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APPLICABLE MATHEMATICS IN A MINIMAL COMPUTATIONAL THEORY OF SETS

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Abstract. In previous papers on this project a general static logical framework for formalizing and mechanizing set theories of different strength was suggested, and the power of some predicatively acceptable theories in that framework was explored. In this work we first improve that framework by enriching it with means for coherently extending by definitions its theories, without destroying its static nature or violating any of the principles on which it is based. Then we turn to investigate within the enriched framework the power of the *minimal* (predicatively acceptable) theory in it that proves the existence of infinite sets. We show that that theory is a computational theory, in the sense that every element of its minimal transitive model is denoted by some of its closed terms. (That model happens to be the second universe in Jensen's hierarchy.) Then we show that already this minimal theory suffices for developing very large portions (if not all) of scientifically applicable mathematics. This requires treating the collection of real numbers as a proper class, that is: a unary predicate which can be introduced in the theory by the static extension method described in the first part of the paper.

1. Introduction

Formalized mathematics and mathematical knowledge management (MKM) are extremely fruitful and quickly expanding fields of research at the intersection of mathematics and computer science (see, e.g., [3, 13, 33]). The declared goal of these fields is to develop computerized systems that effectively represent all important mathematical knowledge and techniques, while conforming to the highest standards of mathematical rigor. At present there is no general agreement what should be the best framework for this task. However, since most mathematicians view *set theory* as the basic foundation of mathematics, formalized set theories should certainly be taken as one of the most natural choice.[1] [2]

[1]Already in [14] it was argued that "a main asset gained from Set theory is the ability to base reasoning on just a handful of axiom schemes which, in addition to being conceptually simple (even though surprisingly expressive), lend themselves to good automated support". More recently, H. Friedman wrote (in a message on FOM on Sep 14, 2015): "I envision a large system and various important weaker subsystems. Since so much math can be done in systems much weaker than ZFC, this should be reflected in the choice of Gold Standards. There should be a few major Gold Standards ranging from Finite Set Theory to full blown ZFC".

[2]Notable set-based automated provers are Mizar [43], Metamath [37], and Referee (aka AetnaNova) [41, 15].

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In [6, 7] a logical framework for developing and mechanizing set theories was introduced. Its key properties are that it is based on the usual (type-free) set theoretic language and makes extensive use of *abstract set terms*. Such terms are extensively used of course in all modern texts in all areas of mathematics (including set theory itself). Therefore their availability is indispensable for the purpose of mechanizing real mathematical practice and for automated or interactive theorem proving in set theories. Accordingly, most of the computerized systems for set theories indeed allow *dynamic* ways of introducing abstract set terms. The great advantage of the framework of [6, 7] is that it does so in a *static* way, so the task of verifying that a given term or formula in it is well-formed is decidable, easily mechanizable, and completely separated from any task connected with proving theorems (like finding proofs or checking validity of given ones). Furthermore, this framework enables the use of different logics and set theories of different strength. This modularity of the system has been exploited in [8], where a hierarchy of set theories for formalizing different levels of mathematics within this framework was presented.

The current paper is mainly devoted to one very basic theory, RST^m_{HF}, from the above-mentioned hierarchy, and to its minimal model. The latter is shown to be the universe J_2 in Jensen's hierarchy [32]. Both RST^m_{HF} and J_2 are *computational* (in a precise sense defined below). With the help of the formal framework of [6, 7, 8] they can therefore be used to make explicit the potential computational content of set theories (first suggested and partially demonstrated in [14]). Here we show that they also suffice for developing large portions of scientifically applicable mathematics [23], especially analysis.[3] In [21, 22, 23] it was forcefully argued by Feferman that scientifically applicable mathematics, the mathematics that is actually indispensable to present-day natural science, can be developed using only predicatively acceptable mathematics. We provide here further support to this claim, using a much simpler framework and by far weaker theory than those employed by Feferman.

The restriction to a minimal framework has of course its price. Not all of the standard mathematical structures can be treated as elements of J_2. (The real line is a case in point.) Hence we have to handle such objects in a different manner. To do this, we first enrich the framework used in [6, 7, 8] with means for coherently extending by definitions theories in it, without destroying its static nature, or violating any of the principles on which it is based. (This step is a very important improvement of the framework on each own right.) This makes it possible to introduce the collection of real numbers in RST^m_{HF} as a *proper class*, that is: a legal defined unary predicate to which no closed term of RST^m_{HF} corresponds. (Classes are introduced here into the formal framework of [6, 7, 8] for the first time.)

The paper is organized as follows: In Section 2 we review the formal framework and the way various standard set theoretical notions have been introduced in in it. We also define in this section the notions of computational theory and universe, and describe the computational theories which are minimal within the framework (as well as the corresponding minimal universes). Section 3 is dedicated to the introduction of standard extensions by definitions of the framework, done in a static way. The notion of a class is then introduced as a particular case, and is used for handling global relations and functions in the system. In Section 4 we introduce the natural numbers in the system. Unlike in [8], this is done here using an absolute characterization of the property of being a natural number, and without any appeal to $\in$-induction. In Section 5 we turn to real analysis, and demonstrate how it can be developed in our minimal computational framework, although the reals are a proper class

[3] The thesis that J_2 is sufficient for core mathematics was first put forward in [46].

in it. This includes the introduction of the real line and real functions, as well as formulating and proving classical results concerning these notions.[4] Section 6 concludes with directions for future continuation of the work.

2. The Formal System and its Minimal Model

2.1. **Preliminaries: the Framework and the Main Formal System.**

Notation 2.1. To avoid confusion, the parentheses $\{\!\!\{\,\}\!\!\}$ are used in our formal languages, for constructing abstract set terms in it, while in the meta-language we use the ordinary $\{\,\}$.[5] We use the letters $X, Y, Z, ...$ for collections; Φ, Θ for finite sets of variables; and $x, y, z, ...$ for variables in the formal language. $Fv(exp)$ denotes the set of free variables of exp, and $\varphi\left[{}^{t_1}/{}_{x_1}, \ldots, {}^{t_n}/{}_{x_n}\right]$ denotes the result of simultaneously substituting t_i for x_i in φ. When the identity of t and x is clear from the context, we just write $\varphi(t)$ instead of $\varphi\left[{}^{t}/{}_{x}\right]$.

One of the foundational questions in set theory is which formulas should be excluded from defining sets by an abstract term of the form $\{x \mid \varphi\}$ in order to avoid the paradoxes of naive set theory. Various set theories provide different answers to this question, which are usually based on *semantical* considerations (such as the limitation of size doctrine [25, 28]). Such an approach is not very useful for the purpose of mechanization. In this work we use instead the general *syntactic* methodology of safety relations developed in [6, 7]. A safety relation is a syntactic relation between formulas and sets of variables. The addition of a safety relation to a logical system allows to use in it statically defined abstract set term of the form $\{x \mid \varphi\}$, provided that φ is safe with respect to $\{x\}$. Intuitively, a statement of the form "φ is *safe* with respect to $\{y_1, ..., y_k\}$", where $Fv(\varphi) = \{x_1, ..., x_n, y_1, ..., y_k\}$, has the meaning that for every "accepted" sets $a_1, ..., a_n$, the collection $\{\langle y_1, ..., y_k\rangle \mid \varphi(a_1, ..., a_n, y_1, ..., y_k)\}$ is an "accepted" set, which is constructed from the previously "accepted" sets $a_1, ..., a_n$ (see discussion below for further details).

Definition 2.2. Let C be a finite set of constants. The language $\mathcal{L}_{RST}^{C}$ and the associated safety relation $\succ$ are simultaneously defined as follows:

- Terms:
  - Every variable is a term.
  - Every $c \in C$ is a term (taken to be a constant).
  - If x is a variable and φ is a formula such that $\varphi \succ \{x\}$, then $\{\!\!\{x \mid \varphi\}\!\!\}$ is a term ($Fv\left(\{\!\!\{x \mid \varphi\}\!\!\}\right) = Fv\left(\varphi\right) - \{x\}$).
- Formulas:
  - If s, t are terms, then $t = s$, $t \in s$ are atomic formulas.
  - If φ, ψ are formulas and x is a variable, then $\neg\varphi, \left(\varphi \wedge \psi\right), \left(\varphi \vee \psi\right), \exists x\varphi$ are formulas.[6]

[4] A few of the claims in Section 5 have counterparts in [8]. However, the models used in that paper are based on universes which are more extensive than the minimal one which is studied here. Hence the development and proofs there were much simpler. In particular: there was no need in [8] to use proper classes, as is essential here. Another, less crucial but still important, difference is that unlike in [8], the use of $\in$-induction is completely avoided at the present paper.

[5] To be extremely precise, we should have also used different notations in the formal languages and in the meta-language for $\in$ and $=$, as well as for many other standard symbols which are used below. However, for readability we shall not do so, and trust the reader to deduce the correct use from the context.

[6] Our official language does not include $\forall$ and $\rightarrow$. However, since the theory studied in this paper is based on classical logic, we take here $\forall x_1 ... \forall x_n\left(\varphi \rightarrow \psi\right)$ as an abbreviation for $\neg\exists x_1 ... \exists x_n\left(\varphi \wedge \neg\psi\right)$.

- The safety relation $\succ$:
  - If φ is an atomic formula, then $\varphi \succ \emptyset$.
  - If t is a term such that $x \notin Fv(t)$, and $\varphi \in \{x \neq x, x \in t, x = t, t = x\}$, then $\varphi \succ \{x\}$.
  - If $\varphi \succ \emptyset$, then $\neg\varphi \succ \emptyset$.
  - If $\varphi \succ \Theta$ and $\psi \succ \Theta$, then $\varphi \vee \psi \succ \Theta$.
  - If $\varphi \succ \Theta$, $\psi \succ \Phi$ and $\Phi \cap Fv(\varphi) = \emptyset$ or $\Theta \cap Fv(\psi) = \emptyset$, then $\varphi \wedge \psi \succ \Theta \cup \Phi$.
  - If $\varphi \succ \Theta$ and $y \in \Theta$, then $\exists y \varphi \succ \Theta - \{y\}$.

Notation. We take the usual definition of $\subseteq$ in terms of $\in$, according to which $t \subseteq s \succ \emptyset$.

Definition 2.3. An RST-theory [7] is a classical first-order system with variable binding term operator ([19]), in a language of the form $\mathcal{L}^C_{RST}$, which includes the following axioms:

- Extensionality: $\forall z\, (z \in x \leftrightarrow z \in y) \rightarrow x = y$
- Comprehension Schema: $\forall x\, (x \in \{\!\!\{x \mid \varphi\}\!\!\} \leftrightarrow \varphi)$

Lemma 2.4. [7] *The following notations are available (i.e. they can be introduced as abbreviations and their basic properties are provable) in every RST-theory:*

- $\emptyset := \{\!\!\{x \mid x \neq x\}\!\!\}$.
- $\{\!\!\{t_1, ..., t_n\}\!\!\} := \{\!\!\{x \mid x = t_1 \vee ... \vee x = t_n\}\!\!\}$, *where x is fresh.*
- $\langle s, t\rangle := \{\!\!\{\{\!\!\{s\}\!\!\}, \{\!\!\{s, t\}\!\!\}\}\!\!\}$. $\langle t_1, ..., t_n\rangle := \langle\langle t_1, ..., t_{n-1}\rangle, t_n\rangle$.
- $\pi_1(t) := \{\!\!\{x \mid \exists y.t = \langle x, y\rangle\}\!\!\}$, $\pi_2(t) := \{\!\!\{y \mid \exists x.t = \langle x, y\rangle\}\!\!\}$.
- $\{\!\!\{x \in t \mid \varphi\}\!\!\} := \{\!\!\{x \mid x \in t \wedge \varphi\}\!\!\}$, *provided $\varphi \succ \emptyset$ and $x \notin Fv(t)$.*
- $\{\!\!\{t \mid x \in s\}\!\!\} := \{\!\!\{y \mid \exists x.x \in s \wedge y = t\}\!\!\}$, *where y is fresh and $x \notin Fv(s)$.*
- $s \times t := \{\!\!\{x \mid \exists a \exists b.a \in s \wedge b \in t \wedge x = \langle a, b\rangle\}\!\!\}$, *where x, a, b are fresh.*
- $s \cup t := \{\!\!\{x \mid x \in s \vee x \in t\}\!\!\}$, *where x is fresh.*
- $s \cap t := \{\!\!\{x \mid x \in s \wedge x \in t\}\!\!\}$, *where x is fresh.*
- $\cup t := \{\!\!\{x \mid \exists y \in t.x \in y\}\!\!\}$, *where x, y are fresh.*
- $\cap t := \{\!\!\{x \mid x \in \cup t \wedge \forall y \in t.x \in y\}\!\!\}$, *where x, y are fresh.*
- $\iota x.\varphi := \bigcup \{\!\!\{x \mid \varphi\}\!\!\}$, *provided $\varphi \succ \{x\}$.*[8]
- $\lambda x \in s.t := \{\!\!\{y \mid \exists x.x \in s \wedge y = \langle x, t\rangle\}\!\!\}$, *provided $x \notin Fv(s)$.*
- $Dom(t) := \{\!\!\{x \mid \exists z \exists v \exists y.z \in t \wedge v \in z \wedge y \in v \wedge x \in v \wedge z = \langle x, y\rangle\}\!\!\}$, *$(z, v, x, y$ fresh$)$.*
- $Im(t) := \{\!\!\{y \mid \exists z \exists v \exists x.z \in t \wedge v \in z \wedge y \in v \wedge x \in v \wedge z = \langle x, y\rangle\}\!\!\}$, *$(z, v, x, y$ fresh$)$.*

Lemma 2.5. [6] *There are formulas, $t \check{=} \langle r, s\rangle$ and $\langle r, s\rangle \check{\in} t$ in $\mathcal{L}^{\{HF\}}_{RST}$ such that:*

(1) $t \check{=} \langle x, s\rangle \succ \{x\}$, $t \check{=} \langle s, x\rangle \succ \{x\}$ *and* $t \check{=} \langle x, y\rangle \succ \{x, y\}$ *for* $x, y \notin Fv(t)$.
(2) $\langle x, s\rangle \check{\in} t \succ \{x\}$, $\langle s, x\rangle \check{\in} t \succ \{x\}$ *and* $\langle x, y\rangle \check{\in} t \succ \{x, y\}$ *for* $x, y \notin Fv(t)$.
(3) $r = \langle s, t\rangle \leftrightarrow r \check{=} \langle s, t\rangle$ *is provable in every RST-theory.*

Definition 2.6.

(1) RST^m is the minimal RST-theory. In other words: RST^m is the theory in $\mathcal{L}^{\emptyset}_{RST}$ whose axioms are those given in Definition 2.3. [9]

[7] 'RST' stands for Rudimentary Set Theory. See Theorem 2.15 below.

[8] Due to the Extensionality Axiom, if $\varphi \succ \{x\}$, then the term above for $\iota x.\varphi$ denotes $\emptyset$ if there is no set which satisfies φ, and it denotes the union of all the sets which satisfy φ otherwise. In particular: this term has the property that if there is exactly one set which satisfies φ, then $\iota x.\varphi$ denotes this unique set since $\cup\{a\} = a$. Note that the definition of $\iota x.\varphi$ taken here is simpler than the definition used in [7], which was $\cap\{\!\!\{x \mid \varphi\}\!\!\}$ (where some caution was taken so that the term is always well defined).

[9] RST^m can be shown to be equivalent to Gandy's basic set theory [27].

(2) RST^m_{HF} is the RST-theory in $\mathcal{L}^{\{HF\}}_{RST}$ in which the following axioms are added to those given in Definition 2.3:
- $\emptyset \in HF$
- $\forall x \forall y\,(x \in HF \wedge y \in HF \rightarrow x \cup \{\!\!\{y\}\!\!\} \in HF)$
- $\forall y\,(\emptyset \in y \wedge \forall v, w \in y.v \cup \{\!\!\{w\}\!\!\} \in y \rightarrow HF \subseteq y)$

Discussion.

- In [6] it was suggested that the computationally meaningful instances of the Comprehension Axiom are those which determine the collections they define in an absolute way, independently of any "surrounding universe". In the context of set theory, a formula φ is "computable" w.r.t. x if the collection $\{x \mid \varphi\,(x, y_1, ..., y_n)\}$ is completely and uniquely determined by the identity of the parameters $y_1, ..., y_n$, and the identity of other objects referred to in the formula (all of which are well-determined beforehand). Note that φ is computable for $\emptyset$ iff it is absolute in the usual sense of set theory. In order to translate this idea into an exact, *syntactic* definition, the safety relation is used. Thus, in an RST-theory only those formulas which are safe with respect to $\{x\}$ are allowed in the Comprehension Scheme. It is not difficult to see that the safety relation $\succ$ used in an RST-theory indeed possesses the above property.[10] Thus the formula $x \in y$ should be safe w.r.t. $\{x\}$ (but not w.r.t. $\{y\}$), since if the identity of y is computationally acceptable as a set, then any of its elements must be previously accepted as a set, and $\{x \,|\, x \in y\} = y$. Another example is given by the clause for negation. The intuitive meaning of $\{x \,|\, \neg\varphi\}$ is the complement (with respect to some universe) of $\{x \,|\, \varphi\}$, which is not in general computationally accepted. However, if φ is absolute, then so is its negation.
- RST^m and RST^m_{HF} differ from the systems RST and RST_{HF} used in [8] with respect to the use of $\in$-induction. In principle, $\in$-induction does not seem to be in any conflict with the notion of a computational theory, since it only imposes further restrictions on the collection of acceptable sets. Accordingly, it was indeed adopted and used in [6, 7, 8]. Nevertheless, in order not to impose unnecessary constraints on our general framework, and in particular to allow to develop in it set theories which adopt the anti-foundation axiom AFA, $\in$-induction is not included in RST^m and RST^m_{HF}.
- It is not difficult to prove that $\mathcal{HF}$, the set of all hereditary finite sets, is a model of RST^m. (In fact, it is the minimal one.) It follows that the set $\mathbb{N}$ of the natural numbers is not definable as a set in RST^m. To solve this problem, the special constant HF was added in RST^m_{HF}, together with appropriate axioms. (These axioms replace in RST^m_{HF} the usual infinity axiom of ZF.) The intended interpretation of the new constant HF is $\mathcal{HF}$, and the axioms for it ensure (as far as it is possible on the first-order level) that HF is indeed to be interpreted as this collection. In particular, we have:

Lemma 2.7. [8] *The followings are provable in* RST^m_{HF}*:*

(1) $x \in HF \leftrightarrow x = \emptyset \vee \exists u, v \in HF.u \cup \{\!\!\{v\}\!\!\} = x.$
(2) $(\psi\,[\emptyset/x] \wedge \forall x \forall y\,(\psi \wedge \psi\,[y/x] \rightarrow \psi\,[x \cup \{\!\!\{y\}\!\!\}/x])) \rightarrow \forall x \in HF.\psi$, *for* $\psi \succ \emptyset$.
(3) $\psi\,[HF/a] \wedge \forall a\,(\psi \rightarrow HF \subseteq a)$, *for* $\psi := \forall x\,(x \in a \leftrightarrow x = \emptyset \vee \exists u, v \in a.u \cup \{\!\!\{v\}\!\!\} = x)$.

[10] Recently it was shown [10] that up to logical equivalence, and as long as we restrict ourselves to the basic first-order language, the converse holds as well. It is not known yet whether this is true also in the presence of abstract set terms.

Remarks 2.8.

(1) An important feature of RST-theories is that their two axioms directly lead (and are equivalent) to the *set-theoretical* β and η reduction rules (see [6]).

(2) While the formal language allows the use of set terms, it also provides a mechanizable static check of their validity due to the syntactic safety relation. To obtain decidable syntax, logically equivalent formulas are not taken to be safe w.r.t. the same set of variables. However, if $\varphi \leftrightarrow \psi$ is provable in some RST-theory, then so is $x \in \{\!\!\{x \mid \varphi\}\!\!\} \leftrightarrow \psi$. Therefore for such φ, ψ we might freely write in what follows $\{\!\!\{x \mid \psi\}\!\!\}$ instead of $\{\!\!\{x \mid \varphi\}\!\!\}$.[11]

(3) It is easy to verify that the system RST^m_{HF} is a proper subsystem of ZF. While the latter is not an RST-theory, in [6] it was shown that it can be obtained from the former by adding the following clauses to the definition of its safety relation:
   - *Separation:* $\varphi \succ \emptyset$ for every formula φ.
   - *Powerset:* $x \subseteq t \succ \{x\}$ if $x \notin Fv(t)$.
   - *Replacement:* $\exists y \varphi \wedge \forall y(\varphi \to \psi) \succ X$, provided $\psi \succ X$, and $X \cap Fv(\varphi) = \emptyset$.

(4) Unlike in this paper, in general the framework for set theories just reviewed is not confined to the first-order level or to classical logic. Thus in [7] it was used together with ancestral logic ([35, 38, 44, 4, 17]). (This involves adding a special clause to the definition of $\succ$ that treats the operation of transitive closure.) Intuitionistic versions have been investigated too.

(5) A safety relation like $\succ$ presents a difficult challenge for mechanized logical frameworks of the Edinburgh LF's type ([29]). First, it is a strictly syntactic *relation* between formulas and variables, whose direct implementation requires the use of meta-variables for the variables of the object language — something which is particularly difficult to handle in this type of logical frameworks ([9]). Second, $\succ$ does not have a fixed arity like all judgements in the Edinburgh LF do: it is actually a relation between formulas and *finite sets* of object-level variables. Therefore it seems that current logical frameworks should be significantly extended and refined in order to be able to handle the syntactic framework for set theories that was proposed in [6] (and is used here).

2.2. **The Minimal Model.** We next recall the definition of rudimentary functions (for more on this topic see [20, 30]).[12] Rudimentary functions are just the functions obtained by omitting the recursion schema from the usual list of schemata for primitive recursive set functions.

Definition 2.9. Every *rudimentary function* is a composition of the following functions:

- $F_0\,(x,y) = \{x,y\}$
- $F_1\,(x,y) = x - y$
- $F_2\,(x,y) = x \times y$
- $F_3\,(x,y) = \{\langle u,z,v\rangle \mid z \in x \wedge \langle u,v\rangle \in y\}$
- $F_4\,(x,y) = \{\langle z,v,u\rangle \mid z \in x \wedge \langle u,v\rangle \in y\}$
- $F_5\,(x,y) = \{Im\,(x|_z) \mid z \in y\}$ where $Im\,(x|_z) = \{w \mid \exists u \in z.\, \langle u,w\rangle \in x\}$
- $F_6\,(x) = \bigcup_{z \in x} z$
- $F_7\,(x) = Dom\,(x) = \{v \mid \exists w.\, \langle v,w\rangle \in x\}$

[11] Further discussion on decidability issues for safety-based languages can be found in [5].

[12] To be precise, the definition we take here is given in The Basis Lemma in [20]. It was shown there that this definition is equivalent to the standard definition of rudimentary functions.

- $F_8(x) = \{\langle u, v\rangle \mid u \in x \wedge v \in x \wedge u \in v\}$

Definition 2.10.

(1) A function is called *HF-rudimentary* if it can be generated by composition of the functions $F_0, ..., F_8$ in Definition 2.9, and the following constant function:
   - $F_9(x) = \mathcal{HF}$ (the set of hereditary finite sets).

(2) An *HF-universe* (*universe* in short) is a transitive collection of sets that is closed under HF-rudimentary functions.

Terminology. In what follows, we do not distinguish between a universe W and the structure for $\mathcal{L}_{RST}^{\{HF\}}$ with domain W and an interpretation function I that assigns the obvious interpretations to the symbols $\in$, $=$, and $\mathcal{HF}$ to HF.

Notation 2.11. We denote by $v[x := a]$ the x-variant of v which assigns a to x. If $\vec{y}, \overrightarrow{a}$ are two vectors of the same length we abbreviate $v[y_1 := a_1, ..., y_n := a_n]$ by $v[\vec{y} := \overrightarrow{a}]$. We denote by $[x_1 := a_1, ..., x_n := a_n]$ any assignment which assigns to each x_i the element a_i.[13]

Definition 2.12. Let W be a universe, v an assignment in W. For any term t and formula φ of $\mathcal{L}_{RST}^{\{HF\}}$, we recursively define a collection $\|t\|_v^W$ and a truth value $\|\varphi\|_v^W \in \{\mathbf{t}, \mathbf{f}\}$ (respectively) by:

- $\|x\|_v^W = v(x)$ for x a variable.
- $\|HF\|_v^W = \mathcal{HF}$
- $\|\{x \mid \varphi\}\|_v^W = \left\{a \in W \mid \|\varphi\|_{v[x:=a]}^W = \mathbf{t}\right\}$
- $\|v\|_v\, t = s = \mathbf{t}$ iff $\|t\|_v^W = \|s\|_v^W$; $\|t \in s\|_v^W = \mathbf{t}$ iff $\|t\|_v^W \in \|s\|_v^W$
- $\|\neg\varphi\|_v^W = \mathbf{t}$ iff $\|\varphi\|_v^W = \mathbf{f}$
- $\|\varphi \wedge \psi\|_v^W = \mathbf{t}$ iff $\|\varphi\|_v^W = \mathbf{t} \wedge \|\psi\|_v^W = \mathbf{t}$
- $\|\varphi \vee \psi\|_v^W = \mathbf{t}$ iff $\|\varphi\|_v^W = \mathbf{t} \vee \|\psi\|_v^W = \mathbf{t}$
- $\|\exists x\varphi\|_v^W = \mathbf{t}$ iff $\exists a \left(a \in W \wedge \|\varphi\|_{v[x:=a]}^W = \mathbf{t}\right)$

Given W and v, we say that the term t *defines* the collection $\|t\|_v^W$.

Remark 2.13. From Theorem 2.16 below it follows that $\|t\|_v^W$ is an element of W (and it denotes the value in W that the term t gets under v), and $\|\varphi\|_v^W$ denotes the truth value of the formula φ under W and v.

Notation 2.14. In case *exp* is a closed term or a closed formula, we denote by $\|exp\|^W$ the value of *exp* in W, and at times we omit the superscript W and simply write $\|exp\|$.

The following theorem is a slight generalization of a theorem proved in [7].

Theorem 2.15.

(1) *If F is an n-ary HF-rudimentary function, then there is a formula φ_F of $\mathcal{L}_{RST}^{\{HF\}}$ s.t.:*
   - $Fv(\varphi_F) \subseteq \{y, x_1, ..., x_n\}$
   - $\varphi_F \succ \{y\}$
   - $F(x_1, ..., x_n) = \{y \mid \varphi_F\}$

(2) *If φ is a formula of $\mathcal{L}_{RST}^{\{HF\}}$ such that:*

[13]As long as we apply $[x_1 := a_1, ..., x_n := a_n]$ to expressions whose set of free variables is contained in $\{x_1, ..., x_n\}$ the exact assignment does not matter.

- $Fv(\varphi) \subseteq \{y_1, ..., y_k, x_1, ..., x_n\}$
- $\varphi \succ \{y_1, ..., y_k\}$

then there exists a HF-rudimentary function F_φ such that:

$$F_\varphi(x_1, ..., x_n) = \{\langle y_1, ..., y_k\rangle \mid \varphi\}$$

(3) *If t is a term of $\mathcal{L}_{RST}^{\{HF\}}$ such that $Fv(t) \subseteq \{x_1, ..., x_n\}$, then there exists a HF-rudimentary function F_t such that $F_t(x_1, ..., x_n) = t$ for every $x_1, ..., x_n$.*

Proof. The corresponding theorem in [7] establishes the connection between $\mathcal{L}_{RST}^{\{HF\}}$ without the constant HF and rudimentary functions. Thus, the only modification required here is the treatment of the new function in (1), and the treatment of the constant HF in (2) and (3) (which are then incorporated in the original proof that was carried out by induction). For (1), it is easy to verify that $\varphi_{F_9} := y = HF$. For (2) and (3) (which are proved by simultaneous induction on the structure of terms and formulas), the case for the constant HF is immediate from the definition of HF-rudimentary functions. $\square$

Theorem 2.16. *Let W be a universe, and v an assignment in W.*

- For t a term of $\mathcal{L}_{RST}^{\{HF\}}$, $\|t\|_v^W \in W$.
- For φ a formula of $\mathcal{L}_{RST}^{\{HF\}}$:
  - If $\varphi \succ \{y_1, ..., y_n\}$ and $n > 0$, then:

$$\left\{\langle a_1, ..., a_n\rangle \in W^n \mid \|\varphi\|_{v[\vec{y}:=\vec{a}]}^W = \mathbf{t}\right\} \in W$$

.

  - If $\varphi \succ \emptyset$ and $\{y_1, ..., y_n\} \subseteq Fv(\varphi)$, then for any $X \in J_2$:

$$\left\{\langle a_1, ..., a_n\rangle \in X^n \mid \|\varphi\|_{v[\vec{y}:=\vec{a}]}^W = \mathbf{t}\right\} \in W$$

Proof. The proof is straightforward using Theorem 2.15. Claims (1) and (2a) are immediate. For (2b) let φ be a formula s.t. $\varphi \succ \emptyset$ and $\{y_1, ..., y_n\} \subseteq Fv(\varphi)$. Using Theorem 2.15 we get that φ defines a HF-rudimentary predicate, P_φ (i.e. one whose characteristic function is HF-rudimentary). Define:

$$H(x_1, ..., x_k, y_1, ..., y_n) = \begin{cases} \{\langle y_1, ..., y_n\rangle\} & \text{if } P_\varphi(x_1, ..., x_k, y_1, ..., y_n) \\ \emptyset & \text{otherwise} \end{cases}$$

H is a HF-rudimentary function (see Lemma 1.1 in [20]). Now, define:

$$F(z, x_1, ..., x_k) = z^n \cap \{\langle y_1, ..., y_n\rangle \mid P_\varphi(x_1, ..., x_k, y_1, ..., y_n)\}$$

F is also HF-rudimentary since

$$F(z, x_1, ..., x_k) = \bigcup_{\langle y_1, ..., y_n\rangle \in z^n} H(x_1, ..., x_k, y_1, ..., y_n)$$

Now, the fact that W is a universe entails that for every assignment v in W and every $X \in W$, $F(X, v(x_1), ..., v(x_k)) \in W$, and so $\left\{\langle a_1, ..., a_n\rangle \in X^n \mid \|\varphi\|_{v[\vec{y}:=\vec{a}]}^W = \mathbf{t}\right\} \in W$. $\square$

Proposition 2.17. *Let W be a universe. Then, W is a model of RST_{HF}^m.*

Proof. The Extensionality axiom is clearly satisfied in any universe. Theorem 2.16 entails that the interpretation of any term is an element of the universe. This immediately implies that the other axioms are satisfied in any universe. It is also straightforward to verify that the interpretation of HF as $\mathcal{HF}$ satisfies the three axioms for HF. □

2.3. **Computational Theories and Universes.**

Computations within a set of objects require concrete representations of these objects. Accordingly, we call a theory *computational* if its set of closed terms induces in a natural way a minimal model of the theory, and it enables the key properties of these elements to be provable within it. Next we provide a more formal definition for the case of set theories which are defined within our general framework. Note that from a Platonist point of view, the set of closed terms of such a theory $\mathcal{T}$ induces some subset $\mathcal{S}_{\mathcal{T}}$ of the cumulative universe of sets V, as well as some subset $\mathcal{M}_{\mathcal{T}}$ of any transitive model $\mathcal{M}$ of $\mathcal{T}$.

Definition 2.18.
(1) A theory $\mathcal{T}$ in the above framework is called *computational* if the set $\mathcal{S}_{\mathcal{T}}$ it induces is a transitive model of $\mathcal{T}$, and the identity of $\mathcal{S}_{\mathcal{T}}$ is absolute in the sense that $\mathcal{M}_{\mathcal{T}} = \mathcal{S}_{\mathcal{T}}$ for any transitive model $\mathcal{M}$ of $\mathcal{T}$ (implying that $\mathcal{S}_{\mathcal{T}}$ is a *minimal* transitive model of $\mathcal{T}$).
(2) A set is called *computational* if it is $\mathcal{S}_{\mathcal{T}}$ for some computational theory $\mathcal{T}$.

The most basic computational theories are RST^m and RST^m_{HF}, which are the two minimal theories in the hierarchy of systems developed in [8]. This fact, as well as the corresponding computational universes, are described in the following three results from [8].

Proposition 2.19. *Let J_1 and J_2 be the first two elements in Jensen's hierarchy* [32].[14]
(1) *J_1 is a model of RST.*
(2) *J_2 with the interpretation of HF as J_1 is a model of RST^m_{HF}.*

Proof. The first claim is trivial. The second claim follows from Corollary 2.17, since J_2 is clearly a universe. □

Theorem 2.20.
(1) *$X \in J_1$ iff there is a closed term t of $\mathcal{L}_{RST}$ s.t. $\|t\|^{J_1} = X$.*
(2) *$X \in J_2$ iff there is a closed term t of $\mathcal{L}^{\{HF\}}_{RST}$ such that $\|t\|^{J_2} = X$.*

Proof. We prove the second item, leaving the easy proof of the first to the reader. Theorem 2.16 entails the right-to-left implication. The converse is proved by induction, using Lemma 2.4. Clearly, $\|\{\!\!\{x \mid x \in x\}\!\!\}\|^{J_2} = \emptyset$ and $\|HF\|^{J_2} = J_1$. Now, suppose that for $A, B \in J_2$ there are closed terms t_A and t_B such that $\|t_A\|^{J_2} = A$ and $\|t_B\|^{J_2} = B$. We show that there are closed terms for any of the results of applications of $F_0, ..., F_8$ to A and B.

- $F_0\left(A, B\right) = \|\{\!\!\{t_A, t_B\}\!\!\}\|^{J_2}$
- $F_1\left(A, B\right) = \|t_A - t_B\|^{J_2}$
- $F_2\left(A, B\right) = \|t_A \times t_B\|^{J_2}$
- $F_3\left(A, B\right) = \|\{\!\!\{x \mid \exists z \in t_A \exists u, v. \left\langle u, v\right\rangle \check{\in} t_B \wedge x = \left\langle u, z, v\right\rangle \}\!\!\}\|^{J_2}$
- $F_4\left(A, B\right) = \|\{\!\!\{x \mid \exists z \in t_A \exists u, v. \left\langle u, v\right\rangle \check{\in} t_B \wedge x = \left\langle z, v, u\right\rangle \}\!\!\}\|^{J_2}$
- $F_5\left(A, B\right) = \|\{\!\!\{Im\left(\{\!\!\{w \mid w \in t_A \wedge \pi_1\left(w\right) \in z\}\!\!\}\right) \mid z \in t_B\}\!\!\}\|^{J_2}$

[14] $J_1 = \mathcal{HF}$ and $J_2 = Rud\left(J_1\right)$, where $Rud\left(x\right)$ denotes the smallest set y such that $x \subseteq y$, $x \in y$, and y is closed under application of all rudimentary functions.

- $F_6(A) = \|\{\!\!\{ x \mid \exists u \in t_A . x \in u \}\!\!\}\|^{J_2}$
- $F_7(A) = \|Dom(t_A)\|^{J_2}$
- $F_8(A) = \|\{\!\!\{ x \mid \exists u \in t_A \exists v \in t_a . u \in v \wedge x \check{=} \langle u, v \rangle \}\!\!\}\|^{J_2}$ □

Corollary 2.21. *RST^m and RST^m_{HF} are computational, and J_1 and J_2 are their computational models.*

Now J_1, the minimal computational set, is the set of hereditary finite sets. Its use captures the standard data structures used in computer science, like strings and lists. However, in order to be able to capture computational structures with infinite objects, we have to move to RST^m_{HF}, whose computational universe, J_2, seems to be the minimal universe that suffices for this purpose. RST^m_{HF} still allows for a very concrete, computationally-oriented interpretation, and it is appropriate for mechanical manipulations and interactive theorem proving. As noted in the introduction, the main goal of this paper is to show that this theory and its corresponding universe J_2 are sufficiently rich for a systematic development of (great parts of) applicable mathematics.

3. Classes and Static Extensions by Definitions

When working in a minimal computational universe such as J_2 (as done in the next section), many of the standard mathematical objects (such as the real line and real functions) are only available in our framework as proper classes. Thus, in order to be able to formalize standard theorems regarding such objects we must enrich our language to include them. However, introducing classes into our framework is a part of the more general method of extensions by definitions, which is an essential part of every mathematical research and its presentation. There are two principles that govern this process in our framework. First, the static nature of our framework demands that conservatively expanding the language of a given theory should be reduced to the use of *abbreviations*. Second, since the introduction of new predicates and function symbols creates new atomic formulas and terms, one should be careful that the basic conditions concerning the underlying safety relation $\succ$ are preserved. Thus only formulas φ s.t. $\varphi \succ \emptyset$ can be used for defining new predicate symbols.

We start with the problem of introducing new predicate symbols. Since n-ary predicates can be reduced in the framework of set theory to unary predicates, we focus on the introduction of new unary predicates. In standard practice such extensions are carried out by introducing a new unary predicate symbol P and either treating $P(t)$ as an abbreviation for $\varphi[{}^{t}/{}_{x}]$ for some formula φ and variable x, or (what is more practical) adding $\forall x (P(x) \leftrightarrow \varphi)$ as an axiom to the (current version of the base) theory, obtaining by this a conservative theory in the extended language. However, in the set theoretical framework it is possible and frequently more convenient to uniformly use class terms, rather than introduce a new predicate symbol each time. Thus, instead of writing "$P(t)$" one uses an appropriate class term S and writes "$t \in S$". Whatever approach is chosen – in order to respect the definition of a safety relation, class terms should be restricted so that "$t \in S$" is safe w.r.t. $\emptyset$. Accordingly, we extend our language by incorporating class terms, which are objects of the form $\{\!\!\{ x \mid \widehat{\ } \varphi \}\!\!\}$, where $\varphi \succ \emptyset$. The use of these terms is done in the standard way. In particular, $t \in \{\!\!\{ x \mid \widehat{\ } \varphi \}\!\!\}$ (where t is free for x in φ) is equivalent to (and may be taken as an abbreviation for) $\varphi[{}^{t}/{}_{x}]$. It should be emphasized that a class term is not a valid term in the language, but only a definable predicate. Thus the addition of the new notation does not enhance the expressive power of languages like $\mathcal{L}^{\{HF\}}_{RST}$, but only increases the ease of using them.

Remark 3.1. Further standard abbreviations (see [34]) are:

- $t \subseteq \{x \mid \varphi\}$ is an abbreviation for $\forall z \left(z \in t \to z \in \{x \mid \varphi\}\right)$.
- $t = \{x \mid \varphi\}$ and $\{x \mid \varphi\} = t$ stand for $\forall z \left(z \in t \leftrightarrow z \in \{x \mid \varphi\}\right)$.
- $\{x \mid \varphi\} = \{y \mid \psi\}$ is an abbreviation for $\forall z \left(z \in \{x \mid \varphi\} \leftrightarrow z \in \{y \mid \psi\}\right)$.
- $\{x \mid \varphi\} \in t$ is an abbreviation for $\exists z.z = \{x \mid \varphi\} \wedge z \in t$.
- $\{x \mid \varphi\} \in \{y \mid \psi\}$ is an abbreviation for $\exists z.z = \{x \mid \varphi\} \wedge z \in \{y \mid \psi\}$.

Note that these formulas are merely abbreviations for formulas which are *not* necessarily atomic (even though, $t \subseteq \{x \mid \varphi\}$ also happens to be safe with respect to $\emptyset$).

A further conservative extension of the language that we shall use incorporates free class variables, $\boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{Z}$, and free function variables, $\boldsymbol{F}, \boldsymbol{G}$, into $\mathcal{L}_{RST}^{\{HF\}}$ (as in free-variable second-order logic [44]). These variables stand for arbitrary class or function terms (the latter is defined in Def. 3.10), and they may only appear as *free* variables, *never to be quantified.* We allow occurrences of such variables inside a formula in a class term or a function term. One may think of a formula with such variables as a schema, where the variables play the role of "place holders", and whose substitution instances abbreviate official formulas of the language. (See Example 5.13.) In effect, a formula $\psi(\boldsymbol{X})$ with free class variable $\boldsymbol{X}$ can be intuitively interpreted as "for any *given* class X, $\psi(X)$ holds". Thus, a free-variable formulation has the flavor of a universal formula. Therefore, this addition allows statements about *all* potential classes and *all* potential functions.

Definition 3.2. Let W be a universe, v an assignment in W, and let $\varphi \succ \emptyset$. Define:

$$\left\|\{x \mid \varphi\}\right\|_v^W := \left\{a \in W \mid \|\varphi\|_{v[x:=a]}^W = \mathbf{t}\right\}$$

Given W and v, we again say that the class term on the left *defines* here the collection on the right (even though it might not be an element of W).

Definition 3.3. Let X be a collection of elements in a universe W.

- X is a $\succ$-*set* (in W) if there is a closed term that defines it. (See Definition 2.12.) If X is a $\succ$-set, $\widetilde{X}$ denotes some closed term that defines it.
- X is a $\succ$-*class* (in W) if there is a closed class term that defines it. If X is a $\succ$-class, $\bar{X}$ denotes some closed class term that defines it.

Note that by Corollary 2.16, if X is a $\succ$-*set* in W then $X \in W$.

Proposition 3.4. *The following holds for every universe W:*

(1) *Every $\succ$-set is a $\succ$-class.*
(2) *The intersection of a $\succ$-class with a $\succ$-set is a $\succ$-set.*
(3) *Every $\succ$-class that is contained in a $\succ$-set is a $\succ$-set.*

Proof.

(1) If X is a $\succ$-set, then $x \in \widetilde{X} \succ \{x\}$. Hence (see [6]) $x \in \widetilde{X} \succ \emptyset$. This implies that $\{x \mid x \in \widetilde{X}\}$ is a class term which defines X, and so X is a $\succ$-class.
(2) Let X be a $\succ$-class and Y be a $\succ$-set. Then, $X \cap Y$ can be defined by the term $\{z \mid z \in \bar{X} \wedge z \in \widetilde{Y}\}$. Since $z \in \bar{X} \succ \emptyset$ and $z \in \widetilde{Y} \succ \{z\}$ we get that $z \in \bar{X} \wedge z \in \widetilde{Y} \succ \{z\}$. Hence $X \cap Y$ is a $\succ$-set.

(3) Follows from (2), since if $X \subseteq Y$ then $X = X \cap Y$. $\square$

Remark 3.5. A semantic counterpart of our notion of a $\succ$-class was used in [46], and is called there an ι-class. It is defined as a definable subset of J_2 whose intersection with any element of J_2 is in J_2. The second condition in this definition seems somewhat ad hoc. More importantly, it is unclear how it can be checked in general, and what kind of set theory is needed to establish that certain collections are ι-classes. In contrast, the definition of a $\succ$-class used here is motivated by, and based on, purely syntactical considerations. It is also a simplification of the notion of ι-class, as by Prop. 3.4(2) every $\succ$-class is an ι-class.[15]

Proposition 3.6. *The following holds for every universe W:*

- *Let Y be a $\succ$-set. If $\varphi \succ \emptyset$ and $Fv(\varphi) \subseteq \{x\}$, then $\{x \in Y \mid \varphi\}$ is a $\succ$-set.*
- *If $\varphi \succ \{x_1, ..., x_n\}$, then $\{\langle x_1, ..., x_n\rangle \mid \varphi\}$ is a $\succ$-set.*

Proof.

- $\{x \in Y \mid \varphi\}$ is defined by $\lbrace\!\!\lbrace x \mid x \in \widetilde{Y} \wedge \varphi \rbrace\!\!\rbrace$.
- $\{\langle x_1, ..., x_n\rangle \mid \varphi\}$ is defined by $\lbrace\!\!\lbrace z \mid \exists x_1 ... \exists x_n \left(\varphi \wedge z = \langle x_1, ..., x_n\rangle\right) \rbrace\!\!\rbrace$, where z is fresh. $\square$

Proposition 3.7. *For every n-ary HF-rudimentary function f there is a term t with $Fv(t) \subseteq \{x_1, ..., x_n\}$ s.t. for any $\langle A_1, ..., A_n\rangle \in W^n$, f returns the $\succ$-set $\|t\|^W_{[x_1:=A_1,...,x_n:=A_n]}$.*

Proof. It is easy to see that if $X_1, ..., X_n$ are $\succ$-sets, and φ is a formula such that $Fv(\varphi) \subseteq \{y, v_1, ...v_n\}$ and $\varphi \succ \{y\}$, then $\left\{y \mid \varphi\left\{\frac{\widetilde{X_1}}{v_1}, ..., \frac{\widetilde{X_n}}{v_n}\right\}\right\}$ is a $\succ$-set. Therefore the proposition easily follows from Theorem 2.15. $\square$

Proposition 3.8. *If X, Y are $\succ$-classes (in a universe W), so are $X \cup Y$, $X \cap Y$, $X \times Y$, $J_2 - X$, and $P_{J_2}(X) = \{z \in J_2 \mid z \subseteq X\}$.*

Proof:

- $\overline{X \cup Y} = \lbrace\!\!\lbrace x \,\hat{|}\, x \in \bar{X} \vee x \in \bar{Y} \rbrace\!\!\rbrace$.
- $\overline{X \cap Y} = \lbrace\!\!\lbrace x \,\hat{|}\, x \in \bar{X} \wedge x \in \bar{Y} \rbrace\!\!\rbrace$.
- $\overline{X \times Y} = \lbrace\!\!\lbrace x \,\hat{|}\, \exists a \exists b \left(a \in \bar{X} \wedge b \in \bar{Y} \wedge x \,\check{=}\, \langle a, b\rangle\right) \rbrace\!\!\rbrace$. (See Lemma 2.5.)
- $\overline{J_2 - X} = \lbrace\!\!\lbrace x \,\hat{|}\, x \notin \bar{X} \rbrace\!\!\rbrace$.
- $P_{J_2}(X) = \lbrace\!\!\lbrace z \,\hat{|}\, z \subseteq \bar{X} \rbrace\!\!\rbrace = \lbrace\!\!\lbrace z \,\hat{|}\, \forall a \left(a \in z \to a \in \bar{X}\right) \rbrace\!\!\rbrace$. (See footnote 6.) $\square$

For a class term s we denote by 2^s the class term $\lbrace\!\!\lbrace z \,\hat{|}\, z \subseteq s \rbrace\!\!\rbrace$. Note that for any assignment v in W and class term s, $\|2^s\|^W_v$ is equal to $P_W\left(\|s\|^W_v\right)$, i.e., the intersection of the power set of $\|s\|^W_v$ and W. This demonstrates the main difference between set terms and class terms. The interpretation of set terms is absolute, whereas the interpretation of class terms might not be (though membership in the interpretation of a class term *is* absolute).

Definition 3.9. Let W be a universe. A *$\succ$-relation (in W) from* a *$\succ$-class X to* a *$\succ$-class Y* is a $\succ$-class A s.t. $A \subseteq X \times Y$. A $\succ$-relation is called *small* if it is a $\succ$-set (of W).

[15] Two other ideas that appear in the sequel were adopted from [46]: treating the collection of reals as a proper class, and the use of codes for handling certain classes. It should nevertheless be emphasized that the framework in [46] is exclusively based on semantical considerations, and it is unclear how it can be turned into a formal (and suitable for mechanization) theory like ZF or PA.

Next we extend our framework by the introduction of new function symbols. This poses a new difficulty. While new relation symbols are commonly introduced in a static way, new function symbols are usually introduced *dynamically*: a new function symbol is made available after appropriate existence and uniqueness theorems had been proved. However, one of the main guiding principles of our framework is that its languages should be treated exclusively in a *static* way. Thus function symbols, too, are introduced here only as abbreviations for definable operations on sets.[16]

Definition 3.10. Let W be a universe. (The various definitions should be taken with respect to W.)

- For a closed class term T and a term t of $\mathcal{L}_{RST}^{\{HF\}}$, $\lambda x \in \mathsf{T}.t$ is a *function term* which is an abbreviation for $\{\!\{ z \,\hat{\lceil}\, \exists x \exists y\, (z \check{\doteq} \langle x, y\rangle \wedge x \in \mathsf{T} \wedge y = t) \}\!\}$.[17]
- A $\succ$-class F is a *$\succ$-function on a $\succ$-class* X if there is a function term $\lambda x \in \mathsf{T}.t$ such that $X = \|\mathsf{T}\|$, $Fv(t) \subseteq \{x\}$ and $F = \|\lambda x \in \mathsf{T}.t\|$. t is called a term which *represents* F.
- A $\succ$-class is called a $\succ$-function if it is a $\succ$-function on some $\succ$-class.
- A $\succ$-function is called *small* if it is a $\succ$-set.

Note that the standard functionality condition is always satisfied by a *$\succ$-function.*

Terminology. In what follows, claiming that an object is *available in RST^m_{HF} as a $\succ$-function ($\succ$-relation)* means that for every universe W, the object is definable in $\mathcal{L}_{RST}^{\{HF\}}$ as a $\succ$-function ($\succ$-relation) of W, and that its basic properties are provable in RST^m_{HF}.[18]

Proposition 3.11. *Let X, Y be $\succ$-classes and R a $\succ$-relation from X to Y.*

(1) *R is small iff $Dom\,(R)$ and $Im\,(R)$ are $\succ$-sets.*
(2) *$R^{-1} = \{\langle y, x\rangle \mid \langle x, y\rangle \in R\}$ is available in RST^m_{HF} as a $\succ$-relation from Y to X. If R is small, then so is R^{-1}.*
(3) *If $Z \subseteq X$ and $U \subseteq Y$ are $\succ$-classes, then $R \cap (Z \times U)$ is available in RST^m_{HF} as a $\succ$-relation from Z to U .*

Proof.

(1) ($\Rightarrow$) If R is a $\succ$-set, then $\exists y.\, \langle x, y\rangle \check{\in} \widetilde{R} \succ \{x\}$ and $\exists x.\, \langle x, y\rangle \check{\in} \widetilde{R} \succ \{y\}$. Thus, $Dom\,(R)$ is defined by $\{\!\{ x \mid \exists y.\, \langle x, y\rangle \check{\in} \widetilde{R} \}\!\}$ and $Im\,(R)$ by $\{\!\{ y \mid \exists x.\, \langle x, y\rangle \check{\in} \widetilde{R} \}\!\}$.
($\Leftarrow$) If $Dom\,(R)$ and $Im\,(R)$ are $\succ$-sets, then $Dom\,(R) \times Im\,(R)$ is a $\succ$-set as $\times$ is a rudimentary function. Since R is a $\succ$-class such that $R \subseteq Dom\,(R) \times Im\,(R)$, Prop. 3.4 entails that R is a $\succ$-set.
(2) Since R is a $\succ$-class, we can define $\overline{R^{-1}} = \{\!\{ z \,\hat{\lceil}\, \exists x \exists y \left(\langle x, y\rangle \in \bar{R} \wedge z \check{\doteq} \langle y, x\rangle\right) \}\!\}$. Now, $\exists x \exists y \left(\langle x, y\rangle \in \bar{R} \wedge z \check{\doteq} \langle y, x\rangle\right) \succ \emptyset$, as $z \check{\doteq} \langle y, x\rangle \succ \{x, y\}$ and $\langle x, y\rangle \in \bar{R} \succ \emptyset$. It is standard to prove in RST^m_{HF} properties such as $\langle x, y\rangle \in R \leftrightarrow \langle y, x\rangle \in R^{-1}$ and $\left(R^{-1}\right)^{-1} = R$. If R is a $\succ$-set, R^{-1} can be defined by $\{\!\{ z \mid \exists x \exists y \left(\langle x, y\rangle \in \widetilde{R} \wedge z = \langle y, x\rangle\right) \}\!\}$, hence R^{-1} is a $\succ$-set.

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[16]In this paper, as in standard mathematical textbooks, the term "function" is used both for collections of ordered pairs and for set-theoretical operations (such as $\cup$).

[17]We abbreviate by $z \check{\doteq} \langle x, y\rangle$ and $\langle x, y\rangle \check{\in} z$ the two formulas that are provably equivalent to $z = \langle x, y\rangle$ and $\langle x, y\rangle \in z$ and are safe w.r.t. $\{x, y\}$ which were introduced in [8].

[18]The "basic properties" of a certain object is of course a fuzzy notion. However, it is not difficult to identify its meaning in each particular case, as will be demonstrated in several examples below.

(3) Surely $R \cap (Z \times U) \subseteq Z \times U$. By Prop.3.8, since R, Z, U are $\succ$-classes, we have that $R \cap (Z \times U)$ is a $\succ$-class. □

Proposition 3.12. *A $\succ$-set is a function according to the standard mathematical definition (a single-valued relation) iff it is a small $\succ$-function.*

Proof. Let A be a $\succ$-set which is a relation that satisfies the functionality condition. Since A is a $\succ$-set, there is a closed term $\widetilde{A}$ that defines it. A is a $\succ$-function on the $\succ$-set $Dom\,(A)$ since the term $t = \iota y.\, \langle x, y\rangle \check{\in} \widetilde{A}$ represents it (see Lemma 2.4). The term t is legal and it represents A since $\langle x, y\rangle \check{\in} \widetilde{A} \succ \{y\}$ and A satisfies the functionality condition. The converse is trivial, since for every small $\succ$-function there is a term representing it, and thus the functionality condition clearly holds by the equality axioms of FOL. □

Notation. Let $F = \left\|\lambda x \in \bar{X}.t\right\|$ be a $\succ$-function. We employ standard β-reduction for λ terms. Thus, we write $F\,(s)$ for $t\,[s/x]$ if s is free for x in t. Hence $F\,(s) = y$ stands for $t\,[s/x] = y$, and so if $y \notin Fv\,[t] \cup Fv\,[s] \setminus \{x\}$, then $F\,(s) = y \succ \{y\}$.

Proposition 3.13 (Replacement axiom in class form)**.** *Let F be a $\succ$-function on a $\succ$-class X. Then for every $\succ$-set $A \subseteq X$, $F\,[A] = \{F\,(a) \mid a \in A\}$ is a $\succ$-set.*

Proof. The term $\lbrace\!\!\lbrace y \mid \exists a \in \widetilde{A}.F\,(a) = y \rbrace\!\!\rbrace$ defines $F\,[A]$. □

Below is a natural generalization of Def. 3.10 to functions of several variables.

Lemma 3.14. *If $X_1, ..., X_n$* are *$\succ$-classes and t is a term s.t. $Fv\,(t) \subseteq \{x_1, ..., x_n\}$, then $F = \left\|\lambda x_1 \in \bar{X}_1, ..., x_n \in \bar{X}_n.t\right\|$ is available in RST^m_{HF} as a $\succ$-function on $X_1 \times ... \times X_n$. (Here $\lambda x_1 \in \bar{X}_1, ..., x_n \in \bar{X}_n.t$ abbreviates $\lbrace\!\!\lbrace \langle\langle x_1, ..., x_n\rangle, t\rangle \mathbin{\hat{\restriction}} \langle x_1, ..., x_n\rangle \in \bar{X}_1 \times ... \times \bar{X}_n \rbrace\!\!\rbrace$).*

Corollary 3.15. *Every HF-rudimentary function is available in RST^m_{HF} as a $\succ$-function.*

Proposition 3.16. *Let F be a $\succ$-function on a $\succ$-class X.*

(1) *F is small iff X is a $\succ$-set.*
(2) *If Y_0 is a $\succ$-class, then $F^{-1}\,[Y_0] = \{a \in X \mid F\,(a) \in Y_0\}$ is a $\succ$-class. If F is small, then $F^{-1}\,[Y_0]$ is a $\succ$-set.*
(3) *If $X_0 \subseteq X$ is a $\succ$-class, then $F\restriction_{X_0}$ is available in RST^m_{HF} as a $\succ$-function.*
(4) *$G \circ F$ is available in RST^m_{HF} as a $\succ$-function on X, in case G is a $\succ$-function on a $\succ$-class Y and $Im(F) \subseteq Y$.*
(5) *If G is a $\succ$-function on a $\succ$-class Y and F and G agree on $X \cap Y$, then $G \cup F$ is available in RST^m_{HF} as a $\succ$-function on $X \cup Y$.*
(6) *If Z is a $\succ$-class then the identity on Z and any constant function on Z are available in RST^m_{HF} as $\succ$-functions.*

Proof.

(1) ($\Rightarrow$) If F is a $\succ$-set, then $Dom\,(F) = X$ is also a $\succ$-set, since Dom is rudimentary.
($\Leftarrow$) Suppose t represents F. If X is a $\succ$-set, then $F = \left\|\lambda x \in \widetilde{X}.t\right\|$ which is a $\succ$-set.
(2) $\overline{F^{-1}\,[Y_0]} = \lbrace\!\!\lbrace a \mathbin{\hat{\restriction}} a \in \bar{X} \wedge F\,(a) \in \bar{Y_0} \rbrace\!\!\rbrace$. Since $a \in \bar{X} \wedge F\,(a) \in \bar{Y_0} \succ \emptyset$, we get that $F^{-1}\,[Y_0]$ is a $\succ$-class. If F is small, then by (1) we have that X is a $\succ$-set. The fact that $F^{-1}\,[Y_0] \subseteq X$ entails that $F^{-1}\,[Y_0]$ is a $\succ$-set.
(3) If t is a term that represents F, it also represents $F\restriction_{X_0}$.

(4) Denote by t_F, t_G terms that represent the $\succ$-functions F, G respectively. Thus, $G \circ F = \left\| \lambda x \in \overline{X}.t_G \left\{ \frac{t_F(x)}{x} \right\} \right\|$. It is easy to see that standard properties, such as the associativity of $\circ$, are provable in RST^m_{HF}.

(5) Denote by t_F, t_G terms that represent the $\succ$-functions F, G respectively. Then:
$$F \cup G = \left\| \lambda x \in \bar{X} \cup \bar{Y}.\iota y. \left(x \in \bar{X} \wedge y = t_F\right) \vee \left(x \in \bar{Y} - \bar{X} \wedge y = t_G\right) \right\|$$
Since each of the disjuncts is safe w.r.t $\{y\}$, we get that the term is valid. It is easy to verify that in RST^m_{HF} basic properties, such as $\forall x \in \bar{X}.\overline{G \cup F}\,(x) = \bar{F}\,(x)$, are provable.

(6) $id_Z = \left\| \lambda z \in \bar{Z}.z \right\|$, and for any $A \in J_2$, $const_A = \left\| \lambda z \in \bar{Z}.\tilde{A} \right\|$. Proving basic properties such as $\forall x, y \in \bar{Z}.const_A\,(x) = const_A\,(y)$ in RST^m_{HF} is routine. □

4. The Natural Numbers

We introduce the natural numbers by following their standard construction in set theory: $0 := \emptyset; \quad n + 1 := S\,(n)$, where $S\,(x) = x \cup \{x\}$. Obviously, each $n \in \mathbb{N}$ is a $\succ$-set in any universe, and $\mathbb{N}$ (the set of natural numbers) is contained in $\mathcal{HF} = J_1$. What is more, the property of being a natural number is defined in any universe by the following formula:
$$N(x) := \ \forall y \in x \cup \{x\}.y = \emptyset \vee \exists w \in x.y = w \cup \{w\}$$
Note that this formula has the same extension in any transitive set which includes $\emptyset$ and is closed under the operation $\lambda x.x \cup \{x\}$.[19] It follows that from a semantic point of view it should be taken as safe with respect to x. However, syntactically we have only that $N(x) \succ \emptyset$ (that is: $N(x)$ is absolute), but not that $N(x) \succ \{x\}$. As a result, $\mathbb{N}$ is available in RST^m (and its minimal model J_1) only as a proper $\succ$-class. In contrast, $\mathbb{N}$ is available as a $\succ$-set in RST^m_{HF}, since it is definable in all the universes (including of course J_2) by the term:
$$\widetilde{\mathbb{N}} := \{\!\!\{ x \mid x \in HF \wedge N(x) \}\!\!\}$$
Now we show that appropriate counterparts of Peano's axioms for $\mathbb{N}$ are provable in RST^m_{HF}. For this we need the following two lemmas.

Lemma 4.1. $\vdash_{RST^m_{HF}} \forall x \in HF \forall y.y \in x \rightarrow x \notin y$.

Proof. Let $\psi := \forall u \forall v.u \in z \wedge v \in u \rightarrow u \notin v$. We first show that $\vdash_{RST^m_{HF}} \forall z \in HF\psi$. By Lemma 2.7(2), it suffices to show that $\psi\,[\emptyset/z]$ and $\forall x \forall y\,(\psi\,[x/z] \wedge \psi\,[y/z] \rightarrow \psi\,[x \cup \{\!\!\{y\}\!\!\}/z])$ are theorems of RST^m_{HF}. This is obvious for $\psi\,[\emptyset/z]$. To prove the other formula, we reason in RST^m_{HF} as follows. Assume $\psi\,[x/z]$ and $\psi\,[y/z]$. We show ψ for $z = x \cup \{\!\!\{y\}\!\!\}$. So suppose that $u \in z$ and $v \in u$. Then either $u \in x$ or $u = y$. In the first case the assumptions $\psi\,[x/z]$ and $v \in u$ implies that $u \notin v$. In the second case $v \in y$, and so it follows from the assumption $\psi\,[y/z]$ that if $u \in v$ then $v \notin u$, contradicting the assumption that $v \in u$. Hence $u \notin v$ in this case as well.

To complete the proof of the lemma, let $x \in HF$. Then $\{x\} \in HF$ as well, and so $\psi\,[\{x\}/z]$. Since $x \in \{x\}$, this implies that $\forall y.y \in x \rightarrow x \notin y$. □

Lemma 4.2. $\vdash_{RST^m_{HF}} \forall x \in HF \forall y.y \subseteq x \rightarrow y \in HF$.

[19] This is a significant improvement on [8], in which another formula, $Ord\,(x)$, has been used for characterizing $\mathbb{N}$ as $\{x \mid x \in HF \wedge Ord(x)\}$. However, $Ord\,(x)$ is actually true for all ordinals, and so it lacks the strong absoluteness property that $N(x)$ has.

Proof. Let $\psi := \forall u.u \subseteq z \to u \in HF$. We have to show that $\vdash_{RST^m_{HF}} \forall z \in HF\psi$. By Lemma 2.7(2), it suffices to show that $\psi\,[{}^{\emptyset}\!/z]$ and $\forall x\forall y\,(\psi\,[{}^{x}\!/z] \wedge \psi\,[{}^{y}\!/z] \to \psi\,[{}^{x\cup\{\!\!\{y\}\!\!\}}\!/z])$ are theorems of RST^m_{HF}. This is obvious for $\psi\,[{}^{\emptyset}\!/z]$. To prove the other formula, we reason in RST^m_{HF} as follows. Assume $\psi\,[{}^{x}\!/z]$ and $\psi\,[{}^{y}\!/z]$. We show ψ for $z = x \cup \{\!\!\{y\}\!\!\}$. So suppose that $u \subseteq z$. Then there exists $v \subseteq x$ such that either $u = v$ or $u = v \cup \{\!\!\{y\}\!\!\}$. Now the assumption that $\psi\,[{}^{x}\!/z]$ implies that $v \in HF$, while the assumption that $\psi\,[{}^{y}\!/z]$ implies that $y \in HF$. Hence Lemma 2.7(1) entails that $v \cup \{\!\!\{y\}\!\!\} \in HF$. It follows that in both cases $u \in HF$. $\square$

Proposition 4.3.

(1) $\vdash_{RST^m_{HF}} 0 \in \widetilde{\mathbb{N}}$
(2) $\vdash_{RST^m_{HF}} \forall x.S(x) \neq 0$
(3) $\vdash_{RST^m_{HF}} \forall x.x \in \widetilde{\mathbb{N}} \leftrightarrow S(x) \in \widetilde{\mathbb{N}}$
(4) $\vdash_{RST^m_{HF}} \forall x \in \widetilde{\mathbb{N}}\forall y \in \widetilde{\mathbb{N}}.S(x) = S(y) \to x = y$

Proof. The first two items, and the fact that RST^m_{HF} proves that if $x \in \widetilde{\mathbb{N}}$ then $S(x) \in \widetilde{\mathbb{N}}$, are very easy, and are left to the reader.

To prove the other direction of (3), assume that $S(x) \in \widetilde{\mathbb{N}}$. We show that $x \in \widetilde{\mathbb{N}}$. Since $S(x) \in \widetilde{\mathbb{N}}$, by definition $S(x) \in HF$. Hence Lemma 4.2 implies that $x \in HF$ as well. To show that also $N(x)$, let $y \in x \cup \{x\}$, and suppose that $y \neq \emptyset$. These assumptions about y, and the assumption that $S(x) \in \widetilde{\mathbb{N}}$, together imply that there is $w \in S(x)$ such that $y = w \cup \{w\}$. It remains to show that actually $w \in x$. This is obvious in case $y = x$, since $w \in y$ (because $y = w \cup \{w\}$). If $y \in x$ then $x \notin y$ by Lemma 4.3 (because $x \in HF$), while the assumption that $y = w \cup \{w\}$ implies that $w \in y$. It follows that $w \neq x$. Since $w \in S(x)$, this implies that $w \in x$ in this case too.

Finally, to prove item (4), we show that $\vdash_{RST^m_{HF}} \forall x \in HF\forall y.S(x) = S(y) \to x = y$. So suppose that $x \in HF$ and $S(x) = S(y)$. Assume for contradiction that $x \neq y$. Since $S(x) = S(y)$, this implies that both $x \in y$ and $y \in x$, which is impossible by Lemma 4.3. $\square$

The induction rule is available in RST^m_{HF} as well, but only for $\varphi \succ \emptyset$.

Proposition 4.4. $\vdash_{RST^m_{HF}} (\varphi\,[{}^{0}\!/x] \wedge \forall x\,(\varphi \to \varphi\,[{}^{S(x)}\!/x])) \to \forall x \in \widetilde{\mathbb{N}}.\varphi$, *for* $\varphi \succ \emptyset$.

Proof. Let $\varphi \succ \emptyset$, and assume that

$$(*)\quad \varphi\,[{}^{0}\!/x] \wedge \forall x\,(\varphi \to \varphi\,[{}^{S(x)}\!/x])$$

We show that $\forall x \in HF.\varphi$. Let ψ be the formula $(x \in \widetilde{\mathbb{N}} \to \varphi) \wedge \left(\forall z \in x.z \in \widetilde{\mathbb{N}} \to \varphi\,[{}^{z}\!/x]\right)$. Since $\varphi \succ \emptyset$, also $\psi \succ \emptyset$. Hence Lemma 2.7(2) implies:

$$(**)\quad (\psi\,[{}^{\emptyset}\!/x] \wedge \forall x\forall y\,(\psi \wedge \psi\,[{}^{y}\!/x] \to \psi\,[{}^{x\cup\{\!\!\{y\}\!\!\}}\!/x])) \to \forall x \in HF.\psi$$

Clearly $\psi\,[\emptyset]$ is provable in RST^m_{HF},[20] since we have $\varphi\,[\emptyset]$ by (*). Now assume $\psi\,[x] \wedge \psi\,[y]$. We show that $\psi\,[x \cup \{\!\!\{y\}\!\!\}]$.

(1) Let $z \in x \cup \{\!\!\{y\}\!\!\}$, and suppose that $z \in \widetilde{\mathbb{N}}$. Then either $z \in x$ or $z = y$, and in both cases the assumptions $\psi\,[x]$, $\psi\,[y]$, and $z \in \widetilde{\mathbb{N}}$ immediately imply that $\varphi\,[z]$.

[20] To make the text more readable, at the rest of the proof we write $\psi\,[t]$ and $\varphi\,[t]$ instead of $\psi\,[{}^{t}\!/x]$ and $\varphi\,[{}^{t}\!/x]$ (respectively).

(2) Let $z = x \cup \{\!\{y\}\!\}$, and suppose that $z \in \widetilde{\mathbb{N}}$. The first of these assumptions implies that $z \neq \emptyset$, and so the second one entails that there exists w such that $z = w \cup \{w\}$. Then $w \in z$, and so $w \in \widetilde{\mathbb{N}} \to \varphi[w]$ by item (1). But $w \in \widetilde{\mathbb{N}}$ by Proposition 4.3(2) and the assumption $z \in \widetilde{\mathbb{N}}$. It follows that $\varphi[w]$. Therefore (*) implies that $\varphi[z]$ in this case too.

From (1) and (2) it follows that indeed $\psi[x \cup \{\!\{y\}\!\}]$ follows from $\psi[x] \wedge \psi[y]$. Therefore (**) implies that $\forall x \in HF.\psi$. This, in turn, implies that $\forall x \in HF.\varphi$. □

Remark 4.5. The restriction to absolute formulas in Proposition 4.4 is not a real problem for developing the theory of natural numbers that we need. With the help of Proposition 4.6 below, Proposition 4.4 easily implies that all the formulas in the language of first-order Peano's arithmetics and proofs in that theory can be translated into RST^m_{HF} and its language. This is because in this translation, all the quantifications are bounded in $\mathbb{N}$, and thus they are safe w.r.t. $\emptyset$. [21]

Next we show that addition and multiplication on $\mathbb{N}$ are available in RST^m_{HF} as small $\succ$-functions. In view of Propositions 4.3 and 4.4, this suffices (as has been shown by Gödel) for having all recursive functions available in RST^m_{HF} as small $\succ$-functions.[22] [23]

Proposition 4.6.
(1) *The standard ordering $<$ on $\mathbb{N}$ is available in RST^m_{HF} as a small $\succ$-relation.*
(2) *The standard addition and multiplication of natural numbers are available in RST^m_{HF} as small $\succ$-functions.*

Proof.
(1) The standard ordering $<$ on $\mathbb{N}$ coincides with $\in$. Thus it is definable by the term $\{\!\{\langle m,n\rangle \check{\in} \widetilde{\mathbb{N}\times\mathbb{N}} \mid m \in n\}\!\}$. Since $\mathbb{N}$ is a $\succ$-set, so is $\mathbb{N}\times\mathbb{N}$. Hence the fact that $m \in n \succ \emptyset$ and Prop. 3.6 imply that $<$ is a $\succ$-set. It is now straightforward to prove in RST^m_{HF} its two characteristic properties: $\forall x.x \not< 0$ and $\forall x\forall y.x < S(y) \leftrightarrow x < y \vee x = y$. Using Proposition 4.4, this suffices (as is well-known) for deriving all the basic properties of $<$, like its being a linear order or the existence of a $<$-successor for each element in $\mathbb{N}$.
(2) Define:
- $Func(f) := \forall a,b,c\,(\langle a,b\rangle \check{\in} f \wedge \langle a,c\rangle \check{\in} f \to b = c)$
- $add(z,u,n,f) := (z = 0 \wedge u = n) \vee \exists z_1,u_1 \in \widetilde{\mathbb{N}}\,(\langle z_1,u_1\rangle \check{\in} f \wedge z = S(z_1) \wedge u = S(u_1))$.
- $\psi_+(n,k,f) := Func(f) \wedge \forall x\left(x \in f \leftrightarrow \exists z,u \in \widetilde{\mathbb{N}}\,(z \leq k \wedge x = \langle z,u\rangle \wedge add(z,u,n,f))\right)$

Intuitively, $\psi_+(n,k.f)$ says that f stands for $\{\langle 0,n\rangle, \langle 1,n+1\rangle, \langle 2,n+2\rangle, ..., \langle k,n+k\rangle\}$. It is easy to check that $\psi_+(n,f) \succ \emptyset$. Hence (Lemma 2.4) addition is definable by:

$$+ := \lambda n \in \widetilde{\mathbb{N}}, k \in \widetilde{\mathbb{N}}.\iota m.\exists f \in HF\,(\psi_+(n,k,f) \wedge \langle k,m\rangle \check{\in} f)\,.$$

$+$ is a valid term since $f \in HF \succ \{f\}$ and $\langle k,m\rangle \check{\in} f \wedge \psi_+(n,f) \succ \{m\}$, and it is a small $\succ$-function, as $\mathbb{N}\times\mathbb{N}$ is a $\succ$-set.

[21] It can be shown that the power of full induction over $\mathbb{N}$ (i.e. for *any* formula φ) can be achieved by adding to RST^m_{HF} the full $\in$-induction scheme.

[22] Using the method given in the proof of Proposition 4.6 below, it is also not difficult to directly show that every primitive recursive function is available in RST^m_{HF} as a small $\succ$-function.

[23] In [8] it was essentially shown (using a different terminology) that every primitive recursive function is available as a small $\succ$-function in *the extension of RST^m_{HF} with $\in$-induction*. Our present results show that $\in$-induction is not really needed for this.

Multiplication is defined similarly. The only difference is that $add\,(z,u,n,f)$ is replaced with $(z=0\wedge u=0)\vee\exists z_1,u_1\in\widetilde{\mathbb{N}}\,(\langle z_1,u_1\rangle\,\check{\in} f\wedge z=S\,(z_1)\wedge u=n+u_1)$. This is legitimate, since $+$ is a small $\succ$-function.

It is not difficult now to prove in RST^m_{HF} the fundamental properties that characterize addition and multiplication in first-order Peano's arithmetics:

$$\forall x.x+0=x \quad\text{and}\quad \forall x,y.x+S\,(y)=S\,(x+y)$$
$$\forall x.x\cdot 0=0 \quad\text{and}\quad \forall x,y.x\cdot S\,(y)=x+(x\cdot y)$$

Once these properties are proved, it is a standard matter to use Prop. 4.4 for proving all the standard properties of addition and multiplication, such as commutativity, associativity, and distributivity. □

5. Real Analysis in J_2

In this section we turn at last to the main goal of this paper: developing real analysis in J_2. Now it is not difficult to formalize the definitions, claims, and proofs of this section in our formal framework. These translations are straightforward, but rather tedious. Hence we shall omit them, with the exception of a few outlined examples.

Notation and Terminology. Henceforth we restrict our attention to the computational theory RST^m_{HF} and its computational universe J_2. Therefore we simply write $\|exp\|_v$ instead of $\|exp\|_v^{J_2}$. Similarly, when we talk about a $\succ$-set or a $\succ$-class, we mean $\succ$-set/$\succ$-class in J_2.

5.1. **The Construction of the Real Line.**

The standard construction of $\mathbb{Z}$, the set of integers, as the set of ordered pairs $(\mathbb{N}\times\{0\})\cup(\{0\}\times\mathbb{N})$ can be easily carried out in RST^m_{HF}, as can the usual construction of $\mathbb{Q}$, the set of rationals, in terms of ordered pairs of relatively prime integers. There is also no difficulty in defining the standard orderings on $\mathbb{Z}$ and $\mathbb{Q}$ as small $\succ$-relations, as well as the standard functions of addition and multiplication as small $\succ$-functions. The main properties of addition and multiplication are provable in RST^m_{HF}, as the standard proofs by induction can be carried out within it. Furthermore, all the basic properties of $\mathbb{Z}$ and $\mathbb{Q}$ (such as $\mathbb{Q}$ being a dense unbounded field) are straightforwardly proved in RST^m_{HF}.

Now we turn to the standard construction of the real line using Dedekind cuts. Since it is well known that the real line and its open segments are not absolute, they cannot be $\succ$-sets. Thus the collection of real numbers in RST^m_{HF} will not be a term but merely a *definable predicate.*, that is: a *$\succ$-class.*[24]

Let $\psi:=\forall x,y\in\widetilde{\mathbb{Q}}.x\in u\wedge y<x\rightarrow y\in u,\quad \varphi:=\neg\exists x\in u\forall y\in u.y\leq x.$

Definition 5.1 (The Reals)**.** $\mathbb{R}$ is $\left\|\{u\in\overline{P_{J_2}(\mathbb{Q})\setminus\{\emptyset,\mathbb{Q}\}}\mid\psi\wedge\varphi\}\right\|$.

[24]As noted in Footnote 4, this is in sharp difference from the development of real analysis in [8].

The above term is a valid class term as $P_{J_2}(\mathbb{Q}) \setminus \{\emptyset, \mathbb{Q}\}$ is a $\succ$-class, and $\varphi, \psi \succ \emptyset$. Note that it does not denote the "real" real-line (if such a thing really exists). However, it does contain all *computable* real numbers, such as $\sqrt{2}$ and π. (This can be shown by the same method that was used in [8].)

Notation. We employ the following notations: $\mathbb{Q}^+ = \{q \in \mathbb{Q} \mid 0 < q\}$, $\mathbb{R}^+ = \{r \in \mathbb{R} \mid 0 < r\}$, $(a, b) = \{r \in \mathbb{R} \mid a < r < b\}$ and $[a, b] = \{r \in \mathbb{R} \mid a \leq r \leq b\}$, for a, b real numbers. [25]

Proposition 5.2. *The following holds:*

(1) *The standard ordering $<$ on $\mathbb{R}$ is available in RST^m_{HF} as a $\succ$-relation.*
(2) *The standard addition and multiplication of reals are available in RST^m_{HF} as $\succ$-functions.*

Proof.

(1) The relation $<$ on $\mathbb{R}$ coincides with $\subset$, thus we can define the relation $<$ by $\{\!\!\{\langle x, y\rangle \in \overline{\mathbb{R} \times \mathbb{R}} \,\hat{\restriction}\, x \subset y\}\!\!\}$. We have that $x \subset y \succ \emptyset$, hence $<$ is a $\succ$-class. It is straightforward to prove in RST^m_{HF} properties concerning $<$, such as it being a total order on $\mathbb{R}$, the density of the rationals in $\mathbb{R}$, the Archimedean Principle, etc.
(2) The $\succ$-function $+$ can be represented (using Lemma 3.14) by the term
$$+ = \lambda x \in \bar{\mathbb{R}}, y \in \bar{\mathbb{R}}.\{\!\!\{z \mid \exists u \in x \exists v \in y. z = u + v\}\!\!\}$$
since $\exists u \in x \exists v \in y. z = u + v \succ \{z\}$.

To define multiplication, let F_1 be the $\succ$-function:
$$F_1 = \left\| \lambda a \in \bar{\mathbb{R}^+}, b \in \bar{\mathbb{R}^+}.\{\!\!\{z \mid z \leq 0 \vee \exists u \in a \exists v \in b\,(0 \leq u \wedge 0 \leq v \wedge z = u \cdot v)\}\!\!\} \right\|$$
Next, define the $\succ$-function $-$ on $\mathbb{R}$ by
$$- = \left\| \lambda x \in \bar{\mathbb{R}}.\{\!\!\{z \,\hat{\restriction}\, \exists u \in \widetilde{\mathbb{Q} \setminus x} \exists a \in \widetilde{\mathbb{Q}}. z + b = a\}\!\!\} \right\|.$$
Then, for $0 \leq a \wedge b < 0$ define $F_2(\langle a, b\rangle) := -F_1(\langle a, -b\rangle)$, for $a < 0 \wedge 0 \leq b$ define $F_3(\langle a, b\rangle) := -F_1(\langle -a, b\rangle)$, and for $a < 0 \wedge b < 0$ define $F_4(\langle a, b\rangle) := F_1(\langle -a, -b\rangle)$. Now the $\succ$-function $\cdot$ on $\mathbb{R} \times \mathbb{R}$ can be defined by $\cdot := F_1 \cup F_2 \cup F_3 \cup F_4$.
Proving in RST^m_{HF} basic properties regarding these $\succ$-functions, such as $\mathbb{R}$ being an ordered field, is again straightforward. $\square$

5.2. **The Least Upper Bound Principle and the Topology of the Reals.**

In this section we examine to what extent the least upper bound principle is available in RST^m_{HF}. We start with the following positive result:

Theorem 5.3. *It is provable in RST^m_{HF} that every nonempty $\succ$-subset of $\mathbb{R}$ that is bounded above has a least upper bound in $\mathbb{R}$. Furthermore, the induced mapping (l.u.b) is available in RST^m_{HF} as a $\succ$-function.*

Proof. Let X be a nonempty $\succ$-subset of $\mathbb{R}$ that is bounded above. $\cup X$ is a $\succ$-set, and since X is bounded above, standard arguments show that $\cup X$ is a Dedekind cut and thus belongs to $\mathbb{R}$. Since the order $\succ$-relation $\leq$ coincides with the inclusion relation, it follows that $\cup X$ is a least upper bound for X. Moreover, the function that maps each X to $\cup X$ is a rudimentary function (from $P_{J_2}(\mathbb{R})$ to $P_{J_2}(\mathbb{Q})$), and hence it is a $\succ$-function. Denote it by F. The desired function $l.u.b$ is $F\restriction_{F^{-1}[\mathbb{R}]}$, which by Proposition 3.16 is a $\succ$-function. $\square$

[25] Notice that $\mathbb{Q}^+$ is a $\succ$-set and $\mathbb{R}^+$ is a $\succ$-class.

Encouraging as it may be, Theorem 5.3 only states that $\succ$-*subsets* of $\mathbb{R}$ have the least upper bound property. Therefore it is insufficient for developing in RST^m_{HF} most of standard mathematics. The reason is that even the most basic substructures of $\mathbb{R}$, like the intervals, are not $\succ$-sets in RST^m_{HF}, but only proper $\succ$-classes. Hence, a stronger version of the theorem, which ensures that the least upper bound property holds for standard $\succ$-subclasses of $\mathbb{R}$, is needed. Theorem 5.16 below provides such an extension, but it requires some additional definitions and propositions.[26]

First we consider $\succ$-classes $U \subseteq \mathbb{R}$ which are open. These $\succ$-classes are generally not $\succ$-sets (unless empty), since they contain an interval of positive length, which is a proper $\succ$-class and thus cannot be contained in a $\succ$-set (see Prop. 3.4(3)). Clearly, there is no such thing as a $\succ$-set of $\succ$-classes, as a proper $\succ$-class can never be an element of another $\succ$-set or $\succ$-class. However, the use of coding (following [45], [46][27]) allows us, for example, to replace the meaningless statement "the union of a $\succ$-set of $\succ$-classes is a $\succ$-class" with "given a $\succ$-set of codes for $\succ$-classes, the union of the corresponding $\succ$-classes is a $\succ$-class".

The coding technique we use is based on the standard mathematical notation for a "family of sets", $(A_i)_{i\in I}$, where I is a set of indices and A_i is a set for each $i \in I$. In RST^m_{HF} we cannot construct the collection of all such A_i's if A_i is a $\succ$-class for some $i \in I$. Thus, we treat the $\succ$-set I as a code for the "family of classes" $(A_i)_{i\in I}$. In fact, we mainly use the union of such families, i.e., $\bigcup\limits_{i\in I} A_i$.

Definition 5.4. For $p, q \in \mathbb{R}$, the open ball $B_q(p)$ is the $\succ$-class $\{r \in \mathbb{R} \mid |r-p| < q\}$.

Definition 5.5. Let $U \subseteq \mathbb{R}$ be a $\succ$-class. If there exists a $\succ$-set $u \subseteq \mathbb{Q} \times \mathbb{Q}^+$ such that $U = \bigcup_{\langle p,q\rangle\in u} B_q(p) = \{r \in \mathbb{R} \mid \exists p, q\,(\langle p,q\rangle \in u \wedge |r-p| < q)\}$, then U is called *open* and u is a *code* for U.

In what follows, the formalizations in RST^m_{HF} are carried out as follows:

- To quantify over open $\succ$-classes: $Qu \subseteq \widetilde{\mathbb{Q} \times \mathbb{Q}^+}$ ($Q \in \{\forall, \exists\}$).
- To decode the open $\succ$-class whose code is u:

$$dec(u) := \{\!\!\{ r \in \bar{\mathbb{R}} \,\hat{|}\, \exists p, q\,(\langle p,q\rangle \,\check{\in}\, u \wedge |r-p| < q) \}\!\!\}$$

- To state that a class variable $\boldsymbol{U}$ is an open $\succ$-class:

$$Open(\boldsymbol{U}) := \exists u \subseteq \widetilde{\mathbb{Q} \times \mathbb{Q}^+}.\boldsymbol{U} = dec(u)$$

Proposition 5.6. *The following are provable in RST^m_{HF}:*

(1) *For any $\succ$-set $u \subseteq \mathbb{R} \times \mathbb{R}^+$, $\{r \in \mathbb{R} \mid \exists p, q\,(\langle p,q\rangle \in u \wedge |r-p| < q)\}$ is an open $\succ$-class.*
(2) *The open ball $B_q(p)$ is an open $\succ$-class for any $p \in \mathbb{R}$ and $q \in \mathbb{R}^+$.*

Proof.

(1) Take w to be $\left\| \{\!\!\{ \langle p,q\rangle \in \widetilde{\mathbb{Q} \times \mathbb{Q}^+} \mid \exists r, s\,(\langle r,s\rangle \,\check{\in}\, u \wedge q + |r-p| \leq s) \}\!\!\} \right\|$. Since $\mathbb{Q} \times \mathbb{Q}^+$ is a $\succ$-set and $\exists r, s\,(\langle r,s\rangle \,\check{\in}\, u \wedge q + |r-p| \leq s) \succ \emptyset$, w is a $\succ$-set that is a code for an open $\succ$-class. It can easily be proved in RST^m_{HF} that w codes U.
(2) $u = \{\langle p,q\rangle\}$ is a code of $B_q(p)$ (by 1.). $\square$

[26] It should be noted that the full least upper bound principle has not been derivable also in Weyl's approach [47]. To obtain the principle for standard mathematical objects, we use in what follows coding techniques that are similar to those employed by Weyl.

[27] In [46] such codings are called "proxies".

Proposition 5.7. *The following are provable in RST^m_{HF}:*

(1) *The union of a $\succ$-set of open $\succ$-classes is an open $\succ$-class. i.e, given a $\succ$-set of codes of open $\succ$-classes, the union of the corresponding open $\succ$-classes is an open $\succ$-class.*
(2) *The intersection of finitely many open $\succ$-classes is an open $\succ$-class.*

Proof. (1) Let X be a $\succ$-set of codes for open $\succ$-classes. Thus, $\cup X$ is a code for the union of the corresponding open $\succ$-classes.
(2) If U and V are open $\succ$-classes, a code for their intersection is obtained by intersecting every ball in a code for U with every ball in a code for V. □

Remark 5.8. *In general,* when we say that a theorem about a $\succ$-class or a $\succ$-function is provable in RST^m_{HF} (as in Prop. 5.7) we mean that it can be formalized and proved as a *scheme*, that is: its proof can be carried out in RST^m_{HF} using a uniform scheme. However, propositions about open $\succ$-classes form an exception, because due to the coding machinery, they can be fully formalized and proved in RST^m_{HF}.

Example 5.9. As an example of the use of the coding technique, we demonstrate the formalization of Prop. 5.7(1):

$$\forall z.(\forall x \in z.x \subseteq \widetilde{\mathbb{Q} \times \mathbb{Q}^+}) \to \exists w \subseteq \widetilde{\mathbb{Q} \times \mathbb{Q}^+}.dec\,(w) = \{\!\!\{ r \mid \exists x \in z.r \in dec\,(x) \}\!\!\}$$

Definition 5.10. A $\succ$-class $X \subseteq \mathbb{R}$ is *closed* if $\mathbb{R} - X$ is open.

Lemma 5.11. *It is provable in RST^m_{HF} that if $U \subseteq \mathbb{R}$ is an open $\succ$-class, then for every $x \in U$ there is an open ball about x which is contained in U.*

Proof. If $x \in U$, then there is some $\langle p, q \rangle$ in the code of U such that $x \in B_q\,(p)$. Take $\varepsilon = |p - x|$. It is straightforward to see that $B_\varepsilon\,(x) \subseteq B_q\,(p) \subseteq U$. □

The proof of the next Lemma is trivial.

Lemma 5.12. *Let $X \subseteq \mathbb{R}$ be a $\succ$-class and $A \subseteq X$ be a $\succ$-set. The following are equivalent in RST^m_{HF}:*

(1) *Every open ball about a point in X intersects A.*
(2) *Every open $\succ$-class that intersects X also intersects A.*

Example 5.13. As an example of a full formalization which uses class variables, the formalization of the Lemma above is:

$$\phi := \boldsymbol{X} \subseteq \bar{\mathbb{R}} \to \forall a \subseteq \boldsymbol{X} \left(\forall x \in \boldsymbol{X} \forall \varepsilon \in \bar{\mathbb{R}^+} \left(B_\varepsilon\,(x) \cap a \neq \emptyset\right) \leftrightarrow \right.$$
$$\left. \forall u \subseteq \widetilde{\mathbb{Q} \times \mathbb{Q}^+} \left(dec\,(u) \cap \boldsymbol{X} \neq \emptyset \to dec\,(u) \cap a \neq \emptyset\right)\right)$$

We now demonstrate how to obtain an equivalent schema in the basic $\mathcal{L}^{\{HF\}}_{RST}$ by replacing each appearance of a class term or a variable with the formula it stands for. First, we explain the translation of $x \in \bar{\mathbb{R}}$ to $\mathcal{L}^{\{HF\}}_{RST}$. One iteration of the translation entails $x \in \overline{P_{J_2}\,(\mathbb{Q}) \setminus \{\emptyset, \mathbb{Q}\}} \wedge \varphi\,(x) \wedge \psi\,(x)$ for φ, ψ as in Def. 5.1. A second iteration yields $R\,(x) := x \subseteq \tilde{\mathbb{Q}} \wedge x \neq \tilde{\mathbb{Q}} \wedge x \neq \emptyset \wedge \varphi\,(x) \wedge \psi\,(x)$ which is in $\mathcal{L}^{\{HF\}}_{RST}$. For the translation of ϕ, first substitute $\{\!\!\{ x \mid \theta \}\!\!\}$ for $\boldsymbol{X}$, where $\theta \succ \emptyset$. Proceeding with the translation steps results in the following formula (scheme) of $\mathcal{L}^{\{HF\}}_{RST}$, for $\theta \succ \emptyset$:

$$\forall b\,(\theta\,(b) \to R\,(b)) \to \forall a\,((\forall z.z \in a \to \theta\,(z)) \to \forall x\,(\theta\,(x) \to \forall \varepsilon\,((R\,(\varepsilon) \wedge 0 < \varepsilon) \to$$
$$\exists w.\,|w - x| < \varepsilon \wedge w \in a \leftrightarrow \forall u \subseteq \widetilde{\mathbb{Q} \times \mathbb{Q}^+}\,(\exists w.R\,(w) \wedge \exists p, q\,(\langle p, q \rangle \check{\in} u \wedge |w - p| < q) \wedge$$
$$\theta\,(w)) \to \exists w.R\,(w) \wedge \exists p, q\,(\langle p, q \rangle \check{\in} u \wedge |w - p| < q) \wedge w \in a)$$

Definition 5.14. Let $X \subseteq \mathbb{R}$ be a $\succ$-class, and $A \subseteq X$ a $\succ$-set. A is called *dense in* X if one of the conditions of Lemma 5.12 holds. X is called *separable* if it contains a dense $\succ$-subset.

Proposition 5.15. *It is provable in* RST^m_{HF} *that an open* $\succ$*-subclass of a separable* $\succ$*-class is separable.*

Proof. Let X be an open $\succ$-subclass of the separable $\succ$-class S, and let D be the dense $\succ$-subset in S. Let B be an open ball with center $x \in X$. By Lemma 5.11 there is a ball B' about x s.t. $B' \subseteq B \cap X$. Since D is dense in S, $B' \cap D \neq \emptyset$. Hence, $B \cap D \cap X \neq \emptyset$, and so $D \cap X$ is dense in X. $\square$

Now we can finally turn to prove a more encompassing least upper bound theorem.

Theorem 5.16. *It is provable in* RST^m_{HF} *that every nonempty separable* $\succ$*-subclass of* $\mathbb{R}$ *that is bounded above has a least upper bound in* $\mathbb{R}$*.*

Proof. Let X be a nonempty separable $\succ$-subclass of $\mathbb{R}$ that is bounded above, and let A be a dense $\succ$-subset in X. By Theorem 5.3, A has a least upper bound, denote it m. Suppose m is not an upper bound of X, i.e. there exists $x \in X - A$ s.t. $x > m$. Take $\varepsilon_m = |x - m|$. Then, there is no $a \in A$ s.t. $a \in A \cap (x - \varepsilon_m, x + \varepsilon_m)$, which contradicts A being dense in X. That m is the least upper bound of X is immediate. $\square$

Example 5.17. To demonstrate the formalization in $\mathcal{L}^{\{HF\}}_{RST}$ of the last theorem, denote by $separ\left(\boldsymbol{U}\right)$ the formula $\exists d.d \subseteq \boldsymbol{U} \wedge \forall x \in \boldsymbol{U} \forall \varepsilon \in \bar{\mathbb{R}}^{+} B_{\varepsilon}\left(x\right) \cap \boldsymbol{U} \neq \emptyset$, and by $bound_{\boldsymbol{U}}\left(w\right)$ the formula $\forall x \in \boldsymbol{U}.x \leq w$. Now, the full formalization is:

$$\left(\boldsymbol{U} \subseteq \mathbb{R} \wedge \boldsymbol{U} \neq \emptyset \wedge separ\left(\boldsymbol{U}\right) \wedge \exists w \in \bar{\mathbb{R}}.bound_{\boldsymbol{U}}\left(w\right)\right) \rightarrow$$
$$\exists v \in \bar{\mathbb{R}} \left(bound_{\boldsymbol{U}}\left(v\right) \wedge \forall w \in \bar{\mathbb{R}} \left(bound_{\boldsymbol{U}}\left(w\right) \rightarrow v \leq w\right)\right)$$

Definition 5.18. A $\succ$-class $X \subseteq \mathbb{R}$ is called an *interval* if for any $a, b \in X$ s.t. $a < b$: if $c \in \mathbb{R} \wedge a < c < b$ then $c \in X$.

Proposition 5.19. *It is provable in* RST^m_{HF} *that any non-degenerate interval is separable.*

Proof. Let X be a non-degenerate interval. Take A to be $X \cap \mathbb{Q}$. By Prop. 3.4(2) A is a $\succ$-set. A standard argument shows that in every open ball about a point in X there is a rational number, and thus it intersects A. $\square$

Corollary 5.20. *It is provable in* RST^m_{HF} *that any non-degenerate interval that is bounded above has a least upper bound.*

Definition 5.21. A $\succ$-class $X \subseteq \mathbb{R}$ is called *connected* if there are no open $\succ$-classes U and V such that $X \subseteq U \cup V$, $U \cap V \neq \emptyset$, $X \cap U \neq \emptyset$ and $X \cap V \neq \emptyset$.

Example 5.22. The formalization of the above definition can be given by:

$$connected\left(\boldsymbol{X}\right) := \neg \exists u, v \subseteq \widetilde{\mathbb{Q} \times \mathbb{Q}^{+}} \left(\boldsymbol{X} \subseteq decode\left(u\right) \cup decode\left(v\right) \wedge\right.$$
$$\left. decode\left(u\right) \cap decode\left(v\right) \neq \emptyset \wedge \boldsymbol{X} \cap decode\left(u\right) \neq \emptyset \wedge \boldsymbol{X} \cap decode\left(v\right) \neq \emptyset\right)$$

Proposition 5.23. *Let* $X \subseteq \mathbb{R}$ *be a* $\succ$*-class. It is provable in* RST^m_{HF} *that* X *is connected if and only if it is an interval.*

Proof. Assume that there are open $\succ$-classes U and V such that $X \subseteq U \cup V$, $U \cap V \neq \emptyset$, $X \cap U \neq \emptyset$, and $X \cap V \neq \emptyset$ (recall that the formalization of the existence of open $\succ$-classes is done using their codes). Choose $u \in U$ and $v \in V$ and assume that $u < v$. Let $U_0 = U \cap \{z \in \mathbb{R} \mid z < v\}$ and $V_0 = V \cap \{z \in \mathbb{R} \mid z > u\}$. Prop. 5.7 and Prop. 5.15 entail that U_0 and V_0 are open, separable $\succ$-subclass of $\mathbb{R}$. Standard arguments show that they are non-empty and bounded above. Thus, by Theorem 5.16, U_0 and V_0 have least upper bounds. Following the standard proof found in ordinary textbooks we can deduce that the least upper bounds are elements in $[u, v]$, but not elements of U_0 or of V_0, which is a contradiction, since $[u, v] \subseteq U_0 \cup V_0$. The classical proof of the converse direction can easily be carried out in RST^m_{HF}. □

5.3. **Real Functions.**

Definition 5.24. Let X be a $\succ$-class. A $\succ$-*sequence* in X is a $\succ$-function on $\mathbb{N}$ whose image is contained in X.

Lemma 5.25. *It is provable in RST^m_{HF} that Cauchy $\succ$-sequences in $\mathbb{R}$ converge to limits in $\mathbb{R}$. The induced map (lim) is available in RST^m_{HF} as a $\succ$-function.*

Proof. Let a be a Cauchy $\succ$-sequence, and let a_k abbreviate $a\,(k)$. For $n \in \mathbb{N}$ define $v_n := \bigcap_{k \geq n} a_k$. The $l.u.b$ of $\lambda n.v_n$ equals the limit of $\lambda n.a_n$. Thus, $lim\, \lambda n.a_n := \bigcup \{\!\!\{ v_n \mid n \in \widetilde{\mathbb{N}} \}\!\!\}$. □

Proposition 5.26. *It is provable in RST^m_{HF} that if $X \subseteq \mathbb{R}$ is closed, then every Cauchy $\succ$-sequence in X converges to a limit in X.*

Proof. Let a be a Cauchy $\succ$-sequence in X, and let a_k abbreviate $a\,(k)$. By Lemma 5.25, $lim\, \lambda n.a_n$ is an element in $\mathbb{R}$, denote it by l. Assume by contradiction that $l \in \mathbb{R} - X$. Since X is closed, $\mathbb{R} - X$ is open, and thus there exists $\varepsilon > 0$ such that $B_\varepsilon\,(l) \subseteq \mathbb{R} - X$. From this follows that for every a_k, $a_k \notin B_\varepsilon\,(l)$, which contradicts the fact that $lim\, \lambda n.a_n = l$. □

Next we want to study sequences of functions, but Def. 5.24 cannot be applied as is, since $\succ$-functions which are proper $\succ$-classes cannot be values of a $\succ$-function (in particular, of a $\succ$-sequence). Instead, we use the standard Un-currying procedure.

Definition 5.27. For X, Y $\succ$-classes, a $\succ$-*sequence of* $\succ$-*functions* on X whose image is contained in Y is a $\succ$-function on $\mathbb{N} \times X$ with image contained in Y.

Proposition 5.28. *Any point-wise limit of a $\succ$-sequence of $\succ$-functions on a $\succ$-class $X \subseteq \mathbb{R}$ whose image is contained in $\mathbb{R}$ is available in RST^m_{HF} as a $\succ$-function.*

Proof. Let F be a $\succ$-sequence of $\succ$-functions on X whose image is contained in $\mathbb{R}$. Suppose that for each $a \in X$ the $\succ$-sequence $\lambda n.F\,(n, a)$ is converging, and so it is Cauchy. Define: $G_a := \{\!\!\{ \langle n, \bar{F}\,(n, a) \rangle \mid n \in \widetilde{\mathbb{N}} \}\!\!\}$. Then, $\left\| \lambda a \in \bar{X}.limG_a \right\|$ is the desired $\succ$-function. □

Next we turn to continuous real $\succ$-functions. One possibility of doing so, adopted e.g., in [45, 47], is to introduce codes for continuous real $\succ$-functions (similar to the use of codes for open $\succ$-classes). This is of course possible as such $\succ$-functions are determined by their values on the $\succ$-set $\mathbb{Q}$. However, we prefer to present here another approach, which allows for almost direct translations of proofs in standard analysis textbook into our system. This is done using free function variables. Accordingly, the theorems which follow are schemes. Implicitly, the previous sections of this paper can also be read and understood as done in this manner. Therefore, in what follows we freely use results from them.

Definition 5.29. Let $X \subseteq \mathbb{R}$ be a $\succ$-class and let F be a $\succ$-function on X whose image is contained in $\mathbb{R}$. F is called a *continuous real $\succ$-function* if:

$$\forall a \in X \forall \varepsilon \in \mathbb{R}^{+} \exists \delta \in \mathbb{R}^{+} \forall x \in X. |x-a| < \delta \to |F(x) - F(a)| < \varepsilon$$

Proposition 5.30. *Let $X \subseteq \mathbb{R}$ be a $\succ$-class and F be a $\succ$-function on X whose image is contained in $\mathbb{R}$. It is provable in RST_{HF}^{m} that if for every open $\succ$-class $B \subseteq \mathbb{R}$, there is an open $\succ$-class A s.t. $F^{-1}[B] = A \cap X$, then F is continuous.*

Proof. Let $a \in X$, $\varepsilon > 0$, and $V = B_{\varepsilon}(F(a))$. Since V is an open $\succ$-class, there is an open $\succ$-class A s.t. $F^{-1}[V] = A \cap X$ (which is a $\succ$-class by Prop. 3.16(3)). Also, $F(a) \in V$ which entails $a \in F^{-1}[V]$, and thus $a \in A$. Since A is open there exists δ_a s.t. $B_{\delta_a}(a) \subseteq A$. Take $\delta = \delta_a$. For any $x \in X$, if $|x-a| < \delta_a$ then $x \in B_{\delta_a}(a) \subseteq A$. Hence $x \in A \cap X = F^{-1}[V]$, and therefore $F(x) \in V = B_{\varepsilon}(F(a))$, i.e. $|F(x) - F(a)| < \varepsilon$. □

Lemma 5.31. *The following are provable in RST_{HF}^{m}:*

(1) *The composition, sum and product of two continuous real $\succ$-functions is a continuous real $\succ$-function.*
(2) *The uniform limit of a $\succ$-sequence of continuous real $\succ$-functions is a continuous real $\succ$-function.*

Proof. The standard proofs of these claims can be easily carried out in RST_{HF}^{m}. Note that they require the triangle inequality which is provable in RST_{HF}^{m}. □

Next we prove, as examples, the Intermediate Value Theorem and the Extreme Value Theorem, which are two key properties of continuous real functions.

Theorem 5.32 (Intermediate Value Theorem). *Let F be a continuous real $\succ$-function on an interval $[a,b]$ with $F(a) < F(b)$. It is provable in RST_{HF}^{m} that for any $d \in \mathbb{R}$ s.t. $F(a) < d < F(b)$, there is $c \in [a,b]$ s.t. $F(c) = d$.*

Proof. Let $d \in \mathbb{R}$ such that $F(a) < d < F(b)$. Define

$$Q_d := \left\| \{\!\!\{ x \in \tilde{\mathbb{Q}} \mid x \in \overline{[a,b]} \wedge F(x) \leq d \}\!\!\} \right\|.$$

Q_d is clearly bounded (e.g. by b). Since $F(a) < d$, standard arguments that use the continuity of F and the denseness of $\mathbb{Q}$ in $\mathbb{R}$ show that there is a rational $a \leq q$ s.t. $F(q) \leq d$. Thus, Q_d is non-empty and by Thm. 5.3 it has a least upper bound, denote it by c. Since Q_d is non-empty and b is an upper bound for it, $c \in [a,b]$. Assume by contradiction that $F(c) < d$, and pick $\varepsilon = d - F(c)$. By the continuity of F there exists $\delta > 0$ s.t. for any $x \in [a,b]$, if $|x-c| < \delta$, then $|F(x) - F(c)| < \varepsilon = d - F(c)$. This yields the existence of a rational $q \in (c, c+\delta)$ (again, by the denseness of $\mathbb{Q}$ in $\mathbb{R}$) s.t. $F(q) < d$, which is a contradiction. Now, assume by contradiction that $F(c) > d$, and pick $\varepsilon = F(c) - d$. In this case there exists $\delta > 0$ s.t. for any $x \in [a,b]$, if $|x-c| < \delta$, then $F(x) > d$. But then $c - \delta$ is also an upper bound for Q_d, which is again a contradiction. Hence, $F(c) = d$. □

Theorem 5.33 (Extreme Value Theorem). *Let F be a continuous real $\succ$- function on a non-degenerate interval $[a,b]$. It is provable in RST_{HF}^{m} that F attains its maximum and minimum.*

Proof. Let Q be the $\succ$-set $[a,b] \cap \mathbb{Q}$. $F[Q]$ is a $\succ$-set by Prop. 3.13, and it is non-empty by the denseness of $\mathbb{Q}$ in $\mathbb{R}$. Assume by contradiction that $F[Q]$ is not bounded, and define for every $n \in \mathbb{N}$ $C_n = \left\| \{\!\!\{ x \in \tilde{Q} \mid F(x) > n \}\!\!\} \right\|$. By the assumption C_n is a non-empty, bounded

$\succ$-set. Therefore, by Thm. 5.3, each C_n has a least upper bound, denote it c_n. It is easy to see that $c_n \in [a,b]$ for each $n \in \mathbb{N}$. Now, define the $\succ$-sequence $\lambda n \in \mathbb{N}.c_n$ (which is indeed a $\succ$-sequence by Thm. 5.3). Standard arguments show that since $[a,b]$ is closed and bounded, there is a subsequence of $\lambda n \in \mathbb{N}.c_n$, $\lambda k \in \mathbb{N}.c_{n_k}$, which converges to a limit, denote it m. By Prop. 5.26 we have that $m \in [a,b]$. Now, since F is continuous, we easily get that $\lambda k \in \mathbb{N}.F(c_{n_k})$ converges to $F(m)$. But, for each $k \in \mathbb{N}$: $F(c_{n_k}) > n_k \geq k$, which contradicts the convergence of the sequence. Hence, $F[Q]$ is bounded, and by Thm. 5.3 it has a least upper bound, denote it by d. Assume by contradiction that there exists $u \in [a,b]$ s.t. $F(u) > d$. Picking $\varepsilon = F(u) - d$, the continuity of F entails that there exists δ s.t. for every $x \in B_\delta(u)$, $F(x) \geq d$. But the denseness of $\mathbb{Q}$ entails that there is a rational number $q \in B_\delta(u)$, and thus $F(q) \geq d$, which is a contradiction. The proof that there exists $x \in [a,b]$ s.t. that $F(x) = d$ uses arguments similar to the ones used in the proof of Thm. 5.32. The proof that F attains its minimum is symmetric. □

The next step is to introduce in RST^m_{HF} the concepts of differentiation, integration, power series, etc, and develop their theories. It should now be clear that there is no difficulty in doing so. Since a thorough exposition obviously could not fit in one paper we omit it here, but use some relevant facts in what follows.

We end this section by showing that all elementary functions that are relevant to J_2 are available in RST^m_{HF} in the sense that they are formalizable as $\succ$-functions and their basic properties are provable in RST^m_{HF}. Of course, not all constant functions on the "real" real line are available in J_2, even though for every y in $\mathbb{R}$, $\lambda x \in \mathbb{R}.y$ is available in RST^m_{HF} as a $\succ$-function. The reason is that $\lambda x \in \mathbb{R}.y$ does not exists in J_2 for *every* "real" number y (for the simple fact that not every "real" real number is available in RST^m_{HF}). Thus we next define what is an "J_2-elementary function" (see, for example, [42] for a standard definition of "elementary function").

Definition 5.34. The collection of J_2-elementary functions is the minimal collection that is closed under addition, subtraction, multiplication, division, and composition, and includes the following:

- J_2-constant functions: $\lambda x \in \mathbb{R}.c$ where c is a real number in J_2.
- Exponential: $\lambda x \in \mathbb{R}.e^x$
- Natural logarithm: $\lambda x \in \mathbb{R}^+.\ln x$
- Trigonometric functions:$\lambda x \in \mathbb{R}.\sin x$.
- Inverse trigonometric functions: $\lambda x \in [-1,1].\arcsin x$.

Proposition 5.35. *All J_2-polynomials (i.e. with coefficients in J_2) on $\mathbb{R}$ are available in RST^m_{HF} as $\succ$-function, and it is provable in RST^m_{HF} that they are continuous.*

Proof. J_2-constant functions and the identity function are available in RST^m_{HF} by Prop. 3.16, and the proofs of their continuity is immediate. Composition of $\succ$-functions is also available in RST^m_{HF}. All J_2-polynomials on $\mathbb{R}$ are therefore available in RST^m_{HF}, since $+$ and $\cdot$ are $\succ$-functions, and they are continuous by Lemma 5.31. □

Proposition 5.36. *The exponential and trigonometric functions are available in RST^m_{HF}, and it is provable in RST^m_{HF} that they are continuous.*

Proof. Since the exponential and the trigonometric functions all have power series, their definability as $\succ$-functions follows from Prop. 5.28. It is straightforward to verify that the basic properties of these $\succ$-functions are provable in RST^m_{HF}. Examples of such properties are: the monotonicity of the exponential, the power rules of the exponential, trigonometric identities like $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \sin\beta\cos\alpha$, the fact that sin has a period of 2π (where π is its first positive root), etc.[28] The continuity of these functions follows from Lemma 5.31 and Prop. 5.28. $\square$

Lemma 5.37. *Let F be a continuous, monotone real $\succ$-function on a real interval $[a,b]$, and suppose $F(a) < F(b)$. It is provable in RST^m_{HF} that*

$$\forall y \in \bar{\mathbb{R}} \left(\exists x \in \overline{[a,b]}.\bar{F}(x) = y \leftrightarrow y \in \overline{[F(a), F(b)]} \right)$$

Proof. The left-to-right implication is immediate from the monotonicity of F. The right-to-left implication follows from Thm. 5.32. $\square$

Proposition 5.38. *Let F be a continuous, strictly monotone real $\succ$-function on a real interval. Then it is provable in RST^m_{HF} that the inverse function F^{-1} is available in RST^m_{HF} as a $\succ$-function, and its continuity is provable in RST^m_{HF}.*

Proof. We here prove the claim for continuous, strictly monotone real $\succ$-function on a finite closed interval $[a,b]$. The extension from finite closed intervals to arbitrary interval is standard. Suppose F is increasing. The proof is similar to the proof of Thm. 5.32. For any $y \in [F(a), F(b)]$ define the $\succ$-set $Q_y := \left\| \lbrace\!\!\lbrace q \in \widetilde{\mathbb{Q}} \mid q \in \overline{[a,b]} \wedge F(q) \leq y \rbrace\!\!\rbrace \right\|$. It is easy to see that Q_y is non-empty and bounded, thus, by Thm. 5.3, Q_y has a least upper bound. Now, $\left\| \lambda y \in \overline{[F(a), F(b)]}.l.u.b(\widetilde{Q_y}) \right\|$ is the desired inverse $\succ$-function. It is not difficult to prove the basic properties of the inverse function in RST^m_{HF}. We demonstrate the proof that $F^{-1} \circ F = id_{[a,b]}$. For this we need to show that for any $x \in [a,b]$, $l.u.b(Q_{F(x)}) = x$. By the monotonicity of F, x is clearly an upper bound for $Q_{F(x)}$. Assume by contradiction that there is a real number $w < x$ which is an upper bound of $Q_{F(x)}$. Thus, in the interval (w,x) there is a rational number q such that $F(q) \leq F(x)$ (by monotonicity). But then, $q \in Q_{F(x)}$ and $w < q$, which is a contradiction. $\square$

Proposition 5.39. *All J_2-elementary functions are available in RST^m_{HF}.*

Proof. Props. 5.35 and 5.36 show that J_2-polynomials on $\mathbb{R}$, the exponential and trigonometric functions are available in RST^m_{HF}. Prop. 5.38 then enables the availability in RST^m_{HF} of the inverse trigonometric functions, and of the natural logarithm as the inverse of the exponential. $\square$

It is not difficult to see that many standard discontinuous functions are also available in RST^m_{HF}, as the next proposition shows.

Proposition 5.40. *Any piecewise defined function with finitely many pieces such that its restriction to any of the pieces is a J_2-elementary function, is available in RST^m_{HF}.*

Proof. If the function has finitely many pieces and each of the pieces is a J_2-elementary function, then it can be constructed in RST^m_{HF} using Prop. 3.16(5). $\square$

[28] We can prove the standard properties of the exponent and the trigonometric functions as listed, e.g., in [2], using the notion of differentiation.

6. Conclusion and Further Research

In this paper we showed that a minimal computational framework is sufficient for the development of applicable mathematics. Of course, a major future research task is to implement and test the framework. A critical component of such implementation will be to scale the cost of checking the safety relation. We then plan to use the implemented framework to formalize even larger portions of mathematics, including first of all more analysis, but also topology and algebra.

Another important task is to fully exploit the computational power of RST^m_{HF} and J_2. This includes finding a good notion of canonical terms, and investigating various reduction properties such as strong normalization. We intend to try also to profit from this computational power in other ways, e.g., by using it for proofs by reflection as supported by well-known proof assistant like Coq [16], Nuprl [18] and Isabelle/HOL [40].

An intuitionistic variant of the system RST^m_{HF} can be also considered. It is based on intuitionistic first-order logic (which underlies constructive counterparts of ZF, like CZF [1] and IZF [12]), and is obtained by adding to RST^m_{HF} the axiom of Restricted Excluded Middle: $\varphi \vee \neg\varphi$, where $\varphi \succ \emptyset$. This axiom is computationally acceptable since it simply asserts the definiteness of absolute formulas. The resulting computational theory should allow for a similar formalization of constructive analysis (e.g., [39]).

Further exploration of the connection between our framework and other related works is also required. This includes works on: computational set theory [1, 12, 14, 26, 39], operational set theory [24, 31], and rudimentary set theory [11, 36].

Another direction for further research is to consider larger computational structures. This includes J_ω or even J_{ω^ω} (which is the minimal model of the minimal computational theory based on ancestral logic [7, 17]). On the one hand, in such universes standard mathematical structures can be treated as sets. On the other hand, they are more comprehensive and less concrete, thus include more objects which may make computations harder.

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