\bigwedge_{i=1}^n x_i \leq g_i(x) \rightarrow f(x_1, \dots, x_n) \leq x)$$

(e) *Extensions of any theory $\mathcal{T}_0$ in (1)–(3) with functions satisfying $\mathsf{SGc}(f, g_1) \wedge \mathsf{Mon}(f, g_1)$.*

All the extensions above satisfy condition (Loc_f).

4.5. **Locality Transfer Results.** In [15] we analyzed the way locality results can be transferred. Property $(\mathsf{EEmb}_{\mathsf{w,f}})$, for instance, is preserved if we enrich the base theory $\mathcal{T}_0$:

Theorem 4.8 ((EEmb) Transfer, [15])**.** *Let $\Pi_0 = (\Sigma_0, \mathsf{Pred})$ be a signature, $\mathcal{T}_0$ a theory in Π_0, Σ_1 and Σ_2 two disjoint sets of new function symbols, $\Pi_i := (\Sigma_0 \cup \Sigma_i, \mathsf{Pred})$, $i = 1, 2$. Assume that $\mathcal{T}_2$ is a Π_2-theory with $\mathcal{T}_0 \subseteq \mathcal{T}_2$, and $\mathcal{K}$ is a set of universally closed Π_1-clauses. If the extension $\mathcal{T}_0 \subseteq \mathcal{T}_0 \cup \mathcal{K}$ enjoys $(\mathsf{EEmb}_{w,f})$ then so does the extension $\mathcal{T}_2 \subseteq \mathcal{T}_2 \cup \mathcal{K}$.*

In particular, if $\mathcal{K}$ is flat and linear then the extension $\mathcal{T}_2 \subseteq \mathcal{T}_2 \cup \mathcal{K}$ satisfies condition (Loc_f)*. If all the variables in clauses in $\mathcal{K}$ occur below Σ_1-functions, and ground satisfiability is decidable in $\mathcal{T}_2$, then ground satisfiability is decidable in $\mathcal{T}_2 \cup \mathcal{K}$.*

The result extends in a natural way to the case of $(\mathsf{EEmb}^{\Psi}_{w,f})$ and Ψ-locality. Theorem 4.8 is a very useful result, which allows us to identify a large number of local extensions. Below we include an example from [15].

Example 4.9 ([15])**.** Let Lat be the theory of lattices and $\mathsf{Mon}_f = \{\forall x, y\ (x \leq y \rightarrow f(x) \leq f(y))\}$ be the axiom expressing monotonicity of a new function symbol f. We can prove that the extension $\mathsf{Lat} \subseteq \mathsf{Lat} \cup \mathsf{Mon}_f$ satisfies condition $(\mathsf{Comp}_{w,f})$ hence also $(\mathsf{EEmb}_{w,f})$. By Theorem 4.8, $\mathcal{T} \subseteq \mathcal{T} \cup \mathsf{Mon}_f$ satisfies condition $(\mathsf{EEmb}_{w,f})$, hence (Loc_f) for any extension $\mathcal{T}$ of the theory of lattices (i.e. for the theory of distributive lattices, Heyting algebras, Boolean algebras, any theory with a total order – e.g. the (ordered) theory of integers or of reals, etc.).

Theorem 4.10. *Let $\Pi_0 = (\Sigma_0, \mathsf{Pred})$ be a signature, $\mathcal{T}_0$ a theory in Π_0, Σ_P and Σ two disjoint sets of new function symbols, $\Pi_P := (\Sigma_0 \cup \Sigma_P, \mathsf{Pred})$, and $\Pi := (\Sigma_0 \cup \Sigma_P \cup \Sigma, \mathsf{Pred})$. Let Γ be a universal Π_P-formula, and $\mathcal{K}$ be a set of Π-clauses. If the extension $\mathcal{T}_0 \subseteq \mathcal{T}_0 \cup \mathcal{K}$ enjoys* (Comp_f) *then so does the extension $\mathcal{T}_0 \cup \Gamma \subseteq \mathcal{T}_0 \cup \Gamma \cup \mathcal{K}$.*

In particular, if $\mathcal{K}$ is flat and linear then the extension $\mathcal{T}_0 \cup \Gamma \subseteq \mathcal{T}_0 \cup \Gamma \cup \mathcal{K}$ satisfies condition (Loc_f)*.*

Proof. Let $\mathcal{A}$ be a weak partial model of $\mathcal{T} \cup \Gamma \cup \mathcal{K}$ in which all Π_P-functions are total. Then $\mathcal{A}$ is a weak partial model of $\mathcal{T} \cup \mathcal{K}$, hence it weakly embeds into a total model $\mathcal{B}$ of $\mathcal{T} \cup \mathcal{K}$ such that $\mathcal{A}$ and $\mathcal{B}$ have the same support.

Let $h : \mathcal{A} \rightarrow \mathcal{B}$ be the weak embedding. As (i) all Π_P-functions are totally defined in $\mathcal{A}$; (ii) $\mathcal{A}$ and $\mathcal{B}$ have the same support, (iii) Γ is a universal Π_P-formula which holds in $\mathcal{A}$, it follows that Γ holds also in $\mathcal{B}$. Thus, $\mathcal{B}$ is a total model of $\mathcal{T}_0 \cup \Gamma \cup \mathcal{K}$. □

Theorem 4.11 ([15])**.** *Let $\mathcal{T}_0$ be a theory. Assume that $\mathcal{T}_0$ has a model completion $\mathcal{T}_0^*$ such that $\mathcal{T}_0 \subseteq \mathcal{T}_0^*$. Let $\mathcal{T} = \mathcal{T}_0 \cup \mathcal{K}$ be an extension of $\mathcal{T}_0$ with new function symbols Σ whose properties are axiomatized by a set of flat and linear clauses $\mathcal{K}$ (all of which contain symbols in Σ).*

(1) *Assume that:*
   (i) *Every model of $\mathcal{T}_0 \cup \mathcal{K}$ embeds into a model of $\mathcal{T}_0^* \cup \mathcal{K}$.*
   (ii) *$\mathcal{T}_0 \cup \mathcal{K}$ is a local extension of $\mathcal{T}_0$.*

   Then $\mathcal{T}_0^* \subseteq \mathcal{T}_0^* \cup \mathcal{K}$ satisfies condition $(\mathsf{EEmb}_{w,f})$*, hence if $\mathcal{K}$ is a set of flat and linear clauses then $\mathcal{T}_0^* \subseteq \mathcal{T}_0^* \cup \mathcal{K}$ is a local extension.*
(2) *If all variables in $\mathcal{K}$ occur below an extension function and $\mathcal{T}_0^* \cup \mathcal{K}$ is a local extension of $\mathcal{T}_0^*$ then $\mathcal{T}_0 \cup \mathcal{K}$ is a local extension of $\mathcal{T}_0$.*

The result extends in a natural way to Ψ-locality. These results were used in [15] to give further examples of local theory extensions:

Example 4.12 ([15])**.** The following hold:

(1) The extension of the theory TOrd of total orderings with a strictly monotone function, i.e. a function f satisfying the axiom:

$$\mathsf{SMon}(f) \qquad \forall x, y (x < y \rightarrow f(x) < f(y))$$

satisfies condition (Loc_f).

To show this, we used the fact that the model completion TOrd^* of TOrd is the theory of dense total orderings without endpoints, and showed that the extension $\mathsf{TOrd}^* \subseteq \mathsf{TOrd}^* \cup \mathsf{SMon}(f)$ satisfies condition $(\mathsf{EEmb}_{w,f})$, hence it satisfies condition (Loc_f).

(2) The extension of the pure theory of equality with a function f satisfying

$$\mathsf{Inj}(f) \qquad \forall x, y\, (x \not\approx y \rightarrow f(x) \not\approx f(y))$$

is local. (This can be proved in a similar way, using the fact that the model completion of the pure theory of equality is the theory of infinite sets.)

4.6. **Hierarchical Reasoning in Local Theory Extensions.** Consider a Ψ-local theory extension $\mathcal{T}_0 \subseteq \mathcal{T}_0 \cup \mathcal{K}$. Condition (Loc_f^Ψ) requires that for every finite set G of ground Π^C clauses:

$$\mathcal{T}_0 \cup \mathcal{K} \cup G \models \perp \text{ if and only if } \mathcal{T}_0 \cup \mathcal{K}[\Psi_{\mathcal{K}}(G)] \cup G \models \perp .$$

In all clauses in $\mathcal{K}[\Psi_{\mathcal{K}}(G)]{\cup}G$ the function symbols in Σ only have ground terms as arguments, so $\mathcal{K}[\Psi_{\mathcal{K}}(G)]{\cup}G$ can be flattened and purified[2] by introducing, in a bottom-up manner, new constants $c_t \in C$ for subterms $t{=}f(c_1, \dots, c_n)$ where $f{\in}\Sigma$ and c_i are constants, together with definitions $c_t{\approx}f(c_1, \dots, c_n)$ which are all included in a set Def. We thus obtain a set of clauses $\mathcal{K}_0{\cup}G_0{\cup}\mathsf{Def}$, where $\mathcal{K}_0$ and G_0 do not contain Σ-function symbols and Def contains clauses of the form $c{\approx}f(c_1, \dots, c_n)$, where $f{\in}\Sigma$, $c, c_1, \dots, c_n$ are constants.

Theorem 4.13 ([28, 29, 13])**.** *Let $\mathcal{K}$ be a set of clauses. Assume that $\mathcal{T}_0 \subseteq \mathcal{T}_1 = \mathcal{T}_0 \cup \mathcal{K}$ is a Ψ-local theory extension. For any finite set G of ground clauses, let $\mathcal{K}_0 \cup G_0 \cup \mathsf{Def}$ be obtained from $\mathcal{K}[\Psi_{\mathcal{K}}(G)] \cup G$ by flattening and purification, as explained above. Then the following are equivalent to $\mathcal{T}_1 \cup G \models \perp$:*

(1) $\mathcal{T}_0{\cup}\mathcal{K}[\Psi_{\mathcal{K}}(G)]{\cup}G \models \perp$.

(2) $\mathcal{T}_0 \cup \mathcal{K}_0 \cup G_0 \cup \mathsf{Con}_0 \models \perp$, *where* $\mathsf{Con}_0 {=} \{ \bigwedge_{i=1}^n c_i{\approx}d_i \rightarrow c{\approx}d \mid \begin{smallmatrix} f(c_1, \dots, c_n) \approx c \in \mathsf{Def} \\ f(d_1, \dots, d_n) \approx d \in \mathsf{Def} \end{smallmatrix} \}$.

We illustrate the ideas on an example first presented in [30]. We chose this example because in Section 6 we will use it to compare the method of computing interpolants in [30] with the method presented in this paper.

Example 4.14 ([30])**.** Let $\mathcal{T}_1 = \mathcal{T}_0 \cup \mathsf{SGc}(f, g) \cup \mathsf{Mon}(f, g)$ be the extension of the theory $\mathcal{T}_0 = \mathsf{SLat}$ of semilattices with two monotone functions f, g satisfying the semi-Galois condition

$$\mathsf{SGc}(f, g): \quad \forall x, y (x \leq g(y) \rightarrow f(x) \leq y).$$

Consider the following ground formulae A, B in the signature of $\mathcal{T}_1$:

$$A: \; d \leq g(a) \; \wedge \; a \leq c \qquad\qquad B: \; b \leq d \; \wedge \; f(b) \not\leq c$$

where c and d are shared constants. Let $G = A \wedge B$. By Theorem 4.7(e), $\mathcal{T}_1$ is a local extension of the theory of semilattices. To prove that $G \models_{\mathcal{T}_1} \perp$ we proceed as follows:

[2] i.e. the function symbols in Σ are separated from the other symbols.

Step 1: *Use locality.* By the locality condition, G is unsatisfiable with respect to $\mathsf{SLat} \wedge \mathsf{SGc}(f,g) \wedge \mathsf{Mon}(f,g)$ iff $\mathsf{SLat} \wedge \mathsf{SGc}(f,g)[G] \wedge \mathsf{Mon}(f,g)[G] \wedge G$ has no weak partial model in which all terms in G are defined. The extension terms occurring in G are $f(b)$ and $g(a)$, hence:

$$\begin{array}{rcl} \mathsf{Mon}(f,g)[G] & = & \{a \leq a \rightarrow g(a) \leq g(a),\ b \leq b \rightarrow f(b) \leq f(b)\} \\ \mathsf{SGc}(f,g)[G] & = & \{b \leq g(a) \rightarrow f(b) \leq a\} \end{array}$$

Step 2: *Flattening and purification.* We purify and flatten the formula $\mathsf{SGc}(f,g) \wedge \mathsf{Mon}(f,g)$ by replacing the ground terms starting with f and g with new constants. We obtain a set of definitions $\mathsf{Def} = \{a_1 \approx g(a), b_1 \approx f(b)\}$, and a conjunction of formulae in the base signature, $G_0 \wedge \mathsf{SGc}_0 \wedge \mathsf{Mon}_0$ (where $G_0 = A_0 \wedge B_0$ is the purified form of $G = A \wedge B$).

Step 3: *Reduction to testing satisfiability in $\mathcal{T}_0$.* As the extension $\mathsf{SLat} \subseteq \mathcal{T}_1$ is local, by Theorem 4.13 we know that

$$G \models_{\mathcal{T}_1} \perp \text{ iff } G_0 \wedge \mathsf{SGc}_0 \wedge \mathsf{Mon}_0 \wedge \mathsf{Con}_0 \text{ is unsatisfiable with respect to } \mathsf{SLat},$$

where $\mathsf{Con}_0 = \mathsf{Con}[G]_0$ consists of the flattened form of those instances of the congruence axioms containing only f- and g-terms which occur in Def.

Extension Def	Base $G_0 \wedge \mathsf{SGc}_0 \wedge \mathsf{Mon}_0$	$\wedge \mathsf{Con}_0$
$D_A = a_1 \approx g(a)$	$A_0 = d \leq a_1 \wedge a \leq c$	$\mathsf{SGc}_0 = b \leq a_1 \rightarrow b_1 \leq a$
$D_B = b_1 \approx f(b)$	$B_0 = b \leq d \wedge b_1 \not\leq c$	$\mathsf{Con}_0 \wedge \mathsf{Mon}_0$:
		$\mathsf{Con}_A \wedge \mathsf{Mon}_A = a \lhd a \rightarrow a_1 \lhd a_1, \lhd \in \{\approx, \leq\}$
		$\mathsf{Con}_B \wedge \mathsf{Mon}_B = b \lhd b \rightarrow b_1 \lhd b_1,\ \lhd \in \{\approx, \leq\}$

It is easy to see that $G_0 \wedge \mathsf{SGc}_0 \wedge \mathsf{Mon}_0 \wedge \mathsf{Con}_0$ is unsatisfiable with respect to $\mathcal{T}_0$: $G_0 = A_0 \wedge B_0$ entails $b \leq a_1$, together with SGc_0 this yields $b_1 \leq a$, which together with $a \leq c$ and $b_1 \not\leq c$ leads to a contradiction.

4.7. **Chains of Theory Extensions.** We can also consider chains of theory extensions:

$$\mathcal{T}_0 \ \subseteq\ \mathcal{T}_1 = \mathcal{T}_0 \cup \mathcal{K}_1 \ \subseteq\ \mathcal{T}_2 = \mathcal{T}_0 \cup \mathcal{K}_1 \cup \mathcal{K}_2 \ \subseteq\ \dots \ \subseteq\ \mathcal{T}_n = \mathcal{T}_0 \cup \mathcal{K}_1 \cup \ldots \cup \mathcal{K}_n$$

in which each theory is a local extension of the preceding one.

For a chain of local extensions a satisfiability check w.r.t. the last extension can be reduced (in n steps) to a satisfiability check w.r.t. $\mathcal{T}_0$. The only restriction we need to impose in order to ensure that such a reduction is possible is that at each step the clauses reduced so far need to be ground. Groundness is assured if each variable in a clause appears at least once under an extension function. This iterated instantiation procedure for chains of local theory extensions has been implemented in H-PILoT [14].[3]

[3]H-PILoT allows the user to specify a chain of extensions by tagging the extension functions with their place in the chain (e.g., if f occurs in $\mathcal{K}_3$ but not in $\mathcal{K}_1 \cup \mathcal{K}_2$ it is declared as level 3).

Example 4.15. Let $\mathcal{T}_0$ be the theory of dense total orderings without endpoints. Consider the extension of $\mathcal{T}_0$ with functions $\Sigma_1 = \{f, g, h, c\}$ whose properties are axiomatized by:

$$\mathcal{K} := \{ \ \forall x(x \leq c \rightarrow g(x) \approx f(x)), \ \forall x(c < x \rightarrow g(x) \approx h(x)) \ \}.$$

The extension $\mathcal{T}_0 \subseteq \mathcal{T}_0 \cup \mathcal{K}$ can be "refined" to the following chain of theory extensions:

$$\mathcal{T}_0 \subseteq \mathcal{T}_0 \cup \mathsf{UIF}_{\{f,h\}} \subseteq (\mathcal{T}_0 \cup \mathsf{UIF}_{\{f,h\}}) \cup \mathcal{K} = \mathcal{T}_0 \cup \mathcal{K}.$$

- The theory $\mathcal{T}_0 \cup \mathsf{UIF}_{\{f,h\}}$ is a local extension of $\mathcal{T}_0$ because extensions with free function symbols are local ([28], see also the comments in Section 4.4).
- $\mathcal{T}_0 \cup \mathcal{K}$ is the extension of $\mathcal{T}_0 \cup \mathsf{UIF}_{\{f,h\}}$ with the function g, defined by case distinction (this is described in the axioms $\mathcal{K}$). By the results in [34] such extensions are also local.

In fact, both extensions satisfy condition Comp_f.

Let G be a set of ground clauses over the signature $\Sigma_0 \cup \Sigma_1$. Then the following are equivalent:

(1) $\mathcal{T}_0 \cup \mathcal{K} \cup G \models \perp$;

(2) $(\mathcal{T}_0 \cup \mathsf{UIF}_{\{f,h\}}) \cup \mathcal{K} \cup G \models \perp$;

(3) $(\mathcal{T}_0 \cup \mathsf{UIF}_{\{f,h\}}) \cup \mathcal{K}[G] \cup G \models \perp$, where $\mathcal{K}[G]$ is the set of all instances of $\mathcal{K}$ in which the terms starting with the function g are ground terms occurring in $\mathcal{K}$ or G;

(4) $(\mathcal{T}_0 \cup \mathsf{UIF}_{\{f,h\}}) \cup G_0^1 \cup \mathsf{Def}^g \models \perp$, where $G^1 = \mathcal{K}[G] \cup G$ is a ground formula, G_0^1 is obtained from G^1 by purification (replacing all ground terms starting with g with a new constant) and Def^g is the corresponding set of definitions, as explained in Theorem 4.13;

(5) $\mathcal{T}_0 \cup G_0 \cup \mathsf{Def}^g \cup \mathsf{Def}^{f,h} \models \perp$, where G_0 is obtained from G_0^1 after one more round of purification in which all ground terms starting with f and h are replaced with new constants and $\mathsf{Def}^{f,h}$ is the corresponding set of definitions, as explained in Theorem 4.13;

(6) $\mathcal{K}_0 \cup G_0 \cup \mathsf{Con}_0$ is unsatisfiable w.r.t. $\mathcal{T}_0$, where Con_0 is the set of congruence axioms corresponding to the set $\mathsf{Def} = \mathsf{Def}^g \cup \mathsf{Def}^{f,h}$ of definitions as explained in Theorem 4.13.

5. Symbol Elimination in Theory Extensions

Let $\Pi_0 = (\Sigma_0, \mathsf{Pred})$. Let $\mathcal{T}_0$ be a Π_0-theory. We consider theory extensions $\mathcal{T}_0 \subseteq \mathcal{T} = \mathcal{T}_0 \cup \mathcal{K}$, in which among the extension functions we identify a set of *parameters* Σ_P (function and constant symbols). Let Σ be a signature consisting of extension symbols which are not parameters (i.e. such that $\Sigma \cap (\Sigma_0 \cup \Sigma_P) = \emptyset$). We assume that $\mathcal{K}$ is a set of clauses in the signature $\Pi_0 \cup \Sigma_P \cup \Sigma$ in which all variables occur also below functions in $\Sigma_1 = \Sigma_P \cup \Sigma$.

We identify situations in which we can generate, for every ground formula G, a (universal) formula Γ representing a family of constraints on the parameters of G such that $\mathcal{T} \cup \Gamma \cup G \models \perp$. We consider base theories $\mathcal{T}_0$ such that $\mathcal{T}_0$ or its model completion $\mathcal{T}_0^*$ allows quantifier elimination, and use quantifier elimination to generate the formula Γ. Thus, we assume that one of the following conditions holds:

(C1): $\mathcal{T}_0$ allows quantifier elimination, or
(C2): $\mathcal{T}_0$ has a model completion $\mathcal{T}_0^*$ which allows quantifier elimination.

Let G be a finite set of ground clauses, and T a finite set of ground terms over the signature $\Pi_0 \cup \Sigma_P \cup \Sigma \cup C$, where C is a set of additional constants. We construct a universal formula $\forall y_1 \dots y_n \Gamma_T(y_1, \dots, y_n)$ over the signature $\Pi_0 \cup \Sigma_P$ by following the Steps 1–5 below:

Step 1: Let $\mathcal{K}_0 \cup G_0 \cup \mathsf{Con}_0$ be the set of Π_0^C clauses obtained from $\mathcal{K}[T] \cup G$ after the purification step described in Theorem 4.13 (with set of extension symbols Σ_1).

Step 2: Let $G_1 = \mathcal{K}_0 \cup G_0 \cup \mathsf{Con}_0$. Among the constants in G_1, we identify
(i) the constants $\overline{c}_f$, $f \in \Sigma_P$, where either $c_f = f \in \Sigma_P$ is a constant parameter, or c_f is introduced by a definition $c_f := f(c_1, \dots, c_k)$ in the hierarchical reasoning method,
(ii) all constants $\overline{c}_p$ occurring as arguments of functions in Σ_P in such definitions.
Let $\overline{c}$ be the remaining constants. We replace the constants in $\overline{c}$ with existentially quantified variables $\overline{x}$ in G_1, i.e. replace $G_1(\overline{c}_p, \overline{c}_f, \overline{c})$ with $G_1(\overline{c}_p, \overline{c}_f, \overline{x})$, and consider the formula $\exists \overline{x} G_1(\overline{c}_p, \overline{c}_f, \overline{x})$.

Step 3: Using the method for quantifier elimination in $\mathcal{T}_0$ (if Condition (C1) holds) or in $\mathcal{T}_0^*$ (if Condition (C2) holds) we can construct a formula $\Gamma_1(\overline{c}_p, \overline{c}_f)$ equivalent to $\exists \overline{x} G_1(\overline{c}_p, \overline{c}_f, \overline{x})$ w.r.t. $\mathcal{T}_0$ (resp. $\mathcal{T}_0^*$).

Step 4: Let $\Gamma_2(\overline{c}_p)$ be the formula obtained by replacing back in $\Gamma_1(\overline{c}_p, \overline{c}_f)$ the constants c_f introduced by definitions $c_f := f(c_1, \dots, c_k)$ with the terms $f(c_1, \dots, c_k)$. We replace $\overline{c}_p$ with existentially quantified variables $\overline{y}$.

Step 5: Let $\forall \overline{y} \Gamma_T(\overline{y})$ be $\forall \overline{y} \neg \Gamma_2(\overline{y})$.

A similar approach is used in [31] for generating constraints on parameters which guarantee safety of parametric systems. We show that $\forall \overline{y} \Gamma_T(\overline{y})$ guarantees unsatisfiability of G and further study the properties of these formulae. At the end of Section 6 we briefly indicate how this can be used for interpolant generation.

5.1. **Case 1: $\mathcal{T}_0$ allows quantifier elimination.** We first analyze the case in which $\mathcal{T}_0$ allows quantifier elimination.

Theorem 5.1. *Assume that $\mathcal{T}_0$ allows quantifier elimination. For every finite set of ground clauses G, and every finite set T of terms over the signature $\Pi_0 \cup \Sigma \cup \Sigma_P \cup C$ with $\mathsf{est}(\mathcal{K}, G) \subseteq T$ we can construct a universally quantified $\Pi_0 \cup \Sigma_P$-formula $\forall \overline{y} \Gamma_T(\overline{y})$ with the following properties:*
(1) *For every structure $\mathcal{A}$ with signature $\Pi_0 \cup \Sigma \cup \Sigma_P \cup C$ which is a model of $\mathcal{T}_0 \cup \mathcal{K}$, if $\mathcal{A} \models \forall \overline{y} \Gamma_T(\overline{y})$ then $\mathcal{A} \models \neg G$.*
(2) *$\mathcal{T}_0 \cup \forall \overline{y} \Gamma_T(\overline{y}) \cup \mathcal{K} \cup G$ is unsatisfiable.*

Proof. Let $\forall \overline{y} \Gamma_T(\overline{y})$ be the formula obtained in Steps 1–5.
(1) Let $\mathcal{A}$ be a $\Pi_0 \cup \Sigma \cup \Sigma_P \cup C$-structure such that $\mathcal{A} \models \mathcal{T}_0 \cup \mathcal{K} \cup G$. Then $\mathcal{A} \models \mathcal{T}_0 \cup \mathcal{K}[T] \cup G$. Let $\mathcal{K}_0 \cup G_0 \cup \mathsf{Con}_0 \cup \mathsf{Def}$ be the formulae obtained from $\mathcal{K}[T] \cup G$ after purification as explained in Theorem 4.13. Clearly, $\mathcal{A} \models \mathcal{T}_0 \cup \mathcal{K}_0 \cup G_0 \cup \mathsf{Con}_0 \cup \mathsf{Def}$.[4]

[4]For simplicity, we here use the same symbol for $\mathcal{A}$ and its expansion with new constants defined as in Def.

Let $G_1 = \mathcal{K}_0 \cup G_0 \cup \mathsf{Con}_0$. Since $\mathcal{A} \models \mathcal{T}_0 \cup G_1 \cup \mathsf{Def}$, we know that $\mathcal{A} \models \mathcal{T}_0 \cup \exists \overline{x} G_1(\overline{c}_p, \overline{c}_f, \overline{x}) \cup \mathsf{Def}$. By quantifier elimination in $\mathcal{T}_0$ we can construct a formula $\Gamma_1(\overline{c}_p, \overline{c}_f)$ equivalent to $\exists \overline{x} G_1(\overline{c}_p, \overline{c}_f, \overline{x})$ w.r.t. $\mathcal{T}_0$. Hence, $\mathcal{A} \models \mathcal{T}_0 \cup \Gamma_1(\overline{c}_p, \overline{c}_f)$. Since $\mathcal{A}$ is also a model for Def we can replace in Γ_1 the constants c_f introduced by definitions $c_f := f(c_1, \dots, c_k)$ with the terms $f(c_1, \dots, c_k)$ they replaced, thus obtaining the formula $\Gamma_2(\overline{c}_p)$, and $\mathcal{A} \models \mathcal{T}_0 \cup \Gamma_2(\overline{c}_p)$. If $\Gamma_2(\overline{y})$ is obtained from $\Gamma_2(\overline{c}_p)$ by replacing the constants in $\overline{c}_p$ with new variables in $\overline{y}$, it follows that $\mathcal{A} \models \mathcal{T}_0 \cup \exists y \Gamma_2(\overline{y})$, hence (as $\Gamma_T = \neg \Gamma_2$), $\mathcal{A} \models \exists \overline{y} \neg \Gamma_T(\overline{y})$, i.e. $\mathcal{A} \models \neg \forall \overline{y} \Gamma_T(\overline{y})$.

We showed that if $\mathcal{A} \models \mathcal{T}_0 \cup \mathcal{K} \cup G$ then $\mathcal{A} \models \neg \forall \overline{y} \Gamma_T(\overline{y})$. Hence, if $\mathcal{A} \models \mathcal{T}_0 \cup \mathcal{K} \cup \forall \overline{y} \Gamma_T(\overline{y})$ then $\mathcal{A} \not\models \mathcal{T}_0 \cup \mathcal{K} \cup G$, hence G is false in $\mathcal{A}$.

(2) The unsatisfiability of $\mathcal{T}_0 \cup \forall \overline{y} \Gamma_T(\overline{y}) \cup \mathcal{K} \cup G$ follows immediately from (1). □

If we analyze the proof of Theorem 5.1 we can make the following observations.

Lemma 5.2. *With the notation used in Steps 1–5 we can show that the formulae* $\Gamma_2(\overline{c}_p)$ *and* $\exists \overline{x} G_1(\overline{c}_p, \overline{c}_f, \overline{x}) \wedge \mathsf{Def}$ *are equivalent modulo* $\mathcal{T}_0 \cup \mathsf{UIF}_{\Sigma_P}$.

Proof. We show that for every $\Pi_0 \cup \Sigma_P \cup \Sigma \cup C$-structure $\mathcal{A}$ which is a model of $\mathcal{T}_0$, $\mathcal{A} \models \exists \overline{x} G_1(\overline{c}_p, \overline{c}_f, \overline{x}) \wedge \mathsf{Def}$ if and only if $\mathcal{A} \models \Gamma_2(\overline{c}_p)$. Assume that $\mathcal{A}$ is a model of $\mathcal{T}_0$ and of $\exists \overline{x} G_1(\overline{c}_p, \overline{c}_f, \overline{x}) \wedge \mathsf{Def}$. As $\exists \overline{x} G_1(\overline{c}_p, \overline{c}_f, \overline{x})$ and $\Gamma_1(\overline{c}_p, \overline{c}_f)$ are equivalent w.r.t. $\mathcal{T}_0$ (the second is obtained from the first by quantifier elimination) it follows that $\mathcal{A}$ is a model of $\Gamma_1(\overline{c}_p, \overline{c}_f) \wedge \mathsf{Def}$, hence it is a model of $\Gamma_2(\overline{c}_p)$.

Assume now that $\mathcal{A} \models \Gamma_2(\overline{c}_p)$. We can purify Γ_2 by introducing new constants renaming the terms $f(c_1, \dots, c_n)$ according to the definitions in Def. The formula obtained this way is $\Gamma_1(\overline{c}_p, \overline{c}_f) \wedge \mathsf{Def}$. As $\exists \overline{x} G_1(\overline{c}_p, \overline{c}_f, \overline{x})$ and $\Gamma_1(\overline{c}_p, \overline{c}_f)$ are equivalent w.r.t. $\mathcal{T}_0$, it follows[5] that $\mathcal{A} \models \exists \overline{x} G_1(\overline{c}_p, \overline{c}_f, \overline{x}) \wedge \mathsf{Def}$. □

Theorem 5.3. *If* $T_1 \subseteq T_2$ *then* $\forall \overline{y} \Gamma_{T_1}(\overline{y})$ *entails* $\forall \overline{y} \Gamma_{T_2}(\overline{y})$ *(modulo* $\mathcal{T}_0$*).*

Proof. Let T_1, T_2 be two finite sets of terms. If $T_1 \subseteq T_2$ then $\mathcal{K}[T_1] \subseteq \mathcal{K}[T_2]$. We denote by $\mathcal{K}_1$ the purified form of $\mathcal{K}[T_1]$ and by $\mathcal{K}_2$ the purified form of $\mathcal{K}[T_2]$, and let Con_i be the set of axioms corresponding to the terms in $\mathcal{K}[T_i] \cup G$, $i = 1, 2$. Then $\mathcal{K}_1 \cup \mathsf{Con}_1 \subseteq \mathcal{K}_2 \cup \mathsf{Con}_2$, hence $\mathcal{K}_2 \wedge G_0 \wedge \mathsf{Con}_2 \models \mathcal{K}_1 \wedge G_0 \wedge \mathsf{Con}_1$. Then every model of $\mathcal{T}_0$ which is a model of $\mathcal{K}_2 \wedge G_0 \wedge \mathsf{Con}_2$ is also a model of $\mathcal{K}_1 \wedge G_0 \wedge \mathsf{Con}_1$.

Let $\overline{c}$ denote the sequence consisting of all constants in $\mathcal{K}_2 \wedge G_0 \wedge \mathsf{Con}_2$ and not in Σ (a superset of the constants occurring in $\mathcal{K}_1 \wedge G_0 \wedge \mathsf{Con}_1$). We regard the elements in $\overline{c}$ as variables.[6]

We first show that $\exists \overline{c}(\mathcal{K}_2 \wedge G \wedge \mathsf{Con}_2) \models \exists \overline{c}(\mathcal{K}_1 \wedge G \wedge \mathsf{Con}_1)$.

Indeed, let $\mathcal{A}$ be a model of $\exists \overline{c}(\mathcal{K}_2 \wedge G \wedge \mathsf{Con}_2)$. Then there exists a valuation β which assigns values in A to the variables in $\overline{c}$ such that $\mathcal{A}, \beta \models \mathcal{K}_2 \wedge G \wedge \mathsf{Con}_2$. Then $\mathcal{A}^{\overline{c}} \models \mathcal{K}_2 \wedge G \wedge \mathsf{Con}_2$ (where $\mathcal{A}^{\overline{c}}$ is the expansion of $\mathcal{A}$ with new constants $\overline{c}$, interpreted as specified by β). Then $\mathcal{A}^{\overline{c}} \models \mathcal{K}_1 \wedge G_0 \wedge \mathsf{Con}_1$, hence $\mathcal{A} \models \exists \overline{c}(\mathcal{K}_1 \wedge G_0 \wedge \mathsf{Con}_1)$. This shows that $\exists \overline{c}(\mathcal{K}_2 \wedge G \wedge \mathsf{Con}_2) \models \exists \overline{c}(\mathcal{K}_1 \wedge G \wedge \mathsf{Con}_1)$.

We show that $\forall \overline{y} \Gamma_{T_1}(\overline{y}) \models \forall \overline{y} \Gamma_{T_2}(\overline{y})$ modulo $\mathcal{T}_0$, i.e. that every model of $\mathcal{T}_0 \cup \forall \overline{y} \Gamma_{T_1}(\overline{y})$ is also a model of $\mathcal{T}_0 \cup \forall \overline{y} \Gamma_{T_2}(\overline{y})$.

[5] For simplicity, we use the same symbol for $\mathcal{A}$ and its expansion with new constants defined as in Def.

[6] Instead of renaming the constants $\overline{c}$ with new variables $\overline{x}$, we here keep their names, but treat them as variables.

Let $\mathcal{A}$ be a model of $\mathcal{T}_0$. Assume that $\mathcal{A} \not\models \forall \overline{y} \Gamma_{T_2}(\overline{y})$. Then $\mathcal{A} \models \exists \overline{y} \neg \Gamma_{T_2}(\overline{y})$, hence (using the chain of arguments used in the previous proofs and the fact that $\mathcal{A}$ is a model of $\mathcal{T}_0$) $\mathcal{A} \models \exists \overline{c}(\mathcal{K}_2 \wedge G_0 \wedge \mathsf{Con}_2)$. But then $\mathcal{A} \models \exists \overline{c}(\mathcal{K}_1 \wedge G_0 \wedge \mathsf{Con}_1)$, hence (again using the chain of arguments used in the previous proofs and the fact that $\mathcal{A}$ is a model of $\mathcal{T}_0$) $\mathcal{A} \models \exists \overline{y} \neg \Gamma_{T_1}(\overline{y})$, so $\mathcal{A} \not\models \forall \overline{y} \Gamma_{T_1}(\overline{y})$. □

We denote by $\forall \overline{y} \Gamma_G(\overline{y})$ the formula obtained when $T = \mathsf{est}(\mathcal{K}, G)$.

Theorem 5.4. *If the extension $\mathcal{T}_0 \subseteq \mathcal{T}_0 \cup \mathcal{K}$ satisfies condition (Comp_f) and $\mathcal{K}$ is flat and linear then $\forall y \Gamma_G(y)$ is entailed by every conjunction Γ of clauses with the property that $\mathcal{T}_0 \cup \Gamma \cup \mathcal{K} \cup G$ is unsatisfiable (i.e. it is the weakest such constraint).*

Proof. We show that for every set Γ of constraints on the parameters, if $\mathcal{T}_0 \cup \Gamma \cup \mathcal{K} \cup G$ is unsatisfiable then every model of $\mathcal{T}_0 \cup \Gamma$ is a model of $\mathcal{T}_0 \cup \forall y \Gamma_G(y)$.

We know, by Theorem 4.10, that if the extension $\mathcal{T}_0 \subseteq \mathcal{T}_0 \cup \mathcal{K}$ satisfies condition (Comp_f) then also the extension $\mathcal{T}_0 \cup \Gamma \subseteq \mathcal{T}_0 \cup \Gamma \cup \mathcal{K}$ satisfies condition (Comp_f). If $\mathcal{K}$ is flat and linear then the extension is local. Let $T = \mathsf{est}(\mathcal{K}, G)$. By locality, $\mathcal{T}_0 \cup \Gamma \cup \mathcal{K} \cup G$ is unsatisfiable if and only if $\mathcal{T}_0 \cup \Gamma \cup \mathcal{K}[T] \cup G$ is unsatisfiable, if and only if (with the notations in Steps 1–5) $\mathcal{T}_0 \cup \Gamma \cup \mathcal{K}_0 \cup G_0 \cup \mathsf{Con}_0 \cup \mathsf{Def}$ is unsatisfiable. Let $\mathcal{A}$ be a model of $\mathcal{T}_0 \cup \Gamma$. Then $\mathcal{A}$ cannot be a model of $\mathcal{K}_0 \cup G_0 \cup \mathsf{Con}_0 \cup \mathsf{Def}$, so (with the notation used when describing Steps 1–5) $\mathcal{A} \not\models \Gamma_2(\overline{c}_p)$, i.e. $\mathcal{A} \not\models \exists \overline{y} \Gamma_2(\overline{y})$. It follows that $\mathcal{A} \models \forall \overline{y} \Gamma_G(\overline{y})$. □

A similar result holds for chains of local theory extensions.

Theorem 5.5. *Assume that we have the following chain of theory extensions:*

$$\mathcal{T}_0 \ \subseteq \ \mathcal{T}_0 \cup \mathcal{K}_1 \ \subseteq \ \mathcal{T}_0 \cup \mathcal{K}_1 \cup \mathcal{K}_2 \ \subseteq \ \dots \ \subseteq \ \mathcal{T}_0 \cup \mathcal{K}_1 \cup \mathcal{K}_2 \cup \dots \cup \mathcal{K}_n$$

where every extension in the chain satisfies condition (Comp_f), $\mathcal{K}_i$ are all flat and linear, and in all $\mathcal{K}_i$ all variables occur below the extension terms on level i.

Let G be a set of ground clauses, and let G_1 be the result of the hierarchical reduction of satisfiability of G to a satisfiability test w.r.t. $\mathcal{T}_0$. Let $T(G)$ be the set of all instances used in the chain of hierarchical reductions and let $\forall y \Gamma_{T(G)}(y)$ be the formula obtained by applying Steps 2–5 to G_1.

Then $\forall y \Gamma_{T(G)}(y)$ is entailed by every conjunction Γ of clauses with the property that $\mathcal{T}_0 \cup \Gamma \cup \mathcal{K}_1 \cup \dots \cup \mathcal{K}_n \cup G$ is unsatisfiable (i.e. it is the weakest such constraint).

Proof. We show that for every set Γ of constraints on the parameters, if $\mathcal{T}_0 \cup \Gamma \cup \mathcal{K}_1 \cup \dots \cup \mathcal{K}_n \cup G$ is unsatisfiable then every model of $\mathcal{T}_0 \cup \Gamma$ is a model of $\mathcal{T}_0 \cup \forall y \Gamma_{T(G)}(y)$.

We know, by Theorem 4.10, that if the extension

$$\mathcal{T}_0 \cup \mathcal{K}_1 \cup \dots \mathcal{K}_{i-1} \subseteq \mathcal{T}_0 \cup \mathcal{K}_1 \cup \dots \mathcal{K}_{i-1} \cup \mathcal{K}_i$$

satisfies condition (Comp_f) then also the extension

$$\mathcal{T}_0 \cup \Gamma \cup \mathcal{K}_1 \cup \dots \mathcal{K}_{i-1} \subseteq \mathcal{T}_0 \cup \Gamma \cup \mathcal{K}_1 \cup \dots \mathcal{K}_{i-1} \cup \mathcal{K}_i$$

satisfies condition (Comp_f). If $\mathcal{K}_i$ is flat and linear then the extension is local. By locality, the following are equivalent:

- $\mathcal{T}_0 \cup \Gamma \cup \mathcal{K}_1 \cup \dots \cup \mathcal{K}_{n-1} \cup \mathcal{K}_n \cup G \models \perp$;
- $\mathcal{T}_0 \cup \Gamma \cup \mathcal{K}_1 \cup \dots \cup \mathcal{K}_{n-1} \cup \mathcal{K}_n[T_n] \cup G^n \models \perp$ where $G^n = G$ and $T_n = \mathsf{est}(\mathcal{K}_n, G^n)$;

- $\mathcal{T}_0 \cup \Gamma \cup \mathcal{K}_1 \cup \cdots \cup \mathcal{K}_{n-1}[T_{n-1}] \cup G^{n-1} \models \perp$ where $G^{n-1} = \mathcal{K}_n[T_n]_0 \cup G_0^n \cup \mathsf{Con}_0^n$ is the set of ground clauses obtained from $\mathcal{K}_n[T_n] \cup G^n$ after purification and adding the corresponding instances of congruence axioms and $T_{n-1} = \mathsf{est}(\mathcal{K}_{n-1}, G^{n-1})$;

...
- $\mathcal{T}_0 \cup \Gamma \cup \mathcal{K}_1[T_1] \cup G^1 \models \perp$ where $G^1 = \mathcal{K}_2[T_2]_0 \cup G^2{}_0 \cup \mathsf{Con}_0^2$ is the set of ground clauses obtained from $\mathcal{K}_2[T_2] \cup G^2$ after purification and adding the corresponding instances of congruence axioms and $T_1 = \mathsf{est}(\mathcal{K}_1, G^1)$;
- $\mathcal{T}_0 \cup \Gamma \cup \mathcal{K}_0 \cup G_0 \cup \mathsf{Con}_0 \cup \mathsf{Def} \models \perp$ where $\mathcal{K}_0 = \mathcal{K}_1[T_1]_0$, and $G_0 = G_0^1$ are the sets of ground clauses obtained from $\mathcal{K}_1[T_1]$ resp. G^1 after purification and Con_0 is the corresponding set of instances of congruence axioms.

Let $\mathcal{A}$ be a model of $\mathcal{T}_0 \cup \Gamma$. Then $\mathcal{A}$ cannot be a model of $\mathcal{K}_0 \cup G_0 \cup \mathsf{Con}_0 \cup \mathsf{Def}$, as $\mathcal{T}_0 \cup \Gamma \cup \mathcal{K}_0 \cup G_0 \cup \mathsf{Con}_0 \cup \mathsf{Def}$ has no models. Therefore (with the notation used when describing Steps 1–5), $\mathcal{A} \not\models \Gamma_2(\overline{c}_p)$, i.e. $\mathcal{A} \not\models \exists \overline{y} \Gamma_2(\overline{y})$. It follows that $\mathcal{A} \models \forall \overline{y} \Gamma_{T(G)}(\overline{y})$. $\square$

Example 5.6. Let $\mathcal{T}_0$ be the theory of dense total orderings without endpoints. Consider the extension of $\mathcal{T}_0$ with functions $\Sigma_1 = \{f, g, h, c\}$ whose properties are axiomatized by

$$\begin{array}{ll} \mathcal{K} := \{ & \forall x(x \leq c \rightarrow g(x) \approx f(x)), \\ & \forall x(c < x \rightarrow g(x) \approx h(x)) \quad \}. \end{array}$$

Assume $\Sigma_P = \{f, h, c\}$ and $\Sigma = \{g\}$. We are interested in generating a set of constraints on the functions f and h which ensure that g is monotone, e.g. satisfies

$$\mathsf{Mon}(g): \ \forall x, y(x \leq y \rightarrow g(x) \leq g(y)),$$

i.e. a set Γ of $\Sigma_0 \cup \Sigma_P$-constraints such that

$$\mathcal{T}_0 \cup \Gamma \cup \mathcal{K} \cup \{c_1 \leq c_2, g(c_1) > g(c_2)\} \text{ is unsatisfiable,}$$

where $G = \{c_1 \leq c_2, g(c_1) > g(c_2)\}$ is the negation of $\mathsf{Mon}(g)$.

As explained in Example 4.15 we have the follwing chain of local theory extensions:

$$\mathcal{T}_0 \subseteq \mathcal{T}_0 \cup \mathsf{UIF}_{\{f,h\}} \subseteq (\mathcal{T}_0 \cup \mathsf{UIF}_{\{f,g\}}) \cup \mathcal{K} = \mathcal{T}_0 \cup \mathcal{K}.$$

Both extensions satisfy the condition Comp_f, and $\mathcal{T}_0 \cup \mathcal{K} \cup G$ is satisfiable iff $\mathcal{T}_0 \cup \mathcal{K}[G] \cup G$ is satisfiable, where $\mathcal{K}[G]$ contains all instances of $\mathcal{K}$ in which the terms starting with the extension symbol g are ground terms in G:

$$\begin{array}{lll} \mathcal{K}[G] := \{ & c_1 \leq c \rightarrow g(c_1) \approx f(c_1), & c_2 \leq c \rightarrow g(c_2) \approx f(c_2), \\ & c < c_1 \rightarrow g(c_1) \approx h(c_1), & c < c_2 \rightarrow g(c_2) \approx h(c_2)) \quad \}. \end{array}$$

We construct Γ as follows:

Step 1: We compute $\mathcal{T}_0 \cup \mathcal{K}[G] \cup G$ as described above, then purify it in two steps, because we have a chain of two local extensions: First we introduce new constants g_1, g_2 for the terms $g(c_1), g(c_2)$, then, in the next step, we introduce new constants f_1, f_2, h_1, h_2 for the terms $f(c_1), f(c_2), h(c_1)$, and $h(c_2)$. We obtain:

$\mathsf{Def} = \{g_1 {\approx} g(c_1), g_2 {\approx} g(c_2), f_1 {=} f(c_1), f_2 {\approx} f(c_2), h_1 {\approx} h(c_1), h_2 {\approx} h(c_2)\}$ and

$$\begin{array}{ll} \mathcal{K}_0 \cup \mathsf{Con}_0 \cup G_0 := \{ & c_1 \leq c \rightarrow g_1 \approx f_1, \quad c_2 \leq c \rightarrow g_2 \approx f_2, \\ & c < c_1 \rightarrow g_1 \approx h_1, \quad c < c_2 \rightarrow g_2 \approx h_2, \\ & c_1 \approx c_2 \rightarrow g_1 \approx g_2, \quad c_1 \approx c_2 \rightarrow f_1 \approx f_2, \quad c_1 \approx c_2 \rightarrow h_1 \approx h_2, \\ & c_1 \leq c_2, \quad g_1 > g_2 \quad \} \end{array}$$

Step 2: The parameters are contained in the set $\Sigma_P = \{f, h, c\}$. We want to eliminate the function symbol g, so we replace g_1, g_2 with existentially quantified variables z_1, z_2. We obtain the existentially quantified formula $\exists z_1, z_2\, G_1(c_1, c_2, c, f_1, f_2, h_1, h_2, z_1, z_2)$:

$$\begin{array}{l} \exists z_1, z_2 (c_1 \leq c \rightarrow z_1 \approx f_1 \quad \wedge \quad c_2 \leq c \rightarrow z_2 \approx f_2 \quad \wedge \quad c_1 \approx c_2 \rightarrow f_1 \approx f_2 \quad \wedge \\ \qquad c < c_1 \rightarrow z_1 \approx h_1 \quad \wedge \quad c < c_2 \rightarrow z_2 \approx h_2 \quad \wedge \quad c_1 \approx c_2 \rightarrow h_1 \approx h_2 \quad \wedge \\ \qquad c_1 \approx c_2 \rightarrow z_1 \approx z_2 \quad \wedge \quad c_1 \leq c_2 \wedge z_1 > z_2) \end{array}$$

We can simplify the formula $G_1(c_1, c_2, c, f_1, f_2, h_1, h_2, z_1, z_2)$ taking into account that in the theory of (dense) total orderings the following equivalences hold:

(1) $(c_1 \approx c_2 \rightarrow z_1 \approx z_2) \wedge z_1 > z_2 \quad \equiv \quad c_1 \not\approx c_2 \wedge z_1 > z_2$
(2) $(c_1 \approx c_2 \rightarrow f_1 \approx f_2) \wedge (c_1 \approx c_2 \rightarrow h_1 \approx h_2) \wedge c_1 \not\approx c_2 \quad \equiv \quad c_1 \not\approx c_2$

We obtain the formula $\exists z_1, z_2\, G'_1(c_1, c_2, c, f_1, f_2, h_1, h_2, z_1, z_2)$:

$$\begin{array}{l} \exists z_1, z_2 [(c_1 \leq c \rightarrow z_1 \approx f_1) \wedge (c < c_1 \rightarrow z_1 \approx h_1) \wedge \\ \qquad (c_2 \leq c \rightarrow z_2 \approx f_2) \wedge (c < c_2 \rightarrow z_2 \approx h_2) \wedge c_1 \not\approx c_2 \ \wedge \ c_1 \leq c_2 \wedge z_1 > z_2] \end{array}$$

Step 3: For quantifier elimination we can use a system such as Mathematica, Redlog or QEPCAD. For convenience, we illustrate how the computations can be done by hand. Note that in propositional logic we have:

$$(P \rightarrow Q) \wedge (\neg P \rightarrow Q') \equiv (P \wedge Q) \vee (\neg P \wedge Q').$$

From this it follows that:

$$\begin{array}{ll} & (P \rightarrow Q) \wedge (\neg P \rightarrow Q') \wedge (R \rightarrow S) \wedge (\neg R \rightarrow S') \wedge W \\ \equiv & [(P \wedge Q) \vee (\neg P \wedge Q')] \wedge [(R \wedge S) \vee (\neg R \wedge S')] \wedge W \\ \equiv & (P \wedge Q \wedge R \wedge S \wedge W) \vee (P \wedge Q \wedge \neg R \wedge S' \wedge W) \vee \\ & (\neg P \wedge Q' \wedge R \wedge S \wedge W) \vee (\neg P \wedge Q' \wedge \neg R \wedge S' \wedge W). \end{array}$$

Therefore, the formula above is equivalent to:

$$\begin{array}{ll} \exists z_1, z_2 (& (c_1 \leq c \wedge z_1 \approx f_1 \wedge c_2 \leq c \wedge z_2 \approx f_2 \wedge c_1 \leq c_2 \wedge z_1 > z_2 \wedge c_1 \not\approx c_2) \vee \\ & (c_1 \leq c \wedge z_1 \approx f_1 \wedge c < c_2 \wedge z_2 \approx h_2 \wedge c_1 \leq c_2 \wedge z_1 > z_2 \wedge c_1 \not\approx c_2) \vee \\ & (c < c_1 \wedge z_1 \approx h_1 \wedge c_2 \leq c \wedge z_2 \approx f_2 \wedge c_1 \leq c_2 \wedge z_1 > z_2 \wedge c_1 \not\approx c_2) \vee \\ & (c < c_1 \wedge z_1 \approx h_1 \wedge c < c_2 \wedge z_2 \approx h_2 \wedge c_1 \leq c_2 \wedge z_1 > z_2 \wedge c_1 \not\approx c_2)) \end{array}$$

Using the method for quantifier elimination for dense total orderings without endpoints for eliminating the existentially quantified variables z_1, z_2 in this last formula we obtain the formula $\Gamma_1(c_1, c_2, c, f_1, f_2, h_1, h_2)$:

$$\begin{array}{ll} (& (c_1 \leq c \ \wedge \ c_2 \leq c \ \wedge \ c_1 \leq c_2 \ \wedge \ f_1 > f_2 \ \wedge \ c_1 \not\approx c_2) \vee \\ & (c_1 \leq c \ \wedge \ c < c_2 \ \wedge \ c_1 \leq c_2 \ \wedge \ f_1 > h_2 \ \wedge \ c_1 \not\approx c_2) \vee \\ & (c < c_1 \ \wedge \ c_2 \leq c \ \wedge \ c_1 \leq c_2 \ \wedge \ h_1 > f_2 \ \wedge \ c_1 \not\approx c_2) \vee \\ & (c < c_1 \ \wedge \ c < c_2 \ \wedge \ c_1 \leq c_2 \ \wedge \ h_1 > h_2 \ \wedge \ c_1 \not\approx c_2) \ ) \end{array}$$

Step 4: We construct the formula $\Gamma_2(c_1, c_2, c)$ obtained from Γ_1 by replacing f_i by $f(c_i)$ and h_i by $h(c_i)$, $i = 1, 2$. We obtain (after further minor simplification and rearrangement for facilitating reading):

$$((c_1 < c_2 \leq c \wedge f(c_1){>}f(c_2)) \vee (c_1{\leq}c < c_2 \wedge f(c_1){>}h(c_2)) \vee (c{<}c_1{<}c_2 \wedge h(c_1){>}h(c_2)))$$

Step 5: Then we obtain the constraint on the parameters $\forall z_1, z_2 \Gamma_T(z_1, z_2)$, i.e.:

$$\begin{array}{ll}\forall z_1, z_2[& (z_1 < z_2 \leq c \rightarrow f(z_1) \leq f(z_2)) \wedge \\ & (z_1 \leq c < z_2 \rightarrow f(z_1) \leq h(z_2)) \wedge \\ & (c < z_1 < z_2 \rightarrow h(z_1) \leq h(z_2)) \quad]\end{array}$$

which guarantees that g is monotone.

5.2. **Case 2: $\mathcal{T}_0$ does not allow quantifier elimination, but its model completion does.** We now analyze the case in which $\mathcal{T}_0$ does not necessarily allow quantifier elimination, but has a model completion which allows quantifier elimination.

Theorem 5.7. *Let $\mathcal{T}_0$ be a theory. Assume that $\mathcal{T}_0$ has a model completion $\mathcal{T}_0^*$ such that $\mathcal{T}_0 \subseteq \mathcal{T}_0^*$. Let $\mathcal{T} = \mathcal{T}_0 \cup \mathcal{K}$ be an extension of $\mathcal{T}_0$ with new function symbols $\Sigma_1 = \Sigma_P \cup \Sigma$ whose properties are axiomatized by a set of clauses $\mathcal{K}$ (all of which contain symbols in Σ) in which all variables occur also below extension functions in Σ_1. Assume that:*
(i) *every model of $\mathcal{T}_0 \cup \mathcal{K}$ embeds into a model of $\mathcal{T}_0^* \cup \mathcal{K}$, and*
(ii) *$\mathcal{T}_0^*$ allows quantifier elimination.*
Then, for every finite set of ground clauses G and every finite set T of ground terms over the signature $\Pi^C = \Pi_0 \cup \Sigma \cup \Sigma_P \cup C$ with $\mathsf{est}(\mathcal{K}, G) \subseteq T$ we can construct a universally quantified $\Pi_0 \cup \Sigma_P$-formula $\forall \overline{x} \Gamma_T(\overline{x})$ such that:
(1) *For every structure $\mathcal{A}$ with signature $\Pi_0 \cup \Sigma \cup \Sigma_P \cup C$ which is a model of $\mathcal{T}_0 \cup \mathcal{K}$, if $\mathcal{A} \models \forall \overline{x} \Gamma_T(\overline{x})$ then $\mathcal{A} \models \neg G$.*
(2) *$\mathcal{T}_0 \cup \forall y \Gamma_T(y) \cup \mathcal{K} \cup G$ is unsatisfiable.*

Proof. Let G be a finite set of Π^C-clauses and T be a finite set of ground Π^C-terms containing $\mathsf{est}(\mathcal{K}, G)$. Since, by assumption (ii), $\mathcal{T}_0^*$ has quantifier elimination, by Theorem 5.1 we know that we can construct a universally quantified $\Pi_0 \cup \Sigma_P$-formula $\forall \overline{x} \Gamma_T(\overline{x})$ (containing some parameters in Σ_P) with the following properties:

- For every structure $\mathcal{A}$ with signature $\Pi_0 \cup \Sigma \cup \Sigma_P \cup C$ which is a model of $\mathcal{T}_0^* \cup \mathcal{K}$, if $\mathcal{A} \models \forall \overline{x} \Gamma_T(\overline{x})$ then $\mathcal{A} \models \neg G$;
- $\mathcal{T}_0^* \cup \forall y \Gamma_T(y) \cup \mathcal{K} \cup G$ is unsatisfiable;

and that $\forall \overline{x} \Gamma_T(\overline{x})$ is constructed using Steps 1–5. We show that (1) and (2) hold.

(1) We prove the contrapositive. Let $\mathcal{A}$ be a structure with signature $\Pi_0 \cup \Sigma \cup \Sigma_P \cup C$ which is a model of $\mathcal{T}_0 \cup \mathcal{K} \cup G$. As $\mathcal{A}$ is a model of $\mathcal{T}_0 \cup \mathcal{K}$, by Assumption (i), $\mathcal{A}$ embeds into a model $\mathcal{B}$ of $\mathcal{T}_0^* \cup \mathcal{K}$. Since G is a set of ground clauses which are true in $\mathcal{A}$ and $\mathcal{A}$ embeds into $\mathcal{B}$, G is also true in $\mathcal{B}$. Thus, $\mathcal{B}$ is a model of $\mathcal{T}_0^* \cup \mathcal{K} \cup G$. By the proof of Theorem 5.1 and with the notation used there it follows that $\mathcal{B} \models \mathcal{T}_0^* \cup \Gamma_2(\overline{c}_p)$. Again, since $\mathcal{A}$ embeds into $\mathcal{B}$, and since $\Gamma_2(\overline{c}_p)$ is a ground formula in the signature of $\mathcal{A}$, $\mathcal{A} \models \Gamma_2(\overline{c}_p)$. It follows (as in the proof of Theorem 5.1) that $\mathcal{A} \models \exists \overline{y} \Gamma_2(\overline{y})$.

We showed that if $\mathcal{A} \models \mathcal{T}_0 \cup \mathcal{K} \cup G$ then $\mathcal{A}$ is a model of $\exists x \neg \Gamma_T(x)$. Hence, if $\mathcal{A} \models \mathcal{T}_0 \cup \mathcal{K} \cup \forall \overline{y} \Gamma_T(\overline{y})$ then $\mathcal{A} \not\models \mathcal{T}_0 \cup \mathcal{K} \cup G$, hence G is false in $\mathcal{A}$.
(2) follows directly from (1). □

Example 5.8. Consider the problem in Example 5.6 when the base theory $\mathcal{T}_0$ is the theory of total orderings. We first show that conditions (i) and (ii) in Theorem 5.7 hold:

$\mathcal{T}_0^*$ is the theory of dense total orderings without endpoints, which allows quantifier elimination, so (ii) holds.

Let $\mathcal{A}$ be a model of $\mathcal{T}_0 \cup \mathcal{K}$, where

$$\mathcal{K} := \{ \quad \begin{array}{l} \forall x(x \leq c \rightarrow g(x) \approx f(x)), \\ \forall x(c < x \rightarrow g(x) \approx h(x)) \quad \}. \end{array}$$

Then $\mathcal{A}$ is a totally ordered set, which clearly embeds into a model $\mathcal{B}$ of $\mathcal{T}_0^*$ (a dense, totally ordered set without endpoints). We can use the definitions of the functions of f, g, h in $\mathcal{A}$ to define a partial $\Sigma_P \cup \Sigma$-structure on $\mathcal{B}$. Due to the form of $\mathcal{K}$ it is easy to see that we can extend this partial structure to a total model of $\mathcal{T}_0^* \cup \mathcal{K}$:

- the functions f, h can be defined arbitrarily wherever they are not defined;
- g is then defined by case distinction, such that for every $b \in \mathcal{B}$, if $b \leq c$ then $g_{\mathcal{B}}(b) = f_{\mathcal{B}}(b)$ and if $b > c$ then $g_{\mathcal{B}}(b) = h_{\mathcal{B}}(b)$.

By Theorem 5.7, the formula $\forall z_1, z_2 \Gamma_T(z_1, z_2)$ constructed in Example 5.6 ensures that g is monotone also in this case.

Unfortunately, under the assumptions of Theorem 5.7, in the case of local theory extensions satisfying the conditions in Theorem 5.4 we cannot guarantee that the formula $\forall y \Gamma_G(y)$ is the weakest among all universal formulae Γ with $\mathcal{T}_0 \cup \Gamma \cup \mathcal{K} \cup G \models \perp$, as is illustrated by the following example.

Example 5.9. Let $\mathcal{T}_0$ be the theory of total orderings and

$$G := \{a < g(a),\ g(a) < h(a)\}.$$

We apply Steps 1–5 for $\mathcal{T}_0^*$, $\mathcal{K} = \emptyset$ and G, with $T = \mathsf{st}(G) = \{a, g(a), h(a)\}$, where $\Sigma_P = \{h\}$:

Step 1: We compute $\mathcal{T}_0 \cup \mathcal{K}[G] \cup G$, then purify it. We obtain:

$$\mathsf{Def} = \{g_1 \approx g(a), h_1 \approx h(a)\} \qquad \mathcal{K}_0 \cup \mathsf{Con}_a \cup G_0 = \{a < g_1, g_1 < h_1\}.$$

Step 2: $\Sigma_P = \{h\}$. We want to eliminate g, so we replace g_1 with the existentially quantified variable z_1. We obtain the existentially quantified formula $\exists z_1(a < z_1 \wedge z_1 < h_1)$.

Step 3: Using a method for quantifier elimination for the theory of dense total orderings without endpoints for eliminating the existentially quantified variable z_1 in this formula we obtain the formula $a < h_1$.

Step 4: We construct the formula $a < h(a)$ from this formula by replacing h_1 back with $h(a)$.

Step 5: By replacing a with an existentially quantified variable and negating we obtain the formula: $\forall y \Gamma_G(y) = \forall y(h(y) \leq y)$.

We argue that this last formula is not the most general universal formula that entails $\neg G$ (w.r.t. $\mathcal{T}_0$). Let $\Gamma := \forall x, y, z(x < y \rightarrow y \geq z)$. Then $\Gamma \wedge G$ is unsatisfiable w.r.t. $\mathcal{T}_0$: Indeed, assume that $\Gamma \wedge G$ has a model $\mathcal{A}$. Then in $\mathcal{A}$, $a < g(a)$ and $g(a) < h(a)$.

- As $a < g(a)$, $g(a) \geq a'$ for every $a' \in A$, so $g(a)$ is a maximal element of $\mathcal{A}$.
- But then $g(a) \geq h(a)$. This contradicts the fact that, in $\mathcal{A}$, $g(a) < h(a)$.

This shows that $\Gamma \wedge G$ is unsatisfiable w.r.t. $\mathcal{T}_0$.
However, there exists a structure $\mathcal{A}_1$ with two elements a_1, a_2 where $a_1 < a_2$ such that $h_{\mathcal{A}_1}(a_1) = a_2$ which satisfies Γ but not Γ_G:

(1) This structure clearly satisfies Γ: for every valuation $\beta : \{x, y, z\} \rightarrow \{a_1, a_2\}$ we have the following situations:
   - $\beta(x) \geq \beta(y)$: Then $\mathcal{A}_1, \beta \models (x < y \rightarrow y \geq z)$ since the premise is false.
   - $\beta(x) < \beta(y)$: Then $\beta(x) = a_1, \beta(y) = a_2$, so $\beta(y) \geq \beta(z)$ no matter what the value of $\beta(z)$ is. Thus, also in this case $\mathcal{A}_1, \beta \models (x < y \rightarrow y \geq z)$.

(2) The structure does not satisfy $\forall y \Gamma_G(y) := \forall y (h(y) \leq y)$: for the valuation with $\beta(y) = a_1$ we have $a_2 = h_{\mathcal{A}_1}(a_1) > a_1$, so $\mathcal{A}_1, \beta \not\models \forall y (h(y) \leq y)$.

Note that this situation cannot occur when $\mathcal{T}_0$ has quantifier elimination: Then the formula $\exists \overline{x} G_1(\overline{x})$ is either true or false in $\mathcal{T}_0$. If it is true then to achieve unsatisfiability we have to add $\Gamma = \perp$, which entails any other constraint. If it is false then we do not need to add any constraints to achieve unsatisfiability, so $\Gamma = \top$, which is entailed by any other constraint.

6. Ground Interpolation in Theory Extensions

In this section we present criteria for recognizing whether a theory extension $\mathcal{T} = \mathcal{T}_0 \cup \mathcal{K}$ has ground interpolation provided that $\mathcal{T}_0$ has (general) ground interpolation. In particular, we are interested in giving criteria for checking whether a theory $\mathcal{T}$ (resp. a theory extension $\mathcal{T} = \mathcal{T}_0 \cup \mathcal{K}$) has a special form of the ground interpolation property, in which for every pair of ground formulae A, B with $A \wedge B \models_{\mathcal{T}} \perp$ there exists an interpolant I of A and B such that the terms (resp. the extension terms) occurring in I are in a set $W(A, B)$ which can be constructed from the set of ground terms of A and B.

6.1. Previous Work.

In [30] we identified classes of local extensions in which ground interpolants can be computed hierarchically (for this, we had to find ways of separating the instances of axioms in $\mathcal{K}$ and of the congruence axioms). We present the ideas below.

Let $\mathcal{T}_0 \subseteq \mathcal{T} = \mathcal{T}_0 \cup \mathcal{K}$ be a local theory extension with function symbols in a set Σ_1 and let $A(\overline{a}, \overline{c}), B(\overline{b}, \overline{c})$ be sets of ground clauses over the signature of $\mathcal{T}$, possibly containing additional constants in a set C, such that $A \wedge B \models_{\mathcal{T}_0 \cup \mathcal{K}} \perp$. From Theorem 4.13 we know that in such extensions hierarchical reasoning is possible: if A and B are sets of ground clauses in a signature Π^C, and $A_0 \wedge D_A$ (resp. $B_0 \wedge D_B$) are obtained from A (resp. B) by purification and flattening – where $\mathsf{Def} = D_A \cup D_B$ the union of the set D_A containing those equalities $c_t \approx t$, where t is an extension term in A and the set D_B containing those equalities $c_t \approx t$, where t is an extension term in B – then the following are equivalent:

- $A \wedge B \models_{\mathcal{T}_1} \perp$;
- $\mathcal{T}_0 \wedge \mathcal{K}[A \wedge B] \wedge (A \wedge B) \models \perp$;
- $\mathcal{T}_0 \wedge \mathcal{K}[A \wedge B] \wedge (A_0 \wedge D_A) \wedge (B_0 \wedge D_B) \models \perp$;
- $\mathcal{K}_0 \wedge A_0 \wedge B_0 \wedge \mathsf{Con}[D_A \wedge D_B]_0 \models_{\mathcal{T}_0} \perp$,

where $\mathcal{K}_0$ is obtained from $\mathcal{K}[A \wedge B]$ by replacing the Σ_1-terms with the corresponding constants contained in the definitions D_A and D_B and

$$\mathsf{Con}[D_A \wedge D_B]_0 = \mathsf{Con}_0 = \bigwedge \{ \bigwedge_{i=1}^n c_i \approx d_i \rightarrow c \approx d \mid \begin{array}{l} f(c_1, \dots, c_n) \approx c \in \mathsf{Def} = D_A \cup D_B \\ f(d_1, \dots, d_n) \approx d \in \mathsf{Def} = D_A \cup D_B \end{array} \}.$$

In general the method for hierarchical reasoning in local theory extensions is not sufficient for computing hierarchically interpolants because:

(i) $\mathcal{K}[A \wedge B]$ may contain free variables.

(ii) If some clause in $\mathcal{K}$ contains two or more different extension functions, these extension functions cannot always be "separated".

(iii) The clauses $\mathcal{K}_0 \wedge \mathsf{Con}[D_A \wedge D_B]_0$ may contain combinations of constants and extension functions from A and B.

In order to avoid problem (i), in [30] we considered only extensions with sets of clauses $\mathcal{K}$ of clauses in which all variables occur below some extension term. To address (ii), we defined an equivalence relation $\sim$ between extension functions, where $f \sim g$ if f and g occur in the same clause in $\mathcal{K}$, and considered that a function $f \in \Sigma_1$ is common to A and B if there exist $g, h \in \Sigma_1$ such that $f \sim g$, $f \sim h$, g occurs in A and h occurs in B.

In order to address (iii), we identified situations in which it is possible to separate mixed instances of axioms in $\mathcal{K}_0$, or of congruence axioms in $\mathsf{Con}[D_A \wedge D_B]_0$, into an A-part and a B-part. We illustrate this on an example discussed in [30].

Example 6.1 ([30])**.** Consider the conjunction $A_0 \wedge D_A \wedge B_0 \wedge D_B \wedge \mathsf{Con}[D_A \wedge D_B]_0 \wedge \mathsf{Mon}_0 \wedge \mathsf{SGc}_0$ in Example 4.14, where $\mathcal{T}_0 = \mathsf{SLat}$. The A and B-part share the constants c and d, and no function symbols. However, as f and g occur together in SGc, $f \sim g$, so they are considered to be all shared. (Thus, the interpolant is allowed to contain both f and g.) We can obtain a separation for the clause $b \leq a_1 \rightarrow b_1 \leq a$ of SGc_0 as follows:

(i) We note that $A_0 \wedge B_0 \models b \leq a_1$.

(ii) We can find an SLat-term t containing only shared constants of A_0 and B_0 such that $A_0 \wedge B_0 \models b \leq t \wedge t \leq a_1$. (Indeed, such a term is $t = d$.)

(iii) We show that, instead of the axiom $b \leq g(a) \rightarrow f(b) \leq a$, whose flattened form is in SGc_0, we can use, without loss of unsatisfiability:

   (1) an instance of the monotonicity axiom for f: $b \leq d \rightarrow f(b) \leq f(d)$,

   (2) another instance of SGc, namely: $d \leq g(a) \rightarrow f(d) \leq a$.

   For this, we introduce a new constant $c_{f(d)}$ for $f(d)$ (its definition, $c_{f(d)} \approx f(d)$, is stored in a set D_T), and the corresponding instances $\mathcal{H}_{\mathsf{sep}} = \mathcal{H}^A_{\mathsf{sep}} \wedge \mathcal{H}^B_{\mathsf{sep}}$ of the congruence, monotonicity and $\mathsf{SGc}(f, g)$-axioms, which are now separated into an A-part ($\mathcal{H}^A_{\mathsf{sep}} : d \leq a_1 \rightarrow c_{f(d)} \leq a$) and a B-part ($\mathcal{H}^B_{\mathsf{sep}} : b \leq d \rightarrow b_1 \leq c_{f(d)}$). We thus obtain a separated conjunction $\overline{A}_0 \wedge \overline{B}_0$ (where $\overline{A}_0 = \mathcal{H}^A_{\mathsf{sep}} \wedge A_0$ and $\overline{B}_0 = \mathcal{H}^B_{\mathsf{sep}} \wedge B_0$), which can be proved to be unsatisfiable in $\mathcal{T}_0 = \mathsf{SLat}$.

(iv) To compute an interpolant in SLat for $\overline{A}_0 \wedge \overline{B}_0$ note that $\overline{A}_0$ is logically equivalent to the conjunction of unit literals $d \leq a_1 \ \wedge \ a \leq c \ \wedge \ c_{f(d)} \leq a$ and $\overline{B}_0$ is logically equivalent to $b \leq d \ \wedge \ b_1 \not\leq c \ \wedge \ b_1 \leq c_{f(d)}$. An interpolant is $I_0 = c_{f(d)} \leq c$.

(v) By replacing the new constants with the terms they denote we obtain the interpolant $I = f(d) \leq c$ for $A \wedge B$.

The same ideas can be used when $\mathcal{T}_0 = \mathsf{TOrd}$.

Criteria linking hierarchical ground interpolation to a notion of "separability" and to an amalgamability property for partial algebras were given in [35, 36]. We present the ideas in [35, 36] and then extend some of the results presented there.

6.2. **W-Separability.**

Definition 6.2 ([35])**.** An amalgamation closure for a theory extension $\mathcal{T} = \mathcal{T}_0 \cup \mathcal{K}$ is a function W associating with finite sets of ground terms T_A and T_B, a finite set $W(T_A, T_B)$ of ground terms such that
(1) all ground subterms in $\mathcal{K}$ and T_A are in $W(T_A, T_B)$;
(2) W is monotone, i.e., for all $T_A \subseteq T'_A$, $T_B \subseteq T'_B$, $W(T_A, T_B) \subseteq W(T'_A, T'_B)$;
(3) W is a closure, i.e., $W(W(T_A, T_B), W(T_B, T_A)) \subseteq W(T_A, T_B)$;
(4) W is compatible with any map h between constants satisfying $h(c_1) \neq h(c_2)$, for all constants $c_1 \in \mathsf{st}(T_A), c_2 \in \mathsf{st}(T_B)$ that are not shared between T_A and T_B, i.e., for any such h we require $W(h(T_A), h(T_B)) = h(W(T_A, T_B))$; and
(5) $W(T_A, T_B)$ only contains T_A-pure terms (i.e. terms containing only constants in C which occur in T_A).

For sets of ground clauses A, B we write $W(A, B)$ for $W(\mathsf{st}(A), \mathsf{st}(B))$. In what follows, when we use a binary function W we always refer to an amalgamation closure.

Definition 6.3 ([35])**.** A theory extension $\mathcal{T} = \mathcal{T}_0 \cup \mathcal{K}$ is W-separable if for all sets of ground clauses A and B,

$$\mathcal{T}_0 \cup \mathcal{K} \cup A \cup B \models \perp \text{ iff } \mathcal{T}_0 \cup \mathcal{K}[W(A, B)] \cup A \cup \mathcal{K}[W(B, A)] \cup B \models \perp .$$

Example 6.4. Let $\mathcal{T}_0$ be the theory TOrd of total orderings[7]. We consider the extension of $\mathcal{T}_0$ with function symbols f, g satisfying the axioms $\mathcal{K} = \{\mathsf{SGC}(f, g), \mathsf{Mon}(f, g)\}$ discussed in Examples 4.14 and 6.1 (cf. also [30]), where:
- $\mathsf{SGC}(f, g) : \forall x, y(x \leq g(y) \rightarrow f(x) \leq y)$;
- $\mathsf{Mon}(f, g) : \forall x, y(x \leq y \rightarrow f(x) \leq f(y)) \wedge \forall x, y(x \leq y \rightarrow g(x) \leq g(y))$.

The theory of total orderings is $\leq$-interpolating (for details cf. [30]): If A_0 and B_0 are sets of ground clauses in the signature of TOrd and $A_0 \wedge B_0 \models_{\mathsf{TOrd}} a \leq b$, where a is a constant in A_0 and b a constant in B_0 then there exists a constant d (common to A_0 and B_0) such that $A_0 \wedge B_0 \models_{\mathsf{TOrd}} a \leq d \wedge d \leq b$. Thus, the terms needed for $\leq$-interpolation are the common constants of A and B.

If C_A (C_B) are the constants in A (B) then, from the form of the clauses in $\mathcal{K}$ and the results in [30] we can show that $\mathcal{T}_0 \cup \mathcal{K}$ is W-separable where $W(A, B) = \mathsf{st}(A) \cup \{f(c), g(c) \mid c \in C_A \cap C_B\}$ and $W(B, A) = \mathsf{st}(B) \cup \{f(c), g(c) \mid c \in C_A \cap C_B\}$.

In fact, the results in [30] show that if $A \wedge B \models \perp$, W can be defined more precisely as $W(A, B) = \mathsf{st}(A) \cup \{f(c), g(c) \mid c \in D_{AB}\}$ and $W(B, A) = \mathsf{st}(B) \cup \{f(c), g(c) \mid c \in D_{AB}\}$, where D_{AB} is the set of constants common to A and B which can be used for $\leq$-interpolation.

6.3. W**-Separability and Partial** W**-Amalgamation.** In [35] it is shown that if $\mathcal{T} = \mathcal{T}_0 \cup \mathcal{K}$ is W-separable, and $\mathcal{K}$ is flat and linear, then the extension $\mathcal{T}_0 \subseteq \mathcal{T} = \mathcal{T}_0 \cup \mathcal{K}$ is Ψ-local where $\Psi(T) = W(T, T)$ for all sets of ground terms T. Then, a notion of partial W-amalgamability is defined as follows:

[7]We chose here $\mathcal{T}_0$ to be the theory of total orderings in order to simplify the example: The signature of TOrd does not contain function symbols, so the amalgamation closure W is easier to describe.

Definition 6.5 ([35])**.** A theory extension $\mathcal{T}_0 \subseteq \mathcal{T} = \mathcal{T}_0 \cup \mathcal{K}$ is said to have the partial amalgamation property with respect to amalgamation closure W (for short: the partial W-amalgamation property) if whenever $M_A, M_B, M_C \in \mathsf{PMod}_{w,f}(\Sigma, \mathcal{T})$ are such that:

(1) M_C is a substructure of M_A and of M_B, i.e. the universe $|M_C|$ of M_C is included in the universes of M_A and M_B and the inclusions into M_A, M_B are weak embeddings;
(2) $|M_C| = |M_A| \cap |M_B|$;
(3) the sets $T_{M_A} = \{f(a_1, \dots, a_n) \mid a_1, \dots, a_n \in M_A, f_{M_A}(a_1, \dots, a_n) \text{ defined}\}$ and $T_{M_B} = \{f(a_1, \dots, a_n) \mid a_1, \dots, a_n \in M_B, f_{M_B}(a_1, \dots, a_n) \text{ defined}\}$ of terms which are defined in M_A resp. M_B are closed under the operator W, i.e. $W(T_{M_A}, T_{M_B}) \subseteq T_{M_A}$ and $W(T_{M_B}, T_{M_A}) \subseteq T_{M_B}$;
(4) $T(M_A) \cap T(M_B) \subseteq T(M_C)$;

there exists a model M_D of $\mathcal{T}$, and weak embeddings $h_A : M_A \rightarrow M_D$ and $h_B : M_B \rightarrow M_D$, such that $h_{A|_{|M_C|}} = h_{B|_{|M_C|}}$.

It is shown that if $\mathcal{T}_0 \subseteq \mathcal{T} = \mathcal{T}_0 \cup \mathcal{K}$ is a theory extension with $\mathcal{K}$ flat and linear and $\mathcal{T}_1$ has the partial W-amalgamation property w.r.t. W, then $\mathcal{T}_1$ is W-separable.

We make this last result more precise by showing that:

- In order to obtain a criterion for W-separability we only need a weak version of partial W-amalgamability, namely partial W-amalgamability for partial structures with the same Π_0-reduct (Definition 6.6).
- We then prove that also the converse holds, i.e. that for extensions $\mathcal{T} = \mathcal{T}_0 \cup \mathcal{K}$ of a first-order theory $\mathcal{T}_0$ if (i) the extension is W-separable and (ii) $\mathcal{T}_0$ allows general ground interpolation then $\mathcal{T}$ has the partial W-amalgamability property for partial structures with the same Π_0-reduct. This implication was not studied in [35].

We will then show that in general partial W-amalgamability implies partial W-amalgamability for partial structures with the same Π_0-reduct, and that the converse implication holds under the additional assumption that $\mathcal{T}_0$ allows general ground interpolation.

Definition 6.6. Let W be an amalgamation closure operator. A theory extension $\mathcal{T} = \mathcal{T}_0 \cup \mathcal{K}$ has the *partial W-amalgamation property for models with the same* Π_0*-reduct* if whenever $M_A, M_B \in \mathsf{PMod}_{w,f}(\Sigma, \mathcal{T})$ are such that:

(1) M_A, M_B have the same reduct M to Π_0;
(2) For all $m_1, \dots, m_n \in |M| = |M_A| = |M_B|$ if $f_{M_A}(m_1, \dots, m_n)$ is defined and is equal to m and $f_{M_B}(m_1, \dots, m_n)$ is defined and is equal to m' then $m = m'$;
(3) The sets $T_{M_A} = \{f(a_1, \dots, a_n) \mid a_1, \dots, a_n \in M_A, f_{M_A}(a_1, \dots, a_n) \text{ defined}\}$ and $T_{M_B} = \{f(a_1, \dots, a_n) \mid a_1, \dots, a_n \in M_B, f_{M_B}(a_1, \dots, a_n) \text{ defined}\}$ of terms which are defined in M_A resp. M_B are closed under the operator W, i.e. $W(T_{M_A}, T_{M_B}) \subseteq T_{M_A}$ and $W(T_{M_B}, T_{M_A}) \subseteq T_{M_B}$;

there exists a model M_D of $\mathcal{T}_0 \cup \mathcal{K}$ and weak embeddings $h_A : M_A \rightarrow M_D, h_B : M_B \rightarrow M_D$ which agree on M (and thus coincide as functions).[8]

[8]The partial W-amalgamation property for models with the same Π_0-reduct can also be regarded as an embeddability property for partial algebras in which the set of terms which are defined can be seen as the union of two sets T_{M_A} and T_{M_B} which are closed under the application of W. We decided to use the notion "partial W-amalgamation property for models with the same Π_0-reduct" in this paper for the sake of consistency with the terminology introduced in [33].

Definition 6.7 ([35])**.** Let M be a model of $\mathcal{T}_0 \cup \mathcal{K}$ and let T be a finite set of ground terms such that $\mathsf{st}(\mathcal{K}) \subseteq T$. We assume that the terms in T are flat (cf. Definition 4.4) or quasi-flat (i.e. for all terms of the form $f(t_1, \dots, t_n)$ of T, where f is an extension function, $t_1, \dots, t_n$ are constants or ground terms over Π_0^C).

We denote by $M_{|T}$ the partial structure which has the same support as M, and in which all symbols in Π_0 are defined as in M, but in which the interpretation of extension symbols $f \in \Sigma$ is restricted as follows: For all elements $a_1, \dots, a_n \in |M|$ if there exist terms $t_1, \dots, t_n$ such that the interpretation $(t_i)_M$ of t_i in M is a_i for all $i \in \{1, \dots, n\}$, and $f(t_1, \dots, t_n) \in T$, then $f_{M_{|T}}(a_1, \dots, a_n) = f_M(a_1, ..., a_n)$; otherwise $f_{M_{|T}}(a_1, \dots, a_n)$ is undefined.

Lemma 6.8. *Let M be a model of $\mathcal{T} = \mathcal{T}_0 \cup \mathcal{K}$ and T be a finite set of ground quasi-flat Π-terms (closed under subterms) with $\mathsf{st}(\mathcal{K}) \subseteq T$. Then the following hold:*
(1) *$M_{|T} \in \mathsf{PMod}_{w,f}(\Sigma, \mathcal{T})$.*
(2) *$M_{|T}$ is a partial model of $\mathcal{K}[T]$ in which all extension terms in $\mathcal{K}[T]$ are defined.*

Proof. (1) The proof proceeds as in [35]. Clearly, $M_{|T}$ is a total model of $\mathcal{T}_0$. Let $C \in \mathcal{K}$ and $\beta : X \to M_{|T}$. If some of the terms in $\beta(C)$ are undefined in $M_{|T}$ then by definition $M_{|T}, \beta \models_w C$. Assume now that all terms in $\beta(C)$ are defined in $M_{|T}$. By the definition of $M_{|T}$ they have the same values as in M. Therefore, as $M, \beta \models C$ it follows that $M_{|T}, \beta \models_w C$ also in this case.

(2) The only extension terms occurring in $\mathcal{K}[T]$ are those in T, and these are defined in $M_{|T}$. Let $D \in \mathcal{K}[T]$ and $\beta : X \to M_{|T}$. Then $D = C\sigma$ where $C \in \mathcal{K}$ and σ is a substitution such that for every term t occurring in C which starts with a function symbol in Σ, $t\sigma \in T$. Thus, in D all terms starting with an extension function $f \in \Sigma$ are ground terms $f(t_1, \dots t_n) \in T$. If D contains variables, these do not occur below extension functions. Thus, $\beta(D)$ is defined in $M_{|T}$. Let $\gamma : X \to M$ be $\gamma(x) = \beta(\sigma(x))$ for every $x \in X$. Then $\beta(D) = \beta(\sigma(C)) = \gamma(C)$. In $\beta(D)$, the Σ-terms are in T, hence they are defined in $M_{|T}$ and have the same value as in M. Since $M \models \mathcal{K}$, $M, \gamma \models C$, so there exists at least one literal in C which is true in M w.r.t. γ, thus there exists at least one literal in D which is true in $M_{|T}$ w.r.t. β. ☐

Remark 6.9. If we impose, in addition, that all variables in $\mathcal{K}$ occur below an extension function, then (2) is immediate: $\mathcal{K}[T]$ is ground and contains only Σ-terms in T; those terms are all defined in $M_{|T}$ and their value is the same as in M. (We do not need to consider variable assignments in that case.)

Lemma 6.10. *Let $\mathcal{T}_0 \cup \mathcal{K}$ be an extension of $\mathcal{T}_0$ with a set $\mathcal{K}$ of flat and linear clauses. Let T be a finite set of ground flat Π-terms (closed under subterms) with $\mathsf{st}(\mathcal{K}) \subseteq T$, and let M be a model of $\mathcal{T}_0 \cup \mathcal{K}[T]$. Then $M_{|T} \in \mathsf{PMod}_{w,f}(\Sigma, \mathcal{T})$.*

Proof. The proof is similar to the beginning of the proof of Theorem 2 in [28] (cf. also [35]). Clearly, $M_{|T}$ is a total model of $\mathcal{T}_0$. To show that $M_{|T} \models_w \mathcal{K}$ we use the fact that if D is a clause in $\mathcal{K}$ and $\beta : X \to M_{|T}$ is an assignment in which $\beta(t)$ is defined for every term t occurring in D, then D is true in $M_{|T}$ w.r.t. β.

For this, note that if for every term $t = f(x_1, \dots, x_n)$ of D, $\beta(t) = f_{M_{|T}}(\beta(x_1), \dots, \beta(x_n))$ is defined in $M_{|T}$ then there exist terms $t_1, \dots, t_n$ such that $(t_i)_M = \beta(x_i)$ and $f(t_1, \dots, t_n) \in T$. Let σ be the substitution with $\sigma(x_i) = t_i$ (σ is well-defined because D is flat and linear). Then $\sigma(D) \in \mathcal{K}[T]$ and $\beta(\sigma(t)) = \beta(t)$. Since M be a model of $\mathcal{T}_0 \cup \mathcal{K}[T]$, $\sigma(D)$ is true in M w.r.t. β, thus (as $\beta(\sigma(t)) = \beta(t)$ for every term in D) D is true in $M_{|T}$ w.r.t. β. ☐

Theorem 6.11. *Let W be an amalgamation closure operator with the additional property that if T_1 and T_2 are sets of flat ground terms, $W(T_1, T_2)$ is a set of flat ground terms*[9]*. Assume that $\mathcal{T}_0$ is a first-order theory and let $\mathcal{K}$ be a set of flat and linear clauses over $\Pi_0 \cup \Sigma$. If $\mathcal{T}_0 \cup \mathcal{K}$ has the partial amalgamation property for models with the same Π_0-reduct then $\mathcal{T}_0 \cup \mathcal{K}$ is W-separable.*

Proof. The proof proceeds like the proof in [35] (but we reformulated some hypotheses). Assume that $\mathcal{T}_0 \cup \mathcal{K}$ has the partial W-amalgamation property for models with the same Π_0-reduct. Let A and B be sets of ground clauses over $\Pi_0 \cup \Sigma \cup C$. Without loss of generality we assume that A and B are flat sets of ground clauses, thus $\mathsf{st}(A)$ and $\mathsf{st}(B)$ consist of flat terms only. We show that $\mathcal{T}_0 \cup \mathcal{K} \cup A \cup B$ is unsatisfiable iff $\mathcal{T}_0 \cup (\mathcal{K}[W(A,B)] \cup A) \cup (\mathcal{K}[W(B,A)] \cup B)$ is unsatisfiable.

The converse implication is obvious. We prove the direct implication. Assume that $\mathcal{T}_0 \cup \mathcal{K} \cup A \cup B$ is unsatisfiable but $\mathcal{T}_0 \cup (\mathcal{K}[W(A,B)] \cup A) \cup (\mathcal{K}[W(B,A)] \cup B)$ has a model M. We define $M_A := M_{|W(A,B)}$, $M_B := M_{|W(B,A)}$. As $\mathsf{st}(\mathcal{K}) \subseteq W(A,B) \cap W(B,A)$, and $\mathcal{K}$ is flat and linear we know, by Lemma 6.8 and Lemma 6.10, that:

(i) $M_A, M_B \in \mathsf{PMod}_{w,f}(\Sigma, \mathcal{T})$.
(ii) M_A is a model of $\mathcal{K}[W(A,B)] \cup A$ and M_B is a model of $\mathcal{K}[W(B,A)] \cup B$, and
(iii) all terms in $W(A,B)$ and A are defined in M_A, and
all terms in $W(B,A)$ and B are defined in M_B.

The models M_A and M_B satisfy the conditions in Definition 6.6:

(1) they clearly have the same reduct to Π_0 (namely $M_{|\Pi_0}$),
(2) $f_{M_A}(m_1, \dots, m_n)$ is defined and equal to m iff there exists ground Π_0-terms $t_1, \dots, t_n$ with $f(t_1, \dots, t_n) \in W(A,B)$, $(t_i)_M = m_i$ for all $i = 1, \dots, n$, and $f_M(m_1, \dots, m_n) = m$; $f_{M_B}(m_1, \dots, m_n)$ is defined and equal to m' iff there exists ground Π_0-terms $t'_1, \dots, t'_n$ with $f(t'_1, \dots, t'_n) \in W(B,A)$, $(t'_i)_M = m_i$ for all $i = 1, \dots, n$ and $f_M(m_1, \dots, m_n) = m'$.
Thus, if $f_{M_A}(m_1, \dots, m_n) = m$ and $f_{M_B}(m_1, \dots, m_n) = m'$ then $f_M(m_1, \dots, m_n) = m = m'$, so $m = m'$.
(3) $W(T_{M_A}, T_{M_B}) \subseteq T_{M_A}$ and $W(T_{M_B}, T_{M_A}) \subseteq T_{M_B}$. Indeed, let h be the map that maps all ground Π_0-terms occurring in $W(A,B)$ resp. $W(B,A)$ (which are constants if $W(A,B)$ and $W(B,A)$ are flat) to the value of these terms in M. Then:
   - $W(T_{M_A}, T_{M_B}) = W(h(W(A,B)), h(W(B,A))) \subseteq h(W(A,B)) = T_{M_A}$ and
   - $W(T_{M_B}, T_{M_A}) = W(h(W(B,A)), h(W(A,B))) \subseteq h(W(B,A)) = T_{M_B}$.

Since $\mathcal{T}_0 \cup \mathcal{K}$ has the partial amalgamation property for models with the same Π_0-reduct, there exists a model M_D of $\mathcal{T}_0 \cup \mathcal{K}$ and weak embeddings $h_A : M_A \to M_D, h_B : M_B \to M_D$ which agree on $M_{|\Pi_0}$.

Clearly, weak embeddings of a partial into a total algebra preserve the truth of ground formulae which are defined in the partial algebra. As all terms in A are defined in M_A and A is true in M_A, and as all terms in B are defined in M_B and B is true in M_B, it follows therefore that M_D is a model of A and B, hence a model of $\mathcal{T}_0 \cup \mathcal{K} \cup A \cup B$. Contradiction.

It follows that $\mathcal{T}_0 \cup (\mathcal{K}[W(A,B)] \cup A) \cup (\mathcal{K}[W(B,A)] \cup B)$ is satisfiable iff $T_0 \cup \mathcal{K} \cup A \cup B$ is satisfiable. $\square$

[9]The result holds also if $W(T_1, T_2)$ is a set of quasi-flat ground terms whenever T_1, T_2 are sets of quasi-flat ground terms, but condition (4) in the definition of an amalgamation closure needs then to be adapted.

Example 6.12. In [35] it was proved that the theory of arrays with difference function and the theory of linked lists with reachability have partial amalgamation, hence are W-separable (for suitable versions of W, described in [35]).

We now prove a converse of Theorem 6.11.

Theorem 6.13. *Let W be an amalgamation closure operator. Assume that $\mathcal{T}_0$ is a first-order theory which allows general ground interpolation and has the property that for every set Σ of additional function symbols and ground $\Sigma_0 \cup \Sigma$-formulae A, B, the interpolant I contains only Σ-terms in $W(A, B) \cap W(B, A)$. Let $\mathcal{K}$ be a set of clauses over $\Pi_0 \cup \Sigma$, such that all variables occur below an extension symbol.*

If $\mathcal{T} = \mathcal{T}_0 \cup \mathcal{K}$ is W-separable then it has the partial W-amalgamation property for models with the same Π_0-reduct.

Proof. Assume that $\mathcal{T}_0$ satisfies the assumptions of this theorem and $\mathcal{T} = \mathcal{T}_0 \cup \mathcal{K}$ is W-separable. We show that $\mathcal{T}$ has the partial W-amalgamation property for models with the same Π_0-reduct. Let $M_A, M_B \in \mathsf{PMod}_{w,f}(\Sigma, \mathcal{T})$ be such that:

(1) M_A, M_B have the same reduct M to Π_0.

(2) For all $m_1, \dots, m_n \in |M|$, if $f_{M_A}(m_1, \dots, m_n)$ and $f_{M_B}(m_1, \dots, m_n)$ are defined and $f_{M_A}(m_1, \dots, m_n) = m$ and $f_{M_B}(m_1, \dots, m_n) = m'$ then $m = m'$.

(3) The sets $T_{M_A} = \{f(a_1, \dots, a_n) \mid a_1, \dots, a_n \in M_A, f_{M_A}(a_1, \dots, a_n) \text{ defined}\}$ and $T_{M_B} = \{f(a_1, \dots, a_n) \mid a_1, \dots, a_n \in M_B, f_{M_B}(a_1, \dots, a_n) \text{ defined}\}$ of terms which are defined in M_A resp. M_B are closed under the closure operator W, i.e. $W(T_{M_A}, T_{M_B}) \subseteq T_{M_A}$ and $W(T_{M_B}, T_{M_A}) \subseteq T_{M_B}$.

In order to show that there exists a model M_D of $\mathcal{T}_0 \cup \mathcal{K}$ and weak embeddings $h_A : M_A \rightarrow M_D, h_B : M_B \rightarrow M_D$ which agree on M we show that $\mathcal{T}_0 \cup \mathcal{K} \cup \mathcal{D}_A \cup \mathcal{D}_B$ is satisfiable, where $\mathcal{D}_A$ is defined by:

$$\begin{array}{rcl} \mathcal{D}_A & = & \{f(a_1, \dots a_n) \approx a \mid \text{ if } f_{M_A}(a_1, \dots, a_n) \text{ is defined and equal to } a\} \\ & & \cup\{f(a_1, \dots a_n) \not\approx a \mid \text{ if } f_{M_A}(a_1, \dots, a_n) \text{ is defined and not equal to } a\} \\ & & \cup\{P(a_1, \dots, a_n) \mid P \in \mathsf{Pred} \text{ and } (a_1, \dots, a_n) \in P_{M_A}\} \\ & & \cup\{\neg P(a_1, \dots, a_n) \mid P \in \mathsf{Pred} \text{ and } (a_1, \dots, a_n) \notin P_{M_A}\} \\ & & \cup\{a \not\approx a' \mid a, a' \in |M_A| \text{ different }\} \end{array}$$

and $\mathcal{D}_B$ is defined analogously (the elements in the universe of M are used as additional constants).

Assume $\mathcal{T}_0 \cup \mathcal{K} \cup \mathcal{D}_A \cup \mathcal{D}_B$ is unsatisfiable. Then, by compactness, there exist finite subsets $A \subseteq \mathcal{D}_A, B \subseteq \mathcal{D}_B$ such that $\mathcal{T}_0 \cup \mathcal{K} \cup A \cup B$ is unsatisfiable. As $\mathcal{T}_1 = \mathcal{T}_0 \cup \mathcal{K}$ is W-separable, it follows that $\mathcal{T}_0 \cup (\mathcal{K}[W(A, B)] \cup A) \cup (\mathcal{K}[W(B, A)] \cup B)$ is unsatisfiable.

On the other hand, A contains only terms in T_{M_A} and B only terms in T_{M_B}. Thus, $W(A, B) \subseteq W(T_{M_A}, T_{M_B}) \subseteq T_{M_A}$ and $W(B, A) \subseteq W(T_{M_B}, T_{M_A}) \subseteq T_{M_B}$.

Therefore, all extension terms in $\mathcal{K}[W(A, B)] \cup A$ are defined in M_A and all extension terms in $\mathcal{K}[W(B, A)] \cup B$ are defined in M_B.

As $\mathcal{T}_0$ has general ground interpolation, $\mathcal{T}_0 \cup \mathsf{UIF}_\Sigma$ has ground interpolation. Thus, there exists an interpolant I for $(\mathcal{K}[W(A, B)] \cup A)$ and $(\mathcal{K}[W(B, A)] \cup B)$ w.r.t. $\mathcal{T}_0 \cup \mathsf{UIF}_\Sigma$ such that all Σ-terms in I are in $W(W(A, B), W(B, A)) \cap W(W(B, A), W(A, B))$, thus as W is a closure, in $W(A, B) \cap W(B, A)$.

As all terms in $\mathcal{K}[W(A,B)] \cup A$ and I are defined in M_A and $\mathcal{K}[W(A,B)] \cup A \models_{\mathcal{T}_0 \cup \mathsf{UIF}_\Sigma} I$, it follows that I is true in M_A. As I contains only terms that are defined in M_B and the definitions of the extension functions in M_A and M_B agree for defined terms, I is also true in M_B. On the other hand, we know that $I \cup \mathcal{K}[W(B,A)] \cup B \models_{\mathcal{T}_0 \cup \mathsf{UIF}_\Sigma} \perp$, hence $\mathcal{K}[W(B,A)] \cup B \models_{\mathcal{T}_0 \cup \mathsf{UIF}_\Sigma} \neg I$. As all terms in $\mathcal{K}[W(B,A)] \cup B$ and of I are defined in M_B, and $\mathcal{K}[W(B,A)] \cup B$ is true in M_B, $\neg I$ must be true in M_B. Contradiction.

It follows that $\mathcal{T}_0 \cup \mathcal{K} \cup \mathcal{D}_A \cup \mathcal{D}_B$ is satisfiable. Let M_D be a model for $\mathcal{T}_0 \cup \mathcal{K} \cup \mathcal{D}_A \cup \mathcal{D}_B$. Define $h_A : M_A \to M_D$ and $h_B : M_B \to M_D$ by $h_A(a) = h_B(a) = a_{M_D}$ for every $a \in M$ (where a_{M_D} is the interpretation of the constant a in M_D).

Then M_D is a model of $\mathcal{T}_0 \cup \mathcal{K}$. h_A and h_B are clearly injective: Assume that $a, a' \in M$ and $a \not\approx a'$. Then $a \not\approx a' \in \mathcal{D}_A$ so this literal is true in M_D, hence $h_A(a) = h_B(a) = a_{M_D} \neq a'_{M_D} = h_B(a') = h_A(a')$. Also, whenever $f_{M_A}(a_1, \dots, a_n)$ is defined and is equal to a, $f(a_1, \dots, a_n) \in \mathcal{D}_A$, hence $f_{M_D}(a_{1M_D}, \dots, a_{nM_D}) = a_{M_D}$. Moreover, $(a_1, \dots, a_n) \in P_{M_A}$ iff $P(a_1, \dots, a_n) \in \mathcal{D}_A$ iff $(a_{1M_D}, \dots, a_{nM_D}) \in P_{M_D}$. Thus, h_A is a weak embedding; the proof that h_B is a weak embedding is similar. These embeddings clearly agree on M. □

We now show that in general partial W-amalgamation implies partial W-amalgamation for models with the same Π_0-reduct, and that if $\mathcal{T}_0$ allows general ground interpolation the two notions are equivalent.

Proposition 6.14. *Let W be an amalgamation closure operator. Let $\mathcal{K}$ be a set of clauses over $\Pi_0 \cup \Sigma$. The following hold:*

(1) *If $\mathcal{T} = \mathcal{T}_0 \cup \mathcal{K}$ has the partial W-amalgamation property then it has the partial W-amalgamation property for models with the same Π_0-reduct.*

(2) *If $\mathcal{T}_0$ is a first-order theory which allows general ground interpolation and $\mathcal{T} = \mathcal{T}_0 \cup \mathcal{K}$ has the partial W-amalgamation property for models with the same Π_0-reduct then $\mathcal{T} = \mathcal{T}_0 \cup \mathcal{K}$ has the partial W-amalgamation property.*

Proof. (1) Assume that $\mathcal{T} = \mathcal{T}_0 \cup \mathcal{K}$ has the partial W-amalgamation property. Let $M_A, M_B \in \mathsf{PMod}_{w,f}(\Sigma, \mathcal{T})$ having the same reduct M to Π_0 and such that:

(i) for all $m_1, \dots, m_n \in |M|$, if $f_{M_A}(m_1, \dots, m_n)$ and $f_{M_B}(m_1, \dots, m_n)$ are both defined and $f_{M_A}(m_1, \dots, m_n) = m$ and $f_{M_B}(m_1, \dots, m_n) = m'$ then $m = m'$, and

(ii) the sets T_{M_A} and T_{M_B} of terms defined in M_A resp. M_B are closed under W.

Let M_C be the partial structure with the same reduct M to Π_0 as M_A and M_B, and where for every $f \in \Sigma$ with arity n, and every $m_1, \dots, m_n \in |M|$, $f_{M_C}(m_1, \dots, m_n)$ is defined iff $f_{M_A}(m_1, \dots, m_n)$ and $f_{M_B}(m_1, \dots, m_n)$ are both defined, and if so $f_{M_C}(m_1, \dots, m_n) = f_{M_A}(m_1, \dots, m_n) = f_{M_B}(m_1, \dots, m_n)$. Clearly, the conditions in Definition 6.5 are satisfied:

- The universe $|M_C|$ of M_C is included in the universes of M_A and M_B and the inclusions into M_A, M_B are weak embeddings;
- $|M_C| = |M| = |M| \cap |M| = |M_A| \cap |M_B|$;
- The sets T_{M_A} and T_{M_B} of terms which are defined in M_A resp. M_B are closed under W.
- $T(M_A) \cap T(M_B) \subseteq T(M_C)$: If $f(a_1, \dots, a_n) \in T(M_A) \cap T(M_B)$ then $f_{M_A}(a_1, \dots, a_n)$ is defined and $f_{M_B}(a_1, \dots, a_n)$ is defined (and they are equal by (i)), so by the definition of M_C $f_{M_C}(a_1, \dots, a_n)$ is also defined.

It follows that there exists a model M_D of $\mathcal{T}$, and weak embeddings $h_A : M_A \to M_D$ and $h_B : M_B \to M_D$, such that $h_{A|_{|M_C|}} = h_{B|_{|M_C|}}$, i.e. h_A and h_B coincide on M.

(2) Assume that $\mathcal{T}_0$ allows general ground interpolation and $\mathcal{T} = \mathcal{T}_0 \cup \mathcal{K}$ has the partial W-amalgamation property for models with the same Π_0-reduct. Let $M_A, M_B, M_C \in \mathsf{PMod}_{w,f}(\Sigma, \mathcal{T})$ satisfying the conditions from Definition 6.5. Then $M_{C|_{\Pi_0}}$ is a substructure of $M_{A|_{\Pi_0}}$ and of $M_{B|_{\Pi_0}}$.

By Theorem 3.10 (cf. also [2]), $\mathcal{T}_0$ allows general ground interpolation iff $\mathcal{T}_0$ is strongly sub-amalgamable (cf. Definition 3.8). Therefore there exists a further model M of $\mathcal{T}_0$ and embeddings $\mu_1 : M_{A|_{\Pi_0}} \to M$ and $\mu_2 : M_{B|_{\Pi_0}} \to M$ whose restrictions to $M_{C|_{\Pi_0}}$ coincide, such that if $\mu_1(m_1) = \mu_2(m_2)$ then there exists $m \in |M_{C|_{\Pi_0}}|$ with $m = m_1 = m_2$.

We use the embeddings μ_1 and μ_2 to construct two partial algebras M'_A and M'_B as follows: M'_A has universe $|M|$, all Π_0 operations are defined as in M, and for every $f \in \Sigma$ with arity m and every $m_1, \dots, m_n \in |M|$, $f_{M'_A}(m_1, \dots, m_n)$ is defined iff there exist $\overline{m}_1, \dots, \overline{m}_n \in |M_A|$ such that $\mu_1(\overline{m}_i) = m_i$ and $f_{M_A}(\overline{m}_1, \dots, \overline{m}_n)$ is defined. If this is the case, then $f_{M'_A}(m_1, \dots, m_n) = \mu_1(f_A(\overline{m}_1, \dots, \overline{m}_n))$. M'_B is defined analogously. We show that M'_A, M'_B satisfy the conditions from Definition 6.6.

- M'_A, M'_B have the same reduct M to Π_0,
- For $m_1, \dots, m_n \in |M|$, if $f_{M'_A}(m_1, \dots, m_n)$ is defined and equal to m and $f_{M'_B}(m_1, \dots, m_n)$ is defined and equal to m' then $m = m'$.
  Indeed, if $f_{M'_A}(m_1, \dots, m_n)$ is defined then there exist $\overline{m}_1, \dots, \overline{m}_n \in |M_A|$ with $\mu_1(\overline{m}_i) = m_i$ such that $f_{M_A}(\overline{m}_1, \dots, \overline{m}_n)$ is defined and $f_{M'_A}(m_1, \dots, m_n) = \mu_1(f_{M_A}(\overline{m}_1, \dots, \overline{m}_n))$. If $f_{M'_B}(m_1, \dots, m_n)$ is defined then there exist $\overline{\overline{m}}_1, \dots, \overline{\overline{m}}_n \in |M_B|$ with $\mu_2(\overline{\overline{m}}_i) = m_i$ such that $f_{M_B}(\overline{\overline{m}}_1, \dots, \overline{\overline{m}}_n)$ is defined and $f_{M'_B}(m_1, \dots, m_n) = \mu_2(f_{M_B}(\overline{\overline{m}}_1, \dots, \overline{\overline{m}}_n))$. Since $\mu_1(\overline{m}_i) = \mu_2(\overline{\overline{m}}_i) = m_i$, there exists $m'_i \in |M_C|$ such that $m'_i = \overline{m}_i = \overline{\overline{m}}_i$.
  As $f_{M_A}(\overline{m}_1, \dots, \overline{m}_n)$ and $f_{M_B}(\overline{\overline{m}}_1, \dots, \overline{\overline{m}}_n)$ are defined, $f(\overline{m}_1, \dots, \overline{m}_n) \in T_{M_A}$ and $f(\overline{\overline{m}}_1, \dots, \overline{\overline{m}}_n) \in T_{M_B}$. Since for every i, $m'_i = \overline{m}_i = \overline{\overline{m}}_i$ and we assumed that $T(M_A) \cap T(M_B) \subseteq T(M_C)$, it follows that $f(m'_1, \dots, m'_n) \in T(M_C)$, so $f_{M_C}(m'_1, \dots, m'_n)$ is defined. Since M_C is a weak substructure of M_A, M_B it follows that $f_{M_A}(m'_1, \dots, m'_n) = f_{M_C}(m'_1, \dots, m'_n) = f_{M_B}(m'_1, \dots, m'_n)$, and therefore (since μ_1 and μ_2 agree on M_C) we have: $m = \mu_1(f_{M_A}(m'_1, \dots, m'_n)) = \mu_1(f_{M_C}(m'_1, \dots, m'_n)) = \mu_2(f_{M_C}(m'_1, \dots, m'_n)) = \mu_2(f_{M_B}(m'_1, \dots, m'_n)) = m'$.
- Up to renaming of constants h defined using μ_1 and μ_2: $T_{M'_A} = h(T_{M_A})$, $T_{M'_B} = h(T_{M_B})$, so $T_{M'_A}$ and $T_{M'_B}$ are closed under W.

It follows that there exists a model M_D of $\mathcal{T}_0 \cup \mathcal{K}$ and weak embeddings $h'_A : M'_A \to M_D, h'_B : M'_B \to M_D$ which agree on M. The maps $h_A = \mu_1 \circ h'_A : M_A \to M_D$ and $h_B = \mu_2 \circ h'_B : M_B \to M_D$ are then weak embeddings such that $h_{A|_{M_C}} = h_{B|_{M_C}}$, thus $\mathcal{T}_0 \cup \mathcal{K}$ has the partial W-amalgamation property. □

6.4. **Separability, Locality and Interpolant Computation.** If the extension $\mathcal{T}_0 \subseteq \mathcal{T}_0 \cup \mathcal{K}$ is W-separable and $\mathcal{T}_0$ has ground interpolation, then we can hierarchically compute interpolants in $\mathcal{T}_0 \subseteq \mathcal{T}_0 \cup \mathcal{K}$ (cf. also [36]).

Theorem 6.15. *Let W be an amalgamation closure operator. Assume that the theory $\mathcal{T}_0$ has general ground interpolation, and there is a method for effectively computing general ground interpolants w.r.t. $\mathcal{T}_0$. Let $\mathcal{T}_0 \cup \mathcal{K}$ be a W-separable extension of $\mathcal{T}_0$ with a set of clauses $\mathcal{K}$ in which every variable occurs below an extension function. Let A and B be two ground $\Sigma_0 \cup \Sigma$-formulae. Assume that $A \wedge B \models_{\mathcal{T}_0 \cup \mathcal{K}} \perp$. Then we can effectively compute* *a ground interpolant for A and B, by computing an interpolant of $\mathcal{K}[W(A,B)] \cup A$ and $\mathcal{K}[W(B,A)] \cup B$.*

Proof. By W-separability $\mathcal{T}_0 \cup \mathcal{K} \cup (A \wedge B) \models \perp$ iff $\mathcal{T}_0 \cup \mathcal{K}[W(A,B)] \cup A \cup \mathcal{K}[W(A,B)] \cup B \models \perp$. As every variable occurs in $\mathcal{K}$ below an extension function, $\mathcal{K}[W(A,B)] \cup A \cup \mathcal{K}[W(B,A)] \cup B$ is a set of ground formulae.

We can use the method for computing general ground interpolants in $\mathcal{T}_0$ for computing the interpolant I_0, which is an interpolant for A and B w.r.t. $\mathcal{T}_0 \cup \mathcal{K}$. □

Corollary 6.16. *Let W be an amalgamation closure operator, and let $\mathcal{T}_0 \cup \mathcal{K}$ be a W-separable extension of $\mathcal{T}_0$ with a set of clauses $\mathcal{K}$ in which every variable occurs below an extension function. Then $\mathcal{T}_0 \cup \mathcal{K}$ has ground interpolation in each of the following cases:*
(1) *$\mathcal{T}_0$ has ground interpolation and is equality interpolating.*
(2) *$\mathcal{T}_0$ allows quantifier elimination and is equality interpolating.*
(3) *$\mathcal{T}_0$ is universal and allows quantifier elimination.*

Proof. (1) If $\mathcal{T}_0$ has ground interpolation and is equality interpolating, then by Theorem 3.10(3) and (2), the extension of $\mathcal{T}_0$ with uninterpreted function symbols in Σ has ground interpolation.
(2) If $\mathcal{T}_0$ allows quantifier elimination then it has ground interpolation, so (1) can be used.
(3) follows from (2) and Theorem 3.10(4). □

6.5. **Ground Interpolation and Model Completions.** It is sometimes difficult to check directly whether the theory $\mathcal{T}_0$ has ground interpolation. If $\mathcal{T}_0$ has a model completion with good properties, this becomes easier to check. In this case, we can use quantifier elimination in the model completion to compute the interpolant.

Theorem 6.17. *Let W be an amalgamation closure operator, and let $\mathcal{T}_0 \cup \mathcal{K}$ be a W-separable extension of $\mathcal{T}_0$ with a set of clauses $\mathcal{K}$ in which every variable occurs below an extension function.*

Assume that $\mathcal{T}_0$ has a model companion $\mathcal{T}_0^*$ with the following properties:
(1) *$\mathcal{T}_0 \subseteq \mathcal{T}_0^*$;*
(2) *$\mathcal{T}_0^*$ has general ground interpolation. (This can happen for instance when $\mathcal{T}_0^*$ allows quantifier elimination and is equality interpolating.)*

Then $\mathcal{T}_0 \cup \mathcal{K}$ has ground interpolation.

Proof. Assume that $\mathcal{T}_0 \cup \mathcal{K} \cup (A \wedge B) \models \perp$. By W-separability $\mathcal{T}_0 \cup \mathcal{K} \cup (A \wedge B) \models \perp$ iff $\mathcal{T}_0 \cup \mathcal{K}[W(A,B)] \cup A \cup \mathcal{K}[W(A,B)] \cup B \models \perp$. As every variable occurs in $\mathcal{K}$ below an extension function, $\mathcal{K}[W(A,B)] \cup A \cup \mathcal{K}[W(A,B)] \cup B$ is a set of ground formulae.

Condition (1) implies that every model of $\mathcal{T}_0^* \cup \mathsf{UIF}_\Sigma$ is a model of $\mathcal{T}_0 \cup \mathsf{UIF}_\Sigma$.

From the assumption that $\mathcal{T}_0^*$ is a model companion of $\mathcal{T}_0$ we know that every model of $\mathcal{T}_0$ embeds into a model of $\mathcal{T}_0^*$. This implies that every model of $\mathcal{T}_0 \cup \mathsf{UIF}_\Sigma$ embeds into a model of $\mathcal{T}_0^* \cup \mathsf{UIF}_\Sigma$. Indeed, let $\mathcal{A}$ be a model of $\mathcal{T}_0 \cup \mathsf{UIF}_\Sigma$. Then $\mathcal{A}_{|\Pi_0}$ is a model of $\mathcal{T}_0$ thus it embeds into a model $\mathcal{B}$ of $\mathcal{T}_0^*$. We define a partial Σ-structure P, having as Π_0-reduct $\mathcal{B}$, and such that for every $f \in \Sigma$ with arity n, $f_P(a_1, \dots, a_n)$ is defined iff $a_1, \dots, a_n \in |\mathcal{A}|$, and if so $f_P(a_1, \dots, a_n) = f_{\mathcal{A}}(a_1, \dots, a_n)$. P can be transformed into a total algebra $\mathcal{P}$ by selecting an element c in its support and setting the values of all function symbols in Σ to c

if they are undefined in P. $\mathcal{P}$ has the same Π_0-reduct as P, namely $\mathcal{B}$, and is thus a model of $\mathcal{T}_0^* \cup \mathsf{UIF}_\Sigma$.

It therefore follows that $\mathcal{T}_0 \cup \mathsf{UIF}_\Sigma$ and $\mathcal{T}_0^* \cup \mathsf{UIF}_\Sigma$ are co-theories, so, by Lemma 2.15, $\mathcal{T}_0^* \cup \mathcal{K}[W(A,B)] \cup A \cup \mathcal{K}[W(A,B)] \cup B \models \perp$. As $\mathcal{T}_0^*$ has general ground interpolation, we know that there exists a ground formula I containing only the common constants and extension functions of $\mathcal{K}[W(A,B)] \cup A$ and $\mathcal{K}[W(A,B)] \cup B$ such that:

(a) $\mathcal{K}[W(A,B)] \cup A \cup \neg I \models_{\mathcal{T}_0^* \cup \mathsf{UIF}_\Sigma} \perp$, hence by Lemma 2.15, $\mathcal{K}[W(A,B)] \cup A \cup \neg I \models_{\mathcal{T}_0 \cup \mathsf{UIF}_\Sigma} \perp$;
(b) $\mathcal{K}[W(B,A)] \cup B \cup I \models_{\mathcal{T}_0^* \cup \mathsf{UIF}_\Sigma} \perp$, hence by Lemma 2.15, $\mathcal{K}[W(B,A)] \cup B \cup I \models_{\mathcal{T}_0 \cup \mathsf{UIF}_\Sigma} \perp$.

Thus, I is an interpolant w.r.t. $\mathcal{T}_0 \cup \mathsf{UIF}_\Sigma$, hence, as can be seen from the proof of Theorem 6.15, it is an interpolant of A and B w.r.t. $\mathcal{T}_0 \cup \mathcal{K}$. $\square$

Example 6.18. Consider the theory extension $\mathcal{T}_0 \cup \mathcal{K}$ in Examples 4.14, 6.1 and 6.4, where $\mathcal{T}_0 = \mathsf{TOrd}$ is the theory of total orderings and $\mathcal{K} = \{\mathsf{SGC}(f,g), \mathsf{Mon}(f,g)\}$. Let A and B be as in Example 4.14:

$$A:\ d \leq g(a)\ \wedge\ a \leq c \qquad B:\ b \leq d\ \wedge\ f(b) \not\leq c.$$

We already proved that $A \wedge B \models_{\mathcal{T}_0 \cup \mathcal{K}} \perp$, by using hierarchical reasoning in local theory extensions; after instantiation and purification we obtained:

Extension	Base	
$D_A \wedge D_B$	$A_0 \wedge B_0 \wedge \mathsf{SGc}_0 \wedge \mathsf{Mon}_0$	$\wedge \mathsf{Con}_0$
$a_1 \approx g(a)$	$A_0 = d \leq a_1 \wedge a \leq c$	$\mathsf{SGc}_0 = b \leq a_1 \rightarrow b_1 \leq a$
$b_1 \approx f(b)$	$B_0 = b \leq d \wedge c < b_1$	$\mathsf{Con}_A \wedge \mathsf{Mon}_A = a \lhd a \rightarrow a_1 \lhd a_1, \lhd \in \{\approx, \leq\}$
		$\mathsf{Con}_B \wedge \mathsf{Mon}_B = b \lhd b \rightarrow b_1 \lhd b_1,\ \lhd \in \{\approx, \leq\}$

As mentioned before, $A_0 \wedge B_0 \models b \leq a_1$ and, in fact: $A_0 \wedge B_0 \models b \leq d \wedge d \leq a_1$ (d is a shared constant). After separation and purification of the newly introduced instances of the axioms SGc and Mon (using d_1 for $f(d)$) as explained in [30] we obtain:

- $\overline{A}_0 = d \leq a_1 \wedge a \leq c \wedge (d \leq a_1 \rightarrow d_1 \leq a)$ equiv. to $(d \leq a_1 \wedge a \leq c \wedge d_1 \leq a)$
- $\overline{B}_0 = b \leq d \wedge c < b_1 \wedge (b \leq d \rightarrow b_1 \leq d_1)$ equiv. to $(b \leq d \wedge b_1 \not\leq c \wedge b_1 \leq d_1)$

We can use a method for ground interpolation in the theory of total orderings to obtain a ground interpolant I_0. However, it might be more efficient to do so by using quantifier elimination in the model completion of $\mathcal{T}_0$ (the theory of dense total orderings without endpoints) to eliminate the constants a, a_1 from $\overline{A}_0$. We can eliminate quantifiers as follows:

$$\begin{array}{rcl} \exists a \exists a_1\, (d \leq a_1 \wedge a \leq c \wedge (d \leq a_1 \rightarrow d_1 \leq a)) & \equiv & \exists a \exists a_1\, (d \leq a_1 \wedge a \leq c \wedge d_1 \leq a) \\ & \equiv & \exists a\, (a \leq c \wedge d_1 \leq a) \\ & \equiv & d_1 \leq c. \end{array}$$

We obtain the interpolant $I_0 = d_1 \leq c$. Since d_1 is used as an abbreviation for $f(d)$, we replace it back and obtain the interpolant $I = f(d) \leq c$.

A similar result can also be obtained using W-separability with the second of the amalgamation operators defined in Example 6.4: If $D_{AB} = \{d\}$ is the set of constants common to A and B which can be used for $\leq$-interpolation then an amalgamation closure which can be used is $W(A,B) = \mathsf{st}(A) \cup \{f(e), g(e) \mid e \in D_{AB}\} = \{a, c, d, g(a)\} \cup \{f(d), g(d)\}$ and $W(B,A) = \mathsf{st}(B) \cup \{f(e), g(e) \mid e \in D_{AB}\} = \{b, c, d, f(b)\} \cup \{f(d), g(d)\}$. The results in [30] show that the smaller sets $W'(A,B) = \{a, c, d, g(a)\} \cup \{f(d)\}$ and $W'(B,A) =$

$\{b, c, d, f(b)\} \cup \{f(d)\}$ are sufficient for this example. After instantiation and purification we obtain the conjunctions $\overline{A}_0$ and $\overline{B}_0$ of ground clauses we considered above.

6.6. **Symbol Elimination and Interpolation.** For W-separable theories we can use the method for symbol elimination in Section 5 for computing interpolants. If $\mathcal{T}_0 \cup \mathcal{K}[W(A, B)] \cup A \cup \mathcal{K}[W(B, A)] \cup B \models \perp$, the formula Γ_2 obtained using Steps 1–4 in Section 5 for $\mathcal{T}_0 \cup \mathcal{K}[W(A, B)] \cup A$ (with Σ_P consisting of the common constants) is an interpolant.

Theorem 6.19. *If $\mathcal{T}_0 \cup \mathcal{K}[W(A, B)] \cup A \cup \mathcal{K}[W(B, A)] \cup B \models \perp$, the formula Γ_2 obtained using Steps 1–4 in Sect. 5 for $\mathcal{T}_0 \cup \mathcal{K}[W(A, B)] \cup A$ (with Σ_P consisting of the common extension functions and constants) is an interpolant of A and B w.r.t. $\mathcal{T}_0 \cup \mathcal{K}$.*

Proof. By Lemma 5.2, we know that the formulae $\exists \overline{x} G_1(\overline{c}_p, \overline{c}_f, \overline{x}) \wedge \mathsf{Def}$ and $\Gamma_2(\overline{c}_p)$ are equivalent w.r.t. $\mathcal{T}_0 \cup \mathsf{UIF}_\Sigma$. In other words, for every model $\mathcal{A}$ of $\mathcal{T}_0 \cup \mathsf{UIF}_\Sigma$, $\mathcal{A} \models \Gamma_2(\overline{c}_p)$ if and only if its extension $\mathcal{A}^{\overline{c}_f}$ with the new constants $\overline{c}_f$ with interpretations defined such that Def are true, is a model of $\exists \overline{x} G_1(\overline{c}_p, \overline{c}_f, \overline{x})$.

We first prove that $A \models_{\mathcal{T}_0 \cup \mathcal{K}} \Gamma_2(\overline{c}_p)$. Let $\mathcal{A}$ be a model of $\mathcal{T}_0 \cup \mathcal{K} \cup A$. Then $\mathcal{A}$ is a model of $\mathcal{T}_0 \cup \mathcal{K}[W(A, B)] \cup A$, so its extension $\mathcal{A}^{\overline{c}_f}$ with the new constants $\overline{c}_f$ with interpretations defined such that Def are true is a model of $\mathcal{T}_0 \cup \mathcal{K}[W(A, B)])_0 \cup A_0 \cup \mathsf{Def}$. It follows therefore that $\mathcal{A}^{\overline{c}_f}$ is a model of $\exists \overline{x} G_1(\overline{c}_p, \overline{c}_f, \overline{x}) \cup \mathsf{Def}$. By Lemma 5.2, $\mathcal{A}$ is then a model of $\Gamma_2(\overline{c}_p)$.

We now show that $\Gamma_2(\overline{c}_p) \cup B \models_{\mathcal{T}_0 \cup \mathcal{K}} \perp$. Assume that this is not the case, i.e. there exists a structure $\mathcal{A}$ which is a model of $\mathcal{T}_0 \cup \mathcal{K}$, of B and of $\Gamma_2(\overline{c}_p)$. Then:

- $\mathcal{A} \models \mathcal{K}[W(B, A)] \cup B$.
- As $\mathcal{A}$ is a model of $\Gamma_2(\overline{c}_p)$, its extension $\mathcal{A}^{\overline{c}_f}$ with the new constants $\overline{c}_f$ with interpretations defined such that Def are true is a model of $\exists \overline{x} G_1(\overline{c}_p, \overline{c}_f, \overline{x})$. Thus, there exists $\beta : X \to |\mathcal{A}|$ such that $A, \beta \models G_1(\overline{c}_p, \overline{c}_f, \overline{x})$.

  Let $G_1(\overline{c}_p, \overline{c}_f, \overline{c})$ be the ground formula obtained from $G_1(\overline{c}_p, \overline{c}_f, \overline{x})$ by replacing every variable x_i in $\overline{x} = x_1, \dots, x_n$ with the constant c_i in $\overline{c} = c_1, \dots, c_n$. Let $\mathcal{A}^{\overline{c}_f, \overline{c}}$ be the structure that coincides with $\mathcal{A}^{\overline{c}_f}$, except for the values of the constants in $\overline{c}$ which are given by the values of the variables $\overline{x}$ w.r.t. β. Then $\mathcal{A}^{\overline{c}_f, \overline{c}} \models G_1(\overline{c}_p, \overline{c}_f, \overline{c})$.

  From the definition of G_1, $\mathcal{A}^{\overline{c}_f, \overline{c}} \models \mathcal{T}_0 \cup \mathcal{K}[W(A, B)])_0 \cup A_0 \cup \mathsf{Def}$, hence
  $\mathcal{A}^{\overline{c}} \models \mathcal{T}_0 \cup \mathcal{K}[W(A, B)] \cup A$.
- As the constants $\overline{c}$ do not occur in B or $\mathcal{K}[W(B, A)]$, $\mathcal{A}^{\overline{c}} \models \mathcal{K}[W(B, A)] \cup B$.

It follows that $\mathcal{K}[W(A, B)] \cup A \cup \mathcal{K}[W(B, A)] \cup B$ is satisfiable w.r.t. $\mathcal{T}_0 \cup \mathsf{UIF}_\Sigma$. Contradiction. This shows that $\Gamma_2(\overline{c}_p) \wedge B \models_{\mathcal{T}_0 \cup \mathcal{K}} \perp$. □

Example 6.20. Consider the theory $\mathcal{T}_0 \cup \mathcal{K}$ in Example 5.6: $\mathcal{T}_0$ is the theory of dense total orderings without endpoints; we consider its extension with functions $\Sigma_1 = \{f, g, h, c\}$ whose properties are axiomatized by

$$\mathcal{K} := \{ \quad \forall x (x \leq c \rightarrow g(x) \approx f(x)), \qquad \forall x (c < x \rightarrow g(x) \approx h(x)) \quad \}.$$

Let A and B be the formulae:

- $A := \{c_1 \leq c_2, \quad g(c_1) \approx a_1, \quad g(c_2) \approx a_2, a_1 > a_2\}$
- $B := \{c_1 \leq c < c_2, \quad f(c_1) \approx b_1, \quad h(c_2) \approx b_2, \quad b_1 \leq b_2\}$.

It is easy to check that $\mathcal{T}_0 \cup \mathcal{K} \cup A \cup B \models \perp$. The common symbols of A and B are c_1 and c_2, as well the constant c which is part of the signature of the theory $\mathcal{T}_0 \cup \mathcal{K}$. All function symbols f, g, h can be considered to be shared because they are used together in the axioms in $\mathcal{K}$. An interpolant of A and B cannot contain the constants a_1, a_2, b_1, b_2.

We can compute an interpolant by eliminating the symbols a_1, a_2 from A with the method described in Section 5. Note that the formula A coincides with the set obtained from the family G of ground clauses considered in Example 5.6 after introducing new constants a_1, a_2 for the extension terms $g(c_1), g(c_2)$. We apply Steps 1–4 to this formula. Let Γ_2 be the formula computed in Step 4 in Example 5.6, namely:

$$\begin{array}{rl} \Gamma_2 = (& (c_1 < c_2 \leq c \ \wedge \ f(c_1) > f(c_2)) \vee \\ & (c_1 \leq c < c_2 \ \wedge \ f(c_1) > h(c_2)) \vee \\ & (c < c_1 < c_2 \ \wedge \ h(c_1) > h(c_2)) \quad) \end{array}$$

By Theorem 6.19, this formula is an interpolant of A and B.

7. Conclusions, Summary of Results

In this paper we studied several problems related to symbol elimination and ground interpolation in theories and theory extensions. We here briefly summarize these results, then discuss some directions in which we would like to extend them.

7.1. **Amalgamation, Ground Interpolation, Quantifier Elimination.** It is well-known that if a theory has quantifier elimination then this can be used for symbol elimination and also for computing ground interpolants of ground formulae. However, the great majority of logical theories do not have quantifier elimination. We showed that if a theory $\mathcal{T}$ has a model completion $\mathcal{T}^*$, then interpolants computed w.r.t. $\mathcal{T}^*$ are also interpolants w.r.t. $\mathcal{T}$. As there are many examples of model completions of theories $\mathcal{T}$ which allow quantifier elimination, this can be used for computing interpolants w.r.t. $\mathcal{T}$.

The links between the different notions amalgamation, quantifier-free interpolation, quantifier elimination and the quality of being equality interpolating are summarized below:

$$\begin{tikzcd}
\mathcal{T}\text{ AP} \arrow[r, Leftrightarrow, "\mathcal{T}\text{universal}", "\text{Thm.3.5 [1]}"'] & \mathcal{T}\text{ QF-Int} \arrow[r, Leftrightarrow, "\text{Lem. 3.6}"'] \arrow[dl, Leftrightarrow, "\text{Thm.3.10(1), [2]}"'] & \mathcal{T}_\forall\text{ QF-Int} \arrow[r, Leftrightarrow, "\text{Thm.3.5}"'] & \mathcal{T}_\forall\text{ AP} \arrow[d, Leftrightarrow] \\
\mathcal{T}\text{ subAP} \arrow[u, Rightarrow] & \mathcal{T}\text{ GQF-Int} \arrow[u, Rightarrow] & \mathcal{T}\text{ QE} \arrow[l, Rightarrow, "\mathcal{T}\text{univ.}"', "\text{Thm.3.14}"] \arrow[ul, Rightarrow] \arrow[dl, Rightarrow, "\mathcal{T}\text{univ.}"', "\text{Thm.3.10(4), [2]}"] & \mathcal{T}_\forall\text{ subAP} \\
\mathcal{T}\text{ strong subAP} \arrow[u, Rightarrow] \arrow[ur, Leftrightarrow, "\text{Thm.3.10(2), [2]}"] \arrow[r, Leftrightarrow, "\text{QF-Int}", "\text{Thm.3.10(2), [2]}"'] & \mathcal{T}\text{ EQ-Int} & &
\end{tikzcd}$$

We used the following abbreviations:

- AP: amalganation property;
- (strong) subAP: (strong) sub-amalgamation property;
- (G)QF-Int: (general) quantifier free interpolation;
- EQ-Int: Equality Interpolating

Let $\mathcal{T}$ be a theory and $\mathcal{T}^*$ a model companion of $\mathcal{T}$. The links between the properties of $\mathcal{T}^*$ and amalgamation in $\mathcal{T}$ and the links between ground interpolation in $\mathcal{T}$ and $\mathcal{T}^*$ are summarized below:

$$\begin{tikzcd}
\mathcal{T}^* \text{ model completion of } \mathcal{T} \arrow[r, Leftrightarrow, "\text{Thm.3.11 [3]}"'] & \mathcal{T} \text{ has AP} \arrow[d, Leftrightarrow, "\mathcal{T}\text{universal}"', "\text{Thm.3.11 [3]}"] & \\
\mathcal{T} \text{ has QF-Int} \arrow[r, Rightarrow, "\mathcal{T}\text{universal}", "\text{Thm.3.16}"'] & \mathcal{T}^* \text{ has QE} \arrow[ul, Rightarrow] \arrow[r, Leftrightarrow, "\text{Thm.3.12}"'] \arrow[d, Rightarrow] & \mathcal{T}_\forall \text{ has AP} \\
\mathcal{T} \text{ has QF-Int} \arrow[r, Leftrightarrow, "\mathcal{T}\text{ universal}", "\text{Cor.3.18}"'] & \mathcal{T}^* \text{ has QF-Int} \arrow[ul, Rightarrow, "\text{Thm.3.17}"'] &
\end{tikzcd}$$

7.2. **Symbol Elimination in Theory Extensions.** We analyzed how this approach can be lifted to *extensions* of a theory $\mathcal{T}$, by identifying situations in which we can use existing methods for symbol elimination in $\mathcal{T}$ for symbol elimination or for ground interpolation in the extension. If $\mathcal{T}$ has a model completion $\mathcal{T}^*$, we analyzed under which conditions we can use possibilities of symbol elimination in $\mathcal{T}^*$ for such tasks.

The results we obtained are schematically presented below: Assume that $\mathcal{T}_0$ is a theory and $\mathcal{T} = \mathcal{T}_0 \cup \mathcal{K}$ is an extension of $\mathcal{T}_0$ with additional function symbols, whose properties are axiomatized by a set $\mathcal{K}$ of clauses, and $\mathcal{T}_0^*$ a model completion of $\mathcal{T}_0$ (when applicable).

Condition	Symbol Elimination
$\mathcal{T}_0$ allows quantifier elimination	For every set G of clauses and every set T of terms there exists a universal formula $\forall y \Gamma_T(y)$ s.t. (*) $\mathcal{T}_0 \cup \mathcal{K} \cup \forall y \Gamma_T(y) \models \neg G$.
+ $\mathcal{T}_0 \subseteq \mathcal{T}_0 \cup \mathcal{K}$ sat. Comp_f + $\mathcal{K}$ flat & linear	$\forall y \Gamma_T(y)$ weakest universal constraint with (*)
$\mathcal{T}_0$ does not allow quantifier elimination + $\mathcal{T}_0^*$ allows quantifier elimination; + every model of $\mathcal{T}_0 \cup \mathcal{K}$ embeds into one of $\mathcal{T}_0^* \cup \mathcal{K}$	$\forall y \Gamma_T(y)$ satisfying (*) can be obtained using quantifier elimination in $\mathcal{T}_0^*$

7.3. **Separability, Amalgamation and Ground Interpolation.** In the study of ground interpolation in extensions $\mathcal{T} \cup \mathcal{K}$ of a theory $\mathcal{T}$ with a set of clauses $\mathcal{K}$ we followed an approach proposed in [35, 36], in which the terms needed to separate the instances of $\mathcal{K}$ are considered explicitly. Our analysis extends both the results in [30] and those in [35] mainly by avoiding the restriction to convex base theories (in [36] the formulation is more general) and by identifying conditions under which W-separability implies an amalgamation property. In addition, when formulating our theorems we explicitly pointed out all conditions needed for hierarchical interpolation which were missing or only implicit in [35].

7.4. **Future Work.** The results we established in this paper have direct applicability to the verification of parametric systems. In the future we plan to further analyze such situations. The results about the links between separability, amalgamation and ground interpolation we established in Theorem 6.13 use the fact that we assume that the sets of terms which need to be used in the separations, for equality interpolation, and in the interpolants themselves can be described using a closure operator. We would like to obtain criteria that guarantee the existence of interpolants containing terms that can be described using such operators. In future work we would like to also extend the approach to interpolation and symbol elimination described here such that it can be used for the study of *uniform interpolation* in logical theories and theory extensions.

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