

## A FINITE-MODEL-THEORETIC VIEW ON PROPOSITIONAL PROOF COMPLEXITY

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**ABSTRACT.** We establish new, and surprisingly tight, connections between propositional proof complexity and finite model theory. Specifically, we show that the power of several propositional proof systems, such as Horn resolution, bounded-width resolution, and the polynomial calculus of bounded degree, can be characterised in a precise sense by variants of fixed-point logics that are of fundamental importance in descriptive complexity theory. Our main results are that *Horn resolution* has the same expressive power as *least fixed-point logic*, that *bounded-width resolution* captures *existential least fixed-point logic*, and that the *monomial calculus with bounded degree over the rationals* solves precisely the problems definable in *fixed-point logic with counting*. We also study the *bounded-degree polynomial calculus*. Over the rationals, it captures fixed-point logic with counting if we restrict the bit-complexity of the coefficients. For unrestricted coefficients, we can only say that the bounded-degree polynomial calculus is at most as powerful as *bounded variable infinitary counting logic*, but a precise logical characterisation of its power remains an open problem. These connections between logics and proof systems allow us to establish finite-model-theoretic tools for proving lower bounds for the polynomial calculus over the rationals and also over finite fields.

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### 1. INTRODUCTION

The question whether there exists an efficient proof system by means of which the validity of *arbitrary propositional formulas* can be verified via *proofs of polynomial size* is equivalent to the closure of NP under complementation. Since Cook and Reckhow [15] made the notion

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of an efficient propositional proof system precise, a huge body of research on the power of various propositional proof system has been established. In particular, we now have super-polynomial lower bounds on the proof complexity for quite strong proof systems, see [9, 41] for surveys on propositional proof complexity.

In this paper we study *polynomial-time variants* of propositional proof systems, which admit efficient proof search, resulting in proofs of polynomial size, such as restricted variants of resolution and the polynomial calculus. To be precise, one of these variants, the bounded-degree polynomial calculus over the rationals, is not known to admit polynomial-time proof search because the proofs may involve very large coefficients. Thankfully, as it turns out, this issue does not prevent a meaningful connection to finite model theory.

Recall that the resolution proof system RES takes as input a propositional formula  $\varphi$  in conjunctive normal form (CNF), and it refutes the satisfiability of  $\varphi$  if there is a derivation of the empty clause from  $\varphi$ . It is well-known that shortest resolution proofs can be of exponential size, so in general, we provably cannot search for resolution proofs in polynomial time. However, there are interesting restrictions of RES, such as HORN-RES (resolution restricted to Horn clauses) and bounded-width resolution  $k$ -RES (resolution restricted to clauses of size  $\leq k$ ) that do admit efficient proof search, that is the existence of refutations can be verified in polynomial time. Of course, unless  $P = NP$ , any proof system that admits efficient proof search is necessarily incomplete for full propositional logic. Nevertheless we can still prove interesting statements in such systems, and usually have completeness for relevant fragments of propositional logic, such as Horn-logic or 2-CNF. We can now try to solve algorithmic problems by reducing them to provability (or refutability) in some specific polynomial-time proof system, which, if it works successfully for all inputs, would give us a polynomial-time algorithm for the problem. Our goal is to understand how powerful this approach can be, depending on the specific proof system that we use.

Let us illustrate this by two concrete problems. First we consider *graph isomorphism*, a problem which is not known to be solvable in polynomial time although there is strong evidence that it is not NP-complete. Given two graphs  $G = (V, E)$  and  $H = (W, F)$  we ask whether there is a bijection  $\pi: V \rightarrow W$  such that  $\pi(E) = F$ . Of course, this can easily be encoded as the satisfiability problem of a propositional CNF-formula. First, for each pair of vertices  $v \in V$  and  $w \in W$  we introduce a variable  $X_{vw}$  with the intended meaning that  $X_{vw} = 1$  if  $\pi(v) = w$ . We add clauses  $\bigvee_{w \in W} X_{vw}$  for every  $v \in V$  and  $\bigvee_{v \in V} X_{vw}$  for every  $w \in W$  to ensure that every  $v \in V$  has an image and every  $w \in W$  has a preimage. Additionally we add for all  $v_1, v_2 \in V$  and  $w_1, w_2 \in W$  a clause  $\neg(X_{v_1 w_1} \wedge X_{v_2 w_2})$  in case that  $\{v_1 \mapsto w_1, v_2 \mapsto w_2\}$  is not a partial isomorphism. The resulting CNF-formula, denoted by  $\text{Iso}(G, H)$ , is satisfiable if, and only if, the two graphs  $G$  and  $H$  are isomorphic. Following our reasoning from above, we can now use an efficient variant of resolution, or of a stronger proof system, and try to refute the satisfiability of the formula  $\text{Iso}(G, H)$ . If this is possible, then  $G$  and  $H$  are not isomorphic. Unfortunately, if we do not find a proof, then we are stuck, because it might still be the case that  $G$  and  $H$  are not isomorphic, but our proof system is just not strong enough to show this. Hence, we get an efficient, sound, but not necessarily complete graph isomorphism test. The question of how successful this approach is when based on resolution was studied by Toran in [42]. Unfortunately, he proved that shortest resolution proofs for graph non-isomorphism can be of exponential size (even for graphs with colour class size four). More recently, Grohe and Berkholz showed that also in the stronger system polynomial calculus (PC) one cannot obtain small proofs for graph non-isomorphism [10, 11] in the general case.

Our second example is directed graph reachability: Given a directed graph  $G = (V, E)$  with two distinguished vertices  $s, t \in V$ , we want to know whether there is a path from  $s$  to  $t$  in  $G$ . Again, it is easy to encode this as a satisfiability problem in propositional logic, by taking the conjunction of all implication clauses  $X_v \rightarrow X_w$ , for all edges  $(v, w) \in E$ , together with the two clauses  $1 \rightarrow X_s$  and  $X_t \rightarrow 0$ . Clearly the resulting formula  $\text{NonReach}(G, s, t)$  is unsatisfiable if, and only if,  $t$  is reachable from  $s$  in  $G$ . However, in clear contrast to the formulas  $\text{Iso}(G, H)$  from above, we can easily prove unsatisfiability for the formulas  $\text{NonReach}(G, s, t)$  in efficient variants of resolution such as HORN-RES and  $k$ -RES for  $k \geq 2$ .

Our two examples demonstrate the following: while certain problems, such as directed graph reachability, allow for small and efficient resolution proofs, other problems, such as the graph isomorphism problem, provably require proofs of super-polynomial size even in quite strong proof systems. This leads to the main question that we want to address in this paper: is there a *classification* for those problems which can be solved in natural restricted versions of propositional proof systems such as HORN-RES,  $k$ -RES and  $\text{PC}_k$  (the degree- $k$  restriction of the polynomial calculus)? It came as a surprise to us that there is, indeed, a very clear and tight classification of the power of all of these proof systems in terms of definability in important fixed-point logics and infinitary logics which are well-studied in the area of descriptive complexity theory.

Before we can state our results in detail, we have to explain what we mean by saying that a problem, such as directed graph reachability, can be solved by a propositional proof system PROP. As usual, each decision problem can be identified with a membership problem “ $\mathfrak{A} \in \mathcal{K}$ ?” for some class of structures  $\mathcal{K}$ . For instance, the graph reachability problem from above is identified with the class  $\mathcal{K}_{\text{Reach}} = \{(V, E, s, t) : \text{there is a path from } s \text{ to } t \text{ in } G = (V, E)\}$ . Then we naturally want to say that a problem  $\mathcal{K}$  can be solved by the proof system PROP if we can find a reduction function  $f$  which maps structures  $\mathfrak{A}$  to inputs  $f(\mathfrak{A})$  for PROP such that  $\mathfrak{A} \in \mathcal{K}$  if, and only if, PROP can prove that  $f(\mathfrak{A})$  is not satisfiable. It is clear that we only want to allow *simple* reduction functions  $f$ , because otherwise the computation of the encoding could already contain part of the work to solve the problem. Coming from the area of finite model theory the obvious and natural formalisation for “ $f$  being simple” is to say that  $f$  is definable in *first-order logic* (FO). We introduce the precise technical definition of such reductions, which is the notion of a *first-order interpretation*, in Section 2. Note that for the two examples we discussed above the encoding functions are clearly FO-definable.

Having established this definition it turns out that our classification problem is really about understanding the expressive power of the *Lindström extensions* of first-order logic by *generalised quantifiers* for propositional proof systems PROP. We denote these logics by  $\text{FO}(\text{PROP})$ . The basic idea of the logic  $\text{FO}(\text{PROP})$  is to extend first-order logic by new quantifiers  $\mathcal{Q}_{\text{PROP}}$  which are capable of simulating PROP. In other words, we just incorporate into first-order logic the power to simulate PROP in an explicit way, that is the logics  $\text{FO}(\text{PROP})$  are a formalisation of the concept of oracle Turing-machines with access to PROP in the world of first-order logic (the oracle calls to the proof system PROP correspond to applications of the new generalised quantifiers). Again, the precise technical definitions of the Lindström extensions  $\text{FO}(\text{PROP})$  can be found in Section 2. We can now say that a problem  $\mathcal{K}$  can be solved in a proof system PROP if, and only if, it is definable in  $\text{FO}(\text{PROP})$ . For instance, we saw that  $\mathcal{K}_{\text{Reach}}$  is definable in the logics  $\text{FO}(\text{HORN-RES})$  and  $\text{FO}(2\text{-RES})$ .

We proceed to describe our main results and give a rough sketch of the structure of this article. This work is based on our conference paper [27]. However, the present article also

contains some new results and substantial generalisations of our results from [27] on the polynomial calculus.

In Section 3, we study the resolution proof system and its aforementioned restrictions Horn-Resolution (HORN-RES) and Bounded-width- $k$  Resolution ( $k$ -RES), for  $k \geq 2$ . It turns out that HORN-RES can express precisely the problems that are definable in least-fixed point logic (LFP), that is  $\text{FO}(\text{HORN-RES}) = \text{LFP}$ . This readily follows by the well-known fact that the problem of computing winning positions in reachability games (known as GAME or alternating reachability) is complete for LFP with respect to FO-reductions. More interestingly, we proceed to show that  $k$ -RES, for every  $k \geq 2$ , is less powerful than HORN-RES. In fact,  $\text{FO}(2\text{-RES}) = \text{FO}(\text{TC})$ , where  $\text{FO}(\text{TC})$  is the extension of first-order logic by a transitive closure operator. Moreover, we prove that, for every  $k \geq 3$ ,  $\text{FO}(k\text{-RES}) = \text{EFP}$ , where EFP is the *existential* fragment of least fixed-point logic which is known to be a strict fragment of full least fixed-point logic. We can also show that the Lindström extensions for Horn resolution and width- $k$  resolution have different structural properties. While for  $\text{FO}(\text{HORN-RES})$  a single application of a  $\mathcal{Q}_{\text{HORN-RES}}$  quantifier suffices to obtain the full expressive power, nesting of  $\mathcal{Q}_{k\text{-RES}}$  quantifiers is needed for the logics  $\text{FO}(k\text{-RES})$ .

In Section 4, we then turn our attention to the polynomial calculus (PC), a propositional proof system which is based on algebraic reasoning techniques. The polynomial calculus manipulates polynomial equations over an underlying field  $\mathbb{F}$ . A PC-refutation is a derivation of the equation  $0 = 1$ . As in the case of bounded-width resolution, if one restricts the degree of the polynomials in all equations to some constant  $k \geq 1$ , then one can search for PC-proofs in polynomial time (when working over the field of rationals, the bit-complexity of the coefficients must also be restricted to binary representations of polynomial length). Besides restricting the degree, one can also vary the underlying field  $\mathbb{F}$ . Specifically, we consider the cases where  $\mathbb{F}$  is the field of rationals (or reals) or a finite field. Moreover, the polynomial calculus can also be restricted by weakening its proof rule for multiplication, which defines a variant known as the monomial-PC (MON-PC). We denote its corresponding restriction to degree  $k$  by  $\text{MON-PC}_k$ .

For the case of the polynomial calculus over  $\mathbb{Q}$  we show the following. First of all, if we consider the monomial-PC restricted to some degree  $k \geq 2$ , then this proof system  $\text{MON-PC}_k$  has precisely the same expressive power as fixed-point logic with counting (FPC), which is a very expressive logic well-studied in descriptive complexity theory [17, 39]; formally, we show that  $\text{FO}^+(\text{MON-PC}_k) = \text{FPC}$  for  $k \geq 2$  where  $\text{FO}^+$  denotes the extension of FO by a numeric sort to match the setting of FPC. In particular, this separates the (monomial-)PC from the resolution proof system since FPC is known to be much stronger than EFP and LFP. In a second step, we generalise this characterisation for the monomial-PC to the full polynomial calculus (PC). To deal with the already mentioned phenomenon of potentially exceedingly large coefficients, we restrict the degree- $k$  PC further and define, for any  $b \in \mathbb{N}$ , the proof system  $\text{PC}_{k,b}$  as the degree- $k$  PC with the limitation that all coefficients occurring in a proof must be representable as fractions of binary numbers with at most  $n^b$  bits each ( $n$  refers to the number of variables in the input polynomials). Then we prove that for any constants  $k, b$ , the proof system  $\text{PC}_{k,b}$  captures FPC, just like  $\text{MON-PC}_k$  does. From there, we move on to the more common  $\text{PC}_k$  with *unrestricted bit-complexity*, and observe that we can define the existence of  $\text{PC}_k$ -proofs in the infinitary counting logic  $\text{C}_{\infty\omega}^k$ . This logic is strictly more expressive than FPC. The question whether  $\text{PC}_k$ -proofs with unbounded rational coefficients are also definable in FPC remains open. Yet, we can say that a positive

answer to it seems unlikely because it is not even clear that this problem is decidable in polynomial time: As shown by Hakoniemi [31], there exists a set  $Q_n$  of polynomials over Boolean variables that has a refutation in the degree-2 polynomial calculus, but none that requires less than exponentially many bits for the coefficients; it is doubtful that such a refutation can be computed in polynomial time.

On our way we prove a result which is of independent interest, namely that FPC can define solution spaces of linear equation systems over the rationals. We need this in order to express  $\text{PC}_{k,b}$  in FPC, and indirectly also to express  $\text{PC}_k$  in  $\text{C}_{\infty\omega}^k$ . The latter result allows us to answer an open question by Grohe and Berkholz from [10] about the relative power of  $\text{MON-PC}_k$  and  $\text{PC}_k$  with respect to the graph isomorphism problem.

In Section 5, we turn our attention to the polynomial calculus over finite fields. It is easy to see that the connection between FPC and the (monomial-)PC breaks down. We set out to establish criteria on the characteristic of the underlying finite field and certain finite-model-theoretic properties of polynomial equation systems that allow us to retain FPC-definability of bounded-degree PC-refutations. This result proves to be very useful in order to derive lower bounds for the polynomial calculus over finite fields. There are also technical results in this section which should be of independent interest. For example, we show that classes of CFI-structures over expander graphs are homogeneous with respect to FPC-definability.

Finally, in Section 6, we discuss how we can apply our FPC-definability results in order to prove lower bounds for the polynomial calculus. We give examples including the graph isomorphism problem and constraint satisfaction problems. Although most (but not all) of the lower bounds have been known before, we present new proofs which only use finite-model-theoretic arguments. Our novel, uniform approach to these lower bounds, also suggests a way to capture a common weakness of many propositional proof systems: whenever a proof system has a stratification which allows for *symmetric* refutations that can be described and verified in counting logic with a bounded number of variables, our lower bounds techniques can be applied. For illustration, we discuss the example of the Positivstellensatz proof system in Section 7.

**Related work.** Let us discuss some related work. The most relevant result to mention here is the characterisation by Atserias and Dalmau of resolution width in terms of the number of pebbles required to win an existential pebble game played on a given CNF-formula and a structural encoding of truth assignments [2, 4]. This resembles our result that bounded-width resolution corresponds to *existential* least fixed-point logic. Using their game-theoretic characterisation, Atserias and Dalmau can reprove many of the known lower bounds on resolution width. Again, this is similar to the applications we give in Section 6.1.

However, what makes our setting different from the approach of Atserias and Dalmau is that we always consider the power of proof systems only *up to logical reductions*. This reflects, for example, in our result saying that  $\text{FO}(3\text{-RES}) = \text{FO}(4\text{-RES})$ , i.e. that 3-RES has the same expressive power as 4-RES. But, certainly, this only holds if we allow first-order reductions to transform inputs between 4-RES and 3-RES. Hence our characterisation of resolution width is “coarser” than that of Atserias and Dalmau. But it has the advantage of being more robust. For instance, in the situation of lower bound proofs, we can avoid playing pebble games directly on the inputs to proof systems, such as CNF-formulas, but instead it suffices to play suitable games on pairs of structures in which these inputs interpret. This can make the description of winning strategies much simpler. Furthermore, our setting

allows us to prove lower bound results not depending on specific encodings of a problem, since our logics are closed under interpretations, see Section 6.

Besides this, we want to mention the series of papers [5, 10, 38, 29] which establish surprisingly tight connections between the equivalence of graphs in counting logic and their indistinguishability by linear programming techniques (Sherali-Adams relaxations of graph isomorphism polytopes) and algebraic propositional proof system. Similar to our applications, these results also allow the transfer of lower bounds from finite model theory to get lower bounds on proof complexity. In particular, we use notions and ideas of [10] in Section 4. Let us also point to the excellent work of Dawar and Wang [20, 21] which connect finite model theory with semi-algebraic proof systems. This work, and our own, has certainly also interesting connections to the very recent work by Atserias and Ochremiak [7] showing by means of finite-model-theoretic arguments that the Sums-of-Squares proof system can be simulated in  $C_{\infty\omega}^\omega$ . Surely these connections deserve to be explored further.

## 2. PRELIMINARIES

This is a paper in finite model theory. All structures are *relational* and *finite* if not explicitly stated otherwise. We assume that the reader has a solid background in logic. To fully understand and appreciate our results, familiarity with the ideas and techniques of finite model theory will be necessary (see [22, 34, 37, 25]). In particular, a good knowledge of fixed-point logic with counting is needed in order to understand our definability results for the polynomial calculus in Sections 4,5,6, see the above references plus [39, 17].

**2.1. Finite Relational Structures.** Given a (finite, relational) *vocabulary* (or *signature*)  $\tau$ , a  $\tau$ -*structure*  $\mathfrak{A}$  consists of a finite *universe*  $A$  and a relation  $R^A \subseteq A^k$  for each  $k$ -ary relation symbol  $R$  in  $\tau$ . If we consider (undirected) graphs, that is structures over the vocabulary  $\tau = \{E\}$ , then we usually use a different notation and denote graphs by  $G = (V, E)$ . In particular, we denote the vertex set of a graph  $G$  by  $V = V(G)$  and the set of edges  $E$  by  $E = E(G)$ . The class of all (finite)  $\tau$ -structures is denoted by  $\text{Str}(\tau)$ . Sometimes we want to distinguish certain constants in  $\tau$ -structures  $\mathfrak{A}$ . For a tuple of parameters  $\vec{z}$  we denote by  $\text{Str}(\tau, \vec{z})$  the class of all pairs  $(\mathfrak{A}, \vec{z} \mapsto \vec{a})$  where  $\mathfrak{A} \in \text{Str}(\tau)$ . For our applications in Section 5 and Section 6, we also fix an encoding of ordered pairs  $(\mathfrak{A}, \mathfrak{B})$  of  $\tau$ -structures as structures  $(\mathfrak{A}, \mathfrak{B})$  of some vocabulary  $\tau_{\text{pair}}$ .

**2.2. Logics without Counting.** We assume that the reader is familiar with *first-order logic* (FO) and *least and inflationary fixed-point logic* (LFP and IFP). *Infinitary finite variable logic*  $L_{\infty\omega}^\omega$  extends FO by infinite conjunctions and disjunctions in formulas, but with the additional requirement that formulas only contain a finite number of variables. More precisely, if we denote by  $L_{\infty\omega}^k$  the  $k$ -*variable fragment* of  $L_{\infty\omega}^\omega$ , then we have  $L_{\infty\omega}^\omega = \bigcup_k L_{\infty\omega}^k$ . Formulas of LFP with  $k$  variables can be translated into equivalent formulas of  $L_{\infty\omega}^k$ . In particular,  $\text{LFP} \leq L_{\infty\omega}^\omega$  (in this article we use the notation  $\mathcal{L}_1 \leq \mathcal{L}_2$  to say that every class of structures that is  $\mathcal{L}_1$ -definable is also  $\mathcal{L}_2$ -definable, that is  $\leq$  refers to semantic inclusion of logics wrt. sentences).

**2.3. Logics with Counting.** (*Infinitary*) *counting logic*  $C_{\infty\omega}^\omega$  is the extension of  $L_{\infty\omega}^\omega$  that allows counting quantifiers  $\exists^{\geq m}x$  (“there exist at least  $m$  values for  $x$ ”) for each  $m$  (with the same restriction on the number of variables as before, that is each  $C_{\infty\omega}^\omega$ -formula only contains a finite number of variables). Note that each individual quantifier  $\exists^{\geq m}x$  can be expressed using  $m$  first-order quantifiers and  $m$  distinct variables for  $x$ . However, the translation leads to formulas with a higher quantifier rank and, moreover, it increases the number of required variables. Analogous to the above, we denote by  $C_{\infty\omega}^k$  the fragment of  $C_{\infty\omega}^\omega$  consisting of all formulas with at most  $k$  (free or bound) variables. Then  $C_{\infty\omega}^\omega = \bigcup_k C_{\infty\omega}^k$ . For two structures  $\mathfrak{A}, \mathfrak{B}$  (of the same vocabulary) we write  $\mathfrak{A} \equiv^k \mathfrak{B}$  if the structures cannot be distinguished by any formula of  $C_{\infty\omega}^k$ .

We now recall the definition of *fixed-point logic with counting* (FPC). In a nutshell, FPC is the extension of inflationary fixed-point logic (IFP) by *counting terms*. Formulas of FPC are evaluated over the *two-sorted extension* of an input structure  $\mathfrak{A}$  by a copy of the natural numbers. Following [18] we denote by  $\mathfrak{A}^\#$  the two-sorted extension of a  $\tau$ -structure  $\mathfrak{A} = (A, R_1, \dots, R_k)$  by  $\mathfrak{N} = (\mathbb{N}, +, \cdot, 0, 1)$ , that is the two-sorted structure  $\mathfrak{A}^\# = (A, R_1, \dots, R_k, \mathbb{N}, +, \cdot, 0, 1)$  where the universe of the first sort (also referred to as *vertex sort*) is  $A$  and the universe of the second sort (also referred to as *number sort* or *counting sort*) is  $\mathbb{N}$ . For both, the vertex and the number sort, we have a collection of *typed first-order variables*, that is the domain of any variable  $x$  (over the input structure  $\mathfrak{A}$ ) is either  $A$  or  $\mathbb{N}$ . Similarly, for second-order variables  $R$  we allow mixed types, that is a relation symbol  $R$  of type  $(k, \ell) \in \mathbb{N} \times \mathbb{N}$  stands for a relation  $R \subseteq A^k \times \mathbb{N}^\ell$ .

Of course, already FO is undecidable over the class of two-sorted structures  $\mathfrak{A}^\#$ . To obtain a logic with polynomial-time data complexity, we have to restrict the range of quantifiers over the numeric sort by fixed polynomials. More precisely, FPC-formulas can use quantifiers over the numeric sort only in the form  $Qx \leq n^q \cdot \varphi$  where  $Q \in \{\exists, \forall\}$  and where  $q \geq 1$  is a fixed constant. The range of the quantifier  $Q$  is  $\{0, \dots, n^q\}$  where  $n$  denotes the size of the input structure  $\mathfrak{A}$ . To simplify notation, we henceforth assume that each numeric variable  $x$  comes with a built-in restricted range polynomial, that is  $x = (x \leq n^q)$ . For better readability, we usually omit this range polynomial in our notation. By this convention, each variable  $x$  has a predefined range in any input structure  $\mathfrak{A}^\#$  of polynomial size (which is either  $A$  or  $\{0, \dots, n^q\}$  for a fixed  $q \geq 1$ ). We denote this range by  $\text{dom}(\mathfrak{A}, x)$  (or just by  $\text{dom}(x)$  if  $\mathfrak{A}$  is clear from the context). Analogously, for a tuple of variables  $\vec{x} = (x_1, \dots, x_k)$  we set  $\text{dom}(\vec{x}) = \text{dom}(x_1) \times \dots \times \text{dom}(x_k)$ . By this, we also obtain polynomial bounds for numeric components in fixed-point definitions [**ifp**  $R\vec{x} \cdot \varphi(R, \vec{x})$ ] ( $\vec{x}$ ). Indeed, the inflationary fixed-point defined by this formula is of the form  $R \subseteq \text{dom}(\vec{x})$ .

The crucial elements of FPC are *counting terms* which allow to define cardinalities of sets. Starting with an arbitrary FPC-formula  $\varphi(x)$  one can form a new *counting term*  $s = [\#x \cdot \varphi]$  whose value in  $\mathfrak{A}$  is just the size of the set defined by  $\varphi$  in  $\mathfrak{A}$ . In particular, the term  $s$  is a *numeric term*, that is  $s$  takes its value in the number sort. More precisely, for an input structure  $\mathfrak{A}$ , the value  $s^{\mathfrak{A}} \in \mathbb{N}$  of  $s$  in  $\mathfrak{A}$  is the number of elements  $a \in A$  such that  $\mathfrak{A} \models \varphi(a)$ . One can allow counting terms of a more general form without increasing the expressive power of FPC. In particular, counting terms  $[\#\vec{x} \cdot \varphi]$  over mixed tuples of variables can be simulated with unary counting terms and fixed-point operators; we refer to [39] for more details and background on fixed-point logic with counting.

An important fact that we are going to use frequently is that formulas of FPC with  $k$  variables can be rewritten as equivalent  $C_{\infty\omega}^k$ -formulas. In particular, we have that  $\text{FPC} \leq C_{\infty\omega}^\omega$ . In Section 5, we also make use of the fact that for every  $k \geq 1$ , there exists

an FPC-formula  $\varphi$  with  $\mathcal{O}(k)$  many variables such that  $(\mathfrak{A}, \mathfrak{B}) \models \varphi$  if, and only if,  $\mathfrak{A} \equiv^k \mathfrak{B}$ , see e.g. [39].

In Section 4, we also make use of the *numeric extension of first-order logic*, denoted by  $\text{FO}^+$ , which is defined as FPC, but without the rule for forming (inflationary) fixed points.

**2.4. Logical Interpretations and Lindström Quantifiers.** The logical counterpart of the notion of an (algorithmic) reduction is the notion of a *logical interpretation*. A logical interpretation  $\mathcal{I}$  transforms an input structure  $\mathfrak{A}$  into a new structure  $\mathfrak{B} = \mathcal{I}(\mathfrak{A})$  and this transformation is defined by formulas of some logic  $\mathcal{L}$ . In this article we consider  $\mathcal{L}$ -interpretations with respect to different underlying logics  $\mathcal{L}$ , such as  $\text{FO}$ ,  $\text{FO}^+$ ,  $\text{LFP}$ ,  $\text{FPC}$ ,  $\text{C}_{\infty\omega}^\omega$ . Basically, the definition of an  $\mathcal{L}$ -interpretation is uniform for all of these logics. However, there is one exception for the case of  $\text{FO}^+$  and  $\text{FPC}$  where we have the special situation that formulas can use numeric variables. As a consequence, the interpreted structures  $\mathcal{I}(\mathfrak{A})$  can contain such numeric elements. In this section, we further introduce *Lindström quantifiers*, also known as *generalised quantifiers*, which capture the notion of *oracles* in the realm of finite model theory.

Let us start with the case of single-sorted logics  $\mathcal{L}$ , such as  $\text{FO}$ ,  $\text{LFP}$ , or  $\text{C}_{\infty\omega}^\omega$ . Let  $\sigma, \tau$  be signatures with  $\tau = \{S_1, \dots, S_\ell\}$ . Let  $s_i$  denote the arity of  $S_i$ . An  $\mathcal{L}[\sigma, \tau]$ -interpretation is a tuple

$$I(\vec{z}) = (\varphi_\delta(\vec{x}, \vec{z}), \varphi_\approx(\vec{x}_1, \vec{x}_2, \vec{z}), \varphi_{S_1}(\vec{x}_1, \dots, \vec{x}_{s_1}, \vec{z}), \dots, \varphi_{S_\ell}(\vec{x}_1, \dots, \vec{x}_{s_\ell}, \vec{z}))$$

where  $\varphi_\delta, \varphi_\approx, \varphi_{S_1}, \dots, \varphi_{S_\ell} \in \mathcal{L}[\sigma]$  and  $\vec{x}, \vec{x}_1, \dots, \vec{x}_{s_\ell}$  are tuples of pairwise distinct variables of the same length  $d$  and  $\vec{z}$  is a tuple of variables pairwise distinct from the  $x$ -variables. We call  $d$  the *dimension* and  $\vec{z}$  the *parameters* of  $\mathcal{I}(\vec{z})$ .

A  $d$ -dimensional  $\mathcal{L}[\sigma, \tau]$ -interpretation  $\mathcal{I}(\vec{z})$  defines a partial mapping  $\mathcal{I}: \text{Str}(\sigma, \vec{z}) \rightarrow \text{Str}(\tau)$  in the following way: For  $(\mathfrak{A}, \vec{z} \mapsto \vec{a}) \in \text{Str}(\sigma, \vec{z})$  we obtain a  $\tau$ -structure  $\mathfrak{B}$  over the universe  $\{\vec{b} \in A^d \mid \mathfrak{A} \models \varphi_\delta(\vec{b}, \vec{a})\}$ , setting  $S_i^{\mathfrak{B}} = \{(\vec{b}_1, \dots, \vec{b}_{s_i}) \in B^{s_i} \mid \mathfrak{A} \models \varphi_{S_i}(\vec{b}_1, \dots, \vec{b}_{s_i}, \vec{a})\}$  for each  $S_i \in \tau$ . Moreover let  $\mathcal{E} = \{(\vec{b}_1, \vec{b}_2) \in A^d \times A^d \mid \mathfrak{A} \models \varphi_\approx(\vec{b}_1, \vec{b}_2, \vec{a})\}$ . Now we define

$$\mathcal{I}(\mathfrak{A}, \vec{z} \mapsto \vec{a}) := \begin{cases} \mathfrak{B}/\mathcal{E} & \text{if } \mathcal{E} \text{ is a congruence relation on } \mathfrak{B} \\ \text{undefined} & \text{otherwise.} \end{cases}$$

We say that  $\mathcal{I}$  interprets  $\mathfrak{B}/\mathcal{E}$  in  $\mathfrak{A}$ .

Let us briefly discuss the case of two-sorted logics. If  $\mathcal{L}$  is  $\text{FPC}$  or  $\text{FO}^+$ , then we have the same definition of an  $\mathcal{L}$ -interpretation as above. However, note that now the variable tuples  $\vec{x}, \vec{z}, \dots$  may contain numeric variables. Recall that each numeric variable  $x$  has an explicit polynomial range bound  $\text{dom}(\mathfrak{A}, x)$  which is either  $A$  or  $\{0, \dots, n^q\}$  for a fixed  $q \geq 1$ . As a consequence, the domain of the structure  $\mathcal{I}(\mathfrak{A}, \vec{z} \mapsto \vec{a})$  does not longer consist of equivalence classes of tuples in  $A^d$  but, more generally, it consists of equivalence classes of elements in  $\text{dom}(\mathfrak{A}, \vec{x})$  (and note that these tuples may contain numeric components).

Next, we introduce Lindström quantifiers. Let  $\mathcal{L}$  be a logic and  $\mathcal{K} \subseteq \text{Str}(\tau)$  a class of  $\tau$ -structures with  $\tau = \{S_1, \dots, S_\ell\}$ . The *Lindström extension*  $\mathcal{L}(\mathcal{Q}_{\mathcal{K}})$  of  $\mathcal{L}$  by *Lindström quantifiers* for the class  $\mathcal{K}$  is obtained by extending the syntax of  $\mathcal{L}$  by the following formula creation rule:

Let  $\varphi_\delta, \varphi_\approx, \varphi_{S_1}, \dots, \varphi_{S_\ell}$  be formulas in  $\mathcal{L}(\mathcal{Q}_{\mathcal{K}})$  that form an  $\mathcal{L}[\sigma, \tau]$ -interpretation  $\mathcal{I}(\vec{z})$ . Then  $\psi(\vec{z}) = \mathcal{Q}_{\mathcal{K}}\mathcal{I}(\vec{z})$  is a formula in  $\mathcal{L}(\mathcal{Q}_{\mathcal{K}})$  over the signature  $\sigma$ ,

with  $(\mathfrak{A}, \vec{z} \mapsto \vec{a}) \models \mathcal{Q}_{\mathcal{K}}\mathcal{I}(\vec{z})$ , if, and only if,  $\mathfrak{B} := \mathcal{I}(\mathfrak{A}, \vec{z} \mapsto \vec{a})$  is defined and  $\mathfrak{B} \in \mathcal{K}$ .

As we see, adding the Lindström quantifier  $\mathcal{Q}$  to the logic  $\mathcal{L}$  is the most direct way to make the class  $\mathcal{K}$  definable in  $\mathcal{L}$ . We use this key notion to capture the power of propositional proof systems up to first-order definable transformations.

**2.5. Representing Propositional Formulas as Relational Structures.** As always when we are dealing with logics in algorithmic contexts, we have to agree on encodings of (non-structural) inputs as relational structures. In this article such inputs are, for instance, propositional formulas, polynomial equation systems, matrices and vectors over fields. In all of these cases, it is straightforward to come up with natural structural representations. Most often, we refrain from describing such encodings explicitly. For it is rather tedious, and, more importantly, the concrete details do not matter too much: all (natural) encodings will be interdefinable in first-order logic.

To get a better intuition, let's go through one encoding explicitly. Let us briefly discuss two ways to represent propositional formulas (in CNF) as finite relational structures. Perhaps the most obvious representation of a CNF-formula  $\psi$  as a structure  $\mathfrak{A}(\psi)$  is based on the vocabulary  $\{C, V, P, N\}$ ; the universe of  $\mathfrak{A}(\psi)$  consists of the variables and the clauses of  $\psi$ , the monadic relations  $V$  and  $C$  identify the variables and clauses, respectively, and the binary relations  $P$  and  $N$  specify which variables appear positively and negatively in which clauses; so  $Pvc$  is true in  $\mathfrak{A}(\psi)$  if the variable  $v$  appears positively in the clause  $c$ , and analogously for  $N$ . A different representation, that sometimes leads to more elegant logical descriptions works with the set  $L$  of literals and with a self-inverse bijection  $\neg : L \rightarrow L$ , so that  $\psi$  would be represented by  $\mathfrak{A}(\psi) = (A, C, L, \neg, \in)$  where  $A$  is the set of clauses and literals,  $\neg(x)$  is the complementary literal to  $x$ , and  $x \in c$  means that the literal  $x$  occurs in the clause  $c$  (note that, formally, we do not allow function symbols in our vocabularies, but, of course, we can substitute function symbols by their graph relations).

### 3. RESOLUTION AND (EXISTENTIAL) LEAST FIXED-POINT LOGIC

In this section we study the resolution proof system. We start by showing that Horn-Resolution (HORN-RES) is complete for least fixed-point logic (LFP) wrt. (many-to-one) first-order interpretations, see Theorem 3.3. In a second step, we consider bounded-width resolution ( $k$ -RES, for  $k \geq 2$ ). We show that bounded-width resolution is strictly weaker than Horn-Resolution from the viewpoint of finite model theory. Specifically, we prove that 2-RES is complete for transitive closure logic FO(TC) (Theorem 3.6) and that for every  $k \geq 3$ ,  $k$ -RES is complete for the existential fragment of least fixed-point logic (EFP), see Theorem 3.7. Since it is known that FO(TC) < EFP < LFP, this separates the power of these polynomial-time restrictions of the resolution proof system.

**3.1. Horn Resolution Captures Least Fixed-Point Logic.** Let posLFP be the fragment of LFP-formulas that are in negation normal form (i.e. negation is applied only to input atoms), in which each fixed-point variable is bound only once, and that do not make use of greatest fixed points. Further, let EFP<sub>0</sub> be the basic existential fragment of LFP; it consists of those formulas in posLFP whose quantifiers are all existential.

It is known that, on finite structures (but not in general), every LFP-formula can be effectively translated into an equivalent one in posLFP. On the other side EFP<sub>0</sub> is strictly weaker; it has the same expressive power as Datalog with negation of input atoms.

**Theorem 3.1.** *For every  $\varphi \in \text{posLFP}[\tau]$  there is a first-order interpretation  $I_\varphi$  that maps finite  $\tau$ -structures to propositional Horn formulas  $\psi_{\mathfrak{A},\varphi}$  such that  $\mathfrak{A} \models \varphi$  if, and only if,  $\psi_{\mathfrak{A},\varphi}$  is unsatisfiable. Further, if  $\varphi$  is in EFP<sub>0</sub> then all clauses in  $\psi_{\mathfrak{A},\varphi}$  have width at most three.*

*Proof.* Fix a formula  $\varphi \in \text{posLFP}[\tau]$ . For every finite  $\tau$ -structure  $\mathfrak{A}$ , with universe  $A$ , we construct the propositional Horn formula  $\psi_{\mathfrak{A},\varphi}$  as follows. An *instantiated subformula* of  $\varphi$  is an expression  $\beta(\bar{a})$  which is obtained by taking some subformula  $\beta(\bar{x})$  of  $\varphi$  and by instantiating every free variable  $x$  by some element  $a \in A$ . We now take for every instantiated subformula  $\beta$  of  $\varphi$  a propositional variable  $X_\beta$ , and inductively define a set  $C(\mathfrak{A}, \varphi)$  of clauses as follows.

- (1) If  $\beta$  is a  $\tau$ -literal, then we add  $1 \rightarrow X_\beta$  in case that  $\mathfrak{A} \models \beta$  and  $X_\beta \rightarrow 0$  in case  $\mathfrak{A} \not\models \beta$ .
- (2) If  $\beta = \eta \vee \vartheta$ , then we add the clauses  $X_\eta \rightarrow X_\beta$  and  $X_\vartheta \rightarrow X_\beta$ .
- (3) If  $\beta = \eta \wedge \vartheta$ , then we add the clause  $X_\eta \wedge X_\vartheta \rightarrow X_\beta$ .
- (4) If  $\beta = \exists x \eta(x)$ , then we add all clauses  $X_{\eta(a)} \rightarrow X_\beta$  for  $a \in A$ .
- (5) If  $\beta = \forall x \eta(x)$ , then we add the clause  $(\bigwedge_{a \in A} X_{\eta(a)}) \rightarrow X_\beta$ .
- (6) If  $\beta = [\mathbf{lfp} R\bar{x}. \eta](\bar{a})$  or  $\beta = R\bar{a}$ , then we add the clause  $X_{\eta(\bar{a})} \rightarrow X_\beta$ .

By induction, it readily follows that the minimal model of all these clauses sets the variable  $X_\beta$  to true if, and only if,  $\mathfrak{A} \models \beta$  (with fixed-point variables interpreted by their least fixed-point on  $\mathfrak{A}$ ). Let now  $\psi_{\mathfrak{A},\varphi}$  be defined as the conjunction of all clauses in  $C(\mathfrak{A}, \varphi)$  together with  $X_\varphi \rightarrow 0$ . Then  $\psi_{\mathfrak{A},\varphi}$  is unsatisfiable if, and only if,  $\mathfrak{A} \models \varphi$ .

We observe that the only clauses of size larger than three are those coming from universal quantifiers. Hence, if there are no universal quantifiers, the formula only has clauses of size at most three. Finally it is clear that, for every fixed  $\varphi \in \text{posLFP}[\tau]$ , we can interpret (a representation of) the formula  $\psi_{\mathfrak{A},\varphi}$  inside  $\mathfrak{A}$ , by using an FO-interpretation  $I_\varphi$ .  $\square$

This shows that  $\text{LFP} \leq \text{FO}(\text{HORN-RES})$ . Actually we established a stronger result.

**Theorem 3.2.** *For every formula  $\varphi \in \text{LFP}$  there exists a first-order interpretation  $J_\varphi$  such that  $\mathcal{Q}_{\text{HORN-RES}}(J_\varphi)$  is equivalent to  $\varphi$  on finite structures. In particular, each LFP-formula can be translated into an equivalent FO(HORN-RES)-formula with a single application of the generalised quantifier  $\mathcal{Q}_{\text{HORN-RES}}$ .*

We are ready to prove that  $\text{FO}(\text{HORN-RES})$  has the same expressive power as LFP.

**Theorem 3.3.** *On finite structures,  $\text{LFP} = \text{FO}(\text{HORN-RES})$ .*

It remains to show that  $\text{FO}(\text{HORN-RES}) \leq \text{LFP}$ , that is we have to express Horn resolution in LFP. Recall that a propositional Horn formula  $\psi$  admits a derivation of the empty clause if, and only if,  $\psi$  contains a clause in which all variables appear negatively, written  $X_1 \wedge \dots \wedge X_k \rightarrow 0$ , such that all unit clauses  $\{X_i\}$  for  $i = 1, \dots, k$  can be derived from  $\psi$  by Horn resolution.

Let  $\psi$  be presented as a structure  $\mathfrak{A}(\psi)$  with universe  $C \cup V$  and vocabulary  $\{C, V, P, N\}$ . Let  $D$  be the set of variables  $v \in V$  such that the clause  $\{v\}$  can be derived from  $\psi$  by Horn resolution. Then  $\psi$  is unsatisfiable if, and only if,  $\mathfrak{A}(\psi) \models \exists c(Cc \wedge \neg \exists x Pxc \wedge \forall x(Nxc \rightarrow Dx))$ . The set  $D$  is definable by the LFP-formula  $[\mathbf{lfp} Dx. \exists c(Pxc \wedge \forall y(Nyc \rightarrow Dy))](x)$ .

**3.2. Bounded-Width Resolution and Existential Least Fixed-Point Logic.** Intuitively, *existential least fixed-point logic* (EFP) extends  $\text{EFP}_0$  by stratified negation. This means that it permits fixed-point formulas over existential formulas which may depend on closed fixed-point relations, defined in a lower stratum, and these can be used also in negated form. Thus, negation (and hence, implicitly, also universal quantifiers) are present in a limited form, but least fixed-point recursions may never go through negation or universal quantification. In fact, EFP is equivalent to Stratified Datalog and is weaker than full LFP [16, 36].

**Definition 3.4.** Existential fixed-point logic  $\text{EFP} := \bigcup_{\ell \geq 0} \text{EFP}_\ell$  generalises  $\text{EFP}_0$  as follows. The stratum  $\text{EFP}_{\ell+1}$  is the closure under disjunction, conjunction and existential quantification of formulas of the form  $[\mathbf{lfp} \ R\bar{x}.\exists\bar{y}\varphi(R, \bar{x}, \bar{y})](\bar{x})$  where  $\varphi(R, \bar{x}, \bar{y})$  is obtained from a quantifier-free formula, that may contain positive and negative occurrences of additional relations  $S_1, \dots, S_m$ , by substituting these relations by formulas from  $\text{EFP}_\ell$ .

Let us remark that the logic EFP is known under different names (we stick to the term EFP which was used in [16, 36]). For instance, in [22], the term *bounded fixed-point logic* (BFP) is used to refer to the same logic. Another common name for this logic is *stratified fixed-point logic* (SFP) in reference to its equivalence with Stratified Datalog.

Notice that first-order logic FO is contained in EFP, but not in any bounded level  $\text{EFP}_\ell$ , because every quantifier alternation in FO must be simulated by an additional level of stratified negation, again see [16, 36]. For the same reason EFP, but none of its levels  $\text{EFP}_\ell$ , is closed under first-order operations. As a consequence of Theorem 3.1 we can infer

**Theorem 3.5.** *On finite structures,  $\text{EFP} \leq \text{FO}(3\text{-RES})$ .*

*Proof.* Theorem 3.1 directly establishes this for  $\text{EFP}_0$ . So assume that the claim is established for  $\text{EFP}_\ell$ . Every formula in  $\text{EFP}_{\ell+1}$  can be written as an  $\text{EFP}_0$ -formula over predicates that are  $\text{EFP}_\ell$ -definable. Hence, by applying Theorem 3.1 once more, it can be rewritten as an  $\text{FO}(3\text{-RES})$ -formula over predicates that are themselves definable in  $\text{FO}(3\text{-RES})$ . Since Lindström extensions of FO are closed under nesting of generalised quantifiers, it follows that also  $\text{EFP}_{\ell+1} \leq \text{FO}(3\text{-RES})$ .  $\square$

We require clauses of width 3 for translating EFP-formulas into Horn formulas. In fact, if we restrict to clauses of width 2, then we obtain the power of first-order logic with a transitive closure operator  $\text{FO}(\text{TC})$ . This immediately follows from the fact that satisfiability of 2-CNF formulas reduces to graph reachability, and from the reduction of graph reachability to the non-satisfiability problem for a 2-CNF formula that we described in the introduction.

**Theorem 3.6.** *On finite structures,  $\text{FO}(2\text{-RES}) = \text{FO}(\text{TC})$ .*

**3.3. Simulating Bounded-Width Resolution in EFP.** To express width- $k$  resolution, for fixed  $k \geq 1$ , in EFP, we shall use the representation of a CNF-formula  $\psi$  by structures  $\mathfrak{A}(\psi) = (A, C, L, \neg, \in)$  where  $C$  is the set of clauses and  $L$  is the set of literals, and the universe is  $A = C \cup L \cup \{0\}$ . Further we shall describe the set of all derivable clauses of size at most  $k$  as a  $k$ -ary relation  $D \subseteq (L \cup \{0\})^k$ , that contains those  $k$ -tuples  $(x_1, \dots, x_k)$  for which  $\{x_i : i \leq k, x_i \neq 0\}$  is a clause that is derivable from  $\psi$ . This relation  $D$  is defined by a fixed-point formula  $[\mathbf{lfp} \ D\vec{x}.\varphi(D, \vec{x})](\vec{x})$  where  $\varphi(D, \vec{x})$  expresses the following. Either

- (1) there exists a clause  $c \in C$  such that  $c = \{x_1, \dots, x_k\} \setminus \{0\}$ , or
- (2) there exist tuples  $\vec{y}, \vec{z} \in D$  such that, for some  $i, j$ , the literal  $z_j$  is the negated literal to  $y_i$ , and  $(\{y_1, \dots, y_k\} \cup \{z_1, \dots, z_k\}) \setminus \{y_i, z_j, 0\} = \{x_1, \dots, x_k\} \setminus \{0\}$ .

When spelling out these equations in first-order logic, we can express  $\varphi(\vec{x}, D)$  by an existential FO-formula  $\exists \vec{y} \alpha(\vec{x}, \vec{y}, D, Q)$  where  $Q$  is FO-definable by a formula (with quantifier prefix  $\exists^* \forall$ ) that does not depend on  $D$ . This yields a formula in  $\text{EFP}_1$ . Since  $\text{EFP}$  is closed under FO-operations, this proves

**Theorem 3.7.** *On finite structures,  $\text{FO}(k\text{-RES}) = \text{EFP}$  for all  $k \geq 3$ .*

Another interesting observation is that if we restrict the nesting depth of  $k\text{-RES}$ -quantifiers in  $\text{FO}(k\text{-RES})$ -formulas to some constant  $d \geq 1$ , then we obtain a fragment  $\text{FO}(k\text{-RES})^{\leq d}$  of  $\text{FO}(k\text{-RES})$  which is strictly less expressive. This follows from the results of Grädel and McColm [26] and the observation that formulas in  $\text{FO}(k\text{-RES})^{\leq d}$  can be written as  $L_{\infty\omega}^\omega$ -formulas with at most  $d$  many nested unbounded quantifier blocks. However, as Grädel and McColm show there are formulas of transitive closure logic  $\text{FO}(\text{TC})$  which require more than  $d$  many such blocks when expressed as equivalent  $L_{\infty\omega}^\omega$ -formulas. Since  $\text{FO}(\text{TC}) \leq \text{FO}(k\text{-RES})$ , it follows that for every  $d \geq 1$  we have  $\text{FO}(k\text{-RES})^{\leq d} < \text{FO}(k\text{-RES})$ . Note that this is different from the case of Horn-Resolution where nesting of  $\text{HORN-RES}$ -quantifiers was not necessary. In other words, while Horn-Resolution  $\text{HORN-RES}$  is many-to-one complete for  $\text{LFP}$  wrt. first-order interpretations,  $k\text{-RES}$  is only complete wrt. first-order Turing reductions for  $\text{FO}(\text{TC})$  and  $\text{EFP}$ , respectively.

#### 4. THE POLYNOMIAL CALCULUS OVER THE FIELD OF RATIONALS AND FIXED-POINT LOGIC WITH COUNTING

We now turn our attention to the *polynomial calculus* (PC). The polynomial calculus is an important and well-studied propositional proof system that is based on algebraic reasoning techniques. The idea is to represent Boolean formulas by polynomial equation systems over some field  $\mathbb{F}$  and to show that, by manipulating these polynomial equations, one can derive an inconsistent equation such as  $0 = 1$ . Analogous to the case of bounded-width resolution  $k\text{-RES}$ , it is possible to stratify the polynomial calculus along a parameter  $k \geq 2$  to obtain polynomial-time fragments. More precisely, if we restrict the degree of all polynomials in PC-refutations to some constant  $k \geq 2$ , then we obtain (incomplete) fragments  $\text{PC}_k$  of the full PC in which proofs can be found in polynomial time (over the rationals, we must also restrict the bit-complexity of the coefficients to ensure this). One can define the polynomial calculus with respect to any underlying field  $\mathbb{F}$ . Throughout this section, this underlying field  $\mathbb{F}$  will always be the field of rationals  $\mathbb{Q}$ . In the following Section 5, we turn our attention to the case of finite fields.

Another important fragment of the polynomial calculus is the so-called *monomial-PC* ( $\text{MON-PC}$ ). This restricted variant of the full PC was introduced by Berkholz and Grohe in [10]. Their intention was to precisely characterise the power of a combinatorial graph isomorphism test, the so-called Weisfeiler-Leman algorithm [13], in terms of propositional proof complexity. Specifically, they proved that two graphs  $G$  and  $H$  can be distinguished by the  $k$ -dimensional Weisfeiler-Leman algorithm if, and only if, the  $k$ -dimensional monomial-PC ( $\text{MON-PC}_k$ ) can refute the solvability of a certain system of polynomial equations  $\text{ISO}(G, H)$  over  $\mathbb{Q}$  which encode the graph isomorphism problem for  $G$  and  $H$ . In this section, we analyse the power of the monomial-PC and the (full) PC from the perspective of finite model

theory. In our main result we are going to show that both proof systems, the bounded-degree monomial-PC and the bounded-degree and bounded bit-complexity PC, have precisely the same expressive power as fixed-point logic with counting (FPC), which is a natural and powerful logic of great importance in the area of descriptive complexity theory (see Theorem 4.11). For the bounded-degree PC over  $\mathbb{Q}$  without any restriction on the complexity of the coefficients, we show that it is contained in finite variable infinitary counting logic. As a consequence, the correspondence between the Weisfeiler-Leman algorithm and the monomial-PC can be generalised to the full PC (though we have to sacrifice the tightness of the connection between the degree of polynomials and the dimension of the Weisfeiler-Leman algorithm), see Theorem 6.6.

Our proof consists of three parts. First of all, we show that proofs in the monomial-PC can be expressed in FPC, see Subsection 4.2. This shows that  $\text{FO}^+(\text{MON-PC}_k) \leq \text{FPC}$ . After that, we show in Subsection 4.3 that FPC-iterations can be simulated using the monomial-PC. Taken together this shows that  $\text{FO}^+(\text{MON-PC}_k) = \text{FPC}$ . As the final step, in Subsection 4.4, we show that FPC can also express degree- $k$  refutations of bounded bit-complexity in the (full) polynomial calculus over  $\mathbb{Q}$ , which also entails that  $\text{PC}_k$  with unbounded coefficients can be simulated in  $C_{\infty\omega}^\omega$ .

**4.1. The Polynomial Calculus.** We start with some background on the polynomial calculus and its restricted variant, the monomial-PC. Both systems refute the solvability of a given set of (*multivariate*) *polynomial equations* over some field  $\mathbb{F}$  using proof rules that manipulate such equations. In this paper,  $\mathbb{F}$  will either be the field of rationals  $\mathbb{Q}$  or a finite field  $\mathbb{F}_{p^n}$  of size  $p^n$  for  $p \in \mathbb{P}$ , where  $\mathbb{P}$  denotes the set of primes and where  $n \geq 1$ . We denote by  $\mathbb{F}[\mathcal{X}]$  the ring of polynomials in variables  $\mathcal{X} = \{X_j : j \in J\}$ , for some index set  $J$  and with coefficients in  $\mathbb{F}$ . For an “exponent”  $\alpha : J \rightarrow \mathbb{N}$  we let the *monomial*  $X^\alpha$  be defined as  $X^\alpha = \prod_{j \in J} X_j^{\alpha(j)}$ . Then polynomials  $f \in \mathbb{F}[\mathcal{X}]$  can be written as  $f = \sum_{\alpha} f_{\alpha} \cdot X^{\alpha}$  where the  $f_{\alpha} \in \mathbb{F}$  are coefficients from the field  $\mathbb{F}$  and such that  $f_{\alpha} \neq 0$  for finitely many  $\alpha$  only. The *degree*  $\text{deg}(X^\alpha)$  of a monomial  $X^\alpha$  is defined as  $|\alpha| = \sum_{j \in J} \alpha(j)$ , and the degree of a polynomial is defined as the maximal degree of its monomials. A *polynomial equation* is an equation of the form  $f = 0$  for a polynomial  $f \in \mathbb{F}[\mathcal{X}]$ . For better readability, we usually omit the equality “ $= 0$ ” when we specify polynomial equations, that is we identify polynomials  $f \in \mathbb{F}[\mathcal{X}]$  with the corresponding normalised polynomial equations  $f = 0$ . A *system of polynomial equations* is a set  $\mathcal{P} = \{f_i : i \in I\}$  consisting of polynomials  $f_i \in \mathbb{F}[\mathcal{X}]$  for all  $i \in I$  where  $I$  is an (unordered) index set. A *solution* of  $\mathcal{P}$  is a common zero  $a \in \mathbb{F}^J$  of all polynomials in  $\mathcal{P}$ . In what follows, we only consider systems  $\mathcal{P} = \{f_i : i \in I\}$  which contain for every variable  $X = X_j$ ,  $j \in J$ , the polynomial equation  $(X^2 - X) = 0$ . The axioms  $(X^2 - X) = 0$  enforce that each variable  $X = X_j$ ,  $j \in J$ , can only take values 0 or 1. These equations encode the Boolean setting (truth values) that we are interested in.

The polynomial calculus is based on the following result from algebra which is known as *Hilbert’s Nullstellensatz*. It says that the non-solvability of the system  $\mathcal{P} = \{f_i : i \in I\}$  of polynomial equations is equivalent to the existence of polynomials  $g_i \in \mathbb{F}[\mathcal{X}]$ ,  $i \in I$ , such that  $\sum_{i \in I} g_i \cdot f_i = 1$ . The polynomials  $g_i$  are called a *Nullstellensatz refutation* for the system  $\mathcal{P}$ . The idea of the polynomial calculus is to search for such polynomials  $g_i$  in a sequential way.

**Definition 4.1.** The inference rules of the *polynomial calculus* (PC) over the ring of polynomials  $\mathbb{F}[\mathcal{X}]$  are as follows:

$$\begin{array}{ll}
\text{(Multiplication)} & \frac{f}{Xf} \quad \text{where } X \in \mathcal{X} \\
\text{(Linear Combination)} & \frac{g, f}{ag + bf} \quad \text{where } a, b \in \mathbb{F}
\end{array}$$

The goal of the polynomial calculus is to derive with these rules, from a collection  $\mathcal{P}$  of axioms  $p \in \mathcal{P}$ , the constant polynomial  $1 \in \mathbb{F}[\mathcal{X}]$ , in order to prove that the polynomials in  $\mathcal{P}$  have no common zero.

The *monomial-PC* (MON-PC) is the restriction of the (full) PC that permits the use of the multiplication rule only in the cases where  $f$  is either a monomial or the product of a monomial and an axiom. A polynomial equation system  $\mathcal{P}$  has a *refutation* of degree  $k \geq 1$  in the PC (or MON-PC) if the constant polynomial  $1 \in \mathbb{F}[\mathcal{X}]$  can be derived from  $\mathcal{P}$  using only polynomials of degree at most  $k$ .

The polynomial calculus, and the monomial-PC, are clearly sound and, by Hilbert’s Nullstellensatz, also complete proof systems. However, completeness requires unbounded degree in refutations. In fact, as we indicated before, the “degree of polynomials” for the PC (MON-PC) is a complexity measure with very similar properties as the “width of clauses” measure for the resolution proof system. If we restrict the PC (MON-PC) to polynomials of degree at most  $k$ , for some fixed  $k \geq 1$ , then the systems become incomplete, but admit proof search in polynomial time (again, for PC over  $\mathbb{Q}$ , we must also restrict the bit-complexity of the coefficients). In what follows, whenever we speak of the monomial-PC or the (full) PC, then we usually refer to a variant with restricted degree  $k \geq 1$ . If we want to make this constant  $k$  explicit, then we denote the corresponding proof system by  $\text{MON-PC}_k$  and  $\text{PC}_k$ , respectively. Another fact which we use implicitly throughout this section is that the axioms  $(X^2 - X)$  guarantee that in (monomial-)PC proofs we can restrict ourselves to *multilinear* polynomials. To see this, say that we were able to derive the polynomial  $p = X^2Y + Z$  within some (monomial-)PC proof. Of course,  $p$  is not multilinear. However, we can use the axiom  $(X^2 - X)$  together with the “linear combination”-rule to reduce this polynomial to the corresponding multilinear polynomial  $p' = XY + Z$ . Indeed,  $p' = p - Y(X^2 - X)$ . Hence, restricting to multilinear polynomials, and modifying the multiplication rule accordingly with implicit linearisation, does not change the power of the corresponding proof systems. For a polynomial  $p \in \mathbb{F}[\mathcal{X}]$  we denote its *multilinearisation* by  $\text{MultLin}(p)$ . So, from now on we stick to the setting of implicitly multilinearising all polynomials which precisely captures the semantics of the polynomial equations  $(X^2 - X) = 0$ .

We remind the reader that in this section the underlying field  $\mathbb{F}$  for the (monomial-)PC is always the field of rationals  $\mathbb{Q}$ .

**4.2. Monomial-PC in Fixed-Point Logic with Counting.** Our first aim is to show that FPC can express  $\text{MON-PC}_k$ -refutations over the rationals using only  $\mathcal{O}(k)$  many variables. Of course, in order to obtain such a definability result, we have to agree on an encoding of sets  $\mathcal{P}$  of rational, multilinear polynomials as finite relational structures. Similar to our representation of CNF-formulas described in Section 2, a natural encoding can be based on a many-sorted structure  $\mathfrak{A}_{\mathcal{P}}$  whose universe is partitioned into sets of polynomials, (multilinear) monomials, variables, and rational coefficients that occur in  $\mathcal{P}$ . As usual, we represent rationals as fractions of integers using binary encoding. Hence,  $\mathfrak{A}_{\mathcal{P}}$  provides a linear order of sufficient length to encode these binary strings. Again, the exact technical details are not important, as long as the encoding has some natural properties, such as FO-definability of

the class of valid encodings. By a slight abuse of notation, we also denote by  $\text{MON-PC}_k$  the class of structures  $\mathfrak{A}_{\mathcal{P}}$  which encode a system  $\mathcal{P}$  of polynomials over  $\mathbb{Q}$  which can be refuted in  $\text{MON-PC}_k$ .

**Theorem 4.2.** *For every  $k \geq 1$ ,  $\text{MON-PC}_k \in \text{FPC}$ .*

Given a set of multilinear polynomials  $\mathcal{P}$  of degree at most  $k$ , we consider the set  $V_{\mathcal{P}} = \text{MON-PC}_k(\mathcal{P})$  of multilinear polynomials which can be derived from  $\mathcal{P}$  using  $\text{MON-PC}_k$ . The first observation is that  $V_{\mathcal{P}}$  is a  $\mathbb{Q}$ -linear space. This easily follows since we can take  $\mathbb{Q}$ -linear combinations of polynomials that we derived. Now, since this vector space  $V_{\mathcal{P}}$  only contains multilinear polynomials of degree at most  $k$ , we can naturally associate polynomials  $p \in V_{\mathcal{P}}$  with vectors  $p \in \mathbb{Q}^{M_k}$  where the index set  $M_k$  denotes the set of all multilinear monomials of degree at most  $k$ . For fixed  $k \geq 1$ , this set  $M_k$  is of polynomial size  $n^{\mathcal{O}(k)}$ .

To prove Theorem 4.2 we are going to express in FPC an inductive algorithm, that is based on a similar algorithm for the full polynomial calculus from [14], for computing a generating set for the  $\mathbb{Q}$ -linear space  $V_{\mathcal{P}}$ . Then, in order to see whether  $\text{MON-PC}_k$  can refute the system  $\mathcal{P}$ , we simply check whether the constant polynomial 1 is contained in  $V_{\mathcal{P}}$ , see Figure 1.

```

Input: Set of multilinear polynomials  $\mathcal{P} \subseteq \mathbb{Q}^{M_k}$ 
Output:  $\mathcal{B} \subseteq \mathbb{Q}^{M_k}$  such that  $\langle \mathcal{B} \rangle = \text{MON-PC}_k(\mathcal{P})$ 
           // where  $\langle \mathcal{B} \rangle$  denotes the  $\mathbb{Q}$ -linear subspace generated by  $\mathcal{B}$ 
 $\mathcal{B} := \{\text{MultLin}(m \cdot p) \mid p \in \mathcal{P}, m \text{ a monomial such that } \deg(\text{MultLin}(m \cdot p)) \leq k\}$ 
           // Initialisation (lift all axioms in  $\mathcal{P}$  up to degree  $k$ )
repeat
  for all monomials  $m \in \langle \mathcal{B} \rangle$ ,  $\deg(m) < k$  do
     $\mathcal{B} := \mathcal{B} \cup \{\text{MultLin}(X \cdot m) : \text{for some variable } X\}$ 
  end for
until  $\mathcal{B}$  remains unchanged
return  $\mathcal{B}$ 

```

Figure 1: FPC-procedure to define generating set for  $V_{\mathcal{P}} = \text{MON-PC}_k(\mathcal{P})$

During the run of the algorithm we iteratively construct a set  $\mathcal{B} \subseteq V_{\mathcal{P}}$  of polynomials such that  $\langle \mathcal{B} \rangle \leq V_{\mathcal{P}}$ . Here,  $\langle \mathcal{B} \rangle$  denotes the  $\mathbb{Q}$ -linear subspace generated by the polynomials in  $\mathcal{B}$  (considered as  $M_k$ -vectors over  $\mathbb{Q}$ ). Moreover, we ensure that at termination we have  $\langle \mathcal{B} \rangle = V_{\mathcal{P}}$ , see Figure 1. One important point to observe is that after the initialisation step we only add *monomials* to the set  $\mathcal{B}$ . This closure operation is sufficient for the monomial-PC, since, except for the given axioms in  $\mathcal{P}$  of which we take care at initialisation, we can only use the multiplication (or lifting) rule for monomials. Since there are only polynomially many different monomials of degree at most  $k$ , for a fixed  $k$ , this means that the algorithm is guaranteed to terminate after a polynomial number of iterations.

It is not obvious how to express this algorithm in FPC. Most steps, such as the representation of the set  $\mathcal{B}$  and the multilinearisation of polynomials, are easy to formalise, but there is a severe obstacle hidden in the condition for the main loop. Here, we want to iterate, in parallel, through all monomials  $m \in \langle \mathcal{B} \rangle$ . This condition “ $m \in \langle \mathcal{B} \rangle$ ” translates to solving a linear equation system over  $\mathbb{Q}$ . Although it is provably impossible to express the method of Gaussian elimination in FPC, since it requires arbitrary choices during its

computation, and although FPC cannot define the solvability of linear equation systems over finite fields [3], it is known [18] that FPC can indeed express solvability of linear equation systems over the rationals, see also Subsection 4.5.

**Theorem 4.3** [18]. *The solvability of linear equation systems over  $\mathbb{Q}$  is definable in FPC.*

Using this result we can express the algorithm from Figure 1 in FPC. In order to complete our proof of Theorem 4.2 we just need to recall that  $\text{MON-PC}_k$  can refute  $\mathcal{P}$  if, and only if,  $1 \in \langle \mathcal{B} \rangle = V_{\mathcal{P}}$ . This last assertion, again, reduces to deciding the solvability of a linear equation system over  $\mathbb{Q}$  and it can thus, by Theorem 4.3, be defined in FPC.

**4.3. Monomial-PC captures Fixed-Point Logic with Counting.** Next we show that the monomial-PC can simulate fixed-point logic with counting. We first observe, however, that the logic  $\text{FO}(\text{MON-PC}_k)$  does *not* suffice for this purpose. This is due to the fact that FPC has access to the second (numeric) sort, on which it can perform arbitrary polynomial-time computations, whereas  $\text{FO}(\text{MON-PC}_k)$  is evaluated over standard single sorted input structures. To overcome this mismatch we have to extend the logic  $\text{FO}(\text{MON-PC}_k)$  to the second-sorted framework as well. We denote this extension of  $\text{FO}(\text{MON-PC}_k)$  by  $\text{FO}^+(\text{MON-PC}_k)$ . As in the case of FPC, this means that formulas are evaluated over extensions  $\mathfrak{A}^+$  of relational structures  $\mathfrak{A}$  by a numeric sort, as defined in Section 2. In particular, interpretations for the Lindström quantifiers can make use of the second numeric sort, and we require this capability in the proof of our following result.

**Theorem 4.4.** *For every  $k \geq 2$ ,  $\text{FPC} \leq \text{FO}^+(\text{MON-PC}_k)$ .*

An elegant way to prove Theorem 4.4 is to use a game-theoretic characterisation of FPC which was recently established in [24]. It is based on the notion of so-called *threshold games*. A *threshold game* is a two-player game played on a directed graph  $G = (V, E)$  that is equipped with a *threshold function*  $\vartheta: V \rightarrow \mathbb{N}$ . This function satisfies that  $\vartheta(v) \leq \delta(v) + 1$  for all  $v \in V$ , where  $\delta(v)$  denotes the out-degree of  $v$  in  $G$ . Moreover, there is a designated vertex  $s \in V$  at which each play starts. A play is a sequence of  $G$ -nodes that arises according to the following rules. At the current position  $v \in V$ , Player 0 first selects a set  $X \subseteq vE = \{w : (v, w) \in E\}$  with  $|X| \geq \vartheta(v)$ . Then Player 1 chooses a node  $w \in X$  and the play moves on to  $w$ . A player who cannot move loses. Hence Player 0 wins at all nodes in  $T_0 := \{v \in V \mid \vartheta(v) = 0\}$  and Player 1 at all nodes in  $T_1 := \{v \in V \mid \delta(v) < \vartheta(v)\}$ .

In [24] it is shown that threshold games provide appropriate model-checking games  $\mathcal{T}(\mathfrak{A}, \varphi)$  for any finite structure  $\mathfrak{A}$  and any formula  $\varphi \in \text{FPC}$ . Since fixed-point evaluations on finite structures can be uniformly unraveled to first-order evaluations, we can in fact assume that the game graphs of these threshold games are acyclic. For any fixed FPC-formula  $\varphi$ , these model checking games are polynomially bounded in the size of the input structure and can, in fact, be interpreted in (two-sorted) input structures using a first-order interpretation. This is related to the transformation of FPC-formulas into uniform families of polynomial-size threshold circuits, as used for instance in [39] and [1].

**Theorem 4.5** [24]. *For every FPC-formula  $\varphi$  there is a first-order interpretation  $I_\varphi$  which, for every finite structure  $\mathfrak{A}$ , interprets in  $\mathfrak{A}^+$  an acyclic threshold game  $\mathcal{G}(\mathfrak{A}, \varphi)$  such that  $\mathfrak{A} \models \varphi$  if, and only if, Player 0 has a winning strategy for  $\mathcal{G}(\mathfrak{A}, \varphi)$ .*

It remains to show that the monomial-PC can define winning regions in acyclic threshold games. Given an acyclic threshold game  $\mathcal{G} = (G = (V, E), \vartheta)$ , we construct an axiom system

$\mathcal{P}(\mathcal{G})$  which consists of polynomial equations of degree at most two. For every node  $v \in V$  in the threshold game  $\mathcal{G}$ , the system  $\mathcal{P}(\mathcal{G})$  contains a variable  $X_v$ . Let us denote by  $W_\sigma^\mathcal{G}$  the winning region of Player  $\sigma$  in  $\mathcal{G}$ . Then  $\mathcal{P}(\mathcal{G})$  satisfies the following:

- if  $v \in W_0^\mathcal{G}$ , then  $X_v = 1$  is derivable from  $\mathcal{P}(\mathcal{G})$  in  $\text{MON-PC}_2$ ;
- if  $v \in W_1^\mathcal{G}$ , then  $X_v = 0$  is derivable from  $\mathcal{P}(\mathcal{G})$  in  $\text{MON-PC}_2$ ;
- $\mathcal{P}(\mathcal{G})$  is consistent; in particular, either  $X_v = 1$  or  $X_v = 0$  is derivable for every  $v \in V$ ;

If we can construct such a system  $\mathcal{P}(\mathcal{G})$  via an FO-interpretation in  $\mathcal{G}$ , then this completes our proof of Theorem 4.4. In fact, it then follows that  $\text{FO}^+(\text{MON-PC}_2)$  can define winning regions in acyclic threshold games: a node  $v \in V$  is in the winning region of Player 0 if, and only if, the system  $\mathcal{P}(\mathcal{G}) \cup \{X_v = 0\}$  can be refuted in  $\text{MON-PC}_2$ .

Recall that  $vE = \{w \in V : (v, w) \in E\}$ , for  $v \in V$ , denotes the set of successors of  $v$ . Further, we let  $s(v)$  denote the number of successors of  $v$ , and we let  $\text{ws}(v)$  denote the number of successors of  $v$  which are in the winning region of Player 0, that is  $s(v) = |vE|$  and  $\text{ws}(v) = |vE \cap W_0^\mathcal{G}|$ . We denote the set of non-terminal positions by  $\text{NonTerm} = \{v \in V : s(v) > 0\}$ . The system  $\mathcal{P}(\mathcal{G})$  uses the following set of variables:

- a variable  $X_v$ , for every  $v \in V$ ,
- a variable  $Y_v^m$  for every  $v \in \text{NonTerm}$ , and  $0 \leq m \leq s(v)$ ,
- a variable  $Z_v^m[u \mapsto j]$  for every  $v \in \text{NonTerm}$ ,  $1 \leq m \leq s(v)$ ,  $1 \leq j \leq m$ ,  $u \in vE$ .

The intuition is that the variables  $X_v$  encode the winning regions of both players, as described above. Moreover, the variables  $Y_v^m$  should indicate whether  $\text{ws}(v) = m$ , in the following way: if  $\text{ws}(v) \neq m$ , then  $Y_v^m = 0$  is derivable, and if  $\text{ws}(v) = m$ , then  $Y_v^m = 1$  is derivable. The variables  $Z_v^m[u \mapsto j]$  are auxiliary variables used to encode this last condition, cf. [10]. The system  $\mathcal{P}(\mathcal{G})$  consists of the following axioms:

$$(T) \quad \text{For } v \in T_0 : X_v = 1 \text{ and for } v \in T_1 : X_v = 0$$

$$(C) \quad \text{For } v \in \text{NonTerm}, 1 \leq m \leq s(v), u \in vE : \sum_{j=1}^m Z_v^m[u \mapsto j] - Y_v^m = 0$$

$$\text{For } v \in \text{NonTerm}, 1 \leq m \leq s(v), 1 \leq j \leq m : \sum_{u \in vE} X_u Z_v^m[u \mapsto j] - Y_v^m = 0$$

$$\text{For } v \in \text{NonTerm} : \sum_{u \in vE} X_u \cdot Y_v^0 = 0$$

$$(E) \quad \text{For } v \in V : (1 - X_v) - \sum_{m=0}^{\vartheta(v)-1} Y_v^m = 0 \text{ and } X_v - \sum_{m=\vartheta(v)}^{s(v)} Y_v^m = 0$$

We also add for each variable  $X = X_v$ ,  $v \in V$ , a syntactic dual variable  $\bar{X}$  together with the axiom

$$(N) \quad 1 - X - \bar{X} = 0.$$

These axioms enforce that each dual variable  $\bar{X}$  takes as value  $1 - X$ . Note that the system  $\mathcal{P}(\mathcal{G})$  only contains axioms of degree at most 2.

**Lemma 4.6.** *The system  $\mathcal{P}(\mathcal{G})$  is consistent.*

*Proof.* We define an intended model of  $\mathcal{P}(\mathcal{G})$ . For  $X$ -variables, we set  $X_v := 1$ , if  $v \in W_0^\mathcal{G}$ , and  $X_v := 0$ , if  $v \in W_1^\mathcal{G}$ . For  $Y$ -variables, we set  $Y_v^m := 1$ , if  $\text{ws}(v) = m$ , and  $Y_v^m := 0$  if  $m \neq \text{ws}(v)$ . For  $Z$ -variables, we set  $Z_v^m[u \mapsto j] := 0$  for all non-terminal positions  $v \in V$ ,  $u \in vE$ ,

and  $j \in \{1, \dots, m\}$ , if  $m \neq \text{ws}(v)$ . For  $m = \text{ws}(v) > 0$ , we let  $vE \cap W_0^{\mathcal{G}} = \{u_1, \dots, u_m\}$ . We then set  $Z_v^m[u_i \mapsto j] = 1$  if  $j = i$ , and  $Z_v^m[u_i \mapsto j] = 0$  for  $j \neq i$ . Moreover, for  $u \in vE \setminus W_0^{\mathcal{G}}$ , we set  $Z_v^m[u \mapsto 1] = 1$ , and  $Z_v^m[u \mapsto j] = 0$  for  $j \in \{2, \dots, m\}$ .  $\square$

**Lemma 4.7.** *If  $v \in W_0^{\mathcal{G}}$ , then we can derive the polynomial  $X_v - 1$  (that is the equation  $X_v = 1$ ) from  $\mathcal{P}(\mathcal{G})$  in MON-PC<sub>2</sub>; and if  $v \in W_1^{\mathcal{G}}$ , then the polynomial  $X_v$  (that is the equation  $X_v = 0$ ) can be derived from  $\mathcal{P}(\mathcal{G})$  in MON-PC<sub>2</sub>.*

*Proof.* We start with a small remark. Assume that we can derive  $(1 - X)$  for a variable  $X = X_v$ ,  $v \in V$ . We show how to derive  $W(1 - X)$  for any variable  $W$ . This is clearly possible in the full polynomial calculus: we just have to multiply by  $W$ . In the monomial-PC, however, we cannot multiply  $(1 - X)$  by  $W$ , since  $(1 - X)$  is neither a monomial nor an axiom. Instead, we use our negation axioms. Starting from  $1 - X$ , we can derive  $\bar{X}$  by subtracting (N) from  $1 - X$ . Since (N) is an axiom, we can multiply it by  $W$ ; also,  $\bar{X}$  is a monomial and so we can multiply it by  $W$ . Thus,  $W(1 - X - \bar{X}) + W\bar{X} = W(1 - X)$  can be derived, as claimed. We make use of this trick in the following.

Our proof is by induction on the height of the subgame rooted at  $v \in V$  (recall that  $\mathcal{G}$  is acyclic). For terminal positions  $v \in V$ , the assertion is immediate from axioms (T).

Assume  $v \in V$  is a non-terminal position. Let  $W_0(v) = vE \cap W_0^{\mathcal{G}}$  and  $W_1(v) = vE \cap W_1^{\mathcal{G}}$ . By the induction hypothesis we know that we can derive in MON-PC<sub>2</sub> for every  $u \in W_0(v)$  the equation  $X_u = 1$  and for every  $u \in W_1(v)$  the equation  $X_u = 0$ .

Let  $m > 0$ . Consider an equation of the form  $\sum_{u \in vE} X_u Z_v^m[u \mapsto j] - Y_v^m = 0$  for  $j \in \{1, \dots, m\}$  of type (C). We have  $vE = W_0(v) \uplus W_1(v)$ . For every  $Z$ -variable and for every  $u \in W_0(v)$  we can derive  $ZX_u = Z$  in MON-PC<sub>2</sub>, and for every  $u \in W_1(v)$  we can derive  $ZX_u = 0$  in MON-PC<sub>2</sub>. Hence, we can simplify these equations of type (C) as  $\sum_{u \in W_0(v)} Z_v^m[u \mapsto j] - Y_v^m = 0$  for  $j \in \{1, \dots, m\}$  in MON-PC<sub>2</sub>.

Next, we consider for every  $u \in W_0(v)$  the equations  $\sum_{j=1}^m Z_v^m[u \mapsto j] - Y_v^m = 0$ , again of type (C). We combine these two sets of equations as follows:

$$\sum_{j=1}^m \left( \sum_{u \in W_0(v)} Z_v^m[u \mapsto j] - Y_v^m \right) - \sum_{u \in W_0(v)} \left( \sum_{j=1}^m Z_v^m[u \mapsto j] - Y_v^m \right) = 0.$$

We can further simplify this equation (the variables  $Z_v^m[u \mapsto j]$  cancel out) and we get

$$(m - \text{ws}(v))Y_v^m = 0.$$

Hence, for every  $m > 0$ ,  $m \neq \text{ws}(v)$ , we can derive  $Y_v^m = 0$  in MON-PC<sub>2</sub>. Indeed, also in the case where  $m = 0 < \text{ws}(v)$  we can derive  $Y_v^m = 0$ . In this case we just use the equation  $\sum_{u \in vE} X_u Y_v^0 = 0$ . Using the same arguments as above, this equation simplifies to  $\text{ws}(v) \cdot Y_v^0 = 0$ . Hence, if  $\text{ws}(v) > 0$ , we can also derive  $Y_v^0 = 0$ . Note that the two equations of type (E) can be combined to the equation  $\sum_{m=0}^{s(v)} Y_v^m = 1$ . Hence, altogether we showed the following. For all  $0 \leq m \leq s(v)$  it holds that:

- if  $m = \text{ws}(v)$ , then we can derive  $Y_v^m = 1$  in MON-PC<sub>2</sub>; and
- if  $m \neq \text{ws}(v)$ , then we can derive  $Y_v^m = 0$  in MON-PC<sub>2</sub>.

Having this, the claim follows immediately by using the equations of type (E).  $\square$

In summary, we have seen that defining the winning regions in acyclic threshold games is an FPC-complete problem, with respect to FO<sup>+</sup>-reductions, and that the winning regions in such games can be defined in FO<sup>+</sup>(MON-PC<sub>2</sub>). Furthermore, it is easy to see that the

system  $\mathcal{P}(\mathcal{G})$  can be obtained from the game  $\mathcal{G}$  by means of an FO-interpretation. This completes the proof of Theorem 4.4 and, together with Theorem 4.2, establishes our first main theorem of this section.

**Theorem 4.8.** *For every  $k \geq 2$ ,  $\text{FPC} = \text{FO}^+(\text{MON-PC}_k)$ .*

**4.4. FPC-Definability of Refutations in the (Full) Polynomial Calculus.** Next, we are going to lift our result concerning the degree- $k$  monomial-PC to the full degree- $k$  polynomial calculus. As we mentioned before, it seems implausible that proof-search for  $\text{PC}_k$  can be implemented in FPC, since there are instances where such refutations necessarily contain polynomials with coefficients of super-polynomial bit-complexity (and  $\text{FPC} \leq \text{PTIME}$ ). Nevertheless, we will provide an FPC-definable proof search procedure, similar to the one in the previous section, but it will only be able to deal with coefficients of restricted size. To this end, we define for each constant  $b \in \mathbb{N}$ ,  $\text{PC}_{k,b}$  as the fragment of degree- $k$  polynomial calculus over  $\mathbb{Q}$  where all coefficients are representable as fractions of binary numbers with at most  $n^b$  many bits. In a next step, we see that, if we drop the restriction on the coefficients, we can still define the proof search algorithm in  $C_{\infty\omega}^k$ . It follows that the degree- $k$  variants  $\text{MON-PC}_k$  and  $\text{PC}_{k,b}$  (for each constant  $b$ ) of the monomial-PC and the full polynomial calculus have the same expressive power (with respect to  $\text{FO}^+$ -interpretations), that is for all  $k \geq 2$  and  $b \in \mathbb{N}$ , we have

$$\text{FO}^+(\text{MON-PC}_k) = \text{FPC} = \text{FO}^+(\text{PC}_{k,b}).$$

Our result provides an interesting new characterisation of the power of the (full) polynomial calculus from the perspective of finite model theory. In particular, it allows us to use techniques from finite model theory to answer open questions about the (relative) power of the two variants of the polynomial calculus. As indicated before, one example is given in Section 6.3 where we use our new characterisation of the polynomial calculus to answer an open question posed by Grohe and Berkholz in [10], see Question 6.5 and Theorem 6.6.

To prove the equivalence of FPC and  $\text{FO}^+(\text{PC}_{k,b})$ , the first important step is to understand why it is more difficult to express  $k$ -dimensional refutations in the (full) polynomial calculus in FPC rather than in its restricted variant  $\text{MON-PC}$ . Basically, this comes down to the following problem: in order to find proofs in the monomial-PC it suffices to decide the *solvability problem* for linear equation systems over  $\mathbb{Q}$  (this is a Boolean decision problem; the output is either *solvable* or *not solvable*). However, in order to search for proofs in the full PC we need to express the functional problem of *computing solution spaces* of linear equation system over  $\mathbb{Q}$  in FPC. However, while it was known that FPC can define the (Boolean) solvability problem over  $\mathbb{Q}$ , it was not known whether solution spaces of linear equation systems over  $\mathbb{Q}$  can be expressed in FPC. Luckily, as we show in Theorem 4.12, this is indeed the case.

Let us now elaborate more on how to find refutations in the full PC. To this end, we recall the procedure from Figure 1 to find  $k$ -dimensional proofs in the monomial-PC. Given a set of multilinear polynomials  $\mathcal{P} \subseteq \mathbb{Q}^{M_k}$  of degree at most  $k$ , the idea is to construct a set  $\mathcal{B} \subseteq \mathbb{Q}^{M_k}$  of (multilinear) polynomials of degree at most  $k$  which generate (as  $\mathbb{Q}$ -linear combinations) the set of *all* polynomials  $\text{MonPC}_k(\mathcal{P})$  that can be derived in the  $k$ -dimensional  $\text{MON-PC}$  (starting from the given set of polynomials  $\mathcal{P}$ ). At the beginning,  $\mathcal{B}$  is set to contain all (linearised and) lifted versions  $\text{MultLin}(m \cdot p)$  of the given polynomials  $p \in \mathcal{P}$  up to degree  $k$ . Subsequently, the set  $\mathcal{B}$  is closed under liftings by variables  $X$ . More precisely, in each

iteration, the set  $\mathcal{B}$  is extended by *all* possible (linearised) liftings  $X \cdot m$  of *monomials*  $m$  of degree at most  $k - 1$  that can be derived up to this stage, i.e. for which  $m \in \langle \mathcal{B} \rangle$  holds (here,  $\langle \mathcal{B} \rangle$  denotes the set of all polynomials that can be derived from polynomials in  $\mathcal{B}$  using  $\mathbb{Q}$ -linear combinations). The crucial observation is that this simple inductive lifting step is sufficient for the monomial-PC, because, indeed, by its rules we are only allowed to lift monomials and the initial polynomials  $p \in \mathcal{P}$ . The set  $\mathcal{B}$  is extended in this way until  $\langle \mathcal{B} \rangle$  remains stable.

In order to adapt this algorithm to the (full) PC, we need to make the following changes. Most importantly, instead of lifting all monomials  $m \in \langle \mathcal{B} \rangle$ ,  $\deg(m) < k$ , during the iteration, for the full PC we have to take all (*multilinear*) *polynomials*  $p \in \langle \mathcal{B} \rangle$ ,  $\deg(p) < k$  into account, and make sure that their liftings  $X \cdot p$  are contained in  $\langle \mathcal{B} \rangle$ . This is more difficult for the following two reasons. First of all, we cannot go through *all* such polynomials  $p$ , simply because their number is exponential in the number of variables. To overcome this obstacle, we have to use linear-algebraic preprocessing which enables us to lift a generating set for the set of polynomials  $p \in \langle \mathcal{B} \rangle$ ,  $\deg(p) < k$ , instead. Note that this was not necessary in the setting of the monomial-PC: here, the number of possible  $k$ -dimensional monomials is bounded polynomially in the number of variables (for fixed  $k \geq 2$ ). There is a second problem. During the iteration, for the monomial-PC we could repeatedly add *all* lifted variants  $X \cdot m$  of all monomials  $m \in \langle \mathcal{B} \rangle$ ,  $\deg(m) < k$  to our partial generating set  $\langle \mathcal{B} \rangle$ . This is because the (linearised) version of a lifted monomial remains a monomial and we just said that the number of all  $k$ -dimensional monomials is polynomially bounded. Hence, we never obtain generating sets  $\mathcal{B}$  of super-polynomial size in this way. In contrast, for the setting of the (full) PC, assume that at some stage during the iteration we have a small generating set  $\mathcal{C}$  for the set of all polynomials  $p \in \langle \mathcal{B} \rangle$  of degree at most  $k - 1$ . If we now lift *all* polynomials  $p \in \mathcal{C}$  in all possible ways  $X \cdot p$ , then clearly the size of the resulting set  $\mathcal{C}'$  increases by a factor which corresponds to the number of variables (and there is no global polynomial upper bound for  $\mathcal{C}'$  as in the case of the monomial-PC). Hence, before each lifting step, we have to ensure that the size of the generating set  $\mathcal{C}$  of polynomials that we lift is (globally) bounded by a polynomial. We can invoke standard linear-algebraic algorithms to achieve this. More specifically, we construct  $\mathcal{C}$  in such a way that its size does not exceed  $|M_k|$ , that is the number of different multilinear monomials of degree at most  $k$ . Note that a generating set of this size exists, since each  $\mathbb{Q}$ -linear subspace of  $\mathbb{Q}^{M_k}$  is of dimension at most  $|M_k|$ . Moreover, as we mentioned before,  $|M_k|$  is of polynomial size for any fixed  $k$ . We summarise the adapted algorithm for finding  $k$ -dimensional refutations in the (full) polynomial calculus in Figure 2.

To see how we can implement the algorithm from Figure 2 in polynomial time, let us have a closer look at the construction of the set  $\mathcal{C}$  during the iteration. First of all note that the set  $\{p \in \langle \mathcal{B} \rangle : \deg(p) < k\}$  is indeed a  $\mathbb{Q}$ -linear subspace of  $\langle \mathcal{B} \rangle$  which, in turn, is a  $\mathbb{Q}$ -linear subspace of  $\mathbb{Q}^{M_k}$ . Hence, it is clear that a generating set  $\mathcal{C}$  of size at most  $|M_k|$  exists. Moreover, we can easily obtain  $\mathcal{C}$  as the solution space of a linear equation system. Indeed, let  $M$  be the  $M_k \times \mathcal{B}$ -matrix over  $\mathbb{Q}$  whose columns correspond to the polynomials in  $\mathcal{B}$ . Then  $\text{im}(M) = \langle \mathcal{B} \rangle$ . Hence, if we let  $x$  and  $p$  denote a  $\mathcal{B}$ -vector and an  $M_k$ -vector of variables ranging over  $\mathbb{Q}$ , respectively, then the solution space of the linear equation system determined by the equation  $Mx = p$  is  $\langle \mathcal{B} \rangle$  when we project it to the variables  $p$ . Hence, by adding extra constraints  $p(m) = 0$  for all monomials  $m \in M_k$  with  $\deg(m) = k$ , we obtain a linear equation system whose solution space, projected to variables in  $p$ , is a generating set

**Input:** Set of multilinear polynomials  $\mathcal{P} \subseteq \mathbb{Q}^{M_k}$   
**Output:**  $\mathcal{B} \subseteq \mathbb{Q}^{M_k}$  such that  $\langle \mathcal{B} \rangle = \text{PC}_k(\mathcal{P})$ .  
 $\mathcal{B} := \{\text{MultLin}(m \cdot p) \mid p \in \mathcal{P}, m \text{ a monomial such that } \deg(\text{MultLin}(m \cdot p)) \leq k\}$   
// Initialisation (lift all axioms in  $\mathcal{P}$ )

**repeat**  
  Construct set  $\mathcal{C} \subseteq \mathbb{Q}^{M_k}$  of size at most  $|M_k|$  such that  $\langle \mathcal{C} \rangle = \{p \in \langle \mathcal{B} \rangle : \deg(p) < k\}$   
  **for all** polynomials  $p \in \mathcal{C}$  **do**  
     $\mathcal{B} := \mathcal{B} \cup \{\text{MultLin}(X \cdot p) : \text{for some variable } X\}$   
  **end for**  
**until**  $\langle \mathcal{B} \rangle$  remains unchanged  
**return**  $\mathcal{B}$

Figure 2: FPC-procedure to construct generating set  $\mathcal{B}$  for the set  $\text{PC}_k(\mathcal{P})$  of all polynomials that can be derived in  $\text{PC}_k$  starting from the given set of polynomials  $\mathcal{P}$

for  $\{p \in \langle \mathcal{B} \rangle : \deg(p) < k\}$ . Clearly, solution spaces for such systems can be computed in polynomial time.

Before we discuss the FPC-definability of this procedure, let us observe that there is a small caveat with the approach above. So far, the generating set for  $\{p \in \langle \mathcal{B} \rangle : \deg(p) < k\}$  that we obtain is not of size at most  $|M_k|$ . Indeed, by our construction, which relies on the final projection step, the size of the generating set depends on  $|\mathcal{B}|$  (because the vector of variables  $x$  is indexed by  $\mathcal{B}$ ). Hence, we need to make a second important observation. Say we were able to construct an  $M_k \times J$ -matrix  $N$  over  $\mathbb{Q}$  with the property that  $\text{im}(N) = \{p \in \langle \mathcal{B} \rangle : \deg(p) < k\}$  (that is the columns of  $N$  form a generating set for the solution space of the above linear equation system projected to  $p$ ). We would like to transform this matrix  $N$  into a “smaller”  $M_k \times M_k$ -matrix  $\hat{N}$  such that  $\text{im}(\hat{N}) = \text{im}(N)$ . This is clearly possible simply because the dimension of the space  $\text{im}(N)$  is at most  $|M_k|$ . However, the question is about how difficult it is to obtain such a “more compact” version  $\hat{N}$  of  $N$ . Specifically, for our FPC-definability proof, we need to express this “compression transformation” in FPC as well.

Fortunately, the step from  $N$  to  $\hat{N}$  is surprisingly easy to realise. As we will see in the following subsection, it holds that the  $(M_k \times M_k)$ -matrix  $\tilde{N} := N \cdot N^T$  has the same image as the matrix  $N$ , see Lemma 4.14. Hence, we obtain a *small* generating set for  $\text{im}(N)$  by taking the columns of  $\hat{N} = N \cdot N^T$ . This shows that we can, in general, quite easily transform an arbitrary generating set for a  $\mathbb{Q}$ -linear subspace of  $\mathbb{Q}^{M_k}$  into a small generating set of size at most  $|M_k|$ . Moreover, this transformation only relies on simple matrix operations, such as transposition and matrix multiplication. As such operations are well-known to be definable in FPC, see e.g. [32], this transformation is FPC-definable. However, it is at this point that the bit-complexity of the coefficients has to be taken into account. Since  $\hat{N}$  is computed by squaring  $N$ , the bit-complexity of the coefficients can increase in this step. If this happens repeatedly, then the required number of bits may become greater than the maximum number of bits that our FPC-sentence can handle. In this case, the computation has to be aborted. This maximum number of bits depends on the number of variables of the FPC-sentence that we are constructing: We want our sentence to be able to find refutations in  $\text{PC}_{k,b}$ , for a fixed value of  $b$ . That is, the coefficients occurring in a refutation can be written as fractions of binary numbers of length  $\leq n^b$ , where  $n$  is the size of the input structure. These coefficients,

i.e. the entries of the matrices that we are manipulating in the fixed-point computation, are represented as follows: We use a tuple of  $b$  variables ranging over the  $n$  ordered elements of the number sort in order to index the positions of a binary string. Relations are used to mark the positions that are 0 and 1, respectively, and to specify the position of the coefficient in the matrix (see [32] for more details). Therefore, it is possible to construct for every fixed  $b$  an FPC-sentence that performs the matrix manipulations mentioned above using binary numbers of length  $n^b$ , but no fixed FPC-sentence can deal with coefficients of unbounded length.

Altogether, this means that the only difficulty we face is to define solution spaces of linear equation system over  $\mathbb{Q}$  in FPC. Recall that by the result of Dawar, Grohe, Holm, and Laubner we know that FPC can express the *Boolean* solvability problem of linear equation systems over  $\mathbb{Q}$ , see Theorem 4.3. However, this does not give direct evidence for FPC being able to express the more general *functional* problem of defining solution spaces over  $\mathbb{Q}$ . For the sake of illustration, consider rank logic over finite fields. Rank logic can define the Boolean solvability problem for linear equation systems over finite fields but it is not to be expected that it can also define vectors in the solution space: This is because any solution vector to a linear equation system obtained from CFI-graphs has an orbit of exponential size, and rank logic is isomorphism-invariant and in PTIME.

Luckily, over  $\mathbb{Q}$ , the situation turns out to be different. Not only can we define the Boolean solvability problem in FPC but we can also define the corresponding solution spaces as we show in this article (see Theorem 4.12). From this result and our preceding discussion it easily follows that the algorithm in Figure 2 (with bounded bit-complexity) is definable in FPC. Beyond this application, we believe that Theorem 4.12 is interesting in its own right and might prove useful in other contexts. Let us conclude by stating our main result of this subsection (where we rely on the yet to be proven Theorem 4.12).

**Theorem 4.9.** *For every  $k \geq 2$  and  $b \in \mathbb{N}$ , there exists an FPC-sentence  $\varphi$  with  $\mathcal{O}(k + b)$  many variables such that given (a structural encoding of) a system  $\mathcal{P}$  of polynomials over  $\mathbb{Q}$  as input for the  $k$ -dimensional polynomial calculus of degree  $k$ ,  $\varphi$  expresses whether  $\mathcal{P}$  can be refuted in  $\text{PC}_{k,b}$ , that is  $\varphi$  defines whether  $1 \in \text{PC}_{k,b}(\mathcal{P})$ .*

A more commonly studied version of the polynomial calculus is  $\text{PC}_k$ , that is, the degree- $k$  PC over  $\mathbb{Q}$  without any restriction on the bit-complexity. The procedure we described above in principle also works for the  $\text{PC}_k$ . It would be FPC-definable, even without a bound on the bit-complexity, if FPC-sentences were evaluated in structures with larger number sorts. Recall that we use the elements of the number sort to index the positions of the binary strings. In FPC, the number sort always has the same size  $n$  as the structure itself, but if we imagine the number sort to be of some size  $f(n)$ , for a sufficiently large function  $f$ , then our sentence can deal with  $f(n)^b$  many bits instead of  $n^b$ . Our algorithm involves squaring a matrix polynomially many times. Hence, if  $f(n)$  is greater than the largest possible growth of the bit-length that can occur in this number of squaring operations, then the procedure could be implemented in FPC with number sorts of size  $f(n)$  and it would always correctly decide the existence of  $\text{PC}_k$ -refutations, regardless of any bit-complexity issues. A fixed-point logic with such big number sorts does not really exist but instead, we can use  $C_{\infty\omega}^k$ . The standard translation of FPC-sentences into  $C_{\infty\omega}^\omega$ -sentences does not increase the number of variables. It simulates the number-sort-variables of the FPC-sentence with large disjunctions or conjunctions over all elements of the number sort. This idea works regardless

of the size of the number sort. These considerations directly lead to the following result for the degree- $k$  polynomial calculus:

**Theorem 4.10.** *For every  $k \geq 2$ , there exists a  $C_{\infty\omega}^\omega$ -sentence  $\varphi$  with  $\mathcal{O}(k)$  many variables such that given (a structural encoding of) a system  $\mathcal{P}$  of polynomials over  $\mathbb{Q}$  as input for the  $k$ -dimensional polynomial calculus of degree  $k$ ,  $\varphi$  expresses whether  $\mathcal{P}$  can be refuted in  $PC_k$ , that is  $\varphi$  defines whether  $1 \in PC_k(\mathcal{P})$ .*

**Theorem 4.11.** *For all  $k \geq 2$ ,  $b \in \mathbb{N}$ :*

$$FO^+(\text{MON-PC}_k) = FO^+(PC_{k,b}) = FPC \stackrel{?}{<} FO^+(PC_k) < C_{\infty\omega}^{\mathcal{O}(k)}.$$

By  $FPC \stackrel{?}{<} FO^+(PC_k)$ , we mean that we do not know whether or not  $FO^+(PC_k) \leq FPC$  holds. However, there are some reasons why we suspect that  $FPC$  is strictly weaker than  $FO^+(PC_k)$ . First of all, to the best of our knowledge, it is an open problem whether or not there exists a PTIME-algorithm that decides the existence of  $PC_k$ -refutations (for unbounded coefficients). The well-known Groebner basis algorithm certainly fails [31], so if this problem is in P, then there must be some way to avoid explicit computation of the coefficients in the refutation. Since  $FPC \leq \text{PTIME}$ , it is “even more open” if the problem is in  $FPC$ .

Secondly, our result  $FO^+(PC_{k,b}) = FPC$  has the following consequence: If it were possible to compute  $PC_k$ -refutations with arbitrarily large coefficients in  $FPC$ , then there would be a numeric FO-interpretation that reduces any input polynomial equation system to one that can be decided in  $PC_{k,b}$ , that is, with small degree and small coefficients. This seems to be a very strong statement because it means that the necessity to use large coefficients in refutations can be circumvented with simple FO-definable preprocessing of the input polynomials. This would be quite surprising, so it seems more reasonable to believe that  $FPC \stackrel{?}{<} FO^+(PC_k)$ .

**4.5. Definability of Solution Spaces of Linear Equation Systems over  $\mathbb{Q}$ .** To complete our proof of Theorem 4.9, we proceed to show that  $FPC$  can define solution spaces of linear equation systems over  $\mathbb{Q}$ . Formally, our main result in this subsection reads as follows.

**Theorem 4.12.** *There exist FPC-formulas which define the following: given (a structural encoding of) a linear equation system  $M \cdot x = b$  over  $\mathbb{Q}$ , for  $M: I \times J \rightarrow \mathbb{Q}$  and  $b: I \rightarrow \mathbb{Q}$ , they express whether  $M \cdot x = b$  is solvable, and in this case, define (structural encodings of) a matrix  $S: J \times J \rightarrow \mathbb{Q}$  and a vector  $c: J \rightarrow \mathbb{Q}$  such that  $\text{im}(S) = \ker(M)$  and  $M \cdot c = b$ , i.e. such that  $\text{im}(S) + c$  is the solution space of  $M \cdot x = b$ .*

In order to prove Theorem 4.12, we make use of the following linear-algebraic properties of matrices over the rationals. For completeness, and since it is central for our application, we present short proofs to recall the underlying algebraic arguments. From now on, let us fix a linear equation system  $M \cdot x = b$  with  $M: I \times J \rightarrow \mathbb{Q}$  and  $b: I \rightarrow \mathbb{Q}$ . The key is to consider the following matrices over  $\mathbb{Q}$ :

$$\begin{aligned} B &:= M \cdot M^T \in \mathbb{Q}^{I \times I} \\ C &:= M^T \cdot M \in \mathbb{Q}^{J \times J}. \end{aligned}$$

In Figure 3 on the next page we summarise what we are going to show.

$$\begin{array}{ccccc}
& \mathbb{Q}^J & & & \mathbb{Q}^I \\
& \parallel & & & \parallel \\
\ker(C) & = & \ker(M) & & \ker(M^T) = \ker(B) \\
\oplus & & \oplus & & \oplus \\
\text{im}(C) & = & \text{im}(M^T) & \xrightarrow{\cong} & \text{im}(M) = \text{im}(B) \\
C \left( \begin{array}{c} \uparrow \\ \cong \end{array} \right) & & & & \left( \begin{array}{c} \uparrow \\ \cong \end{array} \right) B \\
& & M & & \\
& & \cong & & \\
& & M^T & & 
\end{array}$$

Figure 3: Linear-algebraic structure induced by the matrix  $M: I \times J \rightarrow \mathbb{Q}$  where  $B = M \cdot M^T: I \times I \rightarrow \mathbb{Q}$  and  $C = M^T \cdot M: J \times J \rightarrow \mathbb{Q}$

**Lemma 4.13** (Properties of  $B, C$ ).

- (1)  $B^T = B$  and  $C^T = C$ , that is  $B$  and  $C$  are symmetric.
- (2)  $\ker(B) = \ker(B^i)$  and  $\ker(C) = \ker(C^i)$  for all  $i \geq 1$ .
- (3)  $\text{im}(B) = \text{im}(B^i)$  and  $\text{im}(C) = \text{im}(C^i)$  for all  $i \geq 1$ .
- (4)  $\mathbb{Q}^I = \ker(B) \oplus \text{im}(B)$  and  $\mathbb{Q}^J = \ker(C) \oplus \text{im}(C)$ .
- (5)  $B$  is an automorphism of  $\text{im}(B)$  and  $C$  is an automorphism of  $\text{im}(C)$ .

*Proof.* The arguments for  $B$  and  $C$  are completely symmetric, so let us consider the case of  $B: I \times I \rightarrow \mathbb{Q}$ . First of all,  $B^T = (M \cdot M^T)^T = M \cdot M^T = B$ . For the second claim, we proceed by induction on  $i$ . It is clear that  $\ker(B) \subseteq \ker(B^i)$  for all  $i \geq 1$ , so it suffices to show  $\ker(B^i) \subseteq \ker(B)$ . For  $i = 1$ , the claim is trivial, so assume that  $i \geq 2$  and for some  $x \in \mathbb{Q}^I$  we have  $B^i x = 0$ . Then also  $B^{i-1} \cdot B^{i-1} x = 0$ . Since  $B^T = B$ , this means that also  $x^T \cdot (B^{i-1})^T \cdot B^{i-1} x = 0$ . Hence,  $|B^{i-1} x|^2 = 0$ , which implies that  $B^{i-1} x = 0$ . We get  $x \in \ker(B^{i-1})$  and by the induction hypothesis  $x \in \ker(B)$ .

Let's consider (3). Again, it is easy to see that  $\text{im}(B^i) \subseteq \text{im}(B)$  for all  $i \geq 1$ . Now let's choose a basis  $Be_1, \dots, Be_\ell$  for  $\text{im}(B)$  and let  $i \geq 2$ . Then  $B^i e_1, \dots, B^i e_\ell$  is a generating set for  $\text{im}(B^i)$ . We claim that  $B^i e_1, \dots, B^i e_\ell$  is a basis for  $\text{im}(B^i)$  which would prove our claim. Indeed, assume that for some non-zero  $(a_j) \in \mathbb{Q}^\ell$  we had  $\sum_j a_j \cdot B^i e_j = 0$ . Since  $\ker(B^i) = \ker(B)$  it follows that  $\sum_j a_j B e_j = 0$ , a contradiction. We turn our attention to (4). We have to show two things, namely that every vector in  $\mathbb{Q}^I$  can be written as a linear combination of elements in  $\ker(B)$  and  $\text{im}(B)$  and that this expression is unique. Let us start with the latter claim. Assume that  $Bx + y = 0$  for  $x, y \in \mathbb{Q}^I$ ,  $y \in \ker(B)$ . We have to show that  $Bx = y = 0$ . From  $Bx + y = 0$  we can conclude that  $B^2 x = 0$  since  $By = 0$ . Since  $\ker(B^2) = \ker(B)$  it follows that  $Bx = 0$  which yields  $y = 0$ . To complete the proof, let  $x \in \mathbb{Q}^I$ . Consider the cyclic space generated by  $B$ , that is  $\langle \{x, Bx, B^2 x, \dots, B^n x\} \rangle$  where  $n = |I|$ . Note that  $\{x, Bx, B^2 x, \dots, B^n x\}$  is linearly dependent. If  $x \in \text{im}(B)$ , then there is nothing to show. Otherwise, we know that  $x \notin \langle \{Bx, B^2 x, \dots, B^n x\} \rangle$ . We choose a non-zero vector  $(a_j) \in \mathbb{Q}^n$  such that  $\sum_{j=1}^n a_j B^j x = 0$ . Let  $k \geq 1$  be minimal such that  $a_k \neq 0$ . Then  $B^k(a_k x + \sum_{j=k+1}^n a_j B^{j-k} x) = 0$ , hence  $a_k x + z \in \ker(B^k) = \ker(B)$  for some  $z \in \text{im}(B)$ . This is what we wanted to show. Finally, (5) follows from (3).  $\square$

**Lemma 4.14** (Relating  $B, C$  and  $M, M^T$ ).

- (1)  $\ker(B) = \ker(M^T)$  and  $\ker(C) = \ker(M)$ .

- (2)  $\text{im}(B) = \text{im}(M)$  and  $\text{im}(C) = \text{im}(M^T)$ .  
(3)  $M$  is an isomorphism from  $\text{im}(M^T)$  to  $\text{im}(M)$  and  $M^T$  an isomorphism from  $\text{im}(M)$  to  $\text{im}(M^T)$ . In particular,  $\text{rk}(C) = \text{rk}(M^T) = \text{rk}(M) = \text{rk}(B)$ .

*Proof.* Again, as the arguments are symmetric, we only consider the case of  $B$ . For (1), note that  $\ker(M^T) \subseteq \ker(M \cdot M^T) = \ker(B)$  for trivial reasons. Moreover, if  $M \cdot M^T x = 0$  for some  $x \in \mathbb{Q}^I$ , then also  $x^T M M^T x = 0$ , that is  $|M^T x|^2 = 0$  which implies  $M^T x = 0$ . Hence,  $\ker(B) \subseteq \ker(M^T)$ . For (2), note that  $\text{im}(B) = \text{im}(M \cdot M^T) \subseteq \text{im}(M)$ , again for trivial reasons. To verify the other direction, let  $x \in \mathbb{Q}^J$  and consider the element  $Mx \in \text{im}(M)$ . By Lemma 4.13, we can write  $Mx$  as  $Mx = By + z$  for some  $y, z \in \mathbb{Q}^I$  and  $z \in \ker(B) = \ker(M^T)$ . Hence  $M^T Mx = (M^T M)M^T y$ , that is  $Cx = CM^T y$ . From this we get that  $C(x - M^T y) = 0$ . Using  $\ker(C) = \ker(M)$ , we get  $M(x - M^T y) = 0$ . This implies that  $Mx = MM^T y = By$  which proves our claim. Finally, (3) follows immediately from (1),(2) and Lemma 4.13.  $\square$

We are ready to establish the following central criterion for the solvability of linear equation systems over  $\mathbb{Q}$ .

**Lemma 4.15** (Solvability of linear equation system, see also [30]). *Let  $M \cdot x = b$  be a linear equation system over  $\mathbb{Q}$  where with  $M: I \times J \rightarrow \mathbb{Q}$  and  $b: I \rightarrow \mathbb{Q}$ . Let  $B = M \cdot M^T$  as above. Let  $n = \min\{|I|, |J|\}$ . Then the linear equation system  $M \cdot x = b$  is solvable if, and only if,  $b$  can be written as a  $\mathbb{Q}$ -linear combination of vectors in  $\Gamma = \{Bb, B^2b, \dots, B^{n+1}b\}$ , that is if  $b \in \langle \Gamma \rangle$ .*

*Proof.* First, note that  $\langle \Gamma \rangle \subseteq \text{im}(B)$ . Hence, if  $b \in \langle \Gamma \rangle$ , then clearly the linear equation system  $M \cdot x = b$  is solvable. For the other direction, assume that  $b \in \text{im}(M) = \text{im}(B)$ . Then for some  $c \in \mathbb{Q}^J$  we have  $Bc = b$ . Let  $\Delta = \{Bc, B^2c, \dots, B^{n+1}c\} = \{b, Bb, \dots, B^n b\}$ . This set  $\Delta$  is linearly dependent, because it is a subset of  $\text{im}(B)$  and we have established before that the dimension of  $\text{im}(B)$  coincides with  $\text{rk}(M)$  which is at most  $n = \min\{|I|, |J|\}$ . It easily follows that  $\langle \Delta \rangle$  is  $B$ -invariant, that is  $\langle B\Delta \rangle \subseteq \langle \Delta \rangle$ . Moreover, by Lemma 4.13,  $B$  is an automorphism of  $\text{im}(B)$  which implies that  $\langle B\Delta \rangle = \langle \Delta \rangle$ . However, this shows that  $b \in \Delta$  can be written as a linear combination of elements in  $B\Delta = \Gamma$  which proves our claim.  $\square$

Using Lemma 4.15, it is easy to show that FPC can define the solvability problem for linear equation systems over  $\mathbb{Q}$ .

*Proof of Theorem 4.12 - Part 1/2.* Given a linear equation system  $M \cdot x = b$  over  $\mathbb{Q}$  for  $M: I \times J \rightarrow \mathbb{Q}$  and  $b: I \rightarrow \mathbb{Q}$ , we first define  $B = MM^T$  as above and the ordered set of vectors  $\Gamma = \{Bb, \dots, B^{n+1}b\}$  as in Lemma 4.15. This can be done in FPC, since matrix multiplication over  $\mathbb{Q}$  is well-known to be definable in FPC, see e.g. [32].

Since  $\Gamma$  is an ordered set we can use the Immerman-Vardi Theorem to define the problem  $b \in \langle \Gamma \rangle$  in FPC. More precisely, let  $N$  be the  $I \times \{1, \dots, n+1\}$ -matrix over  $\mathbb{Q}$  whose  $i$ -th column is the vector  $B^i b: I \rightarrow \mathbb{Q}$ . Then  $b \in \langle \Gamma \rangle$  if, and only if,  $N \cdot x = b$  is solvable. Note that  $N$  can be written as  $N = B \cdot \hat{N}$  where  $\hat{N}$  is the  $I \times \{1, \dots, n+1\}$  matrix whose  $i$ -th column is  $B^{i-1}b$ . Moreover, since  $B = M \cdot M^T$ , we have transformed our original system  $M \cdot x = b$  into the system  $M \cdot (M^T \cdot \hat{N}) \cdot x = b$  which is solvable if, and only if,  $M \cdot x = b$  is solvable. Furthermore, a solution  $c: \{1, \dots, n+1\} \rightarrow \mathbb{Q}$  for  $M \cdot (M^T \cdot \hat{N}) \cdot x = b$  readily defines the solution  $(M^T \cdot \hat{N} \cdot c): J \rightarrow \mathbb{Q}$  for  $M \cdot x = b$ .

To solve the system  $N \cdot x = b$  in FPC, first note that  $N$  has an ordered set of columns. However, the set of rows  $I$  is not ordered. To obtain an ordered linear equation system, we

can consider the lexicographical ordering on the set of rows of  $N$  induced by the linear order on the set of columns and on  $\mathbb{Q}$ . This results in a linear preorder which merges columns that are identical (such columns correspond to repeated linear equations). By merging identical columns we obtain a fully ordered system. By the Immerman-Vardi Theorem such systems can be solved in FPC.  $\square$

We are left with the second claim of Theorem 4.12, namely that, given  $M \cdot x = b$  with  $M: I \times J \rightarrow \mathbb{Q}$  and  $b: I \rightarrow \mathbb{Q}$ , we can define in FPC a matrix  $S: J \times J \rightarrow \mathbb{Q}$  whose columns form a generating set for  $\ker(M)$ , that is  $\text{im}(S) = \ker(M)$ . For this we make use of the structure induced by the linear transformation  $C = M^T M: J \times J \rightarrow \mathbb{Q}$  on  $\mathbb{Q}^J$ .

*Proof of Theorem 4.12 - Part 2/2.* We established in Lemma 4.13 that  $\mathbb{Q}^J = \ker(C) \oplus \text{im}(C)$  and in Lemma 4.14 that  $\ker(M) = \ker(C)$ . Hence, we can equivalently define a generating set for  $\ker(C)$  in FPC. Let us denote by  $e_j$  the  $j$ -th standard basis vector on  $\mathbb{Q}^J$ . We can clearly define the vector  $e_j: J \rightarrow \mathbb{Q}$  in FPC using  $j \in J$  as a parameter. Since  $\mathbb{Q}^J = \ker(C) \oplus \text{im}(C)$  we can write each  $e_j$  uniquely as  $e_j = k_j + c_j$  for  $k_j \in \ker(C)$  and  $c_j \in \text{im}(C)$ . It is easy to see that the set  $\{k_j : j \in J\}$  forms a generating set for  $\ker(C)$ . Hence, our aim is to define this set in FPC.

To obtain the projections  $k_j$  of  $e_j$  onto  $\ker(C)$ , we make use of the fact that FPC can solve linear equation systems over  $\mathbb{Q}$  and define single solutions. Indeed,  $k_j$  is the *unique* vector  $k_j \in \mathbb{Q}^J$  such that  $e_j = k_j + c_j$  and  $Ck_j = 0$  and  $Cz = c_j$  for some  $c_j, z \in \mathbb{Q}^J$  (where we treat  $k_j, z, c_j$  here as  $J$ -vectors of variables ranging over  $\mathbb{Q}$ ). Since in each solution of this system the projection onto  $k_j$  is unique, we can define  $k_j$  in FPC as we saw before. Note that in order to define these linear equation systems we use  $j \in J$  as a parameter, so we really solve  $|J|$ -many linear equation systems in parallel. Given the vectors  $k_j: J \rightarrow \mathbb{Q}$ , we can define the matrix  $S: J \times J \rightarrow \mathbb{Q}$  as the matrix whose  $j$ -th column is the vector  $k_j$ . Then  $\text{im}(S) = \ker(C) = \ker(M)$ . This completes our proof of Theorem 4.12.  $\square$

**Remark 4.16.** In fact, by going through our proof once again, one can show that Theorem 4.12 can be strengthened to the extent that the FPC-formulas only use fixed-point operators that converge after a polylogarithmic number of steps, cf. [30].

## 5. DEFINABILITY OF POLYNOMIAL CALCULUS REFUTATIONS OVER FINITE FIELDS

In Section 4 we proved that fixed-point logic with counting and the ( $k$ -dimensional) polynomial calculus over  $\mathbb{Q}$  have the same expressive power if we restrict the coefficients that may occur in a refutation. In this section we study the polynomial calculus not over  $\mathbb{Q}$ , but over finite fields. That means we do not need to worry about the representation of coefficients any more. Yet, we cannot hope to express the degree- $k$  PC over finite fields in fixed-point logic with counting in the general case: It is easy to show that the problem of solving linear equation systems over a field  $\mathbb{F}$  can be reduced (in first-order logic) to finding proofs in the ( $k$ -dimensional) PC over  $\mathbb{F}$ . However, FPC cannot define the solvability problem for linear equation systems over finite fields  $\mathbb{F}$ , see [3].

Instead of giving up completely, we set out to explore certain (interesting) situations in which we can establish the same strong connections between FPC and the polynomial calculus that we discovered over  $\mathbb{Q}$  also over finite fields. To identify these, we take a closer look at typical settings where the connection breaks down, that is where we encounter linear equation systems over finite fields that cannot be solved by FPC. To generate such

hard linear equation systems, a common approach is to use the Cai-Fürer-Immerman (CFI) construction [13]. Specifically, the CFI-construction yields for every prime  $p \in \mathbb{P}$  a family of structures  $(\mathfrak{A}_n^p)_{n \geq 1}$  of size  $\mathcal{O}(n)$  such that the solvability problem for linear equation systems over finite fields  $\mathbb{F}$  of characteristic  $p$  that are defined in CFI-structures  $\mathfrak{A}_n^p$  (via FO-interpretations) cannot be expressed in FPC. Clearly, over these families of Cai-Fürer-Immerman-structures  $(\mathfrak{A}_n^p)_{n \geq 1}$  there is no hope to express  $\text{PC}_k$ -proofs in FPC over fields  $\mathbb{F}$  of characteristic  $p$ .

However, what happens if we consider the following slightly more asymmetric situation. As before, we consider equation systems over a finite field  $\mathbb{F}$  that are interpreted in CFI-structures  $\mathfrak{A}_n^p$ . But, in contrast to the above, we make the additional assumption that the characteristic  $q = \text{char}(\mathbb{F})$  of the finite field  $\mathbb{F}$  does not match the prime  $p$  that was used for the Cai-Fürer-Immerman-construction, that is we assume that  $q \neq p$ . In this case, as we show in our main result of this section (Theorem 5.25), we can express  $\text{PC}_k$ -refutations in fixed-point logic with counting. Although this result only gives limited insight into the logical expressiveness of the polynomial calculus over finite fields, it turns out to be extremely useful to prove lower bounds for the polynomial calculus, as we demonstrate in Section 6.

This section is structured as follows. First of all, we recall (a generalised version of) the Cai-Fürer-Immerman construction in Subsection 5.1 and we analyse automorphism groups of Cai-Fürer-Immerman-structures in Subsection 5.2. To unfold its full power, the Cai-Fürer-Immerman-construction relies on an underlying family of highly connected graphs of bounded degree. To this end, we recall the notion of expander graphs in Subsection 5.3. We prove our first main technical result in Subsection 5.4 where we show that Cai-Fürer-Immerman-structures over expander graphs are *homogeneous wrt. FPC*, which means that FPC can describe elements (and tuples of elements) in Cai-Fürer-Immerman-structures up to automorphisms. An important consequence is that FPC can linearly order orbits of elements (and tuples of elements) with a bounded number of variables. In Subsection 5.5 we establish another important property of Cai-Fürer-Immerman-structures which extends homogeneity: we show that Cai-Fürer-Immerman-structures are *cyclic (wrt. to FPC)* which means that FPC can linearly order orbits of elements (and tuples of elements) by fixing a single parameter in this orbit. We also show that this property is closed under taking FPC-interpretations and ordered pairs. Building on this, we establish our key technical result in Subsection 5.6: we show that FPC can define solution spaces of linear equation systems over finite fields  $\mathbb{F}$  that are interpreted in cyclic background structures with the additional assumption that the characteristic of the field  $\mathbb{F}$  does not divide the size of the (Abelian) automorphism group of the cyclic structure (we say that such linear equation systems are *cocyclic*). Finally, in Subsection 5.7 we use this result in order to prove our main Theorem 5.25: FPC can express  $k$ -dimensional PC-refutations for polynomial equation systems that are defined in cyclic structures over finite fields  $\mathbb{F}$  if this same condition on the characteristic for  $\mathbb{F}$  holds.

**5.1. Cai-Fürer-Immerman Construction.** For notational convenience, we introduce the Cai-Fürer-Immerman-construction only for connected (undirected) graphs  $G$  which are *3-regular* and *ordered*. The assumption that  $G$  is ordered means that additional to the set of vertices  $V = V(G)$  and the (symmetric) edge relation  $E = E(G) \subseteq V(G) \times V(G)$  we assume that  $G$  contains a linear order  $\leq = \leq(G)$  on its set of vertices  $V$ . In fact, this is the original setting as it was introduced by Cai, Fürer, and Immerman in [13].

Let  $p \in \mathbb{P}$  be a prime. For every vector  $\lambda \in \mathbb{F}_p^V$  we construct the *CFI-structure*  $\text{CFI}[G; p; \lambda]$  over the (connected and ordered) graph  $G$ , the finite field  $\mathbb{F}_p$ , and with *load*  $\lambda$  as the following relational structure with signature  $\tau_{\text{CFI}} = \{\preceq, R, C, I\}$  where  $R$  is a ternary relation symbol and where  $\preceq, I, C$  are binary relation symbols. The universe  $A$  of the CFI-structure  $\mathfrak{A} = \text{CFI}[G; p; \lambda]$  is  $A = E(G) \times \mathbb{F}_p$ . The linear order  $\leq(G)$  on the vertex set  $V(G)$  of  $G$  extends to a linear order on the edge set  $E(G)$ . We use this linear order on  $E(G)$  to define the following *total preorder*  $\preceq$  on  $A$ :  $(e, x) \preceq (f, y)$  if  $e \leq f$ . Note that  $\preceq$  induces a linear order on the corresponding equivalence classes  $e^p = e \times \mathbb{F}_p$ . Clearly, each of these classes  $e^p$  is of size  $p$ . Since  $G$  is undirected every edge  $e = (v, w) \in E$  comes with its corresponding *dual edge*  $f = (w, v) \in E$ . In what follows, we use the notation  $e^{-1} = f$  to denote the dual of the edge  $e \in E$ . The relations  $I$  and  $C$  are defined follows.

- The *cycle relation*  $C$  defines the cyclic structure of the additive group of  $\mathbb{F}_p$  on each of the equivalence classes  $e^p$ . More precisely,

$$C = \bigcup_{e \in E} \{(e, x), (e, x + 1 \bmod p)\} : x \in \mathbb{F}_p\}.$$

- The *inverse relation*  $I$  relates additive inverses for dual edges. Formally,

$$I = \bigcup_{e \in E} \{(e, x), (e^{-1}, -x)\} : x \in \mathbb{F}_p\}.$$

Note that while the cycle relation  $C$  defines a *directed* cycle, the inverse relation  $I$  is symmetric. Furthermore, observe that the relations  $\preceq, C$  and  $I$  are defined independently of the load vector  $\lambda$  and so do only depend on the underlying graph  $G$  and the prime field  $\mathbb{F}_p$ . In contrast, the *CFI-relation*  $R = R^\lambda$  is defined using the load vector  $\lambda$  as follows. For each  $v \in V$ , we let  $vE \subseteq E$  denote the set of neighbours of  $v$  in  $G$ , that is  $E(v) = \{v\} \times vE \subseteq E$  is the set of edges outgoing from  $v$ . Since  $G$  is 3-regular we have that  $|vE| = 3$  for each  $v \in V$ . For  $v \in V$  let  $E(v) = \{w_1, w_2, w_3\}$  where  $w_1 < w_2 < w_3$ . The *CFI-relation*  $R^\lambda(v)$  at vertex  $v$  is defined as follows:

$$R^\lambda(v) = \{(w_1, x_1), (w_2, x_2), (w_3, x_3)\} : x_1 + x_2 + x_3 = \lambda(v)\}.$$

The full CFI-relation  $R^\lambda$  of the structure  $\text{CFI}[G; p; \lambda]$  is given as  $R^\lambda = \bigcup_{v \in V} R^\lambda(v)$ .

**5.2. Symmetries of Cai-Fürer-Immerman-Structures.** It turns out that the automorphism group  $\Gamma$  of a CFI-structure  $\text{CFI}[G; p; \lambda]$  only depends on  $G$  and  $p$ , but not on  $\lambda$ . To see this, first observe that every automorphism  $\pi \in \Gamma$  has to maintain the linear preorder  $\preceq$  which means that  $\pi(e^p) = e^p$  for all  $e \in E$ . Moreover,  $\pi$  has to maintain the cycle relation  $C$ . This means that the action of  $\pi$  on an edge class  $e^p$  is a cyclic shift in  $\mathbb{F}_p$ . Let us write  $\pi(e) \in \mathbb{F}_p$  to denote this cyclic shift of  $\pi$  on  $e^p$  for  $e \in E$ . Then, because of the inverse relation  $I$ , we have  $\pi(e) + \pi(e^{-1}) = 0$ . Altogether this shows that

$$\Gamma \leq \{\pi \in \mathbb{F}_p^E : \pi(e) + \pi(e^{-1}) = 0 \text{ for } e \in E\} \leq \mathbb{F}_p^E.$$

However, so far we have not taken the CFI-relation  $R^\lambda$  into account. Again, because of the linear preorder  $\preceq$ , for each  $\pi \in \Gamma$  we have  $\pi(R^\lambda(v)) = R^\lambda(v)$  for all  $v \in V$ . Let  $v \in V$  and  $vE = \{w_1, w_2, w_3\}$  and let  $((w_1, x_1), (w_2, x_2), (w_3, x_3)) \in R^\lambda(v)$ , that is  $x_1 + x_2 + x_3 = \lambda(v)$ . From our earlier observations we know that

$$\pi((w_i, x_i)) = (w_i, x_i + \pi(v, w_i)).$$

Hence, the condition  $\pi(R^\lambda(v)) = R^\lambda(v)$  implies that

$$x_1 + \pi(v, w_1) + x_2 + \pi(v, w_2) + x_3 + \pi(v, w_3) = \lambda(v).$$

This implies that  $\pi(v, w_1) + \pi(v, w_2) + \pi(v, w_3) = \sum_{e \in E(v)} \pi(e) = 0$ . In fact, this condition is not only necessary but also sufficient for  $\pi$  to preserve  $R^\lambda(v)$ . Moreover, note that all of this holds independent of what  $\lambda$  is. Altogether this gives us a characterisation of the automorphism group  $\Gamma$  of  $\text{CFI}[G; p; \lambda]$  as a subgroup of the vector space  $\mathbb{F}_p^E$  that is determined by the following set of linear equations in variables  $\pi(e)$  for  $e \in E$ :

$$\pi(e) + \pi(e^{-1}) = 0 \quad \text{for } e \in E \quad (\text{Inv})$$

$$\pi(v) := \sum_{e \in E(v)} \pi(e) = 0 \quad \text{for } v \in V. \quad (\text{CFI})$$

More generally, we can apply each vector  $\pi \in \mathbb{F}_p^E$ , that satisfies the constraints (Inv), to a CFI-structure  $\text{CFI}[G; p; \lambda]$  and obtain a new CFI-structure over the same underlying graph  $G$ . As it turns out the resulting structure is  $\text{CFI}[G; p; \lambda + \pi]$  where  $(\lambda + \pi)(v) = \lambda(v) + \pi(v)$  for all  $v \in V$ . Let us denote by  $\text{Inv}(\mathbb{F}_p^E) \leq \mathbb{F}_p^E$  the set of all vectors  $\pi$  that satisfy the (Inv)-constraints.

**Remark 5.1.** For every  $G = (V, E, \leq)$  (connected, ordered, 3-regular) and each prime  $p \in \mathbb{P}$ , the group  $\Delta = \text{Inv}(\mathbb{F}_p^E) \leq \mathbb{F}_p^E$  acts on the set of CFI-structures over  $G$  that is on  $\text{CFI}[G; p; \star] = \{\text{CFI}[G; p; \lambda] : \lambda \in \mathbb{F}_p^V\}$ , and this action partitions the set into precisely  $p$  orbits (see below).

Clearly, the set  $\text{CFI}[G; p; \star]$  has size  $p^n$  where  $n = |V(G)|$ . However, if we consider this set up to isomorphism, there are only  $p$  different Cai-Fürer-Immerman-structures over a fixed graph  $G$ :

**Theorem 5.2** [13, 32, 40]. *Two CFI-structures  $\text{CFI}[G; p; \lambda], \text{CFI}[G; p; \sigma] \in \text{CFI}[G; p; \star]$  are isomorphic if, and only if,*

$$\sum \lambda = \sum_{v \in V} \lambda(v) = \sum_{v \in V} \sigma(v) = \sum \sigma.$$

Let us remark that, for technical convenience, we have introduced CFI-structures as relational structures. However, it is easy to encode them as usual (unordered) graphs, and, in fact, this is the way in which they were originally defined in [13]. The main step is to introduce for each CFI-constraint  $i = ((e_1, x_1), (e_2, x_2), (e_3, x_3)) \in R^{\lambda(v)}$ ,  $e_i \in E(v)$ ,  $x_i \in \mathbb{F}_p$ , a new node  $i^{\lambda(v)}$  and to connect it to the edge nodes  $(e_i, x_i) \in E(v) \times \mathbb{F}_p$  accordingly (these additional constraint nodes  $i^{\lambda(v)}$  are called *inner nodes* in the original construction of Cai, Fürer, and Immerman). Furthermore, we can replace the linear preorder by a path of the appropriate length and connect vertices in the edge classes to positions on this path accordingly. All of these simple transformation steps are clearly definable in FPC.

**Lemma 5.3.** *There exist FPC-interpretations  $\mathcal{J}$  and  $\mathcal{J}^{-1}$  such that  $\mathcal{J}$  maps CFI-structures  $\mathfrak{A} = \text{CFI}[G; p; \lambda] \in \text{CFI}[\mathcal{F}; p]$  to graphs  $\mathcal{J}(\mathfrak{A})$  of degree  $\mathcal{O}(p^2)$  and with  $\mathcal{O}(p^2 \cdot n)$  many vertices, where  $n = |V(G)|$ , and such that  $\mathcal{J}^{-1}$ , which maps graphs to CFI-structures, is the inverse of  $\mathcal{J}$  in the sense that for all  $\mathfrak{A} \in \text{CFI}[G; p; \lambda]$  we have that  $\mathcal{J}^{-1}(\mathcal{J}(\mathfrak{A}))$  is isomorphic to  $\mathfrak{A}$ , that is  $\mathcal{J}^{-1}(\mathcal{J}(\mathfrak{A})) \cong \mathfrak{A}$ .*

**5.3. Expander Graphs and CFI-Classes.** Let us briefly recall the definition of *expander graphs* based on the exposition in [33]. In the following  $G = (V, E)$  denotes an undirected  $d$ -regular graph (in this paper we only consider the case  $d = 3$ ). For two subsets of vertices  $S, T \subseteq V$  in  $G$  we denote the set of (directed) edges from  $S$  to  $T$  by  $E[S; T] = E \cap (S \times T)$ . The *edge boundary* of a set  $S \subseteq V$  is  $\partial S = E[S; V \setminus S]$  and the *expansion ratio*  $h(G)$  is defined as:

$$h(G) = \min_{\{S: |S| \leq |V|/2\}} \frac{|\partial S|}{|S|}.$$

**Definition 5.4** (Expander graphs). A sequence  $\mathcal{F} = \{G_n : n \in \mathbb{N}\}$  of undirected  $d$ -regular graphs is called a *family of  $d$ -regular expander graphs* if

- $\mathcal{F}$  is *increasing*, that is  $|V(G_n)|$  is monotone and unbounded, and
- $\mathcal{F}$  is *expanding*, that is there exists  $\varepsilon > 0$  such that  $h(G_n) \geq \varepsilon$  for all  $n \in \mathbb{N}$ .

For our applications in this paper we make use of the existence of a family of 3-regular connected expander graphs.

**Theorem 5.5** (see e.g. Example 2.2 in [33]). *There exists a family of 3-regular expander graphs  $\mathcal{F} = \{G_n : n \in \mathbb{N}\}$  such that each graph  $G_n$ ,  $n \in \mathbb{N}$ , is connected and has  $\mathcal{O}(n)$  vertices.*

For the rest of this paper let us fix a family  $\mathcal{F} = \{G_n : n \in \mathbb{N}\}$  of expander graphs as in the previous theorem. Of course, we can also assume that the graphs in  $\mathcal{F}$  are *ordered* just by adding to each graph  $G_n \in \mathcal{F}$  an arbitrary linear order on  $V(G_n)$ . From this family  $\mathcal{F}$  of 3-regular, connected, ordered expander graphs  $G_n$  with  $\mathcal{O}(n)$  many vertices we construct, for every  $p \in \mathbb{P}$ , the CFI-class  $\text{CFI}[\mathcal{F}; p]$  consisting of all CFI-structures over graphs from  $\mathcal{F}$  that is

$$\text{CFI}[\mathcal{F}; p] = \bigcup_{n \in \mathbb{N}} \text{CFI}[G_n; p; \star].$$

The *CFI-problem* (over  $\mathcal{F}$  and  $p \in \mathbb{P}$ ) is to decide, given a structure  $\text{CFI}[G; p; \lambda] \in \text{CFI}[\mathcal{F}; p]$  whether  $\sum \lambda = 0$ . It was shown by Cai, Fürer, and Immerman that this problem is undefinable in counting logic with sublinearly many variables.

**Theorem 5.6** [13]. *For  $\text{CFI}[G_n; p; \lambda], \text{CFI}[G_n; p; \sigma] \in \text{CFI}[\mathcal{F}; p]$  we have*

$$\text{CFI}[G_n; p; \lambda] \equiv^{\Omega(n)} \text{CFI}[G_n; p; \sigma].$$

**5.4. Homogeneity of Cai-Fürer-Immerman-Structures.** In this section we establish a key technical result: We show that CFI-structures (over ordered expander graphs) are (FPC-)homogeneous. This means that the orbits of  $k$ -tuples in CFI-structures can be uniquely described in FPC by using only  $\mathcal{O}(k)$  many variables. This is extremely useful as it implies that we can actually order the set of orbits on  $k$ -tuples in FPC using only  $\mathcal{O}(k)$  many variables. To put it differently, we show that in CFI-structures over expander graphs, FPC can describe tuples up to their automorphism type without using too many resources (variables). We believe that this result is of independent interest and should prove useful in other applications (one example is discussed in Section 7).

**Theorem 5.7.** (*Homogeneity*) *There is a constant  $\ell \geq 1$  such that for every  $k \geq 1$ ,  $p \in \mathbb{P}$ , and every  $\mathfrak{A} = \text{CFI}[G; p; \lambda] \in \text{CFI}[\mathcal{F}; p]$  with automorphism group  $\Gamma \leq \mathbb{F}_p^{E(G)}$  and every  $\vec{a}, \vec{b} \in A^k$  we have that  $(\mathfrak{A}, \vec{a}) \equiv^{\ell \cdot k} (\mathfrak{A}, \vec{b})$  if, and only if,  $\Gamma(\vec{a}) = \Gamma(\vec{b})$ .*

Let  $1 > \varepsilon > 0$  be the expander constant corresponding to the class  $\mathcal{F}$ , that is for all  $G \in \mathcal{F}$  we have that  $h(G) \geq \varepsilon$ . We will prove Theorem 5.7 for

$$\ell \geq \frac{12}{\varepsilon}.$$

To this end, we inductively show the following for all  $k \geq 0$ : For  $\mathfrak{A} = \text{CFI}[G; p; \lambda] \in \text{CFI}[\mathcal{F}; p]$  with automorphism group  $\Gamma \leq \mathbb{F}_p^{E(G)}$ , every  $\vec{a} \in A^k$ , and every  $a, b \in A$  such that  $(\mathfrak{A}, \vec{a}, a) \equiv^{\ell \cdot (k+1)} (\mathfrak{A}, \vec{a}, b)$  we can find an automorphism  $\pi \in \Gamma$  such that  $\pi(\vec{a}, a) = (\vec{a}, b)$ .

If we have shown this, then the above theorem easily follows. Indeed, let  $\vec{a}, \vec{b} \in A^k$  such that  $(\mathfrak{A}, \vec{a}) \equiv^{\ell \cdot k} (\mathfrak{A}, \vec{b})$ . We have to show that in this case  $\vec{a}$  and  $\vec{b}$  are in the same orbit, i.e.  $\Gamma(\vec{a}) = \Gamma(\vec{b})$ . In other words, we have to show that every  $\equiv^{\ell \cdot k}$ -class on  $A^k$  is a single  $\Gamma$ -orbit (clearly, each such class is a union of orbits). For the sake of contradiction, assume that  $\Gamma(\vec{a}) \neq \Gamma(\vec{b})$ . Let  $r \geq 0$ ,  $r < k$  be maximal with respect to the following property: there exists  $\vec{c} \in \Gamma(\vec{a})$  such that  $\vec{b}$  and  $\vec{c}$  share a prefix of length  $r$ , that is  $\vec{b} = (b_1, \dots, b_r, b_{r+1}, \dots, b_k)$ , and  $\vec{c} = (b_1, \dots, b_r, c_{r+1}, \dots, c_k)$ , and  $b_{r+1} \neq c_{r+1}$ . But then  $(\mathfrak{A}, a_1, \dots, a_r, a_{r+1}) \equiv^{\ell \cdot k} (\mathfrak{A}, b_1, \dots, b_r, c_{r+1})$  (since  $\vec{a}$  and  $\vec{c}$  are in the same orbit) and  $(\mathfrak{A}, a_1, \dots, a_r, a_{r+1}) \equiv^{\ell \cdot k} (\mathfrak{A}, b_1, \dots, b_r, b_{r+1})$  (by the assumption). Hence

$$(\mathfrak{A}, b_1, \dots, b_r, b_{r+1}) \equiv^{\ell \cdot k} (\mathfrak{A}, b_1, \dots, b_r, c_{r+1}).$$

By the above proposition, we can find  $\pi \in \Gamma$  such that  $\pi(b_1, \dots, b_r, c_{r+1}) = (b_1, \dots, b_r, b_{r+1})$ . This means that  $\pi(\vec{c}) \in \Gamma(\vec{a})$  is a tuple in  $\Gamma(\vec{a})$  sharing a longer prefix with  $\vec{b}$  than  $\vec{c}$  which leads to the desired contradiction.

Hence, let us now focus on proving the above proposition. For this let  $k \geq 0$ ,  $\vec{a} \in A^k$ , and  $a, b \in A$  such that  $(\mathfrak{A}, \vec{a}, a) \equiv^{\ell \cdot (k+1)} (\mathfrak{A}, \vec{a}, b)$  where  $\mathfrak{A} = \text{CFI}[G; p; \lambda] \in \text{CFI}[\mathcal{F}; p]$ . Recall that  $\Gamma \leq \text{Inv}(\mathbb{F}_p^{E(G)}) \leq \mathbb{F}_p^{E(G)}$  denotes the automorphism group of  $\mathfrak{A}$ . We have to show the existence of some  $\pi \in \Gamma$  such that  $\pi(\vec{a}, a) = (\vec{a}, b)$ . To do this, we establish two key properties of the elements  $a, b \in A$  using the fact that  $(\mathfrak{A}, \vec{a}, a) \equiv^{\ell \cdot (k+1)} (\mathfrak{A}, \vec{a}, b)$ . Clearly, we can assume that  $a \neq b$ , because the claim is trivial otherwise.

(P1) The elements  $a, b$  are in the same edge class in  $\mathfrak{A}$ , i.e. there exists  $e \in E(G)$  such that  $a, b \in e^p = e \times \mathbb{F}_p \subseteq A$ .

This easily follows from the fact that each edge class  $e^p$  can be identified in counting logic by using the preorder  $\preceq$  and at most three variables (which we use to enumerate the edge classes starting from the minimal one). Moreover, this formula does not require access to any of the parameters from  $\vec{a}$ . Hence, a counting logic formula with three variables could distinguish between  $a$  and  $b$  in  $\mathfrak{A}$  if they were in different edge classes. Note that  $\ell \geq 12$ , so we clearly have enough variables available.

The next simple observation is that this edge class  $e^p$  is *free*, a property that we are going to define now. Let  $\text{BL} \subseteq E(G)$  be the smallest set such that

- (i) if  $a_i \in f^p$ , for some  $1 \leq i \leq k$  and for  $f \in E(G)$ , then  $f \in \text{BL}$ , and
- (ii) if  $f \in \text{BL}$ , then  $f^{-1} \in \text{BL}$ ,

We say that the edges in  $\text{BL}$  are *blocked* and the edges  $E(G) \setminus \text{BL}$  are *free*. For blocked edges it is straightforward to define the individual elements in the corresponding blocked edge classes.

**Lemma 5.8.** *For every blocked edge  $f \in \text{BL}$  and every  $c \in f^p = f \times \mathbb{F}_p \subseteq A$ , there exists a formula of counting logic  $\varphi(x_1, \dots, x_k, y)$  with at most  $k + 2$  many variables which defines  $c$  in  $(\mathfrak{A}, a_1, \dots, a_k)$ , i.e. such that for every  $d \in A$  we have that  $\mathfrak{A} \models \varphi(a_1, \dots, a_k, d)$  if, and only if,  $c = d$ .*

*Proof.* First of all, assume that an edge  $f \in \text{BL}$  is marked as blocked because for some  $1 \leq i \leq k$  we have that  $a_i \in f^p = f \times \mathbb{F}_p$ , i.e. we are in case (i). Recall that the cycle relation  $C$  defines a directed cycle on  $f^p$ . Hence, using  $C$  and  $a_i$  as a parameter we can define every other element  $c \in f^p$  in counting logic using the parameter  $a_i$  and one additional auxiliary variable.

Secondly, assume that  $f \in \text{BL}$  is blocked, because  $g = f^{-1} \in \text{BL}$ , i.e. we are in case (ii). By the induction hypothesis we know that we can define in counting logic each element in  $g^p$  using  $k + 2$  many variables (and the parameters in  $\vec{a}$ ). Let  $\varphi(\vec{x}, y)$  be a formula defining such an element  $c \in g^p$ . Then  $\psi(\vec{x}, y) = \exists z(I(z, y) \wedge \varphi(\vec{x}, z))$  is a formula of counting logic with at most  $k + 2$  many variables which defines an element in  $f^p$ . Since  $I$  is a bijection between  $f^p$  and  $g^p$  we can define each element in  $f^p$  in this way.  $\square$

For obvious reasons, the number of blocked edges is linearly bounded in  $k$ :

**Lemma 5.9.** *The number of blocked edges is linear in  $k$ : we have  $|\text{BL}| \leq 2k$  (or  $|\text{BL}| \leq k$  if we count edges as undirected).*

Let us come back to our original goal. Recall that we have to show the existence of some  $\pi \in \Gamma$  such that  $\pi(\vec{a}, a) = (\vec{a}, b)$  where  $(\mathfrak{A}, \vec{a}, a) \equiv^{\ell \cdot (k+1)} (\mathfrak{A}, \vec{a}, b)$ ,  $a \neq b$ . With the above preparation it is now easy to see that  $a$  and  $b$  satisfy the following property.

(P2) The edge class  $e^p$ ,  $e \in E(G)$ , that contains the elements  $a, b$  (see (P1)), is free, that is  $e \in E(G) \setminus \text{BL}$ .

This immediately follows from Lemma 5.8 (note that  $k + 2 \leq \ell \cdot (k + 1)$ ).

Let us fix a free edge class  $e^p$ ,  $e \in E(G) \setminus \text{BL}$ . We are going to construct an automorphism  $\pi \in \Gamma$  such that  $\pi(\vec{a}) = \vec{a}$  and such that  $\pi(e) = 1$ , that is  $\pi$  acts as a cyclic shift by one on the edge class  $e^p$ . If we can show this, then our original claim follows. To this end, we distinguish between the following two cases. We say that the edge  $e$  lies on a *free cycle*, if there exist edges  $e_0 = (v_0, v_1), e_1 = (v_1, v_2), \dots, e_r = (v_r, v_0)$ ,  $r \geq 2$ , such that all  $v_i$ ,  $0 \leq i \leq r$ , are distinct and such that  $e = e_0$  and  $e_i \in E(G) \setminus \text{BL}$ ,  $0 \leq i \leq r$ . Indeed, if  $e$  lies on such a free cycle, then we can construct an automorphism  $\pi \in \Gamma$  with the desired properties as follows: we simply set  $\pi(e_i) = 1$  and  $\pi(e_i^{-1}) = -1$  for all  $0 \leq i \leq r$ . Note that each of the moved edge classes  $e_i^p$  is free, so none of the elements in the tuple  $\vec{a}$  will occur in any of the edge classes moved by  $\pi$ , that is  $\pi(\vec{a}) = \vec{a}$ .

Hence, the interesting case is that  $e$  does not lie on a free cycle. We show that in this case each element in the edge class  $e^p$  can be defined in counting logic by fixing elements in a bounded number of additional edge classes (more precisely, by using at most  $\ell \cdot (k + 1)$  many variables). Hence, for our original setting this would mean that the assumption  $(\mathfrak{A}, \vec{a}, a) \equiv^{\ell \cdot (k+1)} (\mathfrak{A}, \vec{a}, b)$  would already imply that  $a = b$ . To prove this, we strongly make use of the fact that the family  $\mathcal{F}$  of graphs from which we constructed the CFI-class  $\text{CFI}[\mathcal{F}; p]$  is an expander family. Let us formulate our claim precisely.

**Lemma 5.10.** *As above, let  $\mathfrak{A} = \text{CFI}[G; p; \lambda] \in \text{CFI}[\mathcal{F}; p]$ ,  $\vec{a} \in A^k$ ,  $\ell \geq 12/\varepsilon$ , and let  $e \in E(G) \setminus \text{BL}$  be a free edge which does not lie on a free cycle. Then for every  $b \in e^p$  there exists a formula  $\varphi(\vec{x}, y)$  of counting logic with at most  $\ell \cdot (k+1)$  many variables that defines  $b$  in  $(\mathfrak{A}, \vec{a})$ , that is for every  $c \in A$  we have that  $\mathfrak{A} \models \varphi(\vec{a}, c)$  if, and only if,  $c = b$ .*

*Proof.* Let us consider the subgraph  $H = (W, F)$  of  $G$  that is induced by the free edges  $E(G) \setminus \text{BL}$ . Note that since  $\text{BL}$  is symmetric,  $H$  is an undirected graph. Let  $e = (v, w) \in E(G) \setminus \text{BL}$  be the free edge that we consider and let  $X \subseteq V(H)$  and  $Y \subseteq V(H)$  denote the connected components of  $v$  and  $w$  in the graph  $H \setminus e$ , that is in the graph that results from  $H$  by removing the (undirected) edge  $e$ . Since  $e$  does not lie on a free cycle we know that  $X$  and  $Y$  are disjoint. Using the expander property of  $G$ , we now aim to bound the size of  $X$  or  $Y$ . Clearly, at least one of the two sets contains at most  $|V(G)|/2$  many vertices. Without loss of generality, let us assume that  $|X| \leq |V(G)|/2$ . Then  $|\partial X| \geq |X| \cdot \varepsilon$  since  $h(G) \geq \varepsilon$ . The important observation is that we can bound  $|\partial X|$  in terms of  $k$ . Indeed, in  $G$  every edge leaving the set  $X$  (different from  $e$ ) has to be blocked, since  $X$  is a connected component in  $H$ . Hence  $|\partial X| \leq k$ . This yields the bound of  $|X| \leq k/\varepsilon$  on the size of  $X$ .

In conclusion, the set  $X \subseteq V(G)$  in  $G$  is a set of vertices of size at most  $k/\varepsilon$  such that each edge leaving  $X$  is blocked except for the single free edge  $e$ . We now consider the CFI-substructure  $\mathfrak{B}$  of the input CFI-structure  $\mathfrak{A}$  induced on the edge classes incident with vertices in  $X$  where in every blocked edge class  $f^p$  we arbitrarily mark an element  $c \in f^p$  to be  $c = (f, 0) \in f^p$  (this choice depends on the parameters  $\vec{a}$ ). More precisely, let  $E_X = \{e \in E(v) : v \in X, e \notin \text{BL}\}$ , then the universe  $B$  of  $\mathfrak{B}$  is the set  $B = \bigcup_{e \in E_X} e^p$ , and the linear preorder  $\preceq$ , the cycle relation  $C$ , and the inverse relation  $I$  in  $\mathfrak{B}$  are just the restrictions of the corresponding relations in  $\mathfrak{A}$  to the subuniverse  $B$ . To define the CFI-relation  $R_{\mathfrak{B}}^\lambda$  on  $\mathfrak{B}$  we distinguish between the following cases. Recall that  $R^\lambda = \bigcup_{v \in V(G)} R^\lambda(v)$ . First, let us consider vertices  $v \in X$  whose neighbours are all contained in  $X$ . In this case we simply set  $R_{\mathfrak{B}}^\lambda(v) = R^\lambda(v)$ . For vertices  $v \in X$  for which some incident edge (classes) are blocked we define  $R_{\mathfrak{B}}^\lambda(v)$  as follows. Let  $F \subseteq E(v)$  denote the set of blocked edges incident with  $v$ . Note that  $|F| \leq 2$ . We fix  $x^f \in f^p$  for every  $f \in F$  (we can make this choice using the parameters  $\vec{a}$ , see Lemma 5.8). Then we define  $R_{\mathfrak{B}}^\lambda(v)$  to be the restriction of  $R^\lambda(v)$  to those tuples that contain  $x^f$  for every  $f \in F$ . If we recall the definition of  $R^\lambda(v)$ , then this intuitively corresponds to declaring  $x^f = (f, 0)$ . In particular, note that the arity of  $R_{\mathfrak{B}}^\lambda(v)$  is  $3 - |F| \geq 1$ . Finally, the CFI-relation  $R_{\mathfrak{B}}^\lambda$  in  $\mathfrak{B}$  is defined as  $R_{\mathfrak{B}}^\lambda = \bigcup_{v \in X} R_{\mathfrak{B}}^\lambda(v)$ .

It follows from our preparations that  $\mathfrak{B}$  can be defined in  $\mathfrak{A}$  by using a parametrised, one-dimensional interpretation  $\mathcal{I}(\vec{x})$  in counting logic, i.e.  $\mathcal{I}(\mathfrak{A}, \vec{a}) = \mathfrak{B}$ . Moreover,  $\mathcal{I}$  can be constructed by using, as a rough estimate, at most  $k+6$  many variables. The most important thing to observe is that we can use Lemma 5.8 in order to fix elements in all blocked edge classes as required.

We now want to argue that every possible automorphism  $\pi \in \mathbb{F}_p^{E_X}$  of  $\mathfrak{B}$  will fix the edge class  $e^p$  (recall that  $e$  denotes the single free edge  $e$  that leaves the set  $X$ ). Recall that the inverse constraints (Inv) enforce that for each pair of dual edges  $f, g \in E_X$  we have  $\pi(f) + \pi(g) = 0$ . Note that for each edge  $f \in E_X$  we have  $f^{-1} \in E_X$  except for the single edge  $e$  for which  $e^{-1} \notin E_X$ . Hence  $\sum_{f \in E_X} \pi(f) = \pi(e)$ . Moreover, recall that the CFI-constraints (CFI) enforce that for each  $v \in X$  we have  $\sum_{f \in E_X(v)} \pi(f) = 0$ . Hence,  $\sum_{v \in X} \sum_{f \in E_X(v)} \pi(f) = 0$ . Since  $\sum_{f \in E_X} \pi(f) = \sum_{v \in X} \sum_{f \in E_X(v)} \pi(f)$  we conclude that  $\pi(e) = 0$ , that is  $\pi$  fixes the edge class  $e^p$ , as claimed.

It follows that in  $\mathfrak{B}$  every element  $x \in e^p$  can be defined in counting logic by using roughly  $|E_X| + 6$  many variables. Indeed, for some  $c \in e^p$ , we can use  $|E_X|$  many variables to fix elements in all other edge class and then describe the isomorphism type of the structure  $(\mathfrak{B}, c)$ . Since each  $c \in e^p$  is in a singleton orbit, these isomorphism types will be different for all elements  $c \in e^p$ . Note that  $|E_X| + 6 \leq 3k/\varepsilon + 6$ . If we translate the resulting formulas back to  $\mathfrak{A}$  via  $\mathcal{I}(\vec{x})$ , then we obtain a formula in counting logic that defines  $c \in e^p$  in  $\mathfrak{A}$  and which uses at most  $3k/\varepsilon + 6 + k + 6 \leq 4k/\varepsilon + 12 \leq (12k + 12)/\varepsilon \leq \ell \cdot (k + 1)$  many variables.  $\square$

This completes the proof of Theorem 5.7.

**Definition 5.11.** For  $\ell \geq 1$ , we say that a structure  $\mathfrak{A}$  with automorphism group  $\Gamma$  is  $\ell$ -homogeneous if for all  $k \geq 1$  and all  $k$ -tuples  $\vec{a}, \vec{b} \in A^k$  we have that

$$(\mathfrak{A}, \vec{a}) \equiv^{\ell, k} (\mathfrak{A}, \vec{b}) \text{ if, and only if, } \Gamma(\vec{a}) = \Gamma(\vec{b}).$$

Moreover, we say that a class  $\mathcal{K}$  of structures is *homogeneous* if there is an  $\ell \geq 1$  such that each structure  $\mathfrak{A} \in \mathcal{K}$  is  $\ell$ -homogeneous.

**Corollary 5.12.** *The class of CFI-structures  $\text{CFI}[\mathcal{F}; p]$  is homogeneous.*

As mentioned before, an important consequence of homogeneity is that FPC can (uniformly) define a total preorder on the set  $A^k$ , for each  $k \geq 1$ , which orders  $k$ -tuples up to orbits. Moreover, the number of variables required by such an FPC-formula is linear in  $k$ . To see this, we make use of the well-known fact that for every  $\ell \geq 1$  there exists an FPC-formula  $\text{Tp}^\ell(\vec{x}, \vec{y})$  with  $\mathcal{O}(\ell)$  many variables which defines on each input structure  $\mathfrak{A}$  a linear preorder on  $A^\ell$  which distinguishes between all pairs of tuples  $\vec{a}, \vec{b} \in A^\ell$  for which  $(\mathfrak{A}, \vec{a}) \not\equiv^\ell (\mathfrak{A}, \vec{b})$  holds, see e.g. [39]. That is  $\text{Tp}^\ell(\vec{x}, \vec{y})$  defines in each input structure  $\mathfrak{A}$  a linear order on the set  $\{[\vec{a}]_{\equiv^\ell} : \vec{a} \in A^\ell\}$  consisting of  $\equiv^\ell$ -equivalence classes  $[\vec{a}]_{\equiv^\ell} = \{\vec{b} \in A^\ell : (\mathfrak{A}, \vec{a}) \equiv^\ell (\mathfrak{A}, \vec{b})\}$  for  $\vec{a} \in A^\ell$ . Of course, we can also use the formula  $\text{Tp}^\ell(\vec{x}, \vec{y})$  to define the corresponding preorder on  $k$ -tuples for lengths  $1 \leq k < \ell$  (a common approach is to extend  $k$ -tuples to  $\ell$ -tuples by repeating the last component). We denote the corresponding FPC-formula by  $\text{Tp}_k^\ell(\vec{x}, \vec{y}) = \text{Tp}_k^\ell(x_1, \dots, x_k, y_1, \dots, y_k)$ .

**Theorem 5.13.** *Let  $\mathfrak{A}$  be  $\ell$ -homogeneous with automorphism group  $\Gamma$ . Then the FPC-formula  $\text{Tp}_k^{\ell, k}(\vec{x}, \vec{y})$  defines a total preorder  $\preceq$  on  $A^k$  that identifies  $k$ -tuples which are in the same orbit. In particular,  $\text{Tp}_k^{\ell, k}(\vec{x}, \vec{y})$  induces a linear order on the set of orbits of  $k$ -tuples  $\{\Gamma(\vec{a}) : \vec{a} \in A^k\}$ .*

**5.5. CFI-structures are Cyclic.** In Section 5.4 we proved that the CFI-classes  $\text{CFI}[\mathcal{F}; p]$  are homogeneous, which by Theorem 5.13 implies that FPC can order  $k$ -tuples in structures  $\mathfrak{A} \in \text{CFI}[\mathcal{F}; p]$  up to orbits using only  $\mathcal{O}(k)$  many variables. In this subsection we go one step further and show that, as a result of the algebraic properties of the automorphism groups of CFI-structures, each individual orbit of  $k$ -tuples can be linearly ordered in fixed-point logic with counting by fixing a single  $k$ -tuple from this orbit as a parameter (and, again, by using  $\mathcal{O}(k)$  many variables only). Furthermore, we are going to show that this key property of CFI-structures remains intact if we apply logical transformations. Intuitively, our results show that CFI-structures come quite close to ordered structures: in FPC, one can preorder the elements of CFI-structures up to orbits, *and*, secondly, each individual orbit can be totally ordered by fixing a *single* element as a parameter. Note, however, that this does not mean

that we can order the full CFI-structure, since this would require to fix a parameter in *each* of the orbits at the same time. Indeed, Theorem 5.6 implies that CFI-structures can *not* be totally ordered in FPC if we restrict ourselves to formulas with a sublinear number of variables.

Recall that, for  $1 \leq k \leq \ell$ , the formulas  $\text{Tp}_k^\ell = \text{Tp}_k^\ell(\vec{x}, \vec{y})$  define a total preorder that distinguishes  $k$ -tuples up to  $\equiv^\ell$ -equivalence. In what follows we make use of parametrised versions of these formulas. More precisely, for a parameter tuple  $\vec{z}$  of length  $r \geq 0$  we write  $\text{Tp}_k^\ell[\vec{z}](\vec{x}, \vec{y})$  to denote the formula  $\text{Tp}_{r+k}^\ell(\vec{z}\vec{x}, \vec{z}\vec{y})$  (of course this only makes sense if  $r+k \leq \ell$ ). Note that, again, this formula orders  $k$ -tuples up to  $\equiv^\ell$ -equivalence, but now we consider  $\equiv^\ell$ -equivalence with respect to the additional parameter tuple  $\vec{z}$ . Hence for every structure  $\mathfrak{A}$  and every  $\vec{c} \in A^r$  we have that the linear preorder defined by  $\text{Tp}_k^\ell[\vec{c}]$  in  $\mathfrak{A}$  refines the linear preorder defined by  $\text{Tp}_k^\ell$  in  $\mathfrak{A}$ . Note that, in particular, the tuple  $\vec{c}$  will always be in a singleton class according to the preorder  $\text{Tp}_k^\ell[\vec{c}]$ . Note further that for the special case  $r=0$  we just obtain the formula  $\text{Tp}_k^\ell$ .

Given a structure  $\mathfrak{A}$  with automorphism  $\Gamma$ , we denote for a parameter  $\vec{c} \in A^r$  by  $\Gamma_{\vec{c}} \leq \Gamma$  the stabiliser subgroup of the tuple  $\vec{c}$ , i.e. the group of all  $\pi \in \Gamma$  such that  $\pi(\vec{c}) = \vec{c}$ .

**Definition 5.14.** A structure  $\mathfrak{A}$  with automorphism group  $\Gamma$  is called  $(\ell, p)$ -cyclic, for  $\ell \geq 1$  and  $p \in \mathbb{P}$ , if the following holds for every  $k \geq 1$ :

- (C-I)  $\Gamma$  is an Abelian  $p$ -group. In particular, for every  $k$ -tuple  $\vec{a} \in A^k$ , the size of the orbit  $\Gamma(\vec{a})$  of  $\vec{a}$  is a  $p$ -power, that is  $|\Gamma(\vec{a})| = p^n$  for some  $n \geq 0$ .
- (C-II) For every  $\vec{c} \in A^r$ ,  $r \geq 0$ , the FPC-formula  $\text{Tp}_k^{\ell, (k+r)}[\vec{c}]$  defines a total preorder on  $A^k$  such that two tuples  $\vec{a}, \vec{b} \in A^k$  are incomparable if, and only if,  $\Gamma_{\vec{c}}(\vec{a}) = \Gamma_{\vec{c}}(\vec{b})$ . Note that for  $r=0$  we obtain  $\ell$ -homogeneity as a special case.

We say that a class  $\mathcal{K}$  of structures is  $(\ell, p)$ -cyclic if every structure  $\mathfrak{A} \in \mathcal{K}$  is  $(\ell, p)$ -cyclic.

Actually, if in the above definition, we would only include item (C-II), then the resulting notion of  $(\ell, p)$ -cyclic structures would not be very interesting: it would collapse to the notion of  $\ell$ -homogeneity, see Theorem 5.16 below. However, in combination with condition (C-I), we get a remarkable effect:

**Lemma 5.15.** *In Definition 5.14, we can add the following to item (C-II) without changing the resulting notion: Assume that  $r \leq k$ . Then the preorder defined by  $\text{Tp}_k^{\ell, (k+r)}[\vec{c}]$  induces a linear order on the orbit  $\Gamma(\vec{c})$  of the parameter  $\vec{c}$ .*

*Proof.* This follows from the fact that, by (C-I),  $\Gamma$  is an Abelian group (and so the induced group action on the orbit is regular). More explicitly, assume that for some  $\pi \in \Gamma$  and  $(\vec{c}, \vec{a}) \in \{\vec{c}\} \times \Gamma(\vec{c})$  it holds that  $\pi(\vec{c}, \vec{a}) = (\vec{c}, \vec{b})$ . Choose  $\sigma \in \Gamma$  such that  $\sigma(\vec{a}) = \vec{c}$ . Then  $\pi(\sigma(\vec{a})) = \vec{c}$ . Since  $\Gamma$  is Abelian, it follows that  $\sigma(\pi(\vec{a})) = \vec{c}$ . Hence  $\pi(\vec{a}) = \vec{a}$ , which yields  $\vec{a} = \vec{b}$ . Hence, it follows that for every  $\vec{a} \in \Gamma(\vec{c})$  we have  $|\Gamma_{\vec{c}}(\vec{a})| = 1$ . Having this, item (C-II) implies that  $\text{Tp}_k^{\ell, (k+r)}[\vec{c}]$  defines a linear order on  $\{\vec{c}\} \times \Gamma(\vec{c})$ , as claimed.  $\square$

**Theorem 5.16.** *Let  $\mathfrak{A}$  be  $\ell$ -homogeneous and assume that the automorphism group  $\Gamma$  of  $\mathfrak{A}$  is an Abelian  $p$ -group. Then  $\mathfrak{A}$  is  $(\ell, p)$ -cyclic. In particular, there is  $\ell \geq 1$  such that the classes  $\text{CFI}[\mathcal{F}; p]$  are  $(\ell, p)$ -cyclic for all  $p \in \mathbb{P}$ .*

*Proof.* We already analysed the automorphism groups of CFI-structures  $\mathfrak{A} \in \text{CFI}[\mathcal{F}; p]$  in Section 5.2. In particular, we saw that these groups are elementary Abelian  $p$ -groups, so

property (C-I) holds for CFI-structures in  $\text{CFI}[\mathcal{F}; p]$ . Moreover, Corollary 5.12 tells us that classes of CFI-structures are homogeneous.

Now, let  $\ell \geq 1$  and let  $\mathfrak{A}$  be  $\ell$ -homogeneous with automorphism group  $\Gamma$ . Then, by Theorem 5.13, we know that for every  $k \geq 1, r \geq 0$  the formula  $\text{Tp}_{(r+k)}^{\ell \cdot (r+k)}(\vec{x}_1 \vec{x}_2, \vec{y}_1 \vec{y}_2)$  defines in  $\mathfrak{A}$  a total preorder on  $A^{r+k}$  which order  $(r+k)$ -tuples up to  $\Gamma$ -orbits. Since

$$\text{Tp}_k^{\ell \cdot (r+k)}[\vec{z}](\vec{x}, \vec{y}) = \text{Tp}_{(r+k)}^{\ell \cdot (r+k)}(\vec{z}\vec{x}, \vec{z}\vec{y}),$$

we know that for every  $\vec{c} \in A^r$  it holds that the total preorder  $\text{Tp}_k^{\ell \cdot (r+k)}[\vec{c}]$  distinguishes  $k$ -tuples  $\vec{a}, \vec{b} \in A^k$  if, and only if,  $\Gamma(\vec{c}\vec{a}) \neq \Gamma(\vec{c}\vec{b})$ . But this last condition is indeed equivalent to  $\Gamma_{\vec{c}}(\vec{a}) \neq \Gamma_{\vec{c}}(\vec{b})$ , which completes the proof.  $\square$

Our next aim is to show that the class of  $(\ell, p)$ -cyclic structures is closed under FPC-transformations. Unfortunately, stated in this very general form, this claim is clearly wrong. For example, FPC-transformations can easily generate each fixed finite structure (starting from any structure), and so the resulting structures will not have Abelian automorphism groups for instance (which is one of the requirements for being  $(\ell, p)$ -cyclic). However, as we will show next, one can extend each FPC-interpretation  $\mathcal{I}$  to an FPC-interpretation  $\text{Norm}(\mathcal{I})$  in such a way that the original input structure is preserved as a substructure. This will enable us to maintain the property of being  $(\ell, p)$ -cyclic.

Let us be a bit more precise. As said, instead of only interpreting  $\mathcal{I}(\mathfrak{A})$  in  $\mathfrak{A}$  we want to interpret the structure  $\text{Norm}(\mathcal{I})(\mathfrak{A}) = \mathcal{I}(\mathfrak{A}) \uplus \mathfrak{A}$  in  $\mathfrak{A}$ , that is the disjoint union of the original structure  $\mathfrak{A}$  and the interpreted structure  $\mathcal{I}(\mathfrak{A})$ . However, as such, this is not sufficient since we can still get new automorphisms due to the new substructure  $\mathcal{I}(\mathfrak{A})$ . To overcome this problem, we create additional relations that indicate from which elements in  $\mathfrak{A}$  the newly created elements in  $\mathcal{I}(\mathfrak{A})$  originate. Note that the elements in  $\mathcal{I}(\mathfrak{A})$  are equivalence classes of tuples of elements from  $\mathfrak{A}$  and we will encode this information in  $\text{Norm}(\mathcal{I})(\mathfrak{A})$ . Formally, our result is as follows.

**Theorem 5.17.** *Let  $\mathcal{I}(\vec{z}) \in \text{FPC}[\sigma \rightarrow \tau, \vec{z}]$  be an FPC-interpretation of dimension  $d$  and with  $r \geq 0$  parameters  $\vec{z}$ ,  $|\vec{z}| = r$ , that maps  $\sigma$ -structures to  $\tau$ -structures. Let  $\hat{\tau} = \sigma \uplus \tau \uplus \{\in\}$  (where  $\in$  is a fresh binary relation symbol). Then there exists an FPC-interpretation  $\mathcal{J}(\vec{z}) \in \text{FPC}[\sigma \rightarrow \hat{\tau}, \vec{z}]$  such that for every structure  $\mathfrak{A}$  and every  $\vec{a} \in \text{dom}(\mathfrak{A}, \vec{z})$  the following holds for  $\mathfrak{B} = \mathcal{I}(\mathfrak{A}, \vec{a})$  and  $\mathfrak{C} = \mathcal{J}(\mathfrak{A}, \vec{a})$ :*

- (i) *the dimension of  $\mathcal{J}(\vec{z})$  is at most  $d + 3$ , and*
- (ii)  *$\mathfrak{B} \subseteq_{\text{FO}} \mathfrak{C}|_{\tau}$ , that is  $\mathfrak{B}$  is an FO-definable substructure of the reduct of  $\mathfrak{C}$  to  $\tau$ , and*
- (iii) *if  $\Gamma = \text{Aut}(\mathfrak{A}, \vec{a})$  and  $\Delta = \text{Aut}(\mathfrak{C})$ , then  $\Gamma \cong \Delta$  (that is, up to isomorphism, the automorphism group of the input structure  $(\mathfrak{A}, \vec{a})$  is preserved), and*
- (iv) *if  $\mathfrak{A}$  is  $\ell$ -homogeneous (for  $\ell \geq 3$ ), then the structure  $\mathfrak{C}$  is  $\ell \cdot (d + r)$ -homogeneous.*

*Proof.* Let  $\mathcal{I}(\vec{z})$  be  $d$ -dimensional with domain formula  $\varphi_{\delta}(\vec{x}, \vec{z})$  and congruence formula  $\varphi_{\approx}(\vec{x}_1, \vec{x}_2, \vec{z})$ . Let  $\mathfrak{A}$  be a  $\sigma$ -structure and let  $\vec{a} \in \text{dom}(\mathfrak{A}, \vec{z})$ . The elements of the interpreted structure  $\mathfrak{B} = \mathcal{I}(\mathfrak{A}, \vec{a})$  are equivalence classes of tuples in  $\text{dom}(\mathfrak{A}, \vec{z})$ . The idea of the construction of  $\mathfrak{C}$  is as follows. The universe of  $\mathfrak{C}$  consists of four different sorts  $U_A, U_T, U_B, U_N$ . The first sort  $U_A$  contains elements that represent the elements in the universe of the original structure  $\mathfrak{A}$ . The second sort contains elements to represent all elements in  $\text{dom}(\mathfrak{A}, \vec{z})$  that are selected by  $\varphi_{\delta}$  and, furthermore, a unique element that is used in order to encode the parameter tuple  $\vec{a}$ . Also  $U_T$  contains auxiliary elements to encode the structure of tuples, that

is the individual entries. The third sort  $U_B$  contains elements to represent the elements of the structure  $\mathfrak{B}$ , that is the equivalence classes  $[\vec{b}] = \{\vec{c} \in \text{dom}(\mathfrak{A}, \vec{x}) : \mathfrak{A} \models \varphi_\delta(\vec{c}, \vec{a}) \wedge \varphi_\approx(\vec{b}, \vec{c}, \vec{a})\}$  for  $\vec{b} \in \text{dom}(\mathfrak{A}, \vec{x})$  with  $\mathfrak{A} \models \varphi_\delta(\vec{b}, \vec{a})$ . The last sort  $U_N$  is an auxiliary sort which holds a sufficient amount of numbers (that is a linearly ordered set) to represent the different sorts, their relations, the indices for tuples, and so on. The binary relation symbol  $\in$  is used to relate the different sorts and to encode the tuple structure. Relations in  $\sigma$  and  $\tau$  are interpreted on the respective sorts  $U_A$  and  $U_B$  as in  $\mathfrak{A}$  and  $\mathfrak{B}$ , respectively.

Let us elaborate more on some technical details (we remark that, as usual, there are many different ways to formalise an appropriate encoding; in order to verify the properties of  $\mathcal{J}$ , we describe one of them). First of all, we extend the dimension of  $\mathcal{I}$  by three additional components  $(\mu, \nu, x)$  where the first two variables  $\mu, \nu$  range over the number sort and where  $x$  ranges over the vertex sort (we remark that it would be sufficient to increase the dimension by at most one numeric component, but this would unnecessarily make the following description more complicated). In general, we will use the first component  $\mu$  to address different sorts. For instance, let us start with the number sort  $U_N$ . We can use the congruence formula to merge all tuples  $(0, \mu, d, \vec{b})$  and  $(0, \mu, d', \vec{c})$  (for  $d, d' \in A$  and  $\vec{b}, \vec{c} \in \text{dom}(\mathfrak{A}, \vec{x})$ ) and then use the resulting set  $\{(0, 0, \star), (0, 1, \star), (0, 2, \star), \dots\}$  to encode the elements in  $U_N$ . Hereby, we choose the range of the numeric variable  $\nu$  larger than the range of any other numeric variable which occurs in the interpretation  $\mathcal{I}$ . To identify the numeric sort  $U_N$  in the resulting structure we define a linear order on  $U_N$  using the new relation symbol  $\in$ . As a second step, we encode elements  $a \in A$  of the original structure  $\mathfrak{A}$  in  $\mathfrak{C}$  by using elements of the form  $(1, \star, a, \star)$  (as before, the  $\star$ 's in this notation indicate that we use the congruence formula to merge all elements with different  $\star$ -components). To identify the first sort  $U_A$  in the resulting structure  $\mathfrak{C}$ , we draw an  $\in$ -edge from the first element in the number sort  $U_N$  to all elements in the first sort  $U_A$ . Of course, we define all the relation symbols in  $\sigma$  on  $U_A$  by copying their definition from  $\mathfrak{A}$ .

Thirdly, to encode the elements in  $\text{dom}(\mathfrak{A}, \vec{x})$  and the parameter tuple  $\vec{a}$  we proceed in two steps. First of all, for every index  $1 \leq i \leq \max(k, r)$ , we introduce component elements  $(i, a)$ ,  $a \in A$ , and  $(i, m)$ ,  $m < \text{dom}(\nu)$ , to represent all possible components of tuples in  $\text{dom}(\mathfrak{A}, \vec{x})$ . Formally, we encode them in  $\mathfrak{C}$  by using elements of the form  $(2 + i, \star, a, \star)$  and  $(2 + i, m, \star)$ . To identify them in  $\mathfrak{C}$ , we mark them in a similar way as before, i.e. we introduce  $\in$ -edges from position  $2 + i$  in the number sort to all component elements  $(i, a)$  and  $(i, m)$ . We also connect all component elements  $(i, a)$  and  $(i, m)$  to their respective values  $a$  and  $m$  via  $\in$ -edges (which point from component elements to the sorts  $U_A$  and  $U_N$ ). We proceed to represent all tuples in  $\text{dom}(\mathfrak{A}, \vec{x})$  using the original components of the interpretation  $\mathcal{I}$ , that is we use elements  $(2 + \max(k, r) + 1, \star, \star, \vec{b})$  where  $\vec{b} \in \text{dom}(\mathfrak{A}, \vec{x})$  and mark them appropriately. We additionally connect tuples  $(2 + \max(k, r) + 1, \star, \star, \vec{b})$  with their matching component elements, that is with  $(i, b_i)$ . We then use  $\varphi_\delta$  to select those tuples in  $\text{dom}(\mathfrak{A}, \vec{x})$  that are in the domain of  $\mathcal{I}$ . Also, we add one further special tuple element, say encoded as  $(2 + \max(k, r) + 1, 0, \star)$ , which is meant to encode the parameter tuple  $\vec{a}$ . This special element is thus connected to all component elements  $(i, a_i)$ . Finally, we make an additional copy of all tuple elements  $(2 + \max(k, r) + 2, \star, \star, \vec{b})$  that we added, and use  $\varphi_\delta$  to merge them according to  $\mathcal{I}$ . This will give us the sort of elements  $U_B$  that we use in order to represent the elements of  $\mathfrak{B}$ . We mark them appropriately in the same way as we did for the other sorts. Recall that these elements are equivalence classes of elements in  $\text{dom}(\mathfrak{A}, \vec{x})$ , hence we additionally connect them, with  $\in$ -edges, to their representatives in

the tuple sort  $U_T$ . We define the relations in  $\tau$  on  $U_B$  according to  $\mathcal{I}$ , that is we copy them from  $\mathfrak{B} = \mathcal{I}(\mathfrak{A}, \vec{a})$ .

From this description it is easy to see that all of the required transformations can be expressed by an FPC-interpretation  $\mathcal{J}$  with dimension at most  $d + 3$  (and while we increase the number of variables by a constant number only). Also, it should be clear that item (ii) holds as we can very easily define the different sorts in the resulting structure  $\mathfrak{C}$  (in particular, the sort  $U_B$  is the maximal sort according to our linear order on  $U_N$ ). Let us now consider item (iii). The main observation is that each automorphism  $\pi \in \Delta$  of  $\mathfrak{C}$  is uniquely defined by its projection on the sort  $U_A$ . Indeed, this directly follows from the way in which we constructed  $\mathfrak{C}$  using the new relation symbol  $\in$ . First note that no automorphism of  $\mathfrak{C}$  can move elements in the numeric sort  $U_N$ , since  $\in$  defines a linear order on  $U_N$ . In particular it follows that all sorts  $U_N, U_A, U_B, U_T$  are preserved. Secondly, assume that we have a permutation  $\pi$  on  $U_A$  that can be extended to an automorphism of  $\mathfrak{C}$ . Since the  $\sigma$ -relations on  $U_A$  in  $\mathfrak{C}$  coincide with the relations in  $\mathfrak{A}$ , we know that  $\pi$  is an automorphism of  $\mathfrak{A}$ . Moreover, to obtain an automorphism of  $\mathfrak{C}$ , there is only one unique way in which we can extend  $\pi$  to the tuple sort  $U_T$  and the sort  $U_B$  encoding the universe of  $\mathfrak{B}$ . Indeed, the  $\in$ -edges enforce that tuple components  $(i, a)$  are moved to  $(i, \pi(a))$  (and tuple components  $(i, m)$  cannot be moved) and, accordingly, that tuples  $\vec{b}$  in  $U_T$  are moved to  $\pi(\vec{b})$ . In particular, for the special tuple  $\vec{a}$  this means that we have  $\pi(\vec{a}) = \vec{a}$ . Finally, since  $\pi$  extends uniquely to the tuple sort  $U_T$ , it also uniquely extends to the sort  $U_B$  of elements of the interpreted structure  $\mathfrak{B} = \mathcal{I}(\mathfrak{A}, \vec{a})$ . Indeed, the elements in  $U_B$  are sets  $[\vec{b}]$  of tuples  $\vec{b} \in U_T$ , and we have connected these sets with the elements they contain using  $\in$ -edges in  $\mathfrak{C}$ . Hence, for each equivalence class  $[\vec{b}] \in U_B$  for  $\vec{b} \in U_T$  we have  $\pi([\vec{b}]) = [\pi(\vec{b})]$ . So altogether, we can conclude that the extension of  $\pi$  from  $U_A$  to the other sorts  $U_N, U_T, U_B$  is unique. On the other hand, note that if  $\pi$  is an automorphism of  $\mathfrak{A}$  satisfying  $\pi(\vec{a}) = \vec{a}$ , then the resulting extended  $\pi$  is indeed an automorphism of  $\mathfrak{C}$ . To see this, note that all relations in  $\tau$  and  $\sigma$  are preserved under automorphisms of  $(\mathfrak{A}, \vec{a})$ : for the  $\sigma$ -relations this follows from the assumption that  $\pi$  is an automorphism of  $\mathfrak{A}$ , and for the  $\tau$ -relation it follows from the fact that they are defined by the FPC-interpretation  $\mathcal{I}$  in  $(\mathfrak{A}, \vec{a})$ . This shows that  $\text{Aut}(\mathfrak{A}, \vec{a}) \cong \text{Aut}(\mathfrak{C})$ .

Finally, let us consider item (iv). Assume that  $\mathfrak{A}$  is  $\ell$ -homogeneous, for  $\ell \geq 3$ , and let  $k \geq 1$ . Let  $\vec{c} = (c_1, \dots, c_k)$  and  $\vec{d} = (d_1, \dots, d_k)$  be two  $k$ -tuples of elements in  $\mathfrak{C}$ . We have to show that if  $(\mathfrak{C}, \vec{c}) \equiv^{\ell \cdot (d+r) \cdot k} (\mathfrak{C}, \vec{d})$ , then there exists an automorphism  $\pi$  of  $\mathfrak{C}$  such that  $\pi(\vec{c}) = \vec{d}$ . The main observation is that each element in  $\mathfrak{C}$  is  $d$ -supported by elements of  $\mathfrak{A}$ , that is the element can be defined in FPC in  $\mathfrak{C}$  using at most  $d$  parameters from  $U_A$ . More precisely, for every element  $c$  of  $\mathfrak{C}$  there exist at most  $d$ -many elements  $s_1, \dots, s_d \in A = U_A$  for which there exists an FPC-formula  $\psi(x, y_1, \dots, y_d)$  with at most  $(d+3)$  many variables such that  $\psi(x, s_1, \dots, s_d)$  defines  $c$  in the structure  $\mathfrak{C}$ . For instance, for tuples  $c = (b_1, \dots, b_d)$  in  $U_T$ , we can choose  $s_1, \dots, s_d$  to be the components  $b_1, \dots, b_d$  of the tuple, and for elements  $c = [\vec{b}] \in U_B$  we can choose the components of some representative. Hence, if  $(\mathfrak{C}, \vec{c}) \equiv^{\ell \cdot (d+r) \cdot k} (\mathfrak{C}, \vec{d})$ , then in particular we can find two supports  $s(\vec{c}) \in U_A^{d \cdot k}$  for  $\vec{c}$  and  $s(\vec{d}) \in U_A^{d \cdot k}$  for  $\vec{d}$  such that  $(\mathfrak{C}, s(\vec{c})) \equiv^{\ell \cdot (d+r) \cdot k} (\mathfrak{C}, s(\vec{d}))$ . Moreover, since  $\vec{a}$  is FPC-definable in  $\mathfrak{C}$  (each component of the tuple  $\vec{a}$  is definable using at most three variables), we can conclude that  $(\mathfrak{C}, s(\vec{c}), \vec{a}) \equiv^{\ell \cdot (d+r) \cdot k} (\mathfrak{C}, s(\vec{d}), \vec{a})$ . Since  $\mathfrak{A}$  is  $\ell$ -homogeneous, we know that  $\text{Tp}_{d \cdot k}^{\ell \cdot (d+k+r)}[\vec{a}]$  defines a total preorder on  $(d \cdot k)$ -tuples in  $\mathfrak{A}$  which orders tuples up to

orbits with respect to  $\text{Aut}(\mathfrak{A}, \vec{a}) \cong \text{Aut}(\mathfrak{C})$ . Since  $\mathfrak{A}$  is a definable substructure of  $\mathfrak{C}$  we know that  $(\mathfrak{A}, s(\vec{c}), \vec{a}) \equiv^{\ell \cdot (d+r) \cdot k} (\mathfrak{A}, s(\vec{d}), \vec{a})$ . It follows that we can find a  $\pi \in \text{Aut}(\mathfrak{C})$  such that  $\pi(s(\vec{c})) = s(\vec{d})$ . Since supports uniquely describe elements, we conclude that  $\pi(\vec{c}) = \vec{d}$ , as claimed.  $\square$

In what follows, for a given FPC-interpretation  $\mathcal{I}(\vec{z}) \in \text{FPC}[\sigma \rightarrow \tau, \vec{z}]$ , we denote the interpretation  $\mathcal{J}(\vec{z})$  as constructed in Theorem 5.17 by  $\text{Norm}(\mathcal{I})(\vec{z})$ .

**Corollary 5.18.** *For every FPC-interpretation  $\mathcal{I}(\vec{z})$ , there exists  $\ell \geq 1$  such that the class of structures  $\{\text{Norm}(\mathcal{I})(\mathfrak{A}, \vec{a}) : \mathfrak{A} \in \text{CFI}[\mathcal{F}; p], \vec{a} \in A^r\}$  is  $(\ell, p)$ -cyclic.*

Intuitively we showed that the class of  $(\ell, p)$ -cyclic is closed under FPC-interpretations (which, if stated precisely, means that we have to rewrite the interpretations in normal form and we have to increase the homogeneity constant by a factor depending on the dimension and parameter length of the specific interpretation). We end this section by stating a much simpler observation. Assume that we have two  $(\ell, p)$ -cyclic-structures  $\mathfrak{A}$  and  $\mathfrak{B}$  of the same vocabulary  $\tau$ . Then the ordered pair  $(\mathfrak{A}, \mathfrak{B})$  is a  $(\ell, p)$ -cyclic-structure as well. Of course, to some extent this depends on the technical details on how we implement ordered pairs as relational structure. The most important property is that, in the ordered pair  $(\mathfrak{A}, \mathfrak{B})$ , we have a simple means to identify the two substructures  $\mathfrak{A}$  and  $\mathfrak{B}$ , for instance by using additional predicate symbols to identify the two universes  $A$  and  $B$ . In this article, we agree to understand ordered pairs in this way. The consequence is that the automorphism group of  $(\mathfrak{A}, \mathfrak{B})$  is just the direct product of the automorphism groups of  $\mathfrak{A}$  and  $\mathfrak{B}$ . In particular, orbits of (mixed) tuples in  $(\mathfrak{A}, \mathfrak{B})$  can be described in terms of the respective subtuples in  $\mathfrak{A}$  and  $\mathfrak{B}$ . Having this, we can easily see that the following holds.

**Theorem 5.19.** *Let  $\mathfrak{A}$  and  $\mathfrak{B}$  be two  $(\ell, p)$ -cyclic structures of vocabulary  $\tau$ . Then the ordered pair  $(\mathfrak{A}, \mathfrak{B})$  is an  $(\ell, p)$ -cyclic structure as well.*

**5.6. Solving Cocyclic Linear Equation Systems.** If we want to express  $k$ -dimensional PC-refutations over a field  $\mathbb{F}$  in FPC, then we need to be able to define solution spaces of linear equation systems over that field  $\mathbb{F}$  in FPC, see Figure 2. In Section 4.5 we proved that FPC can define solution spaces of linear equation systems over  $\mathbb{Q}$ , and this was the key to showing that FPC can express  $k$ -dimensional PC-refutations over  $\mathbb{Q}$  with polynomial bit-complexity, cf. Theorem 4.9. Hence, in order to prepare our main result of this section (Theorem 5.25), we are now going to show that FPC can define solution spaces of linear equation systems over a finite field  $\mathbb{F}$  of characteristic  $q$  under the assumption that these systems are interpreted in a class of  $(\ell, p)$ -cyclic-structures with  $q \neq p$ . Moreover, we show that the number of required variables is bounded linearly in  $\ell$  (with a constant factor that only depends on the initial interpretation). Note that our assumption  $q \neq p$  is crucial: the CFI-problem over  $\mathbb{F}_p$  cannot be expressed in FPC, but it can be reduced (in first-order logic) to the solvability problem of linear equation systems over  $\mathbb{F}_p$ .

For our proof we make use of a key idea from [28]: in the special situation that we consider here, it turns out that (solvable) linear equation system always have symmetric solutions, that is solutions which are invariant under all automorphisms of the underlying linear equation systems. Together with the property of homogeneity this observation allows us to show that FPC can define such symmetric solutions, see [28]. In this article, we go one important step further. We not only show that, in this particular setting, we can define the

*Boolean* solvability problem for linear equation systems in FPC, but that we can also define the more general *functional* problem of expressing solution spaces of given linear equation systems.

Let us remark that our results here extend our approach from [28] in another crucial way. In [28] we considered the CFI-construction with respect to underlying graphs of unbounded degree. The reason is that, if we work with such underlying graphs, then this considerably simplifies the proof of the homogeneity property for CFI-structures. Here, in contrast, we consider the “full” power of the CFI-construction, that is with respect to a family of underlying three-regular expander graphs. This has the effect that we get much better lower bounds on the number of variables, and this makes our separation results even stronger. That is to say that the techniques that we develop here can readily be used in order to strengthen our separation results from [28] to formulas with a sublinear number of variables (rather than a constant number as we considered in [28]).

As usual, in order to talk about systems of linear equations over finite fields in the context of logical definability, we first have to agree on an encoding of such systems as finite relational structures. Again, the concrete choice does not matter, so we do not specify such an encoding explicitly. Let us rather go through some notation that we use in this section. We consider (unordered) matrices  $M$  over a finite field  $\mathbb{F}$  as mappings  $M: I \times J \rightarrow \mathbb{F}$  for two (non-empty) index sets  $I$  and  $J$ . An (unordered) vector  $v$  over a finite field  $\mathbb{F}$  is a mapping  $v: I \rightarrow \mathbb{F}$ . A linear equation system  $M \cdot x = b$  over a finite field  $\mathbb{F}$  is specified by an  $I \times J$ -coefficient matrix  $M$  over  $\mathbb{F}$  and an  $I$ -constants vector  $b: I \rightarrow \mathbb{F}$ . We usually think of the finite field  $\mathbb{F}$  as being part of the input. We are primarily interested in the setting where the characteristic  $q = \text{char}(\mathbb{F})$  of this field  $\mathbb{F}$  and the prime  $p$  for CFI-class  $\text{CFI}[\mathcal{F}; p]$  are distinct:

**Definition 5.20.** Let  $\ell \geq 1$ . We say that a  $\tau$ -structure  $\mathfrak{A}$  contains an  $\ell$ -cocyclic vector, (or matrix, or linear equation system) over a finite field  $\mathbb{F}$  if

- the structure  $\mathfrak{A}$  is  $(\ell, p)$ -cyclic for some prime  $p \in \mathbb{P}$ , and
- for some distinguished relation symbol  $S \in \tau$ , the substructure of  $\mathfrak{A}$  induced on  $S$  is (the structural encoding of) a vector  $v: I \rightarrow \mathbb{F}$  (or matrix  $M: I \times J \rightarrow \mathbb{F}$ , or linear equation system  $M \cdot x = b$ ) over the finite field  $\mathbb{F}$  with characteristic different from  $p$ , that is  $\text{char}(\mathbb{F}) = q$  for some  $q \in \mathbb{P}$ ,  $p \neq q$ .

We proceed to show that FPC can express solution spaces of  $\ell$ -cocyclic linear equation systems using  $\mathcal{O}(\ell)$  many variables only. The proof consists of two steps. First of all, we show that FPC can define a single solution of a (solvable)  $\ell$ -cocyclic linear equation system (Theorem 5.21). In a second step we then show that FPC can also define (small) generating sets for kernels of  $\ell$ -cocyclic matrices (Theorem 5.22). By putting these two results together, we obtain the desired result. The main idea for this second step is to repeatedly make use of the FPC-formula from Theorem 5.21 for solving  $\ell$ -cocyclic linear equation systems and the fact that  $(\ell, p)$ -cyclic structures can be linearly ordered locally in FPC.

**Theorem 5.21.** *For every  $\ell \geq 1$  there exists an FPC-formula  $\varphi$  with  $\mathcal{O}(\ell)$  many variables such that  $\varphi$  defines in every structure  $\mathfrak{A}$  that contains a solvable  $\ell$ -cocyclic linear equation system  $M \cdot x = b$  over a finite field  $\mathbb{F}$ , where  $M: I \times J \rightarrow \mathbb{F}$  and  $b: I \rightarrow \mathbb{F}$ , a solution to  $M \cdot x = b$ , that is  $\varphi$  defines a vector  $v: J \rightarrow \mathbb{F}$  such that  $M \cdot v = b$  (and, if  $M \cdot x = b$  is not solvable, then, by convention,  $\varphi$  defines the all-0-vector in  $\mathfrak{A}$ ).*

*Proof.* Let  $\mathfrak{A}$  be an  $(\ell, p)$ -cyclic structure which contains a linear equation system  $M \cdot x = b$  for a matrix  $M: I \times J \rightarrow \mathbb{F}$  and a constants vector  $b: I \rightarrow \mathbb{F}$  over a finite field  $\mathbb{F}$  of characteristic  $\text{char}(\mathbb{F}) = q$ ,  $p \neq q$ . Let  $\Gamma = \text{Aut}(\mathfrak{A})$  denote the automorphism group of  $\mathfrak{A}$ . Then  $\Gamma$  acts on the solution space of  $M \cdot x = b$ . We know that this space (in case that it is non-empty) has size  $q^i$  for some  $i \geq 0$ , since we are dealing with a linear equation system over a field of characteristic  $q \in \mathbb{P}$ . On the other hand, recall that  $\Gamma$  is a  $p$ -group which means that each orbit of the action of  $\Gamma$  on the solution space of  $M \cdot x = b$  has size  $p^j$  for some  $j \geq 0$ . We conclude that there has to be at least one orbit of size one. This, however, means that there is a solution  $v: J \rightarrow \mathbb{F}$  such that  $\pi(v) = v$  for all  $\pi \in \Gamma$ . We call a vector  $v: J \rightarrow \mathbb{F}$  which satisfies this property *symmetric*. Note that a symmetric vector  $v: J \rightarrow \mathbb{F}$  is constant on the orbits induced by  $\Gamma$  on the set  $J$  since  $\pi(v)(j) = v(\pi^{-1}(j))$ . By our assumption that  $\mathfrak{A}$  is  $(\ell, p)$ -cyclic, we know that the formula  $\text{Tp}_1^\ell(x, y)$  defines a linear preorder  $\preceq$  on  $J$  which linearly orders  $J$  up to  $\Gamma$ -orbits. Moreover, recall that this FPC-formula  $\text{Tp}_1^\ell$  only uses  $\mathcal{O}(\ell)$  variables. Let  $J = J_0 \preceq J_1 \preceq \dots \preceq J_{n-1}$  denote the  $\Gamma$ -orbit partition of  $J$ .

For  $0 \leq i < n$  let  $t_i: J \rightarrow \mathbb{F}$  denote the  $J$ -vector which is the identity on the  $i$ -th  $J$ -orbit, that is  $t_i(j) = 1$  for  $j \in J_i$  and  $t_i(j) = 0$  for  $j \notin J_i$ . Let  $T$  denote the  $J \times \{0, \dots, n-1\}$ -matrix which has  $t_i$  as its  $i$ -th column. Then for every symmetric  $v: J \rightarrow \mathbb{F}$  we can find a vector  $w: \{0, \dots, n-1\} \rightarrow \mathbb{F}$  such that  $Tw = v$ . Indeed, just choose  $w(i) = v(j)$  for (some)  $j \in J_i$ . We conclude, that the linear equation system  $M \cdot x = b$  is solvable if, and only if, the system  $M \cdot T \cdot x = b$  is solvable. Clearly, every solution of  $M \cdot T \cdot x = b$  gives rise to a solution of  $M \cdot x = b$ . Hence, it suffices to define a solution of  $M \cdot T \cdot x = b$  in fixed-point logic with counting. However, this is very easy because  $M \cdot T$  is an  $I \times \{0, \dots, n-1\}$ -matrix which has a linearly ordered set of columns. Moreover, if we drop duplicates of rows, then the order on the columns also induces a (first-order definable) linear order on the rows, namely the lexicographical ordering (note that there exists an FO-definable order on the finite field  $\mathbb{F}$ ). It follows by the Immerman-Vardi Theorem that fixed-point logic can define a solution of the system  $M \cdot T \cdot x = b$  or determine that the original system was not solvable. This solution can be lifted to a solution of  $M \cdot x = b$  by multiplying by  $T$ . Finally, observe that the number of variables in the resulting formula is independent of  $\ell$  except for the subformula  $\text{Tp}_1^\ell$  which defines the linear order on the orbit-partition of  $J$ . Hence, the required number of variables is indeed  $\mathcal{O}(\ell)$ .  $\square$

**Theorem 5.22.** *For every  $\ell \geq 1$  there exists an FPC-formula  $\varphi$  with  $\mathcal{O}(\ell)$  variables which defines in every structure  $\mathfrak{A}$  that contains an  $\ell$ -cocyclic matrix  $M: I \times J \rightarrow \mathbb{F}$  over a finite field  $\mathbb{F}$ , a matrix  $\varphi^{\mathfrak{A}}: J \times (J \times |J|) \rightarrow \mathbb{F}$  such that  $\text{im}(\varphi^{\mathfrak{A}}) = \ker(M)$ .*

*Proof.* Let  $\mathfrak{A}$  be an  $(\ell, p)$ -cyclic structure with automorphism group  $\Gamma = \text{Aut}(\mathfrak{A})$ , and assume that  $\mathfrak{A}$  contains a matrix  $M: I \times J \rightarrow \mathbb{F}$  over a finite field  $\mathbb{F}$  of characteristic  $\text{char}(\mathbb{F}) = q \neq p$ . First of all, we again use the formula  $\text{Tp}_1^\ell(x, y)$  to define a total preorder  $\preceq$  on  $J$  which orders the indexing elements in  $J$  up to  $\Gamma$ -orbits. Let  $J = J_0 \preceq J_1 \preceq \dots \preceq J_{n-1}$ . Recall that  $\text{Tp}_1^\ell$  is an FPC-formula with  $\mathcal{O}(\ell)$  variables. Our plan is as follows. We aim to define a generating set for  $\ker(M)$  which consists of  $i$ -homogeneous vectors for  $0 \leq i < n-1$ . Here we say that a vector  $v: J \rightarrow \mathbb{F}$  is  $i$ -homogeneous if  $v(j) = 0$  for all  $j \in J_{i'}$  for  $i' < i$ . That is an  $i$ -homogeneous vector is zero on all  $\Gamma$ -orbits on  $J$  which precede the  $i$ -th orbit  $J_i$ . Our plan is to define in FPC, for every  $0 \leq i < n$ , sets  $K_i$  consisting of  $i$ -homogeneous vectors  $v: J \rightarrow \mathbb{F}$ ,  $v \in \ker(M)$ , such that the projections of  $K_i$  to  $J_i$  yield generating sets for the projections of  $\ker(M)$  to  $J_i$ , which means that  $\bigcup_{i < n-1} K_i$  is a generating set for  $\ker(M)$ .

We will index the elements in  $K_i$  by elements in  $J_i \times |J|$ . The crucial insight is that  $(\ell, p)$ -cyclic structures satisfy the additional property that for each fixed parameter  $j \in J_i$  the formula  $\text{Tp}_1^{\ell-2}[j](x, y)$  defines a linear order  $<_j$  on  $J_i$ . Hence, if we have a fixed  $j \in J_i$ , then it makes sense to speak of an  $m$ -th echelon vector  $(0, \dots, 0, 1, \star, \star)$  of the projection of  $\ker(M)$  to  $J_i$ . Here an  $m$ -th echelon vector has entry 1 in the  $m$ -th component and is zero at all preceding components (and its length is  $|J_i|$ ). Clearly, for some  $0 \leq m \leq |J_i|$  such a vector may not exist, but if we collect a set of (existing)  $m$ -th echelon vectors for  $0 \leq m \leq |J_i|$ , then we obtain a generating set for the projection of  $\ker(M)$  to  $J_i$ . With this preparation, we can describe our strategy more precisely. The intention is that the  $J$ -vector  $v \in K_i$  that is indexed by  $(j, m)$ ,  $j \in J_i$ ,  $m < |J|$ , represents an  $i$ -homogeneous vector  $v \in \ker(M)$  with the additional property that the projection of  $v$  to  $J_i$  is the  $m$ -th echelon vector of the projection of  $\ker(M)$  to  $J_i$  (if it exists, otherwise we agree to let  $v = 0$ ). Note that in this way we actually include too many vectors in  $K_i$ . Indeed, it would be sufficient to consider all such vectors indexed by  $j \times |J_i|$  for a single  $j \in J_i$ . However, since we cannot choose a particular  $j \in J_i$  we just add all of these vectors for any  $j \in J_i$ . This does not cause any problems, since we do not aim at defining a basis for  $\ker(M)$ , but just at defining a generating set.

It remains to see how we can define such a vector  $v: J \rightarrow \mathbb{F}$  in FPC given parameters  $(j, m) \in J_i \times |J|$ . To this end we make use of Theorem 5.21 and the formulas  $\text{Tp}_1^\ell$  and  $\text{Tp}_1^{\ell-2}[j]$  (both with  $\mathcal{O}(\ell)$  many variables only). Let us start with the homogeneous linear equation system  $M \cdot x = 0$  which defines  $\ker(M)$ . Given the parameters  $(j, m)$ , we now add extra constraints for the variables  $x = (x_j)_{j \in J}$  as follows:

- for  $j' \in \bigcup_{i' < i} J_{i'}$  we set  $x_{j'} = 0$ ,
- for  $J_i = j_1 <_j \dots <_j j_s$ , we set  $x_j = 0$  for  $j \in \{j_1, \dots, j_{m-1}\}$  and  $x_{j_m} = 1$ .

It is clear that the solution space of this linear equation system consists precisely of the  $i$ -homogeneous vectors  $v: J \rightarrow \mathbb{F}$  in  $\ker(M)$  whose projections to  $J_i$  are  $m$ -th echelon vectors (with respect to the order  $<_j$  defined by  $\text{Tp}_1^{\ell-2}[j]$  on  $J_i$ ). Moreover, this system can easily be defined in  $\mathfrak{A}$  using an FPC-formula which uses  $\text{Tp}_1^{\ell-2}[j]$  and  $\text{Tp}_1^\ell$  as subformulas and parameters  $(j, m)$ . We can now make use of Theorem 5.21 to define a solution  $v: J \rightarrow \mathbb{F}$  of this system (if a solution exists) in FPC using again  $\mathcal{O}(\ell)$  many variables only. This yields the desired vector in  $K_i$ , that is indexed by  $(j, m)$ , and it concludes our proof.  $\square$

By putting Theorem 5.21 and Theorem 5.22 together we arrive at our desired result, namely that FPC is able to express solution spaces of  $\ell$ -cocyclic linear equation systems  $M \cdot x = b$  where  $M: I \times J \rightarrow \mathbb{F}$  and  $b: I \rightarrow \mathbb{F}$  using  $\mathcal{O}(\ell)$  many variables only. Unfortunately, there is still a small problem: according to Theorem 5.22, the index set for the solution space that we get is  $(J \times |J|)$ . However, in general,  $|J|$  can be much larger than  $|I|$ , and we would like to get *small* generating sets for expressing PC-refutations when we think of our procedure from Figure 2. In fact, when we express  $k$ -dimensional PC-refutations in FPC, then for the linear equation systems that we need to solve there, we only have a global polynomial bound on the size of the index  $I$  (the set of  $k$ -dimensional monomials), but not on the size of the index  $J$  (which indexes the generating set for  $\text{PC}_k(\mathcal{P})$  that we have computed up to a certain stage, cf. Figure 2). Fortunately, we can use the same strategy that we used in order to prove Theorem 5.22 in order to convert a (potentially large) generating set for a given linear space into a small one within FPC.

**Theorem 5.23.** *For every  $\ell \geq 1$  there exists an FPC-formula  $\varphi$  with  $\mathcal{O}(\ell)$  variables such that for every structure  $\mathfrak{A}$  which contains an  $\ell$ -cocyclic matrix  $M: I \times J \rightarrow \mathbb{F}$  over a finite field  $\mathbb{F}$ , the formula  $\varphi$  defines in  $\mathfrak{A}$  a matrix  $\varphi^{\mathfrak{A}}: I \times (I \times |I|) \rightarrow \mathbb{F}$  such that  $\text{im}(\varphi^{\mathfrak{A}}) = \text{im}(M)$ .*

*Proof.* The proof is analogous to our proof of Theorem 5.22.  $\square$

**Corollary 5.24.** *For every  $\ell \geq 1$  there exist FPC-formulas with  $\mathcal{O}(\ell)$  variables such that for every structure  $\mathfrak{A}$  which contains an  $\ell$ -cocyclic linear equation system  $M \cdot x = b$  for  $M: I \times J \rightarrow \mathbb{F}$  and  $b: I \rightarrow \mathbb{F}$  over a finite field  $\mathbb{F}$ , the formulas either define a matrix  $N: J \times K \rightarrow \mathbb{F}$  and a  $J$ -vector  $v: J \rightarrow \mathbb{F}$  such that  $\text{im}(N) + v$  is the solution space of  $M \cdot x = b$  where  $K \in \{I \times |I|, J \times |J|\}$  and  $|K| = \min(|I|^2, |J|^2)$ , or, in case that the solution space is empty, they define  $v = \emptyset$ .*

**5.7. Cocyclic PC-Refutations over Finite Fields in FPC.** We can finally come to our main result of this section. We show that FPC can express  $k$ -dimensional PC-refutations over finite fields  $\mathbb{F}$  of characteristic  $q$  if the inputs are polynomial equation systems that are interpreted in a class of  $(\ell, p)$ -cyclic structures, where  $q \neq p$ , using only  $\mathcal{O}(\ell)$  variables. As the prototype example, this situation occurs whenever we interpret polynomial equation systems over a field  $\mathbb{F}$  of characteristic  $\text{char}(\mathbb{F}) = q$  in (disjoint unions) of CFI-structures  $\mathfrak{A} \in \text{CFI}[\mathcal{F}; p]$  over  $\mathbb{F}_p$ . For the proof, recall that for an FPC-interpretation  $\mathcal{I}$ , we denote by  $\text{Norm}(\mathcal{I})$  its normal form according to Theorem 5.17.

**Theorem 5.25.** *Let  $Q \subsetneq \mathbb{P}$  be a (non-trivial) set of primes. Let  $\mathcal{I}(\vec{x})$  be an FPC-interpretation which maps  $\tau$ -structures to polynomial equation systems over finite fields  $\mathbb{F}$  of characteristic  $q \in Q$ . Then for every  $\ell \geq 1$ ,  $k \geq 2$ , and  $p \in \mathbb{P}, p \notin Q$ , there exists an FPC-formula  $\varphi$  with  $\mathcal{O}(k \cdot \ell)$  many variables such that for every  $(\ell, p)$ -cyclic  $\tau$ -structure  $\mathfrak{A}$  and  $\vec{a} \in \text{dom}(\mathfrak{A}, \vec{x})$  we have that  $\mathfrak{A} \models \varphi(\vec{a})$  if, and only if, the polynomial equation system  $\mathcal{I}(\mathfrak{A}, \vec{a})$  has a PC-refutation (over the respective finite field  $\mathbb{F}$ ) of degree at most  $k$ .*

*Proof.* Let  $d \geq 1$  denote the dimension of  $\mathcal{I}(\vec{x})$  and  $r \geq 0$  the number of parameters,  $r = |\vec{x}|$ . We use Theorem 5.17 to transform  $\mathcal{I}(\vec{x})$  into normal form  $\text{Norm}(\mathcal{I})(\vec{x})$ . Then, if  $\mathfrak{A}$  is an  $(\ell, p)$ -cyclic  $\tau$ -structure and  $\vec{a} \in \text{dom}(\mathfrak{A}, \vec{x})$ , then we know that  $\text{Norm}(\mathcal{I})(\mathfrak{A}, \vec{a})$  is  $(\ell \cdot (d + r))$ -homogeneous and that the automorphism group of  $\text{Norm}(\mathcal{I})(\mathfrak{A}, \vec{a})$  is an Abelian  $p$ -group. We conclude, using Theorem 5.16, that  $\text{Norm}(\mathcal{I})(\mathfrak{A}, \vec{a})$  is  $(\ell \cdot (d + r), p)$ -cyclic. Note that  $d, r$  are constants which only depend on the fixed interpretation  $\mathcal{I}$ .

Now, assume that we want to express in FPC, given  $\text{Norm}(\mathcal{I})(\mathfrak{A}, \vec{a})$ , whether the contained polynomial equation system  $\mathcal{P}$  over the finite field  $\mathbb{F}$  of characteristic  $q \neq p$  has a  $k$ -dimensional PC-refutation. In order to do this, we want to express the procedure from Figure 2 in FPC. Recall that the main (and only) difficulty is to (iteratively) define solution spaces of linear equation system over  $\mathbb{F}$  in FPC. However, since  $\mathcal{P}$  is part of an  $(\mathcal{O}(\ell), p)$ -cyclic structure, all linear equation systems that we have to solve are  $\mathcal{O}(\ell)$ -cocyclic systems. Since we can define the index sets for these systems using  $\mathcal{O}(k)$  many variables in FPC (because we basically have to index all degree- $k$  multilinear monomials) it follows from Corollary 5.24 that solution sets can be defined in FPC using at most  $\mathcal{O}(k \cdot \ell)$  many variables. We can now translate the resulting formulas back via  $\text{Norm}(\mathcal{I})$  which adds another constant factor to the number required variables that only depends on  $\mathcal{I}$ . This concludes our proof.  $\square$

As we said, in particular, we can apply this result for polynomial equation systems interpreted in CFI-structures. This will allow us to derive lower bounds for the polynomial calculus just by using finite-model-theoretic arguments in Section 6.

## 6. APPLICATIONS IN PROOF COMPLEXITY

Our model-theoretic characterisations of (bounded-width) resolution and the polynomial calculus via EFP- and FPC-definability allow us to uniformly (re-)prove many lower bounds on the complexity of proofs (size and/or width/degree) for families of propositional formulas using arguments from finite model theory. The basic idea is very simple. We saw that the amount of certain logical resources that are required to express refutations (that is the number of variables) matches the complexity of refutations (width of clauses or degree of polynomials) up to linear factors. It follows that if we exhibit families of propositional formulas  $\Phi_n, \Psi_n$  that cannot be distinguished in EFP (or in FPC or in  $C_{\infty\omega}^\omega$ ) using  $\mathcal{O}(k)$  variables, then also the corresponding propositional proof systems cannot distinguish between these formulas using refutations of width  $k$  or degree  $k$ , respectively. In particular, if one of the formulas in our family  $(\Phi_n, \Psi_n)$ , say  $\Phi_n$ , is satisfiable, then there cannot be a refutation for the indistinguishable formula  $\Psi_n$  (of a certain complexity).

In the conference version of this article [27] we discussed these applications with respect to the resolution proof system. However, given that we extended our definability results for the polynomial calculus in this article, we can basically derive the same lower bounds directly for the full polynomial calculus over the rationals and over finite fields (with the Pigeonhole principle being the only exception). Clearly, this makes the lower bounds more interesting and, for conciseness, we therefore restrict our attention to the polynomial calculus here.

**6.1. Lower Bounds on Degree and Size of Refutations.** In this section we establish our main tool for proving lower bounds for the polynomial calculus. Recall the notion of  $(\ell, p)$ -cyclic structures from Section 5.5.

**Theorem 6.1.** *Let  $\mathbb{F}$  be a finite field or the field of rationals. Moreover, let  $(\mathcal{P}_n)$  and  $(\mathcal{Q}_n)$  be two families of polynomial equation systems over  $\mathbb{F}$  and let  $\mathcal{I}$  be an FPC-interpretation that maps  $\tau$ -structures to polynomial equation systems over  $\mathbb{F}$ . In addition, let  $\ell \geq 1$  and let  $p \in \mathbb{P}$  be such that  $\text{char}(\mathbb{F}) \neq p$  and let  $(\mathfrak{A}_n)$  and  $(\mathfrak{B}_n)$  be two families of  $(\ell, p)$ -cyclic  $\tau$ -structures such that for all  $n \geq 1$ :*

- $\mathcal{I}(\mathfrak{A}_n) = \mathcal{P}_n$  and  $\mathcal{I}(\mathfrak{B}_n) = \mathcal{Q}_n$ ,
- $\mathcal{P}_n$  is satisfiable and  $\mathcal{Q}_n$  is not satisfiable,
- $\mathfrak{A}_n \equiv^{\Omega(n)} \mathfrak{B}_n$ .

Then the following holds:

- (1) Let  $\text{PC-DEGREE}(n)$  denote the minimal degree required to refute the system  $\mathcal{Q}_n$  using the polynomial calculus over  $\mathbb{F}$ . Then  $\text{PC-DEGREE}(n) \in \Omega(n)$ .

Moreover, as a consequence of this, the following holds:

- (2) Let  $\text{PC-SIZE}(n)$  denote the size of a minimal PC-refutation for  $\mathcal{Q}_n$  over  $\mathbb{F}$ . If the systems  $\mathcal{Q}_n$ , for  $n \geq 1$ , only contain  $\mathcal{O}(n)$  many variables, then  $\text{PC-SIZE}(n)$  is bounded from below by  $2^{\Omega(n)}$ .

*Proof.* For any degree  $k \geq 1$ , we know that by Theorem 4.10 (if  $=\mathbb{Q}$ ) or Theorem 5.25 (if  $\mathbb{F}$  is finite and of characteristic  $q \neq p$ ) there exists a  $C_{\infty\omega}^\omega$ -formula  $\varphi_k$  (in the case of finite fields, there even exists an FPC-formula  $\varphi_k$ , but this makes no difference for the argument) with  $\mathcal{O}(k)$  many variables (note that  $\ell$  is fixed) which expresses whether the polynomial equation systems  $(\mathcal{P}_n)$ ,  $(\mathcal{Q}_n)$  have a PC-refutation over  $\mathbb{F}$  of degree at most  $k$ . By translating these formulas back via the fixed FPC-interpretation  $\mathcal{I}$  (where we use  $\text{Norm}(\mathcal{I})$  in case of finite fields) we obtain  $C_{\infty\omega}^\omega$ -formulas  $\psi_k$  with  $\mathcal{O}(k)$  many variables such that  $\mathfrak{A}_n \models \psi_k$  if, and only if,  $\mathcal{P}_n$  has a degree  $k$  PC-refutation over  $\mathbb{F}$ , and likewise for  $\mathfrak{B}_n$  and  $\mathcal{Q}_n$ . However, since  $\mathcal{P}_n$  is satisfiable it has no such refutation for any degree  $k \geq 1$ . Since  $\mathfrak{A}_n \equiv^{\Omega(n)} \mathfrak{B}_n$ , it thus follows that for  $k \in \Omega(n)$ , also  $\mathcal{Q}_n$  has no such degree  $k$ -refutation. This proves our first claim. The second claim follows from the size-degree trade-off for the polynomial calculus, see [35, Corollary 6.3].  $\square$

**6.2. Lower Bounds for the Graph Isomorphism Problem.** We now discuss the prototype example for the lower bound technique on PC-refutations (Theorem 6.1). Specifically, we show that the graph isomorphism problem does not allow small PC-refutations neither over  $\mathbb{Q}$  nor over finite fields. This result has already been established by Berkholz and Grohe in [10, 11] by using known lower bounds for the polynomial calculus. Here, we present an alternative proof of (a generalisation of) their result using only arguments from finite model theory.

Given two graphs  $G = (V, E)$  and  $H = (W, F)$  it is easy to express the graph isomorphism problem for  $G$  and  $H$  as a polynomial equation system  $\text{ISO}(G, H)$  over any field  $\mathbb{F}$  as follows. We use variables  $X[v \mapsto w]$ , for  $v \in V$  and  $w \in W$ , to indicate whether  $v$  is mapped to  $w$  by an isomorphism (that we are going to guess as a solution). We include the Boolean constraints  $X^2 - X = 0$  as usual, i.e.  $X[v \mapsto w] \in \{0, 1\}$  for every solution. Then we just have to express that every vertex  $v \in V$  is mapped to precisely one  $w \in W$ :  $\sum_w X[v \mapsto w] = 1$ , and, dually, that every  $w \in W$  has precisely one preimage  $v \in V$ :  $\sum_v X[v \mapsto w] = 1$ . Finally we want that edges are preserved. We can achieve this by including for each  $v_1, v_2 \in V$  and  $w_1, w_2 \in W$  such that  $(v_1, v_2) \in E$  if, and only if,  $(w_1, w_2) \notin F$  the equation  $X[v_1 \mapsto w_1] \cdot X[v_2 \mapsto w_2] = 0$ . It is this (fixed) encoding that Berkholz and Grohe considered in [10, 11] in order to prove their lower bounds. Interestingly, we can easily lift their result to a more general setting, namely we can allow arbitrary encodings of the graph isomorphism problem that are definable in FPC or even in  $C_{\infty\omega}^\omega$  and still obtain the same lower bounds.

**Theorem 6.2.** *Let  $\mathbb{F}$  be the field of rationals or a finite field. Let  $\mathcal{I}$  be an FPC-interpretation that maps pairs of graphs  $(G, H)$  to polynomial equation systems over  $\mathbb{F}$  such that  $\mathcal{I}(G, H)$  is solvable if, and only if,  $G$  and  $H$  are isomorphic. Then there exists a sequence  $(G_n, H_n)$  of pairs of non-isomorphic graphs  $G_n, H_n$  with bounded degree and of size  $\mathcal{O}(n)$  such that PC-refutations for the systems  $\mathcal{I}(G_n, H_n)$  over  $\mathbb{F}$  require degree  $\Omega(n)$ .*

*Proof.* Choose  $p \in \{2, 3\}$  such that  $\text{char}(\mathbb{F}) \neq p$ . We consider the class of CFI-structures  $\text{CFI}[\mathcal{F}; p]$  over  $\mathbb{F}_p$ . Recall that  $\mathcal{F} = \{F_n : n \geq 1\}$  is a family of 3-regular, connected expander graphs where  $F_n$  has  $\mathcal{O}(n)$  many vertices. In Lemma 5.3 we observed that we can encode such CFI-structures as undirected graphs via FPC-interpretations  $\mathcal{J}$  (with a corresponding inverse interpretation  $\mathcal{J}^{-1}$ ). Moreover, for a CFI-structure  $\mathfrak{A} = \text{CFI}[F_n; p; \lambda] \in \text{CFI}[\mathcal{F}; p]$ , the graph  $\mathcal{J}(\mathfrak{A})$  encoding  $\mathfrak{A}$  has degree  $\mathcal{O}(p^2)$  and contains  $\mathcal{O}(p^2 \cdot n)$  vertices. Since  $p \in \{2, 3\}$ , it follows that the graphs  $\mathcal{J}(\mathfrak{A})$  have bounded degree and contain  $\mathcal{O}(n)$  vertices only.

For  $n \geq 1$  we fix two non-isomorphic CFI-structures  $\mathfrak{A}_n, \mathfrak{B}_n \in \text{CFI}[\mathcal{F}; p]$  with underlying graph  $F_n$  over  $\mathbb{F}_p$ . We let  $G_n = \mathcal{J}(\mathfrak{A}_n)$  and  $H_n = \mathcal{J}(\mathfrak{B}_n)$ . We claim that the resulting sequence  $(G_n, H_n)$  satisfies the above claim. To show this we use Theorem 6.1. Let  $\mathcal{P}_n = \mathcal{I}(G_n, G_n)$  and  $\mathcal{Q}_n = \mathcal{I}(G_n, H_n)$ . Then we observe that  $\mathcal{P}_n$  and  $\mathcal{Q}_n$  can be interpreted in the structures  $(\mathfrak{A}_n, \mathfrak{A}_n)$  and  $(\mathfrak{A}_n, \mathfrak{B}_n)$  via  $(\mathcal{I} \circ \mathcal{J})$  (where, formally, we need to slightly modify  $\mathcal{J}$  to encode ordered pairs of CFI-structures as ordered pairs of graphs). By the CFI-Theorem 5.6 we know that  $(\mathfrak{A}_n, \mathfrak{A}_n) \equiv^{\Omega(n)} (\mathfrak{A}_n, \mathfrak{B}_n)$ . By Theorem 5.19 and Theorem 5.16, we know that the structures  $(\mathfrak{A}_n, \mathfrak{A}_n)$  and  $(\mathfrak{A}_n, \mathfrak{B}_n)$  are  $(\ell, p)$ -cyclic for some fixed  $\ell \geq 1$ . Moreover, by our assumption on  $\mathcal{I}$ , the systems  $\mathcal{P}_n$  are satisfiable and the systems  $\mathcal{Q}_n$  are not satisfiable. Thus, all preconditions of Theorem 6.1 are met, and the lower bound follows.  $\square$

Although Theorem 6.2 gives us the desired linear lower bound on the degree of PC-refutations for the graph isomorphism problem, we can not readily infer the exponential size lower bound from Theorem 6.1. The reason is that the polynomial equation systems which encode the graph isomorphism problem might contain more than a linear number of variables. In fact, the number of variables in the system  $\text{ISO}(G, H)$  that we defined above contains a quadratic number of variables. This means that the size-degree trade-off results for the PC cannot be applied.

However we can fix this as follows. In our proof we used CFI-graphs and these are graphs of *bounded colour class size*. Formally, a *graph with colour class size*  $k \geq 1$  is a structure  $G = (V, E, \preceq)$  where  $(V, E)$  is a graph and where  $\preceq$  is a linear preorder on  $V$  such that every class of  $\preceq$ -incomparable vertices, that is every *colour class*, is of size at most  $k$ . In other words, one can think of the vertices of the graph  $G$  to be coloured while we only allow that at most  $k$  vertices get the same colour. We write  $V = V_0 \preceq \cdots \preceq V_{n-1}$  to denote that  $V$  is linearly ordered by  $\preceq$  into  $n$  colour classes  $V_i$  in the indicated way. We have that  $|V_i| \leq k$  for every  $i < n$ .

For CFI-graphs (that is graphs  $\mathcal{J}(\mathfrak{A})$  for  $\mathfrak{A} \in \text{CFI}[\mathcal{F}; p]$  and where  $\mathcal{J}$  is the graph encoding of CFI-structures from Lemma 5.3) the colour classes are basically given as the edge classes of the underlying graph plus the additional classes of inner nodes which encode the CFI-constraints, see our discussion preceding Lemma 5.3. The size of these classes is at most  $\mathcal{O}(p^2)$ . Since in our proof we can restrict to CFI-graphs over the field  $\mathbb{F}_p$  with  $p \in \{2, 3\}$ , these edge classes are indeed of constant size. Hence, it follows from our proof above that we can require the family of graphs  $(G_n, H_n)$  in Theorem 6.2 to consist of graphs of bounded colour class size.

Now, restricted to graphs of bounded colour class size, our encoding  $\text{ISO}(G, H)$  for the graph isomorphism problem that we introduced above can naturally be simplified resulting in a polynomial equation system that uses linearly many variables only. To see this, we consider pairs of graphs  $G = (V, E, \preceq_V)$  and  $H = (W, F, \preceq_W)$  of colour class size  $k \geq 1$  with the same number of colour classes, that is

$$\begin{aligned} V &= V_0 \preceq_V V_1 \preceq_V \cdots \preceq_V V_{n-1} \\ W &= W_0 \preceq_W W_1 \preceq_W \cdots \preceq_W W_{n-1}. \end{aligned}$$

Then each isomorphism is restricted to map vertices in the  $i$ -th colour class  $V_i$  in  $G$  to the  $i$ -th colour class  $W_i$  in  $H$ . That means that in our system  $\text{ISO}(G, H)$  we only need to include variables  $X[v \mapsto w]$  for all  $v \in V_i, w \in W_i, i < n$ . Since the colour classes are of constant size, this means that the resulting system only contains a linear number of variables. Hence, we obtain the following strengthening of Theorem 6.1 for this setting.

**Theorem 6.3.** *Let  $\mathbb{F}$  be the field of rationals or a finite field. Let  $\mathcal{I}$  be an FPC-interpretation that maps pairs of graphs  $(G, H)$  of bounded colour class size to polynomial equation systems over  $\mathbb{F}$  such that  $\mathcal{I}(G, H)$  is solvable if, and only if,  $G$  and  $H$  are isomorphic, and, moreover  $\mathcal{I}(G, H)$  contains a linear number of variables only (linear with respect to the number of vertices of  $G$  and  $H$ ). Then there exists a sequence  $(G_n, H_n)$  of pairs of non-isomorphic graphs  $G_n, H_n$  with bounded degree, of size  $\mathcal{O}(n)$ , and of bounded colour class size such that PC-refutations for the systems  $\mathcal{I}(G_n, H_n)$  over  $\mathbb{F}$  require degree  $\Omega(n)$  and size  $2^{\Omega(n)}$ .*

**6.3. Monomial-PC versus (Full-)PC over the Field of Rationals.** As mentioned above, in [10] Grohe and Berkholz studied the power of the polynomial calculus with respect to the graph isomorphism problem. One of their main results is that the monomial-PC over  $\mathbb{Q}$  has precisely the same expressive power as the well-known Weisfeiler-Leman graph isomorphism test which, in turn, has the same expressive power as counting logic (with respect to isomorphism testing). However, they left open the question of whether the full polynomial calculus is more expressive than its restricted variant the monomial-PC over  $\mathbb{Q}$  with respect to the graph isomorphism problem.

**Theorem 6.4** [10]. *For all  $k \geq 2$  and graphs  $G, H$  we have that*

$$G \not\equiv^k H \text{ if, and only if, } G \not\equiv^{\text{MON-PC}_k} H$$

In the above theorem,  $G \not\equiv^{\text{MON-PC}_k} H$  means that the monomial-PC (over  $\mathbb{Q}$ ) can refute the system  $\text{ISO}(G, H)$  using degree at most  $k$ . Obviously, this also implies that if  $G \not\equiv^k H$ , then  $G \not\equiv^{\text{PC}_k} H$ , that is  $\text{ISO}(G, H)$  can be refuted in the full-PC with degree at most  $k$ . However, it remained open whether the converse holds as well (in particular, it remained open if the converse holds if we allow to increase the dimension for the Weisfeiler-Leman algorithm by a constant factor).

**Question 6.5** [10]. *Is there a function  $f: \mathbb{N} \rightarrow \mathbb{N}$  such that for all  $k \geq 2$  we have*

$$G \not\equiv^{\text{PC}_k} H \implies G \not\equiv^{f(k)} H?$$

It immediately follows from Theorem 4.10 that the answer is affirmative and that we can choose  $f$  to be linear.

**Theorem 6.6.** *There is a linear function  $f: \mathbb{N} \rightarrow \mathbb{N}$  such that for all  $k \geq 2$  we have*

$$G \not\equiv^{\text{PC}_k} H \implies G \not\equiv^{f(k)} H.$$

*Proof.* Let  $\mathcal{I}$  be an FO-interpretation which interprets the  $\text{ISO}(G, H)$ -formulas as polynomial systems over  $\mathbb{Q}$  in pairs of graphs  $(G, H)$ . Let  $r \geq 1$  be the number of variables in  $\mathcal{I}$  and let  $c \geq 1$  be a constant such that the number of variables in the  $C_{\infty\omega}^\omega$ -formulas  $\varphi_k$ , that express the existence of  $k$ -dimensional PC-proofs according to Theorem 4.10, is bounded by  $c \cdot k$ .

We claim that  $G \equiv^{r \cdot c \cdot k} H \implies G \equiv^{\text{PC}_k} H$ . So let us assume that  $G \equiv^{r \cdot c \cdot k} H$ . First of all it holds that  $G \equiv^{r \cdot c \cdot k} H$  if, and only if,  $(G, G) \equiv^{r \cdot c \cdot k} (G, H)$ . By the closure of  $C_{\infty\omega}^\omega$  under FO-interpretations, it then follows that  $\text{ISO}(G, G) \equiv^{c \cdot k} \text{ISO}(G, H)$ . Since  $\text{ISO}(G, G)$  is clearly satisfiable and since  $\varphi_k$  cannot distinguish between  $\text{ISO}(G, G)$  and  $\text{ISO}(G, H)$  it follows that there does not exist a degree- $k$  PC-refutation of  $\text{ISO}(G, H)$ . Hence  $G \equiv^{\text{PC}_k} H$  as claimed.  $\square$

This shows that  $\text{PC}_k$  over  $\mathbb{Q}$  as a graph distinguishing procedure is not substantially stronger than the  $k$ -dimensional Weisfeiler Leman test, and therefore, with respect to the graph isomorphism problem,  $\text{MON-PC}_{\mathcal{O}(k)}$  and  $\text{PC}_{\mathcal{O}(k)}$  are equally expressive. Generally speaking, though,  $\text{PC}_k$  and  $\text{MON-PC}_k$  differ in so far as  $\text{MON-PC}_k$  proofs over  $\mathbb{Q}$  can be found in polynomial time with the Gröbner basis algorithm (larger coefficients than in the input are never required), whereas for  $\text{PC}_k$ , this is not the case. In light of Hakoniemi’s exponential bit-complexity lower bound for  $\text{PC}_2$  [31], it is in fact plausible that  $\text{PC}_k$  (over  $\mathbb{Q}$ ) is simply not a polynomial-time proof system, and therefore in the general case strictly stronger than  $\text{MON-PC}_k$ .

**6.4. Constraint Satisfaction Problems.** In this section we derive a dichotomy result for constraint satisfaction problems (CSPs) with respect to refutations in the polynomial calculus and the (weaker) resolution proof system. Intuitively, what we are going to show is that each CSP either allows simple proofs of inconsistency, namely such proofs that can be derived in bounded-width resolution, or it requires proofs of very high complexity, that is of linear degree and exponential size, even in the much stronger polynomial calculus proof system.

Let us recall the definition of CSPs. We present the formulation as a homomorphism problem. Let  $\mathfrak{T}$  be a fixed relational  $\tau$ -structure (the *template*). Then the *constraint satisfaction problem* associated with  $\mathfrak{T}$  is the class  $\text{Hom}(\mathfrak{T})$  consisting of all  $\tau$ -structures  $\mathfrak{A}$  for which there exists an homomorphism  $h: \mathfrak{A} \rightarrow \mathfrak{T}$ . Many combinatorial problems can be posed as CSPs. On the other hand, the class of all CSPs is limited in a certain sense: a famous conjecture by Feder and Vardi [23], which was recently confirmed independently by Bulatov [12] and Zhuk [43], says that for each template  $\mathfrak{T}$  the problem  $\text{Hom}(\mathfrak{T})$  is either decidable in polynomial time (P-TIME) or complete for non-deterministic polynomial time (NP-complete). We will make use of a similar definability dichotomy for FPC soon.

But before we do this, let us describe a simple algorithm to (approximately) solve constraint satisfaction problems. This algorithm is known as the *k-consistency test* and it can be phrased as follows. Fix a template  $\mathfrak{T}$  and consider an input structure  $\mathfrak{A}$ . Let us denote by  $\text{Part}^k(\mathfrak{A}, \mathfrak{T})$  the set of all partial homomorphisms  $p$  from  $\mathfrak{A}$  to  $\mathfrak{T}$  whose domain  $\text{dom}(p)$  is of size at most  $k$  (we include the empty homomorphism  $\emptyset$ ). The idea is to iteratively compute restrictions  $T_i \subseteq \text{Part}^k(\mathfrak{A}, \mathfrak{T})$  of  $\text{Part}^k(\mathfrak{A}, \mathfrak{T})$  with respect to the following closure properties. We set  $T_0 = \text{Part}^k(\mathfrak{A}, \mathfrak{T})$ . For  $i \geq 1$  we set

$$T_i = \{p \in T_{i-1} : \text{for all } \text{dom}(p) \subseteq S \subseteq A, |S| \leq k \text{ th. ex. } q \in T_{i-1} \text{ s.th. } p \subseteq q, \text{dom}(q) = S, \\ \text{and for all } q \subseteq p \text{ we have } q \in T_{i-1}\}$$

We output the final set  $T_\infty$ . In other words we iteratively eliminate all partial homomorphisms which cannot be extended to partial homomorphisms of size at most  $k$  with respect to all possible (consistent) domains, and such partial homomorphisms for which we eliminated a restriction in the iteration before. The first observation is that if there exists a homomorphism  $h: \mathfrak{A} \rightarrow \mathfrak{T}$ , then  $T_\infty \neq \emptyset$ , because the set of all restrictions of  $h$  to partial homomorphisms in  $\text{Part}^k(\mathfrak{A}, \mathfrak{T})$  will be contained in each  $T_i$ . Hence, if  $T_\infty = \emptyset$ , then we can correctly conclude that  $\mathfrak{A} \notin \text{Hom}(\mathfrak{T})$ . If, on the other hand,  $T_\infty \neq \emptyset$ , then in the general case we must output “we don’t know”. However, in many cases, depending on the template  $\mathfrak{T}$ , this naive algorithm will work correctly on all inputs, which means that we have an efficient and simple way to decide the problem  $\text{Hom}(\mathfrak{T})$ .

Before we come to this, let us observe that it is very easy to express the  $k$ -consistency test using bounded-width resolution. For every  $p \in \text{Part}^k(\mathfrak{A}, \mathfrak{T})$  consider a Boolean variable  $X_p$  with the intended meaning that  $X_p$  is true if  $p \in T_\infty$ . According to the  $k$ -consistency test, we consider the following set of clauses:

- For every  $p \in \text{Part}^k(\mathfrak{A}, \mathfrak{T})$ , every  $\text{dom}(p) \subseteq S \subseteq A$  with  $|S| \leq k$ :

$$X_p \rightarrow \bigvee_{\substack{p \subseteq q \in \text{Part}^k(\mathfrak{A}, \mathfrak{T}), \\ \text{dom}(q) = S}} X_q.$$

- For every  $p \in \text{Part}^k(\mathfrak{A}, \mathfrak{T})$  and every  $q \subseteq p$ :

$$X_p \rightarrow X_q.$$

We obtain a (dual-)Horn-formula which always has the trivial model where we set every variable to false. Moreover, every non-trivial model is a witness that  $T_\infty \neq \emptyset$ . Since in this case  $\emptyset \in T_\infty$ , this means that if we add the single clause  $X_\emptyset$  to the above formula, then we obtain a (dual-)Horn-formula which is not satisfiable if, and only if,  $T_\infty = \emptyset$ . Moreover, note that since  $\mathfrak{T}$  is a fixed template, the above formula is of constant width. We conclude that  $T_\infty = \emptyset$  if, and only if, bounded-width resolution can refute the above formula. It is obvious that this formula is also interpretable in  $\mathfrak{A}$  using a first-order interpretation.

**Theorem 6.7.** *For every  $k \geq 1$ , the  $k$ -consistency test can be expressed in  $\text{FO}(\text{RES}_{\mathcal{O}(k)})$ .*

Now, what happens if we have a template  $\mathfrak{T}$  for which the  $k$ -consistency test is incomplete for any fixed value of  $k \geq 1$ ? In this case the (descriptive) complexity of the problem  $\text{Hom}(\mathfrak{T})$  is much higher. In fact, it follows from [3] and [8] that in this case, the problem cannot be defined in FPC. This “definability dichotomy” was first explicitly noted, and refined, by Dawar and Wang in [19] and in [21]:

**Theorem 6.8** [8, 3, 19, 21]. *For every template  $\mathfrak{T}$  one of the following is true.*

- (1) *Either there is a  $k \geq 1$  such that the  $k$ -consistency test correctly decides  $\text{Hom}(\mathfrak{T})$ , or*
- (2) *there exists a (non-trivial) finite Abelian group  $G$  such that the problem of deciding the solvability of linear equation systems over  $G$  with at most three variables per equation,  $3\text{LIN}(G)$ , reduces to  $\text{Hom}(\mathfrak{T})$  via an FPC-interpretation of linear size (that is the interpretation only increases the sizes of structures by a constant factor).*

Moreover, it is known that the problem of deciding whether two CFI-structures  $\mathfrak{A}, \mathfrak{B} \in \text{CFI}[\mathcal{F}; p]$  over the same underlying graphs are isomorphic reduces to  $3\text{LIN}(\mathbb{F}_p)$  via an FPC-reduction of linear size. Hence, by applying Theorem 6.1 (and using the same arguments as in the Section 6.2) we get the following dichotomy for the proof systems resolution and polynomial calculus.

**Theorem 6.9.** *For every template  $\mathfrak{T}$  one of the following holds.*

- (1) *Either  $\text{Hom}(\mathfrak{T})$  can be decided using bounded-width resolution, or*
- (2) *there exists a finite set of primes  $P \subseteq \mathbb{P}$  such that for every linear-size FPC-definable encoding of  $\text{Hom}(\mathfrak{T})$  as a system of polynomial equations  $\mathcal{P}(\mathfrak{T})$  over a field  $\mathbb{F}$ , which is either  $\mathbb{Q}$  or a finite field with  $\text{char}(\mathbb{F}) \notin P$ , refutations of  $\mathcal{P}(\mathfrak{T})$  in the polynomial calculus over  $\mathbb{F}$  require degree  $\Omega(n)$  and size  $2^{\Omega(n)}$  (where  $n$  refers to the size of the input structures  $\mathfrak{A}$ ).*

For the case of  $\mathbb{Q}$ , this dichotomy result has been established in [6] via a different proof strategy. Let us remark that, as a result of our approach, we can formulate our dichotomy result with respect to every FPC-definable encoding (of linear size if we want to maintain exponential size lower bounds). Also, to the best of our knowledge, this dichotomy was not known for the case of the polynomial calculus over finite fields.

## 7. DISCUSSION: THE POWER OF THE POLYNOMIAL CALCULUS AND BEYOND

The resolution proof system and the polynomial calculus are two important and well-studied propositional proof systems. In this article we characterised their power from the viewpoint of finite model theory. We proved that bounded-width resolution ( $k$ -RES,  $k \geq 3$ ) is complete for existential fixed-point logic (EFP), that Horn-Resolution (HORN-RES) is complete for least fixed-point logic (LFP), and that the bounded-degree monomial-PC (MON-PC $_k$ ) and the degree- $k$  polynomial calculus over  $\mathbb{Q}$  with bit complexity  $n^b$  (PC $_{k,b}$  for  $k \geq 2, b \geq 1$ ) over  $\mathbb{Q}$  are complete for fixed-point logic with counting (FPC) under (numerical) first-order reductions. Moreover, we showed that the degree- $k$  PC over  $\mathbb{Q}$  without any restriction on the coefficients (PC $_k$ ) can be expressed in  $C_{\infty\omega}^\omega$  with  $\mathcal{O}(k)$  many variables. It remains open if PC $_k$  can also be simulated in the weaker logic FPC, or more generally, in PTIME. However, our result that  $\text{FPC} \equiv \text{FO}^+(\text{PC}_{k,b})$ , and the fact that the proof system PC $_{k,b}$  is strictly weaker than PC $_k$ , suggests that PC $_k$  is really more powerful than FPC.

Interestingly, our Theorem 6.6 implies that for deciding the graph isomorphism problem in the polynomial calculus, using large coefficients in the refutations does not lead to additional power compared to the  $k$ -dimensional Weisfeiler Leman isomorphism test, which can be implemented in FPC. This raises the question what precisely are the problems for which large coefficients in refutations actually take the power of the proof system beyond that of FPC and PC $_{k,b}$ .

Our method that takes definability as the measure for expressive power yields a much finer classification compared to the one that we get by using standard complexity-theoretic notions. Indeed, it is well-known that already 3-RES is PTIME-complete, which means that *all* (fragments of) proof systems that we considered here are equivalent from the viewpoint of (algorithmic) complexity theory. In contrast, as we saw, we obtain a more interesting landscape if we measure their descriptive complexity instead.

On the other hand, compared to the view of proof complexity, our analysis is much coarser. For instance, in our framework there is no explicit difference between width-3 and width- $k$  resolution for any  $k \geq 4$ , while, from the viewpoint of proof complexity, clearly these systems have different power. The reason for this mismatch is that we allow more powerful logical reductions (which are still weak from the viewpoint of finite model theory). We believe that this more general perspective, though not as precise, makes it easier to pin down fundamental differences between, and weaknesses of, the different (fragments) of proof systems. For instance, our results show that the resolution proof system cannot refute the Pigeonhole Principle for any FO-definable encoding, see [27], while the polynomial calculus over  $\mathbb{Q}$  allows simple refutations (with respect to a natural encoding). Our results explain this “counting dichotomy” very clearly: resolution corresponds to EFP, a logic which lacks counting, and the polynomial calculus over  $\mathbb{Q}$  to FPC/ $C_{\infty\omega}^\omega$ , logics which explicitly include a counting ability. Moreover, our results highlight that the polynomial calculus has a severe weakness: it is not able to go beyond FPC (with respect to its bounded-degree

and bounded bit-complexity PTIME-stratification). Since it is known that FPC fails to express all PTIME-properties, this implies that there are certain PTIME-properties which do not have small refutations in the polynomial calculus. The prototype example is solving linear equation systems over finite fields. We can exploit this connection between FPC and the polynomial calculus over  $\mathbb{Q}$  even further to derive yet another characterisation. Indeed, we can show that linear programming is complete for FPC under (numerical) first-order reductions. This means that the power of the polynomial calculus over  $\mathbb{Q}$  corresponds *precisely* to the power of linear programming under numerical FO-reductions. This connects the polynomial calculus with a very natural and significant algorithmic problem in the setting of finite model theory.

Another interesting outcome of our work are the new finite-model theoretic proofs for lower bounds on the complexity of refutations in the polynomial calculus. We saw that, using a uniform finite-model theoretic approach, one can show that many families of propositional formulas require refutations of exponential size. Remarkably, we could obtain these lower bounds not only for the polynomial calculus over  $\mathbb{Q}$ , but also for the polynomial calculus over finite fields. Also, as a result of our approach, our lower bounds are very robust in the sense that they do not rely on any specific encoding of a problem as a propositional formula, but they hold with respect to any (FPC-)definable encoding of the problem. For the case of the polynomial calculus this implies, for example, that all of the aforementioned lower bounds also hold for the *polynomial calculus with resolution* (PCR). This proof system is nothing more than the polynomial calculus, but we include for any variable  $X$  a syntactic dual variable  $\bar{X}$  together with the axiom  $1 - X = \bar{X}$ . Clearly these additional axioms can be defined in FPC, and so, our results do not change in any way by considering the PCR instead of the standard PC.

Let us finally take a look at some future work. We observe that in our lower bound proofs for the polynomial calculus over finite fields we do not require a precise connection with FPC-definability (in fact, as we saw, such a precise match between FPC and the polynomial calculus over finite fields does not exist). Indeed, for proving lower bounds it was sufficient to establish FPC-definability of refutations for families of propositional formulas that are defined in CFI-structures. We then made use of the fact that the CFI-problem is hard for FPC which gave us the lower bounds on the proof complexity. Even more general, we do not need to obtain FPC-definability, but, because of the fact that the CFI-problem is hard already for finite-variable counting logic  $C_{\infty\omega}^\omega$ , it is sufficient to show  $C_{\infty\omega}^\omega$ -definability (recall that  $C_{\infty\omega}^\omega$  is a more powerful logic than FPC, so showing definability is easier). We followed these lines for the case of the polynomial calculus over finite fields in Section 5. For this, we strongly made use of our key technical results which says that CFI-structures over expander graphs are FPC-homogeneous. Recall that this means that we can order orbits of  $k$ -tuples in CFI-structures using FPC-formulas with a linear number of variables only.

In fact, we can use this homogeneity result to develop a much more general strategy for proving lower bounds for certain propositional proof system PROP. As we explain in the following, in certain situations this result allows us to *quantify over refutations in*  $C_{\infty\omega}^\omega$ . More precisely, assume that PROP has a stratification  $\text{PROP} = (\text{PROP}_k)$  along a parameter  $k \geq 1$ . Moreover, assume that whenever a family of propositional formulas  $\mathcal{F}$  that is defined (via a fixed FPC-interpretation) in (pairs of) CFI-structures, has a refutation in  $\text{PROP}_k$ , then it also has a refutation  $\mathfrak{p}$  such that:

- $\mathfrak{p}$  is symmetric, that is invariant under all automorphisms of the underlying CFI-structures, and
- $\mathfrak{p}$  can be encoded as an object that is definable in the logic  $C_{\infty\omega}^\omega$  over the underlying CFI-structures with  $\mathcal{O}(k)$  variables,
- given a description of  $\mathfrak{p}$  as above, it can be verified using a  $C_{\infty\omega}^\omega$ -formula with  $\mathcal{O}(k)$  many variables, that  $\mathfrak{p}$  refutes  $\mathcal{F}$ .

If these (vaguely formulated) conditions are satisfied, then we can basically apply our techniques in order to show that certain families of propositional formulas, namely such formulas which encode the CFI-isomorphism problem, cannot be refuted in  $\text{PROP}_k$  for any sublinear  $k$ . At the moment, we work out the details and study to what extent these conditions can be relaxed.

For now, let us illustrate the usefulness of this approach by means of a simple example. If we take another look at the paper by Grohe and Berkholz [10], then we observe that they do, in fact, not only derive lower bounds on the complexity of refutations for the graph isomorphism problem for the polynomial calculus over  $\mathbb{Q}$ , but also for a stronger proof system which is known as the *Positivstellensatz* (or *Sums-of-Squares Proof System*). Let us briefly introduce this system. The setting is the same as for the polynomial calculus over  $\mathbb{Q}$ , that is our input is a set  $\mathcal{P}$  consisting of multivariate polynomials  $p \in \mathbb{R}[\mathcal{X}]$ , and our aim is to show that the polynomials in  $\mathcal{P}$  do not have a common zero. As before we implicitly assume that the Boolean constraints  $X^2 - X = 0$  are contained in  $\mathcal{P}$  for every variable  $X \in \mathcal{X}$ .

Let us fix a degree  $k \geq 2$  which is even. A degree- $k$  Positivstellensatz refutation of a polynomial equation system  $\mathcal{P}$  over variables  $\mathcal{X}$  consists of polynomials  $f_p \in \mathbb{R}[\mathcal{X}]$  such that

$$\sum_{p \in \mathcal{P}} f_p \cdot p = 1 + s,$$

where  $s$  is a sum-of-squares (sos) polynomial, that is  $s = \sum_{i \in I} q_i^2$  for some polynomials  $q_i \in \mathbb{R}[\mathcal{X}]$ , and such that all polynomials in the above equation have degree at most  $k$ . Since  $s(a) \geq 0$  for every evaluation  $a: \mathcal{X} \rightarrow \mathbb{R}$ , the existence of such a refutation clearly proves that  $\mathcal{P}$  is inconsistent. Now, as in our description above, assume that we have interpreted this system in a (pair) of CFI-structures, and let  $\Gamma$  be the corresponding CFI-automorphism group. Every  $\pi \in \Gamma$  extends (uniquely) to a permutation on  $\mathcal{X}$  and so it defines a unique automorphism of  $\mathbb{R}[\mathcal{X}]$ . Moreover, this automorphism of  $\mathbb{R}[\mathcal{X}]$  stabilises  $\mathcal{P}$ . It follows that if we have a refutation as above, also

$$\sum_{p \in \mathcal{P}} \pi(f_p) \cdot \pi(p) = 1 + \pi(s),$$

is a refutation. Here we are just saying that refutations are mapped to refutations if we permute the variables in such a way that the set of given polynomials remains stable. Clearly, this holds for any reasonable proof system. In particular, note that  $\pi(s)$  is also a sum-of-squares polynomial (because  $\pi$  is an automorphism of  $\mathbb{R}[\mathcal{X}]$ ).

However, in the case of the Positivstellensatz we can go one important step further by summing up over all refutations that we obtain in this way:

$$\sum_{\pi \in \Gamma} \left( \sum_{p \in \mathcal{P}} \pi(f_p) \cdot \pi(p) \right) = |\Gamma| + \sum_{\pi \in \Gamma} \pi(s).$$

The importance of this equation follows from the fact that the sos polynomial on the right-hand side is symmetric with respect to  $\Gamma$ . The simple consequence is that whenever we can derive from  $\mathcal{P}$ , via a degree- $k$  combination of polynomials, a polynomial  $1 + s$ , where  $s$  is an sos polynomial, then we can also derive from  $\mathcal{P}$  a polynomial  $1 + \hat{s}$  where  $\hat{s}$  is a *symmetric* sos polynomial (note that sos polynomials are closed under addition).

This already brings us very close to our proof strategy from above: we saw that whenever there is a degree- $k$  refutation, there is also a symmetric one. Let us now complete our argument for the case of the Positivstellensatz more explicitly. The most important question is how we can obtain the symmetric polynomial  $1 + \hat{s}$  in  $C_{\infty\omega}^\omega$ . The key insight is that we don't have to bother too much about this, because  $1 + \hat{s}$  is symmetric. Clearly, we can describe  $r = 1 + \hat{s}$  as a mapping  $r: M_k \rightarrow \mathbb{R}$  where  $M_k$  denotes the set of all monomials of degree at most  $k$ . Since  $r$  is symmetric,  $r$  is a vector with the same entries on all  $\Gamma$ -orbits on  $M_k$ . We now make use of the fact that CFI-structures are FPC-homogeneous. This allows us to order the  $\Gamma$ -orbits on  $M_k$  in FPC using only  $\mathcal{O}(k)$  many variables. Using this we can see that we can describe the vector  $r$  by using a mapping from an *ordered set* to  $\mathbb{R}$ . This is a quite simple object from the viewpoint of  $C_{\infty\omega}^\omega$  as it has nothing to do with the underlying structure. In particular, we can explicitly quantify over all such mappings, since we have infinite conjunctions and disjunctions available in  $C_{\infty\omega}^\omega$ . The final step is to verify that, having guessed such a vector  $r: M_k \rightarrow \mathbb{R}$  in  $C_{\infty\omega}^\omega$ , this vector is indeed a refutation, that is  $r = 1 + \hat{s}$  for a symmetric sos-polynomial  $\hat{s}$ , and that  $r$  can be derived from  $\mathcal{P}$  using a degree- $k$  polynomial combination. The latter problem is about solving a linear equation system over  $\mathbb{R}$  which can be done in  $C_{\infty\omega}^\omega$  by what we saw in Section 4 (it is not hard to see that dealing with real numbers in this context is easy: since we are working in  $C_{\infty\omega}^\omega$  and not in FPC, we can quantify explicitly over (sets of) real numbers that we can use for our definitions).

The former problem can be reformulated as follows. Let  $M_{k/2}$  denote the set of monomials over  $\mathcal{X}$  of degree at most  $k/2$ . Let  $S$  be the  $M_{k/2} \times M_{k/2}$ -matrix over  $\mathbb{R}$  which is defined by letting  $S(m, n)$  be the leading coefficient of the monomial  $m \cdot n$ ,  $m, n \in M_{k/2}$ , in  $\hat{s}$  that we get when we syntactically expand the sos polynomial  $\hat{s}$ . Then  $S$  is symmetric and, as a consequence of the syntactic form of  $\hat{s}$  ( $\hat{s}$  is an sos polynomial),  $S$  can be written as a sum of matrices  $vv^T$  where the  $v: M_{k/2} \rightarrow \mathbb{R}$  correspond to the summands in  $\hat{s}$ . Vice versa, assume that  $S$  can be written in this form. Let  $z$  be the  $M_{k/2}$ -vector whose entries are the monomials  $m \in M_{k/2}$ , i.e.  $z(m) = m$ . Then it is easy to see that  $z^T S z$  is an sos polynomial. Hence,  $\hat{s}$  is an sos polynomial if, and only if, the corresponding matrix  $S$  can be written as a sum of matrices  $vv^T$  for  $v: M_{k/2} \rightarrow \mathbb{R}$ . This condition is equivalent to saying that  $S$  is positive semi-definite, which, in turn, is equivalent to saying that  $S$  has only non-negative eigenvalues. It is known that the eigenvalues of matrices over  $\mathbb{Q}$  are definable in FPC, see [18]. It is easy to adapt this definability result to our setting which shows that the positive semi-definiteness of  $S$  can be certified in  $C_{\infty\omega}^\omega$  (using  $\mathcal{O}(k)$  many variables) as well.

This proof (sketch) shows that all lower bounds for the polynomial calculus that we obtained in Section 6.1, that is for graph isomorphism refutations and for the CSP dichotomy, remain valid for the Positivstellensatz. These lower bounds have been known before, but it is nice to see how easily they can be derived by using our newly developed finite-model-theoretic tools. Again, let us stress that what makes our arguments particularly simple is the FPC-homogeneity of CFI-structures. As we saw, this result allows us to quantify over refutations in  $C_{\infty\omega}^\omega$  (assuming that symmetric refutations with certain syntactic properties exist), so we are only left with the usually much simpler task of verifying such refutations

in  $C_{\infty\omega}^\omega$ . As indicated above, this line of research is part of an ongoing project where we explore the power of symmetric proof systems from the viewpoint of finite model theory more thoroughly, so we defer the details to this upcoming work.

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