

EQUATIONAL REASONINGS IN WIRELESS NETWORK GOSSIP PROTOCOLS *

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ABSTRACT. *Gossip protocols* have been proposed as a robust and efficient method for disseminating information throughout large-scale networks. In this paper, we propose a compositional analysis technique to study formal probabilistic models of gossip protocols expressed in a simple probabilistic timed process calculus for *wireless sensor networks*. We equip the calculus with a *simulation theory* to compare probabilistic protocols that have similar behaviour up to a certain tolerance. The theory is used to prove a number of algebraic laws which revealed to be very effective to estimate the performances of *gossip networks*, with and without *communication collisions*, and *randomised gossip networks*. Our simulation theory is an asymmetric variant of the *weak bisimulation metric* that maintains most of the properties of the original definition. However, our asymmetric version is particularly suitable to reason on protocols in which the systems under consideration are not approximately equivalent, as in the case of gossip protocols.

INTRODUCTION

Wireless sensor networks (WSNs) are (possibly large-scale) networks of sensor nodes deployed in strategic areas to gather data. Sensor nodes collaborate using wireless communications with an asymmetric many-to-one data transfer model. Typically, they send sensed events or data, by a specific communication protocol, to a specific node called the sink node or base station, which collects the requested information. WSNs are primarily designed for monitoring environments that humans cannot easily reach (*e.g.*, motion, target tracking, fire detection, chemicals, temperature); they are also used as embedded systems (*e.g.*, environmental monitoring, biomedical sensor engineering, smart homes) or mobile applications (*e.g.*, when attached to robots or vehicles). In wireless sensor networks, sensor nodes are usually battery-powered, and the energy expenditure of sensors has to be wisely managed by their architectures and protocols to prolong the overall network lifetime. *Energy conservation* is thus one of the major issues in sensor network communications.

Key words and phrases: Wireless sensor network, gossip protocol, (bi)simulation metric, algebraic theory.

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Flooding is a traditional robust algorithm that delivers data packets in a network from a source to a destination. In the context of WSNs, flooding means that all nodes receiving a message propagate it to all neighbours by broadcast. This causes unnecessary retransmissions, which increase the number of collisions and deprive sensors of valuable battery power. Therefore, flooding algorithms may not be suitable in the context of dense networks like wireless sensor networks.

Gossiping is an algorithm that addresses some critical problems of flooding overhead. Basically, it is based on the repeated probabilistic exchange of information between two members [KvS07]. The goal of gossip protocols is to reduce the number of retransmissions by making some of the nodes discard the message instead of forwarding it. Gossip protocols exhibit both *nondeterministic* and *probabilistic* behaviour. Nondeterminism arises as they deal with distributed networks in which the activities of individual nodes occur nondeterministically. As to the probabilistic behaviour, nodes are required to forward packets with a pre-specified gossip probability p_{gsp} . When a node receives a message, rather than immediately retransmitting it as in flooding, it relies on the probability p_{gsp} to determine whether or not to retransmit. The main benefit is that when p_{gsp} is sufficiently large, the entire network receives the broadcast message with very high probability, even though only a subset of nodes has forwarded the message.

Most of the analyses of protocols for large-scale WSNs are usually based on discrete-event simulators (*e.g.*, ns-2, Opnet and Glomosim). However, different simulators often support different models of the MAC physical-layer yielding different results, even for simple systems (see, *e.g.*, [NCO04]). In principle, as noticed by Bakhshi et al. [BBFH07], due to their relatively simple structure, gossip protocols lend themselves very well to formal analysis, in order to predict their behaviour with high confidence. Formal analysis techniques are supported by (semi-)automated tools. For instance, *probabilistic model checking* has been used in the context of gossip protocols by Fehnker and Gao [FG06] and Kwiatkowska et al. [KNP08] to provide both an exhaustive search of all possible behaviours of the system, and exact, rather than approximate, quantitative results. Of course, a trade-off inevitably exists. Model checking suffers from the so-called state explosion problem, meaning that the state space of a specified system grows exponentially with respect to its number of components. On the other hand, simulation-based approaches are scalable to much larger and more complex systems, at the expense of exhaustiveness and numerical accuracy.

Among the formal techniques for the analysis of complex systems, *behavioural semantics*, such as preorders and equivalences, provide formal instruments to compare the behaviour of *probabilistic systems* [JLY01, Den15]. Preorders allow us to determine whether a system can mimic the stepwise behaviour of another system; whereas equivalences require a sort of mutual simulation between two systems. The most prominent examples are the *simulation preorder* and the *bisimulation equivalence* [LS91, SL95]. Since probability values usually originate from observations (statistical sampling) or from requirements (probabilistic specification), both preorders and equivalences are only partially satisfactory as they can only say whether a system can mimic another one. Any tiny variation of the probabilistic behaviour of a system will break the preorder (or the equivalence) without any further information. In practice, many system implementations can only approximate the system specification; thus, the verification of such implementations requires appropriate instruments to measure the quality of the approximation. To this end, bisimulation-based *metric semantics* [GJS90, DGJP04, vBW05, DCP06, dAFS09] have been successfully employed to formalise the *behavioural distance* between two systems.

Contribution. The *goal* of this paper is to provide a compositional analysis technique to study probabilistic models of gossip protocols in the context of WSNs. We adopt a simple process calculus for wireless sensor networks proposed by Lanotte and Merro [LM11] and called *Probabilistic Timed Calculus of Wireless Systems* (pTCWS). We then introduce the notion of *weak simulation quasimetric* as the asymmetric counterpart of the *weak bisimulation metric* [DJGP02], and the quantitative analogous of the *weak simulation preorder* [BKHH02, BHK04]. We use the definition of weak simulation quasimetric to derive a definition of *weak simulation with tolerance* $p \in [0, 1]$, written \sqsubseteq_p , to compare probabilistic protocols; being 0 and 1 the minimum and the maximum distance, respectively.

The compositionality of a behavioural semantics with respect to the parallel operator is fundamental when reasoning on large-scale systems. We prove that our weak simulation with tolerance matches one of the most restrictive compositional criteria for (bi)simulation metrics, namely *non-expansiveness* [DJGP02, DGJP04] (also known as 1-non-extensiveness [BBLM13]).

We then rely on our simulation with tolerance to develop an *algebraic theory* for pTCWS that allows us to compose the tolerances of different sub-networks with different behaviours. In particular, for a gossip network $\text{GSP}_{p_{\text{gsp}}}$, which transmits with gossip probability p_{gsp} , we are able to estimate an *upper bound* of the tolerance p_{ok} needed to simulate a non-probabilistic network DONE whose target nodes successfully receive the message:

$$\text{DONE} \sqsubseteq_{p_{\text{ok}}} \text{GSP}_{p_{\text{gsp}}}.$$

To this end, we prove and apply a significant number of algebraic laws, whose application can be *mechanised*, to evaluate the performances of gossip networks in terms of message delivery at destination. Our analysis takes into consideration *communication collisions*, to determine the impact of collisions on the performance results, and investigate *randomised gossip protocols*, in which messages may be broadcast in different instants of time according to some uniform distribution probability to mitigate the impact of communication collisions.

Outline. In Section 1, we recall syntax and probabilistic operational semantics of pTCWS. In Section 2, we provide the notions of weak simulation quasimetric and weak simulation with tolerance. In Section 3, we start our analysis on collision-free gossip protocols. In Section 4, we extend our analysis to collision-prone gossip protocols. In Section 5, we study randomised gossip protocols. In Section 6, we draw conclusions and discuss related and future work.

1. A PROBABILISTIC TIMED PROCESS CALCULUS

In Table 1, we provide the syntax of the *Probabilistic Timed Calculus of Wireless Systems*, pTCWS, in a two-level structure, a lower one for *processes*, ranged over by letters P, Q , and R , and an upper one for *networks*, ranged over by letters M, N , and O . We use letters m, n, \dots for logical names, Greek symbols μ, ν, ν_1, \dots for *sets of names*, x, y, z for *variables*, u for *values*, and v and w for *closed values*, *i.e.*, values that do not contain variables. Then, we use p_i for probability weights, hence $p_i \in [0, 1]$.

A network in pTCWS is a (possibly empty) collection of nodes (which represent devices) running in parallel and using a unique common radio channel to communicate with each other. Nodes are also unique; *i.e.*, a node n can occur in a network only once. All nodes are assumed to have the same transmission range (this is a quite common assumption in models for ad hoc networks [MZS09]). The communication paradigm is *local broadcast*;

<i>Networks:</i>			
M, N	$::=$	$\mathbf{0}$	empty network
		$M_1 \mid M_2$	parallel composition
		$n[P]^\nu$	node
		Dead	stuck network
<i>Processes:</i>			
P, Q	$::=$	nil	termination
		$!\langle u \rangle.C$	broadcast
		$[?(x).C]D$	receiver with timeout
		$\tau.C$	internal
		$\sigma.C$	sleep
		X	process variable
		fix $X.P$	recursion
<i>Probabilistic Choice:</i>			
C, D	$::=$	$\bigoplus_{i \in I} p_i : P_i$	

Table 1: Syntax

only nodes located in the range of the transmitter may receive data. We write $n[P]^\nu$ for a node named n (the device network address) executing the sequential process P . The set ν contains (the names of) the neighbours of n . Said in other words, ν contains all nodes laying in the transmission cell of n (except n). In this manner, we model the network topology. Notice that the network topology could have been represented using some kind of restriction operator à la CCS over node names; we preferred our notation to keep at hand the neighbours of a node. Our wireless networks have a fixed topology. Moreover, nodes cannot be created or destroyed. Finally, we write **Dead** to denote a deadlocked network which prevents the execution of parallel components. This is a fictitious network which is introduced for technical convenience in the definition of our metrics (Definition 2.5).

Processes are sequential and live inside the nodes. The symbol **nil** denotes terminated processes. The sender process $!\langle v \rangle.C$ broadcasts the value v , the continuation being C . The process $[?(x).C]D$ denotes a receiver with timeout. Intuitively, this process either receives a value v from a neighbour node in the current time interval, and then continues as C in which the variable x is instantiated with v , or it idles for one time unit and then continues as D , if there are no senders in the neighbourhood. The process $\tau.C$ performs an internal action and then continues as C . The process $\sigma.C$ models sleeping for one time unit. In processes of the form $\sigma.D$ and $[?(x).C]D$ the occurrence of D is said to be *time-guarded*. The process $\text{fix } X.P$ denotes *time-guarded recursion*, as all occurrences of the process variable X may only occur time-guarded in P .

The construct $\bigoplus_{i \in I} p_i : P_i$ denotes *probabilistic choice*, where I is a *finite, non-empty* set of indexes, and $p_i \in (0, 1]$ denotes the probability to execute the process P_i , with $\sum_{i \in I} p_i = 1$. Notice that, as in [DvGHM08], in order to simplify the operational semantics, probabilistic choices occur always underneath prefixing.

In processes of the form $[?(x).C]D$ the variable x is bound in C . Similarly, in process $\text{fix } X.P$ the process variable X is bound in P . This gives rise to the standard notions of *free (process) variables* and *bound (process) variables* and α -conversion. We identify processes and networks up to α -conversion. A process is said to be *closed* if it does not contain free

(process) variables. We always work with closed processes: the absence of free variables is trivially maintained at run-time. We write $\{v/x\}P$ (*resp.*, $\{v/x\}C$) for the substitution of the variable x with the value v in the process P (*resp.*, probabilistic choice C). Similarly, we write $\{P/X\}Q$ (*resp.*, $\{P/X\}C$) for the substitution of the process variable X with the process P in Q (*resp.*, in C).

We report some *notational conventions*. With an abuse of notation, we will write $?(x).C$ as an abbreviation for $\text{fix } X.[?(x).C](1: X)$, where the process variable X does not occur in C . $\prod_{i \in I} M_i$ denotes the parallel composition of all M_i , for $i \in I$. We identify $\prod_{i \in I} M_i$ with $\mathbf{0}$, if $I = \emptyset$. We write $P_1 \oplus_p P_2$ for the probabilistic process $p: P_1 \oplus (1-p): P_2$. We identify $1: P$ with P . We write $!\langle v \rangle$ as an abbreviation for $!\langle v \rangle.1: \text{nil}$. For $k > 0$ we write $\sigma^k.P$ as an abbreviation for $\sigma \dots \sigma.P$, where prefix σ appears k times. Given a network M , $\text{nds}(M)$ returns the names of M . If $m \in \text{nds}(M)$, the function $\text{ngh}(m, M)$ returns the set of the neighbours of m in M . Thus, for $M = M_1 \mid m[P]^\nu \mid M_2$ it holds that $\text{ngh}(m, M) = \nu$. We write $\text{ngh}(M)$ for $\bigcup_{m \in \text{nds}(M)} \text{ngh}(m, M)$.

Definition 1.1 (Structural congruence). *Structural congruence*, written \equiv , is defined as the smallest equivalence relation over networks, preserved by parallel composition, which is a commutative monoid with respect to parallel composition with neutral element $\mathbf{0}$, and for which $n[\text{fix } X.P]^\nu \equiv n[\{\text{fix } X.P/X\}P]^\nu$.

The syntax presented in Table 1 allows us to derive networks which are somehow ill-formed. With the following definition we rule out networks: (i) where nodes can be neighbours of themselves; (ii) with different nodes with the same name; (iii) with non-symmetric neighbouring relations. Finally, in order to guarantee clock synchronisation among nodes, we require network connectivity.

Definition 1.2 (Well-formedness). A network M is said to be *well-formed* if

- whenever $M \equiv M_1 \mid m[P_1]^\nu$ it holds that $m \notin \nu$;
- whenever $M \equiv M_1 \mid m_1[P_1]^{\nu_1} \mid m_2[P_2]^{\nu_2}$ it holds that $m_1 \neq m_2$;
- whenever $M \equiv N \mid m_1[P_1]^{\nu_1} \mid m_2[P_2]^{\nu_2}$ we have $m_1 \in \nu_2$ iff $m_2 \in \nu_1$;
- for all $m, n \in \text{nds}(M)$ there are $m_1, \dots, m_k \in \text{nds}(M)$, such that $m = m_1$, $n = m_k$, and $m_i \in \text{ngh}(m_{i+1}, M)$ for $1 \leq i \leq k-1$.

Henceforth, we will always work with well-formed networks.

1.1. Probabilistic labelled transition semantics. Along the lines of [DvGHM08], we propose an *operational semantics* for pTCWS associating with each network a graph-like structure representing its possible evolutions: we use a generalisation of labelled transition systems that includes probabilities. Below, we report the necessary mathematical machinery.

Definition 1.3. A (discrete) *probability sub-distribution* over a set S is a function $\Delta: S \rightarrow [0, 1]$ with $\sum_{s \in S} \Delta(s) \in (0, 1]$. We denote $\sum_{s \in S} \Delta(s)$ by $|\Delta|$. The *support* of a probability sub-distribution Δ is given by $[\Delta] = \{s \in S : \Delta(s) > 0\}$. We write $\mathcal{D}_{\text{sub}}(S)$, ranged over by Δ, Θ, Φ , for the set of all finite-support probability sub-distributions over S . A probability sub-distribution $\Delta \in \mathcal{D}_{\text{sub}}(S)$ is said to be a *probability distribution* if $|\Delta| = 1$. With $\mathcal{D}(S)$ we denote the set of all probability distributions over S with finite support. For any $s \in S$, the *point (Dirac) distribution at s* , denoted \bar{s} , assigns probability 1 to s and 0 to all others elements of S , so that $[\bar{s}] = \{s\}$.

Let I be a finite index such that (i) Δ_i is a sub-distribution in $\mathcal{D}_{\text{sub}}(S)$ for each $i \in I$, and (ii) $p_i \geq 0$ are probabilities such that $\sum_{i \in I} p_i \in (0, 1]$. Then, the probability sub-distribution $\sum_{i \in I} p_i \cdot \Delta_i \in \mathcal{D}_{\text{sub}}(S)$ is defined as $(\sum_{i \in I} p_i \cdot \Delta_i)(s) \stackrel{\text{def}}{=} \sum_{i \in I} p_i \cdot \Delta_i(s)$, for all $s \in S$. We write a sub-distribution as $p_1 \cdot \Delta_1 + \dots + p_n \cdot \Delta_n$ when the index set I is $\{1, \dots, n\}$. In the following, we will often write $\sum_{i \in I} p_i \Delta_i$ instead of $\sum_{i \in I} p_i \cdot \Delta_i$. Definition 1.1 and Definition 1.2 generalise to sub-distributions in $\mathcal{D}_{\text{sub}}(\text{pTCWS})$. Given two probability sub-distributions Δ and Θ , we write $\Delta \equiv \Theta$ if $\Delta([M]_{\equiv}) = \Theta([M]_{\equiv})$ for all equivalence classes $[M]_{\equiv} \subseteq \text{pTCWS}$ of \equiv . Moreover, a probability sub-distribution $\Delta \in \mathcal{D}_{\text{sub}}(\text{pTCWS})$ is said to be well-formed if its support contains only well-formed networks.

We follow the standard probabilistic generalisation of labelled transition systems [Seg95].

Definition 1.4 (Probabilistic LTS). A *probabilistic labelled transition system* (pLTS) is a triple $\langle S, \mathcal{L}, \rightarrow \rangle$ where (i) S is a set of states; (ii) \mathcal{L} is a set of transition labels; (iii) \rightarrow is a labelled transition relation contained in $S \times \mathcal{L} \times \mathcal{D}(S)$.

The operational semantics of pTCWS is given by a particular pLTS $\langle \text{pTCWS}, \mathcal{L}, \rightarrow \rangle$, where $\mathcal{L} = \{m!v \triangleright \mu, m?v, \tau, \sigma\}$ contains the labels to denote broadcasting, reception, internal actions and time passing, respectively. The definition of the relations $\xrightarrow{\lambda}$, for $\lambda \in \mathcal{L}$, is given by the SOS rules in Table 2. Some of these rules use an obvious notation for distributing parallel composition over a sub-distribution: $(\Delta \mid \Theta)(M) = \Delta(M_1) \cdot \Theta(M_2)$ if $M = M_1 \mid M_2$; $(\Delta \mid \Theta)(M) = \mathbf{0}$ otherwise.

Furthermore, the definition of the labelled transition relation relies on a semantic interpretation of (nodes containing) probabilistic processes in terms of probability distributions.

Definition 1.5. For any probabilistic choice $\bigoplus_{i \in I} p_i : P_i$ over a finite index set I , we write $\llbracket n[\bigoplus_{i \in I} p_i : P_i]^\mu \rrbracket$ to denote the probability distribution $\sum_{i \in I} p_i \cdot \overline{n[P_i]^\mu}$.

Let us comment on the rules of Table 2. In rule (Snd) a node m broadcasts a message v to its neighbours ν , the continuation being the probability distribution associated to $m[C]^\nu$. In the label $m!v \triangleright \nu$ the set ν denotes the neighbours of m : all nodes in ν that are currently listening will receive the message v . In rule (Rcv) a node n gets a message v from a neighbour node m , the continuation being the probability distribution associated to $n[\{v/x\}C]^\nu$. If no message is received in the current time interval then the node n will continue according to D , as specified in rule (Timeout). Rules (Rcv-0) and (RcvEnb) serve to model reception enabling for synchronisation purposes. For instance, rule (RcvEnb) regards nodes n which are not involved in transmissions originating from m . This may happen either because the two nodes are out of range (*i.e.*, $m \notin \nu$) or because n is not willing to receive ($\text{rcv}(P)$ is a boolean predicate that returns true if $n[P]^\nu \equiv n[?[x].C]D]^\nu$, for some x, C, D). In both cases, node n is not affected by the transmission. In rule (RcvPar) we model the composition of two networks receiving the same message from the same transmitter. Rule (Bcast) models the propagation of messages on the broadcast channel. Note that we lose track of those transmitter's neighbours that are in N . Rule (Tau) models internal computations in a single node. The rule (TauPar) propagates internal computations on parallel components different from the Dead network. Rules (σ -nil) and (σ -0) are straightforward as both terms $\mathbf{0}$ and $n[\text{nil}]^\nu$ do not prevent time-passing. Rule (Sleep) models sleeping for one time unit. Rule (σ -Par) models time synchronisation between parallel components (we recall that receiver processes can timeout only if there are no senders in their neighbourhood). Rule (Rec) is standard. For rules (Bcast) and (TauPar) we assume their symmetric counterparts. Finally,

$\text{(Snd)} \frac{\overline{\quad}}{m[!\langle v \rangle.C]^\nu \xrightarrow{m!v \triangleright \nu} \llbracket m[C]^\nu \rrbracket}$	$\text{(Rcv)} \frac{m \in \nu}{n[!?(x).C]D]^\nu \xrightarrow{m?v} \llbracket n[\{v/x\}C]^\nu \rrbracket}$
$\text{(Rcv-0)} \frac{\overline{\quad}}{\mathbf{0} \xrightarrow{m?v} \overline{\mathbf{0}}}$	$\text{(RcvEnb)} \frac{\neg(m \in \nu \wedge \text{rcv}(P)) \wedge m \neq n}{n[P]^\nu \xrightarrow{m?v} \overline{n[P]^\nu}}$
$\text{(RcvPar)} \frac{M \xrightarrow{m?v} \Delta \quad N \xrightarrow{m?v} \Theta}{M \mid N \xrightarrow{m?v} \Delta \mid \Theta}$	$\text{(Bcast)} \frac{M \xrightarrow{m!v \triangleright \nu} \Delta \quad N \xrightarrow{m?v} \Theta \quad \mu := \nu \setminus \text{nds}(N)}{M \mid N \xrightarrow{m!v \triangleright \mu} \Delta \mid \Theta}$
$\text{(Tau)} \frac{\overline{\quad}}{m[\tau.C]^\nu \xrightarrow{\tau} \llbracket m[C]^\nu \rrbracket}$	$\text{(TauPar)} \frac{M \xrightarrow{\tau} \Delta \quad N \neq \text{Dead} \mid N'}{M \mid N \xrightarrow{\tau} \Delta \mid \overline{N'}}$
$\text{(\sigma-0)} \frac{\overline{\quad}}{\mathbf{0} \xrightarrow{\sigma} \overline{\mathbf{0}}}$	$\text{(Timeout)} \frac{\overline{\quad}}{n[!?(x).C]D]^\nu \xrightarrow{\sigma} \llbracket n[D]^\nu \rrbracket}$
$\text{(\sigma-nil)} \frac{\overline{\quad}}{n[\text{nil}]^\nu \xrightarrow{\sigma} \overline{n[\text{nil}]^\nu}}$	$\text{(Sleep)} \frac{\overline{\quad}}{n[\sigma.C]^\nu \xrightarrow{\sigma} \llbracket n[C]^\nu \rrbracket}$
$\text{(\sigma-Par)} \frac{M \xrightarrow{\sigma} \Delta \quad N \xrightarrow{\sigma} \Theta}{M \mid N \xrightarrow{\sigma} \Delta \mid \Theta}$	$\text{(Rec)} \frac{n[\{\text{fix } X.P/X\}P]^\nu \xrightarrow{\lambda} \Delta}{n[\text{fix } X.P]^\nu \xrightarrow{\lambda} \Delta}$

Table 2: Probabilistic Labelled Transition System

note that the semantics of the network **Dead** is different from that of **0**: **Dead** does not perform any action and it prevents the evolution of any parallel component.

1.1.1. *Extensional labelled transition semantics.* Our focus is on weak similarities, which abstract away non-observable actions, *i.e.*, those actions that cannot be detected by a parallel network. The adjective *extensional* is used to stress that here we focus on those activities that require a contribution of the environment. To this end, we extend Table 2 by the following two rules:

$$\text{(ShhSnd)} \frac{M \xrightarrow{m!v \triangleright \emptyset} \Delta}{M \xrightarrow{\tau} \Delta} \qquad \text{(ObsSnd)} \frac{M \xrightarrow{m!v \triangleright \nu} \Delta \quad \nu \neq \emptyset}{M \xrightarrow{!v \triangleright \nu} \Delta}$$

Rule (ShhSnd) models transmissions that cannot be observed because there is no potential receiver outside the network M . Rule (ObsSnd) models transmissions that may be observed by those nodes of the environment contained in ν . Notice that the name of the transmitter is removed from the label. This is motivated by the fact that receiver nodes do not have a direct manner to observe the identity of the transmitter. On the other hand, a network M performing the action $m?v$ can be observed by an external node m which transmits the value v to an appropriate set of nodes in M . Notice that the action $!v \triangleright \nu$ does not propagate over parallel components (there is no rule for that). As a consequence, the rule (ObsSnd) can only be applied to the whole network, never in a sub-network.

In the rest of the paper, the metavariable α will range over the following four kinds of actions: $!v \triangleright \nu$, $m?v$, σ , τ . They denote anonymous broadcast to specific nodes, message reception, time passing, and internal activities, respectively.

Finally, having defined the labelled transitions that can be performed by a network, we can easily concatenate these transitions to define the possible computation traces of a network. A *computation trace* for a network M is a sequence of steps of the form $M_1 \xrightarrow{\alpha_1} \dots \xrightarrow{\alpha_{n-1}} M_n$ where for any i , with $1 \leq i \leq n-1$, we have $M_i \xrightarrow{\alpha_i} \Delta_{i+1}$ such that $M_{i+1} \in [\Delta_{i+1}]$.

1.2. Time properties. The calculus pTCWS enjoys a number of desirable time properties [LM11] that will be very useful for our purposes.

Proposition 1.6 formalises the deterministic nature of time passing: a network can reach at most one new distribution by executing the action σ .

Proposition 1.6 (Time Determinism). *Let M be a well-formed network. If $M \xrightarrow{\sigma} \Delta$ and $M \xrightarrow{\sigma} \Theta$ then Δ and Θ are the same.*

The maximal progress property says that sender nodes transmit immediately. Said in other words, the passage of time cannot block transmissions.

Proposition 1.7 (Maximal Progress). *Let M be a well-formed network. If either $M \equiv m[!\langle v \rangle.C]^\nu \mid N$ or $M \equiv m[\tau.C]^\nu \mid N$ then $M \xrightarrow{\sigma} \Delta$ for no distribution Δ .*

Patience guarantees that a process will wait indefinitely until it can communicate [HR95]. In our setting, this means that if neither transmissions nor internal actions can fire then it must be possible to execute a σ -action to let time pass.

Proposition 1.8 (Patience). *Let $M = \prod_{i \in I} m_i[P_i]^{\nu_i}$ be a well-formed network such that for all $i \in I$ it holds that $P_i \neq !\langle v \rangle.C$ and $P_i \neq \tau.C$, then there is a distribution Δ s.t. $M \xrightarrow{\sigma} \Delta$.*

Finally, our networks satisfy the *well-timedness* (or *finite variability*) property [NS92]. Intuitively, only a finite number of instantaneous actions can fire between two contiguous σ -actions. In pTCWS, this property holds because recursion is defined to be time-guarded.

Proposition 1.9 (Well-Timedness). *For any well-formed network M there is an upper bound $k \in \mathbb{N}$ such that whenever $M \xrightarrow{\alpha_1} \dots \xrightarrow{\alpha_h} N$, with $\alpha_j \notin \{m?v, \sigma\}$ for $1 \leq j \leq h$, then $h \leq k$.*

2. WEAK SIMULATION WITH TOLERANCE

In this section, we introduce *weak simulation quasimetrics* as an instrument to derive a notion of approximate simulation between networks. Our goal is to define a family of relations \sqsubseteq_p over networks, with $p \in [0, 1]$, to formalise the concept of *simulation with a tolerance p* . Intuitively, we wish to write $M \sqsubseteq_p N$ if N can simulate M with a tolerance p . Thus, \sqsubseteq_0 should coincide with the standard weak probabilistic simulation [BKHH02, BHK04], whereas \sqsubseteq_1 should be equal to pTCWS \times pTCWS.

In a probabilistic setting, the definition of weak transition is somewhat complicated by the fact that (strong) transitions take processes (in our case networks) to distributions; consequently if we are to use weak transitions $\xrightarrow{\hat{\alpha}}$ which abstract away from non-observable actions then we need to generalise transitions, so that they take (sub-)distributions to (sub-)distributions.

For a network M and a distribution Δ , we write $M \xrightarrow{\hat{\tau}} \Delta$ if either $M \xrightarrow{\tau} \Delta$ or $\Delta = \overline{M}$. Then, for $\alpha \neq \tau$, we write $M \xrightarrow{\hat{\alpha}} \Delta$ if $M \xrightarrow{\alpha} \Delta$. Relation $\xrightarrow{\hat{\alpha}}$ is extended to model transitions

from sub-distributions to sub-distributions. For a sub-distribution $\Delta = \sum_{i \in I} p_i \overline{M}_i$, we write $\Delta \xrightarrow{\hat{\alpha}} \Theta$ if there is a non-empty set $J \subseteq I$ such that $M_j \xrightarrow{\hat{\alpha}} \Theta_j$ for all $j \in J$, $M_i \not\xrightarrow{\hat{\alpha}}$, for all $i \in I \setminus J$, and $\Theta = \sum_{j \in J} p_j \Theta_j$. Note that if $\alpha \neq \tau$ then this definition admits that only some networks in the support of Δ make the $\hat{\alpha}$ transition. Then, we define the weak transition relation $\hat{\Rightarrow}$ as the transitive and reflexive closure of $\hat{\rightarrow}$, i.e., $\hat{\Rightarrow} = (\hat{\rightarrow})^*$, while for $\alpha \neq \tau$ we let $\hat{\Rightarrow}$ denote $\hat{\rightarrow} \hat{\alpha} \hat{\rightarrow}$.

Example 2.1. Assume a node $n[\tau.(!\langle v_1 \rangle \oplus_{\frac{1}{3}} !\langle v_2 \rangle)]^\nu$ that broadcasts v_1 with probability $\frac{1}{3}$, and v_2 with probability $\frac{2}{3}$. By an application of rule (Tau) we derive the transition

$$n[\tau.(!\langle v_1 \rangle \oplus_{\frac{1}{3}} !\langle v_2 \rangle)]^\nu \xrightarrow{\tau} \frac{1}{3} \cdot \overline{n[!\langle v_1 \rangle]^\nu} + \frac{2}{3} \cdot \overline{n[!\langle v_2 \rangle]^\nu}.$$

Then, by an application of rule (Snd) we can derive:

- either $n[\tau.(!\langle v_1 \rangle \oplus_{\frac{1}{3}} !\langle v_2 \rangle)]^\nu \xrightarrow{n!v_1 \triangleright \nu} \frac{1}{3} \cdot \overline{n[\text{nil}]^\nu}$, because $n[!\langle v_1 \rangle]^\nu \xrightarrow{n!v_1 \triangleright \nu} n[\text{nil}]^\nu$
- or $n[\tau.(!\langle v_1 \rangle \oplus_{\frac{1}{3}} !\langle v_2 \rangle)]^\nu \xrightarrow{n!v_2 \triangleright \nu} \frac{2}{3} \cdot \overline{n[\text{nil}]^\nu}$, because $n[!\langle v_2 \rangle]^\nu \xrightarrow{n!v_2 \triangleright \nu} n[\text{nil}]^\nu$.

In order to define our notion of simulation with tolerance, we adapt the concept of *weak bisimulation metric* of Desharnais et al.'s [DJGP02]. In that paper the behavioural distance between systems is measured by means of suitable *pseudometrics*, namely symmetric functions assigning a numeric value to any pair of systems. Here, we consider asymmetric variants of pseudometrics, called *pseudoquasimetrics*, measuring the tolerance of the *simulation* between networks.

Definition 2.2 (Pseudoquasimetric). A function $d: \text{pTCWS} \times \text{pTCWS} \rightarrow [0, 1]$ is a *1-bounded pseudoquasimetric* over pTCWS if

- $d(M, M) = 0$ for all $M \in \text{pTCWS}$, and
- $d(M, N) \leq d(M, O) + d(O, N)$ for all $M, N, O \in \text{pTCWS}$ (triangle inequality).

Weak simulation quasimetrics provide the quantitative analogous of the weak simulation game: a networks M is simulated by a network N with tolerance p if each transition $M \xrightarrow{\alpha} \Delta$ is mimicked by a transition $N \xrightarrow{\hat{\alpha}} \Theta$ such that the distribution Δ is simulated by the distribution Θ with a tolerance q , with $q \leq p$. As the derivative of a transition is a probability distribution we have to lift the notion of pseudometric from pTCWS to distributions over pTCWS. In [DJGP02], this lifting is realised by means of linear programs, relying on the symmetry of pseudometrics. Since pseudoquasimetrics are not symmetric, we need a different technique. Thus, to this end, we adopt the notions of matching [Vil08] (also known as *coupling*) and Kantorovich lifting [DD09].

Definition 2.3 (Matching). A *matching* for a pair of distributions $(\Delta, \Theta) \in \mathcal{D}(\text{pTCWS}) \times \mathcal{D}(\text{pTCWS})$ is a distribution ω in the network product space $\mathcal{D}(\text{pTCWS} \times \text{pTCWS})$ such that:

- $\sum_{N \in \text{pTCWS}} \omega(M, N) = \Delta(M)$, for all $M \in \text{pTCWS}$, and
- $\sum_{M \in \text{pTCWS}} \omega(M, N) = \Theta(N)$, for all $N \in \text{pTCWS}$.

We write $\Omega(\Delta, \Theta)$ to denote the set of all matchings for (Δ, Θ) .

A matching for (Δ, Θ) may be understood as a transportation schedule for the shipment of probability mass from Δ to Θ [Vil08].

Definition 2.4 (Kantorovich lifting). Let $d: \mathbf{pTCWS} \times \mathbf{pTCWS} \rightarrow [0, 1]$ be a pseudoquasimetric. The *Kantorovich lifting* of d is the function $\mathbf{K}(d): \mathcal{D}(\mathbf{pTCWS}) \times \mathcal{D}(\mathbf{pTCWS}) \rightarrow [0, 1]$ defined as:

$$\mathbf{K}(d)(\Delta, \Theta) \stackrel{\text{def}}{=} \min_{\omega \in \Omega(\Delta, \Theta)} \sum_{M, N \in \mathbf{pTCWS}} \omega(M, N) \cdot d(M, N).$$

Note that since we are considering only distributions with finite support, the minimum over the set of matchings $\Omega(\Delta, \Theta)$ is well defined.

Definition 2.5 (Weak simulation quasimetric). We say that a pseudoquasimetric $d: \mathbf{pTCWS} \times \mathbf{pTCWS} \rightarrow [0, 1]$ is a *weak simulation quasimetric* if for all networks $M, N \in \mathbf{pTCWS}$, with $d(M, N) < 1$, whenever $M \xrightarrow{\alpha} \Delta$ there is a distribution Θ such that $N \xrightarrow{\hat{\alpha}} \Theta$ and $\mathbf{K}(d)(\Delta, \Theta + (1 - |\Theta|)\overline{\text{Dead}}) \leq d(M, N)$.

In the previous definition, if $|\Theta| < 1$ then, with probability $1 - |\Theta|$, there is no way to simulate the behaviour of any network in the support of Δ (the special network `Dead` does not perform any action).

As expected, the kernel of a weak simulation quasimetric is a *weak probabilistic simulation* defined along the lines of [DD11] in terms of the notion of matching.

Definition 2.6. A relation $\mathcal{R} \subseteq \mathbf{pTCWS} \times \mathbf{pTCWS}$ is a weak probabilistic simulation iff whenever $M \mathcal{R} N$ and $M \xrightarrow{\alpha} \Delta$ there exist a transition $N \xrightarrow{\hat{\alpha}} \Theta$ and a matching $\omega \in \Omega(\Delta, \Theta)$ with $\omega(M', N') > 0$ such that $M' \mathcal{R} N'$.

Proposition 2.7. *Let d be a weak simulation quasimetric. The relation $\{(M, N) : d(M, N) = 0\} \subseteq \mathbf{pTCWS} \times \mathbf{pTCWS}$ is a weak probabilistic simulation.*

A crucial result in our construction process is the existence of the minimal weak simulation quasimetric, which can be viewed as the asymmetric counterpart of the minimal weak bisimulation metric [DJGP02].

Theorem 2.8. *There exists a weak simulation quasimetric \mathbf{d} such that $\mathbf{d}(M, N) \leq d(M, N)$ for all weak simulation quasimetrics d and all networks $M, N \in \mathbf{pTCWS}$.*

Now, we have all ingredients to define our simulation with tolerance p .

Definition 2.9 (Weak simulation with tolerance). Let $p \in [0, 1]$, we say that N *simulates* M with tolerance p , written $M \sqsubseteq_p N$, iff $\mathbf{d}(M, N) \leq p$. We write $M \simeq_p N$ if both $M \sqsubseteq_p N$ and $N \sqsubseteq_p M$.

Since the minimum weak simulation quasimetric \mathbf{d} satisfies the triangle inequality, our simulation relation is trivially transitive in an additive sense:

Proposition 2.10 (Transitivity). *$M \sqsubseteq_p N$ and $N \sqsubseteq_q O$ imply $M \sqsubseteq_r O$, with $r = \min(1, p+q)$.*

As expected, if $M \xrightarrow{\hat{\tau}} \Delta$ then M can simulate all networks in $[\Delta]$.

Proposition 2.11. *If $M \xrightarrow{\hat{\tau}} (1-q)\overline{N} + q\Delta$, for some $\Delta \in \mathcal{D}(\mathbf{pTCWS})$, then $N \sqsubseteq_q M$.*

Clearly the transitivity property is quite useful when doing algebraic reasoning. However, we can derive a better tolerance when concatenating two simulations, if one of them is derived by an application of Proposition 2.11.

Proposition 2.12. *If $M \sqsubseteq_p N$ and $O \xrightarrow{\hat{\tau}} (1-q)\overline{N} + q\Delta$, for some $\Delta \in \mathcal{D}(\mathbf{pTCWS})$, then $M \sqsubseteq_{p(1-q)+q} O$.*

Intuitively, in the simulation between M and N the tolerance p must be weighted by taking into consideration that O may evolve into N with a probability $(1 - q)$.

The previous results can be generalised to the case where Proposition 2.11 is applied to networks that are reached after one σ -transition.

Proposition 2.13. *Let M , N and O be networks in \mathbf{pTCWS} such that: (i) $M \sqsubseteq_p N$; (ii) N can only perform a transition of the form $N \xrightarrow{\sigma} \overline{N'}$, for some N' such that $N' \not\stackrel{\tau}{\sim}$; (iii) $O \xrightarrow{\sigma} \overline{O'} \xrightarrow{\hat{\tau}} (1 - q)\overline{N'} + q\Delta$, for some $O' \in \mathbf{pTCWS}$ and $\Delta \in \mathcal{D}(\mathbf{pTCWS})$. Then, $M \sqsubseteq_{p(1-q)+q} O$.*

In order to understand the intuition behind our weak simulation with tolerance, we report here a few simple *algebraic laws* (recall that $1: P = P$).

Proposition 2.14 (Simple algebraic laws).

- (1) $n[P]^\mu \sqsubseteq_{1-p} n[\tau.(P \oplus_p Q)]^\mu$
- (2) $n[Q]^\mu \sqsubseteq_r n[\tau.(\tau.(P \oplus_q Q) \oplus_p R)]^\mu$, with $r = (1 - p) + pq$
- (3) $n[\tau.(\bigoplus_{i \in I, j \in J} (p_i q_j) : P_j^i)]^\mu \sqsubseteq_0 n[\tau.(\bigoplus_{i \in I} p_i : \tau.(\bigoplus_{j \in J} q_j : P_j^i)]^\mu$
- (4) $n[!\langle v \rangle.(\tau.(P \oplus_q \tau.P) \oplus_p Q)]^\mu \simeq_0 n[!\langle v \rangle.(P \oplus_p \tau.Q)]^\mu$
- (5) $n[!\langle v \rangle.!\langle w \rangle]^\mu \sqsubseteq_r n[\tau.(!\langle v \rangle.\tau.(!\langle w \rangle \oplus_q P) \oplus_p Q)]^\mu$, with $r = 1 - pq$.

The first law is straightforward. The second law is a generalisation of the first one where the right-hand side must resolve two probabilistic choices in order to simulate the left-hand side. The third law is an adaptation of the τ -law $\tau.P = P$ of CCS [Mil89] in a distributed and probabilistic setting. Similarly, the fourth law reminds a probabilistic and distributed variant of the τ -law $a.(\tau.(P + \tau.Q)) + a.Q = a.(P + \tau.Q)$. The last law gives an example of a probabilistic simulation involving sequences of actions. Here, pq is the probability that the network on the right-hand side simulates the network on the left-hand side by transmitting both u and v . In fact, we have:

$$n[\tau.(!\langle v \rangle.\tau.(!\langle w \rangle \oplus_q P) \oplus_p Q)]^\mu \xrightarrow{\tau} p \cdot \overline{n[!\langle v \rangle.\tau.(!\langle w \rangle \oplus_q P)]^\mu} + (1 - p) \cdot \overline{n[Q]^\mu}$$

with $n[!\langle v \rangle.!\langle w \rangle]^\mu \sqsubseteq_{1-q} n[!\langle v \rangle.\tau.(!\langle w \rangle \oplus_q P)]^\mu$; the result follows by an application of Proposition 2.12.

A crucial property of our simulation is the possibility to reason on parallel networks in a *compositional* manner. Thus, if $M_1 \sqsubseteq_{p_1} N_1$ and $M_2 \sqsubseteq_{p_2} N_2$ then $M_1 \mid M_2 \sqsubseteq_p N_1 \mid N_2$ for some p depending on p_1 and p_2 ; the intuition being that if one fixes the maximal tolerance p between $M_1 \mid M_2$ and $N_1 \mid N_2$, then there are tolerances p_i between M_i and N_i , $i \in \{1, 2\}$, ensuring that the tolerance p is respected. Following this intuition, several compositional criteria for bisimulation metrics can be found in the literature. Here, we show that our weak simulation with tolerance matches one of the most restrictive among those studied in [GLT16, GT18], namely *non-expansiveness* [DJGP02, DGJP04] (also known as 1-non-extensiveness [BBLM13]) requiring that $p \leq p_1 + p_2$.

Theorem 2.15 (Non-expansiveness law). $M_1 \sqsubseteq_{p_1} N_1$ and $M_2 \sqsubseteq_{p_2} N_2$ entails $M_1 \mid M_2 \sqsubseteq_r N_1 \mid N_2$, with $r = \min(1, p_1 + p_2)$.

Another useful property is that a network can be simulated by means of a probabilistic choice whenever it is simulated by all components.

Proposition 2.16 (Additive law). *Let $M \sqsubseteq_{s_i} n[P_i]^\mu \mid N$, for all $i \in I$, with I a finite index set. Then, $M \sqsubseteq_r n[\tau.(\bigoplus_{i \in I} p_i : P_i)]^\mu \mid N$, for $r = \sum_{i \in I} p_i s_i$.*

Finally, we report a number of algebraic laws that will be useful in the next section, when analysing gossip protocols.

Proposition 2.17 (Further algebraic laws).

- (1) $n[\sigma^k.\text{nil}]^\mu \simeq_0 n[\text{nil}]^\mu$
- (2) $\prod_{i \in I} m_i [P_i]^{\mu_i} \simeq_r \prod_{j \in J} n_j [Q_j]^{\nu_j}$ entails $\prod_{i \in I} m_i [\sigma.P_i]^{\mu_i} \simeq_r \prod_{j \in J} n_j [\sigma.Q_j]^{\nu_j}$
- (3) $n[?(x).C]D]^\mu \simeq_0 n[\sigma.D]^\mu$, if nodes in μ do not send in the current time interval
- (4) $n[?(x).C]^\mu \simeq_0 n[\sigma.?(x).C]^\mu$, if nodes in μ do not send in the current time interval
- (5) $m[\text{nil}]^\mu \mid \prod_{i \in I} n_i [P_i]^{\mu_i} \sqsubseteq_0 m[\tau.(!(v) \oplus_p \text{nil})]^\mu \mid \prod_{i \in I} n_i [P_i]^{\mu_i}$ if $\mu \subseteq \bigcup_{i \in I} n_i$, and for all $n_i \in \mu$ it holds that $P_i \neq [?(x).C]D$.

Please, notice that all laws but last one hold with respect to weak bisimilarity; the last one holds with respect to weak similarity. Intuitively: (1) nil does not prevent time passing; (2) equalities are preserved underneath σ prefixes; (3) receivers timeout if there are not senders around; (4) this law is an instance of (3) because $?(x).C$ is an abbreviation for $\text{fix } X.[?(x).C](1:X) \equiv [?(x).C](1:?(x).C) = ?(x).C$ ($1:P$ is abbreviated with P); (5) broadcasting evolves silently if there are no receivers in the neighbourhood.

3. GOSSIPING WITHOUT COLLISIONS

The baseline model for our case study is gossiping without communication collisions, where all nodes are perfectly synchronised (relying on some clock synchronisation algorithm for WSNs). For the sake of clarity, communication proceeds in synchronous *rounds*: a node can transmit or receive only one message per round. In our implementation, rounds are separated by σ -actions.

The processes involved in the protocol are the following:

$$\text{snd}\langle u \rangle_{p_g} \stackrel{\text{def}}{=} \tau.(!(u) \oplus_{p_g} \text{nil}) \quad \text{fwd}_{p_g} \stackrel{\text{def}}{=} ?(x).\text{resnd}\langle x \rangle_{p_g} \quad \text{resnd}\langle u \rangle_{p_g} \stackrel{\text{def}}{=} \sigma.\text{snd}\langle u \rangle_{p_g}.$$

A sender broadcasts with a gossip probability p_g , whereas a forwarder rebroadcasts the received value, in the subsequent round, with the same probability.

We apply our simulation theory to develop algebraic reasonings on *message propagation*. As an introductory example, let us consider a fragment of a network with two sender nodes, m_1 and m_2 , and two forwarder nodes, n_1 and n_2 which are both neighbours of m_1 and m_2 . Then, assuming $\nu = \{n_1, n_2\}$ and $\{m_1, m_2\} \subseteq \nu_1 \cap \nu_2$, the following algebraic law holds:

$$m_1[\text{snd}\langle v \rangle_{p_1}]^\nu \mid m_2[\text{snd}\langle v \rangle_{p_2}]^\nu \mid n_1[\text{fwd}_{q_1}]^{\nu_1} \mid n_2[\text{fwd}_{q_2}]^{\nu_2} \stackrel{r \sqsupseteq}{=} m_1[\text{nil}]^\nu \mid m_2[\text{nil}]^\nu \mid n_1[\text{resnd}\langle v \rangle_{q_1}]^{\nu_1} \mid n_2[\text{resnd}\langle v \rangle_{q_2}]^{\nu_2} \quad (3.1)$$

with tolerance $r = (1 - p_1)(1 - p_2)$. Here, the network on the left-hand-side evolves to the Dirac distribution of the network on the right-hand-side by performing a sequence of τ -actions. Thus, the algebraic law follows by an application of Proposition 2.11 being $1 - r$ the probability that the message v is broadcast to both forwarder nodes n_1 and n_2 . To explain in detail how the sequence of τ -actions are derived, we first observe that both senders m_1 and m_2 start with the τ -action resolving the probabilistic choice on the transmission. Thus, by an application of rule (Tau) we can derive the transitions $m_i[\text{snd}\langle v \rangle_{p_i}]^\nu \xrightarrow{\tau} p_i \cdot \overline{m_i}[\langle v \rangle]^\nu + (1 - p_i) \cdot \overline{m_i}[\text{nil}]^\nu$, for $i \in \{1, 2\}$, which are propagated to

the whole network by means of the rule (TauPar). At this point, when composing these two transmissions, there are three possible cases.

- With probability $p_1 \cdot p_2$ both senders m_1 and m_2 decide to broadcast (the order of transmission is not relevant). This means that with probability $p_1 \cdot p_2$ the execution of the two τ -transitions above leads to the distribution $\overline{m_1[!\langle v \rangle]^\nu \mid m_2[!\langle v \rangle]^\nu \mid n_1[\text{fwd}_{q_1}]^{\nu_1} \mid n_2[\text{fwd}_{q_2}]^{\nu_2}}$. Now, if we suppose that m_1 transmits first, then we have the transition:

$$\begin{array}{c} m_1[!\langle v \rangle]^\nu \mid m_2[!\langle v \rangle]^\nu \mid n_1[\text{fwd}_{q_1}]^{\nu_1} \mid n_2[\text{fwd}_{q_2}]^{\nu_2} \\ \xrightarrow{\tau} \\ \overline{m_1[\text{nil}]^\nu \mid m_2[!\langle v \rangle]^\nu \mid n_1[\text{resnd}\langle v \rangle_{q_1}]^{\nu_1} \mid n_2[\text{resnd}\langle v \rangle_{q_2}]^{\nu_2}}. \end{array}$$

When also node m_2 broadcasts its value v then we will have:

$$\begin{array}{c} m_1[\text{nil}]^\nu \mid m_2[!\langle v \rangle]^\nu \mid n_1[\text{resnd}\langle v \rangle_{q_1}]^{\nu_1} \mid n_2[\text{resnd}\langle v \rangle_{q_2}]^{\nu_2} \\ \xrightarrow{\tau} \\ \overline{m_1[\text{nil}]^\nu \mid m_2[\text{nil}]^\nu \mid n_1[\text{resnd}\langle v \rangle_{q_1}]^{\nu_1} \mid n_2[\text{resnd}\langle v \rangle_{q_2}]^{\nu_2}}. \end{array}$$

- With probability $p_1 \cdot (1 - p_2) + (1 - p_1) \cdot p_2$ exactly one of the two senders decides to broadcast. This means that the τ -transitions due to the resolution of the probabilistic choice lead either to the distribution $\overline{m_1[!\langle v \rangle]^\nu \mid m_2[\text{nil}]^\nu \mid n_1[\text{fwd}_{q_1}]^{\nu_1} \mid n_2[\text{fwd}_{q_2}]^{\nu_2}}$ or to the distribution $\overline{m_1[\text{nil}]^\nu \mid m_2[!\langle v \rangle]^\nu \mid n_1[\text{fwd}_{q_1}]^{\nu_1} \mid n_2[\text{fwd}_{q_2}]^{\nu_2}}$. In the former case, *i.e.*, when m_1 transmits, we have the transition

$$\begin{array}{c} m_1[!\langle v \rangle]^\nu \mid m_2[\text{nil}]^\nu \mid n_1[\text{resnd}\langle v \rangle_{q_1}]^{\nu_1} \mid n_2[\text{resnd}\langle v \rangle_{q_2}]^{\nu_2} \\ \xrightarrow{\tau} \\ \overline{m_1[\text{nil}]^\nu \mid m_2[\text{nil}]^\nu \mid n_1[\text{resnd}\langle v \rangle_{q_1}]^{\nu_1} \mid n_2[\text{resnd}\langle v \rangle_{q_2}]^{\nu_2}} \end{array}$$

while in the latter case we have the transition

$$\begin{array}{c} m_1[\text{nil}]^\nu \mid m_2[!\langle v \rangle]^\nu \mid n_1[\text{resnd}\langle v \rangle_{q_1}]^{\nu_1} \mid n_2[\text{resnd}\langle v \rangle_{q_2}]^{\nu_2} \\ \xrightarrow{\tau} \\ \overline{m_1[\text{nil}]^\nu \mid m_2[\text{nil}]^\nu \mid n_1[\text{resnd}\langle v \rangle_{q_1}]^{\nu_1} \mid n_2[\text{resnd}\langle v \rangle_{q_2}]^{\nu_2}}. \end{array}$$

- With probability $(1 - p_1)(1 - p_2)$ both senders decide not to transmit. In this case, it is simply not possible to reach the desired network in which both nodes n_1 and n_2 forward the message, as they will never receive anything to forward.

Summarising, the network $m_1[\text{snd}\langle v \rangle_{p_1}]^\nu \mid m_2[\text{snd}\langle v \rangle_{p_2}]^\nu \mid n_1[\text{fwd}_{q_1}]^{\nu_1} \mid n_2[\text{fwd}_{q_2}]^{\nu_2}$ has probability $1 - (1 - p_1)(1 - p_2)$ to reach (the Dirac distribution of) $\overline{m_1[\text{nil}]^\nu \mid m_2[\text{nil}]^\nu \mid n_1[\text{resnd}\langle v \rangle_{q_1}]^{\nu_1} \mid n_2[\text{resnd}\langle v \rangle_{q_2}]^{\nu_2}}$ by the application of a sequence of τ -transitions. This allows us to apply Proposition 2.11 to derive Equation 3.1.

However, Equation 3.1 can be generalised to an arbitrary number of senders and forwarders, provided that parallel contexts are unable to receive messages in the current round.

Theorem 3.1 (Message propagation). *Let I and J be disjoint subsets of \mathbb{N} . Let M be a well-formed network defined as*

$$M \equiv N \mid \prod_{i \in I} m_i[\text{snd}\langle v \rangle_{p_i}]^{\nu_{m_i}} \mid \prod_{j \in J} n_j[\text{fwd}_{q_j}]^{\nu_{n_j}}$$

such that, for all $i \in I$: (1) $\{n_j : j \in J\} \subseteq \nu_{m_i} \subseteq \text{nds}(M)$; (2) the nodes in $\nu_{m_i} \cap \text{nds}(N)$ cannot receive in the current round. Then,

$$M \text{ } r \sqsupseteq N \mid \prod_{i \in I} m_i[\text{nil}]^{\nu_{m_i}} \mid \prod_{j \in J} n_j[\text{resnd}\langle v \rangle_{q_j}]^{\nu_{n_j}}$$

with $r = \prod_{i \in I} (1 - p_i)$.

As an example, consider a simple gossiping network GSP_1 , with gossip probability p , composed by two source nodes s_1 and s_2 and a destination node d . The network is the following:

$$\text{GSP}_1 \stackrel{\text{def}}{=} \prod_{i=1}^2 s_i[\text{snd}\langle v \rangle_p]^{\nu_{s_i}} \mid d[\text{fwd}_1]^{\nu_d}$$

with $\nu_{s_1} = \nu_{s_2} = \{d\}$ and $\nu_d = \{s_1, s_2, \text{tester}\}$, where *tester* is a fresh node of the observer to test successful gossiping. For simplicity, here and in the rest of the paper, the node *tester* can receive messages but it cannot transmit.

We would like to estimate the distance between GSP_1 and a network DONE_1 in which the message v has been delivered to the destination node d :

$$\text{DONE}_1 \stackrel{\text{def}}{=} \prod_{i=1}^2 s_i[\text{nil}]^{\nu_{s_i}} \mid d[\text{resnd}\langle v \rangle_1]^{\nu_d}.$$

By an application of Theorem 3.1 it follows that $\text{GSP}_1 \text{ } (1-p)^2 \sqsupseteq \text{DONE}_1$. Thus, the network GSP_1 succeeds in delivering the message to the destination d with probability (at least) $1 - (1 - p)^2 = 2p - p^2$.

Actually, Theorem 3.1 represents an effective tool to deal with message propagation in gossip networks. However, it requires that all forwarders n_j should be in the neighbouring of all senders m_i (constraint $\{n_j : j \in J\} \subseteq \nu_{m_i}$), which may represent a limitation in many cases. Consider, for example, a gossiping network GSP_2 , with gossip probability p , composed by two source nodes s_1 and s_2 , a destination node d , and three intermediate nodes n_1 , n_2 , and n_3 :

$$\text{GSP}_2 \stackrel{\text{def}}{=} \prod_{i=1}^2 s_i[\text{snd}\langle v \rangle_p]^{\nu_{s_i}} \mid \prod_{i=1}^3 n_i[\text{fwd}_p]^{\nu_{n_i}} \mid d[\text{fwd}_1]^{\nu_d}$$

where $\nu_{s_1} = \{n_1\}$, $\nu_{s_2} = \{n_1, n_2\}$, $\nu_{n_1} = \{s_1, s_2, n_3\}$, $\nu_{n_2} = \{s_2, n_3\}$, $\nu_{n_3} = \{n_1, n_2, d\}$ and $\nu_d = \{n_3, \text{tester}\}$.

Here, we would like to estimate the distance between GSP_2 , and a network DONE_2 , in which the message v propagated somehow up to the destination node d :

$$\text{DONE}_2 \stackrel{\text{def}}{=} \prod_{i=1}^2 s_i[\text{nil}]^{\nu_{s_i}} \mid \prod_{i=1}^3 n_i[\text{nil}]^{\nu_{n_i}} \mid d[\sigma^2.\text{resnd}\langle v \rangle_1]^{\nu_d}.$$

Unfortunately, we cannot directly apply Theorem 3.1 to capture this message propagation because node s_2 , unlike s_1 , can transmit to both n_1 and n_2 . In this case, before applying Theorem 3.1, we would need a result to *compose estimates* of partial networks. More precisely, a result which would allow us to take into account, in the calculation of the tolerance, both the probability that a sender transmits and the probability that the same sender does not transmit.

The additive law of Proposition 2.16 allows us to prove the following result.

Theorem 3.2 (Composing networks). *If $M \xrightarrow{\sigma} \not\approx$ then*

$$N \mid m[\text{snd}\langle v \rangle_p]^{\nu_m} \mid \prod_{j \in J} n_j[[?(x_j).P_j]Q_j]^{\nu_{n_j}} \ r \sqsupseteq M$$

with $r = pq_1 + (1-p)q_2$, whenever

- $N \mid m[\text{nil}]^{\nu_m} \mid \prod_{j \in J} n_j[\{v/x_j\}P_j]^{\nu_{n_j}} \ q_1 \sqsupseteq M$
- $N \mid m[\text{nil}]^{\nu_m} \mid \prod_{j \in J} n_j[[?(x_j).P_j]Q_j]^{\nu_{n_j}} \ q_2 \sqsupseteq M$
- $\{n_j : j \in J\} \subseteq \nu_m \subseteq \{n_j : j \in J\} \cup \text{nds}(N)$
- nodes in $\nu_m \cap \text{nds}(N)$ cannot receive in the current round.¹

Intuitively: (i) in the network $N \mid m[\text{snd}\langle v \rangle_p]^{\nu_m} \mid \prod_{j \in J} n_j[[?(x_j).P_j]Q_j]^{\nu_{n_j}}$ the sender m has not performed yet the τ -action that resolves the probabilistic choice between broadcasting the message v or not; (ii) in the network $N \mid m[\text{nil}]^{\nu_m} \mid \prod_{j \in J} n_j[\{v/x_j\}P_j]^{\nu_{n_j}}$ the sender m has resolved the probabilistic choice deciding to broadcast the message v ; (iii) finally, in the network $N \mid m[\text{nil}]^{\nu_m} \mid \prod_{j \in J} n_j[[?(x_j).P_j]Q_j]^{\nu_{n_j}}$ the sender m has resolved the probabilistic choice deciding not to broadcast v .

Later on we will use the following instance of Theorem 3.2.

Corollary 3.3. *If $M \not\approx \sigma$ then*

$$N \mid m[\text{snd}\langle v \rangle_p]^{\nu_m} \mid \prod_{j \in J} n_j[\text{fwd}_q]^{\nu_{n_j}} \ r \sqsupseteq M$$

with $r = pq_1 + (1-p)q_2$, whenever:

- $N \mid m[\text{nil}]^{\nu_m} \mid \prod_{j \in J} n_j[\text{resnd}\langle v \rangle_q]^{\nu_{n_j}} \ q_1 \sqsupseteq M$
- $N \mid m[\text{nil}]^{\nu_m} \mid \prod_{j \in J} n_j[\text{fwd}_q]^{\nu_{n_j}} \ q_2 \sqsupseteq M$
- $\{n_j \mid j \in J\} \subseteq \nu_m \subseteq \{n_j \mid j \in J\} \cup \text{nds}(M)$
- nodes in $\nu_m \cap \text{nds}(N)$ cannot receive in the current round.

Now, we have all algebraic tools to compute an estimation of the tolerance r , such that $\text{GSP}_2 \ r \sqsupseteq \text{DONE}_2$. Basically, we will compute the tolerances for two partial networks and then will use Theorem 3.2 to compose the two tolerances.

As a first step, we compute an estimation for the network GSP_2 in which the sender s_2 has already broadcast the message v to its neighbours n_1 and n_2 . To this end, we derive the following chain of similarities by applying, in sequence: (i) Proposition 2.17(5) to express that if n_1 receives the message transmitted by s_2 then the message broadcast by s_1 gets lost; (ii) Proposition 2.17(4) to formalise that in the current round both nodes n_3 and d do not receive any message because there are no transmitters in their neighbourhood, as a consequence, they are obliged to timeout; (iii) Theorem 3.1 in which nodes n_1 and n_2 act as senders and n_3 as a forwarder, together with Proposition 2.17(2) to work underneath σ prefixes; (iv) Proposition 2.17(1) to remove σ prefixes in both nodes n_1 and n_2 ; (v) Proposition 2.17(4) because in the next round there are no nodes that can transmit to d ; (vi) Theorem 3.1 in which n_3 acts as sender and d as forwarder, together with Proposition 2.17(2) to work underneath σ prefixes; (vii) Proposition 2.17(1) to remove σ prefixes in n_3 . In all steps, we have reasoned in a compositional manner, up to common parallel

¹We could generalise the result to take into account more senders at the same time. This would not add expressiveness, it would just speed up the reduction process.

components (Theorem 2.15). For convenience, in the chain below we define two auxiliary networks: NET_a and NET_b .

$$\begin{aligned}
& \text{GSP}_2(s_1[v]; s_2 \text{ gossiped}) \\
\stackrel{\text{def}}{=} & s_1[\text{snd}\langle v \rangle_p]^{\nu_{s_1}} \mid s_2[\text{nil}]^{\nu_{s_2}} \mid \prod_{i=1}^2 n_i[\text{resnd}\langle v \rangle_p]^{\nu_{n_i}} \mid n_3[\text{fwd}_p]^{\nu_{n_3}} \mid d[\text{fwd}_1]^{\nu_d} \\
0 \sqsupseteq & \prod_{i=1}^2 s_i[\text{nil}]^{\nu_{s_i}} \mid \prod_{i=1}^2 n_i[\text{resnd}\langle v \rangle_p]^{\nu_{n_i}} \mid n_3[\text{fwd}_p]^{\nu_{n_3}} \mid d[\text{fwd}_1]^{\nu_d} \\
0 \sqsupseteq & \prod_{i=1}^2 s_i[\text{nil}]^{\nu_{s_i}} \mid \prod_{i=1}^2 n_i[\sigma.\text{snd}\langle v \rangle_p]^{\nu_{n_i}} \mid n_3[\sigma.\text{fwd}_p]^{\nu_{n_3}} \mid d[\sigma.\text{fwd}_1]^{\nu_d} \\
\stackrel{\text{def}}{=} & \text{NET}_a \\
(1-p)2 \sqsupseteq & \prod_{i=1}^2 s_i[\text{nil}]^{\nu_{s_i}} \mid \prod_{i=1}^2 n_i[\sigma.\text{nil}]^{\nu_{n_i}} \mid n_3[\sigma.\text{resnd}\langle v \rangle_p]^{\nu_{n_3}} \mid d[\sigma.\text{fwd}_1]^{\nu_d} \\
\stackrel{\text{def}}{=} & \text{NET}_b \\
0 \sqsupseteq & \prod_{i=1}^2 s_i[\text{nil}]^{\nu_{s_i}} \mid \prod_{i=1}^2 n_i[\text{nil}]^{\nu_{n_i}} \mid n_3[\sigma^2.\text{snd}\langle v \rangle_p]^{\nu_{n_3}} \mid d[\sigma.\text{fwd}_1]^{\nu_d} \\
0 \sqsupseteq & \prod_{i=1}^2 s_i[\text{nil}]^{\nu_{s_i}} \mid \prod_{i=1}^2 n_i[\text{nil}]^{\nu_{n_i}} \mid n_3[\sigma^2.\text{snd}\langle v \rangle_p]^{\nu_{n_3}} \mid d[\sigma^2.\text{fwd}_1]^{\nu_d} \\
1-p \sqsupseteq & \prod_{i=1}^2 s_i[\text{nil}]^{\nu_{s_i}} \mid \prod_{i=1}^2 n_i[\text{nil}]^{\nu_{n_i}} \mid n_3[\sigma^2.\text{nil}]^{\nu_{n_3}} \mid d[\sigma^2.\text{resnd}\langle v \rangle_1]^{\nu_d} \\
0 \sqsupseteq & \prod_{i=1}^2 s_i[\text{nil}]^{\nu_{s_i}} \mid \prod_{i=1}^3 n_i[\text{nil}]^{\nu_{n_i}} \mid d[\sigma^3.\text{snd}\langle v \rangle_1]^{\nu_d} \\
= & \text{DONE}_2.
\end{aligned}$$

By transitivity (Proposition 2.10) we can easily derive $\text{NET}_b \ 1-p \sqsupseteq \text{DONE}_2$. This allows us to apply Proposition 2.13, in which M , N , and O are instantiated with DONE_2 , NET_b , and NET_a , respectively, to derive the equation $\text{NET}_a \ 1-2p^2+p^3 \sqsupseteq \text{DONE}_2$ where the tolerance $1-2p^2+p^3$ is obtained by solving the expression $(1-p)(1-(1-p)^2) + (1-p)^2$. By transitivity on the first two steps of the chain above, we can finally derive the equation

$$\text{GSP}_2(s_1[v]; s_2 \text{ gossiped}) \ 1-2p^2+p^3 \sqsupseteq \text{DONE}_2 \quad (3.2)$$

Similarly, we can compute an estimation of the tolerance which allows the network GSP_2 in which the sender s_2 *did not broadcast* the message v to its neighbours to simulate the network DONE_2 . To this end, we derive the following chain of similarities by applying, in sequence: (i) Theorem 3.1; (ii) Proposition 2.17(4); (iii) Theorem 3.1 and Proposition 2.17(2); (iv) Proposition 2.17(1) and Proposition 2.17(4); (v) Theorem 3.1 and Proposition 2.17(2); (vi) Proposition 2.17(1) and Proposition 2.17(5). Again, in all steps, we have reasoned up to common parallel components (Theorem 2.15). For convenience, in the chain below we define

three auxiliary networks: NET_c , NET_d and NET_e .

$$\begin{aligned}
& \text{GSP}_2(s_1[v]; s_2 \text{ did not gossip}) \\
& \stackrel{\text{def}}{=} s_1[\text{snd}\langle v \rangle_p]^{\nu_{s_1}} \mid s_2[\text{nil}]^{\nu_{s_2}} \mid \prod_{i=1}^3 n_i[\text{fwd}_p]^{\nu_{n_i}} \mid d[\text{fwd}_1]^{\nu_d} \\
1-p \sqsupseteq & \prod_{i=1}^2 s_i[\text{nil}]^{\nu_{s_i}} \mid n_1[\text{resnd}\langle v \rangle_p]^{\nu_{n_1}} \mid \prod_{i=2}^3 n_i[\text{fwd}_p]^{\nu_{n_i}} \mid d[\text{fwd}_1]^{\nu_d} \\
& \stackrel{\text{def}}{=} \text{NET}_c \\
0 \sqsupseteq & \prod_{i=1}^2 s_i[\text{nil}]^{\nu_{s_i}} \mid n_1[\sigma.\text{snd}\langle v \rangle_p]^{\nu_{n_1}} \mid \prod_{i=2}^3 n_i[\sigma.\text{fwd}_p]^{\nu_{n_i}} \mid d[\sigma.\text{fwd}_1]^{\nu_d} \\
& \stackrel{\text{def}}{=} \text{NET}_d \\
1-p \sqsupseteq & \prod_{i=1}^2 s_i[\text{nil}]^{\nu_{s_i}} \mid n_1[\sigma.\text{nil}]^{\nu_{n_1}} \mid n_2[\sigma.\text{fwd}_p]^{\nu_{n_2}} \mid n_3[\sigma.\text{resnd}\langle v \rangle_p]^{\nu_{n_3}} \mid d[\sigma.\text{fwd}_1]^{\nu_d} \\
& \stackrel{\text{def}}{=} \text{NET}_e \\
0 \sqsupseteq & \prod_{i=1}^2 s_i[\text{nil}]^{\nu_{s_i}} \mid n_1[\text{nil}]^{\nu_{n_1}} \mid n_2[\sigma^2.\text{fwd}_p]^{\nu_{n_2}} \mid n_3[\sigma^2.\text{snd}\langle v \rangle_p]^{\nu_{n_3}} \mid d[\sigma^2.\text{fwd}_1]^{\nu_d} \\
1-p \sqsupseteq & \prod_{i=1}^2 s_i[\text{nil}]^{\nu_{s_i}} \mid n_1[\text{nil}]^{\nu_{n_1}} \mid n_2[\sigma^2.\text{resnd}\langle v \rangle_p]^{\nu_{n_2}} \mid n_3[\sigma^2.\text{nil}]^{\nu_{n_3}} \mid d[\sigma^2.\text{resnd}\langle v \rangle_1]^{\nu_d} \\
0 \sqsupseteq & \prod_{i=1}^2 s_i[\text{nil}]^{\nu_{s_i}} \mid \prod_{i=1}^3 n_i[\text{nil}]^{\nu_{n_i}} \mid d[\sigma^3.\text{snd}\langle v \rangle_1]^{\nu_d} \\
& = \text{DONE}_2.
\end{aligned}$$

Then, by transitivity (Proposition 2.10) we can derive the equation

$$\text{NET}_e \ 1-p \sqsupseteq \text{DONE}_2.$$

This allows us to apply Proposition 2.13, in which M , N , and O are instantiated with DONE_2 , NET_e , and NET_d , respectively, to derive the equation

$$\text{NET}_d \ 1-p^2 \sqsupseteq \text{DONE}_2$$

where the tolerance $1-p^2$ is obtained by solving the expression $(1-p)(1-(1-p)) + (1-p)$. By transitivity we can easily derive:

$$\text{NET}_c \ 1-p^2 \sqsupseteq \text{DONE}_2.$$

This last equation can be used to apply Proposition 2.12, in which M , N , and O are instantiated with DONE_2 , NET_c , and $\text{GSP}_2(s_1[v]; s_2 \text{ did not gossip})$, respectively, to derive:

$$\text{GSP}_2(s_1[v]; s_2 \text{ did not gossip}) \ 1-p^3 \sqsupseteq \text{DONE}_2 \quad (3.3)$$

where the tolerance $1-p^3$ is obtained by solving the expression $(1-p)(1-(1-p^2)) + (1-p^2)$.

Finally, we can apply Corollary 3.3 to Equations 3.2 and 3.3 to derive:

$$\text{GSP}_2 \ 1-(3p^3-2p^4) \sqsupseteq \text{DONE}_2.$$

Thus, the gossip network GSP_2 will succeed in propagating the messages to the destination d with probability at least $3p^3 - 2p^4$. For instance, with a gossip probability $p = 0.8$ the destination will receive the message with probability 0.716, with a margin of 10%. For $p = 0.85$ the success probability increases to 0.798, with a margin of 6%; while for $p = 0.9$ the probability to propagate the message to the destination rises to 0.88, with a margin of only 2%. As a consequence, $p = 0.9$ can be considered the threshold of our small network.²

Note that in the previous example both messages may reach the destination node in exactly three rounds. However, more generally, we could have different message propagation paths in the same network which might take a different amount of time to be traversed. The

²Had we had more senders we would have estimated a better threshold.

algebraic theory we developed up to now does not allow us to deal with paths of different lengths. As an example, we would like to estimate the distance between the network

$$\text{GSP}_3 \stackrel{\text{def}}{=} s_1[\text{snd}\langle v \rangle_p]^{\nu_{s_1}} \mid s_2[\text{snd}\langle v \rangle_p]^{\nu_{s_2}} \mid n[\text{fwd}_p]^{\nu_n} \mid d[\text{fwd}_1]^{\nu_d}$$

with topology $\nu_{s_1} = \{d\}$, $\nu_{s_2} = \{n\}$, $\nu_n = \{s_2, d\}$ and $\nu_d = \{s_1, n, \text{tester}\}$, and the network

$$\text{DONE}_3 \stackrel{\text{def}}{=} s_1[\text{nil}]^{\nu_{s_1}} \mid s_2[\text{nil}]^{\nu_{s_2}} \mid n[\text{nil}]^{\nu_n} \mid d[\tau.(\sigma.\text{snd}\langle v \rangle_1 \oplus_p \sigma^2.\text{snd}\langle v \rangle_1)]^{\nu_d}$$

in which the message v propagated up to the destination node d following two different paths. Thus, d will probabilistically choose between broadcasting v after one or two rounds.

The following result provides the missing instrument.

Theorem 3.4 (Composing paths). *Let M be a well-formed network. Then,*

$$M \mid m[\tau. \bigoplus_{i \in I} p_i : Q_i]^{\nu_m} \quad r \sqsupseteq \prod_{j \in J} n_j[\text{nil}]^{\nu_{n_j}} \mid d[\tau. \bigoplus_{i \in I} p_i : P_i]^{\nu_d}$$

with $r = \sum_{i \in I} p_i s_i$, whenever $M \mid m[Q_i]^{\nu_m} \quad s_i \sqsupseteq \prod_{j \in J} n_j[\text{nil}]^{\nu_{n_j}} \mid d[P_i]^{\nu_d}$, for any $i \in I$.

Let us provide an estimation of the distance between GSP_3 and DONE_3 . As a first step, we compute an estimation of the tolerance which allows GSP_3 to simulate the first probabilistic behaviour of DONE_3 (broadcast after one round). To this end, we derive the following chain of similarities by applying, in sequence: (i) Theorem 3.1 to model the transmission of s_1 ; (ii) again Theorem 3.1 to model the transmission of s_2 ; (iii) Proposition 2.17(5) to model that the transmission of n gets lost, together with Proposition 2.17(2) to work underneath σ prefixes. In all steps, we reason up to parallel components (Theorem 2.15).

$$\begin{aligned} & \text{GSP}_3(s_1 \text{ gossiped}; s_2[v]) \\ & \stackrel{\text{def}}{=} s_1[\text{snd}\langle v \rangle_1]^{\nu_{s_1}} \mid s_2[\text{snd}\langle v \rangle_p]^{\nu_{s_2}} \mid n[\text{fwd}_p]^{\nu_n} \mid d[\text{fwd}_1]^{\nu_d} \\ & 0 \sqsupseteq s_1[\text{nil}]^{\nu_{s_1}} \mid s_2[\text{snd}\langle v \rangle_p]^{\nu_{s_2}} \mid n[\text{fwd}_p]^{\nu_n} \mid d[\text{resnd}\langle v \rangle_1]^{\nu_d} \\ & 1-p \sqsupseteq s_1[\text{nil}]^{\nu_{s_1}} \mid s_2[\text{nil}]^{\nu_{s_2}} \mid n[\text{resnd}\langle v \rangle_p]^{\nu_n} \mid d[\text{resnd}\langle v \rangle_1]^{\nu_d} \\ & = s_1[\text{nil}]^{\nu_{s_1}} \mid s_2[\text{nil}]^{\nu_{s_2}} \mid n[\sigma.\text{snd}\langle v \rangle_p]^{\nu_n} \mid d[\sigma.\text{snd}\langle v \rangle_1]^{\nu_d} \\ & 0 \sqsupseteq s_1[\text{nil}]^{\nu_{s_1}} \mid s_2[\text{nil}]^{\nu_{s_2}} \mid n[\text{nil}]^{\nu_n} \mid d[\sigma.\text{snd}\langle v \rangle_1]^{\nu_d}. \end{aligned}$$

By transitivity (Proposition 2.10) we derive:

$$\text{GSP}_3(s_1 \text{ gossiped}; s_2[v]) \quad 1-p \sqsupseteq s_1[\text{nil}]^{\nu_{s_1}} \mid s_2[\text{nil}]^{\nu_{s_2}} \mid n[\text{nil}]^{\nu_n} \mid d[\sigma.\text{snd}\langle v \rangle_1]^{\nu_d} \quad (3.4)$$

Then, we compute an estimation of the tolerance which allows the network GSP_3 to simulate the second probabilistic behaviour of DONE_3 . To this end, we derive the following chain of similarities by applying, in sequence: (i) Theorem 3.1 to model the transmission of s_2 ; (ii) Proposition 2.17(4) as without transmitters the node d is obliged to timeout; (iii) Theorem 3.1 as node n may transmit to d , working underneath a σ prefix (Proposition 2.17(2)); (iv) Proposition 2.17(1) to remove a σ prefix in node n . As usual, in all steps, we reason up to parallel components (Theorem 2.15). For convenience, in the

chain below we define two auxiliary networks: NET_f and NET_g .

$$\begin{aligned}
& \text{GSP}_3(s_1 \text{ did not gossip}; s_2[v]) \\
& \stackrel{\text{def}}{=} s_1[\text{nil}]^{\nu_{s_1}} \mid s_2[\text{snd}\langle v \rangle_p]^{\nu_{s_2}} \mid n[\text{fwd}_p]^{\nu_n} \mid d[\text{fwd}_1]^{\nu_d} \\
& \stackrel{1-p\sqsupseteq}{=} s_1[\text{nil}]^{\nu_{s_1}} \mid s_2[\text{nil}]^{\nu_{s_2}} \mid n[\text{resnd}\langle v \rangle_p]^{\nu_n} \mid d[\text{fwd}_1]^{\nu_d} \\
& \stackrel{\text{def}}{=} \text{NET}_f \\
& \stackrel{0\sqsupseteq}{=} s_1[\text{nil}]^{\nu_{s_1}} \mid s_2[\text{nil}]^{\nu_{s_2}} \mid n[\sigma.\text{snd}\langle v \rangle_p]^{\nu_n} \mid d[\sigma.\text{fwd}_1]^{\nu_d} \\
& \stackrel{1-p\sqsupseteq}{=} s_1[\text{nil}]^{\nu_{s_1}} \mid s_2[\text{nil}]^{\nu_{s_2}} \mid n[\sigma.\text{nil}]^{\nu_n} \mid d[\sigma.\text{resnd}\langle v \rangle_1]^{\nu_d} \\
& \stackrel{0\sqsupseteq}{=} s_1[\text{nil}]^{\nu_{s_1}} \mid s_2[\text{nil}]^{\nu_{s_2}} \mid n[\text{nil}]^{\nu_n} \mid d[\sigma^2.\text{snd}\langle v \rangle_1]^{\nu_d} \\
& \stackrel{\text{def}}{=} \text{NET}_g.
\end{aligned}$$

Then, by transitivity (Proposition 2.10) we can derive $\text{NET}_f \stackrel{1-p\sqsupseteq}{=} \text{NET}_g$. This last equation can be used to apply Proposition 2.12, in which M , N , and O are instantiated with NET_g , NET_f , and $\text{GSP}_3(s_1 \text{ did not gossip}; s_2[v])$, respectively, to derive:

$$\text{GSP}_3(s_1 \text{ did not gossip}; s_2[v]) \stackrel{1-p^2\sqsupseteq}{=} \text{NET}_g \quad (3.5)$$

where the tolerance $1-p^2$ is obtained by solving the expression $(1-p)(1-(1-p)) + (1-p)$.

Finally, we can apply Theorem 3.4 to Equations 3.4 and 3.5 to obtain:

$$\text{GSP}_3 \stackrel{r\sqsupseteq}{=} \text{DONE}_3$$

with $r = p(1-p) + (1-p)(1-p^2)$. Thus, the network GSP_3 will succeed in transmitting both messages v to the destination d , with probability at least $1-r$.

We conclude by observing that, in order to deal with paths of different length, one should apply Theorem 3.4 to all possible paths.

4. GOSSIPING WITH COLLISIONS

An important aspect of wireless communication is the presence of collisions: if two or more stations are transmitting over the same channel during the same round then a *communication collision* occurs at the receiver nodes. Collisions in wireless systems cannot be avoided, although there are protocols to reduce their occurrences (see, for instance, the protocol IEEE 802.11 CSMA/CA).

In Section 3, we have reasoned assuming no collisions. Here, we demonstrate that the presence of communication collisions deteriorates the performance of gossip protocols.

The calculus pTCWS allows us to easily express communication collisions. A receiver node experiences a collision if it is exposed to more than one transmission in the same round; as a consequence the reception in that round fails. We can model this behaviour by redefining the forwarder process as follows:

$$\text{resndc}\langle u \rangle_{p_g} \stackrel{\text{def}}{=} [?(x).\text{nil}]\text{snd}\langle u \rangle_{p_g} \quad \text{fwdc}_{p_g} \stackrel{\text{def}}{=} ?(x).\text{resndc}\langle x \rangle_{p_g}.$$

Here, the forwarder fwdc_{p_g} waits for a message in the current round. If a second message is received in the same round then the forwarding is doomed to fail; otherwise if no other messages are received, the process moves to next round and broadcasts the received message

Corollary 4.2. *If $M \xrightarrow{\sigma} \not\rightarrow$ then*

$$N \mid m[\text{snd}\langle v \rangle_p]^{\nu_m} \mid \prod_{j \in J} n_j[\text{resndc}\langle v \rangle_q]^{\nu_{n_j}} \quad r \sqsupseteq M$$

with $r = ps_1 + (1 - p)s_2$, whenever:

- $N \mid m[\text{nil}]^{\nu_m} \mid \prod_{j \in J} n_j[\text{nil}]^{\nu_{n_j}} \quad s_1 \sqsupseteq M$
- $N \mid m[\text{nil}]^{\nu_m} \mid \prod_{j \in J} n_j[\text{resndc}\langle v \rangle_q]^{\nu_{n_j}} \quad s_2 \sqsupseteq M$
- $\{n_j : j \in J\} \subseteq \nu_m \subseteq \{n_j : j \in J\} \cup \text{nds}(M)$
- nodes in $\nu_m \cap \text{nds}(M)$ cannot receive in the current round.

Corollary 4.3. *If $M \xrightarrow{\sigma} \not\rightarrow$ then*

$$N \mid m[\text{snd}\langle v \rangle_p]^{\nu_m} \mid \prod_{j \in J} n_j[\text{fwdc}_q]^{\nu_{n_j}} \quad r \sqsupseteq M$$

with $r = ps_1 + (1 - p)s_2$, whenever:

- $N \mid m[\text{nil}]^{\nu_m} \mid \prod_{j \in J} n_j[\text{resndc}\langle v \rangle_q]^{\nu_{n_j}} \quad s_1 \sqsupseteq M$
- $N \mid m[\text{nil}]^{\nu_m} \mid \prod_{j \in J} n_j[\text{fwdc}_q]^{\nu_{n_j}} \quad s_2 \sqsupseteq M$
- $\{n_j : j \in J\} \subseteq \nu_m \subseteq \{n_j : j \in J\} \cup \text{nds}(M)$
- nodes in $\nu_m \cap \text{nds}(M)$ cannot receive in the current round.

Let us apply the previous results to study how communication collisions degrade the performances of gossip protocols. Let us reformulate the network GSP_2 of the previous section in a scenario with potential communication collisions. Let us define

$$\text{GSP}_5 \stackrel{\text{def}}{=} \prod_{i=1}^2 s_i[\text{snd}\langle v \rangle_p]^{\nu_{s_i}} \mid \prod_{i=1}^3 n_i[\text{fwdc}_p]^{\nu_{n_i}} \mid d[\text{fwdc}_1]^{\nu_d}$$

with the same network topology as GSP_2 : $\nu_{s_1} = \{n_1\}$, $\nu_{s_2} = \{n_1, n_2\}$, $\nu_{n_1} = \{s_1, s_2, n_3\}$, $\nu_{n_2} = \{s_2, n_3\}$, $\nu_{n_3} = \{n_1, n_2, d\}$ and $\nu_d = \{n_3, \text{tester}\}$. Note that, in the same round, the sender s_2 may broadcast to its neighbours n_1 and n_2 , whereas the sender s_1 may broadcast (only) to its neighbour n_1 , causing a failure at n_1 .

As in the previous section, we will quantify the tolerance between the gossiping network GSP_5 and the network $\text{DONE}_5 = \text{DONE}_2$, in which at least one instance of the message v has been successfully propagated up to the destination node.

As a first step, we compute an estimation of the tolerance which allows GSP_5 to simulate DONE_5 , under the hypothesis that the sender s_1 broadcasts to n_1 and the sender s_2 broadcasts to n_1 and n_2 , causing a failure at n_1 . To this end, we derive the following chain of similarities by applying, in sequence: (i) Proposition 2.17(4) to nodes n_3 and d , together to Proposition 2.17(3) applied to node n_2 , to express that these nodes do not receive any message because there are no transmitters in their neighbourhood, as a consequence, they are obliged to timeout; (ii) Theorem 4.1 in which node n_2 acts as a sender and n_3 as a forwarder, together Proposition 2.17(2) to work underneath σ prefixes; (iii) Proposition 2.17(1) to remove a σ prefix in n_2 together with Proposition 2.17(4) as node d cannot receive message and hence it timeouts; (iv) Theorem 4.1 in which node n_3 acts as a sender and d as a forwarder, together with Proposition 2.17(2) to work underneath σ prefixes; (v) Proposition 2.17(1) to remove σ prefixes in n_3 , together with Proposition 2.17(3) as node d is obliged to timeout. In all steps, we reason up to parallel networks (Theorem 2.15). For

convenience, in the chain below we define two auxiliary networks: NET_h and NET_i .

$$\begin{aligned}
& \text{GSP}_5(s_1 \text{ gossiped}; s_2 \text{ gossiped}) \\
& \stackrel{\text{def}}{=} \prod_{i=1}^2 s_i[\text{nil}]^{\nu_{s_i}} \mid n_1[\text{nil}]^{\nu_{n_1}} \mid n_2[\text{resndc}\langle v \rangle_p]^{\nu_{n_2}} \mid n_3[\text{fwdc}_p]^{\nu_{n_3}} \mid d[\text{fwdc}_1]^{\nu_d} \\
& \quad 0 \sqsupseteq \prod_{i=1}^2 s_i[\text{nil}]^{\nu_{s_i}} \mid n_1[\text{nil}]^{\nu_{n_1}} \mid n_2[\sigma.\text{snd}\langle v \rangle_p]^{\nu_{n_2}} \mid n_3[\sigma.\text{fwdc}_p]^{\nu_{n_3}} \mid d[\sigma.\text{fwdc}_1]^{\nu_d} \\
& \stackrel{\text{def}}{=} \text{NET}_h \\
& \quad 1-p \sqsupseteq \prod_{i=1}^2 s_i[\text{nil}]^{\nu_{s_i}} \mid n_1[\text{nil}]^{\nu_{n_1}} \mid n_2[\sigma.\text{nil}]^{\nu_{n_2}} \mid n_3[\sigma.\text{resndc}\langle v \rangle_p]^{\nu_{n_3}} \mid d[\sigma.\text{fwdc}_1]^{\nu_d} \\
& \stackrel{\text{def}}{=} \text{NET}_i \\
& \quad 0 \sqsupseteq \prod_{i=1}^2 s_i[\text{nil}]^{\nu_{s_i}} \mid \prod_{i=1}^2 n_i[\text{nil}]^{\nu_{n_i}} \mid n_3[\sigma^2.\text{snd}\langle v \rangle_p]^{\nu_{n_3}} \mid d[\sigma^2.\text{fwdc}_1]^{\nu_d} \\
& \quad 1-p \sqsupseteq \prod_{i=1}^2 s_i[\text{nil}]^{\nu_{s_i}} \mid \prod_{i=1}^2 n_i[\text{nil}]^{\nu_{n_i}} \mid n_3[\sigma^2.\text{nil}]^{\nu_{n_3}} \mid d[\sigma^2.\text{resndc}\langle v \rangle_1]^{\nu_d} \\
& \quad 0 \sqsupseteq \prod_{i=1}^2 s_i[\text{nil}]^{\nu_{s_i}} \mid \prod_{i=1}^3 n_i[\text{nil}]^{\nu_{n_i}} \mid d[\sigma^3.\text{snd}\langle v \rangle_1]^{\nu_d} \\
& = \text{DONE}_5.
\end{aligned}$$

Now, by transitivity (Proposition 2.10) we can derive the following equation:

$$\text{NET}_i \quad 1-p \sqsupseteq \text{DONE}_5.$$

Thus, we are in condition to apply Proposition 2.13, in which M , N , and O are instantiated with DONE_5 , NET_i , and NET_h , respectively, to derive the equation:

$$\text{NET}_h \quad 1-p^2 \sqsupseteq \text{DONE}_5$$

where the tolerance $1-p^2$ is obtained by solving the expression $(1-p)(1-(1-p)) + (1-p)$. By transitivity on the first step of the chain above, we can finally derive the equation:

$$\text{GSP}_5(s_1 \text{ gossiped}; s_2 \text{ gossiped}) \quad 1-p^2 \sqsupseteq \text{DONE}_5 \tag{4.2}$$

Similarly, we compute an estimation of the tolerance which allows GSP_5 to simulate DONE_5 , under the hypothesis that the sender s_2 broadcasts to n_1 and n_2 , and the sender s_1 does not transmit. To this end, we derive the following chain of similarities by applying, in sequence: (i) Proposition 2.17(3) to timeout nodes n_1 and n_2 , together with Proposition 2.17(4) to timeout nodes n_3 and d ; (ii) Theorem 4.1 in which n_1 and n_2 act as senders and n_3 as forwarder, together with Proposition 2.17(2) to work underneath σ prefixes; (iii) Proposition 2.17(1) to remove σ prefixes from n_1 and n_2 , together with Proposition 2.17(3) to timeout n_3 , together with Proposition 2.17(4) to timeout d , together with Proposition 2.17(2) to work underneath σ prefixes; (iv) Theorem 4.1 where n_3 acts as sender and d as forwarder, together with Proposition 2.17(2) to work underneath σ prefixes; (v) Proposition 2.17(1) to remove σ prefixes in n_3 , together with Proposition 2.17(3) to timeout node d . As usual, in all steps, we reason up to parallel composition (Theorem 2.15).

For convenience, in the chain below we define two auxiliary networks: NET_l and NET_m .

$$\begin{aligned}
& \text{GSP}_5(s_1 \text{ did not gossip; } s_2 \text{ gossiped}) \\
& \stackrel{\text{def}}{=} \prod_{i=1}^2 s_i[\text{nil}]^{\nu_{s_i}} \mid \prod_{i=1}^2 n_i[\text{resndc}\langle v \rangle_p]^{\nu_{n_i}} \mid n_3[\text{fwdc}_p]^{\nu_{n_3}} \mid d[\text{fwdc}_1]^{\nu_d} \\
& 0 \sqsupseteq \prod_{i=1}^2 s_i[\text{nil}]^{\nu_{s_i}} \mid \prod_{i=1}^2 n_i[\sigma.\text{snd}\langle v \rangle_p]^{\nu_{n_i}} \mid n_3[\sigma.\text{fwdc}_p]^{\nu_{n_3}} \mid d[\sigma.\text{fwdc}_1]^{\nu_d} \\
& \stackrel{\text{def}}{=} \text{NET}_l \\
& 1-q \sqsupseteq \prod_{i=1}^2 s_i[\text{nil}]^{\nu_{s_i}} \mid \prod_{i=1}^2 n_i[\sigma.\text{nil}]^{\nu_{n_i}} \mid n_3[\sigma.\text{resndc}\langle v \rangle_p]^{\nu_{n_3}} \mid d[\sigma.\text{fwdc}_1]^{\nu_d} \\
& \stackrel{\text{def}}{=} \text{NET}_m \\
& 0 \sqsupseteq \prod_{i=1}^2 s_i[\text{nil}]^{\nu_{s_i}} \mid \prod_{i=1}^2 n_i[\text{nil}]^{\nu_{n_i}} \mid n_3[\sigma^2.\text{snd}\langle v \rangle_p]^{\nu_{n_3}} \mid d[\sigma^2.\text{fwdc}_1]^{\nu_d} \\
& 1-p \sqsupseteq \prod_{i=1}^2 s_i[\text{nil}]^{\nu_{s_i}} \mid \prod_{i=1}^2 n_i[\text{nil}]^{\nu_{n_i}} \mid n_3[\sigma^2.\text{nil}]^{\nu_{n_3}} \mid d[\sigma^2.\text{resndc}\langle v \rangle_1]^{\nu_d} \\
& 0 \sqsupseteq \prod_{i=1}^2 s_i[\text{nil}]^{\nu_{s_i}} \mid \prod_{i=1}^3 n_i[\text{nil}]^{\nu_{n_i}} \mid d[\sigma^3.\text{snd}\langle v \rangle_1]^{\nu_d} \\
& = \text{DONE}_5
\end{aligned}$$

with $q = 2p(1-p)$. By transitivity (Proposition 2.10) we can derive the equation

$$\text{NET}_m \ 1-p \sqsupseteq \text{DONE}_5.$$

Thus, we are in condition to apply Proposition 2.13, in which M , N , and O are instantiated with DONE_5 , NET_m , and NET_l , respectively, to derive the equation

$$\text{NET}_l \ 1-2(p^2-p^3) \sqsupseteq \text{DONE}_5$$

where the tolerance $1-2(p^2-p^3)$ is obtained by solving the expression $(1-p)(1-(1-q))+(1-q)$ with $q = 2p(1-p)$. By transitivity on the first step of the chain above, we can finally derive the equation

$$\text{GSP}_5(s_1 \text{ did not gossip; } s_2 \text{ gossiped}) \ 1-2(p^2-p^3) \sqsupseteq \text{DONE}_5 \quad (4.3)$$

Now, let us denote with $\text{GSP}_5(s_1[v]; s_2 \text{ gossiped})$ the following network:

$$s_1[\text{snd}\langle v \rangle_p]^{\nu_{s_1}} \mid s_2[\text{nil}]^{\nu_{s_2}} \mid \prod_{i=1}^2 n_i[\text{resndc}\langle v \rangle_p]^{\nu_{n_i}} \mid n_3[\text{fwdc}_p]^{\nu_{n_3}} \mid d[\text{fwdc}_1]^{\nu_d}.$$

This is the gossip network GSP_5 in which the sender s_2 has already broadcast the message v to its neighbours n_1 and n_2 , and the sender s_1 has still to decide whether to broadcast v (to n_1) or not. We can apply Corollary 4.2 to Equation 4.2 and Equation 4.3 to derive

$$\text{GSP}_5(s_1[v]; s_2 \text{ gossiped}) \ r \sqsupseteq \text{DONE}_5 \quad (4.4)$$

with tolerance $r = 1 - 2p^2 + 3p^3 - 2p^4$, obtained by solving $p(1-p^2) + (1-p)(1-2(p^2-p^3))$.

This gives us an estimation of the tolerance which allows the network GSP_5 to simulate the network DONE_5 when the sender s_2 has already broadcast the message v to its neighbours n_1 and n_2 , whereas the sender s_1 has still to decide whether to broadcast v (to n_1) or not.

Similarly, we compute an estimation of the tolerance which allows the network GSP_5 to simulate the network DONE_5 when the sender s_2 has decided not to broadcast the message v , while s_1 has still to decide whether to broadcast v or not. To this end, we derive the following chain of similarities by applying, in sequence: (i) Theorem 4.1 where s_1 broadcasts its message and n_1 receives it; (ii) Proposition 2.17(3) to timeout node n_1 , together with Proposition 2.17(4) to timeout nodes n_2 , n_3 , and d ; (iii) Theorem 4.1 where n_1 acts as transmitter and n_3 as receiver, together with Proposition 2.17(2) to work underneath σ prefixes; (iv) Proposition 2.17(1) to remove the σ prefix in n_1 , together with

Proposition 2.17(4) to timeout nodes n_2 and d , and Proposition 2.17(3) to timeout n_3 ; (v) Theorem 4.1 in which n_3 transmits its value and both n_2 and d receive it, together with Proposition 2.17(2) to work underneath σ prefixes; (vi) Proposition 2.17(1) to remove σ prefixes from n_3 , together with Proposition 2.17(3) to timeout n_2 ; Proposition 2.17(5) to model that the transmission of n gets lost, together with Proposition 2.17(2) to work underneath σ prefixes. As done before, in all steps we use Theorem 2.15 to reason up to parallel components. For convenience, in the chain below we define three auxiliary networks: NET_n , NET_o and NET_t .

$$\begin{aligned}
& \text{GSP}_5(s_1[v]; s_2 \text{ did not gossip}) \\
& \stackrel{\text{def}}{=} s_1[\text{snd}\langle v \rangle_p]^{\nu_{s_1}} \mid s_2[\text{nil}]^{\nu_{s_2}} \mid \prod_{i=1}^3 n_i[\text{fwdc}_p]^{\nu_{n_i}} \mid d[\text{fwdc}_1]^{\nu_d} \\
1-p \sqsupseteq & \prod_{i=1}^2 s_i[\text{nil}]^{\nu_{s_i}} \mid n_1[\text{resndc}\langle v \rangle_p]^{\nu_{n_1}} \mid \prod_{j=2}^3 n_j[\text{fwdc}_p]^{\nu_{n_j}} \mid d[\text{fwdc}_1]^{\nu_d} \\
& \stackrel{\text{def}}{=} \text{NET}_n \\
0 \sqsupseteq & \prod_{i=1}^2 s_i[\text{nil}]^{\nu_{s_i}} \mid n_1[\sigma.\text{snd}\langle v \rangle_p]^{\nu_{n_1}} \mid \prod_{i=2}^3 n_i[\sigma.\text{fwdc}_p]^{\nu_{n_i}} \mid d[\sigma.\text{fwdc}_1]^{\nu_d} \\
& \stackrel{\text{def}}{=} \text{NET}_o \\
1-p \sqsupseteq & \prod_{i=1}^2 s_i[\text{nil}]^{\nu_{s_i}} \mid n_1[\sigma.\text{nil}]^{\nu_{n_1}} \mid n_2[\sigma.\text{fwdc}_p]^{\nu_{n_2}} \mid n_3[\sigma.\text{resndc}\langle v \rangle_p]^{\nu_{n_3}} \mid d[\sigma.\text{fwdc}_1]^{\nu_d} \\
& \stackrel{\text{def}}{=} \text{NET}_t \\
0 \sqsupseteq & \prod_{i=1}^2 s_i[\text{nil}]^{\nu_{s_i}} \mid n_1[\text{nil}]^{\nu_{n_1}} \mid n_2[\sigma^2.\text{fwdc}_p]^{\nu_{n_2}} \mid n_3[\sigma^2.\text{snd}\langle v \rangle_p]^{\nu_{n_3}} \mid d[\sigma^2.\text{fwdc}_1]^{\nu_d} \\
1-p \sqsupseteq & \prod_{i=1}^2 s_i[\text{nil}]^{\nu_{s_i}} \mid n_1[\text{nil}]^{\nu_{n_1}} \mid n_2[\sigma^2.\text{resndc}\langle v \rangle_p]^{\nu_{n_2}} \mid n_3[\sigma^2.\text{nil}]^{\nu_{n_3}} \mid d[\sigma^2.\text{resndc}\langle v \rangle_1]^{\nu_d} \\
0 \sqsupseteq & \prod_{i=1}^2 s_i[\text{nil}]^{\nu_{s_i}} \mid n_1[\text{nil}]^{\nu_{n_1}} \mid n_2[\sigma^3.\text{snd}\langle v \rangle_p]^{\nu_{n_2}} \mid n_3[\text{nil}]^{\nu_{n_3}} \mid d[\sigma^3.\text{snd}\langle v \rangle_1]^{\nu_d} \\
0 \sqsupseteq & \prod_{i=1}^2 s_i[\text{nil}]^{\nu_{s_i}} \mid \prod_{i=1}^3 n_i[\text{nil}]^{\nu_{n_i}} \mid d[\sigma^3.\text{snd}\langle v \rangle_1]^{\nu_d} \\
& = \text{DONE}_5.
\end{aligned}$$

By transitivity (Proposition 2.10) we can derive the equation

$$\text{NET}_t \ 1-p \sqsupseteq \text{DONE}_5.$$

This allows us to apply Proposition 2.13, in which M , N , and O are instantiated with DONE_5 , NET_t , and NET_o , respectively, to derive the equation

$$\text{NET}_o \ 1-p^2 \sqsupseteq \text{DONE}_5$$

where the tolerance $1-p^2$ is obtained by solving the expression $(1-p)(1-(1-p)) + (1-p)$. Then, by transitivity (Proposition 2.10) we can easily derive

$$\text{NET}_n \ 1-p^2 \sqsupseteq \text{DONE}_5.$$

This last equation can be used to apply Proposition 2.12, in which M , N , and O are instantiated with DONE_5 , NET_n , and $\text{GSP}_5(s_1[v]; s_2 \text{ did not gossip})$, respectively, to derive:

$$\text{GSP}_5(s_1[v]; s_2 \text{ did not gossip}) \ 1-p^3 \sqsupseteq \text{DONE}_5 \tag{4.5}$$

where the tolerance $1-p^3$ is obtained by solving the expression $(1-p)(1-(1-p^2)) + (1-p^2)$.

Now, by applying Corollary 4.3 to Equation 4.4 and Equation 4.5 we can combine the two probabilistic behaviours of the sender s_2 , thus obtaining:

$$\text{GSP}_5 \ r \sqsupseteq \text{DONE}_5$$

with tolerance $r = 1 - (3p^3 - 4p^4 + 2p^5)$ obtained by solving $p(1 - 2p^2 + 3p^3 - 2p^4) + (1 - p)(1 - p^3)$. This says that the gossip network GSP_5 will succeed in transmitting the message v to the destination d with probability (at least) $3p^3 - 4p^4 + 2p^5$.

In the previous section, we have seen that in the absence of collisions, the success probability of GSP_2 is (at least) $3p^3 - 2p^4$. Since $3p^3 - 4p^4 + 2p^5 < 3p^3 - 2p^4$, for any $p > 0$, it follows that the presence of collisions downgrades the performance of this small network of $2p^4 - 2p^5$. Thus, for instance, for a gossip probability $p = 0.8$ the destination node in GSP_5 will receive the message with probability (at least) 0.55, whereas in GSP_2 this probability is 0.716; similarly for $p = 0.9$ the probability of success in GSP_5 is about 0.74 while in GSP_2 it is 0.88. In bigger networks, with more collisions, the degradation will be much more evident.

Remark 4.4. Due to its general formulation, Theorem 3.4, dealing with message propagation in paths of different lengths, is valid also in the presence of communication collisions.

5. RANDOM DELAYS TO MITIGATE THE EFFECTS OF COLLISIONS

In this section, we show how pTCWS can be used to model and reason on *randomised gossip protocols*, where messages may be broadcast in different time instants, according to some probability distribution (see, *e.g.*, [FG06]). In this manner, if there is a potential collision at a given round there are still chances that the message will reach the final destination in the following rounds, improving the probability of success of the gossip protocol.

Randomisation may be implemented in different ways. For instance, one may decide to broadcast messages according to a *uniform probability distribution* within a *discrete* time interval $1..k$, for $k \in \mathbb{N}$. This means that the message may be transmitted in each of the k time instants, with a probability $\frac{1}{k}$. In such a scenario, both sender and forwarder processes should be reformulated as follows:

$$\begin{aligned} \text{sndu}\langle v \rangle_{p_g, k} &\stackrel{\text{def}}{=} \tau.(\tau.(\bigoplus_{i=1}^k \frac{1}{k} : \sigma^i.\text{snd}\langle v \rangle_1) \oplus_{p_g} \text{nil}) \\ \text{resndu}\langle v \rangle_{p_g, k} &\stackrel{\text{def}}{=} [?(y).\text{nil}]\text{sndu}\langle v \rangle_{p_g, k} \\ \text{fwdu}_{p_g, k} &\stackrel{\text{def}}{=} ?(x).\text{resndu}\langle x \rangle_{p_g, k}. \end{aligned}$$

The main algebraic instrument to reason on the performance of the protocol remains message propagation. The following result is a straightforward reformulation of Theorem 4.1. In Corollary 5.1, for simplicity, we have only one sender that broadcasts with probability 1. This sender derives by the resolution of the probabilistic choice within any term $\text{sndu}\langle v \rangle_{p_g, k}$.

Corollary 5.1 (Message propagation with collisions and random delays). *Let J be a subset of \mathbb{N} . Let M be a well-formed network defined by*

$$M \equiv N \mid m[\text{snd}\langle v \rangle_1]^{\nu_m} \mid \prod_{j \in J} n_j[\text{fwdu}_{q_j, k}]^{\nu_{n_j}}$$

such that: (1) $\{n_j \mid j \in J\} \subseteq \nu_m \subseteq \text{nds}(M)$, and (2) the nodes in $\nu_m \cap \text{nds}(N)$ cannot receive in the current round. Then,

$$M \text{ }_0\sqsupseteq N \mid m[\text{nil}]^{\nu_m} \mid \prod_{j \in J} n_j[\text{resndu}\langle v \rangle_{q_j, k}]^{\nu_{n_j}}.$$

In the rest of this section, we will apply Corollary 5.1 to reason on a variant of the network GSP_1 of Section 3 where random delays are introduced.

Consider a gossiping network GSP_6 , with gossip probability p , composed by two source nodes s_1 and s_2 that may delay their transmission within a discrete time interval 1..2, and a destination node d . The network is the following:

$$\text{GSP}_6 \stackrel{\text{def}}{=} \prod_{i=1}^2 s_i[\text{sndu}\langle v \rangle_{p,2}]^{\nu_{s_i}} \mid d[\text{fwdu}_{1,1}]^{\nu_d}$$

where $\nu_{s_1} = \nu_{s_2} = \{d\}$ and $\nu_d = \{s_1, s_2, \text{tester}\}$.

Our goal is to estimate the distance between GSP_6 and a network DONE_6 in which the message v propagated up to the destination node d that will rebroadcast the message according to some probabilistic delay depending on the probability p .

$$\text{DONE}_6 \stackrel{\text{def}}{=} \prod_{i=1}^2 s_i[\text{nil}]^{\nu_{s_i}} \mid d[\tau.(\sigma^3.\text{snd}\langle v \rangle_1 \oplus_{(\frac{1}{2} + \frac{p^2}{2})} \sigma^4.\text{snd}\langle v \rangle_1)]^{\nu_d}.$$

At the end of this section, we will show that $\text{GSP}_6 \ r \sqsupseteq \text{DONE}_6$, with tolerance $r = 1 - (2p - \frac{3}{2}p^2)$. In order to estimate the tolerance r we distinguish three possible cases, depending whether the sender nodes s_1 and s_2 eventually transmit or not. More precisely, we estimate the tolerance of the simulation in the following cases: (a) node s_1 transmits while s_2 does not, (b) node s_2 transmits while s_1 does not, (c) both nodes s_1 and s_2 transmit. Notice that we do not analyse the case when neither s_1 nor s_2 transmits, since such a behaviour does not allow GSP_6 to simulate DONE_6 . Then, by applying Theorem 3.4 to compose different paths we combine the tolerances obtained in cases (b) and (c) to get the tolerance when s_2 transmits independently whether s_1 transmits or not. Finally, we use again Theorem 3.4 to combine this tolerance with that obtained from case (a) to infer r .

Let us analyse these three different cases.

(a) *Node s_1 transmits while s_2 does not.* In this case, there are obviously no collisions and the message will eventually reach the destination. Now, if s_1 decides to transmit with a delay of h time units, for $h \in \{1, 2\}$, then, by applying in sequence: (i) Proposition 2.17(4) to timeout node d , as for h rounds it will not get any message; (ii) Corollary 5.1 to propagate the message from the sender s_1 to the node d , together with Proposition 2.17(2) to work underneath σ prefixes; (iii) Proposition 2.17(1) to remove σ prefixes in s_1 ; (iv) Proposition 2.17(3) together with the straightforward equation $d[\text{sndu}\langle v \rangle_{1,1}]^{\nu_d} \ 0 \sqsupseteq d[\sigma.\text{snd}\langle v \rangle_1]^{\nu_d}$; it follows that:

$$\begin{aligned} & \text{GSP}_6(s_1 \text{ sends in } h \text{ rounds; } s_2 \text{ did not gossip}) \\ \stackrel{\text{def}}{=} & s_1[\sigma^h.\text{snd}\langle v \rangle_1]^{\nu_{s_1}} \mid s_2[\text{nil}]^{\nu_{s_2}} \mid d[\text{fwdu}_{1,1}]^{\nu_d} \\ 0 \sqsupseteq & s_1[\sigma^h.\text{snd}\langle v \rangle_1]^{\nu_{s_1}} \mid s_2[\text{nil}]^{\nu_{s_2}} \mid d[\sigma^h.\text{fwdu}_{1,1}]^{\nu_d} \\ 0 \sqsupseteq & s_1[\sigma^h.\text{nil}]^{\nu_{s_1}} \mid s_2[\text{nil}]^{\nu_{s_2}} \mid d[\sigma^h.\text{resndu}\langle v \rangle_{1,1}]^{\nu_d} \\ 0 \sqsupseteq & s_1[\text{nil}]^{\nu_{s_1}} \mid s_2[\text{nil}]^{\nu_{s_2}} \mid d[\sigma^h.\text{resndu}\langle v \rangle_{1,1}]^{\nu_d} \\ 0 \sqsupseteq & \prod_{i=1}^2 s_i[\text{nil}]^{\nu_{s_i}} \mid d[\sigma^{h+2}.\text{snd}\langle v \rangle_1]^{\nu_d}. \end{aligned}$$

By transitivity (Proposition 2.10) we can derive:

$$\text{GSP}_6(s_1 \text{ sends in } h \text{ rounds; } s_2 \text{ did not gossip}) \ 0 \sqsupseteq \prod_{i=1}^2 s_i[\text{nil}]^{\nu_{s_i}} \mid d[\sigma^{h+2}.\text{snd}\langle v \rangle_1]^{\nu_d} \quad (5.1)$$

We know that s_1 transmits in GSP_6 according to a uniform distribution probability in the time interval 1..2. Thus, by an application of Theorem 3.4, we can combine the two instances of Equation 5.1, for $h = 1$ and $h = 2$, to obtain:

$$\begin{aligned} & \text{GSP}_6(s_1 \text{ sends in 1 or 2 rounds}; s_2 \text{ did not gossip}) \\ \stackrel{\text{def}}{=} & s_1[\tau.(\sigma.\text{snd}\langle v \rangle_1 \oplus_{\frac{1}{2}} \sigma^2.\text{snd}\langle v \rangle_1)]^{\nu_{s_1}} \mid s_2[\text{nil}]^{\nu_{s_2}} \mid d[\text{fwd}u_{1,1}]^{\nu_d} \\ \stackrel{0\sqsupseteq}{=} & \prod_{i=1}^2 s_i[\text{nil}]^{\nu_{s_i}} \mid d[\tau.(\sigma^3.\text{snd}\langle v \rangle_1 \oplus_{\frac{1}{2}} \sigma^4.\text{snd}\langle v \rangle_1)]^{\nu_d}. \end{aligned} \quad (5.2)$$

The complete behaviour of s_1 can be derived by the following chain of similarities, obtained by an application of Proposition 2.14(1) and Equation 5.2:

$$\begin{aligned} & \text{GSP}_6(s_1[v]; s_2 \text{ did not gossip}) \\ \stackrel{\text{def}}{=} & s_1[\text{snd}u\langle v \rangle_{p,2}]^{\nu_{s_1}} \mid s_2[\text{nil}]^{\nu_{s_2}} \mid d[\text{fwd}u_{1,1}]^{\nu_d} \\ \stackrel{1-p\sqsupseteq}{=} & s_1[\tau.(\sigma.\text{snd}\langle v \rangle_1 \oplus_{\frac{1}{2}} \sigma^2.\text{snd}\langle v \rangle_1)]^{\nu_{s_1}} \mid s_2[\text{nil}]^{\nu_{s_2}} \mid d[\text{fwd}u_{1,1}]^{\nu_d} \\ \stackrel{0\sqsupseteq}{=} & \prod_{i=1}^2 s_i[\text{nil}]^{\nu_{s_i}} \mid d[\tau.(\sigma^3.\text{snd}\langle v \rangle_1 \oplus_{\frac{1}{2}} \sigma^4.\text{snd}\langle v \rangle_1)]^{\nu_d}. \end{aligned}$$

Finally, by Proposition 2.10 we get:

$$\text{GSP}_6(s_1[v]; s_2 \text{ did not gossip}) \stackrel{1-p\sqsupseteq}{=} \prod_{i=1}^2 s_i[\text{nil}]^{\nu_{s_i}} \mid d[\tau.(\sigma^3.\text{snd}\langle v \rangle_1 \oplus_{\frac{1}{2}} \sigma^4.\text{snd}\langle v \rangle_1)]^{\nu_d} \quad (5.3)$$

Let us consider now the symmetric case.

(b) *Node s_2 transmits while s_1 does not.* This case can be arranged by switching the roles of the nodes s_1 and s_2 in Equation 5.2:

$$\begin{aligned} & \text{GSP}_6(s_1 \text{ did not gossip}; s_2 \text{ sends in 1 or 2 rounds}) \\ \stackrel{\text{def}}{=} & s_1[\text{nil}]^{\nu_{s_1}} \mid s_2[\tau.(\sigma.\text{snd}\langle v \rangle_1 \oplus_{\frac{1}{2}} \sigma^2.\text{snd}\langle v \rangle_1)]^{\nu_{s_2}} \mid d[\text{fwd}u_{1,1}]^{\nu_d} \\ \stackrel{0\sqsupseteq}{=} & \prod_{i=1}^2 s_i[\text{nil}]^{\nu_{s_i}} \mid d[\tau.(\sigma^3.\text{snd}\langle v \rangle_1 \oplus_{\frac{1}{2}} \sigma^4.\text{snd}\langle v \rangle_1)]^{\nu_d} \end{aligned} \quad (5.4)$$

Finally, let us consider the last case.

(c) *Both nodes s_1 and s_2 transmit.* If both senders decide to transmit in the same time instant then there will be a collision and the transmitted message will never reach the destination. If this happens then the network GSP_6 cannot simulate a network in which the message has reached the destination node d . On the other hand, if s_1 broadcasts with a delay of one time unit, and s_2 with a delay of two time units, then there will be no collisions and the message may eventually reach the destination.

Thus, by an application of: (i) Proposition 2.17(4) to timeout node d in the absence of a sender; (ii) Corollary 5.1 to broadcast the message of s_1 to d , together with Proposition 2.17(2) to work underneath σ prefixes; (iii) Proposition 2.17(3) to timeout node d , as after one round there are no senders around; (iv) the straightforward equation $d[\text{snd}u\langle v \rangle_{1,1}]^{\nu_d} \stackrel{0\sqsupseteq}{=} d[\sigma.\text{snd}\langle v \rangle_1]^{\nu_d}$, together with Proposition 2.17(2) to work underneath σ prefixes; (v) Proposition 2.17(5) to model that the message broadcast by s_2 gets lost,

together with Proposition 2.17(2) to work underneath σ prefixes; (vi) Proposition 2.17(1) to remove σ prefixes in s_1 and s_2 ; it follows that:

$$\begin{aligned}
& \text{GSP}_6(s_1 \text{ sends in 1 round; } s_2 \text{ sends in 2 rounds}) \\
& \stackrel{\text{def}}{=} s_1[\sigma.\text{snd}\langle v \rangle_1]^{\nu_{s_1}} \mid s_2[\sigma^2.\text{snd}\langle v \rangle_1]^{\nu_{s_2}} \mid d[\text{fwdu}_{1,1}]^{\nu_d} \\
0 \sqsupseteq & s_1[\sigma.\text{snd}\langle v \rangle_1]^{\nu_{s_1}} \mid s_2[\sigma^2.\text{snd}\langle v \rangle_1]^{\nu_{s_2}} \mid d[\sigma.\text{fwdu}_{1,1}]^{\nu_d} \\
0 \sqsupseteq & s_1[\sigma.\text{nil}]^{\nu_{s_1}} \mid s_2[\sigma^2.\text{snd}\langle v \rangle_1]^{\nu_{s_2}} \mid d[\sigma.\text{resndu}\langle v \rangle_{1,1}]^{\nu_d} \\
0 \sqsupseteq & s_1[\sigma.\text{nil}]^{\nu_{s_1}} \mid s_2[\sigma^2.\text{snd}\langle v \rangle_1]^{\nu_{s_2}} \mid d[\sigma.\sigma.\text{sndu}\langle v \rangle_{1,1}]^{\nu_d} \\
0 \sqsupseteq & s_1[\sigma.\text{nil}]^{\nu_{s_1}} \mid s_2[\sigma^2.\text{snd}\langle v \rangle_1]^{\nu_{s_2}} \mid d[\sigma.\sigma.\sigma.\text{snd}\langle v \rangle_1]^{\nu_d} \\
0 \sqsupseteq & s_1[\sigma.\text{nil}]^{\nu_{s_1}} \mid s_2[\sigma^2.\text{nil}]^{\nu_{s_2}} \mid d[\sigma^3.\text{snd}\langle v \rangle_1]^{\nu_d} \\
0 \sqsupseteq & \prod_{i=1}^2 s_i[\text{nil}]^{\nu_{s_i}} \mid d[\sigma^3.\text{snd}\langle v \rangle_1]^{\nu_d}.
\end{aligned}$$

By transitivity (Proposition 2.10) we derive:

$$\text{GSP}_6(s_1 \text{ sends in 1 round; } s_2 \text{ sends in 2 rounds}) \ 0 \sqsupseteq \prod_{i=1}^2 s_i[\text{nil}]^{\nu_{s_i}} \mid d[\sigma^3.\text{snd}\langle v \rangle_1]^{\nu_d} \quad (5.5)$$

As nodes s_1 and s_2 have the same neighbours, the symmetric case, when s_1 broadcasts with a delay of two rounds and s_2 with a delay of one round, can be captured by switching the roles of s_1 and s_2 in Equation 5.5:

$$\begin{aligned}
& \text{GSP}_6(s_1 \text{ sends in 2 rounds ; } s_2 \text{ sends in 1 round}) \\
& \stackrel{\text{def}}{=} s_1[\sigma^2.\text{snd}\langle v \rangle_1]^{\nu_{s_1}} \mid s_2[\sigma.\text{snd}\langle v \rangle_1]^{\nu_{s_2}} \mid d[\text{fwdu}_{1,1}]^{\nu_d} \quad (5.6) \\
0 \sqsupseteq & \prod_{i=1}^2 s_i[\text{nil}]^{\nu_{s_i}} \mid d[\sigma^3.\text{snd}\langle v \rangle_1]^{\nu_d}.
\end{aligned}$$

Next, we compose the previous estimations to evaluate the tolerance when both senders transmit with a probabilistic delay. Please, notice that the following law holds for any network N (1 is the maximum distance):

$$s_1[\sigma^h.\text{snd}\langle v \rangle_1]^{\nu_{s_1}} \mid s_2[\sigma^h.\text{snd}\langle v \rangle_1]^{\nu_{s_2}} \mid d[\text{fwdu}_{1,1}]^{\nu_d} \ 1 \sqsupseteq N$$

In the rest of this section, we will use the following instances of the law above:

$$s_1[\sigma.\text{snd}\langle v \rangle_1]^{\nu_{s_1}} \mid s_2[\sigma.\text{snd}\langle v \rangle_1]^{\nu_{s_2}} \mid d[\text{fwdu}_{1,1}]^{\nu_d} \ 1 \sqsupseteq \prod_{i=1}^2 s_i[\text{nil}]^{\nu_{s_i}} \mid d[\sigma^3.\text{snd}\langle v \rangle_1]^{\nu_d} \quad (5.7)$$

$$s_1[\sigma^2.\text{snd}\langle v \rangle_1]^{\nu_{s_1}} \mid s_2[\sigma^2.\text{snd}\langle v \rangle_1]^{\nu_{s_2}} \mid d[\text{fwdu}_{1,1}]^{\nu_d} \ 1 \sqsupseteq \prod_{i=1}^2 s_i[\text{nil}]^{\nu_{s_i}} \mid d[\sigma^3.\text{snd}\langle v \rangle_1]^{\nu_d} \quad (5.8)$$

Thus, in the case when s_1 broadcasts with a delay of one round and s_2 with a probabilistic delay of one or two rounds, by applying, in sequence: (i) Theorem 3.4 in which the network M is structural congruent to $s_1[\sigma.\text{snd}\langle v \rangle_1]^{\nu_{s_1}} \mid d[\text{fwdu}_{1,1}]^{\nu_d}$, and the node m is instantiated as s_2 , to compose the paths deriving from Equation 5.5 and Equation 5.7; (ii) the straightforward law $d[\tau.(\sigma^3.\text{snd}\langle v \rangle_1 \oplus_{\frac{1}{2}} \sigma^3.\text{snd}\langle v \rangle_1)]^{\nu_d} \ 0 \sqsupseteq d[\sigma^3.\text{snd}\langle v \rangle_1]^{\nu_d}$; we derive:

$$\begin{aligned}
& \text{GSP}_6(s_1 \text{ sends in 1 round; } s_2 \text{ sends in 1 or 2 rounds}) \\
& \stackrel{\text{def}}{=} s_1[\sigma.\text{snd}\langle v \rangle_1]^{\nu_{s_1}} \mid s_2[\tau.(\sigma.\text{snd}\langle v \rangle_1 \oplus_{\frac{1}{2}} \sigma^2.\text{snd}\langle v \rangle_1)]^{\nu_{s_2}} \mid d[\text{fwdu}_{1,1}]^{\nu_d} \\
\frac{1}{2} \sqsupseteq & \prod_{i=1}^2 s_i[\text{nil}]^{\nu_{s_i}} \mid d[\tau.(\sigma^3.\text{snd}\langle v \rangle_1 \oplus_{\frac{1}{2}} \sigma^3.\text{snd}\langle v \rangle_1)]^{\nu_d} \\
0 \sqsupseteq & \prod_{i=1}^2 s_i[\text{nil}]^{\nu_{s_i}} \mid d[\sigma^3.\text{snd}\langle v \rangle_1]^{\nu_d}.
\end{aligned}$$

By transitivity (Proposition 2.10) we obtain

$$\text{GSP}_6(s_1 \text{ sends in 1 rnd; } s_2 \text{ sends in 1 or 2 rnds)} \stackrel{\frac{1}{2} \sqsupseteq}{=} \prod_{i=1}^2 s_i[\text{nil}]^{\nu_{s_i}} \mid d[\sigma^3.\text{snd}\langle v \rangle_1]^{\nu_d} \quad (5.9)$$

Analogously, in the case when s_1 broadcasts with a delay of two rounds and s_2 with a probabilistic delay of one or two rounds, we can apply in sequence: (i) Theorem 3.4 in which the network M is structural congruent to $s_1[\sigma^2.\text{snd}\langle v \rangle_1]^{\nu_{s_1}} \mid d[\text{fwd}_{u_1,1}]^{\nu_d}$ and the node m is instantiated as s_2 , to compose the paths deriving from Equation 5.6 and Equation 5.8; (ii) the straightforward law $d[\tau.(\sigma^3.\text{snd}\langle v \rangle_1 \oplus_{\frac{1}{2}} \sigma^3.\text{snd}\langle v \rangle_1)]^{\nu_d} \stackrel{0 \sqsupseteq}{=} d[\sigma^3.\text{snd}\langle v \rangle_1]^{\nu_d}$; we derive:

$$\begin{aligned} & \text{GSP}_6(s_1 \text{ sends in 2 rounds; } s_2 \text{ sends in 1 or 2 rounds)} \\ \stackrel{\text{def}}{=} & s_1[\sigma^2.\text{snd}\langle v \rangle_1]^{\nu_{s_1}} \mid s_2[\tau.(\sigma.\text{snd}\langle v \rangle_1 \oplus_{\frac{1}{2}} \sigma^2.\text{snd}\langle v \rangle_1)]^{\nu_{s_2}} \mid d[\text{fwd}_{u_1,1}]^{\nu_d} \\ \stackrel{\frac{1}{2} \sqsupseteq}{=} & \prod_{i=1}^2 s_i[\text{nil}]^{\nu_{s_i}} \mid d[\tau.(\sigma^3.\text{snd}\langle v \rangle_1 \oplus_{\frac{1}{2}} \sigma^3.\text{snd}\langle v \rangle_1)]^{\nu_d} \\ \stackrel{0 \sqsupseteq}{=} & \prod_{i=1}^2 s_i[\text{nil}]^{\nu_{s_i}} \mid d[\sigma^3.\text{snd}\langle v \rangle_1]^{\nu_d}. \end{aligned}$$

By transitivity (Proposition 2.10) we obtain

$$\text{GSP}_6(s_1 \text{ sends in 2 rnds; } s_2 \text{ sends in 1 or 2 rnds)} \stackrel{\frac{1}{2} \sqsupseteq}{=} \prod_{i=1}^2 s_i[\text{nil}]^{\nu_{s_i}} \mid d[\sigma^3.\text{snd}\langle v \rangle_1]^{\nu_d} \quad (5.10)$$

Thus, to estimate the tolerance when both s_1 and s_2 broadcast with a probabilistic delay of one or two steps, we can apply in sequence: (i) Theorem 3.4 in which the network M is structural congruent to $s_2[\tau.(\sigma.\text{snd}\langle v \rangle_1 \oplus_{\frac{1}{2}} \sigma^2.\text{snd}\langle v \rangle_1)]^{\nu_{s_2}} \mid d[\text{fwd}_{u_1,1}]^{\nu_d}$ and the node m is instantiated as s_1 , to compose the paths deriving from Equation 5.9 and Equation 5.10; (ii) the straightforward law $d[\tau.(\sigma^3.\text{snd}\langle v \rangle_1 \oplus_{\frac{1}{2}} \sigma^3.\text{snd}\langle v \rangle_1)]^{\nu_d} \stackrel{0 \sqsupseteq}{=} d[\sigma^3.\text{snd}\langle v \rangle_1]^{\nu_d}$; to obtain:

$$\begin{aligned} & \text{GSP}_6(\text{both } s_1 \text{ and } s_2 \text{ send in 1 or 2 rounds)} \\ \stackrel{\text{def}}{=} & \prod_{i=1}^2 s_i[\tau.(\sigma.\text{snd}\langle v \rangle_1 \oplus_{\frac{1}{2}} \sigma^2.\text{snd}\langle v \rangle_1)]^{\nu_{s_i}} \mid d[\text{fwd}_{u_1,1}]^{\nu_d} \\ \stackrel{\frac{1}{2} \sqsupseteq}{=} & \prod_{i=1}^2 s_i[\text{nil}]^{\nu_{s_i}} \mid d[\tau.(\sigma^3.\text{snd}\langle v \rangle_1 \oplus_{\frac{1}{2}} \sigma^3.\text{snd}\langle v \rangle_1)]^{\nu_d} \\ \stackrel{0 \sqsupseteq}{=} & \prod_{i=1}^2 s_i[\text{nil}]^{\nu_{s_i}} \mid d[\sigma^3.\text{snd}\langle v \rangle_1]^{\nu_d}. \end{aligned}$$

By transitivity (Proposition 2.10) we obtain

$$\text{GSP}_6(\text{both } s_1 \text{ and } s_2 \text{ send in 1 or 2 rounds)} \stackrel{\frac{1}{2} \sqsupseteq}{=} \prod_{i=1}^2 s_i[\text{nil}]^{\nu_{s_i}} \mid d[\sigma^3.\text{snd}\langle v \rangle_1]^{\nu_d} \quad (5.11)$$

By exploiting the three cases (a), (b), (c) analysed above, we can calculate now the tolerance r of GSP_6 $r \sqsupseteq \text{DONE}_6$. First, to estimate the tolerance when s_2 transmits (with a probabilistic delay of one or two steps) independently whether s_1 transmits or not, we can apply: (i) Theorem 3.4 in which the network M is structural congruent to $s_2[\tau.(\sigma.\text{snd}\langle v \rangle_1 \oplus_{\frac{1}{2}} \sigma^2.\text{snd}\langle v \rangle_1)]^{\nu_{s_2}} \mid d[\text{fwd}_{u_1,1}]^{\nu_d}$ and the node m is instantiated as s_1 , to compose the paths deriving from Equation 5.4 and Equation 5.11; (ii) Proposition 2.14(3)

(iii) both additivity and commutativity properties of \oplus ; to obtain:

$$\begin{aligned}
& \text{GSP}_6(s_1[v]; s_2 \text{ sends in 1 or 2 rounds}) \\
\stackrel{\text{def}}{=} & s_1[\text{sndu}\langle v \rangle_{p,2}]^{\nu_{s_1}} \mid s_2[\tau.(\sigma.\text{snd}\langle v \rangle_1 \oplus_{\frac{1}{2}} \sigma^2.\text{snd}\langle v \rangle_1)]^{\nu_{s_2}} \mid d[\text{fwd}_{1,1}]^{\nu_d} \\
\stackrel{\frac{p}{2} \sqsupseteq}{=} & \prod_{i=1}^2 s_i[\text{nil}]^{\nu_{s_i}} \mid d[\tau.(\sigma^3.\text{snd}\langle v \rangle_1 \oplus_p \tau.(\sigma^3.\text{snd}\langle v \rangle_1 \oplus_{\frac{1}{2}} \sigma^4.\text{snd}\langle v \rangle_1))]^{\nu_d} \\
\stackrel{0 \sqsupseteq}{=} & \prod_{i=1}^2 s_i[\text{nil}]^{\nu_{s_i}} \mid d[\tau.(p : \sigma^3.\text{snd}\langle v \rangle_1 \oplus \frac{1}{2}(1-p) : \sigma^3.\text{snd}\langle v \rangle_1 \oplus \frac{1}{2}(1-p) : \sigma^4.\text{snd}\langle v \rangle_1)]^{\nu_d} \\
\stackrel{0 \sqsupseteq}{=} & \prod_{i=1}^2 s_i[\text{nil}]^{\nu_{s_i}} \mid d[\tau.(\sigma^3.\text{snd}\langle v \rangle_1 \oplus_{(\frac{1}{2}+\frac{p}{2})} \sigma^4.\text{snd}\langle v \rangle_1)]^{\nu_d}.
\end{aligned}$$

By transitivity (Proposition 2.10) we obtain

$$\text{GSP}_6(s_1[v]; s_2 \text{ sends in 1 or 2 rnds}) \stackrel{\frac{p}{2} \sqsupseteq}{=} \prod_{i=1}^2 s_i[\text{nil}]^{\nu_{s_i}} \mid d[\tau.(\sigma^3.\text{snd}\langle v \rangle_1 \oplus_q \sigma^4.\text{snd}\langle v \rangle_1)]^{\nu_d} \quad (5.12)$$

with $q = (\frac{1}{2} + \frac{p}{2})$. Finally, in order to estimate the behaviour of the whole network GSP_6 we can apply in sequence: (i) Theorem 3.4 in which the network M is structural congruent to $s_1[\text{sndu}\langle v \rangle_{p,2}]^{\nu_{s_1}} \mid d[\text{fwd}_{1,1}]^{\nu_d}$ and the node m is instantiated as s_2 , to compose the paths deriving from Equation 5.3 and Equation 5.12; (ii) Proposition 2.14(3) together with the additivity and commutativity properties of \oplus ; to obtain:

$$\begin{aligned}
& \text{GSP}_6 \\
\stackrel{\text{def}}{=} & \prod_{i=1}^2 s_i[\text{sndu}\langle v \rangle_{p,2}]^{\nu_{s_i}} \mid d[\text{fwd}_{1,1}]^{\nu_d} \\
\stackrel{r \sqsupseteq}{=} & \prod_{i=1}^2 s_i[\text{nil}]^{\nu_{s_i}} \mid d[\tau.(\tau.(\sigma^3.\text{snd}\langle v \rangle_1 \oplus_q \sigma^4.\text{snd}\langle v \rangle_1) \oplus_p \tau.(\sigma^3.\text{snd}\langle v \rangle_1 \oplus_{\frac{1}{2}} \sigma^4.\text{snd}\langle v \rangle_1))]^{\nu_d} \\
\stackrel{0 \sqsupseteq}{=} & \prod_{i=1}^2 s_i[\text{nil}]^{\nu_{s_i}} \mid d[\tau.(\sigma^3.\text{snd}\langle v \rangle_1 \oplus_{q'} \sigma^4.\text{snd}\langle v \rangle_1)]^{\nu_d} \\
\stackrel{\text{def}}{=} & \text{DONE}_6.
\end{aligned}$$

with $q' = p \cdot q + (1-p) \cdot \frac{1}{2} = \frac{1}{2} + \frac{p^2}{2}$ and $r = p \cdot \frac{p}{2} + (1-p)(1-p) = 1 - (2p - \frac{3}{2}p^2)$. By transitivity (Proposition 2.10) we can finally derive

$$\text{GSP}_6 \stackrel{r \sqsupseteq}{=} \text{DONE}_6.$$

Hence, the network GSP_6 succeeds in delivering the message to the destination node with (at least) probability $2p - \frac{3}{2}p^2$.

Summarising, in the previous sections we have seen that in the absence of collisions the success probability of GSP_1 is (at least) $2p - p^2$. Whereas when collisions are taken into account, the network GSP_4 succeeds in propagating the message with (at least) probability $2p - 2p^2$, where $2p - 2p^2 < 2p - p^2$, for any $p > 0$. Since $2p - 2p^2 < 2p - \frac{3}{2}p^2 < 2p - p^2$, for any $p > 0$, this small example shows that the introduction of random delays may mitigate the effect of collisions.

6. CONCLUSIONS, RELATED AND FUTURE WORK

We have proposed a compositional analysis technique to formally study probabilistic gossip protocols expressed in a simple probabilistic timed process calculus for wireless networks. The calculus is equipped with a notion of *weak simulation quasimetric* which is used to define a *weak simulation with tolerance*, i.e., a *compositional simulation theory* to express that a probabilistic system may be simulated by another one with a given tolerance measuring the distance between the two systems. Basically, weak simulation quasimetric is the asymmetric counterpart of *weak bisimulation metric* [DJGP02], and the quantitative analogous of *weak*

simulation preorder [BKHH02, BHK04]. Based on our simulation quasimetric, we have developed an *algebraic theory* to estimate the performance of *gossip wireless networks* in terms of the probability to successfully propagate messages up to the desired destination. In our study we have considered gossip networks, with and without *communication collisions*, and *randomised gossip networks* adopting a uniform probability distribution to decide whether broadcasting a message.

A nice survey of formal verification techniques for the analysis of gossip protocols appears in Bakhshi et al.'s paper [BBFH07]. Probabilistic model-checking has been used by Fehnker and Gao [FG06] to study the influence of different modelling choices on message propagation in flooding and gossip protocols, and by Kwiatkowska et al. [KNP08] to investigate the expected rounds of gossiping required to form a connected network and how the expected path length between nodes evolves over the execution of the protocol. However, the analysis of gossip protocols in large-scale networks remains beyond the capabilities of current probabilistic model-checking tools. For this reason, Bakhshi et al. [BCFH09] have suggested to apply mean-field analysis for a formal evaluation of gossip protocols. Intuitively, the stochastic process representing the modelled system converges to a deterministic process if the number of nodes goes to infinity, providing an approximation for large numbers of nodes. Finally, Bakhshi et al. [BGFvS09] have developed and validated an analytical model, based on epidemic techniques, for a shuffle protocol, a protocol to disseminate data items to a collection of wireless devices, in a decentralised fashion.

A preliminary version of the current paper has appeared in [LMT17b]. However, in that paper neither communication collisions nor randomised gossip protocols are taken into account. The current paper is the ideal continuation of Lanotte and Merro's work [LM11]. In that paper, the authors developed a notion of *simulation up to probability* to measure the closeness rather than the distance between two probabilistic systems. Then, as in here, simulation up to probability has been used to provide an algebraic theory to evaluate the performance of gossip networks. Despite the similarity of the two simulation theories, the simulation up to probability has a number of limitations that have motivated the current work: (i) the simulation up to probability is not transitive, while simulation quasimetrics are transitive by definition; (ii) in order to work with a transitive relation, paper [LM11] introduces an auxiliary rooted simulation which is much stronger than the main definition; (iii) that rooted simulation (and hence the simulation up to probability) is not suitable to compose estimates originating from paths with different lengths (as we do here by means of Theorem 3.4), and, more generally, to deal with more transmissions; (iv) paper [LM11] does not consider randomised gossip protocols.

Our simulation quasimetric has been inspired by [DGJP04, DJGP02, vBW05, DCP06], where the notion of behavioural distance between two probabilistic systems is formalised in terms of the notion of bisimulation metric. Bisimulation metric works fine for systems being approximately equivalent. Recent applications of weak bisimulation metrics can be found in [LMT18a, LMT18b]. However, when the simulation game works only in one direction, as in the gossip protocols analysed in the current paper, an asymmetric notion of simulation pseudometric is required.

The compositionality criteria proposed in the literature for bisimulation metrics require that operators satisfy different forms of *uniform continuity* property [GT13, GT14, GLT15, GT15, LMT17a]. We proved that simulation quasimetric matches one of the most restrictive, namely *non-expansiveness* [DJGP02, DGJP04] (also known as 1-non-extensiveness [BBLM13]).

Finally, several process calculi for wireless systems have been proposed in the last years [LS10, NH06, God07, SRS10, GFM10, FvGH⁺12, BHJ⁺15]. Among these, Merro et al. [MBS11, CHM15] have proposed two different calculi modelling time-consuming communication to formally represent and study communication collisions in a wireless setting. Song and Godskesen [SG10] have proposed the first probabilistic untimed calculus for wireless systems, where connections are established with a given probability. *pTCWS* is a probabilistic variant of the calculus proposed by Macedonio and Merro [MM14] that takes inspiration from Deng et al.’s probabilistic CSP [DvGHM08].

As future work, we intend to study gossip protocols in the presence of lossy channels. We then plan to apply our metric-based simulation theory in the context of other probabilistic protocols, such as those for *probabilistic anonymity* [Cha88] and *probabilistic non-repudiation* [MR99].

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APPENDIX A. PROOFS

A.1. Proofs of results in Section 2.

Proof of Proposition 2.7. Let \mathcal{R} denote the relation $\mathcal{R} = \{(M, N) : d(M, N) = 0\}$. We have to prove that, given any pair of related networks $M \mathcal{R} N$, whenever $M \xrightarrow{\alpha} \Delta$ there are a transition $N \xrightarrow{\hat{\alpha}} \Theta$ and a matching $\omega \in \Omega(\Delta, \Theta)$ with $\omega(M', N') > 0$ only if $M' \mathcal{R} N'$. By definition of \mathcal{R} , relation $M \mathcal{R} N$ implies $d(M, N) = 0$. Being d a weak simulation quasimetric, $d(M, N) = 0$ implies that whenever $M \xrightarrow{\alpha} \Delta$ there are a transition $N \xrightarrow{\hat{\alpha}} \Theta$ with $\mathbf{K}(d)(\Delta, \Theta + (1 - |\Theta|)\overline{\text{Dead}}) \leq d(M, N) = 0$. From $\mathbf{K}(d)(\Delta, \Theta + (1 - |\Theta|)\overline{\text{Dead}}) \leq 0$ we infer that $|\Theta| = 1$, thus giving $\mathbf{K}(d)(\Delta, \Theta) = 0$. Let $\omega \in \Omega(\Delta, \Theta)$ be one of the optimal matchings realising $\mathbf{K}(d)(\Delta, \Theta)$. From $\mathbf{K}(d)(\Delta, \Theta) = 0$ we infer that $\omega(M', N') > 0$ only if $d(M', N') = 0$. Namely, $\omega(M', N') > 0$ only if $M' \mathcal{R} N'$, which complete the proof. \square

To prove Theorem 2.8 we need two preliminary results. First we show that the pseudoquasimetric property is preserved by function \mathbf{K} , namely $\mathbf{K}(d)$ is a pseudoquasimetric whenever d is a pseudoquasimetric.

Proposition A.1. *If $d: \text{pTCWS} \times \text{pTCWS} \rightarrow [0, 1]$ is a 1-bounded pseudoquasimetric, then $\mathbf{K}(d)$ is a 1-bounded pseudoquasimetric.*

Proof. To show $\mathbf{K}(d)(\Delta, \Delta) = 0$ it is enough to take the matching $\omega \in \Omega(\Delta, \Delta)$ defined by $\omega(M, M) = \Delta(M)$, for all $M \in \text{pTCWS}$, and $\omega(M, N) = 0$, for all $M, N \in \text{pTCWS}$ with $M \neq N$. In fact, we obtain $\mathbf{K}(d)(\Delta, \Delta) = 0$ by $\mathbf{K}(d)(\Delta, \Delta) \leq \sum_{M, N \in \text{pTCWS}} \omega(M, N) \cdot d(M, N) = \sum_{M \in \text{pTCWS}} \Delta(M) \cdot d(M, M) = 0$, because $d(M, M) = 0$.

To show the triangular property $\mathbf{K}(d)(\Delta_1, \Delta_2) \leq \mathbf{K}(d)(\Delta_1, \Delta_3) + \mathbf{K}(d)(\Delta_3, \Delta_2)$, we take the function $\omega: \text{pTCWS} \times \text{pTCWS} \rightarrow [0, 1]$ defined for all networks $M_1, M_2 \in \text{pTCWS}$

as $\omega(M_1, M_2) = \sum_{M_3 \in \text{pTCWS} | \Delta_3(M_3) \neq 0} \frac{\omega_1(M_1, M_3) \cdot \omega_2(M_3, M_2)}{\Delta_3(M_3)}$, with $\omega_1 \in \Omega(\Delta_1, \Delta_3)$ one of the optimal matchings realising $\mathbf{K}(d)(\Delta_1, \Delta_3)$, and $\omega_2 \in \Omega(\Delta_3, \Delta_2)$ one of the optimal matchings realising $\mathbf{K}(d)(\Delta_3, \Delta_2)$. Then, we prove that: (i) ω is a matching in $\Omega(\Delta_1, \Delta_2)$, and (ii) $\sum_{M_1, M_2 \in \text{pTCWS}} \omega(M_1, M_2) \cdot d(M_1, M_2) \leq \mathbf{K}(d)(\Delta_1, \Delta_3) + \mathbf{K}(d)(\Delta_3, \Delta_2)$, which immediately implies $\mathbf{K}(d)(\Delta_1, \Delta_2) \leq \mathbf{K}(d)(\Delta_1, \Delta_3) + \mathbf{K}(d)(\Delta_3, \Delta_2)$. To show (i) we prove that the left marginal of ω is Δ_1 by

$$\begin{aligned} & \sum_{M_2 \in \text{pTCWS}} \omega(M_1, M_2) \\ = & \sum_{M_2 \in \text{pTCWS}} \sum_{M_3 \in \text{pTCWS} | \Delta_3(M_3) \neq 0} \frac{\omega_1(M_1, M_3) \cdot \omega_2(M_3, M_2)}{\Delta_3(M_3)} \\ = & \sum_{M_3 \in \text{pTCWS} | \Delta_3(M_3) \neq 0} \frac{\omega_1(M_1, M_3) \cdot \Delta_3(M_3)}{\Delta_3(M_3)} \\ = & \sum_{M_3 \in \text{pTCWS} | \Delta_3(M_3) \neq 0} \omega_1(M_1, M_3) \\ = & \Delta_1(M_1) \end{aligned}$$

with the second step by $\omega_2 \in \Omega(\Delta_3, \Delta_2)$ and the last step by $\omega_1 \in \Omega(\Delta_1, \Delta_3)$. The proof that the right marginal of ω is Δ_2 is analogous. Then, we show (ii) by

$$\begin{aligned} & \sum_{M_1, M_2 \in \text{pTCWS}} \omega(M_1, M_2) \cdot d(M_1, M_2) \\ = & \sum_{M_1, M_2 \in \text{pTCWS}} \sum_{M_3 \in \text{pTCWS} | \Delta_3(M_3) \neq 0} \frac{\omega_1(M_1, M_3) \cdot \omega_2(M_3, M_2)}{\Delta_3(M_3)} \cdot d(M_1, M_2) \\ \leq & \sum_{M_1, M_2 \in \text{pTCWS}, M_3 \in \text{pTCWS} | \Delta_3(M_3) \neq 0} \frac{\omega_1(M_1, M_3) \cdot \omega_2(M_3, M_2)}{\Delta_3(M_3)} \cdot d(M_1, M_3) + \\ & \sum_{M_1, M_2 \in \text{pTCWS}, M_3 \in \text{pTCWS} | \Delta_3(M_3) \neq 0} \frac{\omega_1(M_1, M_3) \cdot \omega_2(M_3, M_2)}{\Delta_3(M_3)} \cdot d(M_3, M_2) \\ = & \sum_{M_1, M_3 \in \text{pTCWS}} \frac{\omega_1(M_1, M_3) \cdot \Delta_3(M_3)}{\Delta_3(M_3)} \cdot d(M_1, M_3) + \\ & \sum_{M_2, M_3 \in \text{pTCWS}} \frac{\Delta_3(M_3) \cdot \omega_2(M_3, M_2)}{\Delta_3(M_3)} \cdot d(M_3, M_2) \\ = & \sum_{M_1, M_3 \in \text{pTCWS}} \omega_1(M_1, M_3) \cdot d(M_1, M_3) + \sum_{M_2, M_3 \in \text{pTCWS}} \omega_2(M_3, M_2) \cdot d(M_3, M_2) \\ = & \mathbf{K}(d)(\Delta_1, \Delta_3) + \mathbf{K}(d)(\Delta_3, \Delta_2) \end{aligned}$$

where the inequality follows from the triangular property of d and the third last equality follows by $\omega_2 \in \Omega(\Delta_3, \Delta_2)$ and $\omega_1 \in \Omega(\Delta_1, \Delta_2)$. \square

Now we prove that if $d(M, N) < 1$ for any weak simulation quasimetric d , then N is able to simulate transitions of the form $M \xrightarrow{\hat{\alpha}} \Delta$, besides those of the form $M \xrightarrow{\alpha} \Delta$.

Lemma A.2. *Assume any weak simulation quasimetric d and two networks $M, N \in \text{pTCWS}$ with $d(M, N) < 1$. If $M \xrightarrow{\hat{\alpha}} \Delta$ then there is a weak transition $N \xrightarrow{\hat{\alpha}} \Theta$ such that $\mathbf{K}(d)(\Delta + (1 - |\Delta|)\overline{\text{Dead}}, \Theta + (1 - |\Theta|)\overline{\text{Dead}}) \leq d(M, N)$.*

Proof. We proceed by induction on the length n of $M \xrightarrow{\hat{\alpha}} \Delta$.

Base case $n = 1$. We have two cases: The first is $\alpha = \tau$ and $\Delta = \overline{M}$, the second is $M \xrightarrow{\alpha} \Delta$. In the first case, by definition of $\hat{\tau}$ we have $N \xrightarrow{\hat{\tau}} \overline{N}$ and the thesis holds for the distribution $\Theta = \overline{N}$ by $\mathbf{K}(d)(\overline{M} + (1 - |\overline{M}|)\overline{\text{Dead}}, \overline{N} + (1 - |\overline{N}|)\overline{\text{Dead}}) = \mathbf{K}(d)(\overline{M}, \overline{N}) = d(M, N)$. In the second case, the thesis follows directly by the definition of weak simulation quasimetric.

Inductive step $n > 1$. The derivation $M \xrightarrow{\hat{\alpha}} \Delta$ is obtained by $M \xrightarrow{\hat{\beta}_1} \Delta'$ and $\Delta' \xrightarrow{\hat{\beta}_2} \Delta$, for some distribution $\Delta' \in \mathcal{D}(\text{pTCWS})$. The length of the derivation $M \xrightarrow{\hat{\beta}_1} \Delta'$ is $n - 1$ and hence, by the inductive hypothesis, there is a weak transition $N \xrightarrow{\hat{\beta}_1} \Theta'$ such that $\mathbf{K}(d)(\Delta' + (1 - |\Delta'|)\overline{\text{Dead}}, \Theta' + (1 - |\Theta'|)\overline{\text{Dead}}) \leq d(M, N)$. The sub-distributions Δ' and Θ'

are of the form $\Delta' = \sum_{i \in I} p_i M_i$ and $\Theta' = \sum_{j \in J} q_j N_j$, for suitable networks M_i and N_j . We have two sub-cases: The first is $\beta_1 = \tau$ and $\beta_2 = \alpha$, the other $\beta_1 = \alpha$ and $\beta_2 = \tau$.

We consider the case $\beta_1 = \tau$ and $\beta_2 = \alpha$, the other is analogous. In this case we have $|\Delta'| = |\Theta'| = 1$ and $\mathbf{K}(d)(\Delta', \Theta') \leq d(M, N)$. The transition $\Delta' \xrightarrow{\hat{\beta}_2} \Delta$ is derived from a β_2 -transition by some of the networks M_i , namely I is partitioned into sets I_1 and I_2 such that: (i) for all $i \in I_1$ we have $M_i \xrightarrow{\beta_2} \Delta_i$ for suitable distributions Δ_i , (ii) for each $i \in I_2$ we have $M_i \not\xrightarrow{\beta_2}$, (iii) $\Delta = \sum_{i \in I_1} p_i \Delta_i$, (iv) $1 - |\Delta| = \sum_{i \in I_2} p_i$. Analogously, J is partitioned into sets J_1 and J_2 such that for all $j \in J_1$ we have $N_j \xrightarrow{\hat{\beta}_2} \Theta_j$ for suitable distributions Θ_j and for each $j \in J_2$ we have $N_j \not\xrightarrow{\hat{\beta}_2}$. This gives $\Theta' \xrightarrow{\hat{\beta}_2} \Theta$ with $\Theta = \sum_{j \in J_1} q_j \Theta_j$. Since we had $N \xrightarrow{\hat{\beta}_1} \Theta'$, we can conclude $N \xrightarrow{\hat{\alpha}} \Theta$. In the following we will prove that the transitions $N_j \xrightarrow{\hat{\beta}_2} \Theta_j$ can be chosen so that $\mathbf{K}(d)(\Delta + (1 - |\Delta|)\overline{\text{Dead}}, \Theta + (1 - |\Theta|)\overline{\text{Dead}}) \leq d(M, N)$, which will conclude the proof.

Let ω be one of the optimal matchings realising $\mathbf{K}(d)(\Delta', \Theta')$. Since $p_i = \sum_{j \in J} \omega(M_i, N_j)$ for all $i \in I$ and, then, $q_j = \sum_{i \in I} \omega(M_i, N_j)$ for all $j \in J$, we can rewrite the distributions $\Delta' = \sum_{i \in I} p_i M_i$ and $\Theta' = \sum_{j \in J} q_j N_j$ as $\Delta' = \sum_{i \in I, j \in J} \omega(M_i, N_j) M_i$ and $\Theta' = \sum_{i \in I, j \in J} \omega(M_i, N_j) N_j$. For all $i \in I_1$ and $j \in J$, define $\Delta_{i,j} = \Delta_i$. Since $\Delta = \sum_{i \in I_1} p_i \Delta_i$, we can rewrite Δ as $\Delta = \sum_{i \in I_1, j \in J} \omega(M_i, N_j) \Delta_{i,j}$. Analogously, for each $j \in J_1$ and $i \in I$

we note that the transition $q_j \overline{N_j} \xrightarrow{\hat{\beta}_2} \Theta_j$ can always be split into $\sum_{i \in I} \omega(M_i, N_j) \overline{N_j} \xrightarrow{\hat{\beta}_2} \sum_{i \in I} \omega(M_i, N_j) \Theta_{i,j}$ for suitable distributions $\Theta_{i,j}$ so that for all $j \in J_1$ we can rewrite Θ_j as $\Theta_j = \sum_{i \in I} \omega(M_i, N_j) \Theta_{i,j}$, and we can rewrite Θ as $\Theta = \sum_{i \in I, j \in J_1} \omega(M_i, N_j) \Theta_{i,j}$. Then

we note that for all $i \in I_1$ and $j \in J_1$ with $d(M_i, N_j) < 1$, the transition $N_j \xrightarrow{\hat{\beta}_2} \Theta_{i,j}$ can be chosen so that $\mathbf{K}(d)(\Delta_{i,j}, \Theta_{i,j} + (1 - |\Theta_{i,j}|)\overline{\text{Dead}}) \leq d(M_i, N_j)$. For all $i \in I_1$ and $j \in J_1$ with $d(M_i, N_j) < 1$, let $\omega_{i,j}$ be one of the optimal matchings realising $\mathbf{K}(d)(\Delta_{i,j}, \Theta_{i,j} + (1 - |\Theta_{i,j}|)\overline{\text{Dead}})$.

Define $\omega' : \text{pTCWS} \times \text{pTCWS} \rightarrow [0, 1]$ as the function such that $\omega'(M', N')$ is the summation of the following values:

- (1) $\sum_{i \in I_1, j \in J_1} \omega(M_i, N_j) \cdot \omega_{i,j}(M', N')$
- (2) $\sum_{i \in I_1, j \in J_2} \omega(M_i, N_j) \cdot \Delta_{i,j}(M') \cdot \overline{\text{Dead}}(N')$
- (3) $\sum_{i \in I_2, j \in J_1} \omega(M_i, N_j) \cdot (\overline{\text{Dead}}(M') \cdot \Theta_{i,j}(N') + (1 - |\Theta_{i,j}|) \cdot \overline{\text{Dead}}(M') \cdot \overline{\text{Dead}}(N'))$
- (4) $\sum_{i \in I_2, j \in J_2} \omega(M_i, N_j) \cdot \overline{\text{Dead}}(M') \cdot \overline{\text{Dead}}(N')$.

To infer the proof obligation $\mathbf{K}(d)(\Delta + (1 - |\Delta|)\overline{\text{Dead}}, \Theta + (1 - |\Theta|)\overline{\text{Dead}}) \leq d(M, N)$ it is now enough to show that:

- (a) the function ω' is a matching in $\Omega(\Delta + (1 - |\Delta|)\overline{\text{Dead}}, \Theta + (1 - |\Theta|)\overline{\text{Dead}})$, and
- (b) $\sum_{M', N' \in \text{pTCWS}} \omega'(M', N') \cdot d(M', N') \leq d(M, N)$.

To show (a) we prove that the left marginal of ω' is $\Delta + (1 - |\Delta|)\overline{\text{Dead}}$. The proof that the right marginal is $\Theta + (1 - |\Theta|)\overline{\text{Dead}}$ is analogous. For any network $M' \in \text{pTCWS}$ we have that $\sum_{N' \in \text{pTCWS}} \omega'(M', N')$ is the summation of the following values:

- (1) $\sum_{N' \in \text{pTCWS}} \sum_{i \in I_1, j \in J_1} \omega(M_i, N_j) \cdot \omega_{i,j}(M', N') = \sum_{i \in I_1, j \in J_1} \omega(M_i, N_j) \cdot \Delta_{i,j}(M')$
- (2) $\sum_{N' \in \text{pTCWS}} \sum_{i \in I_1, j \in J_2} \omega(M_i, N_j) \cdot \Delta_{i,j}(M') \cdot \overline{\text{Dead}}(N') = \sum_{i \in I_1, j \in J_2} \omega(M_i, N_j) \cdot \Delta_{i,j}(M')$

$$(3) \sum_{N' \in \text{pTCWS}} \sum_{i \in I_2, j \in J_1} \omega(M_i, N_j) (\overline{\text{Dead}}(M') \Theta_{i,j}(N') + (1 - |\Theta_{i,j}|) \overline{\text{Dead}}(M') \overline{\text{Dead}}(N')) \\ = \sum_{i \in I_2, j \in J_1} \omega(M_i, N_j) \cdot \overline{\text{Dead}}(M')$$

$$(4) \sum_{N' \in \text{pTCWS}} \sum_{i \in I_2, j \in J_2} \omega(M_i, N_j) \overline{\text{Dead}}(M') \overline{\text{Dead}}(N') = \sum_{i \in I_2, j \in J_2} \omega(M_i, N_j) \overline{\text{Dead}}(M')$$

with the first equality by $\omega_{i,j} \in \Omega(\Delta_{i,j}, \Theta_{i,j} + (1 - |\Theta_{i,j}|))$, the second by $\sum_{N' \in \text{pTCWS}} \overline{\text{Dead}}(N') = 1$, the third by $\sum_{N' \in \text{pTCWS}} \Theta_{i,j}(N') = |\Theta_{i,j}|$ and $\sum_{N' \in \text{pTCWS}} \overline{\text{Dead}}(N') = 1$, and the last by $\sum_{N' \in \text{pTCWS}} \overline{\text{Dead}}(N') = 1$.

The summation of the first two items gives $\sum_{i \in I_1, j \in J} \omega(M_i, N_j) \cdot \Delta_{i,j}(M')$, which is $\Delta(M')$ by the definition of Δ , and the summation of the last two items gives $\sum_{i \in I_2, j \in J} \omega(M_i, N_j) \cdot \overline{\text{Dead}}(M')$, which is $(1 - |\Delta|) \overline{\text{Dead}}(M')$. Overall, the summation of all items gives $\Delta(M') + (1 - |\Delta|) \overline{\text{Dead}}(M')$, namely $(\Delta + (1 - |\Delta|) \overline{\text{Dead}})(M')$.

To prove (b), by the definition of ω' we have that $\sum_{M', N' \in \text{pTCWS}} \omega'(M', N') \cdot d(M', N')$ is the summation of the following values:

$$(1) \sum_{M', N' \in \text{pTCWS}} \sum_{i \in I_1, j \in J_1} \omega(M_i, N_j) \cdot \omega_{i,j}(M', N') \cdot d(M', N')$$

$$(2) \sum_{M', N' \in \text{pTCWS}} \sum_{i \in I_1, j \in J_2} \omega(M_i, N_j) \cdot \Delta_{i,j}(M') \cdot \overline{\text{Dead}}(N') \cdot d(M', N') \\ = \sum_{M' \in \text{pTCWS}} \sum_{i \in I_1, j \in J_2} \omega(M_i, N_j) \cdot \Delta_{i,j}(M') \cdot d(M', \text{Dead}) \\ \leq \sum_{M' \in \text{pTCWS}} \sum_{i \in I_1, j \in J_2} \omega(M_i, N_j) \cdot \Delta_{i,j}(M')$$

$$(3) \sum_{M', N' \in \text{pTCWS}} \sum_{i \in I_2, j \in J_1} \omega(M_i, N_j) \cdot (\overline{\text{Dead}}(M') \cdot \Theta_{i,j}(N') + (1 - |\Theta_{i,j}|) \cdot \overline{\text{Dead}}(M') \cdot \overline{\text{Dead}}(N')) \cdot d(M', N') = 0$$

$$(4) \sum_{M', N' \in \text{pTCWS}} \sum_{i \in I_2, j \in J_2} \omega(M_i, N_j) \cdot \overline{\text{Dead}}(M') \cdot \overline{\text{Dead}}(N') \cdot d(M', N') = 0.$$

where in the second item the equality follows from $\overline{\text{Dead}}(\text{Dead}) = 1$ and $\overline{\text{Dead}}(N') = 0$ for all $N' \neq \text{Dead}$ and the inequality from $d(M', \text{Dead}) \leq 1$, and in the third and fourth item the equality follows by the fact the only network with $\overline{\text{Dead}}(M') > 0$ is Dead , for which we have $d(\text{Dead}, N') = 0$ for all $N' \in \text{pTCWS}$.

By the definition of $\omega_{i,j}$ the first item is $\sum_{i \in I_1, j \in J_1} \omega(M_i, N_j) \cdot \mathbf{K}(d)(\Delta_{i,j}, \Theta_{i,j})$. If $d(M_i, N_j) < 1$, we argued above that the distribution $\Theta_{i,j}$ satisfies $\mathbf{K}(d)(\Delta_{i,j}, \Theta_{i,j}) \leq d(M_i, N_j)$. If $d(M_i, N_j) = 1$, then $\mathbf{K}(d)(\Delta_{i,j}, \Theta_{i,j}) \leq d(M_i, N_j)$ is immediate. Henceforth we are sure that in all cases the first item is less or equal $\sum_{i \in I_1, j \in J_1} \omega(M_i, N_j) \cdot d(M_i, N_j)$. The second item is clearly less or equal than $\sum_{i \in I_1, j \in J_2} \omega(M_i, N_j)$. Summarising, we have

$$\sum_{M', N' \in \text{pTCWS}} \omega'(M', N') \cdot d(M', N') \leq \sum_{i \in I_1, j \in J_1} \omega(M_i, N_j) \cdot d(M_i, N_j) + \sum_{i \in I_1, j \in J_2} \omega(M_i, N_j).$$

Then, since $\mathbf{K}(d)(\Delta', \Theta')$ is the summation of the following values:

- $\sum_{i \in I_1, j \in J_1} \omega(M_i, N_j) \cdot d(M_i, N_j)$
- $\sum_{i \in I_1, j \in J_2} \omega(M_i, N_j) \cdot d(M_i, N_j) = \sum_{i \in I_1, j \in J_2} \omega(M_i, N_j)$ (since $M_i \xrightarrow{\beta_2}$ and $N_j \xrightarrow{\beta_2} \text{Dead}$ give $d(M_i, N_j) = 1$)
- $\sum_{i \in I_2, j \in J_1} \omega(M_i, N_j) \cdot d(M_i, N_j)$
- $\sum_{i \in I_2, j \in J_2} \omega(M_i, N_j) \cdot d(M_i, N_j)$.

it is immediate that $\sum_{i \in I_1, j \in J_1} \omega(M_i, N_j) \cdot d(M_i, N_j) + \sum_{i \in I_1, j \in J_2} \omega(M_i, N_j) \leq \mathbf{K}(d)(\Delta', \Theta')$. Since we had $\mathbf{K}(d)(\Delta', \Theta') \leq d(M, N)$ we can conclude that $\sum_{M', N' \in \text{pTCWS}} \omega'(M', N') \cdot d(M', N') \leq d(M, N)$, as required. \square

We are now ready to prove Theorem 2.8.

Proof of Theorem 2.8. Let D be the set of all weak simulation quasimetrics. Define the function $\mathbf{d}: \mathbf{pTCWS} \times \mathbf{pTCWS} \rightarrow [0, 1]$ by

$$\mathbf{d}(M, N) = \inf \left\{ \sum_{i=1}^n \inf_{d \in D} d(O_i, O_{i+1}) \mid n \in \mathbb{N}^+, O_1 = M, O_{n+1} = N, O_i \in \mathbf{pTCWS} \right\}$$

for all $M, N \in \mathbf{pTCWS}$. It is immediate that for all $d \in D$ we have $\mathbf{d}(M, N) \leq d(M, N)$ for all $M, N \in \mathbf{pTCWS}$. Therefore, to derive the thesis it is enough to show that \mathbf{d} is a weak simulation quasimetric. We start with showing that \mathbf{d} is a pseudoquasimetric. The property $\mathbf{d}(M, M) = 0$ follows directly by $\mathbf{d}(M, M) \leq d(M, M)$ and $d(M, M) = 0$ for all $d \in D$. Then we show the triangular inequality $\mathbf{d}(M, N) \leq \mathbf{d}(M, O) + \mathbf{d}(O, N)$ by

$$\begin{aligned} & \mathbf{d}(M, N) \\ &= \inf \left\{ \sum_{i=1}^n \inf_{d \in D} d(O_i, O_{i+1}) \mid n \in \mathbb{N}^+, O_1 = M, O_{n+1} = N, O_i \in \mathbf{pTCWS} \right\} \\ &\leq \inf \left\{ \sum_{i=1}^n \inf_{d \in D} d(O_i, O_{i+1}) \mid n \in \mathbb{N}^+, O_1 = M, O_{n+1} = N, O_i \in \mathbf{pTCWS} \right\} \\ &= \inf \left\{ \sum_{i=1}^n \inf_{d \in D} d(O_i, O_{i+1}) \mid n \in \mathbb{N}^+, O_1 = M, O_{n+1} = O, O \in \{O_2, \dots, O_n\}, O_i \in \mathbf{pTCWS} \right\} \\ &+ \\ &\inf \left\{ \sum_{i=1}^n \inf_{d \in D} d(O_i, O_{i+1}) \mid n \in \mathbb{N}^+, O_1 = O, O \in \{O_2, \dots, O_n\}, O_{n+1} = N, O_i \in \mathbf{pTCWS} \right\} \\ &= \mathbf{d}(M, O) + \mathbf{d}(O, N). \end{aligned}$$

We conclude that the pseudometric property of \mathbf{d} is given. It remains to prove that whenever $\mathbf{d}(M, N) < 1$ and $M \xrightarrow{\alpha} \Delta$ there is a transition $N \xrightarrow{\hat{\alpha}} \Theta$ with $\mathbf{K}(\mathbf{d})(\Delta, \Theta + (1 - |\Theta|)\overline{\text{Dead}}) \leq \mathbf{d}(M, N)$. By the finite branching property, it is enough to prove that, fixed $O_1 = M$ and $O_{n+1} = N$ and given arbitrary networks $O_2, \dots, O_n \in \mathbf{pTCWS}$ and arbitrary weak simulation quasimetrics $d_1, \dots, d_n \in D$ such that $\sum_{i=1}^n d_i(O_i, O_{i+1}) \leq 1$, then $M \xrightarrow{\alpha} \Delta$ implies that there is a transition $N \xrightarrow{\hat{\alpha}} \Theta$ with $\mathbf{K}(\mathbf{d})(\Delta, \Theta + (1 - |\Theta|)\overline{\text{Dead}}) \leq \sum_{i=1}^n d_i(O_i, O_{i+1})$. Assume such a $M \xrightarrow{\alpha} \Delta$, and let $\Delta_1 = \Delta$. For $i = 1, \dots, n$ we get $O_i \xrightarrow{\hat{\alpha}} \Delta_i$ with $\mathbf{K}(d_i)(\Delta_i + (1 - |\Delta_i|)\overline{\text{Dead}}, \Delta_{i+1} + (1 - |\Delta_{i+1}|)\overline{\text{Dead}}) \leq d_i(O_i, O_{i+1})$, by exploiting Lemma A.2 in cases $i = 2, \dots, n$. This allows us to infer

$$\begin{aligned} & \mathbf{K}(\mathbf{d})(\Delta_1, \Delta_{n+1} + (1 - |\Delta_{n+1}|)\overline{\text{Dead}}) \\ &\leq \sum_{i=1}^n \mathbf{K}(\mathbf{d})(\Delta_i + (1 - |\Delta_i|)\overline{\text{Dead}}, \Delta_{i+1} + (1 - |\Delta_{i+1}|)\overline{\text{Dead}}) \\ &\leq \sum_{i=1}^n \mathbf{K}(d_i)(\Delta_i + (1 - |\Delta_i|)\overline{\text{Dead}}, \Delta_{i+1} + (1 - |\Delta_{i+1}|)\overline{\text{Dead}}) \\ &\leq \sum_{i=1}^n d_i(O_i, O_{i+1}) \end{aligned}$$

thus confirming that Δ_{n+1} is the distribution Θ we were looking for, where the first step follows by the triangular inequality for $\mathbf{K}(\mathbf{d})$, which is a pseudoquasimetric by Proposition A.1 and the fact that we have already proved that \mathbf{d} is a pseudoquasimetric, and the second step follows by the monotonicity of \mathbf{K} and the fact that we have proved that $\mathbf{d}(M', N') \leq d_i(M', N')$ for all $M', N' \in \mathbf{pTCWS}$. \square

Proof of Proposition 2.11. We have to prove that any transition $N \xrightarrow{\alpha} \Theta$ is simulated by a suitable weak transition $M \xrightarrow{\hat{\alpha}} \Theta'$ with $\mathbf{K}(\mathbf{d})(\Theta, \Theta' + (1 - |\Theta'|)\overline{\text{Dead}}) \leq q$. Assume an arbitrary transition $N \xrightarrow{\alpha} \Theta$. From $N \xrightarrow{\alpha} \Theta$ and the hypothesis $M \xrightarrow{\hat{\tau}} (1 - q)\overline{N} + q\Delta$, we get $M \xrightarrow{\hat{\alpha}} (1 - q)\Theta + q\Delta'$, for a suitable sub-distribution Δ' such that $\Delta \xrightarrow{\hat{\alpha}} \Delta'$. To conclude we have to prove that $\mathbf{K}(\mathbf{d})(\Theta, (1 - q)\Theta + q\Delta' + q(1 - |\Delta'|)\overline{\text{Dead}}) \leq q$, namely $(1 - q)\Theta + q\Delta'$ is the sub-distribution Θ' we were looking for. Let $\omega: \mathbf{pTCWS} \times \mathbf{pTCWS} \rightarrow [0, 1]$ be the function defined by

- $\omega(M', M') = (1 - q) \cdot \Theta(M') + q \cdot \Theta(M') \cdot \Delta'(M') + q \cdot (1 - |\Delta'|) \cdot \Theta(M') \cdot \overline{\text{Dead}}(M')$, and
- $\omega(M', N') = q \cdot \Theta(M') \cdot \Delta'(N') + q \cdot (1 - |\Delta'|) \cdot \Theta(M') \cdot \overline{\text{Dead}}(N')$, for $M' \neq N'$.

We have to prove that:

- (1) $\omega \in \Omega(\Theta, (1 - q)\Theta + q\Delta' + q(1 - |\Delta'|)\overline{\text{Dead}})$, and
- (2) $\sum_{M', N' \in \text{pTCWS}} \omega(M', N') \cdot \mathbf{d}(M', N') \leq q$.

To prove (1) we show that the left marginal of ω is Θ by

$$\begin{aligned}
& \sum_{N' \in \text{pTCWS}} \omega(M', N') \\
&= \omega(M', M') + \sum_{N' \neq M'} \omega(M', N') \\
&= (1 - q) \cdot \Theta(M') + q \cdot \Theta(M') \cdot \Delta'(M') + q \cdot (1 - |\Delta'|) \cdot \Theta(M') \cdot \overline{\text{Dead}}(M') + \\
&\quad \sum_{N' \neq M'} q \cdot \Theta(M') \cdot \Delta'(N') + q \cdot (1 - |\Delta'|) \cdot \Theta(M') \cdot \overline{\text{Dead}}(N') \\
&= (1 - q) \cdot \Theta(M') + \sum_{N' \in \text{pTCWS}} q \cdot \Theta(M') \cdot \Delta'(N') + \\
&\quad \sum_{N' \in \text{pTCWS}} q \cdot (1 - |\Delta'|) \cdot \Theta(M') \cdot \overline{\text{Dead}}(N') \\
&= (1 - q) \cdot \Theta(M') + q \cdot |\Delta'| \cdot \Theta(M') + q \cdot (1 - |\Delta'|) \cdot \Theta(M') \\
&= \Theta(M')
\end{aligned}$$

the proof that the right marginal is $(1 - q)\Theta + q\Delta' + q(1 - |\Delta'|)\overline{\text{Dead}}$ being analogous. We get (2) by

$$\begin{aligned}
& \sum_{M', N'} \omega(M', N') \cdot \mathbf{d}(M', N') \\
&= \sum_{M', M'} \omega(M', M') \cdot \mathbf{d}(M', M') + \sum_{M' \neq N'} \omega(M', N') \cdot \mathbf{d}(M', N') \\
&= \sum_{M' \neq N'} \omega(M', N') \cdot \mathbf{d}(M', N') \\
&= \sum_{M' \neq N'} (q \cdot \Theta(M') \cdot \Delta'(N') + q \cdot (1 - |\Delta'|) \cdot \Theta(M') \cdot \overline{\text{Dead}}(N')) \cdot \mathbf{d}(M', N') \\
&\leq q \cdot |\Delta'| + q \cdot (1 - |\Delta'|) \\
&\leq q
\end{aligned}$$

with the second equality by $\mathbf{d}(M', M') = 0$ and the first inequality by $\mathbf{d}(M', N') \leq 1$, the fact that $\sum_{M' \in \text{pTCWS}} \Theta(M') = 1$ and $\sum_{N' \in \text{pTCWS}} \overline{\text{Dead}}(N') = 1$. \square

Proof of Proposition 2.12. Assume a transition $M \xrightarrow{\alpha} \Theta$. By $M \sqsubseteq_p N$ there is a transition $N \xrightarrow{\hat{\alpha}} \Theta'$ with $\mathbf{K}(\mathbf{d})(\Theta, \Theta' + (1 - |\Theta'|)\overline{\text{Dead}}) \leq p$. By the hypothesis $O \xrightarrow{\hat{\tau}} (1 - q)\overline{N} + q\Delta$ we get $O \xrightarrow{\hat{\alpha}} (1 - q)\Theta' + q\Delta'$, for a suitable distribution Δ' such that $\Delta \xrightarrow{\hat{\alpha}} \Delta'$. To conclude the proof we should show that

$$\mathbf{K}(\mathbf{d})(\Theta, (1 - q)\Theta' + q\Delta' + ((1 - q)(1 - |\Theta'|) + q(1 - |\Delta'|))\overline{\text{Dead}}) \leq p(1 - q) + q.$$

Let us consider $\omega \in \Omega(\Theta, \Theta' + (1 - |\Theta'|)\overline{\text{Dead}})$ as one of the optimal matching realising $\mathbf{K}(\mathbf{d})(\Theta, \Theta' + (1 - |\Theta'|)\overline{\text{Dead}})$. Define the function $\omega': \text{pTCWS} \times \text{pTCWS} \rightarrow [0, 1]$ by

$$\omega'(M', N') = (1 - q) \cdot \omega(M', N') + q \cdot \Theta(M') \cdot \Delta'(N') + q \cdot (1 - |\Delta'|) \cdot \Theta(M') \cdot \overline{\text{Dead}}(N')$$

for all $M', N' \in \text{pTCWS}$. We have to prove that

- (1) $\omega' \in \Omega(\Theta, (1 - q) \cdot \Theta' + q\Delta' + ((1 - q)(1 - |\Theta'|) + q(1 - |\Delta'|))\overline{\text{Dead}})$, and
- (2) $\sum_{M', N' \in \text{pTCWS}} \omega'(M', N') \cdot \mathbf{d}(M', N') \leq p(1 - q) + q$.

To prove (1) we show that the left marginal of ω' is Θ by

$$\begin{aligned}
& \sum_{N' \in \text{pTCWS}} \omega'(M', N') \\
&= (1 - q) \cdot \sum_{N' \in \text{pTCWS}} \omega(M', N') + q \cdot \sum_{N' \in \text{pTCWS}} \Theta(M') \cdot \Delta'(N') + \\
&\quad q \cdot \sum_{N' \in \text{pTCWS}} (1 - |\Delta'|) \cdot \Theta(M') \cdot \overline{\text{Dead}}(N') \\
&= (1 - q) \cdot \Theta(M') + q \cdot \Theta(M') \cdot |\Delta'| + q \cdot \Theta(M') \cdot (1 - |\Delta'|) \\
&= \Theta(M')
\end{aligned}$$

with the second equality by the fact that ω is a matching in $\Omega(\Theta, \Theta' + (1 - |\Theta'|)\overline{\text{Dead}})$ and by the equalities $\sum_{N' \in \text{pTCWS}} \Delta'(N') = |\Delta'|$ and $\sum_{N' \in \text{pTCWS}} \overline{\text{Dead}}(N') = 1$. The proof that the right marginal is $(1 - q)\Theta' + q\Delta' + ((1 - q)(1 - |\Theta'|) + q(1 - |\Delta'|))\overline{\text{Dead}}$ is analogous. We get (2) by the following chain of inequalities:

$$\begin{aligned} & \sum_{M', N'} \omega'(M', N') \cdot \mathbf{d}(M', N') \\ &= (1 - q) \sum_{M', N' \in \text{pTCWS}} \omega(M', N') \cdot \mathbf{d}(M', N') + q \sum_{M', N' \in \text{pTCWS}} \Theta(M') \cdot \Delta'(N') \cdot \mathbf{d}(M', N') \\ &+ q \cdot \sum_{M', N' \in \text{pTCWS}} (1 - |\Delta'|) \cdot \Theta(M') \cdot \overline{\text{Dead}}(N') \cdot \mathbf{d}(M', N') \\ &\leq (1 - q) \cdot \mathbf{K}(\mathbf{d})(\Theta, \Theta' + (1 - |\Theta'|)\overline{\text{Dead}}) + q |\Delta'| + q(1 - |\Delta'|) \\ &\leq p(1 - q) + q \end{aligned}$$

with the first inequality deriving by the fact that ω is a matching realising

$$\mathbf{K}(\mathbf{d})(\Theta, \Theta' + (1 - |\Theta'|)\overline{\text{Dead}}), \mathbf{d}(M', N') \leq 1$$

and the equalities

$$\sum_{M' \in \text{pTCWS}} \Theta(M') = 1, \quad \sum_{N' \in \text{pTCWS}} \Delta'(N') = |\Delta'|, \quad \text{and} \quad \sum_{N' \in \text{pTCWS}} \overline{\text{Dead}}(N') = 1. \quad \square$$

Proof of Proposition 2.13. Assume a transition $M \xrightarrow{\alpha} \Theta$. By $M \sqsubseteq_p N$ there is a transition $N \xrightarrow{\hat{\alpha}} \Theta'$ with $\mathbf{K}(\mathbf{d})(\Theta, \Theta' + (1 - |\Theta'|)\overline{\text{Dead}}) \leq p$. By the hypothesis that N can perform only the transition $N \xrightarrow{\sigma} \overline{N'}$ for a network N' such that $N' \not\xrightarrow{\tau}$, we are sure that $\alpha = \sigma$ and $\Theta' = \overline{N'}$. By the hypothesis $O \xrightarrow{\sigma} O' \xrightarrow{\hat{\tau}} (1 - q)\overline{N'} + q\Delta$ we get $O \xrightarrow{\hat{\alpha}} (1 - q)\Theta' + q\Delta'$, for a suitable distribution Δ' such that $\Delta \xrightarrow{\hat{\alpha}} \Delta'$. To conclude, it remains to prove $\mathbf{K}(\mathbf{d})(\Theta, (1 - q)\Theta' + q\Delta' + ((1 - q)(1 - |\Theta'|) + q(1 - |\Delta'|))\overline{\text{Dead}}) \leq p(1 - q) + q$, which follows as in the proof of Lemma 2.12. \square

To prove Theorem 2.15 we give the following preliminary result.

Lemma A.3. $M \sqsubseteq_p N$ entails $M \mid O \sqsubseteq_p N \mid O$ and $O \mid M \sqsubseteq_p O \mid N$.

Proof. We show that $M \sqsubseteq_p N$ entails $M \mid O \sqsubseteq_p N \mid O$, the property $O \mid M \sqsubseteq_p O \mid N$ can be derived analogously. It is enough to prove that $\mathbf{d}(M, N) = p$ entails $\mathbf{d}(M \mid O, N \mid O) \leq p$. Let us define the function $d' : \text{pTCWS} \times \text{pTCWS} \rightarrow [0, 1]$ by $d'(M \mid O, N \mid O) = \mathbf{d}(M, N)$ for all $M, N, O \in \text{pTCWS}$. To prove the thesis it is enough to show that d' is a weak simulation quasimetric. In fact, since \mathbf{d} is the minimal weak simulation quasimetric, this implies $\mathbf{d} \sqsubseteq d'$, thus giving $\mathbf{d}(M \mid O, N \mid O) \leq d'(M \mid O, N \mid O) = \mathbf{d}(M, N)$. Hence we need to show that $d'(M \mid O, N \mid O) < 1$ and $M \mid O \xrightarrow{\alpha} \Delta$ implies that there is a transition $N \mid O \xrightarrow{\hat{\alpha}} \Theta$ with $\mathbf{K}(d')(\Delta, \Theta + (1 - |\Theta|)\overline{\text{Dead}}) \leq d'(M \mid O, N \mid O)$. We distinguish three cases.

- $M \mid O \xrightarrow{\alpha} \Delta$ follows by $M \xrightarrow{\alpha} \Delta'$ and $\Delta = \Delta' \mid \overline{O}$ by applying rule (TauPar)
- $M \mid O \xrightarrow{\alpha} \Delta$ follows by $O \xrightarrow{\alpha} \Delta'$ and $\Delta = \overline{M} \mid \Delta'$ by applying rule (TauPar)
- $M \mid O \xrightarrow{\alpha} \Delta$ follows by $M \xrightarrow{\alpha_1} \Delta_1$ and $O \xrightarrow{\alpha_2} \Delta_2$ and $\Delta = \Delta_1 \mid \Delta_2$ by applying one of the rules (RcvPar), (Bcast), (σ -Par).

We consider the first case, the others are analogous. Since $M \mid O \xrightarrow{\alpha} \Delta$ is obtained by applying rule (TauPar), we have $\alpha = \tau$. By $d'(M \mid O, N \mid O) < 1$ we get $\mathbf{d}(M, N) < 1$, which, together with $M \xrightarrow{\alpha} \Delta'$ gives $N \xrightarrow{\hat{\alpha}} \Theta'$, with $|\Theta'| = 1$ following by $\alpha = \tau$, and $\mathbf{K}(\mathbf{d})(\Delta', \Theta') \leq \mathbf{d}(M, N)$. By applying the rule (TauPar) possibly several times we can derive $N \mid O \xrightarrow{\hat{\alpha}} \Theta' \mid \overline{O}$. We have now to show that $\mathbf{K}(d')(\Delta' \mid \overline{O}, \Theta' \mid \overline{O}) \leq d'(M \mid O, N \mid O)$,

namely that $\Theta' | \overline{O}$ is the distribution Θ we were looking for. To this end, we take one of the optimal matchings $\omega \in \Omega(\Delta', \Theta')$ realising $\mathbf{K}(\mathbf{d})(\Delta', \Theta')$ and we define $\omega': \text{pTCWS} \times \text{pTCWS} \rightarrow [0, 1]$ by $\omega'(M' | O, N' | O) = \omega(M', N')$ and $\omega'(M'', N'') = 0$ whenever M'' is not of the form $M' | O$ or N'' is not of the form $N' | O$. Thus, we prove:

- (1) ω' is a matching in $\Omega(\Delta' | \overline{O}, \Theta' | \overline{O})$, and
- (2) $\sum_{M'', N'' \in \text{pTCWS}} d'(M'', N'') \cdot \omega'(M'', N'') \leq d'(M' | O, N' | O)$.

To show (1) we prove that the left marginal of ω' is $\Delta' | \overline{O}$ by

$$\begin{aligned} & \sum_{N'' \in \text{pTCWS}} \omega'(M' | O, N'') \\ &= \sum_{N' \in \text{pTCWS}} \omega'(M' | O, N' | O) \\ &= \sum_{N' \in \text{pTCWS}} \omega(M', N') \\ &= \Delta'(M') \\ &= (\Delta' | \overline{O})(M' | O) \end{aligned}$$

with the second last equality by $\omega \in \Omega(\Delta', \Theta')$. The proof that the right marginal of ω' is $\Theta' | \overline{O}$ can be proved analogously. Then, (2) follows by

$$\begin{aligned} & \sum_{M'', N'' \in \text{pTCWS}} d'(M'', N'') \cdot \omega'(M'', N'') \\ &= \sum_{M', N' \in \text{pTCWS}} d'(M' | O, N' | O) \cdot \omega(M', N') \\ &= \sum_{M', N' \in \text{pTCWS}} \mathbf{d}(M', N') \cdot \omega(M', N') \\ &= \mathbf{K}(\mathbf{d})(\Delta', \Theta') \\ &\leq \mathbf{d}(M', N') \\ &= d'(M' | O, N' | O). \end{aligned}$$

with the second last equality by the fact that ω is an optimal matching in $\Omega(\Delta', \Theta')$. \square

Proof of Theorem 2.15. By Lemma A.3 and transitivity (Proposition 2.10). \square

Proof of Proposition 2.16. Assume that $M \xrightarrow{\alpha} \Delta$. By the hypothesis $M \sqsubseteq_{s_i} n[P_i]^\nu | N$ we have $n[P_i]^\nu | N \xrightarrow{\hat{\alpha}} \Delta_i$ with $\mathbf{K}(\mathbf{d})(\Delta, \Delta_i + (1 - |\Delta_i|)\overline{\text{Dead}}) \leq s_i$. Hence we have $n[\tau \cdot \bigoplus_{i \in I} p_i : P_i]^\nu \xrightarrow{\hat{\alpha}} \sum_{i \in I} p_i \Delta_i$. It remains to show the inequality $\mathbf{K}(\mathbf{d})(\Delta, \sum_{i \in I} p_i \Delta_i + (1 - |\sum_{i \in I} p_i \Delta_i|)\overline{\text{Dead}}) \leq \sum_{i \in I} p_i s_i$. Assume that ω_i is one of the optimal matchings realising $\mathbf{K}(\mathbf{d})(\Delta, \Delta_i + (1 - |\Delta_i|)\overline{\text{Dead}})$. Define the function $\omega: \text{pTCWS} \times \text{pTCWS} \rightarrow [0, 1]$ by $\omega(M', N') = \sum_{i \in I} p_i \omega_i(M', N')$ for all networks $M', N' \in \text{pTCWS}$. To conclude that $\mathbf{K}(\mathbf{d})(\Delta, \sum_{i \in I} p_i \Delta_i + (1 - |\sum_{i \in I} p_i \Delta_i|)\overline{\text{Dead}}) \leq \sum_{i \in I} p_i s_i$, we show that:

- (1) ω is a matching in $\Omega(\Delta, \sum_{i \in I} p_i \Delta_i + (1 - |\sum_{i \in I} p_i \Delta_i|)\overline{\text{Dead}})$ and
- (2) $\sum_{M', N' \in \text{pTCWS}} \omega(M', N') \cdot d(M', N') \leq \sum_{i \in I} p_i s_i$.

Let us start with (1). The proof that the left marginal of ω is Δ is immediate. We show that the right marginal of ω is the distribution $\sum_{i \in I} p_i \Delta_i + (1 - |\sum_{i \in I} p_i \Delta_i|)\overline{\text{Dead}}$ by

$$\begin{aligned} & \sum_{M' \in \text{pTCWS}} \omega(M', N') \\ &= \sum_{M' \in \text{pTCWS}} \sum_{i \in I} p_i \cdot \omega_i(M', N') \\ &= \sum_{i \in I} p_i \sum_{M' \in \text{pTCWS}} \omega_i(M', N') \\ &= \sum_{i \in I} p_i \cdot (\Delta_i + (1 - |\Delta_i|)\overline{\text{Dead}})(N') \\ &= (\sum_{i \in I} p_i \Delta_i + (1 - |\sum_{i \in I} p_i \Delta_i|)\overline{\text{Dead}})(N') \end{aligned}$$

for each $N' \in \text{pTCWS}$, with the second last equality by $\omega_i \in \Omega(\Delta, \Delta_i + (1 - |\Delta_i|)\overline{\text{Dead}})$. Consider now (2). We have

$$\begin{aligned}
& \sum_{M', N' \in \text{pTCWS}} \omega(M', N') \cdot d(M', N') \\
= & \sum_{M', N' \in \text{pTCWS}} \sum_{i \in I} p_i \cdot \omega_i(M', N') \cdot d(M', N') \\
= & \sum_{i \in I} p_i \sum_{M', N' \in \text{pTCWS}} \omega_i(M', N') \cdot d(M', N') \\
= & \sum_{i \in I} p_i \cdot \mathbf{K}(\mathbf{d})(\Delta, \Delta_i + (1 - |\Delta_i|)\overline{\text{Dead}}) \\
\leq & \sum_{i \in I} p_i \cdot s_i
\end{aligned}$$

with the last equality by the definition of ω_i . This concludes the proof. \square

Proof of Proposition 2.17.

- (1) We proceed by induction over k . The base case $k = 0$ is immediate. Consider the inductive step $k + 1$. We prove that any transition by $n[\sigma^{k+1}.\text{nil}]^\mu$ can be simulated by $n[\text{nil}]^\mu$, the symmetric case is analogous. The only transition by $n[\sigma^{k+1}.\text{nil}]^\mu$ is the transition $n[\sigma^{k+1}.\text{nil}]^\mu \xrightarrow{\sigma} \overline{n[\sigma^k.\text{nil}]^\mu}$ derived by rule (Sleep). By an application of rule (σ -nil) we get $n[\text{nil}]^\mu \xrightarrow{\sigma} \overline{n[\text{nil}]^\mu}$. We have $\mathbf{K}(\mathbf{d})(\overline{n[\sigma^k.\text{nil}]^\mu}, n[\text{nil}]^\mu) = \mathbf{d}(n[\sigma^k.\text{nil}]^\mu, n[\text{nil}]^\mu) = 0$, where the last equality follows by the inductive hypothesis.
- (2) By the fact that the only transitions that can be made by the two networks $\prod_{i \in I} m_i[\sigma.P_i]^{\nu_{m_i}}$ and $\prod_{j \in J} n_j[\sigma.Q_j]^{\mu_{n_j}}$ are $\prod_{i \in I} m_i[\sigma.P_i]^{\nu_{m_i}} \xrightarrow{\sigma} \overline{\prod_{i \in I} m_i[P_i]^{\nu_{m_i}}}$ and $\prod_{j \in J} n_j[\sigma.Q_j]^{\mu_{n_j}} \xrightarrow{\sigma} \overline{\prod_{j \in J} n_j[Q_j]^{\mu_{n_j}}}$ (see rules (Sleep) and (σ -Par)).
- (3) The hypothesis that nodes in μ do not send in the current time interval imply that the only transition by $n[?(x).C]D]^\mu$ is $n[?(x).C]D]^\mu \xrightarrow{\sigma} \llbracket n[D]^\mu \rrbracket$. On the other hand, the only transition by $n[\sigma.D]^\mu$ is $n[\sigma.D]^\mu \xrightarrow{\sigma} \llbracket n[D]^\mu \rrbracket$. The thesis immediately follows.
- (4) This is an immediate corollary of item 3.
- (5) By Proposition 2.11, it is enough to prove that

$$m[\tau.(!\langle v \rangle \oplus_p \text{nil})]^\nu \mid \prod_{i \in I} n_i[P_i]^{\nu_i} \xrightarrow{\tau} \overline{m[\text{nil}]^\nu \mid \prod_{i \in I} n_i[P_i]^{\nu_i}} \quad (\text{A.1})$$

By rule (Tau) we get

$$m[\tau.(!\langle v \rangle \oplus_p \text{nil})]^\nu \xrightarrow{\tau} \overline{pm[!\langle v \rangle]^\nu} + (1 - p)\overline{m[\text{nil}]^\nu}.$$

Hence, to prove Equation A.1 it suffices to show that

$$m[!\langle v \rangle]^\nu \text{ big} \mid \prod_{i \in I} n_i[P_i]^{\nu_i} \xrightarrow{\tau} \overline{m[\text{nil}]^\nu \mid \prod_{i \in I} n_i[P_i]^{\nu_i}} \quad (\text{A.2})$$

To this end, first of all we note that, since for all $i \in I$ $P_i \neq [?(x).C]D$, by rule (RcvEnb) we have that

$$n_i[P_i]^{\nu_i} \xrightarrow{m?v} \overline{n_i[P_i]^{\nu_i}}$$

and, then, by rule (RcvPar) we get $\prod_{i \in I} n_i[P_i]^{\nu_i} \xrightarrow{m?v} \overline{\prod_{i \in I} n_i[P_i]^{\nu_i}}$. By an application of rule (Snd) we have $m[!\langle v \rangle]^\nu \xrightarrow{m!v \triangleright \nu} \overline{m[\text{nil}]^\nu}$. Finally, we can apply rule (Bcast) to the last two transitions, and, since $\nu = \bigcup_{i \in I} n_i$, by applying rule (Shh) we get the proof obligation Equation A.2. \square

A.2. Proofs of results in Section 3. First we describe how we prove Theorem 3.1. Let O denote the network $O \stackrel{\text{def}}{=} N \mid \prod_{i \in I} m_i[\text{nil}]^{\nu_{m_i}} \mid \prod_{j \in J} n_j[\text{resnd}\langle v \rangle_{q_j}]^{\nu_{n_j}}$. We will prove that for some distribution Δ there is a transition $M \xrightarrow{\hat{\tau}} (1 - \prod_{i \in I} (1 - p_i)) \cdot \overline{O} + \prod_{i \in I} (1 - p_i) \cdot \Delta$, then the thesis follows by Proposition 2.11. The proof that $M \xrightarrow{\hat{\tau}}$

$(1 - \prod_{i \in I} (1 - p_i)) \cdot \bar{O} + \prod_{i \in I} (1 - p_i) \cdot \Delta$ is divided in two parts. First we prove that with probability $(1 - \prod_{i \in I} (1 - p_i))$ the network M can reach through a sequence of τ -transitions a network O_{I_1, I_2, I_3} of the form $O_{I_1, I_2, I_3} = N \mid \prod_{i \in I_1} m_i[!\langle v \rangle]^{\nu_{m_i}} \mid \prod_{i \in I_2} m_i[\text{nil}]^{\nu_{m_i}} \mid \prod_{i \in I_3} m_i[\text{snd}\langle v \rangle_{p_i}]^{\nu_{m_i}} \mid \prod_{j \in J} n_j[\text{resnd}\langle v \rangle_{q_j}]^{\nu_{n_j}}$, for $I_1 \cup I_2 \cup I_3$ some partition of I . In particular, one of the τ -transitions in this sequence is derived via rule (ShhSnd) as a consequence of the broadcast of the message by one of the senders. Then we prove that $O_{I_1, I_2, I_3} \xrightarrow{\tau} \bar{O}$. The first part of the proof is given directly in the proof of Theorem 3.1, whereas $O_{I_1, I_2, I_3} \xrightarrow{\tau} \bar{O}$ is proved in the following Lemma.

Lemma A.4. *Assume the hypothesis of Theorem 3.1. Let O denote the network*

$$O \stackrel{\text{def}}{=} N \mid \prod_{i \in I} m_i[\text{nil}]^{\nu_{m_i}} \mid \prod_{j \in J} n_j[\text{resnd}\langle v \rangle_{q_j}]^{\nu_{n_j}}$$

and O_{I_1, I_2, I_3} denote the network

$$N \mid \prod_{i \in I_1} m_i[!\langle v \rangle]^{\nu_{m_i}} \mid \prod_{i \in I_2} m_i[\text{nil}]^{\nu_{m_i}} \mid \prod_{i \in I_3} m_i[\text{snd}\langle v \rangle_{p_i}]^{\nu_{m_i}} \mid \prod_{j \in J} n_j[\text{resnd}\langle v \rangle_{q_j}]^{\nu_{n_j}}$$

for $I_1 \cup I_2 \cup I_3$ any partition of I . Then, we have $O_{I_1, I_2, I_3} \xrightarrow{\tau} \bar{O}$.

Proof. We reason by induction over $|I_1 \cup I_3|$. The base case $|I_1 \cup I_3| = 0$ is immediate, since $I_2 = I$ and O_{I_1, I_2, I_3} is the network O . Consider the inductive step $|I_1 \cup I_3| = n + 1$. We distinguish two sub-cases, depending whether I_3 is empty or not.

Case $I_3 \neq \emptyset$. In this case the network O_{I_1, I_2, I_3} can perform a τ -step originated by a sender m_k with $k \in I_3$. Formally, by rule (Tau) we get

$$m_k[\text{snd}\langle v \rangle_{p_k}]^{\nu_{m_k}} \xrightarrow{\tau} p_k \cdot \overline{m_k[!\langle v \rangle]^{\nu_{m_k}}} + (1 - p_k) \cdot \overline{m_k[\text{nil}]^{\nu_{m_k}}}$$

and, then, by an application of rule (TauPar) we get

$$O_{I_1, I_2, I_3} \xrightarrow{\tau} \bar{N} \mid \prod_{i \in I_1} \overline{m_i[!\langle v \rangle]^{\nu_{m_i}}} \mid \prod_{i \in I_2} \overline{m_i[\text{nil}]^{\nu_{m_i}}} \mid q \prod_{i \in I_3 \setminus \{k\}} \overline{m_i[\text{snd}\langle v \rangle_{p_i}]^{\nu_{m_i}}} \mid (p_k \cdot \overline{m_k[!\langle v \rangle]^{\nu_{m_k}}} + (1 - p_k) \cdot \overline{m_k[\text{nil}]^{\nu_{m_k}}}) \mid \prod_{j \in J} n_j[\text{resnd}\langle v \rangle_{q_j}]^{\nu_{n_j}}$$

namely

$$O_{I_1, I_2, I_3} \xrightarrow{\tau} p_k \cdot \overline{O_{I_1 \cup \{k\}, I_2, I_3 \setminus \{k\}}} + (1 - p_k) \cdot \overline{O_{I_1, I_2 \cup \{k\}, I_3 \setminus \{k\}}} \quad (\text{A.3})$$

Consider first the network $O_{I_1 \cup \{k\}, I_2, I_3 \setminus \{k\}}$ in Equation A.3. A τ -transition by network $O_{I_1 \cup \{k\}, I_2, I_3 \setminus \{k\}}$ models the broadcasting of v by m_k that is not received by any other node. In detail, by rule (Snd) we get

$$m_k[!\langle v \rangle]^{\nu_{m_k}} \xrightarrow{m_k!v \triangleright \nu_{m_k}} \overline{m_k[\text{nil}]^{\nu_{m_k}}}.$$

Then, by an application of rule (RcvEnb) we infer

$$N \xrightarrow{m_k?v} \bar{N}$$

which can be applied since by the hypothesis we know that no node in N is able to receive, by the same (RcvEnb) we get

$$m_i[!\langle v \rangle]^{\nu_{m_i}} \xrightarrow{m_k?v} \overline{m_i[!\langle v \rangle]^{\nu_{m_i}}}$$

for all $i \in I_1$,

$$m_i[\text{nil}]^{\nu_{m_i}} \xrightarrow{m_k?v} \overline{m_i[\text{nil}]^{\nu_{m_i}}}$$

for all $i \in I_2$, and

$$m_i[\text{snd}\langle v \rangle_{p_i}]^{\nu_{m_i}} \xrightarrow{m_k?v} \overline{m_i[\text{snd}\langle v \rangle_{p_i}]^{\nu_{m_i}}}$$

for all $i \in I_3 \setminus \{k\}$, then again by rule (RcvEnb) we infer

$$n_j[\text{resnd}\langle v \rangle_{q_j}]^{\nu_{n_j}} \xrightarrow{m_k?v} \overline{n_j[\text{resnd}\langle v \rangle_{q_j}]^{\nu_{n_j}}}$$

for all $j \in J$, thus allowing us to apply rules (RcvPar) and (Bcast) and obtain

$$O_{I_1 \cup \{k\}, I_2, I_3 \setminus \{k\}} \xrightarrow{m_k!v \triangleright \nu_{m_k} \setminus \text{nds}(M)} \overline{O_{I_1, I_2 \cup \{k\}, I_3 \setminus \{k\}}}$$

from which we get

$$O_{I_1 \cup \{k\}, I_2, I_3 \setminus \{k\}} \xrightarrow{\tau} \overline{O_{I_1, I_2 \cup \{k\}, I_3 \setminus \{k\}}} \quad (\text{A.4})$$

by rule (ShhSnd), which can be applied since by the hypothesis we know that $\nu_{m_k} \subseteq \text{nds}(M)$. Now, since $|I_1 \cup I_3 \setminus \{k\}| = n$ we can apply the inductive hypothesis thus obtaining $O_{I_1, I_2 \cup \{k\}, I_3 \setminus \{k\}} \xrightarrow{\tau} \overline{O}$. Finally, by Equation A.4 and $O_{I_1, I_2 \cup \{k\}, I_3 \setminus \{k\}} \xrightarrow{\tau} \overline{O}$ we get

$$O_{I_1 \cup \{k\}, I_2, I_3 \setminus \{k\}} \xrightarrow{\tau} \overline{O} \quad (\text{A.5})$$

Consider now the network $O_{I_1, I_2 \cup \{k\}, I_3 \setminus \{k\}}$ in Equation A.3. Since $|I_1 \cup I_3 \setminus \{k\}| = n$ we can apply the inductive hypothesis thus giving $O_{I_1, I_2 \cup \{k\}, I_3 \setminus \{k\}} \xrightarrow{\tau} \overline{O}$, which, together with Equation A.3 and Equation A.5, gives the thesis.

Case $I_3 = \emptyset$. Since $|I_1 \cup I_3| = n + 1$, we are sure that $I_1 \neq \emptyset$. For some $k \in I_1$, we can reason as above to infer that from a broadcasting transition by some m_k with $k \in I_1$ we get $O_{I_1, I_2, I_3} \xrightarrow{\tau} \overline{O_{I_1 \setminus \{k\}, I_2 \cup \{k\}, I_3}}$, then by the inductive hypothesis we derive $O_{I_1 \setminus \{k\}, I_2 \cup \{k\}, I_3} \xrightarrow{\tau} \overline{O}$, and, finally, from these two transitions we conclude $O_{I_1, I_2, I_3} \xrightarrow{\tau} \overline{O}$. \square

We are now ready to prove Theorem 3.1.

Proof of Theorem 3.1. As in the proof of Lemma A.4, we use O to denote the network

$$O \stackrel{\text{def}}{=} N \mid \prod_{i \in I} m_i[\text{nil}]^{\nu_{m_i}} \mid \prod_{j \in J} n_j[\text{resnd}\langle v \rangle_{q_j}]^{\nu_{n_j}} .$$

We prove that $M \xrightarrow{\hat{\tau}} (1 - \prod_{i \in I} (1 - p_i)) \overline{O} + \prod_{i \in I} (1 - p_i) \Delta$, for some distribution Δ , then the thesis follows by Proposition 2.11.

By rule (Tau) we derive the following transition step for any sender m_i with $i \in I$:

$$m_i[\text{snd}\langle v \rangle_{p_i}]^{\nu_{m_i}} \xrightarrow{\tau} p_i \cdot \overline{m_i[!\langle v \rangle]^{\nu_{m_i}}} + (1 - p_i) \cdot \overline{m_i[\text{nil}]^{\nu_{m_i}}} .$$

The network M can start by performing $k \leq |I|$ transitions labelled τ , where each of these transitions comes from rule (Tau) applied as above (namely is the initial τ -step by some sender) and (TauPar). Therefore, we can partition I as $I = I' \cup I''$ such that

$$M \xrightarrow{\tau} \dots \xrightarrow{\tau} \Theta$$

with

$$\Theta = \overline{N} \mid \prod_{i \in I'} (\overline{p_i m_i[!\langle v \rangle]^{\nu_{m_i}}} + (1 - p_i) \overline{m_i[\text{nil}]^{\nu_{m_i}}}) \mid \prod_{i \in I''} \overline{m_i[\text{snd}\langle v \rangle_{p_i}]^{\nu_{m_i}}} \mid \prod_{j \in J} \overline{n_j[\text{fwd}_{q_j}]^{\nu_{n_j}}} \quad (\text{A.6})$$

and now we distinguish two cases.

The first case is that all senders m_i with $i \in I$ reach the state $\overline{m_i[\text{nil}]^{\nu_{m_i}}}$ through their initial τ transition, namely in the Equation A.6 we have that $I' = I$, $I'' = \emptyset$ and the

network in the support of Θ that is reached by M is $N \mid \prod_{i \in I} m_i[\text{nil}]^{\nu_{m_i}} \mid \prod_{j \in J} n_j[\text{fwd}_{q_j}]^{\nu_{n_j}}$. Formally,

$$M \xrightarrow{\hat{\tau}} q \cdot (\overline{N} \mid \prod_{i \in I} \overline{m_i[\text{nil}]^{\nu_{m_i}}} \mid \prod_{j \in J} \overline{n_j[\text{fwd}_{q_j}]^{\nu_{n_j}}}) \quad (\text{A.7})$$

for $q = \prod_{i \in I} (1 - p_i)$. The network $N \mid \prod_{i \in I} m_i[\text{nil}]^{\nu_{m_i}} \mid \prod_{j \in J} n_j[\text{fwd}_{q_j}]^{\nu_{n_j}}$ is then unable to simulate O .

The second case is that at least one of the senders m_i decides to transmit and reaches the state $m_i[!\langle v \rangle]^{\nu_{m_i}}$ through the τ transition, namely there is a partition of $I' = I'_1 \cup I'_2$ such that the state in the support of distribution Θ that is reached by M has the form $N \mid \prod_{i \in I'_1} m_i[!\langle v \rangle]^{\nu_{m_i}} \mid \prod_{i \in I'_2} m_i[\text{nil}]^{\nu_{m_i}} \mid \prod_{i \in I''} m_i[\text{snd}\langle v \rangle_{p_i}]^{\nu_{m_i}} \mid \prod_{j \in J} n_j[\text{fwd}_{q_j}]^{\nu_{n_j}}$ with $I'_1 \neq \emptyset$. Formally,

$$M \xrightarrow{\hat{\tau}} q' \cdot \left(\overline{N} \mid \prod_{i \in I'_1} \overline{m_i[!\langle v \rangle]^{\nu_{m_i}}} \mid \prod_{i \in I'_2} \overline{m_i[\text{nil}]^{\nu_{m_i}}} \mid \prod_{i \in I''} \overline{m_i[\text{snd}\langle v \rangle_{p_i}]^{\nu_{m_i}}} \mid \prod_{j \in J} \overline{n_j[\text{fwd}_{q_j}]^{\nu_{n_j}}} \right) \quad (\text{A.8})$$

with $q' = \prod_{i \in I'_1} p_i \cdot \prod_{i \in I'_2} (1 - p_i)$. In this case, each of the nodes m_i with $i \in I'_1$ can then

broadcasts v . Indeed, by applying the rule (Snd) we get $m_h[!\langle v \rangle]^{\nu_{m_h}} \xrightarrow{m_h!v \triangleright \nu_{m_h}} \overline{m_h[\text{nil}]^{\nu_{m_h}}}$, for some $h \in I'_1$. Since by the hypothesis we know that all nodes in $\nu_{m_h} \cap \text{nds}(N)$ cannot receive in the current round, and, analogously, all nodes n_i with $i \in I \setminus \{h\} = (I'_1 \cup I'_2 \cup I'') \setminus \{h\}$ cannot receive, by applying rule (RcvEnb) we infer: (a) $N \xrightarrow{m_h?v} \overline{N}$, (b) $m_i[\text{snd}\langle v \rangle_{p_i}]^{\nu_{m_i}} \xrightarrow{m_h?v} \overline{m_i[\text{snd}\langle v \rangle_{p_i}]^{\nu_{m_i}}}$, (c) $m_i[\text{nil}]^{\nu_{m_i}} \xrightarrow{m_h?v} \overline{m_i[\text{nil}]^{\nu_{m_i}}}$. Moreover, for all $j \in J$, by rule (Rcv) we infer

$$n_j[\text{fwd}_{q_j}]^{\nu_{n_j}} \xrightarrow{m_h?v} \overline{n_j[\text{resnd}\langle v \rangle_{q_j}]^{\nu_{n_j}}}$$

thus implying that we can apply rules (RecPar) and (Bcast) and obtain

$$N \mid \prod_{i \in I'_1} m_i[!\langle v \rangle]^{\nu_{m_i}} \mid \prod_{i \in I'_2} m_i[\text{nil}]^{\nu_{m_i}} \mid \prod_{i \in I''} m_i[\text{snd}\langle v \rangle_{p_i}]^{\nu_{m_i}} \mid \prod_{j \in J} n_j[\text{fwd}_{q_j}]^{\nu_{n_j}} \xrightarrow{m_h!v \triangleright \nu_{m_h} \setminus \text{nds}(M)}$$

$$\overline{N} \mid \overline{m_h[\text{nil}]^{\nu_{m_h}}} \mid \prod_{i \in I'_1 \setminus \{h\}} \overline{m_i[!\langle v \rangle]^{\nu_{m_i}}} \mid \prod_{i \in I'_2} \overline{m_i[\text{nil}]^{\nu_{m_i}}} \mid \prod_{i \in I''} \overline{m_i[\text{snd}\langle v \rangle_{p_i}]^{\nu_{m_i}}} \mid \prod_{j \in J} \overline{n_j[\text{resnd}\langle v \rangle_{q_j}]^{\nu_{n_j}}}.$$

Since by the hypothesis we have $\nu_{m_h} \subseteq \text{nds}(M)$, we can apply rule (ShhSnd) to infer

$$N \mid \prod_{i \in I'_1} m_i[!\langle v \rangle]^{\nu_{m_i}} \mid \prod_{i \in I'_2} m_i[\text{nil}]^{\nu_{m_i}} \mid \prod_{i \in I''} m_i[\text{snd}\langle v \rangle_{p_i}]^{\nu_{m_i}} \mid \prod_{j \in J} n_j[\text{fwd}_{q_j}]^{\nu_{n_j}} \xrightarrow{\tau} \quad (\text{A.9})$$

$$\overline{N} \mid \overline{m_h[\text{nil}]^{\nu_{m_h}}} \mid \prod_{i \in I'_1 \setminus \{h\}} \overline{m_i[!\langle v \rangle]^{\nu_{m_i}}} \mid \prod_{i \in I'_2} \overline{m_i[\text{nil}]^{\nu_{m_i}}} \mid \prod_{i \in I''} \overline{m_i[\text{snd}\langle v \rangle_{p_i}]^{\nu_{m_i}}} \mid \prod_{j \in J} \overline{n_j[\text{resnd}\langle v \rangle_{q_j}]^{\nu_{n_j}}}$$

Let O' denote the only network in the support of the distribution in Equation A.9, i.e. $N \mid \prod_{i \in I'_1 \setminus \{h\}} m_i[!\langle v \rangle]^{\nu_{m_i}} \mid \prod_{i \in I'_2 \cup \{h\}} m_i[\text{nil}]^{\nu_{m_i}} \mid \prod_{i \in I''} m_i[\text{snd}\langle v \rangle_{p_i}]^{\nu_{m_i}} \mid \prod_{j \in J} n_j[\text{resnd}\langle v \rangle_{q_j}]^{\nu_{n_j}}$. We observe that by adopting the notation in Lemma A.4, O' is in the form O_{I_1, I_2, I_3} with $I_1 = I'_1 \setminus \{h\}$, $I_2 = I'_2 \cup \{h\}$, $I_3 = I''$. By Lemma A.4 we have $O' \xrightarrow{\tau} \overline{O}$. This transition together with Equation A.8 and Equation A.9 gives $M \xrightarrow{\tau} q' \overline{O}$, where we remind that $q' = \prod_{i \in I'_1} p_i \cdot \prod_{i \in I'_2} (1 - p_i)$. This transition is derived for any partition $I'_1 \cup I'_2 \cup I''$ of I with $I'_1 \neq \emptyset$. Hence $M \xrightarrow{\tau} (1 - q) \overline{O}$, with q the probability value in Equation A.7, which gives the thesis. \square

Proof of Theorem 3.2. Let us denote with M_0 , M_1 and M_2 the following networks:

- $M_0 = N \mid m[!\langle v \rangle]^{\nu_m} \mid \prod_{j \in J} n_j[!?(x_j).P_j]Q_j]^{\nu_{n_j}}$
- $M_1 = N \mid m[\text{nil}]^{\nu_m} \mid \prod_{j \in J} n_j[\{v/x_j\}P_j]^{\nu_{n_j}}$
- $M_2 = N \mid m[\text{nil}]^{\nu_m} \mid \prod_{j \in J} n_j[!?(x_j).P_j]Q_j]^{\nu_{n_j}}$.

We recall that $\text{snd}\langle v \rangle_p = \tau.(!\langle v \rangle \oplus_p \text{nil})$. Therefore we infer the thesis from Proposition 2.16 if we show that $M \sqsubseteq_{s_1} M_0$ and $M \sqsubseteq_{s_2} M_2$. The relation $M \sqsubseteq_{s_2} M_2$ holds by the hypothesis, hence we have to prove $M \sqsubseteq_{s_1} M_0$. To this purpose, it suffices to prove $M_1 \sqsubseteq_0 M_0$. In fact, $M_1 \sqsubseteq_0 M_0$ and the hypothesis $M \sqsubseteq_{s_1} M_1$ give, by transitivity (Proposition 2.10), $M \sqsubseteq_{s_1} M_0$. Finally, to prove $M_1 \sqsubseteq_0 M_0$, by Proposition 2.11 it is enough to prove that $M_0 \xrightarrow{\tau} \overline{M_1}$.

We show now that this transition can be derived. By rule (Snd) we get $m[!\langle v \rangle]^{\nu_m} \xrightarrow{m!v \triangleright \nu_m} \overline{m[\text{nil}]^{\nu_m}}$. By an application of rule (Rcv) we get

$$n_j[!?(x_j).P_j]Q_j]^{\nu_{n_j}} \xrightarrow{m?v} \overline{n_j[\{v/x_j\}P_j]^{\nu_{n_j}}}$$

for all $j \in J$, where (Rcv) can be applied since we have $m \in \nu_{n_j}$ for all $j \in J$, which follows by the hypothesis $\{n_j \mid j \in J\} \subseteq \nu_m$ and well-formedness. By rule (RcvEnb) we get $N \xrightarrow{m?v} \overline{N}$, where (RcvEnb) can be applied since by the hypothesis we know that all nodes in $\nu_m \cap \text{nds}(N)$ are not able to receive in the current instant of time. Summarising, we can apply rule (Bcast) to obtain $M_0 \xrightarrow{m!v \triangleright \nu_m \setminus \text{nds}(M)} \overline{M_1}$. Then, since by the hypothesis we have $\nu_m \subseteq \text{nds}(M)$, to conclude the proof, we can apply rule (ShhSnd) to derive the transition $M_0 \xrightarrow{\tau} \overline{M_1}$. \square

Proof of Theorem 3.4. Let us denote with M_0 and M_i , for $i \in I$, the following networks:

- $M_0 = \prod_{j \in J} n_j[\text{nil}]^{\nu_{n_j}} \mid d[\tau. \bigoplus_{i \in I} p_i : P_i]^{\nu_d}$
- $M_i = \prod_{j \in J} n_j[\text{nil}]^{\nu_{n_j}} \mid d[P_i]^{\nu_d}$.

The only admissible transition for the network M_0 is $M_0 \xrightarrow{\tau} \Delta = \sum_{i \in I} p_i \cdot \overline{M_i}$, derived by rules (Tau) and (TauPar). By rules (Tau) and (TauPar), we can derive the transition $M \mid m[\tau. \bigoplus_{i \in I} p_i : Q_i]^{\nu_m} \xrightarrow{\tau} \Theta = \sum_{i \in I} p_i \cdot M \mid m[Q_i]^{\nu_m}$. Since $|\Theta| = 1$, it is enough to prove that $\mathbf{K}(\mathbf{d})(\Delta, \Theta) \leq \sum_{i \in I} p_i s_i$. Define $\omega : \text{pTCWS} \times \text{pTCWS} \rightarrow [0, 1]$ by $\omega(M_i, M \mid m[Q_i]^{\nu_m}) = p_i$, for any $i \in I$, and $\omega(M', N') = 0$ for all other pairs (M', N') . To conclude $\mathbf{K}(\mathbf{d})(\Delta, \Theta) \leq \sum_{i \in I} p_i s_i$, it is enough to show that:

- (1) the function ω is a matching in $\Omega(\Delta, \Theta)$ and
- (2) $\sum_{M', N' \in \text{pTCWS}} \omega(M', N') \cdot \mathbf{d}(M', N') \leq \sum_{i \in I} p_i s_i$.

The case (1) is immediate. Consider now (2). We have that $\sum_{M', N' \in \text{pTCWS}} \omega(M', N') \cdot \mathbf{d}(M', N') = \sum_{i \in I} p_i \mathbf{d}(M_i, M \mid m[Q_i]^{\nu_m}) \leq \sum_{i \in I} p_i s_i$ with the inequality by the hypothesis $M_i \sqsubseteq_{s_i} M \mid m[Q_i]^{\nu_m}$. \square

A.3. Proofs of results in Section 4.

Proof of Theorem 4.1. Let $O = N \mid \prod_{i \in I} m_i[\text{nil}]^{\nu_{m_i}} \mid \prod_{j \in J} n_j[\text{resnd}\langle v \rangle_{q_j}]^{\nu_{n_j}}$. We prove that

$$M \xrightarrow{\hat{\tau}} (\sum_{i \in I} p_i \prod_{j \in I \setminus \{i\}} (1 - p_j)) \overline{O} + (1 - \sum_{i \in I} p_i \prod_{j \in I \setminus \{i\}} (1 - p_j)) \Delta \quad (\text{A.10})$$

for some distribution Δ , then the proof follows by Proposition 2.11. By rule (Tau) we get the following transition step for any sender m_i with $i \in I$:

$$m_i[\text{snd}\langle v \rangle_{p_i}]^{\nu_{m_i}} \xrightarrow{\tau} p_i \cdot \overline{m_i[!\langle v \rangle]^{\nu_{m_i}}} + (1 - p_i) \cdot \overline{m_i[\text{nil}]^{\nu_{m_i}}}.$$

The network M can start by performing $|I|$ transitions labelled τ , where each of these transitions comes from rule (Tau) applied as above (namely is the initial τ -step by some sender) and (TauPar). Therefore, we have

$$M \xrightarrow{\tau} \dots \xrightarrow{\tau} \Theta$$

with $\Theta = \overline{N} \mid \prod_{i \in I} (p_i \cdot \overline{m_i[! \langle v \rangle]^{\nu_{m_i}}} + (1 - p_i) \cdot \overline{m_i[\text{nil}]^{\nu_{m_i}}}) \mid \prod_{j \in J} \overline{n_j[\text{fwd}_{q_j}]^{\nu_{n_j}}}$. In detail, all elements in the support of Θ have the form

$$M_{I', I''} = N \mid \prod_{i \in I'} m_i[! \langle v \rangle]^{\nu_{m_i}} \mid \prod_{i \in I''} m_i[\text{nil}]^{\nu_{m_i}} \mid \prod_{j \in J} n_j[\text{fwd}_{q_j}]^{\nu_{n_j}}$$

for a suitable partition $I' \cup I'' = I$, and we have

$$M \xrightarrow{\hat{\tau}} \left(\prod_{i \in I'} p_i \prod_{j \in I''} (1 - p_j) \right) \cdot M_{I', I''}.$$

Consider any $M_{I', I''}$. If $I' = \emptyset$ then $M_{I', I''}$ cannot simulate N since no forwarder process in node n_j can receive any message. If $I' \neq \emptyset$, each of the m_i with $i \in I'$ can then broadcasts v . Indeed, by applying the rule (Snd) we get

$$m_h[! \langle v \rangle]^{\nu_{m_h}} \xrightarrow{m_h!v \triangleright \nu_{m_h}} \overline{m_h[\text{nil}]^{\nu_{m_h}}}$$

for some $h \in I'$. Since by the hypothesis we know that all nodes in $\nu_{m_h} \cap \text{nds}(M')$ cannot receive in the current round, and, analogously, all nodes n_i with $i \in I \setminus \{h\} = (I' \cup I'') \setminus \{h\}$ cannot receive, by applying rule (RcvEnb) we infer: (i) $N \xrightarrow{m_h?v} \overline{N}$; (ii) $m_i[\text{snd} \langle v \rangle_{p_i}]^{\nu_{m_i}} \xrightarrow{m_h?v} \overline{m_i[\text{snd} \langle v \rangle_{p_i}]^{\nu_{m_i}}}$; (iii) $m_i[\text{nil}]^{\nu_{m_i}} \xrightarrow{m_h?v} \overline{m_i[\text{nil}]^{\nu_{m_i}}}$. Moreover, for all $j \in J$, by rule (Rcv) we infer:

$$n_j[\text{fwd}_{q_j}]^{\nu_{n_j}} \xrightarrow{m_h?v} \overline{n_j[\text{resndc} \langle v \rangle_{q_j}]^{\nu_{n_j}}}$$

thus implying that we can apply rules (RcvPar) and (Bcast) and obtain

$$M_{I', I''} \xrightarrow{\alpha} \overline{N} \mid \prod_{i \in I' \setminus \{h\}} \overline{m_i[! \langle v \rangle]^{\nu_{m_i}}} \mid \prod_{i \in I'' \cup \{h\}} \overline{m_i[\text{nil}]^{\nu_{m_i}}} \mid \prod_{j \in J} \overline{n_j[\text{resndc} \langle v \rangle_{q_j}]^{\nu_{n_j}}}$$

with $\alpha = m_h!v \triangleright \nu_{m_h} \setminus \text{nds}(M)$. Since by the hypothesis we have $\nu_{m_h} \subseteq \text{nds}(M)$, we can apply rule (ShhSnd) to infer

$$M_{I', I''} \xrightarrow{\tau} \overline{N} \mid \prod_{i \in I' \setminus \{h\}} \overline{m_i[! \langle v \rangle]^{\nu_{m_i}}} \mid \prod_{i \in I'' \cup \{h\}} \overline{m_i[\text{nil}]^{\nu_{m_i}}} \mid \prod_{j \in J} \overline{n_j[\text{resndc} \langle v \rangle_{q_j}]^{\nu_{n_j}}} .$$

Now we note that if $I' \setminus \{h\} = \emptyset$ then the target of this transition is the network O and we can derive

$$M \xrightarrow{\tau} p_i \prod_{j \in I \setminus \{i\}} (1 - p_j) \overline{O} \tag{A.11}$$

Otherwise, if $I' \setminus \{h\} \neq \emptyset$ then the network $M_{I', I''}$ cannot simulate O since the nodes $\text{resndc} \langle v \rangle_{q_j}$ fail in the present round since they receive from some sender m_i with $i \in I' \setminus \{h\}$. Overall, we have Equation A.11 for all $i \in I$, from which the proof obligation in Equation A.10 follows. \square