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TERMINATION IN CONVEX SETS OF DISTRIBUTIONS

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Abstract. Convex algebras, also called (semi)convex sets, are at the heart of modelling probabilistic systems including probabilistic automata. Abstractly, they are the Eilenberg-Moore algebras of the finitely supported distribution monad. Concretely, they have been studied for decades within algebra and convex geometry.

In this paper we study the problem of extending a convex algebra by a single point. Such extensions enable the modelling of termination in probabilistic systems. We provide a full description of all possible extensions for a particular class of convex algebras: For a fixed convex subset D of a vector space satisfying additional technical condition, we consider the algebra of convex subsets of D. This class contains the convex algebras of convex subsets of distributions, modelling (nondeterministic) probabilistic automata. We also provide a full description of all possible extensions for the class of free convex algebras, modelling fully probabilistic systems. Finally, we show that there is a unique functorial extension, the so-called black-hole extension.

1. Introduction

In this paper we study the question of how to extend a convex algebra by a single element. Convex algebras have been studied for many decades in the context of algebra, vector spaces, and convex geometry, see e.g. [34, 11, 14] and from a categorical viewpoint, see e.g. [12, 35, 25, 23, 3]. Recently they have attracted more attention in computer science as well, see e.g. [9, 16, 18]. One reason is that probability distributions, the main ingredient for modelling various probabilistic systems, see e.g. [36, 1, 32], have a natural convex algebra structure. Even more than that, the set of distributions over a set S carries the free convex algebra over S. As a consequence, on the concrete side, convexity has notably appeared in the semantics of probabilistic systems, in particular probabilistic automata [29, 28, 20, 15]. One particularly interesting development in the last decade in the theory of probabilistic systems is to consider probabilistic automata as transformers of *belief states*, i.e., probability distributions over states, resulting in semantics on distributions, see [15, 4, 5, 7, 8, 6, 22] to name a few. Convexity is inherent to this modelling and the resulting semantics that we call *distribution bisimilarity*.

Key words and phrases: convex algebra, one-point extensions, convex powerset monad.

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Additionally, on the abstract side, coalgebras over (categories of) algebras have attracted significant attention [31, 17]. They make explicit the algebraic structure that is present in (the states of) transition systems and allow for its utilisation in the notion of semantics. For coalgebras over convex algebras, the most important observation is that convex algebras are the Eilenberg-Moore algebras of the finitely supported distribution monad [35, 9, 10, 16]. The first author, with coauthors, has recently studied the abstract coalgebraic foundation of probabilistic automata as coalgebras over convex algebras in [2], by providing suitable functors and monads on the category $\mathcal{EM}(\mathcal{D}_f)$ of Eilenberg-Moore algebras that model probabilistic automata. As a result, one gets a neat generic treatment and understanding of distribution bisimilarity.

One contribution of [2] is identifying a convex-powerset monad on $\mathcal{EM}(\mathcal{D}_f)$ that together with a constant-exponent functor can be used to model probabilistic automata as coalgebras over $\mathcal{EM}(\mathcal{D}_f)$. However, the convex-powerset monad allows only for *nonempty* convex subsets, and hence there is no notion of termination. As a consequence, one can only model *input enabled* probabilistic automata. Hence, the question arises of how to add termination. One obvious way is to add termination that rules over any other behaviour: Consider a probabilistic automaton with two states s and t; a distribution $ps + \bar{p}t$ with $\bar{p} = 1 - p$ over states s and t terminates if and only if one of the states terminates. We refer to this approach as *black-hole* termination. Several distribution-bisimilarity semantics in the literature disagree on the treatment of termination, see e.g. the discussion in [15] as well as [7, 8, 6, 4]. Understanding termination in probabilistic automata as transformers of belief states is the motivation for this work. On the level of convex algebras, termination amounts to the question of extending a convex algebra by a single element.

Stated algebraically, the questions we address in this paper are:

(1) Given a convex algebra $\mathbb{X}$, is it possible to extend it by a single point?
(2) If yes, what are all possible one-point extensions?
(3) Which one-point extensions are functorial, i.e., do they provide a functor on $\mathcal{EM}(\mathcal{D}_f)$ that could then be used for modelling probabilistic automata as coalgebras over $\mathcal{EM}(\mathcal{D}_f)$?

Observe that extensions by a single point are different from the coproduct $\mathbb{X} + 1$ in $\mathcal{EM}(\mathcal{D}_f)$; the coproduct was concretely described in [18, Lemma 4], and it has a much larger carrier than the set $X + 1$.

Despite a large body of work on convex algebras, to the best of our knowledge, the problem of extending a convex algebra by a single element has not been studied, except for the black-hole extension mentioned above, see [12].

We answer the stated questions, and in particular our answers and main results are:

(1) Yes, it is possible and there are many possible extensions in general. One of them is the mentioned black-hole extension.
(2) We describe all possible extensions for the free convex algebra $\mathbb{D}_S$ of probability distributions over a set S, see Theorem 5.1 in Section 5. Furthermore, we describe all possible extensions of an algebra $\mathcal{P}_c\mathbb{D}$ for $\mathbb{D}$ being a convex subset of a vector space, satisfying a boundedness condition, see Theorem 6.11 in Section 6. As $\mathbb{D}_S$ is a particular subset of a vector space, we get a description of all possible extensions of $\mathcal{P}_c\mathbb{D}_S$ which is exactly what is needed to understand termination in convex sets of distributions.
(3) We prove that only the black-hole extension is functorial, see Theorem 5.3 in Section 5.

In addition, we provide many smaller results, observations, and examples that add to the vast knowledge on convex algebras.

We mention that reading our results and proofs in detail does not require any prior knowledge beyond basics of algebra, with two notable exceptions: (1) We do use some topological and geometric arguments in order to prove claims for the construction of some of our examples; (2) We add small remarks about coalgebras and categories as we already did in this introduction, assuming that readers are familiar with these basic notions (or will otherwise ignore the remarks made).

This paper is an extended version of [33] including all proofs and additional examples (Section 8 and Section 9).

2. Convex Algebras

By $\mathcal{C}$ we denote the signature of convex algebras

$$\mathcal{C} = \{(p_i)_{i=0}^n \mid n \in \mathbb{N}, p_i \in [0,1], \sum_{i=0}^n p_i = 1\}.$$

Intuitively, the $(n+1)$-ary operation symbol $(p_i)_{i=0}^n$ will be interpreted by a convex combination with coefficients p_i for $i = 0, \ldots, n$. For a real number $p \in [0,1]$ we set $\bar{p} = 1 - p$.

Definition 2.1. A *convex algebra* $\mathbb{X}$ is an algebra with signature $\mathcal{C}$, i.e., a set X together with an operation $\sum_{i=0}^n p_i(-)_i$ for each operational symbol $(p_i)_{i=0}^n \in \mathcal{C}$, such that the following two axioms hold:

- Projection: $\sum_{i=0}^n p_i x_i = x_j$ if $p_j = 1$.
- Barycenter: $\sum_{i=0}^n p_i \left(\sum_{j=0}^m q_{i,j} x_j\right) = \sum_{j=0}^m \left(\sum_{i=0}^n p_i q_{i,j}\right) x_j$.

We remark that the terminology in the literature is far from uniform. To give a few examples, convex algebras are called "convex modules in [23], "positive convex structures" in [9] (where X is taken to be endowed with the discrete topology), "finitely positively convex spaces" in [37], and "sets with a convex structure" in [35].

Convex algebras are the Eilenberg-Moore algebras of the finitely-supported distribution monad $\mathcal{D}_f$ on **Sets**, cf. [35, 4.1.3] and [30], see also [9, 10] or [16, Theorem 4] where a concrete and simple proof is given. A convex algebra homomorphism is a morphism in the Eilenberg-Moore category $\mathcal{EM}(\mathcal{D}_f)$. Concretely, a convex algebra homomorphism h from $\mathbb{X}$ to $\mathbb{Y}$ is a *convex* (synonymously, *affine*) map, i.e., $h\colon X \to Y$ with the property $h\left(\sum_{i=0}^n p_i x_i\right) = \sum_{i=0}^n p_i h(x_i)$.

Remark 2.2. Let $\mathbb{X}$ be a convex algebra. Then (for $p_n \neq 1$)

$$\sum_{i=0}^n p_i x_i = \overline{p_n} \left(\sum_{j=0}^{n-1} \frac{p_j}{\overline{p_n}} x_j \right) + p_n x_n. \tag{2.1}$$

Hence, an $(n+1)$-ary convex combination can be written as a binary convex combination using an n-ary convex combination. As a consequence, if X is a set that carries two convex algebras $\mathbb{X}_1$ and $\mathbb{X}_2$ with operations $\sum_{i=0}^n p_i(-)_i$ and $\bigoplus_{i=0}^n p_i(-)_i$, respectively (and binary versions $+$ and $\oplus$, respectively) such that $px + \bar{p}y = px \oplus \bar{p}y$ for all p, x, y, then $\mathbb{X}_1 = \mathbb{X}_2$.

One can also see Equation (2.1) as a definition – the classical definition of Stone [34, Definition 1]. We make the connection explicit with the next proposition.

Proposition 2.3. *Let X be a set with binary operations $px + \bar{p}y$ for $x, y \in X$ and $p \in (0,1)$. Assume*

- *Idempotence: $px + \bar{p}x = x$ for all $x \in X, p \in (0,1)$.*
- *Parametric commutativity: $px + \bar{p}y = \bar{p}y + px$ for all $x, y \in X, p \in (0,1)$.*
- *Parametric associativity: $p(qx + \bar{q}y) + \bar{p}z = pqx + \overline{pq}\left(\frac{p\bar{q}}{\overline{pq}}y + \frac{\bar{p}}{\overline{pq}}z\right)$ for all $x, y, z \in X, p, q, \in (0,1)$.*

Define n-ary convex operations inductively by the projection axiom and the formula (2.1). Then X becomes a convex algebra.

Proof. The proof is carried out by induction along the lines of [34, Lemma 1–Lemma 4]. □

This allows us to focus on binary convex combinations whenever more convenient.

Definition 2.4. Let $\mathbb{X}$ be a convex algebra, and $C \subseteq X$. Then C is called *convex* if it is the carrier of a subalgebra of $\mathbb{X}$, i.e., if $px + \bar{p}y \in C$ for all $x, y \in C$ and $p \in (0,1)$.

Given convex algebras $\mathbb{X}_i$, $i \in I$, the direct product $\prod_{i \in I} X_i$ is a convex algebra $\prod_{i \in I} \mathbb{X}_i$ with operations defined componentwise. We call a relation on the carrier of a convex algebra $\mathbb{X}$ a *convex relation*, if it is a convex subset of $\mathbb{X} \times \mathbb{X}$.

Definition 2.5. Let $\mathbb{X}$ be a convex algebra. An element $z \in X$ is $\mathbb{X}$-*cancellable* if

$$\forall x, y \in X.\ \forall p \in (0,1).\ px + \bar{p}z = py + \bar{p}z \Rightarrow x = y.$$

The convex algebra $\mathbb{X}$ is *cancellative* if every element of X is $\mathbb{X}$-cancellable.

Definition 2.6. Let $\mathbb{X}$ be a convex algebra. An element $x \in X$ *adheres to* an element $y \in X$, notation $x \multimap\!\!\bullet\, y$, if $px + \bar{p}y = y$ for all $p \in (0,1)$.

Observe that for a cancellative algebra the adherence relation equals the identity relation. As examples show, the converse need not hold. The following simple properties of adherence will be needed on multiple occasions.

Lemma 2.7. *Let $\mathbb{X}$ be a convex algebra. The following properties hold.*

1. *For all $x, y \in X$, $x \multimap\!\!\bullet\, y$ if and only if $px + \bar{p}y = y$ for some $p \in (0,1)$.*
2. *The adherence relation is reflexive and convex.*
3. *For all $x, y \in X$, if $x \multimap\!\!\bullet\, y$ then $pz + \bar{p}x \multimap\!\!\bullet\, pz + \bar{p}y$ for all $z \in X$ and $p \in (0,1)$.*
4. *If z is $\mathbb{X}$-cancellable, then for all $x, y \in X$ and $p \in (0,1)$*
$$pz + \bar{p}x \multimap\!\!\bullet\, pz + \bar{p}y \quad \Rightarrow \quad x \multimap\!\!\bullet\, y.$$

Proof.

(1) Let $x, y \in X$. Consider the map $\varphi\colon [0,1] \to \mathbb{X}$ defined by $\varphi(p) = px + \bar{p}y$. Easy calculations show that
$$(qp + \bar{q}r)x + \overline{(qp + \bar{q}r)}y = q(px + \bar{p}y) + \bar{q}(rx + \bar{r}y), \tag{2.2}$$
showing that φ is convex. The implication $\Rightarrow$ trivially holds. For the implication $\Leftarrow$ assume that $rx + \bar{r}y = y$ for some $r \in (0,1)$. Then $\varphi(0) = y = \varphi(r)$ showing that the kernel of φ is a congruence of $[0,1]$ which is not the diagonal. Recall that a congruence on $[0,1]$ is an equivalence R with the property that for all $p \in [0,1]$ and $(x_1, y_1), (x_2, y_2) \in R$, also $(px_1 + \bar{p}x_2, py_1 + \bar{p}y_2) \in R$. Equivalently, congruences are kernels of homomorphisms. By [11, Lemma 3.2], φ is constant on $(0,1)$. This shows that for all $p \in (0,1)$, $px + \bar{p}y = y$ and hence $x \multimap\!\!\bullet\, y$.

(2) Reflexivity is a direct consequence of idempotence. Let $x, y, u, v \in X$ and assume $x \mathrel{\circ\!\!-\!\!\bullet} y$ and $u \mathrel{\circ\!\!-\!\!\bullet} v$. Then

$$q(px + \bar{p}u) + \bar{q}(py + \bar{p}v) = p(qx + \bar{q}y) + \bar{p}(qu + \bar{q}v) = py + \bar{p}v.$$

(3) Direct consequence of reflexivity and convexity of adherence.

(4) Assume $pz + \bar{p}x \mathrel{\circ\!\!-\!\!\bullet} pz + \bar{p}y$ and z is $\mathbb{X}$-cancellable. Let $q \in (0, 1)$. Then

$$pz + \bar{p}y = q(pz + \bar{p}x) + \bar{q}(pz + \bar{p}y) = pz + \bar{p}(qx + \bar{q}y)$$

implies $qx + \bar{q}y = y$, after cancelling z. Hence $x \mathrel{\circ\!\!-\!\!\bullet} y$. □

Example 2.8. Here are two examples of convex algebras.

(1) Let $\mathbb{V}$ be a vector space over $\mathbb{R}$ and $X \subseteq V$ a convex subset. Then X with the operations inherited from $\mathbb{V}$ is a cancellative convex algebra $\mathbb{X}$. Conversely, every cancellative convex algebra is isomorphic to a convex subset of a vector space, cf. [34, Theorem 2] or [19, Satz 3].

(2) In particular, we consider the vector space $\ell^1(S)$ for a set S. Recall, $\ell^1(S) = \{(r_s)_{s \in S} \mid r_s \in \mathbb{R}, \sum_{s \in S} |r_s| < \infty\}$ with the norm $\|(r_s)_{s \in S}\|_1 = \sum_{s \in S} |r_s|$. The set $\mathcal{D}_f S$ of finitely supported probability distributions over S forms a convex subset of $\ell^1(S)$ and hence a cancellative convex algebra $\mathbb{D}_S$. It is shown in [21, Lemma 1] that $\mathbb{D}_S$ is the free convex algebra generated by S, i.e., every map of S into some convex algebra $\mathbb{X}$ has a unique extension to a homomorphism from $\mathbb{D}_S$ into $\mathbb{X}$.

The following construction is basic for our considerations.

Definition 2.9. Let $\mathbb{X}$ be convex algebra. Then $\mathcal{P}_c X$ denotes the set of nonempty convex subsets of X, i.e., carriers of subalgebras of $\mathbb{X}$. We endow $\mathcal{P}_c X$ with the pointwise operations

$$pA + \bar{p}B = \{pa + \bar{p}b \mid a \in A, b \in B\}.$$

Then $\mathcal{P}_c X$ forms a convex algebra, cf. [2], and we write $\mathcal{P}_c \mathbb{X}$ for this algebra. Note that requiring the elements of $\mathcal{P}_c X$ to be nonempty is necessary for the projection axiom to hold.

We note that $\mathcal{P}_c$ is a monad on $\mathcal{EM}(\mathcal{D}_f)$ as shown in [2]. On morphisms, $\mathcal{P}_c$ acts as the powerset monad. The original algebra $\mathbb{X}$ embeds in $\mathcal{P}_c \mathbb{X}$ via the unit of the monad $\eta \colon x \mapsto \{x\}$.

For convex subsets of a finite dimensional vector space, the pointwise operations are known as Minkowski addition and are a basic construction in convex geometry, cf. [27].

The algebra $\mathcal{P}_c \mathbb{X}$ is in general not cancellative and has a nontrivial adherence relation, cf. [12, Example 6.3]. However it contains cancellative elements: It is easy to check that for each $\mathbb{X}$-cancellable element x the element $\{x\}$ is $\mathcal{P}_c \mathbb{X}$-cancellable.

Example 2.10. Here are further two examples of convex algebra which are of particular interst in this paper.

(1) The motivating example for this work is the convex algebra $\mathcal{P}_c \mathbb{D}_S$ of convex subsets of distributions over a set S.

(2) Let X be the carrier of a meet-semilattice and define $px + \bar{p}y = x \wedge y$ for $x, y \in X$ and $p \in (0, 1)$. Then X becomes a convex algebra $\mathbb{X}$ with these operations, cf. [21, §4.5]. This algebra is not cancellative, in fact $\circ\!\!-\!\!\bullet = \{(x, y) \mid x \geq y\}$. For the categorically minded, we remark that behind this construction is the support monad map from $\mathcal{D}_f$ to $\mathcal{P}_f$, the finite powerset monad, and semilattices are the Eilenberg-Moore algebras of $\mathcal{P}_f$.

We now present a construction that provides a beautiful way of constructing convex algebras out of existing ones.

Example 2.11. The *semilattice construction*, cf. [12, p.22f]: Let S be the carrier of a meet-semilattice and let $(\mathbb{X}_s)_{s\in S}$ be an S-indexed family of convex algebras. Moreover, let $(f_s^t)_{\substack{s,t\in S\\ s\leq t}}$ be a family of convex algebra homomorphisms $f_s^t\colon \mathbb{X}_t \to \mathbb{X}_s$ that satisfy $f_s^t \circ f_t^u = f_s^u$ for all $s \leq t \leq u$, and $f_s^s = \mathrm{id}_{X_s}$ for all $s \in S$. Let X be the disjoint union of all X_s for $s \in S$. Then X becomes a convex algebra $\mathbb{X}$ with operations defined by $px+\bar{p}y = pf^s_{s\wedge t}(x)+\bar{p}f^t_{s\wedge t}(y)$ for $x \in X_s$, $y \in X_t$, and $p \in (0, 1)$. The algebra $\mathbb{X}$ obtained in this way is the direct limit of the diagram formed by the algebras $\mathbb{X}_s$ and the maps f_s^t.

Definition 2.12. Let $\mathbb{X}$ be a convex algebra, and $P, Q \subseteq X$.

- P is an *ideal* if $\forall x \in P.\ \forall y \in X.\ \forall p \in (0, 1).\ px + \bar{p}y \in P$.
- P is a *prime ideal* if it is an ideal and its complement $X \setminus P$ is convex.
- Q is an *extremal set* if $px + \bar{p}y \in Q \Rightarrow x, y \in Q$ for all $x, y \in X, p \in (0, 1)$.
- $z \in X$ is an *extremal point* if $\{z\}$ is an extremal set. Explicitly: z is an extremal point if whenever $px + \bar{p}y = z$ for $x, y \in X, p \in (0, 1)$, it follows that $x = y = z$. The set of all extremal points of $\mathbb{X}$ is denoted as $\operatorname{Ext}\mathbb{X}$.

Again, the terminology used in the literature is not uniform: in [16, Definition 7] extremal sets are called *filters*. Moreover, let us remark that the construction Ext is not functorial.

Lemma 2.13. *Let $\mathbb{X}$ be a convex algebra, and $P \subseteq X$. Then P is an ideal if and only if $X \setminus P$ is an extremal set.*

Proof. Assume P is an ideal. Let $x, y \in X, p \in (0, 1)$ such that $px + \bar{p}y \in X \setminus P$. If $x \in P$ or $y \in P$, then since P is an ideal also $px + \bar{p}y \in P$, a contradiction. Hence $x, y \in X \setminus P$.

For the converse, assume $X\setminus P$ is extremal and let $x \in P, y \in X, p \in (0, 1)$. If $px+\bar{p}y \notin P$, then $px + \bar{p}y \in X \setminus P$ which implies $x, y \in X \setminus P$, a contradiction. Hence, $px + \bar{p}y \in P$. □

3. The Problem and Some Example Solutions

> Let $\mathbb{X}$ be a convex algebra. Can one extend it for one element to a convex algebra $\mathbb{X}_*$ with carrier $X \cup \{*\}$ where $* \notin X$? If yes, what are all possible extensions?

We will show that an arbitrary convex algebra $\mathbb{X}$ can be extended in many ways, and describe all possible ways of extending $\mathbb{X} = \mathbb{D}_S$ and $\mathbb{X} = \mathcal{P}_c\mathbb{D}_S$.

First, we provide four examples of extensions, two of which are instances of the semilattice construction from Example 2.11.

Example 3.1. Let $\mathbb{X}$ be a convex algebra and $* \notin X$. We denote the operations of $\mathbb{X}$ as before by $p(-) + \bar{p}(-)$. In each of the examples we construct a convex algebra $\mathbb{X}_*$ with operations denoted by $p(-) \oplus \bar{p}(-)$ satisfying $px \oplus \bar{p}y = px + \bar{p}y$, $x, y \in X$, $p \in [0, 1]$.

(1) Black-hole behaviour, cf. [12, Example 6.1]: In this example, $*$ behaves like a black hole and swallows everything in the sense that $x \circ\!\!-\!\!\bullet *$ for all $x \in X$. To be precise, consider the semilattice $S = \{0, 1\}$ with $0 \leq 1$. Let $\mathbb{X}_0$ be the trivial convex algebra with $X_0 = \{*\}$ and $\mathbb{X}_1 = \mathbb{X}$. Let $f_0^1 : \mathbb{X}_1 \to \mathbb{X}_0$ be the unique homomorphism (mapping

everything to $*$). Then the semilattice construction gives us a convex algebra $\mathbb{X}_*$ with the property

$$px \oplus \bar{p}y = \begin{cases} px + \bar{p}y \,, & x, y \in X, \\ * \quad\quad\;\;, & x = * \text{ or } y = *. \end{cases} \tag{3.1}$$

(2) Imitating behaviour: The intuition behind this construction is that $*$ imitates the behaviour of a given element $w \in X$. Formally, we want to build $\mathbb{X}_*$ in such a way that $px \oplus \bar{p}* = px + \bar{p}w$ for all $p \in (0, 1]$ and $x \in X$.

To do this, consider again the semilattice $S = \{0, 1\}$ with $0 \leq 1$. Let $\mathbb{X}_0 = \mathbb{X}$ and $\mathbb{X}_1$ be the trivial convex algebra with $X_1 = \{*\}$. Let $f_0^1 : \mathbb{X}_1 \to \mathbb{X}_0$ be the homomorphism mapping $*$ to w. Then the semilattice construction gives us a convex algebra $\mathbb{X}_*$ with the property

$$px \oplus \bar{p}y = \begin{cases} px + \bar{p}y \,, & x, y \in X, \\ px + \bar{p}w \,, & x \in X, y = *, \\ pw + \bar{p}y \,, & x = *, y \in X, \\ * \quad\quad\;\;, & x = y = *. \end{cases} \tag{3.2}$$

(3) Imitating an outer element: Assume we are given a convex algebra $\mathbb{Y}$ which contains $\mathbb{X}$ as a subalgebra. Let $w \in Y \setminus X$ be such that $X \cup \{w\}$ is convex. Then we obtain an extension $\mathbb{X}_*$ by identifying $X \cup \{*\}$ with $X \cup \{w\}$ via $x \mapsto x$ for $x \in X$ and $* \mapsto w$. We say that $*$ imitates the outer element w, since $px \oplus \bar{p}* = px + \bar{p}w$ for all $p \in (0, 1]$ and $x \in X$.

This way of defining extensions is of course trivial, but it is useful in presence of a natural larger algebra. For example, we will apply it when D is a convex subset of a vector space $\mathbb{V}$, $\mathbb{X} = \mathcal{P}_c\mathbb{D}$, and $\mathbb{Y} = \mathcal{P}_c\mathbb{V}$.

(4) Mixed behaviour: Let w be an extremal point of $\mathbb{X}$. The intuition in this example is that $*$ imitates $w \in X$ on $X \setminus \{w\}$ and swallows w. That is, we want to build $\mathbb{X}_*$ according to

$$px \oplus \bar{p}y = \begin{cases} px + \bar{p}y \,, & x, y \in X, \\ px + \bar{p}w \,, & x \in X \setminus \{w\}, y = *, \\ pw + \bar{p}y \,, & x = *, y \in X \setminus \{w\}, \\ * \quad\quad\;\;, & \text{otherwise}. \end{cases} \tag{3.3}$$

The fact that this construction indeed produces a convex algebra is not an instance of the semilattice construction and requires a proof, which we give in Section 5 below (p.12).

4. Extensions of Convex Algebras - The Prime Ideal

The following two notions provide a crucial characteristic of an extension $\mathbb{X}_*$ for a convex algebra $\mathbb{X}$.

Definition 4.1. Let $\mathbb{X}$ be a convex algebra, and let $\mathbb{X}_*$ be an extension. Then its *set of adherence* $\mathrm{Ad}(\mathbb{X}_*)$ is $\mathrm{Ad}(\mathbb{X}_*) = \{x \in X \mid x \circ\!\!-\!\!\bullet *\}$ and its *prime ideal* is $\mathrm{P}(\mathbb{X}_*) = X \setminus \mathrm{Ad}(\mathbb{X}_*)$.

Lemma 4.2. *Let $\mathbb{X}$ be a convex algebra, and let $\mathbb{X}_*$ be an extension of $\mathbb{X}$. The set $\mathrm{P}(\mathbb{X}_*)$ is indeed a prime ideal of $\mathbb{X}$.*

Proof. Let $x \in \mathrm{P}(\mathbb{X}_*)$, $y \in X$, $p \in (0,1)$. Then

$$q(px + \bar{p}y) + \bar{q}* = \overline{q\bar{p}}\left(\frac{qp}{\overline{q\bar{p}}}x + \frac{\bar{q}}{\overline{q\bar{p}}}*\right) + q\bar{p}y \in X$$

since $y \in X$ and $\frac{qp}{\overline{q\bar{p}}}x + \frac{\bar{q}}{\overline{q\bar{p}}}* \in X$ due to $x \in \mathrm{P}(\mathbb{X}_*)$ and hence $x \notin \mathrm{Ad}(\mathbb{X}_*)$. Therefore, $px + \bar{p}y \in \mathrm{P}(\mathbb{X}_*)$ proving that $\mathrm{P}(\mathbb{X}_*)$ is an ideal in $\mathbb{X}$. By Lemma 2.7.2 $\mathrm{Ad}(\mathbb{X}_*)$ is convex and hence $\mathrm{P}(\mathbb{X}_*)$ is prime. □

The next lemma gives a way to conclude that $*$ imitates an element.

Lemma 4.3. *Let $\mathbb{Y}$ be a convex algebra, $\mathbb{X} \leq \mathbb{Y}$ a subalgebra, and let $\mathbb{X}_*$ be an extension of $\mathbb{X}$. Further, let $z \in \mathrm{P}(\mathbb{X}_*)$ and assume that z is $\mathbb{Y}$-cancellable. If there exist $w \in \mathbb{Y}$ and $q \in (0,1)$ with $qz + \bar{q}* = qz + \bar{q}w$, then $*$ imitates w on $\mathrm{P}(\mathbb{X}_*)$ and $\mathrm{Ad}(\mathbb{X}_*) \subseteq \{x \in X \mid x \circ\!\!-\!\!\bullet w\}$.*

Proof. Let $x \in \mathrm{P}(\mathbb{X}_*)$, $p \in (0,1)$, and set $s = \frac{\bar{p}}{\overline{pq}}$. Then $s \in (0,1)$ and $\overline{sq} \cdot p = \bar{s}$, $\overline{sq} \cdot \bar{p} = s\bar{q}$, and we have

$$sqz + \overline{sq}(\underbrace{px + \bar{p}*}_{\in \mathrm{P}(\mathbb{X}_*) \subseteq Y}) = s(qz + \bar{q}*) + \bar{s}x = s(qz + \bar{q}w) + \bar{s}x = sqz + \overline{sq}(\underbrace{px + \bar{p}w}_{\in Y}).$$

Cancelling z yields $px + \bar{p}* = px + \bar{p}w$. We conclude that indeed $*$ imitates w on all of $\mathrm{P}(\mathbb{X}_*)$. Assume now that $x \in \mathrm{Ad}(\mathbb{X}_*)$. Then by Lemma 2.7.3.

$$pz + \bar{p}x \circ\!\!-\!\!\bullet pz + \bar{p}* = pz + \bar{p}w, \quad \text{for } p \in (0,1).$$

Again using cancellability of z, it follows that $x \circ\!\!-\!\!\bullet w$ by Lemma 2.7.4. □

5. Extensions of Free Algebras and Functoriality

Let S be a nonempty set and consider the free convex algebra over S. As noted in Example 2.8.2, this is the algebra $\mathbb{D}_S$ of finitely supported distributions on S. In the next theorem we determine all possible one-point extensions of $\mathbb{D}_S$.

Theorem 5.1. *Let S be a nonempty set and consider the free convex algebra $\mathbb{D}_S$. One-point extensions of $\mathbb{D}_S$ can be constructed as follows:*

(1) *The black-hole behaviour, where the set of adherence equals $\mathcal{D}_f S$.*
(2) *Let $w \in \mathcal{D}_f S$, and let $*$ imitate w on all of $\mathcal{D}_f S$.*
(3) *Let w be an extremal point of $\mathbb{D}_S$, and let $*$ imitate w on $\mathcal{D}_f S \setminus \{w\}$ and adhere w.*

Every one-point extension of $\mathbb{D}_S$ can be obtained in this way, and each two of these extensions are different.

Note that $w \in \mathrm{Ext}\,\mathbb{D}_S$ if and only if w is a corner point, in other words, a Dirac measure concentrated at one of the points of S.

The fact that the constructions (1) and (2) give extensions is Example 3.1.1/2. The construction in (3) is Example 3.1.4, for which we will now provide evidence. First, we prove a more general statement that we call the gluing lemma, which will be needed later as well. It gives a way to produce extensions with a prescribed set of adherence.

Lemma 5.2 (Gluing Lemma)**.** *Let $\mathbb{X}$ be a convex algebra, and $P \subseteq X$ a prime ideal. Assume we have convex operations $p(-) \boxplus \bar{p}(-)$ on $\mathbb{P}_*$ that extend $\mathbb{P}$ (whose operations are inherited from $\mathbb{X}$). Assume further that $\mathrm{Ad}(\mathbb{P}_*) = \emptyset$ and that*

$$px + \bar{p}y \mathrel{\circ\!\!-\!\!\bullet} px \boxplus \bar{p}*, \qquad \text{for } x \in P,\ y \in X \setminus P,\ p \in (0,1). \tag{5.1}$$

Then the operations $p(-) \oplus \bar{p}(-)$, $p \in (0,1)$, defined as follows extend $\mathbb{X}$ to a convex algebra $\mathbb{X}_*$ with $\mathrm{Ad}(\mathbb{X}_*) = X \setminus P$:

$$px \oplus \bar{p}y = \begin{cases} px + \bar{p}y, & x, y \in X, \\ px \boxplus py, & x = *, y \in P \text{ or } x \in P, y = *, \\ *, & \text{otherwise.} \end{cases}$$

Proof. We first show two auxilliary properties for $x \in P$ and $y \in X \setminus P$: (5.2) $\Leftrightarrow$ (5.3) and (5.2) $\Leftrightarrow$ (5.4) for

$$\forall p \in (0,1).\ px + \bar{p}y \mathrel{\circ\!\!-\!\!\bullet} px \boxplus \bar{p}* \tag{5.2}$$

$$\forall q, r \in (0,1).\ q(rx \boxplus \bar{r}*) + \bar{q}y = qrx \boxplus \overline{qr}* \tag{5.3}$$

$$\forall q, r \in (0,1).\ q(rx + \bar{r}y) \boxplus \bar{q}* = qrx \boxplus \overline{qr}* \tag{5.4}$$

Let $x \in P, y \in X \setminus P$ and $s, t \in (0,1)$. For (5.2) $\Leftrightarrow$ (5.3) we first compute

$$\begin{aligned} s(tx + \bar{t}y) + \bar{s}(tx \boxplus \bar{t}*) &\overset{(b)}{=} stx + s\bar{t}y + \bar{s}(tx \boxplus \bar{t}*) \\ &\overset{(b)}{=} \overline{s\bar{t}}\left(\frac{st}{\overline{s\bar{t}}}x + \frac{\bar{s}}{\overline{s\bar{t}}}(tx \boxplus \bar{t}*)\right) + s\bar{t}y \\ &\overset{(c)}{=} \overline{s\bar{t}}\left(\frac{st}{\overline{s\bar{t}}}x \boxplus \frac{\bar{s}}{\overline{s\bar{t}}}(tx \boxplus \bar{t}*)\right) + s\bar{t}y \\ &\overset{(d)}{=} \overline{s\bar{t}}\left(\frac{t}{\overline{s\bar{t}}}x \boxplus \frac{\bar{s}\bar{t}}{\overline{s\bar{t}}}*\right) + s\bar{t}y \end{aligned}$$

and refer to this equality as (†). Here, (b) is an application of barycenter in $\mathbb{X}$; (c) holds since $x, tx \boxplus \bar{t}* \in P$; and (d) is an application of barycenter in $\mathbb{P}_*$.

Now assume (5.2), let $q, r \in (0,1)$, and take $s = \frac{\bar{q}}{\overline{qr}}$ and $t = qr$. Then $s, t \in (0,1)$ and

$$q(rx \boxplus \bar{r}*) + \bar{q}y = \overline{s\bar{t}}\left(\frac{t}{\overline{s\bar{t}}}x \boxplus \frac{\bar{s}\bar{t}}{\overline{s\bar{t}}}*\right) + s\bar{t}y \overset{(\dagger)}{=} s(tx + \bar{t}y) + \bar{s}(tx \boxplus \bar{t}*) \overset{(5.2)}{=} tx \boxplus \bar{t}* = qrx \boxplus \overline{qr}*$$

proving (5.3).

For (5.3) $\Rightarrow$ (5.2), assume (5.3), let $p \in (0,1)$, and take any $q, r \in (0,1)$ such that $p = qr$. Set again $s = \frac{\bar{q}}{\overline{qr}}$. Then $s \in (0,1)$ and

$$px \boxplus \bar{p}* = qrx \boxplus \overline{qr}* \overset{(5.3)}{=} q(rx \boxplus \bar{r}*) + \bar{q}y \overset{(\dagger)}{=} s(px + \bar{p}y) + \bar{s}(px \boxplus \bar{p}*)$$

which proves (5.2).

For (5.2) $\Leftrightarrow$ (5.4) we now compute

$$\begin{aligned} s(tx + \bar{t}y) \boxplus \bar{s}(tx \boxplus \bar{t}*) &\overset{(d)}{=} s(tx + \bar{t}y) \boxplus \bar{s}tx \boxplus \bar{s}\bar{t}* \\ &\overset{(d)}{=} \overline{\bar{s}\bar{t}}\left(\frac{s}{\overline{\bar{s}\bar{t}}}(tx + \bar{t}y) \boxplus \frac{\bar{s}t}{\overline{\bar{s}\bar{t}}}x\right) \boxplus \bar{s}\bar{t}* \\ &\overset{(b)}{=} \overline{\bar{s}\bar{t}}\left(\frac{t}{\overline{\bar{s}\bar{t}}}x + \frac{s\bar{t}}{\overline{\bar{s}\bar{t}}}y\right) \boxplus \bar{s}\bar{t}* \end{aligned}$$

and refer to this equality as (‡).

Now assume (5.2), let $q, r \in (0,1)$ be given, consider $p = qr$, and take $s = \frac{q\bar{r}}{\overline{qr}}$ and $t = p$. Hence $\bar{s} = \frac{\bar{q}}{\overline{qr}}$ and $t = qr$. Then $s, t \in (0,1)$, $\frac{t}{\overline{\bar{s}\bar{t}}} = r$, $\overline{\bar{s}\bar{t}} = q$ and hence

$$q(rx + \bar{r}y) \boxplus \bar{q}* \overset{(‡)}{=} \frac{q\bar{r}}{\overline{qr}}(qrx + \overline{qr}y) \boxplus \frac{\bar{q}}{\overline{qr}}(qrx \boxplus \overline{qr}*) \overset{(5.2)}{=} qrx \boxplus \overline{qr} * .$$

For the opposite direction, assume (5.4). Let p be given. Pick $q, r \in (0,1)$ such that $p = qr$ and put $s = \frac{q\bar{r}}{\overline{qr}}$. Then again

$$px + \bar{p}* = qrx + \overline{qr}* \overset{(5.4)}{=} q(rx + \bar{r}y) \boxplus \bar{q}* \overset{(‡)}{=} s(px + \bar{p}y) \boxplus \bar{s}(px \boxplus \bar{p}*)$$

proving (5.2).

Coming back to the actual proof of the lemma, note that (5.2) is satisfied by assumption, hence both (5.3) and (5.4) hold as well. Moreover, recall that $X \setminus P$ is a subalgebra of $\mathbb{X}$, since P is a prime ideal. Write $P' = X \setminus P$ and $\mathbb{P}'$ for the corresponding convex algebra. Further, let $\mathbb{P}'_*$ be the black-hole extension of $\mathbb{P}'$ (operations denoted as $p(-) +' \bar{p}(-)$).

We will show now that $p(-) \oplus \bar{p}(-)$ are indeed convex operations on $\mathbb{X}_*$, i.e., we check idempotence, parametric commutativity, and parametric associativity, cf. Proposition 2.3.

Since the operations of $\mathbb{X}$ and $\mathbb{P}_*$ coincide on P and those of $\mathbb{X}$ and $\mathbb{P}'_*$ coincide on P', the definition of $p(-) \oplus \bar{p}(-)$ gives

$$px \oplus \bar{p}y = \begin{cases} px + \bar{p}y \ , & x, y \in X, \\ px \boxplus \bar{p}y \ , & x, y \in P_*, \\ px +' \bar{p}y \ , & x, y \in P'_*. \end{cases}$$

We have $X_* = P \cup P' \cup \{*\}$ and, by what we just observed, the operations $p(-) \oplus \bar{p}(-)$ coincide on the union of each two of these sets with the operations of a convex algebra (namely with $+$ of $\mathbb{X}$ on $P \cup P'$, with $\boxplus$ of $\mathbb{P}_*$ on $P \cup \{*\}$, and with $+'$ of $\mathbb{P}'_*$ on $P' \cup \{*\}$). The idempotence law involves only one variable and parametric commutativity involves only two variables. We conclude that both of these laws hold for $\oplus$. Parametric associativity is the law

$$p(qx \oplus \bar{q}y) \oplus \bar{p}z = pqx \oplus \overline{pq}(\frac{p\bar{q}}{\overline{pq}}y \oplus \frac{\bar{p}}{\overline{pq}}z).$$

If all three of x, y, z belong to one of $P \cup P'$, $P \cup \{*\}$, and $P' \cup \{*\}$, the above argument shows that this law holds. It remains to check six cases. In the following set $r = \frac{p\bar{q}}{\overline{pq}}$.

(1) $x \in P, y \in P', z = *$.

$$\begin{aligned} p(qx \oplus \bar{q}y) \oplus \bar{p}z &= p(qx + \bar{q}y) \boxplus \bar{p} * \\ &\overset{(5.4)}{=} pqx \boxplus \overline{pq} * \\ &= pqx \boxplus \overline{pq}(ry \oplus \bar{r}z) \\ &= pqx \oplus \overline{pq}(ry \oplus \bar{r}z). \end{aligned}$$

(2) $x \in P, y = *, z \in P'$.

$$\begin{aligned} p(qx \oplus \bar{q}y) \oplus \bar{p}z &= p(qx \oplus \bar{q}*) \oplus \bar{p}z \\ &= p(qx \boxplus \bar{q}*) \oplus \bar{p}z \\ &= p(qx \boxplus \bar{q}*) + \bar{p}z \\ &\overset{(5.3)}{=} pqx \boxplus \overline{pq}\, * \\ &= pqx \boxplus \overline{pq}(r * \oplus \bar{r}z) \\ &= pqx \oplus \overline{pq}(ry \oplus \bar{r}z). \end{aligned}$$

(3) $x \in P', y \in P, z = *$.

$$\begin{aligned} p(qx \oplus \bar{q}y) \oplus \bar{p}z &= p(qx \oplus \bar{q}y) \oplus \bar{p}\, * \\ &\overset{comm}{=} p(\bar{q}y \oplus qx) \oplus \bar{p}\, * \\ &= p(\bar{q}y + qx) \boxplus \bar{p}\, * \\ &\overset{(5.4)}{=} p\bar{q}y \boxplus \overline{p\bar{q}}\, * \\ &= p\bar{q}y \oplus \overline{p\bar{q}}* \end{aligned}$$

and

$$\begin{aligned} pqx \oplus \overline{pq}(ry \oplus \bar{r}z) &\overset{comm}{=} \overline{pq}(ry \oplus \bar{r}z) \oplus pqx \\ &= \overline{pq}(ry \boxplus \bar{r}*) + pqx \\ &\overset{(5.3)}{=} \overline{pq}ry \boxplus \overline{\overline{pq}r}\, * \\ &= \overline{pq}ry \oplus \overline{\overline{pq}r}\, * \\ &= p\bar{q}y \oplus \overline{p\bar{q}}\, * \,. \end{aligned}$$

(4) $x \in P', y = *, z \in P$.

$$\begin{aligned} p(qx \oplus \bar{q}y) \oplus \bar{p}z &= p * \oplus \bar{p}z \\ &\overset{comm}{=} \bar{p}z \oplus p* \end{aligned}$$

and

$$\begin{aligned} pqx \oplus \overline{pq}(ry \oplus \bar{r}z) &\overset{comm}{=} \overline{pq}(\bar{r}z \oplus r*) \oplus pqx \\ &= \overline{pq}(\bar{r}z \boxplus r*) + pqx \\ &\overset{(5.3)}{=} \overline{pq}\bar{r}z \boxplus \overline{\overline{pq}\bar{r}}\, * \\ &= \bar{p}z \oplus p\, * \,. \end{aligned}$$

(5) $x = *, y \in P, z \in P'$.

$$\begin{aligned} p(qx \oplus \bar{q}y) \oplus \bar{p}z &\overset{comm}{=} p(\bar{q}y \oplus q*) \oplus \bar{p}z \\ &= p(\bar{q}y \boxplus q*) + \bar{p}z \\ &\overset{(5.3)}{=} p\bar{q}y \boxplus \overline{p\bar{q}}\, * \\ &= p\bar{q}y \oplus \overline{p\bar{q}}* \end{aligned}$$

and

$$\begin{aligned} pqx \oplus \overline{pq}(ry \oplus \bar{r}z) &\overset{comm}{=} \overline{pq}(ry \oplus \bar{r}z) \oplus pq* \\ &= \overline{pq}(ry + \bar{r}z) \oplus pq* \\ &\overset{(!)}{=} \overline{pq}(ry + \bar{r}z) \boxplus pq* \\ &\overset{(5.4)}{=} \overline{pq}ry \oplus \overline{\overline{pq}r}* \\ &= p\bar{q}y \oplus \overline{p\bar{q}}*, \end{aligned}$$

where the equality marked by (!) holds because $ry + \bar{r}z \in P$.

(6) $x = *, y \in P', z \in P$.

$$\begin{aligned} p(qx \oplus \bar{q}y) \oplus \bar{p}z &= p * \oplus \bar{p}z \\ &\overset{comm}{=} \bar{p}z \oplus p* \end{aligned}$$

and

$$\begin{aligned} pqx \oplus \overline{pq}(ry \oplus \bar{r}z) &\overset{comm}{=} \overline{pq}(\bar{r}z \oplus ry) \oplus pq* \\ &= \overline{pq}(\bar{r}z + ry) \oplus pq* \\ &\overset{(!!)}{=} \overline{pq}(\bar{r}z + ry) \boxplus pq* \\ &\overset{(5.4)}{=} \overline{pq}\bar{r}z \boxplus \overline{\overline{pq}\bar{r}}* \\ &= \bar{p}z \oplus p*, \end{aligned}$$

where the equality marked by (!!) holds because now $\bar{r}z + ry \in P$. □

Proof of Example 3.1.4. Assume we are in the situation of Example 3.1.4, i.e., $\mathbb{X}$ is a convex algebra and w is an extremal point of $\mathbb{X}$. Set $P = X \setminus \{w\}$, then P is a prime ideal. Further, let $\mathbb{P}_*$ be obtained as in Example 3.1.3 with $\mathbb{P} \leq \mathbb{X}$ by letting $*$ imitate w. Condition (5.1) is satisfied with equality, and hence the Gluing Lemma provides $\mathbb{X}_*$. The operations $p(-) \oplus \bar{p}(-)$ obtained in this way coincide with those written in Example 3.1.4. □

Proof of Theorem 5.1. The uniqueness part is easy to see. First, the action of $*$ determines which case of (1)–(3) occurs since $\mathrm{Ad}((\mathbb{D}_S)_*)$ is $\mathcal{D}_f S$ in case (1), $\emptyset$ in case (2), and $\{w\}$ in case (3). Now uniqueness of the point w in (2) and (3) follows since $\mathbb{D}_S$ is cancellative.

We have to show that every extension occurs in one of the described ways. Hence, let an extension $(\mathbb{D}_S)_*$ be given. If $\mathrm{P}((\mathbb{D}_S)_*) = \emptyset$, case (1) takes place. Assume that $\mathrm{P}((\mathbb{D}_S)_*) \neq \emptyset$ and choose $z \in \mathrm{P}((\mathbb{D}_S)_*)$ and $q \in (0,1)$. Set

$$w = \frac{1}{\bar{q}}\big([qz + \bar{q}*] - qz\big) \in \ell^1(S),$$

then $qz + \bar{q}* = qz + \bar{q}w$ by definition. We apply Lemma 4.3 with $\mathbb{D}_S \leq \ell^1(S)$ and z, w, q. This yields

$$px + \bar{p}* = px + \bar{p}w, \quad x \in \mathrm{P}((\mathbb{D}_S)_*), p \in (0,1), \tag{5.5}$$

and $\mathrm{Ad}((\mathbb{D}_S)_*) \subseteq \{x \in \mathcal{D}_f S \mid x \circ\!\!-\!\!\bullet w\} \subseteq \{w\}$.

As a linear combination of two elements of $\mathcal{D}_f S$, the element w is finitely supported. Further, by (5.5),

$$1 = \frac{1}{\bar{p}}\big(\|pz + \bar{p} * \|_1 - p\|z\|_1\big) \leq \|w\|_1 \leq \frac{1}{\bar{p}}\big(\|pz + \bar{p} * \|_1 + p\|z\|_1\big) = \frac{1+p}{1-p}$$

for all $p \in (0,1)$, and we see that $\|w\|_1 = 1$. Together, $w \in \mathcal{D}_f S$.

If $\mathrm{P}((\mathbb{D}_S)_*) = \mathcal{D}_f S$, we are in case (2) of the theorem. Otherwise, $\mathrm{P}((\mathbb{D}_S)_*) = (\mathcal{D}_f S)\setminus\{w\}$. This implies that w is an extremal point of $\mathbb{D}_S$, and we are in case (3). □

Next we investigate functoriality of one-point extensions. We say that a functor $F\colon \mathcal{EM}(\mathcal{D}_f) \to \mathcal{EM}(\mathcal{D}_f)$ naturally provides a one-point extension, if $\mathbb{X} \leq F\mathbb{X}$ and $F\mathbb{X}$ has carrier $X \cup \{*\}$ for $* \notin X$ for every algebra $\mathbb{X}$, and $(Ff)|_X = f$ for every convex map $f\colon \mathbb{X} \to \mathbb{Y}$. The latter property is (literally) a natural property: it says that the family of inclusion maps $\iota_X\colon \mathbb{X} \to F\mathbb{X}$ is a natural transformation of the identity functor to F.

An example of a functor possessing these properties is obtained by the black-hole construction: for an algebra $\mathbb{X}$ let $F\mathbb{X}$ be its black-hole extension, and for a convex map $f : \mathbb{X} \to \mathbb{Y}$ let Ff be the extension of f mapping $*$ (of $F\mathbb{X}$) to $*$ (of $F\mathbb{Y}$).

Theorem 5.3. *Let $F : \mathcal{EM}(\mathcal{D}_f) \to \mathcal{EM}(\mathcal{D}_f)$ be a functor such that for all objects $\mathbb{X}$ and for all morphisms $f : \mathbb{X} \to \mathbb{Y}$*

$$\mathbb{X} \leq F\mathbb{X},\ \textit{the carrier of } F\mathbb{X} \textit{ is } X \cup \{*\} \textit{ with } * \notin X, \qquad \begin{array}{ccc} F\mathbb{X} & \xrightarrow{\ Ff\ } & F\mathbb{Y} \\ {\scriptstyle\iota_X}\uparrow & & \uparrow{\scriptstyle\iota_Y} \\ \mathbb{X} & \xrightarrow[\ f\ ]{} & \mathbb{Y} \end{array} \tag{5.6}$$

Then, for all $\mathbb{X}$, $F\mathbb{X}$ is the black-hole extension, and for all $f\colon \mathbb{X} \to \mathbb{Y}$, Ff is the extension of f mapping $*$ (of $F\mathbb{X}$) to $*$ (of $F\mathbb{Y}$).

We present the proof using two lemmata.

Lemma 5.4. *Assume that $F\colon \mathcal{EM}(\mathcal{D}_f) \to \mathcal{EM}(\mathcal{D}_f)$ satisfies (5.6), and let $f\colon \mathbb{X} \to \mathbb{Y}$ be a convex map. Then $(Ff)(*) = *$ and $f(\mathrm{P}(F\mathbb{X})) \subseteq \mathrm{P}(F\mathbb{Y})$, $f(\mathrm{Ad}(F\mathbb{X})) \subseteq \mathrm{Ad}(F\mathbb{Y})$.*

Proof. For the proof of $(Ff)(*) = *$, note that $(Ff)^{-1}(\{*\}) \subseteq \{*\}$ since $(Ff)|_X = f$. If f has a right inverse, say $g : \mathbb{Y} \to \mathbb{X}$ with $f \circ g = \mathrm{id}_{\mathbb{Y}}$, then $(Ff)((Fg)(*)) = *$, and hence $(Fg)(*) = *$. In turn also $(Ff)(*) = *$. Now let f be arbitrary. Let $\mathbb{Z}$ be an algebra which has only one element, a final object of $\mathcal{EM}(\mathcal{D}_f)$, and let $h : \mathbb{Y} \to \mathbb{Z}$ be the unique convex map. The map $h \circ f$ has a right inverse, and therefore $(Fh)((Ff)(*)) = (F(h \circ f))(*) = *$. Again, we obtain $(Ff)(*) = *$.

It remains to prove that f maps the respective prime ideals (sets of adherence) into each other. Let $x \in X$ and $p \in (0, 1)$. Then

$$pf(x) + \bar{p}* = p(Ff)(x) + \bar{p}(Ff)(*) = (Ff)(px + \bar{p}*) = \begin{cases} f(px + \bar{p}*) \in Y\ , & x \in \mathrm{P}(F\mathbb{X}), \\ (Ff)(*) = * & , \ x \in \mathrm{Ad}(F\mathbb{X}). \end{cases} \tag{5.7}$$

Thus, indeed, $f(x) \in \mathrm{P}(F\mathbb{Y})$ if $x \in \mathrm{P}(F\mathbb{X})$, and $f(x) \in \mathrm{Ad}(F\mathbb{Y})$ if $x \in \mathrm{Ad}(F\mathbb{X})$. □

Lemma 5.5. *Assume that $F\colon \mathcal{EM}(\mathcal{D}_f) \to \mathcal{EM}(\mathcal{D}_f)$ satisfies (5.6), and let S be an infinite set. Then $F\mathbb{D}_S$ is the black-hole extension of $\mathbb{D}_S$.*

Proof. Assume towards a contradiction that $\mathrm{P}(F\mathbb{D}_S) \neq \emptyset$. By Theorem 5.1 we find $w \in \mathcal{D}_f S$ such that $px + \bar{p}* = px + \bar{p}w$, $x \in \mathrm{P}(F\mathbb{D}_S)$, $p \in (0, 1)$. Fix $x \in \mathrm{P}(F\mathbb{D}_S)$ and $p \in (0, 1)$, and let $f\colon \mathbb{D}_S \to \mathbb{D}_S$ be an automorphism. Then $f(x) \in \mathrm{P}(F\mathbb{D}_S)$ by Lemma 5.4, and we can compute

$$pf(x) + \bar{p}w = pf(x) + \bar{p}* \overset{(5.7)}{=} f(px + \bar{p}*) = f(px + \bar{p}w) = pf(x) + \bar{p}f(w).$$

Cancelling $f(x)$ gives $w = f(w)$. Hence w is a fixpoint of every automorphism.

Since S is infinite, we can choose a point $s_1 \in S$ which lies outside of the support of w. Further, let $s_2 \in S$ be in the support of w, and let $\sigma\colon S \to S$ be the permutation of S which exchanges s_1 and s_2 and leaves all other points fixed. Since $\mathbb{D}_S$ is free with basis S, this permutation extends to an automorphism f of $\mathbb{D}_S$. But now $f(w) \neq w$, a contradiction. □

Proof of Theorem 5.3. The fact that Ff is the extension of f mapping $*$ to $*$ was shown in Lemma 5.4. It remains to show that, for every algebra $\mathbb{X}$, $F\mathbb{X}$ is the black-hole extension. Given $\mathbb{X}$, choose an infinite set S and a surjective convex map $f : \mathbb{D}_S \to \mathbb{X}$. This is possible since every convex algebra is the image of a free convex algebra, and if $S \supseteq S'$ then there is a surjective homomorphism from $\mathbb{D}_S$ to $\mathbb{D}_{S'}$. Then, by Lemma 5.4 and Lemma 5.5, $\mathrm{Ad}(F\mathbb{X}) \supseteq f(\mathrm{Ad}(F\mathbb{D}_S)) = f(\mathbb{D}_S) = \mathbb{X}$. □

6. Extensions of $\mathcal{P}_c\mathbb{D}$

In this section we formulate and prove Theorem 6.11 where we describe the set of all extensions $(\mathcal{P}_c\mathbb{D})_*$ for convex algebras $\mathbb{D}$ which are convex subsets of a vector space (equivalently, cancellative) and satisfy a certain linear boundedness condition. Theorem 6.11 applies in particular to the algebra $\mathbb{D} = \mathbb{D}_S$ of finitely supported distributions over S.

We start with some algebraic preliminaries. First, we recall the notion of linear boundedness, see e.g. [3, Definition 1.1].

Definition 6.1. A convex algebra $\mathbb{X}$ is *linearly bounded*, if every homomorphism of the convex algebra $(0, \infty)$ into $\mathbb{X}$ is constant.

Intuitively, a convex algebra is linearly bounded if it does not contain an infinite ray. A large class of examples of linearly bounded algebras is given by topologically bounded subsets of a topological vector space. Recall that a topological vector space is a vector space endowed with a topology such that addition and scalar multiplication are continuous. Our standard reference for the theory of topological vector spaces is [26].

Definition 6.2. Let $\mathbb{V}$ be a topological vector space. A subset $D \subseteq V$ is *bounded*, if for every neighbourhood U of 0 there exists $r_0 > 0$ such that $D \subseteq rU$, $r > r_0$ (cf. [26, p.8]).

For example, if $\mathbb{V}$ is a normed space (with a norm denoted by $\|.\|$), then a subset D is bounded in this sense if and only if $\sup_{x\in D} \|x\| < \infty$.

We could not find an explicit reference for the following (intuitive) fact, and hence provide a complete proof.

Lemma 6.3. *Let $\mathbb{V}$ be a topological vector space. Then for every bounded convex subset D of $\mathbb{V}$, the convex algebra $\mathbb{D}$ is linearly bounded.*

Proof. Let D be a bounded convex and nonempty subset of a topological vector space $\mathbb{V}$, and let $\varphi\colon (0, \infty) \to \mathbb{D}$ be a convex map. By [11, Proposition 2.7] we find a convex extension $\Phi : \mathbb{R} \to \mathbb{V}$ of φ. Set $\Psi = \Phi - \Phi(0)$, then Ψ is convex and $\Psi(0) = 0$. The purpose of this normalisation is that it allows us to conclude $\Psi(tx) = t\Psi(x)$, $t > 0$, $x \in \mathbb{R}$: If $t = 1$ this is trivial. If $t < 1$ use convexity to compute $\Psi(tx) = \Psi(tx + (1 - t)0) = t\Psi(x) + (1 - t)\Psi(0) = t\Psi(x)$. If $t > 1$, use the already known to compute $\Psi(x) = \Psi(\frac{1}{t} \cdot tx) = \frac{1}{t}\Psi(tx)$.

Assume now towards a contradiction that φ is not constant. Then we can choose $s \in (0, \infty)$ with $\varphi(s) \neq \Phi(0)$, i.e., $\Psi(s) \neq 0$. Choose a neighbourhood U of 0 such that

$\Psi(s) \notin U$. Since D is bounded, also its translate $D - \Phi(0)$ is bounded. Hence, we find $r > 0$ with $D - \Phi(0) \subseteq rU$. From

$$r\Psi(s) = \Psi(rs) = \varphi(rs) - \Phi(0) \in D - \Phi(0) \subseteq rU$$

we obtain $\Psi(s) \in U$, and reached a contradiction. □

Remark 6.4. Let $\mathbb{V}$ be a vector space over $\mathbb{R}$. Then, for each fixed $w \in \mathbb{V}$ and $t \in \mathbb{R}\setminus\{0\}$, we have the translation map $x \mapsto x + w$ and the scaling map $x \mapsto tx$. They are bijective convex maps on $\mathbb{V}$. Applying $\mathcal{P}_c$ on these maps gives bijective convex maps on $\mathcal{P}_c\mathbb{V}$. Moreover, a subset $A \in \mathcal{P}_c\mathbb{V}$ is linearly bounded if and only if $t(A + w)$ is linearly bounded.

The following observation holds for all cancellative convex algebras $\mathbb{D}$.

Lemma 6.5. *Let $\mathbb{D}$ be a convex algebra and consider $\mathbb{X} = \mathcal{P}_c\mathbb{D}$. If $\mathbb{D}$ is cancellative, then $A \circ\!\!-\!\!\bullet \{x\} \Rightarrow A = \{x\}$ for all $A \in X$, $x \in D$.*

Proof. Let $a \in A$. Then $pa + \bar{p}x = x = px + \bar{p}x$ which after cancelling with x yields $a = x$. Since A is nonempty, as it belongs to $\mathcal{P}_c\mathbb{D}$, we get $A = \{x\}$. □

Under a linear boundedness condition, the roles of A and $\{x\}$ can be exchanged.

Lemma 6.6. *Let $\mathbb{V}$ be a vector space over $\mathbb{R}$, let $A \in \mathcal{P}_c\mathbb{V}$, and assume that $A - A$ is linearly bounded. Then*

$$\bigcap_{p\in[0,1)} \big(p\{x\} + \bar{p}A\big) \subseteq \{x\}, \qquad \text{for } x \in V.$$

In particular, $\{x\} \circ\!\!-\!\!\bullet A \Rightarrow A = \{x\}$ for all $x \in V$.

Proof. Note first that $A - A$ is convex. Let y belong to the intersection. Then $y \in A$ and for each $p \in (0, 1)$ we find $a_p \in A$ with $y = px + \bar{p}a_p$. This implies

$$\frac{p}{\bar{p}}(x - y) = y - a_p \in A - A, \quad \text{for } p \in (0, 1).$$

Any positive real number t can be represented as $\frac{p}{\bar{p}}$, namely with $p = \frac{t}{1+t} \in (0, 1)$. It is easy to check then that $\varphi\colon t \mapsto t(x - y)$ is a convex homomorphism from $(0, \infty)$ to $A - A$. Since $A - A$ is linearly bounded, φ is constant, which further implies $x = y$. □

In order to construct extensions where $*$ imitates an outer element, we need the following notion of visibility closure.

Definition 6.7. Let $\mathbb{X}$ be a convex algebra and $A \in \mathcal{P}_c\mathbb{X}$. The *visibility hull* of A is

$$\mathrm{Vis}(A) = \big\{x \in X \mid \forall a \in A.\ \forall p \in (0, 1).\ px + \bar{p}a \in A\big\}.$$

The set A is *visibility closed* if $A = \mathrm{Vis}(A)$.

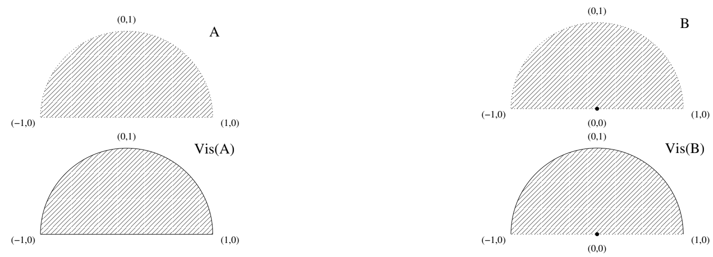
Example 6.8. Let $A \subseteq \mathbb{R}^2$ be the open half-disk $A = \{(t_1, t_2) \in \mathbb{R}^2 \mid t_1^2 + t_2^2 < 1, t_2 > 0\}$. Then $\mathrm{Vis}(A)$ is the closed half disk, shown in Figure 1a.

Now consider $B = A \cup \{(0, 0)\}$. Then the part of the boundary of B located on the t_1-axis does not belong to $\mathrm{Vis}(B)$, see Figure 1b.

Let $\mathbb{V}$ be a vector space over $\mathbb{R}$. The *affine hull* of a subset $A \subseteq V$ is

$$\mathrm{aff}(A) = \{\sum_{i=1}^{n} t_i x_i \mid n \geq 1, x_i \in A, t_i \in \mathbb{R}, \sum_{i=1}^{n} t_i = 1\}.$$

The affine hull of A is the smallest affine subspace of $\mathbb{V}$ containing A, see e.g. [24, p.6].



(A) A and $\mathrm{Vis}(A)$ (B) B and $\mathrm{Vis}(B)$

Figure 1. Visibility hulls

Lemma 6.9. *Let $\mathbb{V}$ be a vector space over $\mathbb{R}$, and $A \in \mathcal{P}_c\mathbb{V}$. Then*

(1) $\mathrm{Vis}(A) = \bigcap_{\substack{a\in A\\ p\in(0,1)}} \frac{1}{p}(A - \bar{p}a) \subseteq \mathrm{aff}(A)$.
(2) $\mathrm{Vis}(A)$ *is convex.*
(3) $A \subseteq \mathrm{Vis}(A)$ *and* $\mathrm{Vis}(\mathrm{Vis}(A)) = \mathrm{Vis}(A)$.
(4) $\mathrm{Vis}(\{z\}) = \{z\}$ *for all* $z \in V$.
(5) *If $\mathbb{V}$ is a topological vector space, then* $\mathrm{Vis}(A) \subseteq \overline{A}$, $\overline{A}$ *being the topological closure of* A.

Proof.
(1) We have

$$x \in \mathrm{Vis}(A) \;\Leftrightarrow\; \forall a \in A.\, \forall p \in (0,1).\; px + \bar{p}a \in A \;\Leftrightarrow\; \forall a \in A.\, \forall p \in (0,1).\; x \in \frac{1}{p}(A - \bar{p}a)$$

(2) By 1., the set $\mathrm{Vis}(A)$ is the intersection of convex sets.
(3) Let $x \in A$. Then $px + \bar{p}a \in A$, $a \in A$, $p \in (0,1)$, since A is convex. Thus $A \subseteq \mathrm{Vis}(A)$. Assume that $x \in \mathrm{Vis}(\mathrm{Vis}(A))$, and let $a \in A$, $p, q \in (0,1)$. Then $px + \bar{p}a \in \mathrm{Vis}(A)$, since $a \in A \subseteq \mathrm{Vis}(A)$, and hence $qpx + \overline{qp}a = q(px + \bar{p}a) + \bar{q}a \in A$. Every number $r \in (0,1)$ can be represented as $r = pq$ with some $p, q \in (0,1)$, and we conclude that $x \in \mathrm{Vis}(A)$.
(4) We have $\frac{1}{p}(\{z\} - \bar{p}z) = \{z\}$, $p \in (0,1)$. By 1., $\mathrm{Vis}(\{z\}) = \{z\}$.
(5) Let $x \in \mathrm{Vis}(A)$ and $a \in A$. Then $x = \lim_{p\to 1}(px + \bar{p}a) \in \overline{A}$. □

The operator $\mathrm{Vis}\colon \mathcal{P}_c\mathbb{V} \to \mathcal{P}_c\mathbb{V}$ is not monotone, as demonstrated in Example 6.8. Hence, it is not the restriction of a topological closure operator to $\mathcal{P}_c\mathbb{V}$. Still, it is related to topological closures:

Remark 6.10. Let $\mathbb{V}$ be a topological vector space and $A \in \mathcal{P}_c\mathbb{V}$ *relatively closed*, i.e., closed in $\mathrm{aff}(A)$ w.r.t. the subspace topology. Then A is visibility closed. This follows by putting together Lemma 6.9.1 and 5. The converse does not hold, as demonstrated by the set $\mathrm{Vis}(B)$ from Example 6.8. This observation shows for example that $\mathrm{Vis}(\mathcal{D}_f S) = \mathcal{D}_f S$.

We can now formulate our description of extensions of $\mathcal{P}_c\mathbb{D}$.

Theorem 6.11. *Let $\mathbb{V}$ be a vector space over $\mathbb{R}$, let D be a convex subset of V with more than one element, and consider the convex algebra $\mathbb{X} = \mathcal{P}_c\mathbb{D}$. One-point extensions of $\mathbb{X}$ can be constructed as follows:*

(1) *The black-hole behaviour, where the set of adherence equals X.*
(2) *Let $C \in \mathcal{P}_c(\mathrm{Vis}(D))$, and let $*$ imitate C on all of X.*

(3) *Let w be an extremal point of $\mathbb{D}$, and let $*$ imitate $\{w\}$ on $X \setminus \{\{w\}\}$ and adhere $\{w\}$.*
(4) *Let $C \in \mathcal{P}_c(\mathrm{Vis}(D))$ with at least two elements, assume $\mathrm{conv}\{A \in X \mid A \not\multimap C\} \neq X$, and let $I = \mathrm{conv}\{A \in X \mid A \not\multimap C\}$. Let $P \neq X$ be a prime ideal in $\mathbb{X}$ with $I \subseteq P$, and let $*$ imitate C on P and adhere $X \setminus P$.*

Assume in addition that $D - D$ is linearly bounded. Then every one-point extension of $\mathbb{X}$ can be obtained in this way. Each two of these extensions are different: the point w in case (3), the set C in cases (2), (4), and the prime ideal P in case (4), are uniquely determined by a given extension.

We are familiar with the constructions (1)–(3) from Example 3.1 and Theorem 5.1. That (4) gives extensions follows from the Gluing Lemma, Lemma 5.2.

Proof of Theorem 6.11; constructions. The black-hole behaviour is always possible, cf. Example 3.1.1. Assume we are given $C \in \mathcal{P}_c(\mathrm{Vis}(D))$. If $C \subseteq D$ use Example 3.1.2. Otherwise, use Example 3.1.3 with the algebra $\mathbb{Y} = \mathcal{P}_c\mathbb{V}$ and its element C. The necessary hypothesis, that $X \cup \{C\}$ is convex, is satisfied since $C \subseteq \mathrm{Vis}(D)$ and hence $pA + \bar{p}C \subseteq D$ for all $A \subseteq D$, $p \in (0,1)$. Assume we are given an extremal point w of $\mathbb{D}$. Then $\{w\}$ is an extremal point of $\mathcal{P}_c\mathbb{D}$. The construction (3) is exactly Example 3.1.4 applied with this extremal point.

Now consider the construction (4). Assume C and P have the properties stated in (4). We first show

$$pA + \bar{p}C \in P, \qquad \text{for } A \in P,\ p \in (0,1). \tag{6.1}$$

Assume towards a contradiction that $pA + \bar{p}C \notin P$ for some $A \in P$, $p \in (0,1)$. Since $P \supseteq I$ we have $pA + \bar{p}C \notin I$ and hence $pA + \bar{p}C \multimap C$. Since $C \subseteq \mathrm{Vis}(D)$ we have $sA + \bar{s}C \in X$, for $s \in (0,1)$. Choose $q \in (0,1)$, then

$$C = q(pA + \bar{p}C) + \bar{q}C = qpA + \overline{qp}C \in X.$$

Since P is an ideal in $\mathbb{X}$, we get $pA + \bar{p}C \in P$, a contradiction.

The relation (6.1) implies that $P \cup \{C\}$ is a convex subset of $\mathcal{P}_c\mathbb{V}$, and Examples 3.1.2/3 provide an extension $\mathbb{P}_*$. Again by (6.1), it holds that $\mathrm{Ad}(\mathbb{P}_*) = \emptyset$. To apply the Gluing Lemma, we need to check (5.1). Let $A \in P$, $B \in X \setminus P$, $p \in (0,1)$. Since $I \subseteq P$ we have $B \multimap C$, and hence $pA + \bar{p}B \multimap pA + \bar{p}C = pA + \bar{p}*$ by Lemma 2.7.3. ☐

Assume that $D - D$ is linearly bounded. Our task is to show that every given extension $\mathbb{X}_*$ can be realised as described in (1)–(4) of the theorem, and show uniqueness. The proof relies on the following lemma.

Lemma 6.12. *Assume $\mathbb{X}_*$ is an extension with $\mathrm{P}(\mathbb{X}_*) \neq \emptyset$. Then $\mathrm{Ad}(\mathbb{X}_*)$ contains at most one singleton set.*

Proof. Assume that $\{x\}, \{y\} \in \mathrm{Ad}(\mathbb{X}_*)$, and choose $A \in \mathrm{P}(\mathbb{X}_*)$. Then, for each $p \in (0,1)$, by Lemma 2.7.3

$$pA + \bar{p}\{x\} \multimap pA + \bar{p}*, \quad pA + \bar{p}\{y\} \multimap pA + \bar{p}*.$$

Set $C = pA + \bar{p}*$. Then $C \in \mathrm{P}(\mathbb{X}_*)$ and for each $q \in (0,1)$

$$q(pA + \bar{p}\{x\}) + \bar{q}C = C = q(pA + \bar{p}\{y\}) + \bar{q}C.$$

Thus, for each $a \in A, c \in C$ we find $a_1 \in A, c_1 \in C$ with

$$q\bar{p}x + \overline{q\bar{p}}\Big(\frac{qp}{\overline{q\bar{p}}}a + \frac{\bar{q}}{\overline{q\bar{p}}}c\Big) = q\bar{p}y + \overline{q\bar{p}}\Big(\frac{qp}{\overline{q\bar{p}}}a_1 + \frac{\bar{q}}{\overline{q\bar{p}}}c_1\Big),$$

and hence

$$\frac{q\bar{p}}{\overline{q\bar{p}}}(x-y) \in D-D.$$

Any positive real number t can be represented as $\frac{q\bar{p}}{\overline{q\bar{p}}}$ with some $p,q \in (0,1)$, for example use $p=\frac{1}{2t+1}, q=\frac{2t+1}{2t+2}$. Thus $\varphi\colon t \mapsto t(x-y)$ is a homomorphism of $(0,\infty)$ to $D-D$. Since $D-D$ is linearly bounded, φ is constant, and hence $x=y$. □

Proof of Theorem 6.11; all $\mathbb{X}_*$ are obtained. Let an extension $\mathbb{X}_*$ of $\mathbb{X}$ be given. If $\mathrm{P}(\mathbb{X}_*)=\emptyset$ then case (1) of the theorem holds. Assume in the following that $\mathrm{P}(\mathbb{X}_*) \neq \emptyset$.

By Lemma 6.12, $\mathrm{Ad}(\mathbb{X}_*)$ contains at most one singleton set. Since D has more than one element, we find $z \in D$ with $\{z\} \in \mathrm{P}(\mathbb{X}_*)$. Choose $q \in (0,1)$. Then $q\{z\}+\bar{q}* \in \mathrm{P}(\mathbb{X}_*) \subseteq \mathcal{P}_{\mathrm{c}}\mathbb{V}$. We will show that $*$ imitates the convex set

$$C=\frac{1}{\bar{q}}\big([q\{z\}+\bar{q}*]-qz\big) \in \mathcal{P}_{\mathrm{c}}\mathbb{V}.$$

By definition, C satisfies $q\{z\}+\bar{q}*=q\{z\}+\bar{q}C$. Since singletons are $\mathcal{P}_{\mathrm{c}}\mathbb{V}$-cancellable, as noted after Definition 2.9, the hypothesis of Lemma 4.3 are fulfilled. We conclude that $*$ imitates C on $\mathrm{P}(\mathbb{X}_*)$ and that $\mathrm{Ad}(\mathbb{X}_*) \subseteq \big\{A \in X \mid A \circ\!\!-\!\!\bullet C\big\}$.

Consider the case that $\mathrm{Ad}(\mathbb{X}_*)$ contains a singleton, say $\{w\} \in \mathrm{Ad}(\mathbb{X}_*)$. Since $C \subseteq \frac{1}{\bar{q}}(D-qz)$, the set $C-C$ is linearly bounded, cf. Remark 6.4. Lemma 6.6 implies that $C=\{w\}$ and Lemma 6.5 that $\{A \in X \mid A \circ\!\!-\!\!\bullet C\}=\{\{w\}\}$. We see that $\mathrm{P}(\mathbb{X}_*)=X \setminus \{\{w\}\}$ and that $*$ imitates $\{w\}$ on $\mathrm{P}(\mathbb{X}_*)$. Since $X \setminus \{\{w\}\}$ is an ideal in $\mathbb{X}$, also $D \setminus \{w\}$ is an ideal in $\mathbb{D}$, i.e., w is an extremal point of $\mathbb{D}$. Thus $\mathbb{X}_*$ has the form described in case (3).

Consider the case that $\mathrm{Ad}(\mathbb{X}_*)$ contains no singleton. Hence all singletons are in $\mathrm{P}(\mathbb{X}_*)$. Then

$$p\{y\}+\bar{p}C=p\{y\}+\bar{p}* \subseteq X, \qquad \text{for } y \in D,\ p \in (0,1).$$

Thus $C \subseteq \bigcap_{\substack{y \in D\\ p \in (0,1)}} \frac{1}{\bar{p}}(D-py)=\mathrm{Vis}(D)$, by Lemma 6.9.1. If $\mathrm{Ad}(\mathbb{X}_*)=\emptyset$, case (2) of the theorem holds. Assume that $\mathrm{Ad}(\mathbb{X}_*) \neq \emptyset$. If C contains only one element, say $C=\{w\}$, we would have

$$\emptyset \neq \mathrm{Ad}(\mathbb{X}_*) \subseteq \{A \in X \mid A \circ\!\!-\!\!\bullet \{w\}\}=\{\{w\}\}.$$

From this $\mathrm{Ad}(\mathbb{X}_*)=\{\{w\}\}$, a contradiction. Thus C has at least two elements. Since

$$X \neq \mathrm{P}(\mathbb{X}_*)=X \setminus \mathrm{Ad}(\mathbb{X}_*) \supseteq \{A \in X \mid A \circ\!\!\not-\!\!\bullet C\},$$

the convex hull of $\{A \in X \mid A \circ\!\!\not-\!\!\bullet C\}$ is not X and case (4) of the theorem holds. □

Proof of Theorem 6.11; uniqueness. The uniqueness assertion of the theorem follows since $\mathrm{P}(\mathbb{X}_*)$ always contains singletons by Lemma 6.12, and singletons are cancellable in $\mathcal{P}_{\mathrm{c}}\mathbb{V}$. □

The following example shows that the linear boundedness condition in Theorem 6.11 cannot be dropped without admitting other types of constructions.

Example 6.13. Let $\mathbb{V}=\mathbb{R}^2$ and $D=\{(t_1,t_2) \in \mathbb{R}^2 \mid t_2>0\} \cup \{(0,0)\}$. Set $P=\{A \in \mathcal{P}_{\mathrm{c}}\mathbb{D} \mid (0,0) \notin A\}$. First we show that P is a prime ideal of $\mathcal{P}_{\mathrm{c}}\mathbb{D}$. Denote by $f\colon \mathbb{R}^2 \to \mathbb{R}$ the projection onto the second coordinate. Let $A \in P$, $B \in \mathcal{P}_{\mathrm{c}}\mathbb{D}$, and $p \in (0,1)$. Then, for each $a \in A$ and $b \in B$,

$$f(pa+\bar{p}b)=p\underbrace{f(a)}_{>0}+\bar{p}\underbrace{f(b)}_{\geq 0}>0.$$

Thus $pA + \bar{p}B \in P$, and we see that P is an ideal. If $A, B \in (\mathcal{P}_c\mathbb{D}) \setminus P$, the point $(0,0)$ belongs to both of A and B and hence also to each convex combination of A and B. Thus $(\mathcal{P}_c\mathbb{D}) \setminus P$ is convex, and P is prime.

Set $C = D \cup \{(t_1, t_2) \in \mathbb{R}^2 \mid t_2 = 0, t_1 > 0\}$. Then $C \subseteq \mathrm{Vis}(D \setminus \{(0,0)\})$, and we can define an extension $\mathbb{P}_*$ by letting $*$ imitate C on all of P. We check that the compatibility condition (5.1) of the gluing lemma is satisfied. To this end we show that every element of $(\mathcal{P}_c\mathbb{D}) \setminus P$ adheres to C (in $\mathcal{P}_c\mathbb{V}$), and refer to Lemma 2.7.3. Let $B \in (\mathcal{P}_c\mathbb{D}) \setminus P$ and $p \in (0,1)$. Since $B \subseteq D \subseteq C$ and C is convex, we have $pB + \bar{p}C \subseteq C$. For the reverse inclusion, observe that $tC = C$ for all $t > 0$. We can thus write any element $c \in C$ as

$$c = p \cdot (0,0) + \bar{p} \cdot \big(\frac{1}{\bar{p}}c\big) \in pB + \bar{p}C.$$

Applying the Gluing Lemma we obtain an extension $(\mathcal{P}_c\mathbb{D})_*$. This extension is not among the ones listed in Theorem 6.11, since $C \not\subseteq \mathrm{Vis}(D)$.

Unboundedness of D enters in this example in the way that it enables us to let $*$ imitate a cone. In fact, dropping linear boundedness, one can still show that an extension which is not of type (1)–(4) of the theorem, must be such that $*$ imitates some cone whose apex lies in D. However, we have no description of which cones occur that way.

7. Termination in Probabilistic Automata

A (simple) *probabilistic automaton* (PA, for short) [29, 28] is a triple $M = (S, A, \to)$ where S is the set of states, A the set of actions, and $\to \subseteq S \times A \times \mathcal{D}_f S$ the transition relation. As usual, we write $a \xrightarrow{a} \xi$ for $(s, a, \xi) \in \to$ and say that s makes an a-step to ξ.

A probabilistic automaton $M = (S, A, \to)$ can be identified [1, 32] with a $(\mathcal{P}\mathcal{D}_f)^A$-coalgebra (S, c_M) where $c_M \colon S \to (\mathcal{P}\mathcal{D}_f S)^A$ on **Sets** and $s \xrightarrow{a} \xi$ in M iff $\xi \in c_M(s)(a)$.

We say that a PA M is *input enabled*, if for all $s \in S$ and all $a \in A$, the set $\{\xi \in \mathcal{D}_f S \mid s \xrightarrow{s} \xi\}$ is nonempty. Input-enabledness implies that the associated coalgebra c_M is actually a $(\mathcal{P}_{\neq\emptyset}\mathcal{D}_f)^A$-coalgebra where $\mathcal{P}_{\neq\emptyset}$ denotes the nonempty powerset functor.

We call this view on PA (their standard definition and the observation that they are coalgebras on **Sets**) the classical view.

In the classical view, the canonical semantics for PA is bisimilarity. However, it is possible to give PA more intricate and useful *distribution* semantics. This was noticed in the last decade by many authors, and studied from a coalgebraic perspective in [2], by viewing them as *belief-state transformers* and making the underlaying convex algebra structure explicit. To be precise, we state the following property, which is an instance of [2, Lemma 25]. By U we denote the forgetful functor from $\mathcal{EM}(\mathcal{D}_f)$ to **Sets**.

Lemma 7.1. *There is a 1-1 correspondence between input enabled PA, i.e., $(\mathcal{P}_{\neq\emptyset}\mathcal{D}_f)^A$-coalgebras on* **Sets***, and $\mathcal{P}_c^A$-coalgebras on $\mathcal{EM}(\mathcal{D}_f)$ with carriers free algebras:*

$$\frac{c_M \colon S \to (\mathcal{P}_{\neq\emptyset}\mathcal{D}_f S)^A \quad \textit{in} \quad \textbf{Sets}}{c_M^{\#} \colon \mathbb{D}_S \to (\mathcal{P}_c\mathbb{D}_S)^A \quad \textit{in} \quad \mathcal{EM}(\mathcal{D}_f)}$$

Given c_M, $c_M^{\#} = \alpha \circ \mathcal{D}_f c$ for α being the convex algebra structure on $(\mathcal{P}_c \mathbb{D}_S)^A$. Concretely, for $\xi = \sum p_i s_i \in \mathbb{D}_S$ and $a \in A$ we have

$$c_M^{\#}(\xi)(a) = \Big\{ \sum p_i \xi_i \mid s_i \xrightarrow{a} \xi_i \Big\}.$$

Given $c_M^{\#}$, $c_M = U c_M^{\#} \circ \eta$ where η is the unit of the distribution monad, i.e., $\eta(s) = \delta_s$ is the Dirac distribution that assigns probability 1 to the state s. Concretely, for $s \in S$ and $a \in A$ we have $c_M(s)(a) = c_M^{\#}(\delta_s)$. □

The fact that input is always enabled is critical here, as $\mathcal{P}_c$ on $\mathcal{EM}(\mathcal{D}_f)$ is the *nonempty* convex powerset and, as discussed above after Definition 2.9, this nonemptiness is crucial in order to get a convex algebra structure.

Our original motivation when starting the work was to answer the question: What are all possible one-point extensions of $\mathcal{P}_c \mathbb{D}_S$ in order to be able to allow for termination in a belief-state transformer, i.e., in order to overcome the restriction of input-enabledness.

Theorem 6.11 answers this question fully: All one-point extensions of $\mathbb{X} = \mathcal{P}_c \mathbb{D}_S$ with $X = \mathcal{P}_c \mathcal{D}_f S$ are given by (1)-(4) below and they are all different.

(1) The black-hole extension, where the set of adherence equals X.
(2) Let $C \in X$ and let $*$ imitate C on all of X.
(3) Let $w = \delta_s$ be a Dirac distribution for $s \in S$, and let $*$ imitate $\{w\}$ on $X \setminus \{\{w\}\}$ and adhere $\{w\}$.
(4) Let $C \in X$ with at least two elements, assume $\operatorname{conv}\{A \in X \mid A \not\multimap C\} \neq X$, and let $I = \operatorname{conv}\{A \in X \mid A \not\multimap C\}$. Let $P \neq X$ be a prime ideal in $\mathbb{X}$ with $I \subseteq P$, and let $*$ imitate C on P and adhere $X \setminus P$.

Here, we get a simplification of the formulation of Theorem 6.11 as (as noted above Theorem 6.11) $\operatorname{Vis}(\mathbb{D}_S) = \mathbb{D}_S$ and the extremal points of $\mathbb{D}_S$ are the Dirac distributions on S. The case (4) does not simplify further, which we elaborate on some examples in Section 8.

Moreover, by Theorem 5.3, we know that only the black-hole extension is functorial, and a 1-1 correspondence between (not necessarily input-enabled) PA, $(\mathcal{P}\mathcal{D}_f)^A$-coalgebras on **Sets**, and $(\mathcal{P}_c + 1)^A$-coalgebras on $\mathcal{EM}(\mathcal{D}_f)$ with carriers free algebras, where $- + 1$ is the black-hole extension functor on $\mathcal{EM}(\mathcal{D}_f)$, is also an instance of [2, Lemma 25].

8. Illustrative Examples of Type (4) Extensions

In what follows, in particular in Example 8.4 and Example 8.6, we discuss the possible extensions of $\mathcal{P}_c \mathbb{D}$ when D is a compact convex subset of $\mathbb{R}^n$ (which is not a singleton), and in particular when $D = \mathcal{D}_f S$ when S is a finite set. The proofs of the stated facts depend heavily on arguments from convex geometry and topology and are deferred to Section 9. Here we only recall the necessary terminology and a few facts taken from the textbooks [24] and [27].

Definition 8.1. Let $A \in \mathcal{P}_c \mathbb{R}^n$. The *relative interior* $\operatorname{ri}(A)$ of A is the interior of A as a subset of $\operatorname{aff}(A)$, where $\operatorname{aff}(A)$ is endowed with the subspace topology inherited from $\mathbb{R}^n$. The set A is *relatively open*, if $A = \operatorname{ri}(A)$ (cf. [24, p.44]).

Definition 8.2.

(1) A *convex body* is a compact, convex, and nonempty subset of an euclidean space $\mathbb{R}^n$. The set of all convex bodies in $\mathbb{R}^n$ is denoted as $\mathcal{K}^n$ (cf. [27, p.8]).

(2) For given $D \in \mathcal{K}^n$, by $\mathcal{K}(D)$ we denote the set $\mathcal{K}(D) = \{A \in \mathcal{K}^n \mid A \subseteq D\}$ (cf. [27, p.157]).
(3) Two convex bodies $A, B \in \mathbb{R}^n$ are *homothetic*, if either one of them is a singleton or there exist $s > 0, x \in \mathbb{R}^n$ with $B = sA + x$ ([27, p.xii]).
(4) A convex body $A \in \mathcal{K}^n$ is *indecomposable*, if a representation $A = B + C$ is only possible with B and C both homothetic to A (cf. [27, p.150]).

Remark 8.3.

(1) When endowed with the Minkowski sum and (pointwise declared) multiplication with positive scalars, $\mathcal{K}^n$ is a convex cone, in particular a convex algebra. These operations are continuous w.r.t. the Hausdorff metric. See [27, p.42, p.126]. Recall in this place that the Hausdorff metric d_H is defined as
$$d_H(A, B) = \max\big\{\sup_{a\in A}\inf_{b\in B} d(a, b), \sup_{b\in B}\inf_{a\in A} d(a, b)\big\}$$
where d denotes the euclidean metric of $\mathbb{R}^n$.
(2) The set $\mathcal{K}(D)$ carries a subalgebra of $\mathcal{P}_c\mathbb{D}$. It is compact w.r.t. the Hausdorff metric. See [27, Theorem 1.8.6].
(3) On the set of all convex bodies having more than one point, homothety induces an equivalence relation.

Example 8.4. Let $n \geq 1$, let D be a compact convex subset of $\mathbb{R}^n$ with more than one point, and consider $\mathbb{X} = \mathcal{P}_c\mathbb{D}$. The first observation is that $\mathrm{Vis}(D) = D$ since D is closed. Hence, the extensions $\mathbb{X}_*$ described in (2) of Theorem 6.11 are in one-to-one correspondence with X itself. The extensions described in (3) correspond to the extremal points $\mathrm{Ext}\, D$ of D. Since $D = \mathrm{conv}(\mathrm{Ext}\, D)$ by Minkowski's theorem, cf. [27, Corollary 1.4.5], this set is certainly not empty. However, it is also not too large, e.g., $\mathrm{ri}(D)$ does not contain any extremal points and is a dense subset of D, cf. [27, Theorem 1.1.14].

Case (4) of the theorem is the most intriguing. To explain that the set of extensions occurring in this way has a complicated structure, we consider two particular situations. Namely, that the element C is either closed or relatively open.

(1) Let $C \in \mathcal{K}(D)$ with more than one point. Then (by Lemma 9.2 and Lemma 9.3 in Section 9)
$$\{A \in X \mid A \circ\!\!-\!\!\bullet C\} = \{C\}, \quad I = \mathrm{conv}\big(X \setminus \{C\}\big) = \begin{cases} X \setminus \{C\}\,, & C \in \mathrm{Ext}\, \mathcal{K}(D), \\ X \quad\quad\quad, & \text{otherwise.} \end{cases}$$
Hence, C is eligible for the construction in (4) if and only if $C \in \mathrm{Ext}\, \mathcal{K}(D)$. If C is an extremal point there is a unique choice for the prime ideal P, namely $P = X \setminus \{C\}$.
(2) Let $C \in \mathcal{K}(D)$ be relatively open with more than one point. Then (by Lemma 9.1 in Section 9)
$$\{A \in X \mid A \circ\!\!-\!\!\bullet C\} = \{A \in X \mid \overline{A} = \overline{C}\}, \quad I = \mathrm{conv}(\{A \in X \mid \overline{A} \neq \overline{C}\}).$$
Assume that $\overline{C} \in \mathrm{Ext}\, \mathcal{K}(D)$. Then I itself is a prime ideal, and we may choose $P = I$ in (4). There are also other possible choices for P. For example, $P = X \setminus \{\overline{C}\}$ is a prime ideal.

If $\overline{C} \notin \mathrm{Ext}\, \mathcal{K}(D)$, we have no general result telling how large I will be. In some concrete low-dimensional examples, we saw that $I \neq X$ may happen and there are many prime ideals admissible for (4).

To summarize, we exhibited two families of different extensions: 1. A unique extension is associated with each extremal point C of $\mathcal{K}(D)$. Points $C \in \mathcal{K}(D) \setminus \operatorname{Ext}\mathcal{K}(D)$ are not eligible; 2. Every relatively open subset C with $\overline{C} \in \operatorname{Ext}\mathcal{K}(D)$ is eligible and each such set gives rise to several extensions. If $\overline{C} \notin \operatorname{Ext}\mathcal{K}(D)$, examples indicate that C can be eligible and give rise to different extensions.

Remark 8.5. Describing $\operatorname{Ext}\mathcal{K}(D)$ is an open problem in convex geometry. $\operatorname{Ext}\mathcal{K}(D)$ must be large in the sense that $\operatorname{conv}\operatorname{Ext}\mathcal{K}(D)$ is dense in $\mathcal{K}(D)$ w.r.t. the Hausdorff metric. This fact is shown by considering $\mathcal{K}(D)$ as a compact convex subset of the space of continuous functions on the n-dimensional sphere via passing to support functions (e.g. [27, Theorems 1.7.1,1.8.11 and p.150]), and applying the Krein-Milman theorem (e.g. [26, Theorem 3.23]).

The only situation we know where an explicit description of $\operatorname{Ext}\mathcal{K}(D)$ is possible is [13, Theorem] where D is a strictly convex subset of $\mathbb{R}^2$ with nonempty interior.

When D is a simplex (defined below), a bit more can be said about $\operatorname{Ext}\mathcal{K}(D)$.

Example 8.6. Let $n \geq 2$, let S be an n-element set, and consider $\mathcal{K}(\mathcal{D}_f S)$. The algebra $\mathcal{D}_f S$ is isomorphic to the standard $(n-1)$-simplex

$$\Delta^{n-1} := \big\{(t_1, \dots, t_n) \in \mathbb{R}^n \mid t_i \geq 0, \sum_{i=1}^{n} t_i = 1\big\}.$$

The extremal points of Δ^{n-1} are its corner points, the canonical basis vectors e_i in $\mathbb{R}^n$. Hence a singleton $\{\xi\}$ belongs to $\operatorname{Ext}\mathcal{K}(\mathcal{D}_f S)$ if and only if ξ is a Dirac measure δ_x concentrated at one of the points x of S. The set

$$\mathcal{E} := \{A \in \operatorname{Ext}\mathcal{K}(\mathcal{D}_f S) \mid A \text{ is not singleton}\} \tag{8.1}$$

corresponds bijectively to the homothety classes of indecomposable convex bodies in $\mathbb{R}^{n-1}$ which are not singletons (see Corollary 9.7, Lemma 9.8 in Section 9). Let us point out that this property is particular for the algebra $\mathcal{K}(\Delta^{n-1})$. For example, it does not hold for $\mathcal{K}(D)$ when D is a square in $\mathbb{R}^2$, cf. [13, Remark 2].

For $n = 2$ and $n = 3$ the set $\mathcal{E}$ is known explicitly; the case $n = 2$ is trivial and the case $n = 3$ is elaborated in [27, Theorem 3.2.11]. Concretely, for $n = 2$ we have

$$\operatorname{Ext}\mathcal{K}(\mathcal{D}_f\{1,2\}) = \big\{\{\delta_1\}, \{\delta_2\}\big\} \cup \big\{\operatorname{conv}\{\delta_1, \delta_2\}\big\} = \big\{\{\delta_1\}, \{\delta_2\}, \mathcal{D}_f\{1,2\}\big\}.$$

For $n = 3$, $\operatorname{Ext}\mathcal{K}(\mathcal{D}_f\{1,2,3\})$ consists of the singletons $\{\delta_1\}, \{\delta_2\}, \{\delta_3\}$, the line segments connecting one Dirac measure with any convex combination of the other two, and the triangles having at least one corner point in each of $\operatorname{conv}\{\delta_2, \delta_3\}, \operatorname{conv}\{\delta_1, \delta_3\}, \operatorname{conv}\{\delta_1, \delta_2\}$.

For $n \geq 4$, $\operatorname{Ext}\mathcal{K}(\mathcal{D}_f S)$ is dense in $\mathcal{K}(\mathcal{D}_f S)$, cf. [27, Theorem 3.2.14].

Remark 8.7. Giving description of indecomposable convex bodies in $\mathbb{R}^n$ with $n \geq 3$ is an open problem in convex geometry. By Baire's category theorem most convex bodies are indecomposable (being a dense set). However, almost no indecomposable bodies are explicitly known., cf. the discussion [27, p.153]. For polytopes, however, there are easy to check conditions for indecomposability. For example, if all 2-dimensional faces of a polytope are triangles, then it is indecomposable, cf. [27, Corollary 3.2.13].

9. Proofs of Geometric Arguments in Example 8.4 and Example 8.6

In this section we provide evidence for the statements made in Example 8.4 and Example 8.6.

Example 8.4 relies on knowing the sets $\{A \in \mathcal{P}_{\mathrm{c}}\mathbb{D} \mid A \mathrel{\circ\!\!-\!\!\bullet} C\}$ when C is either closed or relatively open, and on the fact that an extremal point of $\mathcal{K}(D)$ is automatically extremal in the larger algebra $\mathcal{P}_{\mathrm{c}}\mathbb{D}$. The next three results contain all the details.

Lemma 9.1. *Let $D \in \mathcal{K}^n$.*

(1) *The closure operator $\bar{\cdot} : \mathcal{P}_{\mathrm{c}}\mathbb{D} \to \mathcal{K}(D)$ is a homomorphism of convex algebras.*
(2) *Let $A, B \in \mathcal{P}_{\mathrm{c}}\mathbb{D}$. If $A \mathrel{\circ\!\!-\!\!\bullet} B$, then $\overline{A} = \overline{B}$.*
(3) *Let $C \in \mathcal{P}_{\mathrm{c}}\mathbb{D}$ be relatively open. Then*

$$\{A \in \mathcal{P}_{\mathrm{c}}\mathbb{D} \mid A \mathrel{\circ\!\!-\!\!\bullet} C\} = \{A \in \mathcal{P}_{\mathrm{c}}\mathbb{D} \mid \overline{A} = \overline{C}\}.$$

Proof.

(1) Let $A, B \in \mathcal{P}_{\mathrm{c}}\mathbb{D}$, $p \in (0,1)$, and set $C = pA + \bar{p}B$. Continuity of linear operations ensures that $p\overline{A} + \bar{p}\overline{B} \subseteq \overline{C}$. However, $\overline{A}$ and $\overline{B}$ are compact, so also $p\overline{A} + \bar{p}\overline{B}$ is compact, hence, in particular, closed. We conclude that $\overline{C} \subseteq p\overline{A} + \bar{p}\overline{B}$.
(2) If $A \mathrel{\circ\!\!-\!\!\bullet} B$, by 1., also $\overline{A} \mathrel{\circ\!\!-\!\!\bullet} \overline{B}$, i.e., $p\overline{A} + \bar{p}\overline{B} = \overline{B}$ for all $p \in (0,1)$. This implies $\overline{A} = \lim_{p \to 1}(p\overline{A} + \bar{p}\overline{B}) = \overline{B}$, where the limit is understood w.r.t. the Hausdorff metric. For the first equality recall that the operations are continuous, cf. Remark 8.3.1.
(3) The inclusion "$\subseteq$" holds by 2. For the reverse inclusion, let $A \in \mathcal{P}_{\mathrm{c}}\mathbb{D}$ with $\overline{A} = \overline{C}$ be given. Then $C = \mathrm{ri}(\overline{C}) = \mathrm{ri}(\overline{A}) = \mathrm{ri}\, A \subseteq A$, cf. [24, Theorem 6.3]. Thus $pA + \bar{p}C \supseteq C$. The inclusion $pA + \bar{p}C \subseteq C$ holds by [24, Theorem 6.1]. □

Lemma 9.2. *Let $C \in \mathcal{K}^n$.*

(1) *If $A, B \in \mathcal{P}_{\mathrm{c}}\mathbb{R}^n$, $p \in (0,1)$, with $A, B \subseteq C$ and $pA + \bar{p}B = C$, then $A = B = C$.*
(2) *$\{A \in \mathcal{P}_{\mathrm{c}}\mathbb{R}^n \mid A \mathrel{\circ\!\!-\!\!\bullet} C\} = \{C\}$.*

Proof.

(1) Let $x \in \mathrm{Ext}\, C$, and choose $a \in A$, $b \in B$, with $x = pa + \bar{p}b$. Since $a, b \in C$, it follows that $a = b = x$. We see that $\mathrm{Ext}\, C \subseteq A \cap B$. Using Minkowski's theorem, cf. [27, Corollary 1.4.5], yields $C = \mathrm{conv}(\mathrm{Ext}\, C) \subseteq A \cap B$, from which $A = B = C$ follows.
(2) Let $a \in A$ and choose $c \in C$. Then, for each $p \in (0,1)$, the element $pa + \bar{p}c$ belongs to C. Since C is closed,

$$a = \lim_{p \to 1}(pa + \bar{p}c) \in C.$$

Thus $A \subseteq C$, and 1. shows that $A = C$. □

Lemma 9.3. *Let $D \in \mathcal{K}^n$ and $C \in \mathcal{K}(D)$. Then C is an extremal point of $\mathcal{K}(D)$ if and only if it is an extremal point of $\mathcal{P}_{\mathrm{c}}\mathbb{D}$.*

Proof. Since $\mathcal{K}(D)$ is a subalgebra of $\mathcal{P}_{\mathrm{c}}\mathbb{D}$, we have $\mathrm{Ext}(\mathcal{P}_{\mathrm{c}}\mathbb{D}) \cap \mathcal{K}(D) \subseteq \mathrm{Ext}\, \mathcal{K}(D)$. Assume $C \in \mathrm{Ext}\, \mathcal{K}(D)$ and that $C = pA + \bar{p}B$ with some $A, B \in \mathcal{P}_{\mathrm{c}}\mathbb{D}$ and $p \in (0,1)$. Then $p\overline{A} + \bar{p}\overline{B} = \overline{C} = C$, and we obtain $\overline{A} = \overline{B} = C$. In particular, $A, B \subseteq C$, and Lemma 9.2.1 shows $A = B = C$. □

We turn to the proof of the statement made in Example 8.6, that the set $\mathcal{E}$ from (8.1) corresponds to (classes of) indecomposable bodies.

First, let us introduce some notation. Let $\pi_i\colon \mathbb{R}^n \to \mathbb{R}$ denote the i-th projection, and set $\sigma = \sum_{i=1}^n \pi_i$. Denote by e_i the i-th canonical basis vector of $\mathbb{R}^n$. Then the standard n-simplex can be written as

$$\Delta^n = \operatorname{conv}\{e_1, \dots, e_n\} = \sigma^{-1}(\{1\}) \cap \bigcap_{i=1}^n \pi_i^{-1}([0,1]).$$

A central role is played by the subset of $\mathcal{K}(\Delta^n)$ of all bodies which touch each face of Δ^n.

Definition 9.4. Set $\mathcal{L}(\Delta^n) = \{A \in \mathcal{K}(\Delta^n) \mid \min \pi_i(A) = 0, i = 1, \dots, n\}$.

Lemma 9.5. *The set $\mathcal{L}(\Delta^n)$ contains no singletons. It is a convex and extreme subset of $\mathcal{K}(\Delta^n)$.*

Proof. Assume $\{x\} \in \mathcal{L}(\Delta^n)$. Since $\min \pi_i(\{x\}) = \pi_i(x)$, it follows that $x = 0$ which contradicts $\sigma(x) = \max \sigma(\{x\}) = 1$.

It holds that $\min \pi_i(pA + \bar{p}B) = p \min \pi_i(A) + \bar{p} \min \pi_i(B)$. The fact that $\mathcal{L}(\Delta^n)$ is convex follows immediately. To show that $\mathcal{L}(\Delta^n)$ is extreme, let $A \in \mathcal{K}(\Delta^n) \setminus \mathcal{L}(\Delta^n)$. Choose $j \in \{1, \dots, n\}$ with $\min \pi_j(A) > 0$. Then, for each $p \in (0,1)$ and $B \in \mathcal{K}(\Delta^n)$ we have $\min \pi_i(pA + \bar{p}B) \geq p \min \pi_i(A) > 0$, and conclude that $pA + \bar{p}B \notin \mathcal{L}(\Delta^n)$. □

Next we prove that each extremal point of $\mathcal{K}(\Delta^n)$ which is not a singleton must touch each face of Δ^n. This property fits our intuition, but surprisingly it only holds for the algebra $\mathcal{K}(\Delta^n)$. For example, it does not hold in $\mathcal{K}(D)$ for D a square in $\mathbb{R}^2$, cf. [13, Remark 2].

Lemma 9.6. *We have* $\big\{A \in \operatorname{Ext} \mathcal{K}(\Delta^n) \mid A \textit{ is not singleton}\big\} \subseteq \mathcal{L}(\Delta^n)$.

Proof. Let $A \in \mathcal{K}(\Delta^n) \setminus \mathcal{L}(\Delta^n)$ have more than one point. Our aim is to show that A can be written as a convex combination of two elements of $\mathcal{K}(\Delta^n)$ not both equal to A. Choose $j \in \{1, \dots, n\}$ with $\epsilon = \min \pi_j(A) > 0$. Since A is not a singleton, we have $\epsilon < 1$. Consider the convex map $f\colon x \mapsto (1-\epsilon)x + \epsilon e_j$ and its inverse $f^{-1}\colon x \mapsto \frac{1}{1-\epsilon}x - \frac{\epsilon}{1-\epsilon}e_j$. Obviously, $f(\Delta^n) \subseteq \Delta^n$. We prove that $f^{-1}(A) \subseteq \Delta^n$. Let $x \in A$, then

$$\sigma(f^{-1}(x)) = \frac{1}{1-\epsilon} - \frac{\epsilon}{1-\epsilon} = 1,$$
$$\pi_i(f^{-1}(x)) = \frac{1}{1-\epsilon}\pi_i(x) \geq 0,\ i \neq j, \quad \pi_j(f^{-1}(x)) = \frac{1}{1-\epsilon}\big(\pi_i(x) - \epsilon\big) \geq 0.$$

Thus $f^{-1}(x) \in \Delta^n$.

Set $p = \frac{1}{2-\epsilon}$. Then $p \in (0,1)$, $\bar{p} = \frac{1-\epsilon}{2-\epsilon}$, and

$$pf(x) + \bar{p}f^{-1}(y) = \Big(\frac{1-\epsilon}{2-\epsilon}x + \frac{\epsilon}{2-\epsilon}e_j\Big) + \Big(\frac{1}{2-\epsilon}x - \frac{\epsilon}{2-\epsilon}e_j\Big) = px + \bar{p}y.$$

It follows that $pf(A) + \bar{p}f^{-1}(A) = A$. Since $\min \pi_j(f^{-1}(A)) = 0$, we have $f^{-1}(A) \neq A$. □

Corollary 9.7. *We have* $\operatorname{Ext} \mathcal{K}(\Delta^n) = \{\{e_1\}, \dots, \{e_n\}\} \cup \operatorname{Ext} \mathcal{L}(\Delta^n)$. *The union is disjoint.*

Proof. We already determined the extremal points of $\mathcal{K}(\Delta^n)$ which are singletons. Clearly, $\mathcal{L}(\Delta^n) \cap \operatorname{Ext} \mathcal{K}(\Delta^n) \subseteq \operatorname{Ext} \mathcal{L}(\Delta^n)$. Since $\mathcal{L}(\Delta^n)$ is an extremal set, $\operatorname{Ext} \mathcal{L}(\Delta^n) \subseteq \operatorname{Ext} \mathcal{K}(\Delta^n)$. If $A \in \operatorname{Ext} \mathcal{K}(\Delta^n)$ is not a singleton, the above lemma says $A \in \mathcal{L}(\Delta^n)$. □

To make the connection with indecomposable bodies, we pass to an isomorphic situation. Slightly overloading notation, let again $\pi_i : \mathbb{R}^{n-1} \to \mathbb{R}$ be the i-th projection, and σ and e_i be defined as above (with $n-1$ instead of n). Set

$$D^{n-1} = \big\{(t_1, \ldots, t_n) \in \mathbb{R}^{n-1} \mid t_i \geq 0, \sum_{i=1}^{n-1} t_i \leq 1\big\}.$$

Then $D^{n-1} = \operatorname{conv}\{0, e_1, \ldots, e_{n-1}\} = \sigma^{-1}([0,1]) \cap \bigcap_{i=1}^{n-1} \pi_i^{-1}([0,1])$. The standard simplex Δ^n is isomorphic to D^{n-1} via the convex map ϕ taking the basis vector e_i of $\mathbb{R}^n$ to the corresponding basis vector in $\mathbb{R}^{n-1}$ if $i < n$ and mapping e_n to 0. This map lifts in the natural (pointwise) way to the isomorphism $\Phi\colon A \mapsto \phi(A)$ of the convex algebra $\mathcal{K}(\Delta^n)$ onto $\mathcal{K}(D^{n-1})$. The image of $\mathcal{L}(\Delta^n)$ under this isomorphism is

$$\Phi\big(\mathcal{L}(\Delta^n)\big) = \big\{A \in \mathcal{K}(D^{n-1}) \mid \max \sigma(A) = 1,\ \min \pi_i(A) = 0, i = 1, \ldots, n-1\big\},$$

which we shall denote as $\mathcal{L}(D^{n-1})$. Being an isomorphism, Φ maps extremal points to extremal points, and we obtain

$$\Phi\big(\operatorname{Ext} \mathcal{L}(\Delta^n)\big) = \operatorname{Ext} \mathcal{L}(D^{n-1}).$$

Further note that also $\mathcal{L}(D^{n-1})$ contains no singletons.

Lemma 9.8.

(1) *The set $\mathcal{L}(D^{n-1})$ is a complete system of representatives modulo homothety of the set of convex bodies in $\mathbb{R}^{n-1}$ having more than one point.*

(2) *Let $A \in \mathcal{L}(D^{n-1})$. Then $A \in \operatorname{Ext} \mathcal{L}(D^{n-1})$ if and only if A is indecomposable in $\mathcal{K}^{n-1}$.*

Proof.

(1) Let $A \in \mathcal{K}^{n-1}$ with more than one point, and set $t_i = \min \pi_i(A)$, $x = \sum_{i=1}^{n-1} t_i e_i$. Since A is not a singleton, $\max \sigma(A) > \sum_{i=1}^{n-1} t_i$. The body

$$\Psi(A) = \big(\max \sigma(A) - \sum_{i=1}^{n-1} t_i\big)^{-1}\big(A - x\big)$$

is homothetic to A and belongs to $\mathcal{L}(D^{n-1})$.

Now assume that $s > 0$, $x \in \mathbb{R}^{n-1}$, and that both A and $sA + x$ belong to $\mathcal{L}(D^{n-1})$. Then

$$0 = \min \pi_i(sA + x) = s[\min \pi_i(A)] + \pi_i(x) = \pi_i(x),$$

whence $x = 0$. Now $1 = \max \sigma(sA) = s \max \sigma(A) = s$.

(2) Assume $A \in \mathcal{L}(D^{n-1}) \setminus \operatorname{Ext} \mathcal{L}(D^{n-1})$, then A has a representation $A = pB + \bar{p}C$ with some $p \in (0,1)$ and $B, C \in \mathcal{L}(D^{n-1})$ where not both of B, C equal A. Then pB and $\bar{p}C$ are not both homothetic to A by 1., and we conclude that A is decomposable in $\mathcal{K}^{n-1}$.

To show the converse, we first establish the following: If $B, C \in \mathcal{K}^{n-1}$ are not singletons, then there exists $p \in (0,1)$ with

$$\Psi(B + C) = p\Psi(B) + \bar{p}\Psi(C).$$

To see this, denote

$$t_i^B = \min \pi_i(B), s^B = \max \sigma(B) - \sum_{i=1}^{n-1} t_i^B, \quad t_i^C = \min \pi_i(C), s^C = \max \sigma(C) - \sum_{i=1}^{n-1} t_i^C,$$

$$t_i = \min \pi_i(B+C), s = \max \sigma(B+C) - \sum_{i=1}^{n-1} t_i.$$

Then $t_i = t_i^B + t_i^C$ and $s = s^B + s^C$, which gives

$$\Psi(B+C) = \frac{1}{s^B + s^C}\Big[(B+C) + \sum_{i=1}^{n-1}(t_i^B + t_i^C)\Big] = \frac{s^B}{s^B + s^C}\Psi(B) + \frac{s^C}{s^B + s^C}\Psi(C).$$

Since B and C are not singletons, s^B and s^C are both nonzero. Thus $p = \frac{s^B}{s^B+s^C} \in (0,1)$.

Now let $A \in \mathcal{L}(D^{n-1})$, and assume that $A = B + C$ with $B, C \in \mathcal{K}^{n-1}$ not both homothetic to A. Then

$$A = \Psi(A) = p\Psi(B) + \bar{p}\Psi(C),$$

and not both of $\Psi(B)$ and $\Psi(C)$ are equal to A by 1. □

Putting together Corollary 9.7 and Lemma 9.8, we see that indeed the set $\mathcal{E}$ from (8.1) corresponds bijectively to the homothety classes of indecomposable convex bodies in $\mathbb{R}^{n-1}$ which are not singletons.

10. Conclusions

We have studied the possibility of extending a convex algebra by a single element. We have proven that many different extensions are possible of which only one gives rise to a functor on $\mathcal{EM}(\mathcal{D}_f)$. We have described all extensions of $\mathbb{D}_S$, the free convex algebra of probability distributions over a set S, and of $\mathcal{P}_c\mathbb{D}$, the convex algebra of convex subsets of a particular kind of convex subset of a vector space. As a consequence of the latter result, we have described all extensions of $\mathcal{P}_c\mathbb{D}_S$ used for modelling probabilistic automata.

It would be interesting to investigate whether the methods developed here could be useful in the study of Eilenberg-Moore algebras of the Giry monad on measurable spaces, or on subcategories of measurable spaces like Polish or analytic spaces.

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