

CODENSITY LIFTING OF MONADS AND ITS DUAL

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ABSTRACT. We introduce a method to lift monads on the base category of a fibration to its total category. This method, which we call *codensity lifting*, is applicable to various fibrations which were not supported by its precursor, categorical $\top\top$ -lifting. After introducing the codensity lifting, we illustrate some examples of codensity liftings of monads along the fibrations from the category of preorders, topological spaces and extended pseudometric spaces to the category of sets, and also the fibration from the category of binary relations between measurable spaces. We also introduce the dual method called *density lifting* of comonads. We next study the liftings of *algebraic operations* to the codensity liftings of monads. We also give a characterisation of the class of liftings of monads along posetal fibrations with fibred small meets as a limit of a certain large diagram.

1. INTRODUCTION

Inspired by Lindley and Stark’s work on extending the concept of reducibility candidates to monadic types [Lin05, LS05], the first author previously introduced its semantic analogue called *categorical $\top\top$ -lifting* in [Kat05]. It constructs a lifting of a strong monad \mathcal{T} on the base category of a closed-structure preserving fibration $p : \mathbb{E} \rightarrow \mathbb{B}$ to its total category. The construction takes the inverse image of the continuation monad on the total category along the canonical monad morphism $\sigma : \mathcal{T} \rightarrow (- \Rightarrow TR) \Rightarrow TR$ in the base category, which exists

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for any strong monad \mathcal{T} :

$$\mathcal{T}^{\top\top} \dashrightarrow (- \Rightarrow S) \Rightarrow S \quad (1.1)$$

$$\mathcal{T} \xrightarrow{\sigma} (- \Rightarrow TR) \Rightarrow TR$$

The objects R and S (such that $TR = pS$) are presupposed parameters of this $\top\top$ -lifting, and by varying them we can derive various liftings of \mathcal{T} . The categorical $\top\top$ -lifting has been used to construct logical relations for monads [Kat13] and to analyse the concept of preorders on monads [KS13].

One key assumption for the $\top\top$ -lifting to work is that the fibration p preserves the *closed structure*, so that the continuation monad $(- \Rightarrow S) \Rightarrow S$ on the total category becomes a lifting of the continuation monad $(- \Rightarrow TR) \Rightarrow TR$ on the base category. Although many such fibrations are seen in the categorical formulations of logical relations [MS93, Her93, Kat13], requiring fibrations to preserve the closed structure of the total category imposes a technical limitation to the applicability of the categorical $\top\top$ -lifting. Indeed, outside the categorical semantics of type theories, it is common to work with the categories that are not closed. In the study of coalgebras, *predicate / relational liftings* of functors and monads are fundamental structures to formulate modal operators and (bi)simulation relations, and the underlying categories of them are not necessarily closed. For instance, the category **Meas** of measurable spaces, which is not cartesian closed, is used to host labelled Markov processes [vBMOW05]. The categorical $\top\top$ -lifting does not work in such situations.

To overcome this technical limitation, in this paper we introduce an alternative lifting method called *codensity lifting*. The idea is to replace the continuation monad $(- \Rightarrow S) \Rightarrow S$ with the *codensity monad* $\mathbf{Ran}_S S$ given by a right Kan extension. We then ask fibrations to preserve the right Kan extension, which is often fulfilled when \mathbb{E} has and p preserves limits. We demonstrate that the codensity lifting is applicable to lift monads on the base categories of the following fibrations:

$$\begin{array}{ccccccc} \mathbf{Pre} & \mathbf{Top} & \mathbf{ERel}(\mathbf{Meas}) & \longrightarrow & \mathbf{BRel}(\mathbf{Meas}) & \longrightarrow & \mathbf{Pred} & U^*\mathbf{EPMet} & \longrightarrow & \mathbf{EPMet} \\ \downarrow & \downarrow & \downarrow \lrcorner & & \downarrow \lrcorner & & \downarrow & \downarrow \lrcorner & & \downarrow \\ \mathbf{Set} & \mathbf{Set} & \mathbf{Meas} & \xrightarrow{\Delta} & \mathbf{Meas}^2 & \xrightarrow{(\times) \circ U^2} & \mathbf{Set} & \mathbf{Meas} & \xrightarrow{U} & \mathbf{Set} \end{array}$$

The description of these fibrations are in order:

- Functors $\mathbf{Pre} \rightarrow \mathbf{Set}$ and $\mathbf{Top} \rightarrow \mathbf{Set}$ are the forgetful functors from the category of preorders and that of topological spaces to **Set**.
- Functor $\mathbf{BRel}(\mathbf{Meas}) \rightarrow \mathbf{Meas}$ is the fibration for binary relations between (the carrier sets of) two measurable spaces. Functor $\mathbf{ERel}(\mathbf{Meas}) \rightarrow \mathbf{Meas}$ is the subfibration of $\mathbf{BRel}(\mathbf{Meas}) \rightarrow \mathbf{Meas}$ obtained by restricting objects in $\mathbf{BRel}(\mathbf{Meas})$ to the binary relations over the same measurable spaces.
- Functor $\mathbf{EPMet} \rightarrow \mathbf{Set}$ is the forgetful functor from the category of extended pseudometric spaces and non-expansive functions between them. We then apply the change-of-base to it to overlay extended pseudometrics on measurable spaces. This yields a fibration $U^*\mathbf{EPMet} \rightarrow \mathbf{Set}$.

By taking the categorical dual of the codensity lifting, we obtain the method to lift comonads on the base category of a cofibration to its total category. This method, which we call the *density lifting* of comonads, is newly added to the conference version of this paper [KS15]. We illustrate two examples of the density lifting of **Set**-comonads along the subobject fibration of **Set**.

Another issue when we have a lifting $\dot{\mathcal{T}}$ of a monad \mathcal{T} is the liftability of *algebraic operations* for \mathcal{T} to the lifting $\dot{\mathcal{T}}$. For instance, let $\dot{\mathcal{T}}$ be a lifting of the powerset monad \mathcal{T}_p on **Set** along the forgetful functor $p : \mathbf{Top} \rightarrow \mathbf{Set}$, which is a fibration. A typical algebraic operation for \mathcal{T}_p is the union of A -indexed families of sets:

$$\text{union}_X^A : A \pitchfork T_p X \rightarrow T_p X, \quad \text{union}_X^A(f) = \bigcup_{a \in A} f(a);$$

here \pitchfork denotes the *power*. Then the question is whether we can “lift” the function union_X^A to a continuous function of type $A \pitchfork \dot{T}(X, \mathcal{O}_X) \rightarrow \dot{T}(X, \mathcal{O}_X)$ for every topological space (X, \mathcal{O}_X) . We show that the liftability of algebraic operations to the codensity liftings of monads has a good characterisation in terms of the parameters supplied to the codensity liftings.

We are also interested in the categorical property of the *collection* of liftings of a monad \mathcal{T} (along a limited class of fibrations). We show a characterisation of the class of liftings of \mathcal{T} as a limit of a large diagram of partial orders.

1.1. Related Work. This paper is the journal version of [KS15]. We add an elementary introduction to the $\top\top$ -lifting and the codensity lifting (Section 2), and the section about the density lifting of comonads (Section 5).

In the semantics of programming languages based on typed λ -calculi, *logical predicates* and *logical relations* [Plo80] have been extensively used for establishing (relational) properties of programs. The categorical analysis of logical relations emerged around the 90’s [MS93, MR92], and its fibrational account was given by Hermida [Her93]. In these works, the essence of logical predicates and relations is identified as predicate / relational liftings of the categorical structures corresponding to type constructors; especially Hermida studied the construction of such liftings in fibred category theory. Liftings of categorical structures along a functor are later employed in a categorical treatment of refinement types [MZ15].

One of the earliest work that introduced logical relations (i.e., relational liftings) for monads is Filinski’s PhD thesis [Fil96]. They play a central role in establishing relationships between two monadic semantics of programming languages with computational effects [Fil94, WV04, Fil07, FS07, Kat13]. Larrecq, Lasota and Nowak gave a systematic method to lift monads based on mono-factorisation systems [LLN08]. Their method is fundamentally different from the codensity lifting, and the relationship between these lifting methods is still not clear.

The origin of the codensity lifting goes back to the *biorthogonality technique* developed in proof theory. Girard used this technique in various contexts, such as 1) the phase space semantics of linear logic [Gir87], 2) the proof of the strong normalisation of cut-elimination in proof nets [Gir87] and 3) the definition of types in the geometry of interaction [Gir89]. Krivine also used biorthogonally-closed sets of terms and stacks in his realisability semantics of classical logic [Kri09]. Pitts introduced a similar technique called $\top\top$ -closure operator, and used $\top\top$ -closed relations as a substitute for admissible relations in the operational semantics of a functional language. Abadi considered a domain-theoretic analogue of Pitts’

$\top\top$ -closure operator, and compared $\top\top$ -closed relations and admissible relations [Aba00]. Lindley and Stark’s leapfrog method extends Pitts’ $\top\top$ -closure operator to the construction of logical predicates for monadic types [Lin05, LS05]. The first author gave a categorical analogue of the leapfrog method [Kat05], which constructs a lifting of a strong monad along a closed-structure preserving fibration.

In the coalgebraic study of state transition systems and process calculi, one way to represent a (bi)simulation relation between coalgebras is to give a relational coalgebra with respect to a relational lifting of the coalgebra functor [HJ98, HT00, JH03, Lev11]. When the coalgebra functor preserves weak pullbacks, *Barr extension* [Bar70] is often employed to derive a relational lifting of the coalgebra functor. In a recent work [SKDH18], Sprunger et al. introduced the codensity lifting of **Set**-endofunctors along a partial order fibration over **Set** with fibred small meets. The basic lifting strategy is the same as this paper; we derive a lifting of an endofunctor by pulling back a codensity monad along a canonical morphism. One notable difference is that the parameter to lift an endofunctor F refers to the category of F -algebras. This is because we need a substitute for the Kleisli category used in the codensity lifting of monads.

In Section 4, we illustrate a couple of definitions of (bi)simulation relations that can be naturally expressed by the codensity lifting of monads: one is the definition of simulation relation on a single labelled Markov process [vBMOW05], and the other is the definition of bisimulation relation between two labelled Markov processes [BBLM14].

1.2. Preliminaries. We use white bold letters $\mathbb{B}, \mathbb{C}, \mathbb{E}, \dots$ to range over locally small categories. We sometimes identify an object in a category \mathbb{C} and a functor of type $1 \rightarrow \mathbb{C}$.

We do a lot of 2-categorical calculations in **CAT**. To reduce the notational burden, we omit writing the composition operator \circ between functors, or a functor and a natural transformation. For instance, for functors G, F, P, Q and a natural transformation $\alpha : P \rightarrow Q$, by $G\alpha F$ we mean the natural transformation $G(\alpha_{FI}) : G \circ P \circ F(I) \rightarrow G \circ Q \circ F(I)$. We use \bullet and $*$ for the vertical and horizontal compositions of natural transformations, respectively.

Let A be a set and X be an object of a category \mathbb{C} . A *power of X by A* is a pair of an object $A \pitchfork X$ and an A -indexed family of projection morphisms $\{\pi_a : A \pitchfork X \rightarrow X\}_{a \in A}$. They satisfy the following universal property: for any A -indexed family of morphisms $\{f_a : B \rightarrow X\}_{a \in A}$, there exists a unique morphism $m : B \rightarrow A \pitchfork X$ such that $\pi_a \circ m = f_a$ holds for all $a \in A$. Here are some examples of powers:

- (1) When $\mathbb{C} = \mathbf{Set}$, the function space $A \Rightarrow X$ and the evaluation function $\pi_a(f) = f(a)$ give a power of X by A .
- (2) When \mathbb{C} has small products, the product of A -fold copies of X and the associated projections give a power of X by A .
- (3) When \mathbb{C} has powers by $A \in \mathbf{Set}$, any functor category $[\mathbb{D}, \mathbb{C}]$ also has powers by A , which can be given pointwisely: $(A \pitchfork F)X = A \pitchfork (FX)$.

A *right Kan extension* of $F : \mathbb{A} \rightarrow \mathbb{C}$ along $G : \mathbb{A} \rightarrow \mathbb{D}$ is a pair of a functor $\mathbf{Ran}_G F : \mathbb{D} \rightarrow \mathbb{C}$ and a natural transformation $c : (\mathbf{Ran}_G F)G \rightarrow F$ making the mapping $\underline{(-)} : [\mathbb{D}, \mathbb{C}](H, \mathbf{Ran}_G F) \rightarrow [\mathbb{A}, \mathbb{C}](HG, F)$ defined by

$$\underline{(-)}(\alpha) = c \bullet (\alpha G)$$

bijjective and natural in $H \in [\mathbb{D}, \mathbb{C}]$. A functor $p : \mathbb{C} \rightarrow \mathbb{C}'$ *preserves* a right Kan extension $(\mathbf{Ran}_G F, c)$ if $(p(\mathbf{Ran}_G F), pc)$ is a right Kan extension of pF along G . Thus for any right

Kan extension ($\mathbf{Ran}_G(pF), c'$) of pF along G , we have $p(\mathbf{Ran}_G F) \simeq \mathbf{Ran}_G(pF)$ by the universal property.

Let \mathcal{T} be a monad on a category \mathbb{C} . Its components are denoted by (T, η, μ) . The monad induces the *Kleisli lifting* $(-)^{\#} : \mathbb{C}(I, TJ) \rightarrow \mathbb{C}(TI, TJ)$ defined by $f^{\#} = \mu_J \circ Tf$. We write $J : \mathbb{C} \rightarrow \mathbb{C}_{\mathcal{T}}$ and $K : \mathbb{C}_{\mathcal{T}} \rightarrow \mathbb{C}$ for the left and right adjoint of the Kleisli resolution of \mathcal{T} , respectively. We also write $\epsilon : JK \rightarrow \text{Id}_{\mathbb{C}}$ for the counit of this adjunction. When \mathcal{T} is decorated with an extra symbol, like $\dot{\mathcal{T}}$, the same decoration is applied to the components of \mathcal{T} and the notation for the Kleisli adjunction, like $\dot{\eta}, \dot{J}, \dot{\epsilon}$, etc.

For the definition of fibrations and related concepts, see [Jac99].

2. FROM $\top\top$ -LIFTING TO CODENSITY LIFTING

Before introducing the codensity lifting, we first briefly review its precursor, the *semantic $\top\top$ -lifting* [Kat05]. It is a semantic analogue of Lindley and Stark's *leapfrog method* [LS05, Lin05], and constructs a *logical predicate* for a monad $\mathcal{T} = (T, \eta, \mu)$ on \mathbf{Set} . Below, by a *predicate* we mean a pair (X, I) of sets such that $X \subseteq I$.

The semantic $\top\top$ -lifting takes a *lifting parameter*, which is a pair of a set R and a predicate (S, TR) . Fix such a parameter. The semantic $\top\top$ -lifting is defined as a mapping of a predicate (X, I) to the predicate $(T^{\top\top} X, TI)$, where $T^{\top\top} X$ is constructed in two steps:

$$\begin{aligned} T^{\top} X &= \{k \in I \Rightarrow R \mid \forall x \in X . k(x) \in S\} \\ T^{\top\top} X &= \{c \in TI \mid \forall k \in T^{\top} X . k^{\#}(c) \in S\}. \end{aligned} \quad (2.1)$$

Regarding monads as models of computational effects [Mog91], the above two steps may be intuitively understood as follows. We think of the parameter R as the type of return values of *continuations* (which corresponds to *stack frames* in operational semantics [Pit00, LS05]), and the parameter S as a specification of “good computations” over R . Now let (X, I) be a predicate, which we regard as a set I of values with a specification X of “good values”. Then $T^{\top} X$ collects all the continuations that send good values to good computations, and $T^{\top\top} X$ collects all the computations over I that yield good computations when passed to continuations in $T^{\top} X$. Overall, we regard the semantic $\top\top$ -lifting as a process to collect a set $T^{\top\top} X$ of good computations from a given set X of good values. The semantic $\top\top$ -lifting is suitable for the construction of logical predicates for monadic types [Kat05, Theorem 3.8].

The semantic $\top\top$ -lifting can further be formulated in fibred category theory. To illustrate this, let us introduce the category \mathbf{Pred} , where an object is a predicate and a morphism from (X, I) to (Y, J) is a function $f : I \rightarrow J$ that maps elements in X to those in Y . The evident forgetful functor $p : \mathbf{Pred} \rightarrow \mathbf{Set}$ is a partial order fibration: the inverse image of a predicate (X, I) along a function $f : J \rightarrow I$ is the predicate $(\{j \mid f(j) \in X\}, J)$, which we denote by $f^*(X, I)$; see [Jac99, Chapter 0] for more detail. Moreover, the following facts are known: 1) The category \mathbf{Pred} is cartesian closed. The following gives exponentials in \mathbf{Pred} :

$$(X, I) \Rightarrow (Y, J) = (\{f \mid \forall x \in X . f(x) \in Y\}, I \Rightarrow J),$$

and they are strictly preserved by p [Her93]. 2) Monads on \mathbf{Set} are always equipped with the *bind morphism* $\sigma : TX \rightarrow (X \Rightarrow TR) \Rightarrow TR$ given by $\sigma(c) = \lambda k . k^{\#}(c)$, which is derivable from the *canonical strength* of monads on \mathbf{Set} [Mog91]. Then both $T^{\top} X$ and $T^{\top\top} X$ can be

computed using these categorical facts:

$$\begin{aligned} (T^\top X, I \Rightarrow TR) &= (X, I) \dot{\Rightarrow} (S, TR) \\ (T^{\top\top} X, TR) &= \sigma^*((X, I) \dot{\Rightarrow} (S, TR)) \dot{\Rightarrow} (S, TR). \end{aligned} \quad (2.2)$$

We can also deduce from this characterisation that $T^{\top\top}$ extends to a *lifting* (Definition 3.1) of the monad \mathcal{T} along the fibration $p : \mathbf{Pred} \rightarrow \mathbf{Set}$. A more conceptual reading of (2.2) is that the semantic $\top\top$ -lifting is the inverse image of the continuation monad on \mathbf{Pred} along σ , as depicted in (1.1). The right hand side of (2.2) can be computed in a more general situation where p is a symmetric monoidal closed fibration of type $\mathbb{E} \rightarrow \mathbb{B}$ ¹ and \mathcal{T} is a strong monad on \mathbb{B} . This is the *categorical $\top\top$ -lifting* in [Kat05, Section 4].

In this paper, we pursue the lifting method based on an alternative characterisation of the semantic $\top\top$ -lifting. We observe that: 1) the set $T^\top X$ is identical to the homset $\mathbf{Pred}((X, I), (S, TR))$, and 2) the universal quantification $\forall k \in T^\top X$ in the definition of $T^{\top\top} X$ can be extracted as the intersection of predicates, which corresponds to the *fibred meet* of the partial order fibration $p : \mathbf{Pred} \rightarrow \mathbf{Set}$. From these observations, we can characterise $T^{\top\top} X$ as the fibred meet of inverse images:

$$(T^{\top\top} X, TR) = \bigwedge_{k \in \mathbf{Pred}((X, I), (S, TR))} ((pk)^\#)^*(S, TR); \quad (2.3)$$

here \bigwedge stands for the fibred meet.

The above presentation of $T^{\top\top} X$ in the language of fibred category theory leads us to *adopt* (2.3) as the generalised definition of $T^{\top\top} X$ for any partial order fibration $p : \mathbb{E} \rightarrow \mathbb{B}$ with fibred meets and monad \mathcal{T} on \mathbb{B} . Since p need not to be closed-structure preserving, the generalised definition makes sense in a wide range of such fibrations, including forgetful functors from the category of preorders, topological spaces and metric spaces to \mathbf{Set} . However, unlike the categorical $\top\top$ -lifting, the meaning of the generalised definition is not very clear, because it is a direct encoding of the right hand side of (2.1) in fibred category theory. In the next section, we introduce the *codensity lifting* for general fibrations, which has a conceptually clear definition using the *codensity monad*. Then in Proposition 4.1 we show that the codensity lifting reduces to the right hand side of (2.3) when the lifting parameter is single, and the fibration has sufficient limits.

3. CODENSITY LIFTING OF MONADS

Fix a fibration $p : \mathbb{E} \rightarrow \mathbb{B}$ and a monad \mathcal{T} on \mathbb{B} . We formally introduce the main subject of this study, *liftings* of \mathcal{T} .

Definition 3.1. A *lifting* of \mathcal{T} (along p) is a monad $\dot{\mathcal{T}}$ on \mathbb{E} such that $p\dot{T} = Tp$, $p\dot{\eta} = \eta p$ and $p\dot{\mu} = \mu p$.

We do not require *fibredness* on $\dot{\mathcal{T}}$. The codensity lifting is a method to construct a lifting of \mathcal{T} from the following data called *lifting parameter*.

Definition 3.2. A *lifting parameter* (for \mathcal{T}) is a span $\mathbb{B}_\mathcal{T} \xleftarrow{R} \mathbb{A} \xrightarrow{S} \mathbb{E}$ of functors such that $KR = pS$. We say that it is *single* if $\mathbb{A} = 1$.

¹ That is, \mathbb{E}, \mathbb{B} are symmetric monoidal closed, and p strictly preserves the symmetric monoidal closed structure.

Any single lifting parameter can be written as (JR, S) for some $R \in \mathbb{B}$. We therefore call a pair (R, S) of $R \in \mathbb{B}$ and $S \in \mathbb{E}_{TR}$ a single lifting parameter too. This is the same data used in the single-result categorical $\top\top$ -lifting in [Kat05].

Fix a lifting parameter $\mathbb{B}_{\mathcal{T}} \xleftarrow{R} \mathbb{A} \xrightarrow{S} \mathbb{E}$. In this section we introduce the *codensity lifting* under the assumption that the fibration $p : \mathbb{E} \rightarrow \mathbb{B}$ and the functor S of the lifting parameter satisfy the following *codensity condition*.

Definition 3.3. We say that a fibration $p : \mathbb{E} \rightarrow \mathbb{B}$ and a functor $S : \mathbb{A} \rightarrow \mathbb{E}$ satisfy the *codensity condition* if

- (1) a right Kan extension of S along S exists, and
- (2) $p : \mathbb{E} \rightarrow \mathbb{B}$ preserves this right Kan extension.

We give some sufficient conditions for (p, S) to satisfy the codensity condition.

Proposition 3.4. *Let $p : \mathbb{E} \rightarrow \mathbb{B}$ be a fibration and $S : \mathbb{A} \rightarrow \mathbb{E}$ be a functor. The following are sufficient conditions for (p, S) to satisfy the codensity condition:*

- (1) \mathbb{E} has, and p preserves powers, and $\mathbb{A} = 1$.
- (2) \mathbb{E} has, and p preserves small products, and \mathbb{A} is small discrete.
- (3) \mathbb{E} has, and p preserves small limits, and \mathbb{A} is small.
- (4) S is a right adjoint.

Proof. (1-3) are immediate. (4) Let P be a left adjoint of S . Then the assignment $F \mapsto FP$ extends to a right Kan extension of F along S . This Kan extension is *absolute* [Mac98, Proposition X.7.3]. \square

Assume that (p, S) satisfies the codensity condition. We take a right Kan extension $(\mathbf{Ran}_S S, c : (\mathbf{Ran}_S S)S \rightarrow S)$, and equip it with the following monad structure: the unit $u : \text{Id} \rightarrow \mathbf{Ran}_S S$ and multiplication $m : (\mathbf{Ran}_S S)(\mathbf{Ran}_S S) \rightarrow \mathbf{Ran}_S S$ are respectively unique natural transformations such that $c \bullet uS = \text{id}_S$ and $c \bullet mS = c \bullet (\mathbf{Ran}_S S)c$. This is the *codensity monad* [Mac98, Exercise X.7.3].

Since p preserves the right Kan extension $\mathbf{Ran}_S S$, $(p(\mathbf{Ran}_S S), pc)$ is a right Kan extension of pS along S . Thus the mapping $\overline{(-)} : [\mathbb{E}, \mathbb{B}](H, p(\mathbf{Ran}_S S)) \rightarrow [\mathbb{A}, \mathbb{B}](HS, pS)$ defined by

$$\overline{(-)} = pc \bullet -S$$

is bijective and natural on $H : \mathbb{E} \rightarrow \mathbb{B}$. We denote its inverse by $\overline{(-)}$, and call \bar{f} the *mate* of f .

The codensity lifting constructs a lifting $\mathcal{T}^{\top\top} = (T^{\top\top}, \eta^{\top\top}, \mu^{\top\top})$ of \mathcal{T} along p as follows.

Lifting the Endofunctor T . We first regard $K\epsilon R$ as a natural transformation of type $TpS \rightarrow pS$. We then apply the function $\overline{(-)}$ to it:

$$\frac{K\epsilon R : TpS = KJpS = KJKR \rightarrow KR = pS}{\overline{K\epsilon R} : Tp \rightarrow p(\mathbf{Ran}_S S)}$$

We next take its cartesian lifting with respect to $\mathbf{Ran}_S S$.

$$\begin{array}{ccc} T^{\top\top} & \xrightarrow{\sigma} & \mathbf{Ran}_S S \\ & & \downarrow [\mathbb{E}, p] \\ Tp & \xrightarrow{\overline{K\epsilon R}} & p(\mathbf{Ran}_S S) \end{array} \quad \begin{array}{c} [\mathbb{E}, \mathbb{E}] \\ \downarrow [\mathbb{E}, p] \\ [\mathbb{E}, \mathbb{B}] \end{array}$$

This is possible because $[\mathbb{E}, p] : [\mathbb{E}, \mathbb{E}] \rightarrow [\mathbb{E}, \mathbb{B}]$ is again a fibration. We name the cartesian lifting σ , and its domain $T^{\top\top}$. We have $pT^{\top\top} = [\mathbb{E}, p]T^{\top\top} = Tp$.

Lifting the Unit η . Consider the following diagram:

$$\begin{array}{ccc} \text{Id}_{\mathbb{E}} & \xrightarrow{u} & \mathbf{Ran}_S S \\ \eta^{\top\top} \searrow & & \downarrow [\mathbb{E}, p] \\ T^{\top\top} & \xrightarrow{\sigma} & \mathbf{Ran}_S S \\ & & \downarrow [\mathbb{E}, p] \\ p & \xrightarrow{pu} & p(\mathbf{Ran}_S S) \\ \eta p \searrow & & \downarrow [\mathbb{E}, p] \\ Tp & \xrightarrow{\overline{K\epsilon R}} & p(\mathbf{Ran}_S S) \end{array}$$

The triangle in the base category commutes by:

$$\overline{K\epsilon R} \bullet \eta p = \overline{K\epsilon R} \bullet \eta p S = \overline{K\epsilon R} \bullet \eta \overline{KR} = \overline{\text{id}_{KR}} = \overline{\text{id}_{pS}} = pu.$$

Therefore from the universal property of σ , we obtain the unique natural transformation $\eta^{\top\top}$ above ηp making the triangle in the total category commute.

Lifting the Multiplication μ . Consider the following diagram.

$$\begin{array}{ccc} T^{\top\top} T^{\top\top} & \xrightarrow{T^{\top\top} \sigma} & T^{\top\top} \mathbf{Ran}_S S \xrightarrow{\sigma \mathbf{Ran}_S S} & (\mathbf{Ran}_S S) \mathbf{Ran}_S S & \xrightarrow{m} & \mathbf{Ran}_S S \\ \mu^{\top\top} \searrow & & & & & \downarrow [\mathbb{E}, p] \\ T^{\top\top} & \xrightarrow{\sigma} & \mathbf{Ran}_S S & & & \downarrow [\mathbb{E}, p] \\ T T p & \xrightarrow{T \overline{K\epsilon R}} & T p(\mathbf{Ran}_S S) \xrightarrow{\overline{K\epsilon R} \mathbf{Ran}_S S} & p(\mathbf{Ran}_S S) \mathbf{Ran}_S S & \xrightarrow{pm} & p(\mathbf{Ran}_S S) \\ \mu p \searrow & & & & & \downarrow [\mathbb{E}, p] \\ Tp & \xrightarrow{\overline{K\epsilon R}} & p(\mathbf{Ran}_S S) & & & \downarrow [\mathbb{E}, p] \end{array}$$

The pentagon in the base category commutes by:

$$\begin{aligned} pm \bullet \overline{K\epsilon R} \mathbf{Ran}_S S \bullet T \overline{K\epsilon R} &= pc \bullet p(\mathbf{Ran}_S S)c \bullet \overline{K\epsilon R}(\mathbf{Ran}_S S)S \bullet T \overline{K\epsilon R}S \\ (\text{interchange law}) &= pc \bullet \overline{K\epsilon R}S \bullet Tpc \bullet T \overline{K\epsilon R}S = K\epsilon R \bullet KJK\epsilon R \\ &= K\epsilon R \bullet \mu KR = K\epsilon R \bullet \mu pS = \overline{K\epsilon R} \bullet \mu p. \end{aligned}$$

Therefore from the universal property of σ , we obtain the unique morphism $\mu^{\top\top}$ above μp making the pentagon in the total category commute.

Theorem 3.5. *Let $p : \mathbb{E} \rightarrow \mathbb{B}$ be a fibration, \mathcal{T} be a monad on \mathbb{B} , $\mathbb{B}_{\mathcal{T}} \xleftarrow{R} \mathbb{A} \xrightarrow{S} \mathbb{E}$ be a lifting parameter for \mathcal{T} , and assume that (p, S) satisfies the codensity condition. The tuple $\mathcal{T}^{\top\top} = (T^{\top\top}, \eta^{\top\top}, \mu^{\top\top})$ constructed as above is a lifting of \mathcal{T} along p .*

Proof. From the universal property of the cartesian morphism σ , it suffices to show the following three equalities:

$$\sigma \bullet \mu^{\top\top} \bullet T^{\top\top} \eta^{\top\top} = \sigma, \quad \sigma \bullet \mu^{\top\top} \bullet \eta^{\top\top} T^{\top\top} = \sigma, \quad \sigma \bullet \mu^{\top\top} \bullet \mu^{\top\top} T^{\top\top} = \sigma \bullet \mu^{\top\top} \bullet T^{\top\top} \mu^{\top\top}.$$

They are easily shown from the definition of $\eta^{\top\top}$ and $\mu^{\top\top}$. For instance,

$$\begin{aligned} \sigma \bullet \mu^{\top\top} \bullet T^{\top\top} \eta^{\top\top} &= m \bullet \sigma(\mathbf{Ran}_S S) \bullet T^{\top\top} \sigma \bullet T^{\top\top} \eta^{\top\top} = m \bullet \sigma(\mathbf{Ran}_S S) \bullet T^{\top\top} u \\ (\text{interchange law}) &= m \bullet (\mathbf{Ran}_S S) u \bullet \sigma = \sigma. \end{aligned} \quad \square$$

Corollary 3.6. *The cartesian morphism $\sigma : T^{\top\top} \rightarrow \mathbf{Ran}_S S$ is a monad morphism.*

Any lifting of \mathcal{T} along p can be obtained by the codensity lifting, although the choice of the lifting parameter is rather canonical.

Theorem 3.7. *Let $p : \mathbb{E} \rightarrow \mathbb{B}$ be a fibration, \mathcal{T} be a monad on \mathbb{B} and $\dot{\mathcal{T}}$ be a lifting of \mathcal{T} . Then there exists a lifting parameter R, S such that (p, S) satisfies the codensity condition and $\dot{\mathcal{T}} \simeq \mathcal{T}^{\top\top}$.*

Proof. We write $p_k : \mathbb{E}_{\dot{\mathcal{T}}} \rightarrow \mathbb{B}_{\mathcal{T}}$ for the canonical functor extending $p : \mathbb{E} \rightarrow \mathbb{B}$ to Kleisli categories. Then the span $\mathbb{B}_{\mathcal{T}} \xleftarrow{p_k} \mathbb{E}_{\dot{\mathcal{T}}} \xrightarrow{\dot{K}} \mathbb{E}$ is a lifting parameter that satisfies the codensity condition by Proposition 3.4. Since $\dot{T} = \dot{K} \dot{J}$, $(\dot{T}, \dot{K} \dot{\epsilon})$ is a right Kan extension of \dot{K} along \dot{K} , and this is preserved by p . Moreover, the morphism $\overline{K} \epsilon p_k : T p \rightarrow p \dot{T} = T p$ becomes the identity morphism. Hence $\dot{\mathcal{T}}$ is isomorphic to $\mathcal{T}^{\top\top}$. \square

The codensity lifting is given with respect to the Kleisli resolution $J \dashv K : \mathbb{C}_{\mathcal{T}} \rightarrow \mathbb{C}$ of \mathcal{T} . In fact, we can replace it with the Eilenberg-Moore resolution $J' \dashv K' : \mathbb{C}^{\mathcal{T}} \rightarrow \mathbb{C}$ of \mathcal{T} , because the initiality of the Kleisli resolution of \mathcal{T} is irrelevant in the codensity lifting. This replacement affects the argument in this section as follows:

- A lifting parameter becomes a pair $\mathbb{B}_{\mathcal{T}} \xleftarrow{R} \mathbb{A} \xrightarrow{S} \mathbb{E}$ of functors R, S such that $pS = K'R$. The codensity condition remains the same.
- When lifting the components of \mathcal{T} , we replace J with J' , K with K' , and ϵ with the counit ϵ' of $J' \dashv K'$.
- Theorem 3.5 and Corollary 3.6 remains the same.
- In the proof of Theorem 3.7, we use the Eilenberg-Moore resolution $\mathbb{E}^{\dot{\mathcal{T}}}$ of $\dot{\mathcal{T}}$ instead of $\mathbb{E}_{\dot{\mathcal{T}}}$. The remaining part is the same.

At this moment we do not know which resolution is better for the codensity lifting. Theorem 3.7 shows that for deriving any lifting of \mathcal{T} , it is enough to use $\mathbb{C}_{\mathcal{T}}$ -valued lifting parameters; enlarging $\mathbb{C}_{\mathcal{T}}$ to $\mathbb{C}^{\mathcal{T}}$ does not increase the expressiveness of the codensity lifting. In this paper we use the Kleisli resolution in the codensity lifting.

4. EXAMPLES OF CODENSITY LIFTINGS WITH SINGLE LIFTING PARAMETERS

We illustrate codensity liftings of monads with single lifting parameters where 1) the fibration $p : \mathbb{E} \rightarrow \mathbb{B}$ has fibred small products and 2) \mathbb{B} has small products. In this situation, \mathbb{E} also has small products that are preserved by p [Jac99, Exercise 9.2.4], and any single lifting parameter satisfies the codensity condition. We below give a formula to compute the codensity lifting of a monad with a single lifting parameters.

Proposition 4.1. *Let $p : \mathbb{E} \rightarrow \mathbb{B}$ a fibration such that p has fibred small products and \mathbb{B} has small products, and let \mathcal{T} be a monad on \mathbb{B} . Then the functor part of the codensity lifting \mathcal{T}^{\top} of \mathcal{T} with a single lifting parameter $R \in \mathbb{B}$ and $S \in \mathbb{E}_{TR}$ satisfies*

$$T^{\top} X \simeq \bigwedge_{f \in \mathbb{E}(X, S)} ((pf)^{\#})^{-1}(S), \quad (4.1)$$

where \bigwedge stands for the fibred product in $\mathbb{E}_{T(pX)}$.

Proof. We supply the lifting parameter (JR, S) to the codensity lifting (see the convention after Definition 3.2). Let $X \in \mathbb{E}$. We take the power $(\mathbb{E}(X, S) \pitchfork pS, \pi)$ in \mathbb{B} . From [Jac99, Exercise 9.2.4], the object

$$\bigwedge_{f \in \mathbb{E}(X, S)} (\pi_f)^{-1}(S)$$

together with an appropriate projection morphism is a power of S by $\mathbb{E}(X, S)$, and p sends it to the power $(\mathbb{E}(X, S) \pitchfork pS, \pi)$. Therefore the pair $(\mathbb{E}(-, S) \pitchfork pS, \pi_{\text{id}_S})$ is a right Kan extension of pS along S , and the mate function $\overline{(-)} : \mathbb{E}(FS, pS) \rightarrow [\mathbb{E}, \mathbb{B}](F, \mathbb{E}(-, S) \pitchfork pS)$ of this right Kan extension is given by

$$(\overline{f})_X = \langle f \circ Fg \rangle_{g \in \mathbb{E}(X, S)} : FX \rightarrow \mathbb{E}(X, S) \pitchfork pS.$$

From this, we have $(\overline{K \epsilon JR})_X = \langle K \epsilon JR \circ K Jpg \rangle_{g \in \mathbb{E}(X, S)} = \langle (pg)^{\#} \rangle_{g \in \mathbb{E}(X, S)}$. Therefore

$$\begin{aligned} T^{\top} X &= (\langle (pg)^{\#} \rangle_{g \in \mathbb{E}(X, S)})^{-1} \left(\bigwedge_{f \in \mathbb{E}(X, S)} (\pi_f)^{-1}(S) \right) \\ &\simeq \bigwedge_{f \in \mathbb{E}(X, S)} (\langle (pg)^{\#} \rangle_{g \in \mathbb{E}(X, S)})^{-1} (\pi_f)^{-1}(S) \\ &\simeq \bigwedge_{f \in \mathbb{E}(X, S)} (\pi_f \circ \langle (pg)^{\#} \rangle_{g \in \mathbb{E}(X, S)})^{-1}(S) \\ &= \bigwedge_{f \in \mathbb{E}(X, S)} ((pf)^{\#})^{-1}(S). \quad \square \end{aligned}$$

In the rest of this section, we instantiate the parameters of Proposition 4.1 and identify the right hand side of (4.1). All the fibrations appearing in this section have fibred small limits, and are over categories with small limits.

4.1. Lifting Set-Monads to the Category of Preorders. The forgetful functor $p : \mathbf{Pre} \rightarrow \mathbf{Set}$ from the category \mathbf{Pre} of preorders and monotone functions is a fibration with fibred small limits. The inverse image of a preorder (J, \leq_J) along a function $f : I \rightarrow J$ is the preorder (I, \leq_I) given by $i \leq_I i' \iff f(i) \leq_J f(i')$. The fibred small limits are given by the set-theoretic intersections of preorder relations. Although the category \mathbf{Pre} is cartesian closed, p does not preserve exponentials. Hence the categorical $\top\top$ -lifting [Kat05] is not applicable for lifting \mathbf{Set} -monads along p .

We consider the codensity lifting of a monad \mathcal{T} on \mathbf{Set} along $p : \mathbf{Pre} \rightarrow \mathbf{Set}$ with a single lifting parameter: a pair of $R \in \mathbf{Set}$ and $S = (TR, \leq) \in \mathbf{Pre}$. By instantiating (4.1), for every preorder $(X, \leq_X) \in \mathbf{Pre}$ (X for short), $T^\top X$ is the preorder (TX, \leq_X^\top) where \leq_X^\top is given by

$$x \leq_X^\top y \iff \forall f \in \mathbf{Pre}(X, S) . (pf)^\#(x) \leq (pf)^\#(y). \quad (4.2)$$

We further instantiate this by letting \mathcal{T} be the powerset monad \mathcal{T}_p , $R = 1$ and \leq be one of the following partial orders on $T_p 1 = \{\emptyset, 1\}$:

- (1) Case $\leq = \{\emptyset \leq 1\}$. The homset $\mathbf{Pre}(X, S)$ is isomorphic to the set $\mathbf{Up}(X)$ of upward closed subsets of X , and (4.2) is rewritten to:

$$\begin{aligned} x \leq_X^\top y &\iff (\forall F \in \mathbf{Up}(X) . x \cap F \neq \emptyset \implies y \cap F \neq \emptyset) \\ &\iff \forall i \in x . \exists j \in y . i \leq_X j, \end{aligned}$$

that is, \leq_X^\top is the lower preorder.

- (2) Case $\leq = \{1 \leq \emptyset\}$. By the similar argument, \leq^\top is the upper preorder:

$$x \leq_X^\top y \iff \forall j \in y . \exists i \in x . i \leq_X j.$$

In order to make \leq^\top the *convex preorder* on \mathcal{T}_p :

$$x \leq_X^\top y \iff (\forall i \in x . \exists j \in y . i \leq_X j) \wedge (\forall j \in y . \exists i \in x . i \leq_X j),$$

it suffices to supply the cotupling $\mathbf{Set}_{\mathcal{T}_p} \leftarrow 1 + 1 \rightarrow \mathbf{Pre}$ of the above two lifting parameters to the codensity lifting.

4.2. Lifting Set-Monads to the Category of Topological Spaces. The forgetful functor $p : \mathbf{Top} \rightarrow \mathbf{Set}$ from the category \mathbf{Top} of topological spaces and continuous functions is a fibration with fibred small limits. For a topological space (X, \mathcal{O}_X) and a function $f : Y \rightarrow X$, the inverse image topological space $f^{-1}(X, \mathcal{O}_X)$ is given by $(Y, \{f^{-1}(U) \mid U \in \mathcal{O}_X\})$. We note that each fibre category \mathbf{Top}_X on a set X is the poset of topologies on X ordered by the coarseness, that is, $(X, \mathcal{O}_1) \leq (X, \mathcal{O}_2)$ holds if and only if $\mathcal{O}_2 \subseteq \mathcal{O}_1$.

We consider the codensity lifting of a monad \mathcal{T} on \mathbf{Set} along $p : \mathbf{Top} \rightarrow \mathbf{Set}$ with a single lifting parameter: a pair of $R \in \mathbf{Set}$ and $S = (TR, \mathcal{O}_S) \in \mathbf{Top}$. By instantiating (4.1), for every $(X, \mathcal{O}_X) \in \mathbf{Top}$ (X for short), $T^\top X$ is the topological space $(TX, T^\top \mathcal{O}_X)$ whose topology $T^\top \mathcal{O}_X$ is the coarsest one making every set $((pf)^\#)^{-1}(U)$ open, where f and U range over $\mathbf{Top}(X, S)$ and \mathcal{O}_S , respectively.

We further instantiate this by letting $\mathcal{T} = \mathcal{T}_p$, $R = 1$, and \mathcal{O}_S be one of the following topologies on $T_p 1$. The resulting codensity liftings respectively equip $T_p X$ with the same topologies as *lower* and *upper Vietoris topologies*, which appear in the construction of *hyperspace* [Nad78].

- (1) Case $\mathcal{O}_S = \{\emptyset, \{1\}, \{\emptyset, 1\}\}$. The topology $T_p^{\text{TT}} \mathcal{O}_X$ is the coarsest one making every set $\{V \subseteq pX \mid V \cap U \neq \emptyset\}$ open, where U ranges over \mathcal{O}_X . We call this *lower Vietoris lifting*.
- (2) Case $\mathcal{O}_S = \{\emptyset, \{\emptyset\}, \{\emptyset, 1\}\}$. The topology $T_p^{\text{TT}} \mathcal{O}_X$ is the coarsest one making every set $\{V \subseteq pX \mid V \subseteq U\}$ open, where U ranges over \mathcal{O}_X . We call this *upper Vietoris lifting*.

We note that the hyperspace of $X \in \mathbf{Top}$ has closed subsets of X as points, and therefore is *not* a lifting of the powerset monad \mathcal{T}_p .

4.3. Simulations on LMPs by Codensity Lifting. We next move on to the category \mathbf{Meas} of measurable spaces and measurable functions between them. Recall that \mathbf{Meas} has small limits. We introduce some notations: For $X \in \mathbf{Meas}$, by Σ_X we mean the σ -algebra of X . For a topological space $X \in \mathbf{Top}$, by $\mathcal{B}X \in \mathbf{Meas}$ we mean the Borel space associated to X .

Let $p : \mathbf{Pred} \rightarrow \mathbf{Set}$ be the subobject fibration of \mathbf{Set} . We here explicitly give \mathbf{Pred} as follows: an object of \mathbf{Pred} is a pair of sets (X, I) such that X is a subset of I , and a morphism from (X, I) to (Y, J) is a function $f : I \rightarrow J$ such that $f(X) \subseteq Y$. The first and second component of $X \in \mathbf{Pred}$ is denoted by X_0 and X_1 , respectively. The fibration p has fibred small limits. We then consider the following two fibrations q, r obtained by the change-of-base of the subobject fibration $p : \mathbf{Pred} \rightarrow \mathbf{Set}$:

$$\begin{array}{ccccc}
 \mathbf{ERel}(\mathbf{Meas}) & \longrightarrow & \mathbf{BRel}(\mathbf{Meas}) & \longrightarrow & \mathbf{Pred} \\
 r \downarrow \lrcorner & & q \downarrow \lrcorner & & \downarrow p \\
 \mathbf{Meas} & \xrightarrow{\Delta} & \mathbf{Meas}^2 & \xrightarrow{U^2} & \mathbf{Set}^2 \xrightarrow{\text{Prod}} \mathbf{Set}
 \end{array}$$

Here, Δ is the diagonal functor and Prod is the binary product functor. The derived legs q and r are again fibrations with fibred small limits.² An explicit description of $\mathbf{BRel}(\mathbf{Meas})$ is:

- An object X is a triple, whose components are denoted by X_0, X_1, X_2 , such that X_1, X_2 are measurable spaces and $X_0 \subseteq UX_1 \times UX_2$ is a binary relation between the carrier sets of X_1 and X_2 .
- A morphism $(f_1, f_2) : X \rightarrow Y$ is a pair of measurable functions $f_1 : X_1 \rightarrow Y_1$ and $f_2 : X_2 \rightarrow Y_2$ such that $(Uf_1 \times Uf_2)(X_0) \subseteq Y_0$.

An explicit definition of $\mathbf{ERel}(\mathbf{Meas})$ is:

- An object X is a pair, whose components are denoted by X_0, X_1 , such that X_1 is a measurable space and $X_0 \subseteq UX_1 \times UX_1$ is a binary relation on the carrier set of X_1 .
- A morphism $f : X \rightarrow Y$ is a measurable function $f : X_1 \rightarrow Y_1$ such that $Uf_1(X_0) \subseteq Y_0$.

Before proceeding further, we introduce some concepts and notations about binary relations. For a binary relation $R \subseteq X \times Y$ and a subset $A \subseteq X$, the image of A by R is defined to be the set $\{y \in Y \mid \exists x \in A. (x, y) \in R\}$, and is denoted by $R[A]$. For a subset $V \subseteq I$, by $\chi_V : I \rightarrow [0, 1]$ we mean the indicator function defined by: $\chi_V(i) = 1$ when $i \in V$ and $\chi_V(i) = 0$ when $i \notin V$. For two binary relations $P \subseteq I \times J$ and $Q \subseteq J \times K$, by $P; Q \subseteq I \times K$ we mean the relational composition of P followed by Q . We extend this operation to $\mathbf{BRel}(\mathbf{Meas})$ -objects X, Y such that $X_2 = Y_1$ in the evident way.

² \mathbf{BRel} and \mathbf{ERel} stand for binary relations and endo-relations, respectively.

The target of the codensity lifting in this section is the *sub-Giry monad* [Gir82], which we recall below. For a measurable space $X \in \mathbf{Meas}$, by $\mathbf{SPMsr}(X)$ we mean the set of sub-probability measures on X . We equip it with the σ -algebra generated from the sets of the following form:

$$\{\mu \in \mathbf{SPMsr}(X) \mid \mu(V) \in W\} \quad (V \in \Sigma_X, W \in \mathcal{O}_{[0,1]}),$$

and denote this measurable space by GX . The assignment $X \mapsto GX$ can be extended to a monad \mathcal{G} on \mathbf{Meas} , called the *sub-Giry monad* [Gir82]. Notice that $G1$ is the Borel space $\mathcal{B}[0, 1]$ associated to the unit interval $[0, 1]$ with the subspace topology induced from the real line.

4.3.1. *Liftings of Sub-Giry Monad to $\mathbf{BRel}(\mathbf{Meas})$ and $\mathbf{ERel}(\mathbf{Meas})$.* We first consider the codensity lifting of the product sub-Giry monad \mathcal{G}^2 along the fibration $q : \mathbf{BRel}(\mathbf{Meas}) \rightarrow \mathbf{Meas}^2$ with a single lifting parameter $R = (1, 1)$ (the pair of one-point measurable space) and $S = (S_0, G1, G1)$; here S_0 is a binary relation over the unit interval $[0, 1] = U(G1)$. From (4.1), the codensity lifting sends an object $X \in \mathbf{BRel}(\mathbf{Meas})$ to the object $G^\top X \in \mathbf{BRel}(\mathbf{Meas})$, whose relation part is given by

$$(G^\top X)_0 = \left\{ (v_1, v_2) \mid \forall (f, g) \in \mathbf{BRel}(\mathbf{Meas})(X, S). \left(\int_{X_1} f \, dv_1, \int_{X_2} g \, dv_2 \right) \in S_0 \right\}.$$

When the binary relation S_0 satisfies certain closure properties, we can simplify the right hand side of the above equality.

Proposition 4.2. *Suppose that $S = (S_0, G1, G1) \in \mathbf{BRel}(\mathbf{Meas})$ satisfies the following conditions:*

- (1) *For any object $X \in \mathbf{BRel}(\mathbf{Meas})$, morphism $(f, g) : X \rightarrow S$, and $0 \leq \beta \leq 1$, the pair of indicator functions $(\chi_{f^{-1}([\beta, 1])}, \chi_{g^{-1}([\beta, 1])})$ is a morphism from X to S in $\mathbf{BRel}(\mathbf{Meas})$.*
- (2) *The binary relation S_0 is closed under taking convex hulls.*
- (3) *The binary relation S_0 is closed under taking pointwise suprema.*

Then the codensity lifting of \mathcal{G}^2 along $q : \mathbf{BRel}(\mathbf{Meas}) \rightarrow \mathbf{Meas}^2$ with the single lifting parameter $R = (1, 1)$ and S satisfies:

$$(G^\top X)_0 = \left\{ (v_1, v_2) \mid \begin{array}{l} \forall V \in \Sigma_{X_1}, W \in \Sigma_{X_2} . \\ (\chi_V, \chi_W) \in \mathbf{BRel}(\mathbf{Meas})(X, S) \implies (v_1(V), v_2(W)) \in S_0 \end{array} \right\}.$$

Proof. (\supseteq) Obvious. (\subseteq) Let (v_1, v_2) be a pair in the right hand side relation of the above equation. Take an arbitrary pair $(f, g) \in \mathbf{BRel}(\mathbf{Meas})(X, S)$. From the definition of Lebesgue integral we obtain

$$\int_{X_1} f \, dv_1 = \sup \left\{ \sum_{n=0}^N \beta_n v_1(f^{-1}([\sum_{i=0}^n \beta_i, 1])) \mid \sum_{n=0}^N \beta_n = 1, \beta_n > 0 \right\}$$

From the first and second condition

$$\left(\sum_{n=0}^N \beta_n v_1(f^{-1}([\sum_{i=0}^n \beta_i, 1])), \sum_{n=0}^N \beta_n v_2(g^{-1}([\sum_{i=0}^n \beta_i, 1])) \right) \in S_0$$

holds for each $\{\beta_n\}_{n=0}^N$ such that $\sum_{n=0}^N \beta_n = 1$ and $\beta_n > 0$. From the third condition, we conclude $(\int_{X_1} f \, dv_1, \int_{X_2} g \, dv_2) \in S_0$. \square

We next consider the codensity lifting of \mathcal{G} along the fibration $r : \mathbf{ERel}(\mathbf{Meas}) \rightarrow \mathbf{Meas}$ with $R = 1$ and $S = (S_0, G1)$; here S_0 is again a binary relation over $[0, 1]$. By instantiating (4.1), we obtain

$$(G^{\top} X)_0 = \left\{ (v_1, v_2) \mid \forall f \in \mathbf{ERel}(\mathbf{Meas})(X, S) . \left(\int_{X_1} f \, dv_1, \int_{X_1} f \, dv_2 \right) \in S_0 \right\}.$$

The following proposition, which is analogous to Proposition 4.2, holds for the above codensity lifting.

Proposition 4.3. *Suppose that $S = (S_0, G1) \in \mathbf{ERel}(\mathbf{Meas})$ satisfies the following conditions:*

- (1) *For any object $X \in \mathbf{ERel}(\mathbf{Meas})$, morphism $f : X \rightarrow S$ and $0 \leq \beta \leq 1$, the indicator function $\chi_{f^{-1}([\beta, 1])}$ is a morphism from X to S in $\mathbf{ERel}(\mathbf{Meas})$.*
- (2) *The binary relation S_0 is closed under taking convex hulls and pointwise supremums.*

Then the codensity lifting \mathcal{G}^{\top} of \mathcal{G} along $r : \mathbf{ERel}(\mathbf{Meas}) \rightarrow \mathbf{Meas}$ with the single lifting parameter $R = 1$ and $S = (S_0, G1)$ satisfies

$$(G^{\top} X)_0 = \{(v_1, v_2) \mid \forall V \in \Sigma_{X_1} . \chi_V \in \mathbf{ERel}(\mathbf{Meas})(X, S) \implies (v_1(V), v_2(V)) \in S_0\}.$$

4.3.2. Simulations on a Single LMP by Codensity Lifting. We further instantiate $S_0 \subseteq [0, 1]^2$ in Proposition 4.3 with the numerical order \leq . The codensity lifting with this single lifting parameter is simplified as follows:

Proposition 4.4. *The codensity lifting \mathcal{G}^{\top} of \mathcal{G} along $r : \mathbf{ERel}(\mathbf{Meas}) \rightarrow \mathbf{Meas}$ with the single lifting parameter $R = 1$ and $(\leq, G1)$ satisfies:*

$$(v_1, v_2) \in (G^{\top} X)_0 \iff (\forall U \in \Sigma_{X_1} . X_0[U] \subseteq U \implies v_1(U) \leq v_2(U)).$$

Proof. We apply Proposition 4.3: the binary relation $S_0 = \leq$ is obviously closed under taking convex hulls and pointwise supremums, and for any $f \in \mathbf{ERel}(\mathbf{Meas})(X, S)$ and $\beta \in [0, 1]$ we have $\chi_{f^{-1}([\beta, 1])} \in \mathbf{ERel}(\mathbf{Meas})(X, S)$, because $\beta \leq f(x) \implies \beta \leq f(y)$ holds for each $(x, y) \in X_0$. We then conclude the above equivalence because the condition $X_0[U] \subseteq U$ is equivalent to $\chi_U \in \mathbf{ERel}(\mathbf{Meas})(X, S)$. \square

With the above lifting of sub-Giry monad, we can coalgebraically formulate *simulation relations* on a single *labelled Markov process (LMP)* proposed in [vBMOW05]. Fix a set \mathbf{Act} of actions. The following is a coalgebraic definition of LMPs [vBMOW05]:

Definition 4.5. An LMP is an $(\mathbf{Act} \pitchfork G-)$ -coalgebra in \mathbf{Meas} .

We omit the proof of the equivalence between this coalgebraic definition of LMPs and [vBMOW05, Definition 1]. The concept of simulation relation on an LMP is proposed in [vBMOW05]:

Definition 4.6 [vBMOW05, Definition 3]. Let (X, x) be an LMP and $R \subseteq UX \times UX$ be a reflexive relation. We say that R is a *simulation relation* on (X, x) if

$$\forall (s_1, s_2) \in R . \forall a \in \mathbf{Act} . \forall U \in \Sigma_X . R[U] = U \implies \pi_a(x(s_1))(U) \leq \pi_a(x(s_2))(U).$$

In this definition, the formula after “ $\forall U \in \Sigma_X$ ” is similar to the right hand side of the equivalence proved in Proposition 4.4. Actually, as R is assumed to be reflexive, $R[U] = U$ is equivalent to $R[U] \subseteq U$. Then the above formula defining simulation relations can be folded into the existence of a coalgebra in $\mathbf{ERel}(\mathbf{Meas})$:

Theorem 4.7. *Let (X, x) be an LMP and $R \subseteq UX \times UX$ be a reflexive relation. The following are equivalent:*

- (1) R is a simulation relation on (X, x) .
- (2) x is a morphism of type $(R, X) \rightarrow \text{Act} \pitchfork G^\top(R, X)$ in $\mathbf{ERel}(\mathbf{Meas})$, where G^\top is the lifting given in Proposition 4.4.

4.3.3. *Bisimulations between Two LMPs by Codensity Lifting.* We next instantiate $S_0 \subseteq [0, 1]^2$ in Proposition 4.2 with the equality relation $=$ on $[0, 1]$. We first introduce an auxiliary concept, which appears in [BBLM14].

Definition 4.8. Let $R \subseteq I \times J$ be a binary relation and $U \subseteq I$ and $V \subseteq J$ be subsets. We say that the pair (V, W) is R -closed if $\forall(x, y) \in R . (x \in V) \iff (y \in W)$ holds.

Lemma 4.9. *Let $X \in \mathbf{BRel}(\mathbf{Meas})$ be an object and $V \subseteq UX_1$ and $W \subseteq UX_2$ be arbitrary subsets. Then the following are equivalent:*

- (1) (V, W) is X_0 -closed.
- (2) $X_0 \cap (V \times J) = X_0 \cap (I \times W)$.
- (3) $(\chi_V, \chi_W) \in \mathbf{BRel}(\mathbf{Meas})(X, S)$.

Proposition 4.10. *The codensity lifting \mathcal{G}^\top of \mathcal{G}^2 along $q : \mathbf{BRel}(\mathbf{Meas}) \rightarrow \mathbf{Meas}^2$ with the single lifting parameter $R = 1$ and $(=, G1, G1)$ satisfies:*

$$(G^\top X)_0 = \{(v_1, v_2) \mid \forall V \in \Sigma_{X_1}, W \in \Sigma_{X_2} . (V, W) : X_0\text{-closed} \implies v_1(V) = v_2(W)\}.$$

We point out a relationship between the above codensity lifting and the concept of bisimulation relation between two LMPs introduced by Bacci et al [BBLM14].

Definition 4.11 [BBLM14, Definition 5]. Let (X_1, x_1) and (X_2, x_2) be two LMPs. A binary relation $R \subseteq UX_1 \times UX_2$ is a bisimulation relation if the following holds:

$$\begin{aligned} & \forall(s_1, s_2) \in R . \forall a \in \text{Act} . \forall V \in \Sigma_{X_1}, W \in \Sigma_{X_2} . \\ & (V, W) : R\text{-closed} \implies \pi_a(x_1(s_1))(V) = \pi_a(x_2(s_2))(W). \end{aligned}$$

By folding the above defining formula with the characterisation of G^\top given in Theorem 4.10, we obtain the following coalgebraic reformulation of Bacci et al's bisimulation relation:

Theorem 4.12. *Let (X_i, x_i) be LMPs ($i = 1, 2$) and $R \subseteq UX_1 \times UX_2$ be a binary relation. Then the following are equivalent:*

- (1) R is a bisimulation relation between (X_1, x_1) and (X_2, x_2) .
- (2) (x_1, x_2) is a morphism of type $(R, X_1, X_2) \rightarrow \text{Act} \pitchfork G^\top(R, X_1, X_2)$ in $\mathbf{BRel}(\mathbf{Meas})$, where G^\top is the lifting given in Proposition 4.10.

4.3.4. *Codensity Lifting of \mathcal{G}^2 by the Inequality Relation.* From Proposition 4.4 and Theorem 4.7, we naturally speculate that the codensity lifting of the product sub-Giry monad \mathcal{G}^2 using the inequality relation \leq yields the lifting that may be used for the definition of simulation relations between two LMPs. Below we try this, and discuss the problem of the composability of simulation relations.

We instantiate $S_0 \subseteq [0, 1]^2$ in Proposition 4.2 with the numerical order \leq .

Proposition 4.13. *The codensity lifting \mathcal{G}^\top of \mathcal{G}^2 along $q: \mathbf{BRel}(\mathbf{Meas}) \rightarrow \mathbf{Meas}^2$ with the single lifting parameter $R = (1, 1)$ and $S = (\leq, G1, G1)$ satisfies*

$$(G^\top X)_0 = \{(v_1, v_2) \mid (\forall V \in \Sigma_{X_1}, W \in \Sigma_{X_2} . X_0[V] \subseteq W \implies v_1(V) \leq v_2(W))\}$$

Following Theorem 4.7 and Theorem 4.12, we define simulation relations between two LMPs as coalgebras in $\mathbf{BRel}(\mathbf{Meas})$.

Definition 4.14. We define a *simulation relation* from an LMP (X_1, x_1) to an LMP (X_2, x_2) to be a binary relation $R \subseteq UX_1 \times UX_2$ such that (x_1, x_2) is a morphism of type $(R, X_1, X_2) \rightarrow \text{Act} \pitchfork G^\top(R, X_1, X_2)$ in $\mathbf{BRel}(\mathbf{Meas})$, where G^\top is the lifting given in Proposition 4.13. We moreover say that R *preserves measurable sets* if for any measurable set $V \in \Sigma_{X_1}$, we have $R[V] \in \Sigma_{X_2}$.

Unfolding the definition, R is a simulation relation from (X_1, x_1) to (X_2, x_2) if and only if the following holds:

$$\begin{aligned} & \forall (s_1, s_2) \in R . \forall a \in \text{Act} . \forall V \in \Sigma_{X_1}, W \in \Sigma_{X_2} . \\ & R[V] \subseteq W \implies \pi_a(x_1(s_1))(V) \leq \pi_a(x_2(s_2))(W). \end{aligned}$$

However, simulation relations defined as above are *not* closed under the relational composition. A counterexample can be found when $\text{Act} = 1$.

Example 4.15. Let A and B be the discrete and indiscrete spaces over a two-point set $2 = \{0, 1\}$, respectively. We define three probability measures $v_1, v_3 \in GA$ and $v_2 \in GB$ by:

$$v_1(\{0\}) = v_1(\{1\}) = 1/2, \quad v_2(\{0, 1\}) = 1, \quad v_3(\{0\}) = 1/3, \quad v_3(\{1\}) = 2/3.$$

We consider three constant functions $k(v_i)$ from 2 returning v_i for $i = 1, 2, 3$. They are clearly measurable functions of the following type:

$$k(v_1) : A \rightarrow GA, \quad k(v_2) : B \rightarrow GB, \quad k(v_3) : A \rightarrow GA,$$

hence they are LMPs (recall $\text{Act} = 1$). We can then easily check that $\text{Eq}_2 \subseteq 2 \times 2$ is a simulation relation from $k(v_1)$ to $k(v_2)$, and also from $k(v_2)$ to $k(v_3)$. However, Eq_2 is not a simulation relation from $k(v_1)$ to $k(v_3)$.

This problem stems from the fact that \mathcal{G}^\top given in Proposition 4.13 does *not* satisfy the following property (which is seen in the definition of *relators* in [Lev11] and *lax extensions* in [MV15]):

$$\forall X, Y \in \mathbf{BRel}(\mathbf{Meas}) . X_2 = Y_1 \implies (G^\top X; G^\top Y)_0 \subseteq (G^\top(X; Y))_0,$$

This is a sufficient condition for the composability of simulation relations. Example 4.15 is actually constructed using a counterexample to the above property.

A work-around is to require simulation relations to preserve measurable sets.

Proposition 4.16. *Let (X_i, x_i) be LMPs for $i = 1, 2, 3$, and $R_1 \subseteq UX_1 \times UX_2$ and $R_2 \subseteq UX_2 \times UX_3$ be measurable-set preserving simulation relations from (X_1, x_1) to (X_2, x_2) and (X_2, x_2) to (X_3, x_3) , respectively. Then $R_1; R_2$ is a measurable-set preserving simulation relation from (X_1, x_1) to (X_3, x_3) .*

On the other hand, we do not know whether there is a largest measurable-set preserving simulation relation. We leave this point to the future work, and move onto other examples of the codensity lifting.

4.4. Kantorovich Metric by Codensity Lifting. An *extended pseudometric space* (we omit “extended” hereafter) is a pair (X, d) of a set X and a *pseudometric* $d : X \times X \rightarrow [0, \infty]$ taking values in the extended nonnegative real numbers. The pseudometric should satisfy

$$d(x, x) = 0, \quad d(x, y) = d(y, x), \quad d(x, y) + d(y, z) \geq d(x, z).$$

For pseudometric spaces (X, d) and (Y, e) , a function $f : X \rightarrow Y$ is *non-expansive* if for any $x, x' \in X$, $d(x, x') \geq e(f(x), f(x'))$ holds. We define **EPMet** to be the category of pseudometric spaces and non-expansive functions. The forgetful functor $p : \mathbf{EPMet} \rightarrow \mathbf{Set}$ is a fibration with fibred small limits. The inverse image of a pseudometric (Y, d) along a function $f : X \rightarrow Y$ is given by $f^{-1}(Y, d) = (X, d \circ (f \times f))$. The fibred small limit of pseudometric spaces $\{(X, d_i)\}_{i \in I}$ above the same set X is given by the pointwise sup of pseudometrics: $\bigwedge_{i \in I} (X, d_i) = (X, \sup_{i \in I} d_i)$.

We first consider the codensity lifting of a monad \mathcal{T} on **Set** along $p : \mathbf{EPMet} \rightarrow \mathbf{Set}$ with a single lifting parameter: a pair of $R \in \mathbf{Set}$ and $S = (TR, s) \in \mathbf{EPMet}$. By instantiating (4.1), for every $(X, d) \in \mathbf{EPMet}$ (X for short), the pseudometric space $T^\top X$ is of the form $(TX, T^\top d)$ where the pseudometric $T^\top d$ is given by

$$T^\top d(c, c') = \sup_{f \in \mathbf{EPMet}(X, S)} s(f^\#(c), f^\#(c')).$$

We next derive the *Kantorovich metric* [Kan42] on subprobability measures by the codensity lifting. We perform the following change-of-base of the fibration

$$\begin{array}{ccc} U^*(\mathbf{EPMet}) & \longrightarrow & \mathbf{EPMet} \\ q \downarrow \lrcorner & & \downarrow p \\ \mathbf{Meas} & \xrightarrow{U} & \mathbf{Set} \end{array}$$

and obtain a new fibration q with fibred small limits. An object in $U^*(\mathbf{EPMet})$ is a pair of a measurable space (X, Σ_X) and a pseudometric d on X . A morphism from $((X, \Sigma_X), d)$ to $((Y, \Sigma_Y), e)$ in $U^*(\mathbf{EPMet})$ is a measurable function $f : (X, \Sigma_X) \rightarrow (Y, \Sigma_Y)$ that is also non-expansive with respect to pseudometrics d and e .

We consider the codensity lifting of \mathcal{G} along $q : U^*\mathbf{EPMet} \rightarrow \mathbf{Meas}$ with the following single lifting parameter: the pair of $R = 1$ and $S = (G1, s) = (\mathcal{B}[0, 1], s)$, where $s(x, y) = |x - y|$. By instantiating (4.1), for every $(X, d) \in \mathbf{EPMet}$ (X for short), $G^\top X$ is the pair of the measurable space GX and the following pseudometric $G^\top d$ on the set $\mathbf{SPMsr}(X)$ of subprobability measures on X :

$$G^\top d(v_1, v_2) = \sup_f \left| \int_X f dv_1 - \int_X f dv_2 \right|;$$

in the above sup, f ranges over $U^*\mathbf{EPMet}(X, S)$, the set of measurable functions of type $X \rightarrow \mathcal{B}[0, 1]$ that are also non-expansive, that is, $\forall x, y \in UX \cdot d(x, y) \geq |f(x) - f(y)|$. This pseudometric $G^\top d$ between subprobability measures is called the *Kantorovich metric* [Kan42].

5. DENSITY LIFTING OF COMONADS

The categorical dual of the codensity lifting of monads along fibrations is the *density lifting of comonads* along cofibrations.

Fix a cofibration $p : \mathbb{E} \rightarrow \mathbb{B}$ and a comonad $\mathcal{D} = (D, \epsilon, \delta)$ on \mathbb{B} . We take the co-Kleisli resolution $(K \dashv J, \eta)$ of the comonad \mathcal{D} . A *lifting parameter* for \mathcal{D} is a span of functors $\mathbb{B}_{\mathcal{D}} \xleftarrow{R} \mathbb{A} \xrightarrow{S} \mathbb{E}$ such that $pS = KR$. We also assume that (p, S) satisfies the *density condition*: $\mathbf{Lan}_S S$ exists and p preserves it.

Fix a lifting parameter $\mathbb{B}_{\mathcal{D}} \xleftarrow{R} \mathbb{A} \xrightarrow{S} \mathbb{E}$ and assume that (p, S) satisfies the density condition. The density lifting of \mathcal{D} proceeds as follows. From $pS = KR$, we have the following natural transformation:

$$K\eta R : pS = KR \rightarrow KJKR = DKR = DpS.$$

As p preserves the left Kan extension $\mathbf{Lan}_S S$, we obtain a left Kan extension $p(\mathbf{Lan}_S S)$ of pS along S . With this left Kan extension, we take the mate of the above natural transformation, and obtain

$$\underline{K\eta R} : p(\mathbf{Lan}_S S) \rightarrow Dp.$$

In the cofibration $[\mathbb{E}, p] : [\mathbb{E}, \mathbb{E}] \rightarrow [\mathbb{E}, \mathbb{B}]$, we take the co-cartesian lifting of this natural transformation with respect to $\mathbf{Lan}_S S$:

$$\begin{array}{ccc} \mathbf{Lan}_S S & \xrightarrow{\quad\quad\quad} & D^{\top\top} & [\mathbb{E}, \mathbb{E}] \\ & & & \downarrow [\mathbb{E}, p] \\ p(\mathbf{Lan}_S S) & \xrightarrow{\quad\quad\quad} & Dp & [\mathbb{E}, \mathbb{B}] \end{array}$$

It yields an endofunctor $D^{\top\top}$ above Dp , that is, a lifting of the endofunctor D . The counit and multiplication for $D^{\top\top}$ can be dually constructed as done in Section 3.

Let us see the density lifting of a comonad \mathcal{D} on \mathbf{Set} along the subobject cofibration $p : \mathbf{Pred} \rightarrow \mathbf{Set}$ with a single lifting parameter $R \in \mathbf{Set}$ and $S \in \mathbf{Pred}_{DR}$. It yields a comonad $\mathcal{D}^{\top\top}$ whose functor part is given by

$$D^{\top\top} X = (\{(pf)^{\flat}(x) \mid f \in \mathbf{Pred}(S, X), x \in S_0\}, DI) \quad (X \in \mathbf{Pred}); \quad (5.1)$$

here $(-)^{\flat}$ is the co-Kleisli lifting of the comonad \mathcal{D} . Below we instantiate \mathcal{D} with the *product comonad* and the *stream comonad* [UV08].

5.1. Density Lifting of Product Comonad. Fix a set A . We consider the product comonad \mathcal{D}_A on \mathbf{Set} , whose functor part is given by $D_A I = I \times A$. The co-Kleisli lifting of this comonad extends a function $f : D_A I \rightarrow J$ to the function $f^{\flat} : D_A I \rightarrow D_A J$ given by $f^{\flat}(i, a) = (f(i, a), a)$.

We instantiate the density lifting (5.1) with the product comonad, and obtain

$$D_A^{\top\top} X = (\{(f(i, a), a) \mid f \in \mathbf{Pred}(S, X), (i, a) \in S_0\}, X_1 \times A).$$

This density lifting is actually a product comonad on \mathbf{Pred} .

Theorem 5.1. *Let $R \in \mathbf{Set}$ and $S \in \mathbf{Pred}_{R \times A}$ be a single lifting parameter for the product comonad \mathcal{D}_A . Then the density lifting of \mathcal{D}_A satisfies*

$$D_A^{\top\top} X = X \dot{\times} (S_0[R], A) \quad (X \in \mathbf{Pred}),$$

where $\dot{\times}$ is the binary product in **Pred** given by $(X, I) \dot{\times} (Y, J) = (X \times Y, I \times J)$.

Proof. (\subseteq) easy. (\supseteq) Let $(i, a) \in X \times S_0[R]$. There exists $r \in R$ such that $(r, a) \in S_0$. We take the constant function $k_i : D_A R \rightarrow I$ returning i . This belongs to **Pred**(S, X) as $i \in X_0$. Then we obtain $(k_i(r, a), a) = (i, a) \in (D_A^\top X)_0$. \square

5.2. Density Lifting of Stream Comonad. We next consider the stream comonad \mathcal{D} on **Set**. Its functor part sends a set I to the function space $\mathbb{N} \Rightarrow I$ from the set \mathbb{N} of natural numbers. We regard functions in the space as infinite sequences of elements in I . For an infinite sequence $x \in \mathbb{N} \Rightarrow I$ and a natural number $i \in \mathbb{N}$, by x/i we mean the infinite sequence x_i, x_{i+1}, \dots , that is, $x/i = \lambda j . x(i+j)$. The counit and the comultiplication of the stream comonad are given by

$$\epsilon_I(l) = l(0), \quad \delta_I(l)(m) = l/m.$$

We instantiate the density lifting (5.1) with the stream comonad, and obtain

$$D^\top X = (\{\lambda m . f(s/m) \mid f \in \mathbf{Pred}(S, X), s \in S_0\}, \mathbb{N} \Rightarrow X_1) \quad (X \in \mathbf{Pred}).$$

We note that $D^\top(\emptyset, I) = (\emptyset, DI)$.

Theorem 5.2. *Let $R \in \mathbf{Set}$ and $S \in \mathbf{Pred}_{\mathbb{N} \Rightarrow R}$ be a single lifting parameter for the stream comonad \mathcal{D} . For any $X \in \mathbf{Pred}$, we have the following equivalence:*

$$x \in (D^\top X)_0 \iff \exists v \in S_0 . (\forall i \in \mathbb{N} . v/i \in S_0 \implies x(i) \in X_0) \wedge \\ (\forall m, n \in \mathbb{N} . v/n = v/m \implies x(n) = x(m)).$$

Proof. It is easy to check that this equivalence holds when $X_0 = \emptyset$. We thus show this equivalence under the assumption that $X_0 \neq \emptyset$.

(\implies) Take $f \in \mathbf{Pred}(S, X)$ and $s \in S_0$ and assume $x = \lambda m . f(s/m)$. We show that s satisfies the two conditions on the right hand side:

- (1) Let $i \in \mathbb{N}$ and assume $s/i \in S_0$. From $f \in \mathbf{Pred}(S, X)$, we have $f(s/i) = x(i) \in X_0$.
- (2) Let $m, n \in \mathbb{N}$ and assume $s/n = s/m$. Then $x(n) = f(s/n) = f(s/m) = x(m)$.

(\impliedby) Let $x \in DI$ and $v \in S_0$. We consider the following binary relation $F \subseteq DR \times I$:

$$F = \{(v/i, x(i)) \mid i \in \mathbb{N}\}.$$

From the condition $\forall n, m \in \mathbb{N} . v/n = v/m \implies x(n) = x(m)$, this binary relation is actually a (graph of a) partial function from DR to I . Moreover, for any $a \in S_0$, if $F(a)$ is defined, then $F(a) \in X_0$, because of the condition $\forall i \in \mathbb{N} . v/i \in S_0 \implies x(i) \in X_0$. We now pick an element $y \in X_0$, and extend F to a total function $F' : DR \rightarrow I$ such that $F'(a) = y$ when a is not in the domain of F . Clearly $F' \in \mathbf{Pred}(S, X)$. Now for any $m \in \mathbb{N}$, we have $F'(v/m) = x(m)$. Hence $x \in (D^\top X)_0$. \square

6. LIFTING ALGEBRAIC OPERATIONS TO CODENSITY-LIFTED MONADS

We introduce the concept of *algebraic operation* [PP03] for general monads, and discuss their liftings to codensity liftings of monads. The following definition is a modification of [PP03, Proposition 2] for non-strong monads, and coincides with the original one when $\mathbb{C} = \mathbf{Set}$.

Definition 6.1. Let \mathbb{C} be a category, \mathcal{T} be a monad on \mathbb{C} , A be a set and assume that \mathbb{C} has powers by A . An *A-ary algebraic operation* for \mathcal{T} is a natural transformation $\alpha : A \pitchfork K \rightarrow K$, where K is the right adjoint of the Kleisli resolution of \mathcal{T} . We write $\mathbf{Alg}(\mathcal{T}, A)$ for the class of A -ary algebraic operations for \mathcal{T} .

Example 6.2. For each set A , the powerset monad \mathcal{T}_p has the algebraic operation of *A-ary set-union* $\text{union}_X^A : A \pitchfork T_p X \rightarrow T_p X$ given by $\text{union}_X^A(V) = \bigcup_{a \in A} V_a$.

Fix a fibration $p : \mathbb{E} \rightarrow \mathbb{B}$, a monad \mathcal{T} on \mathbb{B} , a set A and assume that \mathbb{E} has and p preserves powers by A .

Definition 6.3. Let $\dot{\mathcal{T}}$ be a lifting of \mathcal{T} along p and $\alpha \in \mathbf{Alg}(\mathcal{T}, A)$ be an A -ary algebraic operation for \mathcal{T} . A *lifting* of α to $\dot{\mathcal{T}}$ is an algebraic operation $\dot{\alpha} \in \mathbf{Alg}(\dot{\mathcal{T}}, A)$ such that $p\dot{\alpha} = \alpha p_k$; here $p_k : \mathbb{E}_{\dot{\mathcal{T}}} \rightarrow \mathbb{B}_{\mathcal{T}}$ is the canonical extension of p to Kleisli categories. We write $\mathbf{Alg}_{\alpha}(\dot{\mathcal{T}}, A)$ for the class $\{\dot{\alpha} \in \mathbf{Alg}(\dot{\mathcal{T}}, A) \mid p\dot{\alpha} = \alpha p_k\}$ of liftings of α to $\dot{\mathcal{T}}$.

Example 6.4 (Continued from Example 6.2). Let $\dot{\mathcal{T}}$ be a lifting of \mathcal{T}_p along $p : \mathbf{Top} \rightarrow \mathbf{Set}$. Since p is faithful, there is at most one lifting of union^A to $\dot{\mathcal{T}}$. It exists if and only if for every $(X, \mathcal{O}_X) \in \mathbf{Top}$, union_X^A is a continuous function of type $A \pitchfork \dot{T}(X, \mathcal{O}_X) \rightarrow \dot{T}(X, \mathcal{O}_X)$.

We give a characterisation of the liftings of algebraic operations to codensity liftings of monads. Fix a lifting parameter $\mathbb{B}_{\mathcal{T}} \xleftarrow{R} \mathbb{A} \xrightarrow{S} \mathbb{E}$ and assume that (p, S) satisfies the codensity condition. We perform the codensity lifting of \mathcal{T} along p with the lifting parameter (R, S) , and consider the Kleisli resolution of \mathcal{T}^{TT} . The functor $p : \mathbb{E} \rightarrow \mathbb{B}$ extends to $p_k : \mathbb{E}_{\mathcal{T}^{\text{TT}}} \rightarrow \mathbb{B}_{\mathcal{T}}$, and it satisfies

$$p_k J^{\text{TT}} = Jp, \quad p_k K^{\text{TT}} = Kp_k, \quad p\eta^{\text{TT}} = \eta p, \quad p_k \epsilon^{\text{TT}} = \epsilon p_k.$$

Starting from a natural transformation $\alpha_0 : A \pitchfork S \rightarrow S$ such that $p\alpha_0 = \alpha R$, we construct a lifting $\phi(\alpha_0) \in \mathbf{Alg}_{\alpha}(\mathcal{T}^{\text{TT}}, A)$ of α as follows.

From $A \pitchfork S = (A \pitchfork \text{Id}_{\mathbb{E}})S$, the natural transformation α_0 induces the mate $\overline{\alpha_0} : A \pitchfork \text{Id}_{\mathbb{E}} \rightarrow \mathbf{Ran}_S S$. We then obtain the following situation:

$$\begin{array}{ccc}
 A \pitchfork \text{Id}_{\mathbb{E}} & \xrightarrow{\overline{\alpha_0}} & \mathbf{Ran}_S S \\
 \downarrow \beta & & \downarrow \sigma \\
 T^{\text{TT}} & \xrightarrow{\sigma} & \mathbf{Ran}_S S
 \end{array}
 \qquad
 \begin{array}{c}
 [\mathbb{E}, \mathbb{E}] \\
 \downarrow [\mathbb{E}, p] \\
 [\mathbb{E}, \mathbb{B}]
 \end{array}$$

$$\begin{array}{ccc}
 A \pitchfork p & \xrightarrow{\overline{\alpha R}} & p(\mathbf{Ran}_S S) \\
 \downarrow \alpha Jp \bullet A \pitchfork \eta p & & \downarrow K \epsilon R \\
 Tp & \xrightarrow{K \epsilon R} & p(\mathbf{Ran}_S S)
 \end{array}$$

The triangle in the base category commutes by:

$$\begin{aligned} \overline{K\epsilon R} \bullet \alpha J p \bullet A \dashv \eta p &= \overline{K\epsilon R \bullet \alpha J p S \bullet A \dashv \eta p S} = \overline{K\epsilon R \bullet \alpha J K R \bullet A \dashv \eta K R} \\ &= \overline{(K\epsilon \bullet \alpha J K \bullet A \dashv \eta K) R} = \overline{(\alpha \bullet A \dashv K\epsilon \bullet A \dashv \eta K) R} = p\overline{\alpha_0}. \end{aligned}$$

We thus obtain the unique morphism β above $\alpha J p \bullet A \dashv \eta p$ making the triangle in the total category commute. Using this β , we define $\phi(\alpha_0) : A \dashv K^{\top\top} \rightarrow K^{\top\top}$ by

$$\phi(\alpha_0) = K^{\top\top} \epsilon^{\top\top} \bullet \beta K^{\top\top} : A \dashv K^{\top\top} \rightarrow K^{\top\top}.$$

This algebraic operation is a lifting of α to $T^{\top\top}$:

$$p\phi(\alpha_0) = p(K^{\top\top} \epsilon^{\top\top} \bullet \beta K^{\top\top}) = (K\epsilon \bullet \alpha J K \bullet A \dashv \eta K)p_k = (\alpha \bullet A \dashv K\epsilon \bullet A \dashv \eta K)p_k = \alpha p_k.$$

The following theorem shows that ϕ characterises the class of liftings of α to the codensity liftings of monads. It is an analogue of Theorem 11 in [Kat13], which is stated for the categorical $\top\top$ -lifting.

Theorem 6.5. *Let $p : \mathbb{E} \rightarrow \mathbb{B}$ be a fibration, \mathcal{T} be a monad on \mathbb{B} , and $\mathbb{B}_{\mathcal{T}} \xleftarrow{R} \mathbb{A} \xrightarrow{S} \mathbb{E}$ be a lifting parameter, and A be a set. Suppose that (p, S) satisfies the codensity condition, and \mathbb{E} has, and p preserves powers by A . Then for any $\alpha \in \mathbf{Alg}(\mathcal{T}, A)$, the mapping ϕ constructed as above has the following type and is bijective:*

$$\phi : [\mathbb{A}, \mathbb{E}]_{\alpha R}(A \dashv S, S) \rightarrow \mathbf{Alg}_{\alpha}(\mathcal{T}^{\top\top}, A).$$

Proof. The candidate ψ of the inverse of ϕ is explicitly given as follows: it maps $\dot{\alpha} \in \mathbf{Alg}_{\alpha}(\mathcal{T}^{\top\top}, A)$ to a morphism of type $A \dashv S \rightarrow S$ by the following mate:

$$\begin{aligned} \sigma \bullet \dot{\alpha} J^{\top\top} \bullet A \dashv \eta^{\top\top} &: A \dashv \text{Id}_{\mathbb{E}} \rightarrow \mathbf{Ran}_S S \\ \psi(\dot{\alpha}) &= \underline{\sigma \bullet \dot{\alpha} J^{\top\top} \bullet A \dashv \eta^{\top\top}} : A \dashv S \rightarrow S \end{aligned}$$

We first show $\psi \circ \phi = \text{id}$:

$$\psi(\phi(\alpha_0)) = \underline{\sigma \bullet \mu^{\top\top} \bullet \beta T^{\top\top} \bullet A \dashv \eta^{\top\top}} = \underline{\sigma \bullet \mu^{\top\top} \bullet T^{\top\top} \eta^{\top\top} \bullet \beta} = \alpha_0.$$

We next show $\phi \circ \psi = \text{id}$. By definition, $\phi(\psi(\dot{\alpha})) = K^{\top\top} \epsilon^{\top\top} \bullet \beta K^{\top\top}$ where β is the unique morphism above $(\alpha J \bullet A \dashv \eta)p$ such that $\sigma \bullet \beta = \psi(\dot{\alpha}) = \sigma \bullet \dot{\alpha} J^{\top\top} \bullet A \dashv \eta^{\top\top}$. The morphism $\dot{\alpha} J^{\top\top} \bullet A \dashv \eta^{\top\top}$ is exactly such one. Therefore

$$\begin{aligned} \phi(\psi(\dot{\alpha})) &= K^{\top\top} \epsilon^{\top\top} \bullet (\dot{\alpha} J^{\top\top} \bullet A \dashv \eta^{\top\top}) K^{\top\top} = K^{\top\top} \epsilon^{\top\top} \bullet \dot{\alpha} J^{\top\top} K^{\top\top} \bullet A \dashv \eta^{\top\top} K^{\top\top} \\ &= \dot{\alpha} \bullet A \dashv K^{\top\top} \epsilon^{\top\top} \bullet A \dashv \eta^{\top\top} K^{\top\top} = \dot{\alpha}. \end{aligned} \quad \square$$

Example 6.6 (Continued from Example 6.4). We look at liftings of $\text{union}^A \in \mathbf{Alg}(\mathcal{T}_p, A)$ to the codensity liftings of \mathcal{T}_p along $p : \mathbf{Top} \rightarrow \mathbf{Set}$ with some single lifting parameters.

Let $R \in \mathbf{Set}$ and $S = (T_p R, \mathcal{O}_S) \in \mathbf{Top}$ be a single lifting parameter. Theorem 6.5 is instantiated to the following statement: a lifting of union^A to $\mathcal{T}_p^{\top\top}$ exists if and only if $\text{union}_R^A : A \dashv T_p R \rightarrow T_p R$ is a continuous function of type $A \dashv S \rightarrow S$. Here, $A \dashv S$ is the product of A -fold copies of S , and its topology $\mathcal{O}_{A \dashv S}$ is generated from all the sets of the form $\pi_a^{-1}(U)$, where a and U range over A and \mathcal{O}_S , respectively. We further instantiate the single lifting parameter (R, S) as follows (see Section 4.2):

- (1) Case $R = 1, \mathcal{O}_S = \{\emptyset, \{1\}, \{\emptyset, 1\}\}$. For any set A , union_1^A is a continuous function of type $A \dashv S \rightarrow S$ because $(\text{union}_1^A)^{-1}(\{1\}) = \bigcup_{a \in A} \pi_a^{-1}(\{1\}) \in \mathcal{O}_{A \dashv S}$. From Theorem 6.5, for any set A , union^A lifts to the lower Vietoris lifting $\mathcal{T}_p^{\top\top}$.

- (2) Case $R = 1, \mathcal{O}_S = \{\emptyset, \{\emptyset\}, \{\emptyset, 1\}\}$. For any finite set A , union_1^A is a continuous function of type $A \pitchfork S \rightarrow S$ because $(\text{union}_1^A)^{-1}(\{\emptyset\}) = \bigcap_{a \in A} \pi_a^{-1}(\{\emptyset\}) \overset{*}{\in} \mathcal{O}_{A \pitchfork S}$. On the other hand, the membership $\overset{*}{\in}$ does not hold when A is infinite. From Theorem 6.5, for any set A , union^A lifts to the upper Vietoris lifting $\mathcal{T}_p^{\text{VI}}$ if and only if A is finite.

7. POINTWISE CODENSITY LIFTING

Fix a fibration $p : \mathbb{E} \rightarrow \mathbb{B}$, a monad \mathcal{T} on \mathbb{B} and a lifting parameter $\mathbb{B}_{\mathcal{T}} \xleftarrow{R} \mathbb{A} \xrightarrow{S} \mathbb{E}$. When \mathbb{A} is a large category, or \mathbb{B}, \mathbb{E} are not very complete, the right Kan extension $\mathbf{Ran}_S S$ may not exist, hence the codensity lifting in Section 3 is not applicable to lift \mathcal{T} . In this section we introduce an alternative method, called *pointwise codensity lifting*, that relies on fibred limits of p . The trick is to swap the order of computation: instead of taking the inverse image after computing $\mathbf{Ran}_S S$, we first take the inverse image of the components of $\mathbf{Ran}_S S$, bringing everything inside a fibre, then compute the right Kan extension as a fibred limit.

We assume that \mathbb{A} is small (resp. large) and p has fibred small (resp. large) limits. The pointwise codensity lifting lifts \mathcal{T} as follows. What we actually construct below is a Kleisli triple over \mathbb{E} which corresponds to a lifting of \mathcal{T} .

Lifting Object Assignment. We first lift T to an object mapping $\dot{T} : |\mathbb{E}| \rightarrow |\mathbb{E}|$. Let $X \in \mathbb{E}$. Consider the following diagram:

$$\begin{array}{ccccccc}
 X \downarrow S & \xrightarrow{\pi_X} & \mathbb{A} & \xrightarrow{R} & \mathbb{B}_{\mathcal{T}} & \xlongequal{\quad} & \mathbb{B}_{\mathcal{T}} \\
 \downarrow !_{X \downarrow S} & \Rightarrow \gamma_X & \downarrow S & & \downarrow K & \Rightarrow \epsilon & \downarrow K \\
 1 & \xrightarrow{X} & \mathbb{E} & \xrightarrow{p} & \mathbb{B} & \xrightarrow{T} & \mathbb{B} \\
 & & & & & \nearrow J & \\
 & & & & & \searrow &
 \end{array}$$

where $(X \downarrow S, \pi_X, !_{X \downarrow S}, \gamma_X)$ is the *comma category*. The middle square commutes as R, S is a lifting parameter. We let $\delta_X = K \epsilon R \pi_X \bullet T p \gamma_X$ be the composite natural transformation, and take the inverse image of $S \pi_X$ along δ_X :

$$\begin{array}{ccc}
 \delta_X^{-1}(S \pi_X) & \xrightarrow{\overline{\delta_X}(S \pi_X)} & S \pi_X & [X \downarrow S, \mathbb{E}] \\
 & & & \downarrow [X \downarrow S, p] \\
 T p X !_{X \downarrow S} & \xrightarrow{\delta_X} & K R \pi_X & [X \downarrow S, \mathbb{B}]
 \end{array}$$

We obtain a functor $\delta_X^{-1}(S \pi_X) : X \downarrow S \rightarrow \mathbb{E}$ such that $p \delta_X^{-1}(S \pi_X) = T p X !_{X \downarrow S}$. We then define $T^{\text{VI}} X$ by $T^{\text{VI}} X = \lim(\delta_X^{-1}(S \pi_X))$, where right hand side is the fibred limit. In the following calculations we will use the vertical projection and the tupling operation of this fibred limit, denoted by

$$\begin{aligned}
 P_X &: (T^{\text{VI}} X) !_{X \downarrow S} \rightarrow \delta_X^{-1}(S \pi_X), \\
 \langle - \rangle &: [X \downarrow S, \mathbb{E}]_{f !_{X \downarrow S}}(Y !_{X \downarrow S}, \delta_X^{-1}(S \pi_X)) \rightarrow \mathbb{E}_f(Y, T^{\text{VI}} X) \quad (f \in \mathbb{E}(Y, T p X)).
 \end{aligned}$$

Lifting the Unit. We next lift η . Consider the following diagram:

$$\begin{array}{ccc}
 X!_{X \downarrow S} & \xrightarrow{\gamma_X} & S\pi_X \\
 \text{\scriptsize η'_X} \swarrow & & \downarrow \text{\scriptsize $\overline{\delta}_X(S\pi_X)$} \\
 \delta_X^{-1}(S\pi_X) & \xrightarrow{\quad} & S\pi_X \\
 & & [X \downarrow S, \mathbb{E}] \\
 & & \downarrow \text{\scriptsize $[X \downarrow S, p]$} \\
 & & [X \downarrow S, \mathbb{B}]
 \end{array}$$

$$\begin{array}{ccc}
 pX!_{X \downarrow S} & \xrightarrow{p\gamma_X} & KR\pi_X \\
 \text{\scriptsize $\eta pX!_{X \downarrow S}$} \swarrow & & \downarrow \text{\scriptsize δ_X} \\
 TpX!_{X \downarrow S} & \xrightarrow{\quad} & KR\pi_X \\
 & & [X \downarrow S, \mathbb{B}]
 \end{array}$$

where the lower triangle commute by:

$$\delta_X \bullet \eta pX!_{X \downarrow S} = K\epsilon R\pi_X \bullet \eta pS\pi_X \bullet p\gamma_X = K\epsilon R\pi_X \bullet \eta KR\pi_X \bullet p\gamma_X = p\gamma_X.$$

Therefore there exists the unique natural transformation η'_X above $\eta pX!_{X \downarrow S}$ making the upper triangle commute. We define $\eta_X^{\top} = \langle \eta'_X \rangle$, which is above ηpX .

Lifting the Kleisli lifting. We finally lift the Kleisli lifting $(-)^{\#}$ of \mathcal{T} . Let $g : X \rightarrow T^{\top}Y$ be a morphism in \mathbb{E} , and $f = P_Y \bullet g!_{Y \downarrow S} : X!_{Y \downarrow S} \rightarrow \delta_Y^{-1}(S\pi_Y)$ be a morphism, which is above $pg!_{Y \downarrow S}$ and satisfies $g = \langle f \rangle$. We obtain the composite natural transformation $\overline{\delta}_Y(S\pi_Y) \bullet f : X!_{Y \downarrow S} \rightarrow \delta_Y^{-1}(S\pi_Y) \rightarrow S\pi_Y$. From the universal property of the comma category, we obtain the unique functor $M_f : Y \downarrow S \rightarrow X \downarrow S$ such that $\pi_X M_f = \pi_Y$ and $\gamma_X M_f = \overline{\delta}_Y(S\pi_Y) \bullet f$. We next consider the following diagram:

$$\begin{array}{ccc}
 \delta_X^{-1}(S\pi_X)M_f & \xrightarrow{\overline{\delta}_X(S\pi_X)M_f} & S\pi_Y \\
 \text{\scriptsize f^b} \swarrow & & \downarrow \text{\scriptsize $\overline{\delta}_Y(S\pi_Y)$} \\
 \delta_Y^{-1}(S\pi_Y) & \xrightarrow{\quad} & S\pi_Y \\
 & & [Y \downarrow S, \mathbb{E}] \\
 & & \downarrow \text{\scriptsize $[Y \downarrow S, p]$} \\
 & & [Y \downarrow S, \mathbb{B}]
 \end{array}$$

$$\begin{array}{ccc}
 TpX!_{Y \downarrow S} & \xrightarrow{\delta_X M_f} & KR\pi_Y \\
 \text{\scriptsize $\mu pY!_{Y \downarrow S} \bullet Tpf$} \swarrow & & \downarrow \text{\scriptsize δ_Y} \\
 TpY!_{Y \downarrow S} & \xrightarrow{\quad} & KR\pi_Y \\
 & & [Y \downarrow S, \mathbb{B}]
 \end{array}$$

where the lower triangle commutes. Therefore there exists the unique natural transformation f^b above $\mu pY!_{Y \downarrow S} \bullet Tpf = \mu pY!_{Y \downarrow S} \bullet Tpg!_{Y \downarrow S} = (pg)^{\#}!_{Y \downarrow S}$ making the upper triangle commute. Then we define $g^{\# \top} = \langle f^b \bullet P_X M_f \rangle$, which is above $(pg)^{\#}$.

Theorem 7.1. *Let $p : \mathbb{E} \rightarrow \mathbb{B}$ be a fibration with fibred small (resp. large) limits, \mathcal{T} be a monad on \mathbb{B} , $\mathbb{B}_{\mathcal{T}} \xleftarrow{R} \mathbb{A} \xrightarrow{S} \mathbb{E}$ be a lifting parameter for \mathcal{T} and assume that \mathbb{A} is small (resp. large). The tuple $(T^{\top}, \eta^{\top}, (-)^{\# \top})$ constructed as above is a Kleisli triple on \mathbb{E} , and the corresponding monad is a lifting of \mathcal{T} .*

Proof. We first show $(\eta_X^{\top})^\# = \langle \eta'_X \rangle^\# = \text{id}$. The composite natural transformation is $\overline{\delta_X}(S\pi_X) \bullet \eta'_X = \gamma_X$ by definition. Therefore $M_f = \text{Id}_{X \downarrow S}$. Hence f^b is also the identity morphism. Therefore the above composite is also the identity morphism.

We next show $f^\# \circ \eta_X^{\top} = f$:

$$\langle f^b \bullet PM_f \rangle \circ \eta_X^{\top} = \langle f^b \bullet PM_f \bullet \langle \eta'_X \rangle!_{Y \downarrow S} \rangle = \langle f^b \bullet PM_f \bullet \langle \eta'_X \rangle!_{X \downarrow S} M_f \rangle = \langle f^b \bullet \eta'_X M_f \rangle \stackrel{*}{=} \langle f \rangle.$$

The last equation $\stackrel{*}{=}$ holds because the morphisms on both sides are above the same morphism:

$$\begin{aligned} p(f^b \bullet \eta'_X M_f) &= p f^b \bullet p \eta'_X M_f = \mu p Y!_{Y \downarrow S} \bullet T p f \bullet \eta p X!_{X \downarrow S} M_f \\ &= \mu p Y!_{Y \downarrow S} \bullet T p f \bullet \eta p X!_{Y \downarrow S} = \mu p Y!_{Y \downarrow S} \bullet \eta p \delta_Y^{-1}(S\pi_Y) \bullet p f \\ &= \mu p Y!_{Y \downarrow S} \bullet \eta T p Y!_{Y \downarrow S} \bullet p f = p f \end{aligned}$$

and are equalised by the cartesian morphism $\overline{\delta_Y}(S\pi_Y) : \delta_Y^{-1}(S\pi_Y) \rightarrow S\pi_Y$:

$$\overline{\delta_Y}(S\pi_Y) \bullet f^b \bullet \eta'_X M_f = \overline{\delta_X}(S\pi_X) M_f \bullet \eta'_X M_f = \gamma_X M_f = \overline{\delta_Y}(S\pi_Y) \bullet f.$$

We finally show $(\langle g \rangle^\# \circ \langle f \rangle^\#)^\# = \langle g \rangle^\# \circ \langle f \rangle^\#$ for $\langle f \rangle : X \rightarrow T^{\top} Y$ and $\langle g \rangle : Y \rightarrow T^{\top} Z$. Let $h = g^b \bullet f M_g$.

(1) We show $M_f M_g = M_h$. From

$$\pi_X M_h = \pi_Z = \pi_Y M_g = \pi_X M_f M_g$$

and

$$\gamma_X M_h = \overline{\delta_Z}(S\pi_Z) \bullet g^b \bullet f M_g = \overline{\delta_Y}(S\pi_Y) M_g \bullet f M_g = (\overline{\delta_Y}(S\pi_Y) \bullet f) M_g = \gamma_X M_f M_g,$$

the universal property of the comma object makes $M_h = M_f M_g$.

(2) We show $h^b = g^b \bullet f^b M_g$. First the following calculation shows $ph^b = p(g^b \bullet f^b M_g)$:

$$\begin{aligned} ph^b &= \mu p Z!_{Z \downarrow S} \bullet T p h \\ &= \mu p Z!_{Z \downarrow S} \bullet T p g^b \bullet T p f M_g \\ &= \mu p Z!_{Z \downarrow S} \bullet T(\mu p Z!_{Z \downarrow S} \bullet T p g) \bullet T p f M_g \\ &= \mu p Z!_{Z \downarrow S} \bullet T \mu p Z!_{Z \downarrow S} \bullet T^2 p g \bullet T p f M_g \\ &= \mu p Z!_{Z \downarrow S} \bullet \mu T p Z!_{Z \downarrow S} \bullet T^2 p g \bullet T p f M_g \\ &= \mu p Z!_{Z \downarrow S} \bullet T p g \bullet \mu p Y!_{Z \downarrow S} \bullet T p f M_g \\ &= \mu p Z!_{Z \downarrow S} \bullet T p g \bullet \mu p Y!_{Y \downarrow S} M_g \bullet T p f M_g. \\ p(g^b \bullet f^b M_g) &= p g^b \bullet p f^b M_g \\ &= \mu p Z!_{Z \downarrow S} \bullet T p g \bullet (\mu p Y!_{Y \downarrow S} \bullet T p f) M_g \\ &= \mu p Z!_{Z \downarrow S} \bullet T p g \bullet \mu p Y!_{Y \downarrow S} M_g \bullet T p f M_g. \end{aligned}$$

Second, the cartesian morphism $\overline{\delta_Z}(S\pi_Z)$ equalise h^b and $g^b \bullet f^b M_g$:

$$\begin{aligned} \overline{\delta_Z}(S\pi_Z) \bullet h^b &= \overline{\delta_X}(S\pi_X) M_h \\ \overline{\delta_Z}(S\pi_Z) \bullet g^b \bullet f^b M_g &= \overline{\delta_Y}(S\pi_Y) M_g \bullet f^b M_g \\ &= (\overline{\delta_Y}(S\pi_Y) \bullet f^b) M_g \\ &= \overline{\delta_X}(S\pi_X) M_f M_g \\ &= \overline{\delta_X}(S\pi_X) M_h. \end{aligned}$$

Therefore $ph^b = p(g^b \bullet f^b M_g)$.

(3) Finally, we show $\langle g \rangle^\# \circ \langle f \rangle^\# = (\langle g \rangle^\# \circ \langle f \rangle)^\#$.

$$\begin{aligned}
 \langle g \rangle^\# \circ \langle f \rangle^\# &= \langle g^b \bullet P_Y M_g \rangle \circ \langle f^b \bullet P_X M_f \rangle \\
 &= \langle g^b \bullet P_Y M_g \bullet \langle f^b \bullet P_X M_f \rangle!_{Z \downarrow S} \rangle \\
 &= \langle g^b \bullet P_Y M_g \bullet \langle f^b \bullet P_X M_f \rangle!_{Y \downarrow S} M_g \rangle \\
 &= \langle g^b \bullet f^b M_g \bullet P_X M_f M_g \rangle \\
 &= \langle h^b \bullet P_X M_h \rangle \\
 (\langle g \rangle^\# \circ \langle f \rangle)^\# &= (\langle g^b \bullet P_Y M_g \rangle \circ \langle f \rangle)^\# \\
 &= (\langle g^b \bullet P_Y M_g \bullet \langle f \rangle!_{Z \downarrow S} \rangle)^\# \\
 &= (\langle g^b \bullet P_Y M_g \bullet \langle f \rangle!_{Y \downarrow S} M_g \rangle)^\# \\
 &= (\langle h \rangle)^\# \\
 &= \langle h^b \bullet P_X M_h \rangle. \quad \square
 \end{aligned}$$

The pointwise codensity lifting coincides with the codensity lifting in Section 3, provided that $\mathbf{Ran}_S S$ and $p(\mathbf{Ran}_S S)$ are both pointwise.

Theorem 7.2. *Let $p : \mathbb{E} \rightarrow \mathbb{B}$ be a fibration, \mathcal{T} be a monad on \mathbb{B} and $\mathbb{B}_{\mathcal{T}} \xleftarrow{R} \mathbb{A} \xrightarrow{S} \mathbb{E}$ be a lifting parameter. Assume that p, S satisfies the codensity condition, and moreover $\mathbf{Ran}_S S$ and $p(\mathbf{Ran}_S S)$ are both pointwise. Then $((\overline{K\epsilon R})^{-1}(\mathbf{Ran}_S S))X \simeq \lim(\delta_X^{-1}(S\pi_X))$.*

Proof. Let $(c_S, \mathbf{Ran}_S S)$ be the pointwise right Kan extension. Because its image by p is also assumed to be a pointwise Kan extension, the following diagram is a right Kan extension of $pS\pi_X$ along $!_{X \downarrow S}$:

$$\begin{array}{ccccc}
 X \downarrow S & \xrightarrow{\pi_X} & \mathbb{A} & \xrightarrow{S} & \mathbb{E} & \xrightarrow{p} & \mathbb{B} \\
 \downarrow !_{X \downarrow S} & \Rightarrow \gamma_X & \downarrow S & \Rightarrow \epsilon & & \nearrow & \\
 1 & \xrightarrow{X} & \mathbb{E} & \xrightarrow{p\mathbf{Ran}_S S} & & &
 \end{array}$$

That is, the pair $(L, P) = (p(\mathbf{Ran}_S S)X, pc_S \pi_X \bullet p\mathbf{Ran}_S S \gamma_X)$ is a limit of $pS\pi_X$. Then

$$\begin{aligned}
 ((\overline{K\epsilon R})^{-1}(\mathbf{Ran}_S S))X &= (\overline{K\epsilon R X})^{-1}((\mathbf{Ran}_S S)X) \\
 (\mathbf{Ran}_S S \text{ pointwise}) &\simeq (\overline{K\epsilon R X})^{-1}(\lim S\pi_X) \\
 (\text{limits by fibred limits}) &= (\overline{K\epsilon R X})^{-1}(\lim P^{-1}(S\pi_X)) \\
 (\text{preservation of fibred limits}) &\simeq \lim(\overline{K\epsilon R X!_{X \downarrow S}})^{-1}(P^{-1}(S\pi_X)) \\
 &\simeq \lim(P \bullet \overline{K\epsilon R X!_{X \downarrow S}})^{-1}(S\pi_X)
 \end{aligned}$$

By expanding P ,

$$\begin{aligned}
 P \bullet \overline{K\epsilon R X!_{X \downarrow S}} &= pc_S \pi_X \bullet p\mathbf{Ran}_S S \gamma_X \bullet \overline{K\epsilon R X!_{X \downarrow S}} \\
 &= pc_S \pi_X \bullet \overline{K\epsilon R S \pi_X} \bullet Tp\gamma_X \\
 &= (pc_S \bullet \overline{K\epsilon R S})\pi_X \bullet Tp\gamma_X \\
 &= K\epsilon R \pi_X \bullet Tp\gamma_X \\
 &= \delta_X. \quad \square
 \end{aligned}$$

8. CHARACTERISING THE COLLECTION OF LIFTINGS AS A LIMIT

We give a characterisation of the class of liftings of a monad on the base category of a *posetal* fibration with fibred small limits. We show that the class of liftings of \mathcal{T} is the vertex of a certain type of limiting cone.

Fix a posetal fibration $p : \mathbb{E} \rightarrow \mathbb{B}$ with fibred small limits and a monad \mathcal{T} on \mathbb{B} . Notice that each fibre actually admits *large* limits computed by meets. Since p is posetal, p is faithful. Without loss of generality, we regard each homset $\mathbb{E}(X, Y)$ as a subset of $\mathbb{B}(pX, pY)$.

Definition 8.1. We define $\mathbf{Lift}(\mathcal{T})$ to be the class of liftings of \mathcal{T} along p . We introduce a partial order \preceq on them by

$$\dot{T} \preceq \dot{T}' \iff \forall X \in \mathbb{E} . \dot{T}X \leq \dot{T}'X \quad (\text{in } \mathbb{E}_{T(pX)}).$$

The partially ordered class $(\mathbf{Lift}(\mathcal{T}), \preceq)$ admits arbitrary large meets given by the pointwise meet.

We introduce a convenient notation for the codensity liftings of \mathcal{T} . By $[S]^R$ we mean the pointwise codensity lifting \mathcal{T}^{\top} of \mathcal{T} with a single lifting parameter $R \in \mathbb{B}$ and $S \in \mathbb{E}_{TR}$. By expanding the definition, we have

$$[S]^R X = \bigwedge_{f \in \mathbb{E}(X, S)} (f^\#)^{-1}(S);$$

see also (4.1).

Definition 8.2. Let $X \in \mathbb{E}$ be an object. An object $S \in \mathbb{E}_{T(pX)}$ is *closed* with respect to X if 1) $\eta_{pX} \in \mathbb{E}(X, S)$ and 2) for any $f \in \mathbb{E}(X, S)$, we have $f^\# \in \mathbb{E}(S, S)$.

Proposition 8.3. *Let $X \in \mathbb{E}$ be an object. Then an object $S \in \mathbb{E}_{T(pX)}$ is closed with respect to X if and only if $S = [S]^{pX} X$.*

Proof. We first show that $\eta_{pX} \in \mathbb{E}(X, S)$ if and only if $[S]^{pX} X \leq S$. (only if) We have

$$[S]^{pX} X \leq ((\eta_{pX})^\#)^{-1}(S) = (\text{id}_{T(pX)})^{-1}(S) = S.$$

(if) As $[S]^{pX}$ is a lifting of T , we have $\eta_{pX} \in \mathbb{E}(X, [S]^{pX} X) \subseteq \mathbb{E}(X, S)$.

We next show that $\forall f \in \mathbb{E}(X, S) . f^\# \in \mathbb{E}(S, S)$ holds if and only if $S \leq [S]^{pX} X$.

$$S \leq [S]^{pX} X \iff \forall f \in \mathbb{E}(X, S) . S \leq (f^\#)^{-1}S \iff \forall f \in \mathbb{E}(X, S) . f^\# \in \mathbb{E}(S, S). \quad \square$$

Definition 8.4. Let $X \in \mathbb{E}$ be an object.

- (1) We define $\mathbf{Cls}(\mathcal{T}, X)$ to be the set $\{S \in \mathbb{E}_{T(pX)} \mid S = [S]^{pX} X\}$ of closed objects with respect to X .
- (2) We regard the codensity lifting $[-]^{pX}$ as a function of type $\mathbf{Cls}(\mathcal{T}, X) \rightarrow \mathbf{Lift}(\mathcal{T})$.
- (3) We define the monotone function $q_X : (\mathbf{Lift}(\mathcal{T}), \preceq) \rightarrow (\mathbf{Cls}(\mathcal{T}, X), \leq)$ to be the evaluation of a given lifting at X , that is, $q_X(\dot{T}) = \dot{T}X$. Here, the order \leq on $\mathbf{Cls}(\mathcal{T}, X)$ is the one inherited from $\mathbb{E}_{T(pX)}$.
- (4) We extend the order \leq on $\mathbf{Cls}(\mathcal{T}, X)$ to the pointwise order between parallel pairs of functions into $\mathbf{Cls}(\mathcal{T}, X)$.

We note that $[-]^{pX}$ cannot be monotone, because its argument is used both in a positive and a negative way. Still, we have the following adjoint-like relationship:

Theorem 8.5. *For any $X \in \mathbb{E}$, we have $q_X \circ [-]^{pX} = \text{id}_{\mathbf{Cls}(\mathcal{T}, X)}$ and $\text{id}_{\mathbf{Lift}(\mathcal{T})} \preceq [-]^{pX} \circ q_X$.*

Proof. We already have $q_X([S]^{pX}) = [S]^{pX}X = S$ from the definition of $\mathbf{Cls}(\mathcal{T}, X)$.

We show $\dot{T} \preceq [\dot{T}X]^{pX}$. We have the following equivalence:

$$\begin{aligned} \dot{T} \preceq [\dot{T}X]^{pX} &\iff \forall Y \in \mathbb{E}, f \in \mathbb{E}(Y, \dot{T}X) . \dot{T}Y \leq (f^\#)^{-1}(\dot{T}X) \\ &\iff \forall Y \in \mathbb{E}, f \in \mathbb{E}(Y, \dot{T}X) . f^\# \in \mathbb{E}(\dot{T}Y, \dot{T}X) \end{aligned}$$

and the last line always holds as \dot{T} is a lifting of T . \square

We define a function $\phi_{X,Y} : \mathbf{Cls}(\mathcal{T}, X) \rightarrow \mathbf{Cls}(\mathcal{T}, Y)$ by

$$\phi_{X,Y}(S) = q_Y \circ [-]^{pX}(S) = [S]^{pX}Y.$$

Theorem 8.5 asserts that $\phi_{X,X} = \text{id}_{\mathbf{Cls}(\mathcal{T}, X)}$. Using the second inequality of Theorem 8.5, for any $X, Y \in \mathbb{E}$, we have

$$q_X \leq q_X \circ [-]^{pY} \circ q_Y = \phi_{Y,X} \circ q_Y \tag{8.1}$$

$$[S]^{pX} \preceq [[S]^{pX}Y]^{pY} = [\phi_{X,Y}(S)]^{pY}. \tag{8.2}$$

From Theorem 8.5, \dot{T} is a lower bound of the class $\{[q_X(\dot{T})]^{pX} \mid X \in \mathbb{E}\}$. In fact, \dot{T} is the *greatest* lower bound:

Theorem 8.6. *For any lifting \dot{T} of \mathcal{T} , we have $\dot{T} = \bigwedge_{X \in \mathbb{E}} [q_X(\dot{T})]^{pX}$.*

Proof. It suffices to show $\bigwedge_{X \in \mathbb{E}} [q_X(\dot{T})]^{pX} \preceq \dot{T}$. For any $Y \in \mathbb{E}$,

$$\bigwedge_{X \in \mathbb{E}} [q_X(\dot{T})]^{pX}Y = \bigwedge_{X \in \mathbb{E}} \phi_{X,Y}(q_X(\dot{T})) \leq \phi_{Y,Y}(q_Y(\dot{T})) = q_Y(\dot{T}) = \dot{T}Y. \quad \square$$

This theorem also states that any lifting of a monad \mathcal{T} is an intersection of a class of single lifting parameter codensity liftings; see also Theorem 3.7. From this, we obtain the following corollary:

Corollary 8.7. *Let $\mathcal{X} \subseteq \mathbf{Lift}(\mathcal{T})$ be a class of liftings of \mathcal{T} . If 1) for any $X \in \mathbb{E}$ and $S \in \mathbf{Cls}(\mathcal{T}, X)$, $[S]^{pX} \in \mathcal{X}$, and 2) \mathcal{X} is closed under class-size intersection, then $\mathcal{X} = \mathbf{Lift}(\mathcal{T})$.*

Definition 8.8. We say that an object $X \in \mathbb{E}$ is a *split subobject* of an object $Y \in \mathbb{E}$ (denoted by $X \triangleleft Y$) if there is a split monomorphism $m : X \rightarrow Y$.

One easily sees that the binary relation \triangleleft on $\mathbf{Obj}(\mathbb{E})$ is reflexive and transitive. We define $\mathbf{Split}(\mathbb{E})$ to be the preordered class $(\mathbf{Obj}(\mathbb{E}), \triangleleft)$.

Lemma 8.9. *Suppose that $X \triangleleft Y$ holds for objects $X, Y \in \mathbb{E}$. The following holds:*

- (1) $\phi_{Y,X} \circ q_Y = q_X$.
- (2) For any object $Z \in \mathbb{E}$, $\phi_{Y,X} \circ \phi_{Z,Y} = \phi_{Z,X}$.

Proof. Assume $X \triangleleft Y$ in \mathbb{E} .

- (1) Let $(\dot{T}, \dot{\eta}, \dot{\mu}) \in \mathbf{Lift}(\mathcal{T})$. From (8.1) we have $q_X(\dot{T}) \leq \phi_{Y,X}(q_Y(\dot{T}))$. To show the opposite inequality, take a split monomorphism $m : X \rightarrow Y$ in \mathbb{E} . It comes with $e : Y \rightarrow X$ such

that $e \circ m = \text{id}_X$. We then consider the following diagram:

$$\begin{array}{ccccc}
[\dot{T}Y]^{pY} X & & & & \\
\leq \downarrow & & & & \\
((\dot{\eta}_Y \circ m)^\#)^{-1}(\dot{T}Y) & \cdots \cdots \rightarrow & \dot{T}Y & \xrightarrow{\dot{T}e} & \dot{T}X & \mathbb{E} \\
& & & & & \downarrow p \\
T(pX) & \xrightarrow{T(pm)} & T(pY) & \xrightarrow{T(pe)} & T(pX) & \mathbb{B}
\end{array}$$

The composite of the morphisms in the base category is $T(e \circ m) = \text{id}_{T(pX)}$. Therefore $[\dot{T}Y]^{pY} X \leq \dot{T}X$.

(2) From the previous equality, we have

$$\phi_{Y,X} \circ \phi_{Z,Y}(S) = \phi_{Y,X}(q_Y([S]^{pZ})) = q_X([S]^{pZ}) = \phi_{Z,X}(S). \quad \square$$

Following this lemma, we extend $\mathbf{Cls}(\mathcal{T}, -)$ to a functor of type $\mathbf{Split}(\mathbb{E})^{op} \rightarrow \mathbf{Set}$ by

$$\mathbf{Cls}(\mathcal{T}, X \triangleleft Y) = \phi_{Y,X} : \mathbf{Cls}(\mathcal{T}, Y) \rightarrow \mathbf{Cls}(\mathcal{T}, X).$$

This is indeed a functor thanks to Theorem 8.5 (for the preservation of the identity) and Lemma 8.9-2 (for the preservation of the composition).

We establish a universal property of $\mathbf{Lift}(\mathcal{T})$ with respect to a restricted class of cones over $\mathbf{Cls}(\mathcal{T}, -)$.

Definition 8.10. Let V be a class and $\{r_X : V \rightarrow \mathbf{Cls}(\mathcal{T}, X)\}_{X \in \mathbb{E}}$ be a cone from V over $\mathbf{Cls}(\mathcal{T}, -)$. We say that the cone r satisfies ϕ -inequality if $\phi_{Y,X} \circ r_Y \geq r_X$ holds for any $X, Y \in \mathbb{E}$.

From Lemma 8.9-1, $\{q_X : \mathbf{Lift}(\mathcal{T}) \rightarrow \mathbf{Cls}(\mathcal{T}, X)\}_{X \in \mathbb{E}}$ is a cone from $\mathbf{Lift}(\mathcal{T})$ over $\mathbf{Cls}(\mathcal{T}, -)$, and moreover it satisfies the ϕ -inequality by (8.1).

Theorem 8.11. For any class V and cone r from V over $\mathbf{Cls}(\mathcal{T}, -)$ satisfying ϕ -inequality, there exists a unique function $m : V \rightarrow \mathbf{Lift}(\mathcal{T})$ such that $r_X = q_X \circ m$ holds for any $X \in \mathbb{E}$.

Proof. Let V be a class and r be a cone from V over $\mathbf{Cls}(\mathcal{T}, -)$ satisfying ϕ -inequality. We define the function $m : V \rightarrow \mathbf{Lift}(\mathcal{T})$ by

$$m(v) = \bigwedge_{X \in \mathbb{E}} [r_X(v)]^{pX}.$$

We show that m satisfies $q_Y \circ m = r_Y$ for any $Y \in \mathbb{E}$.

$$q_Y(m(v)) = m(v)(Y) = \bigwedge_{X \in \mathbb{E}} \phi_{X,Y}(r_X(v)) = r_Y(v) \wedge \bigwedge_{X \in \mathbb{E}, Y \not\triangleleft X} \phi_{X,Y}(r_X(v)) = r_Y(v).$$

If there is another function $m' : V \rightarrow \mathbf{Lift}(\mathcal{T})$ such that $q_Y \circ m' = r_Y$ then from Theorem 8.6, we have

$$m'(v) = \bigwedge_{X \in \mathbb{E}} [q_X(m'(v))]^{pX} = \bigwedge_{X \in \mathbb{E}} [r_X(v)]^{pX} = m(v).$$

Thus $m = m'$. □

In Theorem 29 of the conference version of this paper [KS15], we showed a different universal property about the cone q from $\mathbf{Lift}(\mathcal{T})$. There, we considered all cones (which may not satisfy ϕ -inequality), while we restricted $\mathbf{Split}(\mathbb{E})$ to be directed. We later realised that 1) $\mathbf{Split}(\mathbb{E})$ is often not directed due to the initial object in \mathbb{E} (this happens, for instance, when $\mathbb{E} = \mathbf{Pred}, \mathbf{Top}, \mathbf{Pre}, \mathbf{EPMet}$), and 2) the ϕ -inequality property makes the proof of the universal property of q work. We therefore changed the claim of [KS15, Theorem 29] to Theorem 8.11.

9. CONCLUSION AND FUTURE WORK

We introduced the codensity lifting of monads along the fibrations that preserve the right Kan extensions giving codensity monads (this codensity condition was relaxed later in Section 7). The codensity lifting allows us to lift various monads on non-closed base categories, which was not possible by its precursor, $\top\top$ -lifting [Kat05]. The categorical dual of the codensity lifting is also given, which lifts comonads along cofibrations.

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