

A LIGHT MODALITY FOR RECURSION

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ABSTRACT. We investigate a modality for controlling the behaviour of recursive functional programs on infinite structures which is completely silent in the syntax. The latter means that programs do not contain “marks” showing the application of the introduction and elimination rules for the modality. This shifts the burden of controlling recursion from the programmer to the compiler. To do this, we introduce a typed lambda calculus à la Curry with a silent modality and guarded recursive types. The typing discipline guarantees normalisation and can be transformed into an algorithm which infers the type of a program.

1. INTRODUCTION

The quest of finding a typing discipline that guarantees that functions on coinductive data types are productive has prompted a variety of works that rely on a modal operator [23, 18, 30, 7, 1, 8, 4, 13]. Typability in these systems guarantees normalization and non-normalizing programs such as `fix I` where $I = \lambda x.x$ are not typable. All these works except for Nakano’s [23] have explicit constructors and destructors in the syntax of programs [18, 30, 7, 8, 13]. This has the advantage that type checking and/or type inference are easy but it has the disadvantage that they do not really liberate the programmer from the task of controlling recursion since one has to know when to apply the introduction and elimination rules for the modal operator.

We present a typed lambda calculus with a temporal modal operator \bullet called $\lambda_{\bullet}^{\circ}$, which has the advantage of having the modal operator silent in the syntax of programs. The system $\lambda_{\bullet}^{\circ}$ does not need to make use of a subtyping relation as Nakano’s. Even without a subtyping relation, the modal operator is still challenging to deal with because types are intrinsically non-structural, not corresponding to any expression form in the calculus.

The modal operator \bullet , also called *the delay operator*, indicates that the data on the recursive calls will only be available later. Apart from the modal operator, we also include guarded recursive types which generalize the recursive equation $\text{Str1}_t = t \times \bullet \text{Str1}_t$ for streams [18, 30]. This allows us to type productive functions on streams such as

$$\text{skip } xs = \langle \text{fst } xs, \text{skip } (\text{snd } (\text{snd } xs)) \rangle$$

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which deletes the elements at even positions of a stream using the type $\mathbf{Str1}_{\mathbf{Nat}} \rightarrow \mathbf{Str2}_{\mathbf{Nat}}$ where $\mathbf{Str2}_t = t \times \bullet \bullet \mathbf{Str2}_t$. The temporal modal operator \bullet allows to type many productive functions which are rejected by the proof assistant Coq [9].

Lazy functional programming is acknowledged as a paradigm that fosters software modularity [14] and enables programmers to specify computations over possibly infinite data structures in elegant and concise ways. We give some examples that show how modularization and compositionality can be achieved using the modal operator. An important recursive pattern used in functional programming for modularisation is `foldr` defined by

$$\mathbf{foldr} \ f \ xs = f \ (\mathbf{fst} \ xs) \ (\mathbf{foldr} \ f \ (\mathbf{snd} \ xs))$$

The type of `foldr` which is $(t \rightarrow \bullet s \rightarrow s) \rightarrow \mathbf{Str1}_t \rightarrow s$, is telling us what is the safe way to build functions. While it is possible to type

$$\mathbf{iterate} \ f = \mathbf{foldr} \ (\lambda xy. \langle f \ x, y \rangle)$$

by assigning the type $t \rightarrow t$ to f , and assuming $s = \mathbf{Str1}_t$, it is not possible to type the unproductive function `foldr` $(\lambda xy. y)$. This is because $\lambda xy. y$ does not have type $(t \rightarrow \bullet s \rightarrow s)$.

In spite of the fact that $\mathbf{Str1}_{\mathbf{Nat}} \rightarrow \mathbf{Str2}_{\mathbf{Nat}}$ is the type of `skip` with the minimal number of bullets, it does not give enough information to the programmer to know that the composition of `skip` with itself is still typeable. The type inference algorithm for $\lambda_{\rightarrow}^{\bullet}$ can infer a more general type for `skip`, which looks like

$$\mathbf{Stream}(N, X) \rightarrow \mathbf{Stream}(N + 1, X)$$

where $\mathbf{Stream}(N, X) = X \times \bullet^N \mathbf{Stream}(N, X)$ and N and X are integer and type variable, respectively. It is clear now that from the above type, a programmer can deduce that it is possible to do the composition of `skip` with itself.

1.1. Outline. Section 2 defines the typed lambda calculus $\lambda_{\rightarrow}^{\bullet}$, with the silent modal operator and shows some examples. Section 3 proves subject reduction and normalisation. It also shows that the Lèvy-Longo and Böhm trees of a typeable expression have no \perp for \bullet^{∞} -free and tail finite types respectively. Section 4 compares $\lambda_{\rightarrow}^{\bullet}$ with the type systems of [23] and [18]. Section 5 gives an adequate denotational semantics for $\lambda_{\rightarrow}^{\bullet}$ in the topos of trees as a way of linking our system to the work by Birkedal et al [3]. Section 6 shows decidability of the type inference problem for $\lambda_{\rightarrow}^{\bullet}$. This problem is solved by an algorithm which has the interesting feature of combining unification of types with integer linear programming. Sections 7 and 8 discuss related and future work, respectively.

1.2. Contribution. This paper improves and extends [29] in several ways:

- (1) We prove that the typed lambda calculus of Krishnaswami and Benton [18] is included in $\lambda_{\rightarrow}^{\bullet}$, and that $\lambda_{\rightarrow}^{\bullet}$ is included in the one of Nakano [23].
- (2) We give conditions on the types that guarantee that the Lèvy-Longo and Böhm trees of a typable expression have no \perp .
- (3) Besides proving soundness for the denotational semantics, we also prove that it is adequate with respect to an observational equivalence.
- (4) We obtain automatic ways of guaranteeing that the program satisfies the properties of normalization, having Lèvy-Longo and Böhm tree without \perp .

2. THE TYPED LAMBDA CALCULUS WITH THE DELAY MODALITY

This section defines the type discipline for $\lambda_{\bullet}^{\bullet}$. Following [23], we introduce the *delay* type constructor \bullet , so that an expression of type $\bullet t$ denotes a value of type t that is available “at the next moment in time”. This constructor is key to control recursion and attain normalisation of expressions.

2.1. Syntax. The syntax for *expressions* and *types* is given by the following grammars.

$e ::=^{ind}$	Expression	$t ::=^{coind}$	Pseudo-type
	x		X
	(variable)		(type variable)
	k		\mathbf{Nat}
	(constant)		(natural type)
	$\lambda x.e$		$t \times t$
	(abstraction)		(product)
	ee		$t \rightarrow t$
	(application)		(arrow)
			$\bullet t$
			(delay)

In addition to the usual constructs of the λ -calculus, expressions include constants, ranged over by k . Constants are the constructor for pairs \mathbf{pair} , the projections \mathbf{fst} and \mathbf{snd} and $\mathbf{0}$ and \mathbf{succ} . We do not need a primitive constant for the fixed point operator because it can be expressed and typed inside the language. Expressions are subject to the usual conventions of the λ -calculus. In particular, we assume that the bodies of abstractions extend as much as possible to the right, that applications associate to the left, and we use parentheses to disambiguate the notation when necessary. We write $\langle e_1, e_2 \rangle$ in place of $\mathbf{pair} \ e_1 \ e_2$.

The syntax of *pseudo-types* is defined co-inductively. A type is a possibly infinite tree, where each internal node is labelled by a type constructor \rightarrow , \times or \bullet and has as many children as the arity of the constructor. The leaves of the tree (if any) are labelled by basic types which in this case are type variables or \mathbf{Nat} . The type variables are needed in $\lambda_{\bullet}^{\bullet}$, to express the general type of functions such as the identity. We use a co-inductive syntax to describe infinite data structures (such as streams). The syntax for pseudo-types include the types of the simply typed lambda calculus (*arrows* and *products*) and the *delay* type constructor \bullet [23].

For simplicity, we only include \mathbf{Nat} and type variables as basic types. One could easily add other basic types such as \mathbf{Bool} and \mathbf{Unit} together with constants for their values.

Definition 2.1 (Types). We say that a pseudo-type t is

- (1) *regular* if its tree representation has finitely many distinct sub-trees;
- (2) *guarded* if every infinite path in its tree representation has infinitely many \bullet 's;
- (3) a *type* if it is regular and guarded.

The regularity condition implies that we only consider types admitting a finite representation. It is equivalent to representing types with μ -notation and a strong equality which allows for an infinite number of unfoldings. This is also called the *equirecursive approach* since it views types as the unique solutions of recursive equations [11] [25, Section 20.2]. The existence and uniqueness of a solution for a pseudo-type satisfying condition 1 follow from known results (see [10] and also Theorem 7.5.34 of [6]). For example, there are unique pseudo-types $\mathbf{Str}'_{\mathbf{Nat}}$, $\mathbf{Str}1_{\mathbf{Nat}}$, and \bullet^{∞} that respectively satisfy the equations $\mathbf{Str}'_{\mathbf{Nat}} = \mathbf{Nat} \times \mathbf{Str}'_{\mathbf{Nat}}$, $\mathbf{Str}1_{\mathbf{Nat}} = \mathbf{Nat} \times \bullet \mathbf{Str}1_{\mathbf{Nat}}$, and $\bullet^{\infty} = \bullet \bullet^{\infty}$.

The guardedness condition intuitively means that not all parts of an infinite data structure can be available at once: those whose type is prefixed by a \bullet are necessarily “delayed” in

the sense that recursive calls on them must be deeper. For example, Str1_{Nat} is a type that denotes streams of natural numbers, where each subsequent element of the stream is delayed by one \bullet compared to its predecessor. Instead Str'_{Nat} is not a type: it would denote an infinite stream of natural numbers, whose elements are all available right away. If the types are written in μ -notation, the guardedness condition means that all occurrences of X in the body t of $\mu X.t$ are in the scope of a \bullet .

The type \bullet^∞ is somehow degenerated in that it contains no actual data constructors. Unsurprisingly, we will see that non-normalising terms such as $\Omega = (\lambda x.x x)(\lambda x.x x)$ can only be typed with \bullet^∞ (see Theorem 3.13). Without condition 2, Ω could be given any type since the recursive pseudo-type $\mathbf{D} = \mathbf{D} \rightarrow t$ would become a type.

We adopt the usual conventions regarding arrow types (which associate to the right) and assume the following precedence among type constructors: \rightarrow , \times and \bullet with \bullet having the highest precedence. Sometimes we will write $\bullet^n t$ in place of $\underbrace{\bullet \cdots \bullet}_n t$.

n-times

2.2. Operational Semantics. Expressions reduce according to a standard *call-by-name* semantics:

$$\begin{array}{c} \text{[R-BETA]} \quad \text{[R-FIRST]} \quad \text{[R-SECOND]} \quad \text{[R-CTXT]} \\ (\lambda x.e_1) e_2 \longrightarrow e_1[e_2/x] \quad \mathbf{fst} \langle e_1, e_2 \rangle \longrightarrow e_1 \quad \mathbf{snd} \langle e_1, e_2 \rangle \longrightarrow e_2 \quad \frac{e \longrightarrow f}{\mathbf{E}[e] \longrightarrow \mathbf{E}[f]} \end{array}$$

where the *evaluation contexts* are $\mathbf{E} ::= [] \mid (\mathbf{E} e) \mid (\mathbf{fst} \mathbf{E}) \mid (\mathbf{snd} \mathbf{E}) \mid (\mathbf{succ} \mathbf{E})$. *Normal forms* are defined as usual as expressions that do not reduce. The reflexive and transitive closure of \longrightarrow is denoted by \longrightarrow^* .

Note that \longrightarrow does not allow to reduce expressions which are inside the body of an abstraction or component of a pair as the usual β -reduction of lambda calculus does. In the standard terminology of lambda calculus, \longrightarrow actually corresponds to *weak head reduction* and the normal forms of \longrightarrow are actually called *weak head normal forms*.

2.3. Type System. First we assume a set κ containing the types for the constants:

$$\begin{array}{ll} \mathbf{pair} : t \rightarrow s \rightarrow t \times s & \mathbf{0} : \text{Nat} \\ \mathbf{fst} : t \times s \rightarrow t & \mathbf{succ} : \text{Nat} \rightarrow \text{Nat} \\ \mathbf{snd} : t \times s \rightarrow s & \end{array}$$

The *type assignment system* λ_\bullet^* is defined by the following rules.

$$\begin{array}{c} \text{[AXIOM]} \quad \text{[CONST]} \quad \text{[}\bullet\text{I]} \quad \text{[}\rightarrow\text{I]} \quad \text{[}\rightarrow\text{E]} \\ \frac{x : t \in \Gamma}{\Gamma \vdash x : t} \quad \frac{\mathbf{k} : t \in \kappa}{\Gamma \vdash \mathbf{k} : t} \quad \frac{\Gamma \vdash e : t}{\Gamma \vdash e : \bullet t} \quad \frac{\Gamma, x : \bullet^n t \vdash e : \bullet^n s}{\Gamma \vdash \lambda x.e : \bullet^n(t \rightarrow s)} \quad \frac{\Gamma \vdash e_1 : \bullet^n(t \rightarrow s) \quad \Gamma \vdash e_2 : \bullet^n t}{\Gamma \vdash e_1 e_2 : \bullet^n s} \end{array}$$

The rule $[\bullet\text{I}]$ introduces the modality. The rule is unusual in the sense that the expression remains the same. We do not have a constructor for \bullet at the level of expressions.

The rule $[\bullet\text{I}]$ does not have any restrictions, i.e. if it is known that a value will only be available with delay n , then it will also be available with any delay $m \geq n$, but not earlier. Using rule $[\bullet\text{I}]$ and the recursive type $\mathbf{D} = \bullet\mathbf{D} \rightarrow t$, we can derive that the fixed point combinator $\mathbf{fix} = \lambda y.(\lambda x.y (x x))(\lambda x.y (x x))$ has type $(\bullet t \rightarrow t) \rightarrow t$ by assigning the type $\mathbf{D} \rightarrow t$ to the first occurrence of $\lambda x.y (x x)$ and $\bullet\mathbf{D} \rightarrow t$ to the second one [23].

The rules $[\rightarrow I]$ and $[\rightarrow E]$ allow for an arbitrary delay in front of the types of the entities involved. Intuitively, the number of \bullet 's represents the delay at which a value becomes available. So for example, rule $[\rightarrow I]$ says that a function which accepts an argument x of type t delayed by n and produces a result of type s delayed by the same n has type $\bullet^n(t \rightarrow s)$, that is a function delayed by n that maps elements of t into elements of s .

2.4. Inductive data types. The temporal modal operator can only be used for defining co-inductive data types. Inductive data types need to be introduced separately into the calculus. For example, we can introduce primitive recursion on \mathbf{Nat} by means of a function $\mathbf{natrec} : t \rightarrow (\mathbf{Nat} \rightarrow t \rightarrow t) \rightarrow \mathbf{Nat} \rightarrow t$ with the reduction rule:

$$\mathbf{natrec} \ f_1 \ f_2 \ 0 \longrightarrow f_1 \quad \mathbf{natrec} \ f_1 \ f_2 \ (\mathbf{succ} \ e) \longrightarrow f_2 \ e \ (\mathbf{natrec} \ f_1 \ f_2 \ e)$$

We can also introduce a type \mathbf{List}_t for the set of finite lists as an inductive data type with $\mathbf{nil} : \mathbf{List}_t$ and $\mathbf{consl} : t \rightarrow \mathbf{List}_t \rightarrow \mathbf{List}_t$ and $\mathbf{lrec} : s \rightarrow (t \rightarrow \mathbf{List}_t \rightarrow s \rightarrow s) \rightarrow \mathbf{List}_t \rightarrow s$ with the reduction rule:

$$\mathbf{lrec} \ f_1 \ f_2 \ \mathbf{nil} \longrightarrow f_1 \quad \mathbf{lrec} \ f_1 \ f_2 \ (\mathbf{consl} \ e_1 \ e_2) \longrightarrow f_2 \ e_1 \ e_2 \ (\mathbf{lrec} \ f_1 \ f_2 \ e_2)$$

2.5. Disjoint union. We could add the disjoint union $t_1 + t_2$ of two types t_1 and t_2 and add constants:

$$\begin{aligned} \mathbf{inl} &: t_1 \rightarrow t_1 + t_2 \\ \mathbf{inr} &: t_2 \rightarrow t_1 + t_2 \\ \mathbf{case} &: (t_1 + t_2) \rightarrow (t_1 \rightarrow s) \rightarrow (t_2 \rightarrow s) \rightarrow s \end{aligned}$$

the reduction rules:

$$\mathbf{case} \ (\mathbf{inl} \ e) \ f_1 f_2 \longrightarrow f_1 \ e \quad \mathbf{case} \ (\mathbf{inr} \ e) \ f_1 f_2 \longrightarrow f_2 \ e$$

and the evaluation contexts are extended with $E ::= \dots \mid \mathbf{case} \ E \ f_1 f_2$.

Using the disjoint union, we can define the type of finite and infinite lists as:

$$\mathbf{coList}_t = \mathbf{Unit} + \bullet t \times \mathbf{coList}_t$$

2.6. Type of conatural numbers. We can also define the type of conatural numbers as $\mathbf{coNat} = \mathbf{Unit} + \bullet \mathbf{coNat}$. Then $\mathbf{z} = \mathbf{inl} \ \mathbf{unit}$ has type \mathbf{coNat} and $\mathbf{s} = \mathbf{inr}$ has type $\bullet \mathbf{coNat} \rightarrow \mathbf{coNat}$. The set of conatural numbers include $\mathbf{s}^\infty = \mathbf{fix} \ \mathbf{s}$. The usual definition of addition given by the equations:

$$n + 0 = n \quad n + (m + 1) = (n + m) + 1$$

can be written as

$$\mathbf{add} \ x \ y = \mathbf{case} \ y \ (\lambda y_1. x) \ (\lambda y_1. \mathbf{s} \ (\mathbf{add} \ x \ y_1))$$

and it can be typed in $\lambda_{\rightarrow}^\bullet$. However, the usual definition of subtraction given by the equations

$$n - 0 = n \quad 0 - n = 0 \quad (n + 1) - (m + 1) = n - m$$

which is expressed as:

$$\mathbf{minus} \ x \ y = \mathbf{case} \ x \ (\lambda x_1. \mathbf{z}) \ (\lambda x_1. \mathbf{case} \ y \ (\lambda y_1. \mathbf{z}) (\lambda y_1. \mathbf{minus} \ x_1 \ y_1))$$

is not typable because $(\mathbf{minus} \ \mathbf{s}^\infty \ \mathbf{s}^\infty)$ is not normalizing.

2.7. Examples of Typable Expressions in $\lambda_{\rightarrow}^{\bullet}$. We consider the function `skip` of the introduction that deletes the elements at even positions of a stream.

$$\text{skip} = \text{fix } \lambda f x. \langle (\text{fst } x), f (\text{snd } (\text{snd } x)) \rangle \quad (2.1)$$

In order to assign the type $\text{Str1}_{\text{Nat}} \rightarrow \text{Str2}_{\text{Nat}}$ to `skip`, the variable f has to be delayed once and the first occurrence of `snd` has to be delayed twice. Note also that when typing the application $f (\text{snd } (\text{snd } x))$ the rule $[\rightarrow E]$ is used with $n = 2$.

The following two functions have type $(t \rightarrow s) \rightarrow \text{Str1}_t \rightarrow \text{Str1}_s$ in $\lambda_{\rightarrow}^{\bullet}$.

$$\begin{aligned} \text{map } f x &= \langle f (\text{fst } x), (\text{map } f (\text{snd } x)) \rangle \\ \text{maap } f x &= \langle f (\text{fst } x), \langle f (\text{fst } (\text{snd } x)), (\text{maap } f (\text{snd } (\text{snd } x))) \rangle \rangle \end{aligned}$$

The following function has type $\text{Str1}_{\text{Nat}} \rightarrow \text{Str1}_{\text{Nat}} \rightarrow \text{Str1}_{\text{Nat}}$ in $\lambda_{\rightarrow}^{\bullet}$.

$$\text{sum } x y = \langle (\text{fst } x) + (\text{fst } y), \text{sum } (\text{snd } x) (\text{snd } y) \rangle$$

The following two functions have type $\text{Str1}_t \rightarrow \text{Str1}_t \rightarrow \text{Str1}_t$ in $\lambda_{\rightarrow}^{\bullet}$.

$$\begin{aligned} \text{interleave } x y &= \langle \text{fst } x, (\text{interleave } y (\text{snd } x)) \rangle \\ \text{merge } x y &= \text{if } (\text{fst } x) \leq (\text{fst } y) \text{ then } \langle \text{fst } x, \text{merge } (\text{snd } x) y \rangle \text{ else } \langle \text{fst } y, \text{merge } x (\text{snd } y) \rangle \end{aligned}$$

The following six functions have type Str1_{Nat} in $\lambda_{\rightarrow}^{\bullet}$.

$$\begin{aligned} \text{ones} &= \langle 1, \text{interleave ones ones} \rangle \\ \text{nats} &= \langle 0, \text{map succ nats} \rangle \\ \text{fib} &= \langle 0, \text{sum fib } \langle 1, \text{fib} \rangle \rangle \\ \text{fib}' &= \langle 0, \langle 1, \text{sum fib}' (\text{snd fib}') \rangle \rangle \\ \text{naats} &= \langle 0, \text{maap succ nats} \rangle \\ \text{ham} &= \langle 1, \text{merge } (\text{map } \lambda x. 2 * x \text{ ham}) (\text{merge } (\text{map } \lambda x. 3 * x \text{ ham}) (\text{map } \lambda x. 5 * x \text{ ham})) \rangle \end{aligned}$$

The functions `merge`, `ones`, `fib`, `fib'`, `naats` and `ham` cannot be typed by the proof assistant Coq because they do not satisfy the syntactic guardness condition (all recursive calls should be guarded by constructors) [12]. The function `fib'` cannot be typed with sized types [28].

It is possible to type in $\lambda_{\rightarrow}^{\bullet}$ other programs shown typable in other papers on the modal operator such as `toggle`, `paperfolds` [4].

2.8. Untypable programs in $\lambda_{\rightarrow}^{\bullet}$. We define the function `get` as follows:

$$\text{get} = \text{natrec fst } \lambda xyz. y (\text{snd } z) \quad (2.2)$$

The first argument of `natrec` has type $\text{Str1}_t \rightarrow t$ while the second argument has type $\text{Nat} \rightarrow (\text{Str1}_t \rightarrow t) \rightarrow \text{Str1}_t \rightarrow \bullet t$. It is not difficult to see that `get` is not typable in $\lambda_{\rightarrow}^{\bullet}$, unless $t = \bullet t$. Note that if we re-define `get` on the set of conatural numbers and use `fix` instead of `natrec` then we obtain a function `get'` defined by:

$$\text{get}' = \text{fix}(\lambda g. \lambda xy. \text{case } x (\lambda x_1. \text{fst } y) (\lambda x_1. g x_1 (\text{snd } y)))$$

which is not typable in $\lambda_{\rightarrow}^{\bullet}$, either.

We now consider the function `take` that takes the first n elements of a stream and returns a finite list in \mathbf{List}_t defined by:

$$\begin{aligned} \mathbf{take} \ 0 \ x &= \mathbf{nil} \\ \mathbf{take} \ (\mathbf{succ} \ n) \ x &= \mathbf{consl} \ (\mathbf{fst} \ x) \ (\mathbf{take} \ n \ (\mathbf{snd} \ x)) \end{aligned}$$

It is expressed using `natrec` as follows:

$$\mathbf{take} = \mathbf{natrec} \ \lambda x.\mathbf{nil} \ \lambda xyz.(\mathbf{consl} \ (\mathbf{fst} \ z) \ (y \ (\mathbf{snd} \ z))) \quad (2.3)$$

The first argument of `natrec` has type $\mathbf{Str1}_t \rightarrow \mathbf{List}_t$ and the second argument has type $\mathbf{Nat} \rightarrow (\mathbf{Str1}_t \rightarrow \mathbf{List}_t) \rightarrow \mathbf{Str1}_t \rightarrow \bullet\mathbf{List}_t$. It is not difficult to see that `take` is not typable in $\lambda_{\rightarrow}^{\bullet}$.

Though the type systems by Clouston et al [8] and the one by Atkey et al [1] have to introduce explicit constants for the introduction and elimination of \bullet , they can type programs such as `get` and `take` which $\lambda_{\rightarrow}^{\bullet}$ cannot do.

3. PROPERTIES OF TYPABLE EXPRESSIONS

This section proves the two most relevant properties of typable expressions, which are subject reduction (reduction of expressions preserves their types) and normalisation. As informally motivated in Section 2, the type constructor \bullet controls recursion and guarantees normalisation of any expression that has a type different from \bullet^{∞} . This section also proves that any typable expression has a Lévy-Longo and Böhm tree without \perp if the type is \bullet^{∞} -free and tail finite respectively.

3.1. Inversion and Subject Reduction.

Lemma 3.1 (Weakening). If $\Gamma \vdash e : t$ and $\Gamma \subseteq \Gamma'$ then $\Gamma' \vdash e : t$.

Proof. By an easy induction on the derivation. □

Lemma 3.2 (Delay). If $\Gamma_1, \Gamma_2 \vdash e : t$ then $\Gamma_1, \bullet\Gamma_2 \vdash e : \bullet t$.

Proof. By induction on the derivation. We only show the case for the rule $[\rightarrow\mathbf{I}]$.

$$\frac{[\rightarrow\mathbf{I}] \quad \Gamma_1, \Gamma_2, x : \bullet^n t_1 \vdash e : \bullet^n t_2}{\Gamma_1, \Gamma_2 \vdash \lambda x.e : \bullet^n (t_1 \rightarrow t_2)}$$

By induction hypothesis, $\Gamma_1, \bullet\Gamma_2, x : \bullet^{n+1} t_1 \vdash e : \bullet^{n+1} t_2$. By applying the rule $[\rightarrow\mathbf{I}]$, we conclude that $\Gamma_1, \bullet\Gamma_2 \vdash \lambda x.e : \bullet^{n+1} (t_1 \rightarrow t_2)$. □

Lemma 3.3 (Inversion).

- (1) If $\Gamma \vdash \mathbf{k} : t$ then $t = \bullet^n t'$ and $\mathbf{k} : t' \in \kappa$.
- (2) If $\Gamma \vdash x : t$ then $t = \bullet^n t'$ and $x : t' \in \Gamma$.
- (3) If $\Gamma \vdash \lambda x.e : t$ then $t = \bullet^n (t_1 \rightarrow t_2)$ and $\Gamma, x : \bullet^n t_1 \vdash e : \bullet^n t_2$.
- (4) If $\Gamma \vdash e_1 e_2 : t$ then $t = \bullet^n t_2$ and $\Gamma \vdash e_2 : \bullet^n t_1$ and $\Gamma \vdash e_1 : \bullet^n (t_1 \rightarrow t_2)$.

Proof. By case analysis and induction on the derivation. We only show Item 3. A derivation of $\Gamma \vdash \lambda x.e : t$ ends with an application of either $[\rightarrow\mathbf{I}]$ or $[\bullet\mathbf{I}]$. The proofs for $[\rightarrow\mathbf{I}]$ is immediate. If the last applied rule is $[\bullet\mathbf{I}]$ then $t = \bullet t'$ and $\Gamma \vdash \lambda x.e : t'$. By induction, $t' = \bullet^n (t_1 \rightarrow t_2)$ and $\Gamma, x : \bullet^n t_1 \vdash e : \bullet^n t_2$. Hence, $t = \bullet t' = \bullet^{n+1} (t_1 \rightarrow t_2)$ and $\Gamma, x : \bullet^{n+1} t_1 \vdash e : \bullet^{n+1} t_2$ by Lemma 3.2. □

Lemma 3.4 (Substitution). If $\Gamma, x : s \vdash e : t$ and $\Gamma \vdash f : s$ then $\Gamma \vdash e[f/x] : t$.

Proof. By induction on the structure of expressions. We only show the case when $e = \lambda y. e'$. It follows from Item 3 of Lemma 3.3 that $t = \bullet^n(t_1 \rightarrow t_2)$ and $\Gamma, x : s, y : \bullet^n t_1 \vdash e' : \bullet^n t_2$. By induction hypothesis, $\Gamma, y : \bullet^n t_1 \vdash e'[f/x] : \bullet^n t_2$. By applying the rule $[\rightarrow]$, $\Gamma \vdash \lambda y. e'[f/x] : \bullet^n(t_1 \rightarrow t_2)$. \square

Lemma 3.5 (Subject Reduction). If $\Gamma \vdash e : t$ and $e \longrightarrow e'$ then $\Gamma \vdash e' : t$.

Proof. By induction on the definition of \longrightarrow . We only do the case $(\lambda x. e_1) e_2 \longrightarrow e_1[e_2/x]$. Suppose $\Gamma \vdash (\lambda x. e_1) e_2 : t$. By Item 4 of Lemma 3.3 we have that:

$$t = \bullet^n t_2 \quad \Gamma \vdash e_2 : \bullet^n t_1 \quad \Gamma \vdash (\lambda x. e_1) : \bullet^n(t_1 \rightarrow t_2)$$

It follows from Item 3 of Lemma 3.3 that $\Gamma, x : \bullet^n t_1 \vdash e_1 : \bullet^n t_2$. By applying Substitution Lemma, we deduce that $\Gamma \vdash e_1[e_2/x] : \bullet^n t_2$. \square

Neither Nakano's type system nor ours is closed under η -reduction. For example, $y : \bullet(t \rightarrow s) \vdash \lambda x. (y x) : (t \rightarrow \bullet s)$ but $y : \bullet(t \rightarrow s) \not\vdash y : (t \rightarrow \bullet s)$. The lack of subject reduction for η -reduction is natural in the context of lazy evaluation where programs are closed terms and only weak head normalised.

3.2. Normalization. We prove that any expression which has a type t such that $t \neq \bullet^\infty$ reduces to a normal form (Theorem 3.13). For this, we define a type interpretation indexed on the set of natural numbers for dealing with the temporal operator \bullet . The time is discrete and represented using the set of natural numbers. The semantics reflects the fact that one \bullet corresponds to one unit of time by shifting the interpretation from i to $i + 1$.

Before introducing the type interpretation, we give a few definitions. Let \mathbb{E} be the set of expressions. We define the following subsets of \mathbb{E} :

$$\begin{aligned} \text{WN} &= \{e \mid e \longrightarrow^* f \ \& \ f \text{ is a normal form}\} \\ \text{HV} &= \{e \mid e \longrightarrow^* E[x] \ \& \ x \text{ is a variable}\} \end{aligned}$$

We will do induction on the rank of types. For **Nat** and type variables, the rank is always 0. For the other types, the rank measures the depth of all what we can observe at time 0. We could also compute it by taking the maximal 0-length of all the paths in the tree representation of the type, where the 0-length of a path is the number of type constructors different from \bullet from the root to a leaf or \bullet .

Definition 3.6 (Rank of a Type). The rank of a type t (notation $\text{rank}(t)$) is defined as follows.

$$\begin{aligned} \text{rank}(\mathbf{Nat}) &= \text{rank}(X) = \text{rank}(\bullet t) = 0 \\ \text{rank}(t \times s) &= \max(\text{rank}(t), \text{rank}(s)) + 1 \\ \text{rank}(t \rightarrow s) &= \max(\text{rank}(t), \text{rank}(s)) + 1 \end{aligned}$$

The rank is well defined (and finite) because the tree representation of a type cannot have an infinite branch with no \bullet 's at all (Condition 2 in Definition 2.1) and $\text{rank}(\bullet t)$ is set to 0.

We now define the type interpretation $\llbracket t \rrbracket \in \mathbb{N} \rightarrow \mathcal{P}(\mathbb{E})$, which is an indexed set, where \mathbb{N} is the set of natural numbers and \mathcal{P} is the powerset constructor.

Definition 3.7 (Type Interpretation). We define $([t])_i \subseteq \mathbb{E}$ by induction on $(i, \text{rank}(t))$.

$$\begin{aligned}
([\mathbf{Nat}])_i &= \mathbb{H}\mathbb{V} \cup \{e \mid e \longrightarrow^* n\} \\
([X])_i &= \mathbb{H}\mathbb{V} \\
([t \times s])_i &= \mathbb{H}\mathbb{V} \cup \{e \mid e \longrightarrow^* \langle e_1, e_2 \rangle \text{ and } e_1 \in ([t])_i \text{ and } e_2 \in ([s])_i\} \\
([t \rightarrow s])_i &= \mathbb{H}\mathbb{V} \cup \{e \mid e \longrightarrow^* \lambda x. f \text{ and } ee' \in ([s])_j \ \forall e' \in ([t])_j, j \leq i\} \\
&\quad \cup \{e \mid e \longrightarrow^* E[\mathbf{k}] \text{ and } ee' \in ([s])_j \ \forall e' \in ([t])_j, j \leq i\} \\
([\bullet t])_0 &= \mathbb{E} \\
([\bullet t])_{i+1} &= ([t])_i
\end{aligned}$$

Note that $([\bullet^\infty])_i = \mathbb{E}$ for all $i \in \mathbb{N}$. In the interpretation of the arrow type, the requirement “for all $j \leq i$ ” (and not just “for all i ”) is crucial for dealing with the contra-variance of the arrow type in the proof of Item 2 of Lemma 3.9.

Lemma 3.8.

- (1) $([\bullet^n t])_i = \mathbb{E}$ if $i < n$.
- (2) $([\bullet^n t])_i = ([t])_{i-n}$ if $i \geq n$.

Proof. Both items are proved by induction on n . □

Lemma 3.9. For all types t and $i \in \mathbb{N}$,

- (1) $\mathbb{H}\mathbb{V} \subseteq ([t])_i$.
- (2) $([t])_{i+1} \subseteq ([t])_i$.

Proof. (Item 1) follows by induction on i and doing case analysis on the shape of the type. (Item 2) follows by induction on $(i, \text{rank}(t))$. Suppose $e \in ([t \rightarrow s])_{i+1}$. Then, $ee' \in ([s])_j$ for $j \leq i + 1$. This is equivalent to saying that $ee' \in ([s])_{j'+1}$ for $j' \leq i$. By induction hypothesis, $([s])_{j'+1} \subseteq ([s])_{j'}$. Hence, $e \in ([t \rightarrow s])_i$. □

Lemma 3.10. If $t \neq \bullet^\infty$ then $\bigcap_{i \in \mathbb{N}} ([t])_i \subseteq \mathbb{W}\mathbb{N}$.

Proof. Suppose $t = \bullet^n t_0$ and t_0 is either \mathbf{Nat} , X , $t_1 \rightarrow t_2$ or $t_1 \times t_2$. Then, for all $i \geq n$,

$$([\bullet^n t_0])_i = ([t_0])_{i-n} \subseteq \mathbb{W}\mathbb{N}$$

by Lemma 3.8 and the definition of type interpretation. □

In order to deal with open expressions we resort to substitution functions, as usual. A substitution function is a mapping from (a finite set of) variables to \mathbb{E} . We use ρ to range over substitution functions. Substitution functions allows us to extend the semantics to typing judgements (notation $\Gamma \models_i e : t$).

Definition 3.11 (Typing Judgement Interpretation). Let ρ be a substitution function.

- (1) $\rho \models_i \Gamma$ if $\rho(x) \in ([t])_i$ for all $x : t \in \Gamma$.
- (2) $\Gamma \models_i e : t$ if $\rho(e) \in ([t])_i$ for all $\rho \models_i \Gamma$.

As expected we can show the soundness of our type system with respect to the indexed semantics.

Theorem 3.12 (Soundness). If $\Gamma \vdash e : t$ then $\Gamma \models_i e : t$ for all $i \in \mathbb{N}$.

The proof of the above theorem can be found in Appendix A.

Theorem 3.13 (Normalisation of Typable Expressions). If $\Gamma \vdash e : t$ and $t \neq \bullet^\infty$ then e reduces (in zero or more steps) to a (weak head) normal form.

Proof. It follows from Theorem 3.12 that

$$\Gamma \models_i e : t \quad (3.1)$$

for all $i \in \mathbb{N}$. Let id be the identity substitution and suppose $x : s \in \Gamma$. Then

$$\begin{aligned} id(x) = x &\in \mathbb{H}\mathbb{V} \\ &\subseteq \llbracket s \rrbracket_i \text{ by Item 1 of Lemma 3.9.} \end{aligned}$$

This means that $id \models_i \Gamma$ for all $i \in \mathbb{N}$. From Eq. (3.1) we have that $id(e) = e \in \llbracket t \rrbracket_i$ for all i . Hence,

$$e \in \bigcap_{i \in \mathbb{N}} \llbracket t \rrbracket_i$$

It follows from Lemma 3.10 that $e \in \mathbb{W}\mathbb{N}$. □

Notice that there are normalising expressions that cannot be typed, for example $\lambda x. \Omega \mathbf{I}$, where $\Omega = (\lambda y. y y)(\lambda y. y y)$ and $\mathbf{I} = \lambda z. z$. It is easy to show that $\mathbf{fix} \mathbf{I}$ has type \bullet^∞ and so does Ω by Lemma 3.5. By Theorem 3.13, it cannot have other types, and this implies that the application $\Omega \mathbf{I}$ has no type.

Notice also that there are normalizing expressions of type \bullet^∞ , e.g. $x : \bullet^\infty \vdash x : \bullet^\infty$. However, there are no normalizing closed expressions of type \bullet^∞ as the next lemma shows.

Lemma 3.14. If $\vdash e : t$ and e is normalizing then $t \neq \bullet^\infty$.

Proof. By Theorems 3.13 and 3.5, we can assume that e is a closed expression in normal form. Then, e could be either $\lambda x. f$ or $\langle e_1, e_2 \rangle$ or \mathbf{fst} or \mathbf{snd} or \mathbf{succ} or $\mathbf{succ}^n 0$. The type of all these expressions is different from \bullet^∞ . □

The function \mathbf{filter} that selects the elements of a list xs that satisfy p defined as

$$\mathbf{filter} p xs = \mathbf{if} p (\mathbf{fst} xs) \mathbf{then} \langle (\mathbf{fst} xs), (\mathbf{filter} p xs) \rangle \mathbf{else} (\mathbf{filter} p xs)$$

is normalizing (actually it is in normal form) and it can not be typed in λ_{\rightarrow}^* . Suppose towards a contradiction that \mathbf{filter} is typable. Then, $(\mathbf{filter} \mathbf{ones} (\lambda x. \mathbf{eq?} x 0))$ is also typable. But $(\mathbf{filter} \mathbf{ones} (\lambda x. \mathbf{eq?} x 0))$ is not normalizing contradicting Theorem 3.13.

3.3. Lévy-Longo and Böhm Trees. A simple way to give meaning to a computation of lambda calculus is to consider Lévy-Longo and Böhm trees [2, 21, 22]. Consider, for example, a procedure for computing the decimal expansions of π ; if implemented appropriately, it can provide partial output as it runs and this ongoing output is a natural way to assign meaning to the computation. This is in contrast to a program that loops infinitely without ever providing an output. These two procedures have very different intuitive meanings. This section gives a nice characterization of computations that never produce meaningless information by means of types.

In lambda calculus, expressions such as $\mathbf{fix} x$ or $\mathbf{fix} (\lambda xy. x)$ have no finite normal form though they are intuitively meaningful and should be distinguished from meaningless expressions such as $\mathbf{fix} \mathbf{I}$. By analogy with π , the Lévy-Longo (as well as the Böhm) tree of a term is obtained as the limit of these partial outputs. If in the process of computing the Lévy-Longo or Böhm tree of a term, we find a subexpression that has no meaning such as $\mathbf{fix} \mathbf{I}$ then this is recorded by replacing $\mathbf{fix} \mathbf{I}$ by \perp . Lévy-Longo and Böhm trees differ

on the notion of meaningless expressions: weak head normal forms for the former and head normal forms for the latter.

Definition 3.15 (Lèvy-Longo Tree). Let e be an expression (it may be untypable). The Lèvy-Longo tree of e , denoted as $LLT(e)$ is defined coinductively as follows.

$$LLT(e) = \begin{cases} x \ LLT(e_1) \dots LLT(e_n) & \text{if } e \longrightarrow^* x \ e_1 \dots e_n \\ \mathbf{k} \ LLT(e_1) \dots LLT(e_n) & \text{if } e \longrightarrow^* \mathbf{k} \ e_1 \dots e_n \\ \lambda x. LLT(e') & \text{if } e \longrightarrow^* \lambda x. e' \\ \perp & \text{otherwise, i.e. } e \text{ has no (weak head) normal form} \end{cases}$$

Definition 3.16 (Böhm Tree). Let e be an expression (it may be untypable). The Böhm tree of e , denoted as $BT(e)$ is defined coinductively as follows.

$$BT(e) = \begin{cases} \lambda x_1 \dots x_k. x \ BT(f_1) \dots BT(f_n) & \text{if } hnf(e) = \lambda x_1 \dots x_k. x \ f_1 \dots f_n \\ \lambda x_1 \dots x_k. \mathbf{k} \ BT(f_1) \dots BT(f_n) & \text{if } hnf(e) = \lambda x_1 \dots x_k. \mathbf{k} \ f_1 \dots f_n \\ \perp & \text{otherwise} \end{cases}$$

where $hnf(e)$ is the *head normal form* of e defined as follows:

$$hnf(e) = \begin{cases} \lambda x_1 \dots x_k. x \ f_1 \dots f_n & \text{if } e = e_0 \text{ and } e_k \longrightarrow^* x \ f_1 \dots f_n \\ & \text{and } e_i \longrightarrow^* \lambda x_{i+1}. e_{i+1} \text{ for } 0 \leq i \leq k-1 \\ \lambda x_1 \dots x_k. \mathbf{k} \ f_1 \dots f_n & \text{if } e = e_0 \text{ and } e_k \longrightarrow^* \mathbf{k} \ f_1 \dots f_n \\ & \text{and } e_i \longrightarrow^* \lambda x_{i+1}. e_{i+1} \text{ for } 0 \leq i \leq k-1 \\ \perp & \text{otherwise} \end{cases}$$

Lèvy-Longo and Böhm trees are obtained as possible infinite normal forms of possible infinite β -reduction sequences. Infinitary normalization formalizes the idea of productivity, i.e. an infinite reduction sequence which always produces part of the result [16, 17, 30, 20]. Besides the β -rule, the infinitary lambda calculus of Lèvy-Longo trees has a reduction rule defined as $M \longrightarrow \perp$ if M has no weak head normal form while the one of Böhm trees has a similar rule using the condition that M has no head normal form.

Definition 3.17. We say that t is \bullet^∞ -free if it does not contain \bullet^∞ .

For example, $\mathbf{Str1}_{\mathbf{Nat}}$ and $\mathbf{E}_{\mathbf{Nat}}$ are \bullet^∞ -free where $\mathbf{E}_t = t \rightarrow \bullet \mathbf{E}_t$. But $\mathbf{Str1}_{\bullet^\infty}$ and $\mathbf{Nat} \rightarrow \bullet^\infty$ are not \bullet^∞ -free.

Theorem 3.18 (Productivity I: Lèvy-Longo Trees without \perp). Let t and all types in Γ be \bullet^∞ -free. If $\Gamma \vdash e : t$ then the Lèvy-Longo tree of e has no \perp 's.

Proof. We construct the Lèvy-Longo tree depth by depth. By Theorem 3.13, e reduces to a (weak head) normal form e_0 which is typable by Lemma 3.5. Suppose e_0 is $\lambda x. e'$ (the cases when $e_0 = x \ e_1 \dots e_n$ or $e_0 = \mathbf{k} \ e_1 \dots e_n$ are similar). The Lèvy-Longo tree of e contains λx at depth 0. It only remains to show that we can construct the Lèvy-Longo tree of e' which will be at depth $n > 0$. It follows from Inversion Lemma that $\Gamma, x : t_1 \vdash e' : t_2$ and both t_1 and t_2 are \bullet^∞ -free. Hence, we can repeat the process to obtain the Lèvy-Longo tree of e' . \square

Definition 3.19.

- (1) An *infinite alternation of \bullet 's and \rightarrow 's* is a type of the form: $t = \bullet^{n_1}(t_1 \rightarrow \bullet^{n_2}(t_2 \rightarrow \dots))$.
- (2) We say that t is *tail finite* if it is \bullet^∞ -free and it does not contain an infinite alternation of \bullet 's and \rightarrow 's.

For example, $\mathbf{G} = \mathbf{Nat} \rightarrow \bullet(\mathbf{Nat} \times \mathbf{G})$ is tail finite but $\mathbf{F} = \mathbf{Nat} \rightarrow \bullet \mathbf{F}$ is not.

Theorem 3.20 (Productivity II: Böhm Trees without \perp). Let all the types in Γ and t be tail finite. If $\Gamma \vdash e : t$ then the Böhm tree of e has no \perp 's.

Proof. By Theorem 3.18, the Lévy-Longo tree of e has no \perp 's. Suppose the Lévy-Longo tree of e contains a subtree of the form $\lambda x_1 \lambda x_2 \lambda x_3 \dots$. It is not difficult to prove using Lemma 3.3 and Lemma 3.5 that the type should contain an infinite alternation of \bullet 's and \rightarrow 's. \square

Let $e = \mathbf{fix}(\lambda xy.x)$. Then e has type \mathbf{ENat} which is \bullet^∞ -free and $LLT(e) = \lambda x.\lambda x\dots$ has no \perp . For example, \mathbf{fib} has type $\mathbf{Str1Nat}$ which is tail finite and $BT(\mathbf{fib}) = \langle 0, \langle 1, \langle 1, \langle 2, \dots \rangle \rangle \rangle \rangle$ has no \perp 's.

4. FORMAL COMPARISON WITH OTHER TYPE SYSTEMS

This section establishes a formal relation between λ_\bullet^\bullet and the typed lambda calculi of Krishnaswami and Benton [18] and Nakano [23].

4.1. Embedding the Type System by Krishnaswami and Benton into λ_\bullet^\bullet . The typing rules of λ^{KB} defined by Krishnaswami and Benton [18] are:

$$\begin{array}{c} \frac{[\text{AXIOM}]}{\Gamma, x :_i t \vdash x :_j t} \quad j \geq i \quad \frac{[\text{CONST}]}{\Gamma \vdash \mathbf{k} :_i t} \quad \frac{[\bullet\text{I}]}{\Gamma \vdash e :_{i+1} t} \quad \frac{[\bullet\text{E}]}{\Gamma \vdash e :_i \bullet t} \\ \frac{[\rightarrow\text{I}]}{\Gamma, x :_i t \vdash e :_i s} \quad \frac{[\rightarrow\text{E}]}{\Gamma \vdash e_1 :_i (t \rightarrow s) \quad \Gamma \vdash e_2 :_i t} \\ \Gamma \vdash \lambda x.e :_i (t \rightarrow s) \quad \Gamma \vdash e_1 e_2 :_i s \end{array}$$

λ^{KB} has only the recursive types $\mathbf{Str1}_t = t \times \bullet \mathbf{Str1}_t$ for infinite lists with only one \bullet . Since \mathbf{fix} cannot be expressed in λ^{KB} , we need to add it to the set κ of constants:

$$\mathbf{fix} : (\bullet t \rightarrow t) \rightarrow t$$

The mapping $itob$ replaces the subindex i by \bullet^i in typing contexts.

$$\begin{array}{l} itob(\emptyset) = \emptyset \\ itob(\Gamma, x :_i t) = itob(\Gamma), x : \bullet^i t \end{array}$$

The function $clear$ removes the constructor and destructor of \bullet from e , i.e. $clear(\bullet e') = e'$ and $clear(\mathbf{await} e') = e'$.

Lemma 4.1 (Embedding λ^{KB} into λ_\bullet^\bullet). If $\Gamma \vdash e :_i t$ in λ^{KB} then $itob(\Gamma) \vdash clear(e) : \bullet^i t$ in λ_\bullet^\bullet .

Proof. By an easy induction on the derivation. \square

If only recursive types for lists are available, one cannot create other recursive types such as trees but having only one bullet also limits the amount of functions on streams we can type. For example, the functions \mathbf{skip} (see Eq. (2.1)), \mathbf{ones}' and \mathbf{pairup} are not typable in λ^{KB} where

$$\begin{array}{l} \mathbf{ones}' = \langle 1, \mathbf{interleave} \mathbf{ones}' (\mathbf{snd} \mathbf{ones}') \rangle \\ \mathbf{pairup} \ xs = \langle \langle (\mathbf{fst} \ xs), (\mathbf{fst} (\mathbf{snd} \ xs)) \rangle, (\mathbf{snd} (\mathbf{snd} \ xs)) \rangle \end{array}$$

The functions `skip`, `ones'` and `pairup` are all typeable in $\lambda_{\bullet}^{\circ}$, because $\lambda_{\bullet}^{\circ}$ has more flexibility in the location and number of \bullet 's which can be inserted in the types. We can derive that `ones'` has type $\mathbf{Str1}_X$ in $\lambda_{\bullet}^{\circ}$, by assigning the type $\mathbf{Str1}_X \rightarrow \bullet\mathbf{Str1}_X \rightarrow \mathbf{Str1}_X$ to `interleave` and `pairup` has type $\mathbf{StrHalf}_X \rightarrow \mathbf{Str1}_X$ where $\mathbf{StrHalf}_X = X \times X \times \bullet\mathbf{StrHalf}_X$.

4.2. Embedding $\lambda_{\bullet}^{\circ}$ into the Type System by Nakano. We consider the type system by Nakano of [23] without \top and call it λ_{Nak} .

Types in λ_{Nak} are finite, written in μ -notation and denoted by T, S . Types in λ_{Nak} are *guarded* which means that the type variable X of $\mu X.T$ can occur free in T only under the scope of a \bullet .

Let \simeq be the equivalence between types which allows for an infinite number of unfolding and Σ be a set $\{X_1 \leq Y_1, \dots, X_n \leq Y_n\}$ of subtype assumptions between type variables which cannot contain twice the same variable. The subtyping relation of λ_{Nak} is defined as the smallest relation on types closed under the following rules:

$$\begin{array}{c} [\leq \bullet] \quad [\bullet \rightarrow] \quad [\rightarrow \bullet] \quad [\text{VAR}] \\ \Sigma \vdash T \leq \bullet T \quad \Sigma \vdash T \rightarrow S \leq \bullet T \rightarrow \bullet S \quad \Sigma \vdash \bullet T \rightarrow \bullet S \leq \bullet(T \rightarrow S) \quad \Sigma, X \leq Y \vdash X \leq Y \end{array}$$

$$\begin{array}{c} [\simeq] \quad [\mu] \quad [\bullet] \quad [\rightarrow] \\ \frac{T \simeq S}{\Sigma \vdash T \leq S} \quad \frac{[\mu] \quad \Sigma, X \leq Y \vdash T \leq S}{\Sigma \vdash \mu X.T \leq \mu Y.S} \quad \frac{[\bullet] \quad \Sigma \vdash T \leq S}{\Sigma \vdash \bullet T \leq \bullet S} \quad \frac{[\rightarrow] \quad \Sigma_1 \vdash S_1 \leq T_1 \quad \Sigma_2 \vdash T_2 \leq S_2}{\Sigma_1 \cup \Sigma_2 \vdash T_1 \rightarrow T_2 \leq S_1 \rightarrow S_2} \end{array}$$

In $\Sigma \vdash T \leq S$, we assume that all the free type variables of T and S are in Σ and that the X 's cannot occur on the right hand side of \leq and the Y 's cannot occur on the left. The typing rules of λ_{Nak} are defined as follows.

$$\begin{array}{c} [\text{AXIOM}] \quad [\leq] \\ \frac{x : T \in \Gamma}{\Gamma \vdash x : T} \quad \frac{\Gamma \vdash e : T \quad T \leq T'}{\Gamma \vdash e : T'} \\ [\bullet \text{ E}] \quad [\rightarrow \text{ I}] \quad [\rightarrow \text{ E}] \\ \frac{\bullet \Gamma \vdash e : \bullet T \quad \Gamma, x : T \vdash e : S}{\Gamma \vdash e : T \quad \Gamma \vdash \lambda x.e : (T \rightarrow S)} \quad \frac{\Gamma \vdash e_1 : \bullet^n(T \rightarrow S) \quad \Gamma \vdash e_2 : \bullet^n T}{\Gamma \vdash e_1 e_2 : \bullet^n S} \end{array}$$

We define the function $n_{f_{\mu}}$ on guarded μ -types as follows.

$$\begin{array}{l} n_{f_{\mu}}(X) = X \quad n_{f_{\mu}}(T \rightarrow S) = n_{f_{\mu}}(T) \rightarrow n_{f_{\mu}}(TS) \\ n_{f_{\mu}}(\bullet T) = \bullet T \quad n_{f_{\mu}}(\mu X.T) = n_{f_{\mu}}(T)\{X \mapsto T\} \end{array}$$

We define the translation *tree* from μ -types to types in $\lambda_{\bullet}^{\circ}$, by coinduction as follows:

$$\text{tree}(T) = \begin{cases} \text{tree}(X) & \text{if } n_{f_{\mu}}(T) = X \\ \bullet \text{tree}(S) & \text{if } n_{f_{\mu}}(T) = \bullet S \\ \text{tree}(T_1) \rightarrow \text{tree}(T_2) & \text{if } n_{f_{\mu}}(T) = T_1 \rightarrow T_2 \end{cases}$$

The translation $\text{tree}(\Sigma)$ is extended to typing contexts in the obvious way.

Lemma 4.2 (Embedding $\lambda_{\bullet}^{\circ}$ into λ_{Nak}). If $\text{tree}(\Gamma) \vdash e : \text{tree}(T)$ in $\lambda_{\bullet}^{\circ}$, without using rule $[\text{CONST}]$ then $\Gamma \vdash e : T$ in λ_{Nak} .

Proof. It follows by induction on the derivation. It is easy to see that the typing rules $[\rightarrow \text{ I}]$ and $[\bullet \text{ I}]$ of $\lambda_{\bullet}^{\circ}$, are admissible in λ_{Nak} by making use of the typing rule $[\leq]$. \square

Programs in λ_{Nak} have more types than in $\lambda_{\bullet}^{\circ}$. For example, $\vdash \lambda x.x : (T \rightarrow S) \rightarrow (\bullet T \rightarrow \bullet S)$ in λ_{Nak} but $\not\vdash \lambda x.x : (T \rightarrow S) \rightarrow (\bullet T \rightarrow \bullet S)$ in $\lambda_{\bullet}^{\circ}$.

5. DENOTATIONAL SEMANTICS

This section gives a denotational semantics for $\lambda_{\bullet}^{\circ}$, where types and expressions are interpreted as objects and morphisms in the *topos of trees* [3]. We give a self-contained description of this topos as a cartesian closed category for a reader familiar with λ -calculus.

The *topos \mathcal{S} of trees* has as objects A families of sets A_1, A_2, \dots indexed by positive integers, equipped with family of restrictions $r_i^A : A_{i+1} \rightarrow A_i$. Types will be interpreted as family of sets (not just sets). Intuitively the family represents better and better sets of approximants for the values of that type. Arrows $f : A \rightarrow B$ are families of functions $f_i : A_i \rightarrow B_i$ obeying the naturality condition $f_i \circ r_i^A = r_i^B \circ f_{i+1}$.

$$\begin{array}{ccccc} A_1 & \xleftarrow{r_1^A} & A_2 & \xleftarrow{r_2^A} & A_3 & \dots \\ f_1 \downarrow & & f_2 \downarrow & & f_3 \downarrow & \\ B_1 & \xleftarrow{r_1^B} & B_2 & \xleftarrow{r_2^B} & B_3 & \dots \end{array}$$

We define $r_{ii}^A = id_A$ and $r_{ij}^A = r_j \circ \dots \circ r_{i-1}$ for $1 \leq j < i$. Products are defined pointwise.

Exponentials B^A have as components the sets:

$$(B^A)_i = \{(f_1, \dots, f_i) \mid f_j : A_j \rightarrow B_j \text{ and } f_j \circ r_j^A = r_j^B \circ f_{j+1}\}$$

and as restrictions $r_i^{A \Rightarrow B}(f_1, \dots, f_{i+1}) = (f_1, \dots, f_i)$.

We define $eval : B^A \times A \rightarrow B$ as $eval_i((f_1, \dots, f_i), a) = f_i(a)$ and $curry(f) : C \rightarrow B^A$ for $f : C \times A \rightarrow B$ as $curry(f)_i(c) = (g_1, \dots, g_i)$ where $g_j(a) = f_j(r_{ij}^A(c), a)$ for all $a \in A_j$ and $1 \leq j \leq i$. The functor $\blacktriangleright : \mathcal{S} \rightarrow \mathcal{S}$ is defined on objects as $(\blacktriangleright A)_1 = \{*\}$ and $(\blacktriangleright A)_{i+1} = A_i$ where $r_1^{\blacktriangleright A} = !$ and $r_{i+1}^{\blacktriangleright A} = r_i^A$ and on arrows $(\blacktriangleright f)_1 = id_{\{*\}}$ and $(\blacktriangleright f)_{i+1} = f_i$. We write \blacktriangleright^n for the n -times composition of \blacktriangleright .

The natural transformation $next_A : A \rightarrow \blacktriangleright A$ is given by $(next_A)_1 = !$ and $(next_A)_{i+1} = r_i^A$ which can easily be extended to a natural transformation $next_A^n : A \rightarrow \blacktriangleright^n A$. It is not difficult to see that there are isomorphisms $\theta : \blacktriangleright A \times \blacktriangleright B \rightarrow \blacktriangleright (A \times B)$ and $\xi : (\blacktriangleright B)^{(\blacktriangleright A)} \rightarrow \blacktriangleright (B^A)$ which are also natural. These can also be easily extended to isomorphisms $\theta^n : \blacktriangleright^n A \times \blacktriangleright^n B \rightarrow \blacktriangleright^n (A \times B)$ and $\xi^n : (\blacktriangleright^n B)^{(\blacktriangleright^n A)} \rightarrow \blacktriangleright^n (B^A)$.

The category Set is a full subcategory of \mathcal{S} via the functor \mathcal{I} with $(\mathcal{I}(X))_i = X$ and $r^{\mathcal{I}(X)} = id_X$ and $(\mathcal{I}(f))_i = f$. The unit 1 of \mathcal{S} is $\mathcal{I}(*)$.

A type t is interpreted as an object in \mathcal{S} (for simplicity, we omit type variables).

$$\begin{array}{ll} [\mathbf{Nat}] & = \mathcal{I}(\mathbf{Nat}) & [\bullet t] & = \blacktriangleright \circ [t] \\ [t \times s] & = [t] \times [s] & [t \rightarrow s] & = [s]^{[t]} \end{array}$$

In order to justify that the interpretation is well-defined, it is necessary to view the above definition as indexed sets and we do induction on $(i, rank(t))$ taking the lexicographic order where $rank(t)$ is defined in Section 3.2. By writing the indices explicitly in the definition of

$[t]$, we obtain:

$$\begin{aligned}
[\mathbf{Nat}]_i &= \mathbf{Nat} \\
[t \times s]_i &= [t]_i \times [s]_i \\
[t \rightarrow s]_i &= \{(f_1, \dots, f_i) \mid f_j : [t]_j \rightarrow [s]_j \text{ and } f_j \circ r_j^{[t]} = r_j^{[s]} \circ f_{j+1}\} \\
[\bullet t]_1 &= \{*\} \\
[\bullet t]_{i+1} &= [t]_i
\end{aligned}$$

As it is common in categorical semantics, the interpretation is not defined on lambda terms in isolation but on typing judgements. In order to define terms as morphisms, we need the context and the type to specify their domain and co-domain. Typing contexts $\Gamma = x_1 : t_1, \dots, x_k : t_k$ are interpreted as $[t_1] \times \dots \times [t_k]$. The interpretation of typed expressions $[\Gamma \vdash e : t] : [\Gamma] \rightarrow [t]$ is defined by induction on e (using Inversion Lemma):

$$\begin{aligned}
[\Gamma \vdash x : \bullet^n t] &= \text{next}^n \circ \pi_j \text{ if } x = x_j \text{ and } t_j = t \text{ and } 1 \leq j \leq k \\
[\Gamma \vdash \lambda x. e : \bullet^n (t_1 \rightarrow t_2)] &= \xi^n \circ \text{curry}([\Gamma, x : \bullet^n t_1 \vdash e : \bullet^n t_2]) \\
[\Gamma \vdash e_1 e_2 : \bullet^n s] &= \text{eval} \circ \langle (\xi^n)^{-1} \circ [\Gamma \vdash e_1 : \bullet^n (t \rightarrow s)], [\Gamma \vdash e_2 : \bullet^n t] \rangle \\
[\Gamma \vdash \text{pair} : (t \rightarrow s \rightarrow (t \times s))] &= \text{curry}(\text{curry}(!_{[\Gamma]} \times \text{id}_{[t \times s]})) \\
[\Gamma \vdash \text{fst} : (t \times s \rightarrow t)] &= \text{curry}(!_{[\Gamma]} \times \pi_1) \\
[\Gamma \vdash \text{snd} : (t \times s \rightarrow s)] &= \text{curry}(!_{[\Gamma]} \times \pi_2) \\
[\Gamma \vdash \mathbf{0} : \mathbf{Nat}] &= \mathcal{I}(* \mapsto 0) \\
[\Gamma \vdash \text{succ} : (\mathbf{Nat} \rightarrow \mathbf{Nat})] &= \text{succ}
\end{aligned}$$

Lemma 5.1 (Semantic Substitution).

$$[\Gamma, x : t \vdash e_1 : s] \circ \langle \text{id}, [\Gamma \vdash e_2 : t] \rangle = [\Gamma \vdash e_1[e_2/x] : s]$$

Proof. This follows by induction on e_1 . We only sketch the proof for the case $e_1 = x$.

$$[\Gamma, x : t \vdash x : \bullet^n t] \circ \langle \text{id}, [\Gamma \vdash e_2 : t] \rangle = \text{next}^n \circ [\Gamma \vdash e_2 : t] = [\Gamma \vdash e_2 : \bullet^n t] \quad \square$$

Theorem 5.2 (Soundness). If $\Gamma \vdash e : t$ and $e \longrightarrow e'$ then $[\Gamma \vdash e : t] = [\Gamma \vdash e' : t]$.

Proof. We show the case of $[\text{R-BETA}]$. Let $v_1 = [\Gamma, x : \bullet^n t_1 \vdash e_1 : \bullet^n t_2]$ and $v_2 = [\Gamma \vdash e_2 : \bullet^n t_1]$.

$$\begin{aligned}
[\Gamma \vdash (\lambda x. e_1) e_2 : \bullet^n t_2] &= \text{eval} \circ \langle (\xi^n)^{-1} \circ \xi^n \circ \text{curry}(v_1), v_2 \rangle = v_1 \circ \langle \text{id}, v_2 \rangle \\
&= [\Gamma \vdash e_1[e_2/x] : \bullet^n t_2] \text{ by Lemma 5.1} \quad \square
\end{aligned}$$

We consider contexts C without any restriction in the position of the hole $[\]$. We say that C is a *closing context of type t* if $x : s \vdash C[x] : t$ for some variable x that does not occur in C .

Definition 5.3. Let $\Gamma \vdash e_1 : t$ and $\Gamma \vdash e_2 : t$. We say that e_1 and e_2 are *observationally equivalent*, denoted by $e_1 \approx e_2$, if $C[e_1] \longrightarrow^* \text{succ}^n \mathbf{0}$ iff $C[e_2] \longrightarrow^* \text{succ}^n \mathbf{0}$ for all closing contexts C of type \mathbf{Nat} .

Corollary 5.4. If $[\Gamma \vdash e_1 : t] = [\Gamma \vdash e_2 : t]$ then, $e_1 \approx e_2$.

Proof. By compositionality of the denotational semantics, $[\vdash C[e_1] : \mathbf{Nat}] = [\vdash C[e_2] : \mathbf{Nat}]$. Suppose $C[e_1] \longrightarrow^* \text{succ}^n \mathbf{0}$. By Theorem 5.2, $[\vdash C[e_1] : \mathbf{Nat}] = [\text{succ}^n \mathbf{0}]$. By Theorem 3.13, $C[e_2] \longrightarrow^* \text{succ}^m \mathbf{0}$. It is not difficult to show using the definition of interpretation that $m = n$. \square

6. A TYPE INFERENCE ALGORITHM

In this section, we define a type inference algorithm for $\lambda_{\rightarrow}^{\bullet}$. Apart from the usual complications that come from having no type declarations, the difficulty of finding an appropriate type inference algorithm for $\lambda_{\rightarrow}^{\bullet}$ is due to the fact that the expressions do not have a constructor and destructor for \bullet . We do not know which sub-expressions need to be delayed as illustrated by the type derivation of `fix` where the first occurrence of $(\lambda x.y (x x))$ has a different derivation from the second one since $[\bullet I]$ is applied in different places [23]. Even worse, in case a sub-expression has to be delayed, we do not know how many times needs to be delayed to be able to type the whole expression.

6.1. A Syntax-directed Type System. We obtain a syntax-directed type system by eliminating the rule $[\bullet I]$ from $\lambda_{\rightarrow}^{\bullet}$. The *type assignment system* $(\lambda_{\rightarrow}^{\bullet})^{-}$ is defined by the following rules.

$$\begin{array}{c} \text{[AXIOM]} \\ \frac{x : t \in \Gamma}{\Gamma \vdash x : \bullet^n t} \end{array} \quad \begin{array}{c} \text{[CONST]} \\ \frac{\mathbf{k} : t \in \kappa}{\Gamma \vdash \mathbf{k} : \bullet^n t} \end{array} \quad \begin{array}{c} \text{[}\rightarrow\text{I]} \\ \frac{\Gamma, x : \bullet^n t \vdash e : \bullet^n s}{\Gamma \vdash \lambda x.e : \bullet^n (t \rightarrow s)} \end{array} \quad \begin{array}{c} \text{[}\rightarrow\text{E]} \\ \frac{\Gamma \vdash e_1 : \bullet^n (t \rightarrow s) \quad \Gamma \vdash e_2 : \bullet^n t}{\Gamma \vdash e_1 e_2 : \bullet^n s} \end{array}$$

Lemma 6.1 (Equivalence between $\lambda_{\rightarrow}^{\bullet}$ and $(\lambda_{\rightarrow}^{\bullet})^{-}$). $\Gamma \vdash e : t$ in $(\lambda_{\rightarrow}^{\bullet})^{-}$ iff $\Gamma \vdash e : t$ in $\lambda_{\rightarrow}^{\bullet}$.

Proof. The direction from left to right follows by induction on the derivation. The direction from right to left follows by induction on e using Lemma 3.3. \square

Though $\lambda_{\rightarrow}^{\bullet}$ and $(\lambda_{\rightarrow}^{\bullet})^{-}$ are equivalent in the sense they type the same expressions, they do not have the same type derivations. The derivations of $(\lambda_{\rightarrow}^{\bullet})^{-}$ are more restrictive than the ones of $\lambda_{\rightarrow}^{\bullet}$ since one can only introduce \bullet 's at the beginning of the derivation with the rules $[\text{AXIOM}]$ and $[\text{CONST}]$.

6.2. Meta-types. The type inference algorithm infers *meta-types* which are a generalization of types where the \bullet can be exponentiated with integer expressions, e.g. $\bullet^{N_1} X \rightarrow \bullet^{N_2 - N_1} X$. The syntax for *meta-types* is defined below. A meta-type can contain type variables and (non-negative) integer expressions with variables. In this syntax, $\bullet T$ is written as $\bullet^1 T$. We identify $\bullet^0 T$ with T and $\bullet^E \bullet^{E'} T$ with $\bullet^{E+E'} T$.

$E ::= \textit{ind}$	Integer Expression	$T ::= \textit{coind}$	Pseudo Meta-type
N	(integer variable)	X	(type variable)
\mathbf{n}	(integer number)	\mathbf{Nat}	(natural numbers)
$E + E$	(addition)	$T \times T$	(product)
$E - E$	(subtraction)	$T \rightarrow T$	(arrow)
		$\bullet^E T$	(delay)

Definition 6.2. We say that T is *positive* if all the exponents of \bullet are natural numbers.

We identify a positive pseudo meta-type with the pseudo-type obtained from replacing the type constructor \bullet^n in the syntax of pseudo meta-types with n consecutive \bullet 's in the syntax of pseudo-types.

Definition 6.3 (Integer Expression and Type Substitution). Let $\tau = \{X_1 \mapsto T_1, \dots, X_n \mapsto T_n\}$ be a finite mapping from type variables to pseudo meta-types where $\text{dom}(\tau) = \{X_1, \dots, X_n\}$. Let $\theta = \{N_1 \mapsto E_1, \dots, N_m \mapsto E_m\}$ be a finite mapping from integer variables to integer expressions where $\text{dom}(\theta) = \{N_1, \dots, N_m\}$. Let also $\sigma = \tau \cup \theta$. We define the substitutions $\theta(E)$ on an integer expression E and $\sigma(T)$ on a pseudo meta-type T as follows.

$$\begin{array}{ll} \theta(N) & = \begin{cases} E & \text{if } N \mapsto E \in \theta \\ N & \text{otherwise} \end{cases} & \sigma(X) & = \begin{cases} T & \text{if } X \mapsto T \in \tau \\ X & \text{otherwise} \end{cases} \\ \theta(\mathbf{n}) & = \mathbf{n} & \sigma(\mathbf{Nat}) & = \mathbf{Nat} \\ \theta(E_1 + E_2) & = \theta(E_1) + \theta(E_2) & \sigma(T_1 \rightarrow T_2) & = \sigma(T_1) \rightarrow \sigma(T_2) \\ \theta(E_1 - E_2) & = \theta(E_1) - \theta(E_2) & \sigma(T_1 \times T_2) & = \sigma(T_1) \times \sigma(T_2) \\ & & \sigma(\bullet^E T) & = \bullet^{\theta(E)} \sigma(T) \end{array}$$

If the substitution τ is $\{X \mapsto T\}$ then $\tau(T)$ is also denoted as $T\{X \mapsto T\}$ as usual.

Composition of substitutions is defined as usual. Note that the empty set is the identity substitution and $\sigma = \theta \circ \tau$ if $\sigma = \tau \cup \theta$.

Definition 6.4. We say that θ is *natural* if it maps all integer variables into natural numbers, i.e. $\theta(N) \in \mathbb{N}$ for all $N \in \text{dom}(\tau)$.

For example, the substitution $\theta = \{N \mapsto 3, M \mapsto 5\}$ is natural but $\theta(\bullet^{N-M} \mathbf{Nat})$ is not positive.

We can define *regular* pseudo meta-types similar to Definition 2.1. We say that a pseudo meta-type T is *guarded* if $\theta(T)$ is positive and guarded (as a pseudo type) for all natural θ .

The pseudo meta-type which is solution of the recursive equation $T = \mathbf{Nat} \times \bullet^N T$ is not guarded but it is guarded if we substitute N by $M + 1$.

Definition 6.5. A *meta-type* is a pseudo meta-type that is regular and guarded.

Note that finite pseudo meta-types are always meta-types, e.g. $\mathbf{Nat} \times \bullet^N X$ is a meta-type.

Definition 6.6. We say that the substitution $\sigma = \tau \cup \theta$ is:

- (1) *guarded* if $\sigma(X)$ is guarded for all $X \in \text{dom}(\tau)$.
- (2) *meta-type* if it is a substitution from type variables to meta-types.
- (3) *positive* if θ is natural and $\theta(\tau(X))$ is positive for all $X \in \text{dom}(\tau)$.
- (4) *type* if it is meta-type and positive.

Note that a type substitution $\sigma = \tau \cup \theta$ is a substitution from type variables to types where θ is natural.

Definition 6.7 (Constraints). There are two types of *constraints*.

- (1) A *meta-type constraint* is $T \stackrel{?}{=} T'$ for finite meta-types T and T' .
- (2) An *integer constraint* is either $E \stackrel{?}{=} E'$ or $E \stackrel{?}{\geq} E'$ or $E \stackrel{?}{<} E'$.

Sets of constraints are denoted by \mathcal{C}, \mathcal{E} , etc. If \mathcal{C} is a set of meta-type constraints then,

- (1) $\mathcal{C}^=$ denotes the subset of equality constraints from \mathcal{C} ,
- (2) $\mathcal{C}^{\mathbb{N}}$ denotes the set of integer constraints from \mathcal{C} .

Moreover, $\sigma \models T \stackrel{?}{=} T'$ means that $\sigma(T) = \sigma(T')$. Similarly, we define $\theta \models E \stackrel{?}{=} E'$, $\theta \models E \stackrel{?}{\geq} E'$ and $\theta \models E \stackrel{?}{<} E'$. This notation extends to sets \mathcal{C} of constraints in the obvious way. When $\sigma \models \mathcal{C}$, we say that σ is a *solution* (*unifier*) for \mathcal{C} .

Definition 6.8. We say that τ is:

- (1) \mathcal{E} -*guarded* if $\theta \circ \tau$ is guarded for all natural substitution θ such that $\theta \models \mathcal{E}$.
- (2) \mathcal{E} -*positive* if $\theta \circ \tau$ is positive for all natural substitution θ such that $\theta \models \mathcal{E}$.

Recall that $\mathbf{Stream}(N, Y) = Y \times \bullet^N \mathbf{Stream}(N, Y)$. For example,

- (1) $\tau = \{X \mapsto \mathbf{Stream}(N, Y)\}$ is \mathcal{E} -guarded but it is not \emptyset -guarded for $\mathcal{E} = \{N \stackrel{?}{\geq} 1\}$ and,
- (2) $\tau = \{X \mapsto \bullet^{M-N} \mathbf{Nat}\}$ is \mathcal{E} -positive but it is not \emptyset -positive for $\mathcal{E} = \{M \stackrel{?}{\geq} N\}$.

Definition 6.9. We say that \mathcal{C} is *substitutional* if $\mathcal{C}^= = \{X_1 \stackrel{?}{=} T_1, \dots, X_n \stackrel{?}{=} T_n\}$, all variables X_1, \dots, X_n are pairwise different.

Since a substitutional \mathcal{C} corresponds to a set of recursive equations where T_1, \dots, T_n are finite meta-types, there exists a unique solution $\tau_{\mathcal{C}}$ such that $\tau_{\mathcal{C}}(X_i)$ is regular for all $1 \leq i \leq n$ [10], [6, Theorem 7.5.34]. The mapping $\tau_{\mathcal{C}}$ is the *most general unifier*, i.e. if $\sigma \models \mathcal{C}$ then there exists σ' such that $\sigma = \sigma' \circ \tau_{\mathcal{C}}$. Moreover, if $\sigma = \tau \cup \theta$ then $\sigma' = \tau' \cup \theta$ where

$$\tau'(X) = \begin{cases} \tau(X) & \text{if } X \neq X_i \text{ for all } 1 \leq i \leq n \\ X & \text{otherwise} \end{cases}$$

For example, if $\mathcal{C} = \{X \stackrel{?}{=} Y \times \bullet^N X\}$ then $\tau_{\mathcal{C}}(X) = \mathbf{Stream}(N, Y)$. If $\sigma \models \mathcal{C}$ and $\sigma = \tau \cup \theta$ then $\sigma = \sigma' \circ \tau_{\mathcal{C}}$ and $\sigma' = \tau' \cup \theta$ and $\tau' = \{Y \mapsto \tau(Y)\}$.

Note that the unicity of the solution of a set of recursive equations would not be guaranteed if we were following the iso-recursive approach which allows only for a finite number of unfoldings $\mu X.t = t\{X \mapsto \mu X.t\}$ [5, 6].

For $1 \leq i \leq n$, we know that $\tau_{\mathcal{C}}(X_i)$ is regular but we do not know yet if $\tau_{\mathcal{C}}$ is guarded or positive. Below, we show some examples:

- (1) Let $\mathcal{C} = \{X_1 \stackrel{?}{=} X_2 \rightarrow \bullet X_1, X_2 \stackrel{?}{=} \bullet X_1 \rightarrow \bullet X_2\}$. Then, $\tau_{\mathcal{C}} = \{X_1 \mapsto t_1, X_2 \mapsto t_2\}$ is a type substitution where t_1 and t_2 satisfy the recursive equations $t_1 = t_2 \rightarrow \bullet t_1$ and $t_2 = \bullet t_1 \rightarrow \bullet t_2$.
- (2) Let $\mathcal{C} = \{X_1 \stackrel{?}{=} X_2 \rightarrow \bullet X_1, X_2 \stackrel{?}{=} X_1 \rightarrow \bullet X_2\}$. Then, $\tau_{\mathcal{C}} = \{X_1 \mapsto t_1, X_2 \mapsto t_2\}$ is positive where t_1 and t_2 satisfy the recursive equations $t_1 = t_2 \rightarrow \bullet t_1$ and $t_2 = t_1 \rightarrow \bullet t_2$. But $\tau_{\mathcal{C}}$ is not guarded since there is an infinite branch in the tree representations of t_1 and t_2 that has no \bullet 's. Observe that $t_1 = (t_1 \rightarrow \bullet t_2) \rightarrow \bullet t_1$ and $t_2 = (t_2 \rightarrow \bullet t_1) \rightarrow \bullet t_2$.
- (3) Let $\mathcal{C} = \{X_1 \stackrel{?}{=} \bullet^{N+1}(X_2 \rightarrow \bullet X_1), X_2 \stackrel{?}{=} \bullet^M X_1\}$. Then, $\tau_{\mathcal{C}} = \{X_1 \mapsto T_1, X_2 \mapsto T_2\}$ is a meta-type substitution where T_1 and T_2 satisfy the recursive equations $T_1 = \bullet^{N+1}(T_2 \rightarrow \bullet T_1)$ and $T_2 = \bullet^M T_1$. But $\tau_{\mathcal{C}}$ is not positive since N and M are integer variables.

Definition 6.10. We say that \mathcal{C} is

- (1) *always positive* if all the meta-types in $\theta(\mathcal{C}^=)$ are positive for all natural substitution θ such that $\theta \models \mathcal{C}^{\mathbb{N}}$.
- (2) *simple* if $0 \stackrel{?}{<} E \in \mathcal{C}$ for all $X \stackrel{?}{=} \bullet^E X' \in \mathcal{C}$.

For example, $\mathcal{C} = \{X \stackrel{?}{=} \bullet^N(X_1 \rightarrow X_2), N \stackrel{?}{\geq} 1\}$ is always positive while $\mathcal{C} = \{X \stackrel{?}{=} \bullet^N(X_1 \rightarrow X_2)\}$ is not.

Table 1: Constraint Typing Rules for $\lambda_{\rightarrow}^{\bullet}$

$\frac{[\text{AXIOM}] \quad N \text{ is fresh}}{\Delta, x : X \vdash x : \bullet^N X \mid \emptyset}$	$\frac{[\times \text{CONST}] \quad N, X_1, X_2 \text{ are fresh}}{\Delta \vdash \mathbf{k} : \bullet^N \text{typeOf}(\mathbf{k}, X_1, X_2) \mid \emptyset} \quad \mathbf{k} \in \{\text{pair}, \text{fst}, \text{snd}\}$
$\frac{[\text{Nat0}] \quad N \text{ is fresh}}{\Delta \vdash \mathbf{0} : \bullet^N \text{Nat} \mid \emptyset}$	$\frac{[\text{Natsucc}] \quad N \text{ is fresh}}{\Delta \vdash \text{succ} : \bullet^N (\text{Nat} \rightarrow \text{Nat}) \mid \emptyset}$
$\frac{[\rightarrow\text{I}] \quad \Delta, x : X \vdash e : T \mid \mathcal{C} \quad X, X_1, X_2, N \text{ are fresh}}{\Delta \vdash \lambda x. e : \bullet^N (X_1 \rightarrow X_2) \mid \mathcal{C} \cup \{X \stackrel{?}{=} \bullet^N X_1, T \stackrel{?}{=} \bullet^N X_2\}}$	
$\frac{[\rightarrow\text{E}] \quad \Delta \vdash e_1 : T_1 \mid \mathcal{C}_1 \quad \Delta \vdash e_2 : T_2 \mid \mathcal{C}_2 \quad X_1, X_2, N \text{ are fresh}}{\Delta \vdash e_1 e_2 : \bullet^N X_2 \mid \mathcal{C}_1 \cup \mathcal{C}_2 \cup \{\bullet^N (X_1 \rightarrow X_2) \stackrel{?}{=} T_1, \bullet^N X_1 \stackrel{?}{=} T_2\}}$	

6.3. Constraint Typing Rules. The *constraint typing rules* for $\lambda_{\rightarrow}^{\bullet}$ are given in Table 1. Since `pair`, `fst` and `snd` are actually ‘polymorphic’, we need to generate different fresh variables depending on their position. For convenience, we define the following function:

$$\begin{aligned} \text{typeOf}(\text{pair}, X_1, X_2) &= X_1 \rightarrow X_2 \rightarrow X_1 \times X_2 \\ \text{typeOf}(\text{fst}, X_1, X_2) &= X_1 \times X_2 \rightarrow X_1 \\ \text{typeOf}(\text{snd}, X_1, X_2) &= X_1 \times X_2 \rightarrow X_2 \end{aligned}$$

The typing rules should be read bottom-up. We start from an empty context and a closed expression and we build the typing context Δ and the set \mathcal{C} of constraints. We can, then, assume that the typing context Δ only contains declarations of the form $x : X$ and the set \mathcal{C} contains only equality constraints between finite meta-types. The type variables in the contexts Δ are all fresh and they are created by the rule $[\rightarrow\text{I}]$ for the abstraction.

If instead of generating the fresh variable X in $[\rightarrow\text{I}]$ we directly put $\bullet^N X_1$ in the context, then the algorithm would not be complete. For example, consider $x : \bullet^5 \text{Nat} \vdash x : \bullet^7 \text{Nat}$. Then N should be assigned the value 3 and not 5 if later we derive that $\lambda x. x : \bullet^2 (\bullet^3 \text{Nat} \rightarrow \bullet^5 \text{Nat})$.

Theorem 6.11 (Soundness of Constraint Typing). Let σ be a type substitution. If $\Delta \vdash e : T \mid \mathcal{C}$ and $\sigma \models \mathcal{C}$ then $\sigma(\Delta) \vdash e : \sigma(T)$ in $(\lambda_{\rightarrow}^{\bullet})^-$ and $\sigma(T)$ is a type.

Proof. We proceed by induction on the derivation of $\Delta \vdash e : T \mid \mathcal{C}$. Suppose the last rule in the derivation of $\Delta \vdash e : T \mid \mathcal{C}$ is:

$$\frac{[\rightarrow\text{I}] \quad \Delta, x : X \vdash e : T' \mid \mathcal{C} \quad X, X_1, X_2, N \text{ are fresh}}{\Delta \vdash \lambda x. e : \bullet^N (X_1 \rightarrow X_2) \mid \mathcal{C} \cup \{X \stackrel{?}{=} \bullet^N X_1, T' \stackrel{?}{=} \bullet^N X_2\}}$$

It follows from induction hypothesis that $\sigma(\Delta), x : \sigma(X) \vdash e : \sigma(T')$. Since $\sigma \models \{X \stackrel{?}{=} \bullet^N X_1, T' \stackrel{?}{=} \bullet^N X_2\}$, we also have that $\sigma(\Delta), x : \bullet^{\sigma(N)} \sigma(X_1) \vdash e : \bullet^{\sigma(N)} \sigma(X_2)$. We can, then,

apply $[\rightarrow I]$ of $(\lambda_{\rightarrow}^{\bullet})^{-}$ to conclude that $\sigma(\Delta) \vdash \lambda x.e : \bullet^{\sigma(N)}(\sigma(X_1) \rightarrow \sigma(X_2))$. It follows from the fact that σ is a type substitution that $\sigma(T) = \bullet^{\sigma(N)}(\sigma(X_1) \rightarrow \sigma(X_2))$ is a type. \square

Theorem 6.12 (Completeness of Constraint Typing). Let $\Gamma = x_1 : t_1, \dots, x_m : t_m$ and $\Delta = x_1 : X_1, \dots, x_m : X_m$. If $\Gamma \vdash e : t$ in $(\lambda_{\rightarrow}^{\bullet})^{-}$ then there are T and \mathcal{C} such that

- (1) $\Delta \vdash e : T \mid \mathcal{C}$
- (2) there exists a type substitution $\sigma \supseteq \{X_1 \mapsto t_1, \dots, X_m \mapsto t_m\}$ such that $\sigma \models \mathcal{C}$ and $\sigma(T) = t$ and $\text{dom}(\sigma) \setminus \{X_1, \dots, X_m\}$ is the set of fresh variables in the derivation of $\Delta \vdash e : T \mid \mathcal{C}$.

Proof. We proceed by induction on the derivation. Suppose the last typing rule in the derivation is:

$$\frac{[\text{AXIOM}] \quad x_i : t_i \in \Gamma}{\Gamma \vdash x_i : \bullet^n t_i}$$

Then, $\Delta \vdash x_i : \bullet^N X_i \mid \emptyset$. We define $\sigma = \{X_1 \mapsto t_1, \dots, X_m \mapsto t_m\} \cup \{N \mapsto n\}$. It is easy to see that $\sigma(\Delta) = \Gamma$ and $t = \sigma(\bullet^N X_i)$.

Suppose the last typing rule in the derivation is:

$$\frac{[\rightarrow E] \quad \Gamma \vdash e_1 : \bullet^n (t \rightarrow s) \quad \Gamma \vdash e_2 : \bullet^n t}{\Gamma \vdash e_1 e_2 : \bullet^n s}$$

By induction hypotheses,

- $\Delta \vdash e_1 : T_1 \mid \mathcal{C}_1$ and $\Delta \vdash e_2 : T_2 \mid \mathcal{C}_2$,
- $\sigma_1 \models \mathcal{C}_1$ and $\sigma_2 \models \mathcal{C}_2$ and $\sigma_1(T_1) = \bullet^n (t \rightarrow s)$ and $\sigma_2(T_2) = \bullet^n s$ for $\sigma_1, \sigma_2 \supseteq \{X_1 \mapsto t_1, \dots, X_m \mapsto t_m\}$,
- $\text{dom}(\sigma_1) \setminus \{X_1, \dots, X_m\}$ and $\text{dom}(\sigma_2) \setminus \{X_1, \dots, X_m\}$ are the sets of fresh variables in the derivation of $\Delta \vdash e_1 : T_1 \mid \mathcal{C}_1$ and $\Delta \vdash e_2 : T_2 \mid \mathcal{C}_2$, respectively.

It follows from the latter that $\sigma_1 \cup \sigma_2$ is a function and we can define the substitution σ as $(\sigma_1 \cup \sigma_2) \setminus \{N \mapsto n\} \setminus \{X_1 \mapsto t\} \setminus \{X_2 \mapsto s\}$. \square

6.4. Unification Algorithm. Algorithm 1 solves the unification problem. The input consists of a set \mathcal{C} of constraints and an extra argument \mathcal{V} that keeps tracks of the “visited equality constraints” and guarantees termination.

The base case on Line 1 is when the set $\mathcal{C} = \mathcal{C}^= \cup \mathcal{C}^{\mathbb{N}}$ is substitutional and simple (see Definitions 6.7, 6.9 and 6.10). The set $\mathcal{C}^{\mathbb{N}}$ guarantee that the exponents of the \bullet ’s are positive. Line 2 invokes the function `guard` which creates a set of constraints to ensure guardedness. Line 3 checks whether $\mathcal{E} = \mathcal{C}^{\mathbb{N}} \cup \text{guard}(\mathcal{C})$ has a solution. Altogether, this implies that $\theta \circ \tau_{\mathcal{C}}$ is guarded and positive for all natural $\theta \models \mathcal{E}$. The function `guard` is defined in Algorithm 3 (Appendix B). In order to check that \mathcal{E} has a solution of non-negative integers (Line 3), we can use any algorithm for linear integer programming [24].

If \mathcal{C} is neither substitutional nor simple, we examine the equations in $\mathcal{C}^=$ (Line 4 onwards).

The case that eliminates two equality constraints for the same variable (Line 7) removes $X \stackrel{?}{=} S$ from \mathcal{C} but it adds $S \stackrel{?}{=} S'$. If \mathcal{V} already contains $X \stackrel{?}{=} S$, it means that $X \stackrel{?}{=} S$ has already been processed and it will not be added again avoiding non-termination. The unification algorithm for the simply typed lambda calculus solves the problem of termination in this case by reducing the number of variables, i.e. checks if $X \notin S$ and then, substitutes

Algorithm 1: Unification Algorithm

```

Function unify( $\mathcal{C}, \mathcal{V}$ )
  input :  $\mathcal{C}$  is a set of constraints and  $\mathcal{V}$  is the set of visited equality constraints
  output :  $\{\langle \tau, \mathcal{E} \rangle \mid \tau \cup \theta \models \mathcal{C} \ \forall \theta \models \mathcal{E}\}$ 
1  if  $\mathcal{C}$  is substitutional and simple then
2     $\mathcal{E} \leftarrow \mathcal{C}^{\mathbb{N}} \cup \text{guard}(\mathcal{C})$ ;
3    if  $\mathcal{E}$  has a solution of non-negative integers then return  $\{\langle \tau_{\mathcal{C}}, \mathcal{E} \rangle\}$  else
   |   return  $\emptyset$ ;
4  else if  $T \stackrel{?}{=} S \in \mathcal{C}$  then
5     $\mathcal{C}' \leftarrow \mathcal{C} \setminus \{T \stackrel{?}{=} S\}$ ;  $\mathcal{V}' \leftarrow \mathcal{V} \cup \{T \stackrel{?}{=} S\}$ 
6    case  $S = T$  do unify( $\mathcal{C}', \mathcal{V}'$ ) /*  $T \stackrel{?}{=} T$  */;
7    case  $T = X$  and  $X \stackrel{?}{=} S' \in \mathcal{C}'$  /*  $X \stackrel{?}{=} S$  and  $X \stackrel{?}{=} S'$  */ do
   |   if  $T \stackrel{?}{=} S \notin \mathcal{V}$  then return unify( $\mathcal{C}' \cup \{S \stackrel{?}{=} S'\}, \mathcal{V}'$ );
   |   else return unify( $\mathcal{C}', \mathcal{V}'$ );
10   case  $T = T_1 \text{ op } T_2$  and  $S = S_1 \text{ op } S_2$  /*  $T_1 \text{ op } T_2 \stackrel{?}{=} S_1 \text{ op } S_2$  */ do
   |   return unify( $\mathcal{C}' \cup \{T_1 \stackrel{?}{=} S_1, T_2 \stackrel{?}{=} S_2\}, \mathcal{V}'$ )
12   case  $T = X$  and  $S = \bullet^E X'$  and  $X \stackrel{?}{=} \bullet^E X' \notin \mathcal{V}$  /*  $X \stackrel{?}{=} \bullet^E X'$  */ do
   |    $\mathcal{C}_1 \leftarrow \mathcal{C}' \cup \{X \stackrel{?}{=} X', E \stackrel{?}{=} 0\}$ ; /* 1st option */
   |    $\mathcal{C}_2 \leftarrow \mathcal{C}' \cup \{X \stackrel{?}{=} \bullet^E X', 0 \stackrel{?}{<} E\}$  /* 2nd option */
   |   return unify( $\mathcal{C}_1, \mathcal{V}'$ )  $\cup$  unify( $\mathcal{C}_2, \mathcal{V}'$ )
16   case  $T = \bullet^E X$  and  $S = \bullet^{E'} X'$  /*  $\bullet^E X \stackrel{?}{=} \bullet^{E'} X'$  */ do
   |    $\mathcal{C}_1 \leftarrow \mathcal{C}' \cup \{X \stackrel{?}{=} \bullet^{E'-E} X', E' \stackrel{?}{\geq} E, E \stackrel{?}{\geq} 0\}$ ; /* 1st option */
   |    $\mathcal{C}_2 \leftarrow \mathcal{C}' \cup \{X' \stackrel{?}{=} \bullet^{E-E'} X, E' \stackrel{?}{<} E, E' \stackrel{?}{\geq} 0\}$  /* 2nd option */
   |   return unify( $\mathcal{C}_1, \mathcal{V}'$ )  $\cup$  unify( $\mathcal{C}_2, \mathcal{V}'$ )
20   case  $T = \bullet^E X$  and  $S = \bullet^{E'} S'$  and isNatorOp?( $S'$ ) /*  $\bullet^E X \stackrel{?}{=} \bullet^{E'} S'$  */ do
   |   return unify( $\mathcal{C}' \cup \{X \stackrel{?}{=} \bullet^{E'-E} S', E' \stackrel{?}{\geq} E, E \stackrel{?}{\geq} 0\}, \mathcal{V}'$ )
22   case  $T = \bullet^E T'$  and  $S = \bullet^{E'} S'$  and EqualCons?( $T', S'$ ) /*  $\bullet^E T' \stackrel{?}{=} \bullet^{E'} S'$  */
   |   do
23     return unify( $\mathcal{C}' \cup \{T' \stackrel{?}{=} S', E \stackrel{?}{=} E'\}, \mathcal{V}'$ )
24   case  $T = \bullet^E T'$  and EqualCons?( $T', S$ ) /*  $\bullet^E T' \stackrel{?}{=} S$  */ do
   |   return unify( $\mathcal{C}' \cup \{T' \stackrel{?}{=} S, E \stackrel{?}{=} 0\}, \mathcal{V}'$ )
26   case  $T \stackrel{?}{=} S \notin \mathcal{V}$  /*  $T \stackrel{?}{=} S$  */ do
   |   return unify( $\mathcal{C}' \cup \{S \stackrel{?}{=} T\}, \mathcal{V}'$ ) /* swap metatypes (symmetry) */
28   otherwise do return  $\emptyset$  /* failure */;

```

the variable X by S in the remaining set of constraints. With recursive types, however, we do not perform the occur check and the number of variables may not decrease since the variable X may not disappear after substituting X by S (because X occurs in S). In order to decrease the number of variables, we could perhaps substitute X by the solution of the recursive equation $X = S$ (which may be *infinite*). But the problem of guaranteeing termination would still be present because the meta-types in the constraints can be infinite and the size of the equality constraints may not decrease. We would also have a similar problem with termination if we use multi-equations and a rewrite relation instead of giving a function such as `unify` [26].

The function `unify` does not return only one solution but a set of solutions. This is because there are two ways of solving the equality constraint $\bullet^E X \stackrel{?}{=} \bullet^{E'} X'$ (Line 16). If $\bullet^E X \stackrel{?}{=} \bullet^{E'} X'$, we solve either $E \stackrel{?}{\geq} E'$ and $X' \stackrel{?}{=} \bullet^{E-E'} X$ or $E' \stackrel{?}{\geq} E$ and $X \stackrel{?}{=} \bullet^{E'-E} X'$.

Line 12 is similar to Line 16, In order to solve $X \stackrel{?}{=} \bullet^E X'$, we solve either $E \stackrel{?}{=} 0$ and $X \stackrel{?}{=} X'$ or $0 \stackrel{?}{<} E$ and $X \stackrel{?}{=} \bullet^E X'$. This distinction is needed because in the particular case when $X = X'$, the solution on the first case is any meta-type while the solution on the second case is \bullet^∞ . Termination is guaranteed in this case by checking that the constraint $X \stackrel{?}{=} \bullet^E X'$ has not been visited before.

The case on Lines 20 uses the function `isNatorOp?(T)` which returns true iff T is either `Nat` or $(S_1 \text{ op } S_2)$ (see Algorithm 4 in Appendix B). It is clear that if $E' < E$, then the equation $\bullet^E X \stackrel{?}{=} \bullet^{E'} S'$ does not have a positive solution, e.g. $\bullet^5 X \stackrel{?}{=} \bullet^2 (T_1 \times T_2)$ then the solution is $\{X \mapsto \bullet^{-3} (T_1 \times T_2)\}$ which is not positive.

The cases on Lines 22 and 24 use the function `EqualCons?(T, S)` which returns true iff both types are `Nat` or built from the same type constructor (see Algorithm 5 in Appendix B).

The algorithm analyses the form of the equation $T \stackrel{?}{=} S$ and the cases are done in order which means that at Line 26, the equation does not have any of the forms described in the above cases but it could be that $S \stackrel{?}{=} T$ does. For this, the algorithm swaps the equation and calls `unify` again. Termination is guaranteed by checking that the constraint does not belong to \mathcal{V} . In the last case (Line 28), the algorithm returns the empty set of solutions when all the previous cases have failed.

The *size* $|\mathcal{C}|$ of a set of meta-type constraints is the sum of the number of type variables and type constructors in the left hand side of the equality constraints. Since the meta-types in \mathcal{C} are finite, the size is always finite. We define

$$\begin{aligned} \text{SubT}(\mathcal{C}) &= \{T \mid S_1 \stackrel{?}{=} S_2 \in \mathcal{C} \text{ and either } S_1 \text{ or } S_2 \text{ contains } T\} \\ \text{SubC}(\mathcal{C}) &= \{T_1 \stackrel{?}{=} T_2 \mid T_1, T_2 \in \text{SubT}(\mathcal{C})\} \end{aligned}$$

Theorem 6.13 (Termination). Let \mathcal{C} and \mathcal{V} be sets of constraints. Then, `unify`(\mathcal{C}, \mathcal{V}) terminates

Proof. Let $\mathcal{C}_0 = \mathcal{C}$ and $\mathcal{V}_0 = \mathcal{V}$. Suppose towards a contradiction that there is an infinite sequence of recursive calls:

$$\text{unify}(\mathcal{C}_0, \mathcal{V}_0), \text{unify}(\mathcal{C}_1, \mathcal{V}_1), \text{unify}(\mathcal{C}_2, \mathcal{V}_2), \dots \quad (6.1)$$

This means that `unify`($\mathcal{C}_i, \mathcal{V}_i$) recursively calls `unify`($\mathcal{C}_{i+1}, \mathcal{V}_{i+1}$) for all $i \geq 0$. From the definition of the algorithm, one can see that $\mathcal{C}_i \subseteq \text{SubC}(\mathcal{C}_0)$ and $\mathcal{V}_i \subseteq \mathcal{V}_{i+1} \subseteq \mathcal{V}_0 \cup \text{SubC}(\mathcal{C}_0)$

for all $i \geq 0$. We define now a measure that decreases with each recursive call:

$$\mathbf{r}_i = (|\mathbf{SubC}(\mathcal{C}_0)| + |\mathcal{V}_0| - |\mathcal{V}_i|, |\mathcal{C}_i|)$$

It is not difficult to show that $\mathbf{r}_i > \mathbf{r}_{i+1}$ contradicting the fact that the sequence in Eq. (6.1) is infinite. \square

Theorem 6.14 (Completeness of Unification). Let σ be a type substitution. If $\sigma \models \mathcal{C}$ then there exists $\langle \tau, \mathcal{E} \rangle \in \mathbf{unify}(\mathcal{C}, \mathcal{V})$ such that $\sigma = \sigma' \circ \tau \upharpoonright_{\text{dom}(\sigma)}$ and $\sigma' = \tau' \cup \theta'$ and $\theta' \models \mathcal{E}$.

Proof. By induction on the number of recursive calls which is finite by Theorem 6.13. We do only two cases.

Suppose \mathcal{C} is substitutional and simple. Since $\tau = \tau_{\mathcal{C}}$ is the most general unifier, there exists $\sigma' = \tau' \cup \theta'$ such that $\sigma = \sigma' \circ \tau_{\mathcal{C}}$. Since σ is guarded, so is $\theta' \circ \tau_{\mathcal{C}}$ and $\theta' \models \mathcal{E} = \mathbf{guard}(\mathcal{C})$.

Suppose the constraint of \mathcal{C} which is being processed is $\bullet^E X \stackrel{?}{=} \bullet^{E'} X'$. Then,

$$\bullet^{\sigma(E)} \sigma(X) = \bullet^{\sigma(E')} \sigma(X')$$

Then, there are two cases:

- (1) Case $\sigma(E') \geq \sigma(E)$ and $\sigma(X) = \bullet^{\sigma(E') - \sigma(E)} \sigma(X')$. Then, $\sigma \models \mathcal{C}_1$ where \mathcal{C}_1 is the set of constraints of Line 17. It follows by induction hypothesis that there exists $\langle \tau, \mathcal{E} \rangle \in \mathbf{unify}(\mathcal{C}_1, \mathcal{V})$ such that $\sigma = \sigma' \circ \tau \upharpoonright_{\text{dom}(\sigma)}$ and $\sigma' = \tau' \cup \theta'$ and $\theta' \models \mathcal{E}$. We have that $\langle \tau, \mathcal{E} \rangle \in \mathbf{unify}(\mathcal{C}, \mathcal{V})$ because $\mathbf{unify}(\mathcal{C}_1, \mathcal{V}) \subseteq \mathbf{unify}(\mathcal{C}, \mathcal{V})$.
- (2) Case $\sigma(E') < \sigma(E)$ and $\sigma(X) = \bullet^{\sigma(E) - \sigma(E')} \sigma(X')$. Then, $\sigma \models \mathcal{C}_2$ where \mathcal{C}_2 is the set of constraints of Line 18. The proof of this case is similar to the first one. \square

Theorem 6.15 (Soundness of Unification). Let \mathcal{C} be always positive. If $\langle \tau, \mathcal{E} \rangle \in \mathbf{unify}(\mathcal{C}, \mathcal{V})$ then

- (1) there exists a natural θ such that $\theta \models \mathcal{E}$ and,
- (2) for all natural θ such that $\theta \models \mathcal{E}$,
 - (i) $\tau \cup \theta \models \mathcal{C}$ and,
 - (ii) $\tau \cup \theta$ is a type substitution.

Proof. By induction on the number of recursive calls which is finite by Theorem 6.13. We do only two cases.

Suppose first that \mathcal{C} is substitutional and simple. Then, $\tau = \tau_{\mathcal{C}}$ and $\mathcal{E} = \mathcal{C}^{\mathbb{N}} \cup \mathbf{guard}(\mathcal{C})$. Item 1 follows from the check on Line 3. Assume that θ is natural and $\theta \models \mathcal{E}$. Part (i) of Item 2 holds since $\tau_{\mathcal{C}} \cup \theta \models \mathcal{C}$. We also have that $\theta \models \mathcal{C}^{\mathbb{N}}$ and $\theta \models \mathbf{guard}(\mathcal{C})$. Since \mathcal{C} is always positive, all the meta-types in $\theta(\mathcal{C}^{\mathbb{N}})$ are positive and hence, $\tau_{\mathcal{C}} \cup \theta$ is positive. We also have that $\tau_{\mathcal{C}} \cup \theta$ is guarded because $\theta \models \mathbf{guard}(\mathcal{C})$. \square

6.5. Type Inference Algorithm. Algorithm 2 solves the type inference problem in $\lambda_{\rightarrow}^{\bullet}$. Its inputs are a decidable function p from meta-types to booleans and a closed expression e and the output is a finite set of meta-types that cover all the possible types of e , i.e. any type of e is an instantiation of one of those meta-types.

Line 1 computes T and \mathcal{C} such that $\emptyset \vdash e : T \mid \mathcal{C}$ from the constraint typing rules of Section 6.3. Line 3 calls the unification algorithm. The first argument of \mathbf{unify} is the set \mathcal{C} and the second one is \emptyset for the set of visited equality constraints. Line 4 checks that the meta-type $\tau(T)$ satisfies the property p . We are interested in certain properties $p(T)$ given below:

- (1) $\mathbf{True}(T)$ which returns true for all T .
- (2) $\mathbf{Diff}_{\bullet^\infty?}(T)$ returns true iff T is different from \bullet^∞ . The function $\mathbf{Diff}_{\bullet^\infty?}(T)$ is defined in Algorithm 6. The inference algorithm can be used in this case for excluding non-normalizing programs.
- (3) $\bullet^\infty\text{-free?}(T)$ is true iff T is \bullet^∞ -free. The function $\bullet^\infty\text{-free?}(T)$ is defined in Algorithm 7. The inference algorithm can be used in this case for excluding programs that have Lèvy-Longo trees with \perp .
- (4) $\mathbf{tailfinite?}(T)$ which checks whether T is tail finite. The function $\mathbf{tailfinite?}(T)$ is defined in Algorithm 8. The inference algorithm can be used in this case for excluding programs that have Böhm trees with \perp .

Algorithm 2: Type Inference Algorithm

Function $\mathbf{infer}(p, e)$

input : p is a decidable function from meta-types to booleans and e is a closed expression

output : set of pairs $\langle T, \mathcal{E} \rangle$ such that $t = \theta(T)$ and $\vdash e : t$ for all $\theta \models \mathcal{E}$

- 1 Compute T and \mathcal{C} such that $\emptyset \vdash e : T \mid \mathcal{C}$
- 2 $\mathcal{T} \leftarrow \emptyset$
- 3 **foreach** $\langle \tau, \mathcal{E} \rangle \in \mathbf{unify}(\mathcal{C}, \emptyset)$ **do**
- 4 **if** $p(\tau(T))$ **then** $\mathcal{T} \leftarrow \mathcal{T} \cup \langle \tau(T), \mathcal{E} \rangle$;
- 5 **return** \mathcal{T}

Theorem 6.16 (Completeness of Type Inference). Let $\vdash e : t$ in $\lambda_{\perp}^{\bullet}$. Then,

- (1) there exists $\langle T, \mathcal{E} \rangle \in \mathbf{infer}(\mathbf{True}, e)$ such that $t = \sigma(T)$ for some $\sigma = \tau \cup \theta$ and $\theta \models \mathcal{E}$,
- (2) $\langle T, \mathcal{E} \rangle \in \mathbf{infer}(\mathbf{Diff}_{\bullet^\infty?}, e)$ if $t \neq \bullet^\infty$,
- (3) $\langle T, \mathcal{E} \rangle \in \mathbf{infer}(\bullet^\infty\text{-free?}, e)$ if the Lèvy-Longo tree of e has no \perp 's and
- (4) $\langle T, \mathcal{E} \rangle \in \mathbf{infer}(\bullet^\infty\text{-free?}, e)$ if the Böhm tree of e has no \perp 's.

Proof. The first part follows from Lemma 6.1, Theorem 6.12 and Theorem 6.14. We show only the second part since the remaining parts can be proved similarly. Suppose $\sigma(T) = t \neq \bullet^\infty$. But this means that T cannot be \bullet^∞ and $\langle T, \mathcal{E} \rangle \in \mathbf{infer}(\mathbf{Diff}_{\bullet^\infty?}, e)$. \square

Theorem 6.17 (Soundness of Type Inference). If $\langle T, \mathcal{E} \rangle \in \mathbf{infer}(p, e)$ then

- (1) $p(T)$ is true,
- (2) there exists a natural θ such that $\theta \models \mathcal{E}$ and
- (3) for all natural θ such that $\theta \models \mathcal{E}$,
 - (i) $\theta(T) = t$ is a type and $\vdash e : t$ in $\lambda_{\perp}^{\bullet}$,
 - (ii) If p is $\mathbf{Diff}_{\bullet^\infty?}$ then $t \neq \bullet^\infty$.
 - (iii) If p is $\bullet^\infty\text{-free?}$ then t is \bullet^∞ -free.
 - (iv) If p is $\mathbf{Diff}_{\bullet^\infty?}$ then t is tail finite.

Proof. Suppose $\langle T, \mathcal{E} \rangle \in \mathbf{infer}(e)$. It follows from the definition of Algorithm 2 that $T = \tau(S)$ for some $\emptyset \vdash e : S \mid \mathcal{C}$ and $\langle \tau, \mathcal{E} \rangle \in \mathbf{unify}(\mathcal{C}, \emptyset)$.

Item 1 follows from the definition of Algorithm 2.

Item 2 follows from Item 1 of Theorem 6.15 and from the fact that \mathcal{C} is always positive. One can show by induction on the rules of Table 1 that \mathcal{C} is always positive.

In order to prove Item 3, assume that θ is natural and $\theta \models \mathcal{E}$ and take $\sigma = \tau \cup \theta$. We only prove (i) since Items (ii), (iii) and (iv) follow from Items 1 and (i). It follows from Item 2 of Theorem 6.15 that $\sigma \models \mathcal{C}$ and σ is a type substitution. By Theorem 6.11, $\vdash e : \sigma(S)$ in $(\lambda_{\rightarrow}^{\bullet})^{-}$ and $\sigma(S) = \theta(T) = t$ is a type. By Lemma 6.1, we have that $\vdash e : \sigma(S)$ in $\lambda_{\rightarrow}^{\bullet}$. \square

6.6. Examples. We now illustrate the type inference algorithm through some examples.

Consider $e = \lambda x.x$. The result of $\mathbf{infer}(\mathbf{True}, \lambda x.x)$ is a set that contains only one meta-type:

$$S = \bullet^N(X_1 \rightarrow \bullet^M X_1)$$

with the implicit set of integer constraints $N, M \geq 0$. This meta-type covers all solutions, i.e. any type of $e = \lambda x.x$ in $\lambda_{\rightarrow}^{\bullet}$ is an instantiation of S . How is this meta-type obtained by the algorithm? We can apply the rule $[\rightarrow]$ of Table 1 which gives the type derivation $\emptyset \vdash e : T \mid \mathcal{C}$ where $T = \bullet^N(X_1 \rightarrow X_2)$ and $\mathcal{C} = \{X \stackrel{?}{=} \bullet^N X_1, \bullet^M X \stackrel{?}{=} \bullet^N X_2\}$. The result of $\mathbf{unify}(\mathcal{C}, \emptyset)$ is $\tau = \{X \mapsto \bullet^N X_1, X_2 \mapsto \bullet^M X_1\}$ and $\tau(T) = \bullet^N(X_1 \rightarrow \bullet^M X_1) = S$.

Consider now $\lambda x.xx$. Then, the result of $\mathbf{infer}(\mathbf{True}, \lambda x.xx)$ is a set that contains two meta-types. The first meta-type is $S_1 = \bullet^{N_1}(X_1 \rightarrow \bullet^{N_2-N_1} X_4)$ where X_1 should satisfy the recursive equation:

$$X_1 \stackrel{?}{=} \bullet^{N_2-(N_1+N_3)}(\bullet^{N_1+N_4-N_2} X_1 \rightarrow X_4) \quad (6.2)$$

and the set of integer constraints for S_1 is:

$$\mathcal{E}_1 = \{N_1 + N_4 \geq N_2 \geq N_1 + N_3, N_4 - N_3 \geq 1\} \quad (6.3)$$

The constraints $N_1 + N_4 \geq N_2 \geq N_1 + N_3$ guarantee that the exponents of the \bullet 's are all positive. The constraint $N_4 - N_3 \geq 1$ forces the recursive type to be guarded.

The second meta-type is $S_2 = \bullet^{N_1}(\bullet^{N_2-(N_1+N_4)} X_3 \rightarrow \bullet^{N_2-N_1} X_4)$ where X_3 should satisfy the recursive equation:

$$X_3 \stackrel{?}{=} \bullet^{N_4-N_3}(X_3 \rightarrow X_4) \quad (6.4)$$

and the set of integer constraints for S_2 is:

$$\mathcal{E}_2 = \{N_2 > N_1 + N_4, N_4 \geq N_3, N_4 - N_3 \geq 1\} \quad (6.5)$$

The constraints $N_2 > N_1 + N_4$ and $N_4 \geq N_3$ guarantee that the exponents of the \bullet 's are all positive. The constraint $N_4 - N_3 \geq 1$ forces the recursive type to be guarded.

6.7. Typability and Type Checking. Typability (finding out if the program is typable or not) in $\lambda_{\rightarrow}^{\bullet}$, can be solved by checking whether $\mathbf{infer}(\mathbf{True}, e) \neq \emptyset$. By Theorem 6.16, if $\mathbf{infer}(\mathbf{True}, e) = \emptyset$ then there is no t such that $\vdash e : t$ in $\lambda_{\rightarrow}^{\bullet}$. By Theorem 6.16, if $\mathbf{infer}(e) \neq \emptyset$ then there exists t such that $\vdash e : t$ in $\lambda_{\rightarrow}^{\bullet}$.

Type checking is usually an easier problem than type inference. In the case of $\lambda_{\rightarrow}^{\bullet}$ (given an expression e and a type t check if $\vdash e : t$), type checking can easily be solved by inferring the (finite) set of meta-types for e and checking whether one of these meta-types unifies with t .

The type inference algorithm is exponential in the size of the input because of the unification algorithm which branches (case 16 of Algorithm 1). If we are only interested in typability (and not type inference), then the complexity could be reduced to **NP** by “guessing

the solutions”. Instead of giving the union of the results in Line 19 of Algorithm 1, we could choose one of the options non-deterministically and call `unify` only “once”.

7. RELATED WORK

This paper is an improvement over past typed lambda calculi with a temporal modal operator like \bullet in two respects. Firstly, we do not need any subtyping relation as in [23] and secondly programs are not cluttered with constructs for the introduction and elimination of individuals of type \bullet as in [18, 30, 19, 1, 7, 8, 4, 13].

Another type-based approach for ensuring productivity are sized types [15]. Type systems using size types do not always have neat properties: strong normalisation is gained by contracting the fixed point operator inside a case and they lack the property of subject reduction [28]. Another disadvantage of size types is that they do not include negative occurrences of the recursion variable [15] which are useful for some applications [31].

The proof assistant Coq does not ensure productivity through typing but by means of a syntactic guardedness condition (the recursive calls should be guarded by constructors) [12, 9] which is somewhat restrictive since it rules out some interesting functions [30, 8].

A sound but not complete type inference algorithm for Nakano’s type system is presented in [27]. This means that the expressions typable by the algorithm are also typable in Nakano’s system but the converse is not true. Though this algorithm is tractable, it is not clear to which type system it corresponds.

8. CONCLUSIONS AND FUTURE WORK

The typability problem is trivial in $\lambda\mu$ because all expressions are typable using $\mu X.(X \rightarrow X)$. In $\lambda_{\perp}^{\bullet}$, this problem turns out to be interesting because it gives us a way of filtering programs that do not satisfy certain properties, i.e. normalization, having a Lèvy-Longo or a Böhm tree without \perp . It is also challenging because it involves the generation of integer constraints.

Due to the high complexity of the type inference algorithm, it will be important to find optimization techniques (heuristics, use of concurrency, etc) to make it feasible.

It will be interesting to investigate the interaction of this modal operator with dependent types. As observed in Section 2, our type system is not closed under η -reduction and this property may be useful for establishing a better link with the denotational semantics. We leave the challenge of attaining a normalising and decidable type system closed under $\beta\eta$ -reduction for future research. It will also be interesting to investigate ways of extending $\lambda_{\perp}^{\bullet}$ to be able to type the programs `get` and `take` of Section 2.8.

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APPENDIX A. PROOF OF THEOREM 3.12

Lemma A.1. (1) Let $e \longrightarrow e'$. Then, $e \in \llbracket t \rrbracket_i$ iff $e' \in \llbracket t \rrbracket_i$ for all $i \in \mathbb{N}$ and type t .

(2) If $\mathbf{k} : t \in \kappa$, then $\mathbf{k} \in \bigcap_{i \in \mathbb{N}} \llbracket t \rrbracket_i$.

Proof. (Item 1). By induction on $(i, \text{rank}(t))$.

(Item 2). Using the definition of type interpretation. \square

Lemma A.2. If $\rho \models_i \Gamma$, then $\rho \models_j \Gamma$ for all $j \leq i$.

Proof. It follows from Item 2 of Lemma 3.9. \square

Proof of Theorem 3.12.

Proof. We prove that $\Gamma \models_i e : t$ for all $i \in \mathbb{N}$ by induction on $\Gamma \vdash e : t$.

Rule [CONST] .

It follows from Item 2 of Lemma A.1.

Rule [•] .

Suppose $i = 0$. Then,

$$\rho(e) \in \llbracket t \rrbracket_0 = \mathbb{E}$$

Suppose $i > 0$ and $\rho \models_i \Gamma$. If $x : s \in \Gamma$ then $\rho(x) \in \llbracket s \rrbracket_i \subseteq \llbracket s \rrbracket_{i-1}$ by Item 2 of Lemma 3.9. Hence, $\rho \models_{i-1} \Gamma$. By induction hypothesis, $\Gamma \models_j e : t$ for all $j \in \mathbb{N}$. Hence, $\rho(e) \in \llbracket t \rrbracket_{i-1}$ and

$$\rho(e) \in \llbracket t \rrbracket_{i-1} = \llbracket \bullet t \rrbracket_i$$

Rule [→E] .

The derivations ends with the rule:

$$\frac{\Gamma \vdash e_1 : \bullet^n(s \rightarrow t) \quad \Gamma \vdash e_2 : \bullet^n s}{\Gamma \vdash e_1 e_2 : \bullet^n t}$$

with $e = e_1 e_2$. By induction hypothesis, for all $i \in \mathbb{N}$

$$\Gamma \models_i e_1 : \bullet^n(s \rightarrow t) \tag{A.1}$$

$$\Gamma \models_i e_2 : \bullet^n s \tag{A.2}$$

We have two cases:

(1) Case $i < n$. By Item 1 of Lemma 3.8, $\llbracket \bullet^n t \rrbracket_i = \mathbb{E}$. We trivially get

$$\rho(e_1 e_2) \in \llbracket \bullet^n t \rrbracket_i$$

(2) Case $i \geq n$. Suppose that $\rho \models_i \Gamma$.

$$\begin{aligned} \rho(e_1) &\in \llbracket \bullet^n(s \rightarrow t) \rrbracket_i \text{ by (A.1)} \\ &= \llbracket (s \rightarrow t) \rrbracket_{i-n} \text{ by Item 2 of Lemma 3.8} \end{aligned} \tag{A.3}$$

$$\begin{aligned} \rho(e_2) &\in \llbracket \bullet^n s \rrbracket_i \text{ by (A.2)} \\ &= \llbracket s \rrbracket_{i-n} \text{ by Item 2 of Lemma 3.8} \end{aligned} \tag{A.4}$$

By Definition of $\llbracket (s \rightarrow t) \rrbracket_{i-n}$ and (A.3), there are two possibilities:

(a) Case $\rho(e_1) \in \mathbb{H}\mathbb{V}$. Then,

$$\rho(e_1 e_2) = \rho(e_1) \rho(e_2) \longrightarrow^* \mathbb{E}[x] \rho(e_2) \tag{A.5}$$

Hence,

$$\begin{aligned} \rho(e_1 e_2) &\in \mathbb{H}\mathbb{V} \quad \text{by (A.5)} \\ &\subseteq \llbracket \bullet^n t \rrbracket_i \quad \text{by Item 1 of Lemma 3.9.} \end{aligned}$$

(b) Case $\rho(e_1) \longrightarrow^* \lambda x.e'$ or $\rho(e_1) \longrightarrow^* E[\mathbf{k}]$. We also have that

$$\rho(e_1)e'' \in ([t]_{i-n} \quad \forall e'' \in ([s]_{i-n}))$$

In particular (A.4), implies

$$\rho(e_1e_2) = \rho(e_1)\rho(e_2) \in ([t]_{i-n})$$

Since $([t]_{i-n} = ([\bullet^n t]_i)$ by Item 2 of Lemma 3.8, we are done.

Rule $[\rightarrow \mathbf{I}]$.

The derivation ends with the rule:

$$\frac{\Gamma, x : \bullet^n t \vdash e : \bullet^n s}{\Gamma \vdash \lambda x.e : \bullet^n (t \rightarrow s)}$$

By induction hypothesis, for all $i \in \mathbb{N}$

$$\Gamma, x : \bullet^n t \models_i e : \bullet^n s \tag{A.6}$$

We have two cases:

(1) Case $i < n$. By Item 1 of Lemma 3.8, $([\bullet^n (t \rightarrow s)]_i = \mathbb{E}$. We trivially get

$$\rho(\lambda x.e) \in ([\bullet^n (t \rightarrow s)]_i)$$

(2) Case $i \geq n$. Suppose that $\rho \models_i \Gamma$. By Item 2 of Lemma 3.8, it is enough to prove that

$$\rho(\lambda x.e) \in ([t \rightarrow s]_{i-n})$$

For this, suppose $f \in ([t]_j)$ for $j \leq i - n$. We consider the substitution function defined as $\rho_0 = \rho \cup \{(x, f)\}$. We have that

$$\rho_0 \models_{j+n} \Gamma, x : \bullet^n t \tag{A.7}$$

because

(a) $\rho_0(x) = f \in ([t]_j = ([\bullet^n t]_{j+n})$ by Item 2 of Lemma 3.8.

(b) $\rho_0 \models_{j+n} \Gamma$ by Lemma A.2 and the fact that $\rho_0 \models_i \Gamma$.

It follows from (A.6) and (A.7) that

$$\rho_0(e) \in ([\bullet^n s]_{j+n}) \tag{A.8}$$

Therefore, we obtain

$$\begin{aligned} (\lambda x.e)f \longrightarrow \rho(e)[f/x] &= \rho_0(e) \in ([\bullet^n s]_{j+n}) && \text{by (A.8)} \\ &= ([s]_j) && \text{by Item 2 of Lemma 3.8} \end{aligned}$$

By Item 1 of Lemma A.1, we conclude

$$(\lambda x.e)f \in ([s]_j). \quad \square$$

APPENDIX B. AUXILIARY FUNCTIONS FOR THE TYPE INFERENCE ALGORITHM

This section gives several algorithms (functions) that are used in the unification and type inference algorithms.

Let $\mathbf{trees}(T)$ denote the set of subtrees of T and $\mathbf{ptrees}(T)$ denote the set of proper subtrees of T . If T is a regular pseudo meta-type then, $\mathbf{trees}(T)$ and $\mathbf{ptrees}(T)$ are finite.

B.1. Guardness. Algorithm 3 defines the function \mathbf{guard} that given a set \mathcal{C} of meta-type constraints returns a set \mathcal{E} of integers constraints that guarantees guardedness.

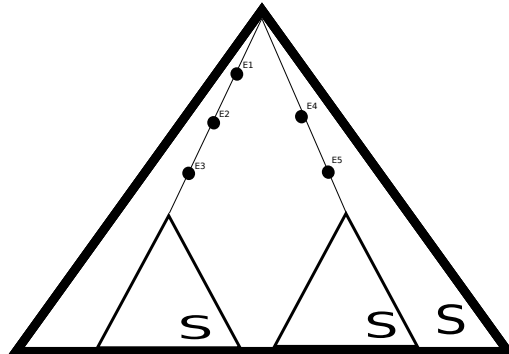
Algorithm 3: Integer constraints to guarantee guardedness

Function $\mathbf{guard}(\mathcal{C})$ **input** : \mathcal{C} is a set of substitutive constraints**output** : $\mathbf{guard}(\mathcal{C}) = \mathcal{E}$ such that $\theta \circ \tau_{\mathcal{C}}$ is guarded iff $\theta \models \mathcal{E}$ for all natural θ .**return** $\bigcup_{X \in \text{dom}(\tau_{\mathcal{C}})} \mathbf{guard}(\tau_{\mathcal{C}}(X))$ **Function** $\mathbf{guard}(T)$ **input** : T regular pseudo meta-type**output** : $\mathbf{guard}(T) = \mathcal{E}$ such that $\theta(T)$ is guarded iff $\theta \models \mathcal{E}$ for all natural θ . $\mathcal{E} \leftarrow \emptyset$;**foreach** $S \in \mathbf{ptrees}(T)$ such that $S \in \mathbf{ptrees}(S)$ /* $S = \dots S \dots$ */ **do**

$\mathcal{E} \leftarrow \mathcal{E} \cup \{E \geq 1 \mid E \in \text{count}(S, S)\}$
--

return \mathcal{E} **Function** $\mathbf{count}(S, T)$ **input** : S, T regular pseudo meta-types**output** : the set of $E_1 + \dots + E_n$ such that $T = \bullet^{E_1}(\dots(\bullet^{E_n}S)\dots)$ **if** $S \notin \mathbf{ptrees}(T)$ **then return** \emptyset ;**case** $T = S$ **do return** $\{0\}$;**case** $T = \bullet^{E}T'$ **do return** $\{E + E' \mid E' \in \text{count}(S, T')\}$;**case** $T = (T_1 \text{ op } T_2)$ **do return** $\text{count}(S, T_1) \cup \text{count}(S, T_2)$;

The function $\mathbf{count}(S, S)$ counts the number of bullets from the root of S to each recursive occurrence of S , i.e. it gives the set of $E_1 + \dots + E_n$ such that $S = \bullet^{E_1}(\dots(\bullet^{E_n}S)\dots)$ In the figure below, $\mathbf{count}(S, S)$ returns $\{E_1 + E_2 + E_3, E_4 + E_5\}$.



For example,

- (1) Let $\mathcal{C} = \{X \stackrel{?}{=} \bullet^N(\mathbf{Nat} \rightarrow \bullet^M X)\}$. The set $\mathcal{E} = \{N + M \stackrel{?}{\geq} 1\}$ guarantees that $\tau_{\mathcal{C}}$ is guarded, i.e. $\theta \circ \tau_{\mathcal{C}}$ is guarded for all θ such that $\theta \models \mathcal{E}$.
- (2) Let $\mathcal{C} = \{X_1 \stackrel{?}{=} \bullet^{N_1}(\bullet^{N_2} X_1 \rightarrow \bullet^{N_3} X_2), X_2 \stackrel{?}{=} \bullet^{M_1}(\bullet^{M_2} X_2 \rightarrow \bullet^{M_3} X_3), X_3 \stackrel{?}{=} \bullet^{K_1}(\bullet^{K_2} X_1 \rightarrow \bullet^{K_3} X_3)\}$. The set $\mathcal{E} = \{N_1 + N_2 \geq 1, N_1 + N_3 + M_1 + M_3 + K_1 + K_2 \geq 1, M_1 + M_2 \geq 1, K_1 + K_3 \geq 1\}$ enforces that $\tau_{\mathcal{C}}$ is guarded, i.e. $\theta \circ \tau_{\mathcal{C}}$ is guarded for all θ such that $\theta \models \mathcal{E}$.

B.2. Other Auxiliary Functions. Algorithm 4 checks whether a given meta-type has certain shape and Algorithm 5 checks whether two meta-types have the same shape ‘at depth 0’. Algorithm 6 defines two functions for checking whether a meta-type is equal to \bullet^∞ . Algorithm 7 checks whether the meta-type is \bullet^∞ -free and Algorithm 8 checks whether the meta-type is tail finite.

Algorithm 4: Checking the shape of the meta-type

Function `isNatorOp?(T)`
 \lfloor return $T = \mathbf{Nat}$ or $T = S_1 \text{ op } S_2$

Algorithm 5: Checking that meta-types are equally constructed

Function `EqualCons?(T, S)`
 \lfloor return $T = \mathbf{Nat}$ and $S = \mathbf{Nat}$ or $T = T_1 \text{ op } T_2$ and $S = S_1 \text{ op } S_2$

Algorithm 6: Checking that a meta-type is equal to \bullet^∞

Function `Diff $_{\bullet^\infty}$?(T)`

output: `Diff $_{\bullet^\infty}$?(T)` is true iff T is different from \bullet^∞ .
if $T = \bullet^E S$ and $(S = X$ or $S = \mathbf{Nat}$ or $S = (t_1 \rightarrow t_2)$ or $S = (t_1 \times t_2)$) **then**
 \lfloor return true
else return false;

Algorithm 7: Checking that a meta-type is \bullet^∞ -free

Function `\bullet^∞ -free?(T)`

output: `\bullet^∞ -free?(T)` is true iff T is \bullet^∞ -free.
foreach $S \in \text{trees}(T)$ **do**
 \lfloor **if** $\neg \text{Diff}_{\bullet^\infty}(S)$ **then** return false;
return true

Algorithm 8: Checking that a meta-type is tail finite

Function $\text{tailfinite?}(T)$
 input : T regular meta-type
 output : $\text{tailfinite?}(T)$ is true iff T is tail finite.
 if $\neg \bullet^\infty\text{-free?}(T)$ then return false;
 $\mathcal{E} \leftarrow \emptyset$;
 foreach $S \in \text{ptrees}(T)$ such that $S \in \text{ptrees}(S)$ /* $S = \dots S \dots$ */ do
 if $\text{arrows}(S, S)$ then return false;
 return true

Function $\text{arrows}(S, T)$
 input : S, T regular pseudo meta-types
 output : $\text{arrows}(S, T)$ true iff $T = \bullet^{E_1}(T_1 \rightarrow \dots (T_n \rightarrow \bullet^{E_n} S) \dots)$
 if $T = \bullet^E(T_1 \rightarrow T_2)$ and $S \in \text{ptrees}(T_2)$ then return $\text{arrows}(S, T_2)$;
 else return false;
