

ELEMENTARY QUOTIENT COMPLETIONS, CHURCH’S THESIS, AND PARTITIONED ASSEMBLIES

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ABSTRACT. Hyland’s effective topos offers an important realizability model for constructive mathematics in the form of a category whose internal logic validates Church’s Thesis. It also contains a boolean full sub-quasitopos of “assemblies” where only a restricted form of Church’s Thesis survives. In the present paper we compare the effective topos and the quasitopos of assemblies each as the elementary quotient completions of a Lawvere doctrine based on the partitioned assemblies. In that way we can explain why the two forms of Church’s Thesis each category satisfies differ by the way each is inherited from specific properties of the doctrine which determines the elementary quotient completion.

INTRODUCTION

Hyland’s paper “The Effective Topos” [Hyl82], introducing and studying the category $\mathcal{E}ff$ in the title of the paper, opened a new way to apply techniques developed in realizability to analyse extensively various aspects of constructive mathematics and of computer science, combining them with the essential use of category theory, see [vO02, HRR90b, HRR90a, FMRS92, FRR92]. The effective topos is the first example of an elementary non-Grothendieck topos with a natural number object. It also provides a computational interpretation of the logic of a topos, see [BJ81], and [Mai05] for a dependent type-theoretic version of it. Indeed the interpretation of the internal logic in $\mathcal{E}ff$ extends Kleene’s realizability interpretation of Intuitionistic Arithmetic [Kle45], validates formal Church’s Thesis \mathbf{CT} , and the statement that every Cauchy real is computable, see [Hyl82].

In *loc. cit.*, the full subcategory $\mathcal{A}sm$ on the $\neg\neg$ -separated objects of $\mathcal{E}ff$ is also introduced and studied—those objects have later been christened “assemblies”, hence the shorthand $\mathcal{A}sm$ for the full subcategory they determine, see [CFS88, vO08]. In the category $\mathcal{A}sm$ the

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endoarrows on the natural number object correspond exactly to the computable functions. But in the (boolean) logic of the strong subobjects of the quasitopos $\mathcal{A}sm$, not all Cauchy reals defined as (equivalence classes of certain) functional relations are computable and **CT** does not hold. In that logic, only a restricted form of **CT**—expressing internally that the arrows on the natural numbers in $\mathcal{A}sm$ are computable—survives and it is called Type-Theoretic Church’s Thesis, in short **TCT**, see Definition 3.9 for the precise forms of these principles. That in turn implies that the Axiom of Unique Choice, even on the natural numbers, does not hold in the logic of strong subobjects in $\mathcal{A}sm$. Instead the Axiom of Unique Choice, and even the Axiom of Countable Choice and **CT**, hold in the internal logic of subobjects of $\mathcal{A}sm$, see Remark 4.12.

In this paper we show that each of the categories $\mathcal{E}ff$ and $\mathcal{A}sm$ can be viewed as the domain Q_P of the “elementary quotient completion”

$$\widehat{P}: Q_P^{\text{op}} \longrightarrow \mathbf{InfSL}$$

of a doctrine $P: \mathcal{C}^{\text{op}} \longrightarrow \mathbf{InfSL}$, as introduced in [MR13b, MR13a]. Intuitively, Q_P is obtained from $P: \mathcal{C}^{\text{op}} \longrightarrow \mathbf{InfSL}$ by freely adding quotients of the equivalence relations specified by P , while \widehat{P} extends P to the new sorts of Q_P in an appropriate way. The two doctrines giving rise to $\mathcal{E}ff$ and $\mathcal{A}sm$ have the same domain $\mathcal{P}\mathcal{A}sm$, the full subcategory of $\mathcal{A}sm$ (therefore of $\mathcal{E}ff$) on the partitioned assemblies. Specifically:

- (1) The doctrine $\text{Sub}_{\mathcal{E}ff}$ of the subobjects on $\mathcal{E}ff$ is the elementary quotient completion of the doctrine $\Psi_{\mathcal{P}\mathcal{A}sm}$ of variations on $\mathcal{P}\mathcal{A}sm$. The intuition about a doctrine of the form $\Psi_{\mathcal{A}}$ for a category \mathcal{A} dates back to the original paper [Law69] and the term “variation” was introduced in [Gra00], see Example 1.1(b).
- (2) The doctrine $\text{Sts}_{\mathcal{A}sm}$ of strong subobjects on $\mathcal{A}sm$ is the elementary quotient completion of the boolean doctrine $\mathbb{P}\Gamma$ on $\mathcal{P}\mathcal{A}sm$ which is the composition of the powerset functor with the global section functor $\Gamma: \mathcal{P}\mathcal{A}sm \longrightarrow \mathit{Set}$.

After showing some general transfer principles describing how the validity of choice principles and the principles **CT** and **TCT** transfers from a doctrine P to its elementary quotient completion \widehat{P} we conclude that:

- (1) The doctrine $\text{Sts}_{\mathcal{A}sm}$ satisfies only **TCT** (but not **CT**) as a direct consequence of the validity of **TCT** in $\mathbb{P}\Gamma$. Thanks to an adjoint situation between $\mathbb{P}\Gamma$ and $\Psi_{\mathcal{P}\mathcal{A}sm}$, we can prove that **TCT** is inherited by $\Psi_{\mathcal{P}\mathcal{A}sm}$. And this is strengthened to the full validity of **CT** in $\Psi_{\mathcal{P}\mathcal{A}sm}$ by choice principles.
- (2) In the logic of $\mathcal{E}ff$ the validity of **CT** and of choice principles on partitioned assemblies is a direct consequence of the validity of corresponding principles on the doctrine $\Psi_{\mathcal{P}\mathcal{A}sm}$. Also the fact that the logic of $\mathcal{E}ff$ extends Kleene’s realizability interpretation of Intuitionistic Arithmetic [Kle45] is again inherited by $\Psi_{\mathcal{P}\mathcal{A}sm}$.

These results on $\mathcal{E}ff$ are to be compared with the original construction of $\mathcal{E}ff$ via the tripos-to-topos construction in [HJP80] applied to a hyperdoctrine with domain the category Set of sets and functions. That hyperdoctrine does not validate Intuitionistic Arithmetic (neither does it extend Kleene’s realizability interpretation of Intuitionistic Arithmetic!) but nevertheless it produces the topos $\mathcal{E}ff$ whose subobject doctrine does.

Section 1 collects basic notions about elementary doctrines $P: \mathcal{C}^{\text{op}} \longrightarrow \mathbf{InfSL}$, introduced in [Law69, Law70] as well as the construction of the elementary quotient completion $\widehat{P}: Q_P^{\text{op}} \longrightarrow \mathbf{InfSL}$. In section 2 we recall some transfer results for some logical principles

between an elementary doctrine and its elementary quotient completion including a characterization of the doctrine of variations via a choice principle. In section 3 we introduce arithmetic doctrines, which are doctrines with a parameterized natural number object which satisfy induction in the sense of the logic determined by P , and we prove that the property of being arithmetic transfers from suitable doctrines to their elementary quotient completions. We also prove transfer results for **CT** and **TCT**. In section 4 we show that the doctrine of subobjects on $\mathcal{E}ff$ is the quotient completion of the doctrine of variations on $\mathcal{P}A\mathit{sm}$, and that the doctrine of strong subobjects on $\mathcal{A}sm$ is the quotient completion of the doctrine $\mathbb{P}\Gamma$ on $\mathcal{P}A\mathit{sm}$. We then apply the general transfer principles proved before to deduce the validity of **CT** in $\text{Sub}_{\mathcal{E}ff}$ and the validity of **TCT** in $\text{Sts}_{\mathcal{A}sm}$. Finally in section 5 we justify why $\text{Sub}_{\mathcal{E}ff}$ extends Kleene's realisability interpretation of Intuitionistic Arithmetic as a consequence of the facts that $\Psi_{\mathcal{P}A\mathit{sm}}$ does so and that $\text{Sub}_{\mathcal{E}ff}$ is the elementary quotient completion of $\Psi_{\mathcal{P}A\mathit{sm}}$.

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1. ELEMENTARY QUOTIENT COMPLETIONS: A BRIEF RECAP

In this section we review some notions and results about elementary doctrines and their elementary quotient completion, which was introduced in [MR13b, MR13a] and studied extensively in a series of papers which will be mentioned in due course.

Let \mathcal{C} be a category with binary products

$$A_1 \xleftarrow{\text{pr}_1} A_1 \times A_2 \xrightarrow{\text{pr}_2} A_2$$

for every pair of objects A_1 and A_2 in \mathcal{C} , and a terminal object T . Recall from [MR13b, MR13a] that a **primary doctrine** on \mathcal{C} is an indexed inf-semilattice $P: \mathcal{C}^{\text{op}} \rightarrow \mathbf{InfSL}$, *i.e.* a (contravariant) functor $P: \mathcal{C}^{\text{op}} \rightarrow \mathbf{Pos}$ such that each poset $P(C)$ is an \wedge -semilattice and for every arrow $f: A \rightarrow B$ in \mathcal{C} the monotone map $P_f: P(B) \rightarrow P(A)$ is a \wedge -homomorphism—note the reversed direction!—and one declares a primary doctrine **elementary** when, for every object A in \mathcal{C} , there is an object δ_A in $P(A \times A)$ such that for every arrow e of the form $\langle \text{pr}_1, \text{pr}_2, \text{pr}_2 \rangle: X \times A \rightarrow X \times A \times A$ in \mathcal{C} , the assignment

$$\mathcal{F}_e(\alpha) := P_{\langle \text{pr}_1, \text{pr}_2 \rangle}(\alpha) \wedge P_{\langle \text{pr}_2, \text{pr}_3 \rangle}(\delta_A)$$

for α in $P(X \times A)$ determines a left adjoint to the map $P_e: P(X \times A \times A) \rightarrow P(X \times A)$.

Elementary doctrines are the cloven Eq-fibrations of [Jac99] and, as explained in *loc.cit.*, there is a deductive logical calculus associated with them: it is the fragment of Intuitionistic Logic with conjunctions and equality over a type theory with a unit type and the binary product type constructor. From now on, we shall employ the logical language introduced in *loc.cit.* and often write

$$a_1: A_1, \dots, a_k: A_k \mid \phi_1(a_1, \dots, a_k), \dots, \phi_n(a_1, \dots, a_k) \vdash \psi(a_1, \dots, a_k)$$

in place of

$$\phi_1 \wedge \dots \wedge \phi_n \leq \psi$$

in $P(A_1 \times \dots \times A_k)$. Note that, in line with *loc.cit.*, $\delta_A(a, a')$ will be written as $a: A, a': A \mid a =_A a'$. Also we write $a: A \mid \alpha(a) \dashv\vdash \beta(a)$ to abbreviate the two facts that $a: A \mid \alpha(a) \vdash \beta(a)$ and $a: A \mid \beta(a) \vdash \alpha(a)$.

Examples 1.1. (a) The doctrine of subobjects on a category \mathcal{C} with finite limits will be denoted as $\text{Sub}_{\mathcal{C}}: \mathcal{C}^{\text{op}} \longrightarrow \mathbf{InfSL}$ —the elementary structure is provided by the diagonal arrows.

(b) Another example is provided by the *doctrine of variations* $\Psi_{\mathcal{S}}: \mathcal{S}^{\text{op}} \longrightarrow \mathbf{InfSL}$ of \mathcal{S} , where \mathcal{S} is a category with binary products and weak pullbacks. The fibre on the object A in \mathcal{S} is the poset reflection of the comma category \mathcal{S}/A , see [Gra00], the action on arrows is given by weak pullbacks.

The categorical approach makes it possible to express precisely how the doctrines are related as category theory suggests directly what “homomorphisms of doctrines” should be. In fact, one introduces the 2-category **ED** of elementary doctrines which has

1-arrows $(F, b): P \rightarrow R$: pairs (F, b) where $F: \mathcal{C} \rightarrow \mathcal{D}$ is a functor and $b: P \rightarrow R \circ F^{\text{op}}$ is a natural transformation as in the diagram

$$\begin{array}{ccc} \mathcal{C}^{\text{op}} & \xrightarrow{P} & \mathbf{InfSL} \\ F^{\text{op}} \downarrow & \searrow b \cdot & \downarrow \\ \mathcal{D}^{\text{op}} & \xrightarrow{R} & \mathbf{InfSL} \end{array}$$

where the functor F preserves products and, for every object A in \mathcal{C} , the functor $b_A: P(A) \rightarrow R(F(A))$ preserves finite infima and

$$b_{A \times A}(\delta_A) = R_{(F(\text{pr}_1), F(\text{pr}_2))}(\delta_{F(A)});$$

2-arrows $\theta: (F, b) \rightarrow (G, c)$: natural transformations $\theta: F \rightarrow G$ such that for every A in \mathcal{C} and every α in $P(A)$, one has that $b_A(\alpha) \leq_{F(A)} R_{\theta_A}(c_A(\alpha))$.

Examples 1.2. Given a category \mathcal{C} with products and pullbacks, one can consider the two indexed posets: that of subobjects $\text{Sub}_{\mathcal{C}}: \mathcal{C}^{\text{op}} \longrightarrow \mathbf{InfSL}$ and that of variations $\Psi_{\mathcal{C}}: \mathcal{C}^{\text{op}} \longrightarrow \mathbf{InfSL}$. Recall that $\Psi_{\mathcal{C}}(A)$ is the poset reflection of the comma category \mathcal{C}/A . Its inclusion in $\text{Sub}_{\mathcal{C}}(A)$ extends to a 1-arrow from $\text{Sub}_{\mathcal{C}}$ to $\Psi_{\mathcal{C}}$.

Recall that a category \mathcal{C} with binary products is *weakly cartesian closed* if for every pair of objects A and B there is an object W and an arrow $\text{ev}: W \times A \rightarrow B$ such that for every $f: C \times A \rightarrow B$ there is $g: C \rightarrow W$ with $\text{ev}(g \times \text{id}_A) = f$. Since a category \mathcal{C} is cartesian closed when every mediating arrow g in the condition above is unique, we refer to W as a *weak exponential* of B with A and to the arrow $\text{ev}: W \times A \rightarrow B$ as a *weak evaluation*.

The category *Set* is cartesian closed, while the category of topological spaces and continuous functions is notoriously not cartesian closed, but it is weakly cartesian closed, see [CR00].

A *weak hyperdoctrine* $P: \mathcal{C}^{\text{op}} \longrightarrow \mathbf{InfSL}$ is an elementary doctrine such that

- (i) \mathcal{C} is weakly cartesian closed;
- (ii) P factors through the category **Heyt** of Heyting algebras and Heyting algebras homomorphisms;
- (iii) for every product projection $\text{pr}_1: A \times B \rightarrow A$ the monotone map P_{pr_1} has a left adjoint $\mathcal{E}_{\text{pr}_1}: P(A) \rightarrow P(A \times B)$ and a right adjoint $\mathcal{V}_{\text{pr}_1}: P(A) \rightarrow P(A \times B)$
- (iv) for every arrow $f: X \rightarrow A$ the canonical inequalities $\mathcal{E}_{\text{pr}_1} P_f \leq P_f \mathcal{E}_{\text{pr}_1}$ and $P_f \mathcal{V}_{\text{pr}_1} \leq \mathcal{V}_{\text{pr}_1} P_f$, where $\text{pr}_1: A \times B \rightarrow A$ and $\text{pr}': X \times B \rightarrow X$ are projections, are equalities.

A weak hyperdoctrine $P: \mathcal{C}^{\text{op}} \longrightarrow \mathbf{InfSL}$ is a *hyperdoctrine* if \mathcal{C} is cartesian closed.

When $P: \mathcal{C}^{\text{op}} \longrightarrow \mathbf{InfSL}$ is a weak hyperdoctrine we may write that $P: \mathcal{C}^{\text{op}} \longrightarrow \mathbf{Heyt}$. Also we shall refer to condition (iv) as the *Beck-Chevalley condition*.

Similarly to the case of elementary doctrines, in line with [Jac99] and [Pit00], one can associate a deductive logical calculus to a hyperdoctrine $P: \mathcal{C}^{\text{op}} \longrightarrow \mathbf{Heyt}$: it is a predicate calculus with equality and a lambda constructor over a type theory with a unit type, a binary-product type constructor and a function type constructor. We shall employ the following notation: Given a term $(a: A, c: C \mid t: B)$ in \mathcal{C} , the term $(c: C \mid \lambda a: A.t: W)$ is such that the terms $(a: A, c: C \mid t: B)$ and $(a: A, c: C \mid \text{ev}(\lambda a: A.t, a): B)$ are equal. Also a term $(c: C \mid s: W)$ in \mathcal{C} is equal to the term $(c: C \mid \lambda a: A.\text{ev}(s, a): W)$. Given the well formed formulas $a: A \mid \phi(a)$ and $a: A \mid \psi(a)$ we write

$$a: A \mid \phi(a) \vee \psi(a) \quad a: A \mid \phi(a) \Rightarrow \psi(a)$$

to denote joins and Heyting implication. The least element will be $a: A \mid \perp$. As is customary, we abbreviate $a: A \mid \phi(a) \Rightarrow \perp$ with $a: A \mid \neg\phi(a)$. For a projection $\text{pr}_1: A \times B \rightarrow A$ and for ϕ in $P(A \times B)$ we shall write $\mathcal{E}_{\text{pr}_1}(\phi)$ and $\mathcal{V}_{\text{pr}_1}(\phi)$ in $P(A)$ as

$$a: A \mid \exists_{b: B} \phi(a, b) \quad a: A \mid \forall_{b: B} \phi(a, b).$$

Remark 1.3. There is instead a radical difference in case $P: \mathcal{C}^{\text{op}} \longrightarrow \mathbf{Heyt}$ is a weak hyperdoctrine—and in some sense this shows the usefulness of the categorical presentation. The weakened condition, stripped of uniqueness, allows to introduce a λ -notation, but in general the terms $(c: C \mid s: W)$ and $(c: C \mid \lambda a: A.\text{ev}(s, a): W)$ do not coincide, and more importantly, it is not possible to substitute *inside* a λ -term. So for a weak hyperdoctrine we shall use all the above but with no reference to λ -terms, namely concerning function types we just use the evaluation constructor.

Remark 1.4. If $\text{ev}: W \times A \rightarrow B$ and $\text{ev}': W' \times A \rightarrow B$ are two weak evaluation maps, then

$$f: W \vdash \exists_{f': W'} \forall_{a: A} [\text{ev}(f, a) = \text{ev}'(f', a)]$$

It is easy to see that, for $P: \mathcal{C}^{\text{op}} \longrightarrow \mathbf{Heyt}$ a weak hyperdoctrine on \mathcal{C} , for every arrow $f: A \rightarrow B$ the monotone map $P_f: P(B) \rightarrow P(A)$ has a left adjoint $\mathcal{E}_f: P(A) \rightarrow P(B)$ and a right adjoint \mathcal{V}_f that send $a: A \mid \alpha(a)$ in $P(A)$ respectively to

$$b: B \mid \exists_{a: A} [[f(a) =_B b] \wedge \alpha(a)] \quad \text{and} \quad b: B \mid \forall_{a: A} [[f(a) =_B b] \Rightarrow \alpha(a)]$$

We shall employ logical wording to mark certain situations in an elementary doctrine $P: \mathcal{C}^{\text{op}} \longrightarrow \mathbf{InfSL}$. For the terminal object 1 in \mathcal{C} , we call an element of $P(1)$ a *sentence*. For a sentence α in P such that $\top \leq \alpha$ we write $\vdash \alpha$.

Examples 1.5. (a) The doctrine $\text{Sub}_{\mathcal{C}}$ in Example 1.1(a) is a (weak) hyperdoctrine if and only if \mathcal{C} is a (weakly) cartesian closed Heyting category.

(b) If \mathcal{C} is (weakly) locally cartesian closed with finite (weak) coproducts, the doctrine $\Psi_{\mathcal{C}}$ in Example 1.1(b) is a (weak) hyperdoctrine. Since weak hyperdoctrines of the form $\Psi_{\mathcal{C}}$ play a central role in the paper, we find it convenient to denote the left adjoint along $\Psi_{\mathcal{C}}(f)$ as Σ_f and the right adjoint as Π_f .

Let $P: \mathcal{C}^{\text{op}} \longrightarrow \mathbf{Heyt}$ and $R: \mathcal{C}^{\text{op}} \longrightarrow \mathbf{Heyt}$ be weak hyperdoctrines. Suppose the natural transformations $r: P \rightarrow R$ is a 1-arrow of doctrines in **ED**; r is a right adjoint if there is a 1-arrow of doctrines $l: R \rightarrow P$ such that $\text{Id}_R \leq r \circ l$ and $l \circ r \leq \text{Id}_P$. This adjoint pair satisfies the *Frobenius reciprocity* if, for all A in \mathcal{C} , for all α in $P(A)$ and all β in $R(A)$ it holds that $l_A(\beta) \wedge \alpha = l_A(\beta \wedge r_A(\alpha))$. In the following proposition we use superscript to distinguish operations in P from the corresponding operations in R .

Proposition 1.6. *If a 1-arrow of doctrine $r: P \rightarrow R$ is a right adjoint, then for every α in $P(X \times Y)$*

$$r_X \mathbb{V}_{\text{pr}_1}^P(\alpha) = \mathbb{V}_{\text{pr}_1}^R r_{X \times Y}(\alpha).$$

Moreover for every γ and β in $P(A)$ it holds

$$r_A(\gamma \Rightarrow^P \beta) = r_A(\alpha) \Rightarrow^R r_A(\beta)$$

if and only if the adjoint pair satisfies the Frobenius reciprocity.

Note that if the adjoint pair is such that $l \circ r = \text{Id}_P$, then it satisfies the Frobenius reciprocity.

Proposition 1.6 proves that right adjoints commute with right adjoints. Oppositely to right adjoints, left adjoints do not commute with respect to $r: P \rightarrow R$. We shall see that in our case of interest r commutes with \mathcal{I} exactly when P satisfies a form of choice that we call (RC), see Theorem 2.7. Of course, it might be the case that for some specific formula the property holds, though P does not satisfy (RC). This motivates the following definition, which is instrumental for the proofs of the main theorems in section 4.

Definition 1.7. Suppose P is an elementary existential doctrine, α is in $P(Y \times B)$ and $\epsilon: Y \rightarrow B$ is an arrow in \mathcal{C} . We say that ϵ is a **Skolem arrow for B in α** if $\mathcal{I}_{\text{pr}_1} \alpha = P_{(\text{id}_Y, \epsilon)}(\alpha)$, i.e. if $y: Y \mid \exists_{b: B} \alpha(y, b) \dashv\vdash \alpha(y, \epsilon(y))$.

We use the Greek letter ϵ to denote a Skolem arrow in view of the strict connection between Skolem terms and ϵ -terms of Hilbert's ϵ -calculus. Here we observe that, if B has Skolem arrows for all formulas, then B is endowed with an ϵ -operator as defined in [MPR17], which is a stronger property than the Rule of Choice on B introduced in Definition 2.1 (see also [Pas16b, Pas18b]).

Doctrines of subobjects are characterized via the notion of comprehension. Though very general, we shall recall this notion in the particular case of an elementary doctrine $P: \mathcal{C}^{\text{op}} \rightarrow \mathbf{InfSL}$. For a given object A in \mathcal{C} and an object α in $P(A)$, a **weak comprehension** of α is an arrow $\{\alpha\}: X \rightarrow A$ in \mathcal{C} such that

$$x: X \mid \top \vdash \alpha(\{\alpha\}(x))$$

and, for every $f: Z \rightarrow A$ such that $z: Z \mid \top \vdash \alpha(f(z))$, there is an arrow $f': Z \rightarrow X$ such that $f = \{\alpha\}f'$. The arrow $\{\alpha\}$ is the **strong comprehension** or simply **comprehension** of α if $\{\alpha\}$ is monic, making the required f' the unique such.

Intuitively, the comprehension arrow represents the inclusion of the object obtained by comprehending the predicate α over A into A itself as a form of subtype.

We simply say that the doctrine $P: \mathcal{C}^{\text{op}} \rightarrow \mathbf{InfSL}$ **has (weak) comprehensions** when every α has a (weak) comprehension arrow. And P **has full (weak) comprehensions** if $\alpha \leq \beta$ in $P(A)$ whenever $\{\alpha\}$ factors through $\{\beta\}$.

Example 1.8. Doctrines of the form $\Psi_{\mathcal{C}}$ have full weak comprehensions: if $[f]$ is in $\Psi_{\mathcal{C}}(A)$ then the representative f is a weak full comprehension of $[f]$: it is strong if and only if f is monic. So doctrines of the form $\text{Sub}_{\mathcal{C}}$ have full comprehensions.

An elementary doctrine $P: \mathcal{C}^{\text{op}} \rightarrow \mathbf{InfSL}$ **has comprehensive diagonals** if for every A in \mathcal{C} the diagonal $\Delta_A: A \rightarrow A \times A$ is the full comprehension of δ_A . It is straightforward to verify that an elementary doctrine has comprehensive diagonals if and only if any two parallel arrows of \mathcal{C} , say $f, g: X \rightarrow Y$, are equal whenever $x: X \mid \top \vdash f(x) =_Y g(x)$.

The intuition underlying the construction of the elementary quotient completion is to add quotients to the domain of the elementary doctrine with respect to equivalence relations in the fibres of the doctrine.

In an elementary doctrine $P: \mathcal{C}^{\text{op}} \rightarrow \mathbf{InfSL}$, if A is an object in \mathcal{C} , an object ρ in $P(A \times A)$ is a ***P-equivalence relation on A*** if it satisfies

reflexivity: $a: A, a': A \mid a =_A a' \vdash \rho(a, a')$;

symmetry: $a: A, a': A \mid \rho(a, a') \vdash \rho(a', a)$;

transitivity: $a: A, a': A, a'': A \mid \rho(a, a') \wedge \rho(a', a'') \vdash \rho(a, a'')$.

Examples 1.9. (a) For a category \mathcal{D} with products and pullbacks, consider the elementary doctrine $\text{Sub}_{\mathcal{D}}: \mathcal{D}^{\text{op}} \rightarrow \mathbf{InfSL}$ of the subobjects of \mathcal{D} . A $\text{Sub}_{\mathcal{D}}$ -equivalence relation is an equivalence relation in \mathcal{D} . In particular, Sub_{Set} -equivalence relations coincide with the usual notion of equivalence relations.

(b) For a category \mathcal{C} with products and weak pullbacks, consider the elementary doctrine $\Psi_{\mathcal{C}}: \mathcal{C}^{\text{op}} \rightarrow \mathbf{InfSL}$ of the variations. A $\Psi_{\mathcal{C}}$ -equivalence relation is a pseudo-equivalence relation in \mathcal{C} , see [CC82].

Given a P -equivalence relation ρ on A , a ***P-quotient of ρ*** , or simply a ***quotient*** when the doctrine is clear from the context, is an arrow $q: A \rightarrow A/\rho$ in \mathcal{C} such that

$$a; A, a': A \mid \rho(a, a') \vdash q(a) =_{A/\rho} q(a')$$

and, for every arrow $g: A \rightarrow Z$ such that

$$a; A, a': A \mid \rho(a, a') \vdash g(a) =_Z g(a'),$$

there is a unique arrow $\bar{g}: A/\rho \rightarrow Z$ such that $g = \bar{g}q$.

We say that such a P -quotient is ***stable*** if, for every pullback

$$\begin{array}{ccc} B & \xrightarrow{q'} & C \\ f' \downarrow & & \downarrow f \\ A & \xrightarrow{q} & A/\rho \end{array}$$

in \mathcal{C} , the arrow q' is a P -quotient.

For an equivalence relation ρ on A , the poset \mathcal{Des}_{ρ} of ***descent data*** is the sub-poset of $P(A)$ on those α such that

$$a: A, a': A \mid \alpha(a) \wedge \rho(a, a') \vdash \alpha(a').$$

Like for comprehension, it is possible to complete an elementary doctrine $P: \mathcal{C}^{\text{op}} \rightarrow \mathbf{InfSL}$ to one with stable quotients of equivalence relations: the ***elementary quotient completion*** $\hat{P}: Q_P^{\text{op}} \rightarrow \mathbf{InfSL}$ of P which was introduced and studied in [MR13a, MR13b, MR15, MR16]. It is defined as follows

Objects of Q_P : (A, ρ) such that ρ is a P -equivalence relation on A .

Arrows of Q_P : an arrow $[f]: (A, \rho) \rightarrow (B, \sigma)$ is an equivalence class of arrows $f: A \rightarrow B$ in \mathcal{C} such that

$$a: A, a': A \mid \rho(a, a') \vdash \sigma(f(a), f(a'))$$

in $P(A \times A)$ with respect to the relation $f \sim g$ which holds if and only if

$$a: A, a': A \mid \rho(a, a') \vdash \sigma(f(a), g(a')).$$

Composition of Q_P : that of \mathcal{C} on representatives.

Identities of Q_P : are represented by identities of \mathcal{C} .

The functor $\widehat{P}: Q_P^{\text{op}} \rightarrow \mathbf{InfSL}$: is defined as

$$\widehat{P}(A, \rho) := \mathcal{D}es_\rho.$$

We refer the reader to [MR13a] for all the details. We just note that the exact completion in [Car95] has a description in terms of the elementary quotient completion of doctrines: given a category \mathcal{C} with finite products and weak pullbacks, the doctrine $\text{Sub}_{\mathcal{C}_{\text{ex/lex}}}$ is equivalent to the doctrine $\widehat{\Psi}_{\mathcal{C}}$.

Here we limit ourselves to recall a few properties of the constructions:

- the elementary quotients completion has effective quotients: for an equivalence relation σ on (A, ρ) , the quotient is given by

$$[\text{id}_A]: (A, \rho) \rightarrow (A, \sigma);$$

- the equality predicate over (A, ρ) is ρ itself, *i.e.* $\delta_{(A, \rho)} = \rho$;
- in case P is a weak hyperdoctrine, the evaluation in Q_P $[\text{ev}]: (B, \delta_B)^{(A, \delta_A)} \times (A, \delta_A) \rightarrow (B, \delta_B)$ can be chosen as a weak evaluation $\text{ev}: W \times A \rightarrow B$ in \mathcal{C} , and $(B, \delta_B)^{(A, \delta_A)}$ is (W, θ) where θ in $P(W \times W)$ is the formula $t: W, t': W \mid \forall_{a:A} \forall_{a':A} a =_A a' \Rightarrow \text{ev}(t, a) =_B \text{ev}(t', a')$.

It is quite apparent that the elementary structure plays no role in the definitions of \widehat{P} . We refer the reader to [Pas15, Pas16a] for an analysis of that.

2. SOME CHOICE PRINCIPLES

In this section we analyse various forms of choice principle in the context of existential elementary doctrines.

Let $P: \mathcal{C}^{\text{op}} \rightarrow \mathbf{Heyt}$ be a weak hyperdoctrine. An element R of $P(A \times B)$ is often called a **relation**. We say that a relation R is **entire** if

$$a: A \mid \top \vdash \exists_{b:B} R(a, b)$$

and that it is **functional** if

$$a: A, b: B, b': B \mid R(a, b) \wedge R(a, b') \vdash b =_B b'.$$

For every arrow $f: A \rightarrow B$ the formula in $P(A \times B)$ determined by

$$a: A, b: B \mid f(a) =_B b$$

is an entire functional relation, called the **P -graph** of f .

Definition 2.1. Let $P: \mathcal{C}^{\text{op}} \rightarrow \mathbf{Heyt}$ be a weak hyperdoctrine.

The Rule of Unique Choice (RUC) holds in P : if for every entire functional relation R in $P(A \times B)$ there is an arrow $f: A \rightarrow B$ whose P -graph is R .

The Rule of Choice (RC) holds in P : if for every entire relation R in $P(A \times B)$ there is an arrow $f: A \rightarrow B$ such that

$$a: A \mid \top \vdash R(a, f(a)).$$

The Rule of Choice holds on A in P : if for every entire relation R in $P(A \times A)$ there is an arrow $f: A \rightarrow A$ such that

$$a: A \mid \top \vdash R(a, f(a)).$$

There are axioms that correspond to (RUC) and to (RC) respectively.

Definition 2.2. Let P be a weak hyperdoctrine. Let A be an object of \mathcal{C} . We say that *the Axiom of Unique Choice* (AUC) *holds on* A if, for every object B in \mathcal{C} , for every relation R in $P(A \times B)$ it is

$$\forall_{a:A} \exists!_{b:B} R(a, b) \vdash \exists_{f:W} \forall_{a:A} R(a, \text{ev}(f, a))$$

where $\text{ev}: W \times A \rightarrow B$ is a weak evaluation map. We say that *the Axiom of Choice* (AC) *holds on* A if, for every object B in \mathcal{C} , for every relation R in $P(A \times B)$ it is

$$\forall_{a:A} \exists_{b:B} R(a, b) \vdash \exists_{f:W} \forall_{a:A} R(a, \text{ev}(f, a))$$

where $\text{ev}: W \times A \rightarrow B$ is a weak evaluation map. When the Axiom of (Unique) Choice holds on every object A in \mathcal{C} , we say that the *Axiom of (Unique) Choice holds in* P .

Clearly, if (AC) holds on A , then (AUC) holds on A .

Those choice principles are useful to characterize variational doctrines as shown in [MPR17]. That characterization employs also an adjunction between variational doctrines and an elementary existential doctrine P with full weak comprehensions as stated in the following proposition from *loc.cit.*.

Proposition 2.3. *Suppose P is an elementary existential doctrine on \mathcal{C} with full weak comprehensions and comprehensive diagonals. There are arrows of doctrines*

$$\begin{array}{ccc} \mathcal{C}^{op} & \xrightarrow{P} & \\ \text{id}_{\mathcal{C}}^{op} \left(\begin{array}{c} \mathcal{E}_-(\top) \cdot \left(\begin{array}{c} \uparrow \\ \downarrow \end{array} \right) \cdot [\{\!-\!\}] \\ \Psi_{\mathcal{C}} \end{array} \right) & & \text{InfSL} \\ \mathcal{C}^{op} & \xrightarrow{\Psi_{\mathcal{C}}} & \end{array}$$

such that $\mathcal{E}_-(\top) \circ [\{\!-\!\}] = \text{id}_P$ and $\text{id}_{\Psi_{\mathcal{C}}} \leq [\{\!-\!\}] \circ \mathcal{E}_-(\top)$.

For clarity, we recall the construction of the two natural transformations: For an object A of \mathcal{C} , $[f: X \rightarrow A]$ in $\Psi_{\mathcal{C}}$, and α in $P(A)$ it is

$$\mathcal{E}_{[f]}(\top) := \mathcal{E}_f(\top_X) \quad \text{and} \quad [\{\!-\!\}]_A(\alpha) = [\{\!\alpha\!\}].$$

Note that the conditions $\text{id}_{\Psi_{\mathcal{C}}} \leq [\{\!-\!\}] \circ \mathcal{E}_-(\top)$ and $\mathcal{E}_-(\top) \circ [\{\!-\!\}] = \text{id}_P$ establish that $[\{\!-\!\}]$ and $\mathcal{E}_-(\top)$ form an adjoint pair satisfying Frobenius reciprocity. This will be useful to prove commutativity of $[\{\!-\!\}]$ and Σ for some formulas of P .

Remark 2.4. Note that if a weak hyperdoctrine on \mathcal{C} has comprehensive diagonals and full weak comprehensions then \mathcal{C} has weak pullbacks whereas, if comprehensions are strong, then \mathcal{C} has pullbacks, see [MPR17]. For this reason we did not assume pullbacks or weak pullbacks in the formulation of Proposition 2.3.

Remark 2.5. Suppose P is a weak hyperdoctrine on \mathcal{C} with full comprehensions and comprehensive diagonals. There is adjunction situation analogous to the one described in Proposition 2.3 between P and $\text{Sub}_{\mathcal{C}}$, *i.e.* $\mathcal{E}_-(\top): \text{Sub}_{\mathcal{C}} \rightarrow P$ and $[\{\!-\!\}]: P \rightarrow \text{Sub}_{\mathcal{C}}$ are such that $\mathcal{E}_-(\top) \circ [\{\!-\!\}] = \text{id}_P$ and $\text{id}_{\text{Sub}_{\mathcal{C}}} \leq [\{\!-\!\}] \circ \mathcal{E}_-(\top)$.

Proposition 2.3 and Remark 2.5 together with Proposition 1.6 prove the following corollary.

Corollary 2.6. *Suppose P is a weak hyperdoctrine on \mathcal{C} with full weak comprehensions and comprehensive diagonals, then for every α in $P(X \times Y)$ and every γ and β in $P(A)$ it is*

$$\{\mathbb{V}_{\text{pr}_1}^P(\alpha)\} = \mathbb{V}_{\text{pr}_1}^{\Psi_{\mathcal{C}}} \{\alpha\} \quad \{\gamma \Rightarrow^P \beta\} = \{\alpha\} \Rightarrow^{\Psi_{\mathcal{C}}} \{\beta\}.$$

Moreover, if comprehensions are strong, it also holds that

$$\{\mathbb{V}_{\text{pr}_1}^P(\alpha)\} = \mathbb{V}_{\text{pr}_1}^{\text{Sub}_{\mathcal{C}}} \{\alpha\} \quad \{\gamma \Rightarrow^P \beta\} = \{\alpha\} \Rightarrow^{\text{Sub}_{\mathcal{C}}} \{\beta\}$$

where superscripts distinguish operations between P and $\Psi_{\mathcal{C}}$ and between P and $\text{Sub}_{\mathcal{C}}$.

Among all doctrines, the subobject doctrines of the form $\text{Sub}_{\mathcal{C}}$ are characterized by the fact that they satisfy (RUC) (see [Jac99]), while variational doctrines of the form $\Psi_{\mathcal{C}}$ are characterized by the fact that they satisfy (RC) (see [MPR17]). Since we shall refer to this characterization repeatedly in the special case of elementary existential doctrines, we state it explicitly in the next theorem. We refer the reader to [MPR17] for a proof.

Theorem 2.7. *Suppose $P: \mathcal{C}^{op} \rightarrow \mathbf{InfSL}$ is a weak hyperdoctrine.*

- (i) *The doctrine P is equivalent to $\text{Sub}_{\mathcal{C}}$ if and only if P has comprehensive diagonals, full weak comprehensions and (RUC) holds in P .*
- (ii) *The doctrine P is equivalent to $\Psi_{\mathcal{C}}$ if and only if P has comprehensive diagonals, full weak comprehensions and (RC) holds in P .*
- (iii) *If the doctrine P has comprehensive diagonals, full weak comprehensions, then it is equivalent to $\Psi_{\mathcal{C}}$ if and only if the inequality $\text{id}_{\Psi_{\mathcal{C}}} \leq [\{-\}] \circ \mathcal{E}_-(\top)$ is in fact an equality.*

Proof. See [Jac99] for the proof of (i); see [MPR17] for those of (ii) and (iii). \square

Proposition 4.11 in [MR16] states that in any weak hyperdoctrine P with comprehension (RC) holds if and only if (RUC) holds in \widehat{P} . So Theorem 2.7 immediately gives the following result.

Corollary 2.8. *Let $P: \mathcal{C}^{op} \rightarrow \mathbf{InfSL}$ be a weak hyperdoctrine with full weak comprehensions and comprehensive diagonals. The doctrine P is equivalent to $\Psi_{\mathcal{C}}$ if and only if the doctrine \widehat{P} is equivalent to $\text{Sub}_{\mathcal{Q}_\bullet}$.*

Observe that in a weak hyperdoctrine with full weak comprehensions the validity of (RUC) implies that of (AUC), as well as the validity of (RC) implies that of (AC). This can be proved by translating in the internal language of weak hyperdoctrines the proofs in [Mai17]. Therefore, if $\text{Sub}_{\mathcal{C}}$ is a weak hyperdoctrine then (AUC) holds in $\text{Sub}_{\mathcal{C}}$, and if $\Psi_{\mathcal{C}}$ is a weak hyperdoctrine, then (AC) holds in $\Psi_{\mathcal{C}}$. Moreover the proof in Proposition 6.5 in [MR16] proves also the following.

Proposition 2.9. *Let $P: \mathcal{C}^{op} \rightarrow \mathbf{InfSL}$ be a weak hyperdoctrine.*

- (i) *If (AC) holds in P , then (AUC) holds in the quotient completion \widehat{P} .*
- (ii) *If (AUC) holds in \widehat{P} and P has full weak comprehensions, then (AC) holds in P .*

3. ARITHMETIC DOCTRINES

The aim of this section is to show that the elementary quotient completion inherits the validity of Formal Church's Thesis from the doctrine on which it is performed. To this purpose we first briefly show some preliminary results concerning primary doctrines equipped with a natural numbers object.

Recall [LS86] that, in a category \mathcal{C} with binary products, a **parameterized natural number object** (pnno) is an object \mathbf{N} together with two arrows $0: 1 \rightarrow \mathbf{N}$ and $s: \mathbf{N} \rightarrow \mathbf{N}$ such that for every A and X and every pair of arrows $a: A \rightarrow X$ and $f: X \rightarrow X$ there is a unique arrow $k: A \times \mathbf{N} \rightarrow X$ such that the following diagram

$$\begin{array}{ccccc}
 A & \xrightarrow{\langle \text{id}_A, 0 \rangle} & A \times \mathbf{N} & \xrightarrow{\text{id}_A \times s} & A \times \mathbf{N} \\
 & \searrow a & \downarrow k & & \downarrow k \\
 & & X & \xrightarrow{f} & X
 \end{array} \tag{3.1}$$

commutes.

Let $P: \mathcal{C}^{\text{op}} \rightarrow \mathbf{InfSL}$ be a primary doctrine, and suppose that $(\mathbf{N}, 0, s)$ is a pnno in \mathcal{C} . We say that the pnno satisfies **induction in P** when for every A in \mathcal{C} and ϕ in $P(A \times \mathbf{N})$, if $a: A \vdash \phi(0)$ and $a: A, m: \mathbf{N} \mid \phi(m) \vdash \phi(s(m))$, then also

$$a: A, n: \mathbf{N} \vdash \phi(n).$$

Remark 3.1. There is a weakened version of the notion of pnno when, for pairs (a, f) , the mediating arrow k is not necessarily unique with the commutation property. There is no point to consider the weak version here because, if $P: \mathcal{C}^{\text{op}} \rightarrow \mathbf{InfSL}$ is a weak hyperdoctrine with comprehensive diagonals, and $(\mathbf{N}, 0, s)$ is a wpnno which satisfies induction in P , then it is a pnno in \mathcal{C} . To see this, given arrows $a: 1 \rightarrow A$ and $f: A \rightarrow A$, suppose that $k: A \times \mathbf{N} \rightarrow A$ and $h: A \times \mathbf{N} \rightarrow A$ make the diagram (3.1) commute. So

$$\vdash k(0) =_{\mathbf{N}} h(0) \quad \text{and} \quad a: A, n: \mathbf{N} \mid k(a, n) =_A h(a, n) \vdash k(a, s(n)) =_A h(a, s(n)).$$

By induction $a: A, n: \mathbf{N} \vdash k(a, n) =_A h(a, n)$, and $k = h$ since diagonals are comprehensive.

We are interested in studying the behavior of arithmetic doctrines with respect to the notion of elementary quotient completion. Since all our examples and applications concern elementary doctrines with comprehensive diagonals, from now on we will consider only this class of doctrines, and arithmetic doctrines within.

Remark 3.2. Induction takes a more familiar form when the doctrine P bears sufficient structure to express it. In case P is a weak hyperdoctrine, the pnno $(\mathbf{N}, 0, s)$ satisfies induction if and only if, for every A in \mathcal{C} and ϕ in $P(A \times \mathbf{N})$,

$$\vdash \forall_{a:A} [[\phi(0) \wedge \forall_{m:\mathbf{N}} [\phi(m) \Rightarrow \phi(s(m))]] \Rightarrow \forall_{n:\mathbf{N}} \phi(n)]$$

A weak hyperdoctrine $P: \mathcal{C}^{\text{op}} \rightarrow \mathbf{InfSL}$ with a pnno which satisfies induction is said **arithmetic**.

Proposition 3.3. *Suppose \mathcal{C} has a pnno. If $P: \mathcal{C}^{\text{op}} \rightarrow \mathbf{InfSL}$ is a weak hyperdoctrine with full weak comprehensions and comprehensive diagonals, then P is arithmetic.*

Proof. Suppose that ϕ in $P(A \times \mathbf{N})$ is such that $a: A \vdash \phi(0)$ and $a: A, m: \mathbf{N} \mid \phi(m) \vdash \phi(s(m))$. Consider a weak comprehension $\{\!\{ \phi \}\!\}: X \rightarrow A \times \mathbf{N}$ of ϕ . By the property of weak comprehension, the condition $a: A \vdash \phi(0)$ implies that id_A factors through the weak pullback of $\{\!\{ \phi \}\!\}$ along $\langle \text{id}_A, 0 \rangle: A \rightarrow \mathbf{N}$, while the condition $a: A, m: \mathbf{N} \mid \phi(m) \vdash \phi(s(m))$ implies that $\{\!\{ \phi \}\!\}$ factors through the weak pullback of $\{\!\{ \phi \}\!\}$ along $\text{id}_A \times s: A \times \mathbf{N} \rightarrow A \times \mathbf{N}$. The resulting

commutative diagram is

$$\begin{array}{ccccc}
 1 & \longrightarrow & X & \longrightarrow & X \\
 & \searrow & \downarrow \{\!\!\{ \phi \}\!\!\} & & \downarrow \{\!\!\{ \phi \}\!\!\} \\
 \langle \text{id}_A, 0 \rangle & & A \times \mathbf{N} & \xrightarrow{\text{id}_A \times s} & A \times \mathbf{N}
 \end{array}$$

The universal property of \mathbf{N} , gives a section of $\{\!\!\{ \phi \}\!\!\}$. Fullness of comprehensions completes the proof. \square

Example 3.4. Suppose that \mathcal{C} has a pnno. If $\text{Sub}_{\mathcal{C}}$ is a weak hyperdoctrine, then it is also arithmetic. If $\Psi_{\mathcal{C}}$ is a weak hyperdoctrine, then is also arithmetic.

Lemma 3.5. *If P is an elementary doctrine with comprehensive diagonals, then \mathcal{C} has a pnno if and only if \mathcal{Q}_P has a pnno.*

Proof. We shall employ Lemma 5.7 in [MR13b] and see \mathcal{C} as the full subcategory of \mathcal{Q}_P on the objects of the form (A, δ_A) . Suppose $((\mathbf{N}, \rho), [0], [s])$ is a pnno in \mathcal{Q}_P . The arrow $[\text{id}_{\mathbf{N}}]: (\mathbf{N}, \delta_{\mathbf{N}}) \rightarrow (\mathbf{N}, \rho)$ makes $((\mathbf{N}, \delta_{\mathbf{N}}), [0], [s])$ a pnno in \mathcal{Q}_P . Conversely, suppose $(\mathbf{N}, 0, s)$ is a pnno in \mathcal{C} . Then it is easy to check that $((\mathbf{N}, \delta_{\mathbf{N}}), [0], [s])$ is a pnno in \mathcal{Q}_P . \square

Corollary 3.6. *Suppose P is an elementary doctrine with comprehensive diagonals. The doctrine P is arithmetic if and only if the doctrine \widehat{P} is arithmetic.*

Proof. Immediate from Lemma 5.7 in [MR13b] and Lemma 3.5. \square

Let P be an arithmetic weak hyperdoctrine and let W be a weak exponential of \mathbf{N} over \mathbf{N} with weak evaluation $\text{ev}: W \times \mathbf{N} \rightarrow \mathbf{N}$. One can develop standard recursion theory as the operations of sum and product of pair of natural numbers can be introduced using weak exponentials and the pnno structure. So one can introduce the standard Kleene primitive recursive arrows for test and output $T: \mathbf{N} \times \mathbf{N} \times \mathbf{N} \rightarrow \mathbf{N}$ and $U: \mathbf{N} \rightarrow \mathbf{N}$.

For the rest of the section, P is assumed to be an arithmetic weak hyperdoctrine on \mathcal{C} . So in particular \mathcal{C} is weakly cartesian closed.

Notation 3.7. For R is in $P(\mathbf{N} \times \mathbf{N} \times A)$, write $\mathbf{K}_R(e, x, y, a)$ in $P(\mathbf{N} \times \mathbf{N} \times \mathbf{N} \times A)$ for the formula

$$T(e, x, y) =_{\mathbf{N}} s(0) \wedge R(x, U(y), a).$$

Write \mathbf{K}_{ev} in $P(\mathbf{N} \times \mathbf{N} \times \mathbf{N} \times W)$ for \mathbf{K}_S where $S(x, n, f)$ is

$$n =_{\mathbf{N}} \text{ev}(f, x).$$

And write $\mathbf{Rec}_{\text{ev}}(f)$ in $P(W)$ for the formula

$$\exists e: \mathbf{N} \forall x: \mathbf{N} \exists y: \mathbf{N} \mathbf{K}_{\text{ev}}(e, x, y, f)$$

Lemma 3.8. *In every arithmetic doctrine, if W and W' are weak exponentials of \mathbf{N} with \mathbf{N} , with corresponding weak evaluations $\text{ev}: W \times \mathbf{N} \rightarrow \mathbf{N}$ and $\text{ev}': W' \times \mathbf{N} \rightarrow \mathbf{N}$, then*

$$\vdash \forall f: W \mathbf{Rec}_w(f) \Leftrightarrow \forall g: W' \mathbf{Rec}_{w'}(g).$$

Proof. Immediate consequence of Remark 1.4. \square

By Lemma 3.8, we can discard the index in \mathbf{Rec}_{ev} and simply write \mathbf{Rec} .

Definition 3.9. Let $P: \mathcal{C}^{\text{op}} \rightarrow \mathbf{InfSL}$ be an arithmetic doctrine with comprehensive diagonals. We say that

- (1) the (**formal**) *Type-theoretic Church's Thesis* holds in P if for some weak evaluation $ev: \mathbb{N} \times W \rightarrow \mathbb{N}$ it is

$$\vdash \forall_{f:W} \mathbf{Rec}_{ev}(f);$$

- (2) the (**formal**) *Church's Thesis* holds in P if, for every R in $P(\mathbb{N} \times \mathbb{N})$,

$$\vdash \forall_{x:\mathbb{N}} \exists_{n:\mathbb{N}} R(x, n) \Rightarrow \exists_{e:\mathbb{N}} \forall_{x:\mathbb{N}} \exists_{y:\mathbb{N}} \mathbf{K}_R(e, x, y)$$

Remark 3.10. Note that, by Lemma 3.8, any evaluation can be chosen in the formula $\forall_{f:W} \mathbf{Rec}_{ev}(f)$. Because of that, we shall refer to such a sentence as **(TCT)**. On the other hand, Church's Thesis is a schema of formulas \mathbf{CT}_R as R varies in $P(\mathbb{N} \times \mathbb{N})$; and we may abbreviate the statement that Church's Thesis holds in P by writing that **(CT)** holds in P .

Proposition 3.11. *Suppose $P: \mathcal{C}^{op} \rightarrow \mathbf{InfSL}$ is an arithmetic doctrine with comprehensive diagonals. The following hold:*

- (i) *The schema **(CT)** holds in P if and only if the schema **(CT)** holds in \widehat{P} .*
- (ii) *The sentence **(TCT)** holds in P if and only if the sentence **(TCT)** holds in \widehat{P} .*

Proof. By Lemma 3.6, $(\mathbb{N}, 0, s)$ is a pnno in \mathcal{C} if and only if $(\mathbb{N}, \delta_{\mathbb{N}}, [0], [s])$ is a pnno in \mathcal{Q}_P .
 (i) The claim is proved since, for a fixed R in $P(\mathbb{N} \times \mathbb{N})$ the formula \mathbf{CT}_R in it is built using only quantifications, finite conjunctions and the equality predicate over \mathbb{N} , and these operations of \widehat{P} over any finite power of $(\mathbb{N}, \delta_{\mathbb{N}})$ are the restriction of those of P over the corresponding finite power of \mathbb{N} .

(ii) The (\Leftarrow) direction follows from Lemma 5.7 in [MR13b]: P is equivalent to the restriction of \widehat{P} to the subcategory of \mathcal{Q}_P on objects of the form (A, δ_A) . For the other direction, note that by Proposition 6.7 in [MR13b] an arrow $w: \mathbb{N} \times W \rightarrow \mathbb{N}$ is a weak evaluation in \mathcal{C} if and only if $[w]: (\mathbb{N}, \delta_{\mathbb{N}}) \times (W, \theta) \rightarrow (\mathbb{N}, \delta_{\mathbb{N}})$ is an evaluation map in \mathcal{Q}_P where θ is an appropriate P -equivalence relation over W . The claim is proved since the formula **TCT** is built using only the universal quantification, finite conjunctions and the equality predicate over $(\mathbb{N}, \delta_{\mathbb{N}})$ and (W, θ) , and these operations of \widehat{P} are the restriction of those of P as $\widehat{P}(W, \theta) \subseteq P(W)$. \square

Corollary 3.12. *Suppose \mathcal{C} is such that $\Psi_{\mathcal{C}}$ is arithmetic.*

- (i) *The schema **(CT)** holds in $\Psi_{\mathcal{C}}$ if and only if the schema **(CT)** holds in $\text{Sub}_{\mathcal{C}_{\text{ex/lex}}}$.*
- (ii) *The sentence **(TCT)** holds in $\Psi_{\mathcal{C}}$ if and only if the sentence **(TCT)** holds in $\text{Sub}_{\mathcal{C}_{\text{ex/lex}}}$.*

Proof. It is a direct consequence of Corollary 2.8 and Proposition 3.11. \square

It is well known that if the validity of **(TCT)** in a theory implies the validity of **(CT)** in the presence of choice principles for total relations on natural numbers.

Some forms of choice are transferred via the elementary quotient completion under suitable assumptions on the doctrine.

Let $P: \mathcal{C}^{op} \rightarrow \mathbf{Heyt}$ be a hyperdoctrine with comprehensive diagonals. Proposition 2.9 says that the elementary quotient completion necessarily transfers **(AC)** to **(AUC)**. Thus **(AC)** is in general not preserved by the completions discussed so far. Nevertheless there are some instances of **(AC)** restricted to specific objects of the domain of the doctrine as in Definition 2.2.

Proposition 3.13. *Let $P: \mathcal{C}^{op} \rightarrow \mathbf{Heyt}$ be a weak hyperdoctrine with comprehensive diagonals and let A be an object of \mathcal{C} . The doctrine P satisfies **(AC)** on A if and only if the doctrine \widehat{P} satisfies **(AC)** on (A, δ_A)*

Proof. It follows from Lemma 5.7 in [MR13b] and from the fact that, if $w: A \times W \rightarrow A$ is a weak evaluation in \mathcal{C} , then $[w]: (A, \delta_A) \times (W, \rho) \rightarrow (A, \delta_A)$ is an evaluation map in \mathcal{Q}_P , where ρ is a suitable P -equivalence relation. Moreover quantifiers of \widehat{P} are those of P and $\mathcal{D}es_\rho \subseteq P(W)$. \square

When $P: \mathcal{C}^{\text{op}} \rightarrow \mathbf{InfSL}$ is arithmetic, we say that P satisfies the **Countable Axiom of Choice** ($\text{AC}_{\mathbb{N}}$) when P satisfies the (AC) on the pnno of \mathcal{C} .

As an immediate corollary of Proposition 3.13 and Lemma 3.6 we have the following.

Corollary 3.14. *Let $P: \mathcal{C}^{\text{op}} \rightarrow \mathbf{InfSL}$ be an arithmetic doctrine. The doctrine P satisfies ($\text{AC}_{\mathbb{N}}$) if and only if \widehat{P} satisfies ($\text{AC}_{\mathbb{N}}$).*

Proposition 3.15. *Suppose P is an arithmetic doctrine on \mathcal{C} . If P satisfies both ($\text{AC}_{\mathbb{N}}$) and (TCT), then P satisfies (CT).*

4. ELEMENTARY QUOTIENT COMPLETIONS ON PARTITIONED ASSEMBLIES

We are finally in a position to analyze the realizability model offered by the effective topos and various doctrines related to it. For a detailed presentation of the categorical structure of realizability we refer the reader to [vO08]; here we restrict ourselves to give just the essential details needed for our purposes.

Although most of the development could be performed relative to an arbitrary partial combinatory algebra, we shall refer only to the partial combinatory algebra which is Kleene's first model \mathbb{K}_1 on the natural numbers, with the usual notation φ_e for the e -th partial recursive function. We shall write $\langle\langle n, m \rangle\rangle$ for a fixed recursive encoding of pairs and k_0 and k_1 for the (unique) pair of numbers such that $k = \langle\langle k_0, k_1 \rangle\rangle$.

Recall the category $\mathcal{A}sm$ of assemblies and its full subcategory of partitioned assemblies from [CFS88]. An **assembly** is a pair (P, T) where P is a set and $T \subseteq P \times \mathbb{N}$ is a total relation from P to \mathbb{N} , *i.e.* for every element $x \in P$ there is a number $n \in \mathbb{N}$ such that $x T n$.¹ An arrow $f: (P, T) \rightarrow (P', T')$ of assemblies is a function $f: P \rightarrow P'$ such that for some $t \in \mathbb{N}$ t **tracks** f , *i.e.* for every $x \in P$ and every $n \in \mathbb{N}$, if $x T n$, then $f(x) T' \varphi_t(n)$. Arrows compose as functions. The category $\mathcal{A}sm$ is a quasitopos, see [Hyl82]. In particular, a strong subobject of (P, T) in $\mathcal{A}sm$ is represented by an inclusion $\text{id}_P \upharpoonright_X: (X, T \cap (X \times \mathbb{N})) \hookrightarrow (P, T)$ for some (unique) subset X of P . Also, since the terminal assembly $\mathbf{1} = (\{0\}, \{(0, 0)\})$ is a generator in $\mathcal{A}sm$, the global-section functor $\Gamma = \text{hom}_{\mathcal{P}\mathcal{A}sm}(\mathbf{1}, -): \mathcal{P}\mathcal{A}sm \rightarrow \mathbf{Set}$ is (isomorphic to) the forgetful functor that sends (P, T) to P and f to itself.

An assembly (P, T) is **partitioned** if T is single-valued (hence T is a function from P to \mathbb{N}).² The full subcategory of $\mathcal{A}sm$ on partitioned assemblies is written $\mathcal{P}\mathcal{A}sm$.

The category $\mathcal{P}\mathcal{A}sm$ of partitioned assemblies has finite limits, finite coproducts, weak exponentials, and a pnno, see [Car95, vO08].

The terminal assembly $\mathbf{1}$ is partitioned. The product of the two partitioned assemblies (P, T) and (M, S) can be chosen as $(P \times M, T \otimes S)$ where $(T \otimes S)(x, y) := \langle\langle T(x), S(y) \rangle\rangle$. A weak exponential of (P, T) with (M, S) is (W, V) where

$$W := \{(f, t) \in P^M \times \mathbb{N} \mid t \text{ tracks } f\}$$

¹The name assembly refers to the way the relation T “assembles” the elements of P within subsets $T_n := \{x \in P \mid x T n\}$, possibly overlapping.

²The past participle partitioned refers to the fact that the assembled subsets T_n of P are disjoint, hence form a partition of P .

and $V(f, t) := t$; the weak evaluation $\text{ev}: (W, V) \times (P, T) \rightarrow (M, S)$ is given by the function $\text{ev}: W \times P \rightarrow M$ defined as $\text{ev}((f, t), x) := f(x)$ which is tracked by a code for the recursive function $k \mapsto \varphi_{k_0}(k_1)$. The pno is determined on the partitioned assembly $(\mathbb{N}, \text{id}_{\mathbb{N}})$.

Remark 4.1. Recall that $\mathcal{A}sm \equiv \mathcal{P}\mathcal{A}sm_{\text{reg/lex}}$ and that the exact completion $\mathcal{P}\mathcal{A}sm_{\text{ex/lex}}$ is the effective topos $\mathcal{E}ff$, see [RR90, Car95, vO08]. A crucial point to see this is that every partitioned assembly is projective with respect to regular epis in $\mathcal{A}sm$ [Car95].

Consider the doctrines on $\mathcal{P}\mathcal{A}sm$

$$\Psi_{\mathcal{P}\mathcal{A}sm}: \mathcal{P}\mathcal{A}sm^{\text{op}} \longrightarrow \mathbf{InfSL} \quad \text{Sub}_{\mathcal{P}\mathcal{A}sm}: \mathcal{P}\mathcal{A}sm^{\text{op}} \longrightarrow \mathbf{InfSL} \quad \mathbb{P}\Gamma: \mathcal{P}\mathcal{A}sm^{\text{op}} \longrightarrow \mathbf{InfSL},$$

respectively the doctrine of variations, that of subobjects and that obtained as the composite of Γ with the contravariant powerset functor. Clearly $\mathbb{P}\Gamma$ is a boolean arithmetic hyperdoctrine.

Lemma 4.2. *The doctrine $\Psi_{\mathcal{P}\mathcal{A}sm}: \mathcal{P}\mathcal{A}sm^{\text{op}} \longrightarrow \mathbf{InfSL}$ is a weak hyperdoctrine which satisfies (RC) and (AC) on each objects and is arithmetic.*

Proof. After Theorem 2.7 (ii), we only need to show that each functor

$$(\Psi_{\mathcal{P}\mathcal{A}sm})_{\text{pr}_1}: \Psi_{\mathcal{P}\mathcal{A}sm}(P, T) \longrightarrow \Psi_{\mathcal{P}\mathcal{A}sm}((P, T) \times (M, S))$$

has a right adjoint. Consider $[f: (Y, Z) \rightarrow (P, T) \times (M, S)]$, say that d tracks f , let

$$Q := \{\langle x, h, t \rangle \in P \times Y^M \times \mathbb{N} \mid t \text{ tracks } h \text{ and for all } m \in M, f(h(m)) = \langle x, m \rangle\}$$

and let $R: \langle x, h, t \rangle \mapsto \langle T(x), t \rangle: Q \rightarrow \mathbb{N}$. Define $V_{\text{pr}_1}([f]) := \text{pr}_1: (Q, R) \rightarrow (P, T)$ which is tracked by a code of the function $(-)_0$. The function $((x, h, t), m) \mapsto h(m): Q \times M \rightarrow Y$ is tracked by a code for the recursive function $k \mapsto \varphi_{(k_0)_1}(k_1)$, thus producing an arrow $(Q, R) \times (M, S) \rightarrow (Y, Z)$ which shows that $(\Psi_{\mathcal{P}\mathcal{A}sm})_{\text{pr}_1}(V_{\text{pr}_1}([f]) \leq [f]$. The conclusion is now straightforward. \square

There are obvious 1-arrows of elementary doctrines $\mathbb{P}\Gamma \rightarrow \text{Sub}_{\mathcal{P}\mathcal{A}sm}$ and $\text{Sub}_{\mathcal{P}\mathcal{A}sm} \rightarrow \Psi_{\mathcal{P}\mathcal{A}sm}$ which are the identity on the domain of the doctrine and monotone inclusions on the fibres.

Theorem 4.3. *The doctrine $\text{Sts}_{\mathcal{A}sm}: \mathcal{A}sm^{\text{op}} \longrightarrow \mathbf{InfSL}$ of strong subobjects on $\mathcal{A}sm$ is the elementary quotient completion of the doctrine $\mathbb{P}\Gamma: \mathcal{P}\mathcal{A}sm^{\text{op}} \longrightarrow \mathbf{InfSL}$.*

Proof. By the universal property of the elementary quotient completion, a 1-arrow of doctrines with stable quotients as in the diagram on the right

$$\begin{array}{ccc} \mathcal{P}\mathcal{A}sm^{\text{op}} & \xrightarrow{\mathbb{P}\Gamma} & \mathbf{InfSL} \\ G^{\text{op}} \downarrow & \searrow c \cdot & \downarrow \\ \mathcal{A}sm^{\text{op}} & \xrightarrow{\text{Sts}_{\mathcal{A}sm}} & \mathbf{InfSL} \end{array} \quad \begin{array}{ccc} Q_{\mathbb{P}\Gamma}^{\text{op}} & \xrightarrow{\widehat{\mathbb{P}\Gamma}} & \mathbf{InfSL} \\ F^{\text{op}} \downarrow & \searrow b \cdot & \downarrow \\ \mathcal{A}sm^{\text{op}} & \xrightarrow{\text{Sts}_{\mathcal{A}sm}} & \mathbf{InfSL} \end{array}$$

is completely determined by a 1-arrow of elementary doctrines as in the diagram on the left. So take G as the inclusion of $\mathcal{P}\mathcal{A}sm$ into $\mathcal{A}sm$, that preserves all finite limits. The (P, T) -component of the transformation c takes a subset X of $\Gamma(P, T)$ to the strong subobject $\text{id}_P \upharpoonright_X: (X, T \cap (X \times \mathbb{N})) \rightarrow (P, T)$; it is an isomorphism because of the characterization of strong subobjects in $\mathcal{A}sm$. The induced functor $F: Q_{\mathbb{P}\Gamma}^{\text{op}} \rightarrow \mathcal{A}sm$ is faithful. It is also full as partitioned assemblies are regular projective (see Remark 4.1). Finally, F is essential surjective because $\mathcal{A}sm$ has enough regular projectives (see again Remark 4.1). \square

Corollary 4.4. *The doctrine $\text{Sts}_{\mathcal{A}sm}$ is arithmetic.*

Proof. It follows from Theorem 4.3 since $\mathbb{P}\Gamma$ is arithmetic by Lemma 4.2. Thus its elementary quotient completion is arithmetic by Lemma 3.6. \square

Proposition 4.5. *The doctrine $\mathbb{P}\Gamma$ satisfies (TCT).*

Proof. Consider the weak exponential (W, V) of $(\mathbb{N}, \text{id}_{\mathbb{N}})$ to its power as

$$W := \{(g, t) \in \mathbb{N}^{\mathbb{N}} \times \mathbb{N} \mid t \text{ tracks } g\}$$

and V the (restriction of the) second projection. But t tracks g exactly when $g = \varphi_t$. So $\Gamma(W, V)$ is (in bijection) with the set of total recursive functions on \mathbb{N} , and $\vdash \forall_{f:(W,V)} \text{Rec}(f)$ in $\mathbb{P}\Gamma$ because for all $(g, t) \in |W|$ there is $e := t \in \mathbb{N}$ such that for all $x \in \mathbb{N}$ there is $y \in \mathbb{N}$ such that $T(e, x, y) = 1 \wedge U(y) = f(x)$. \square

Lemma 4.6. *The formula $f: (W, V), e: \mathbb{N}, x: \mathbb{N}, y: \mathbb{N} \mid \mathbf{K}_w(e, x, y, f)$ in $\mathbb{P}\Gamma(W \times \mathbb{N} \times \mathbb{N} \times \mathbb{N})$ has a Skolem arrow for (the third occurrence of) \mathbb{N} .*

Proof. Consider the function $W \times \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$ defined as

$$((g, t), e, x) \mapsto \min\{y \in \mathbb{N} \mid T(e, x, y) = 1 \wedge U(y) = g(x)\}.$$

This function is tracked by (a code of) the partial recursive function

$$(t, e, x) \mapsto \min\{y \in \mathbb{N} \mid T(e, x, y) = 1 \wedge U(y) = \varphi_t(x)\}.$$

For every element $(g, t) \in W$ the set of numbers on the right-hand side is non-empty since t belongs to it. So it defines an arrow $\gamma: (W, V) \times \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$ in $\mathcal{P}\mathcal{A}sm$ which is clearly the required Skolem arrow. \square

Lemma 4.7. *The formula $f: (W, V), e: \mathbb{N} \mid \forall_{x:\mathbb{N}} \exists_{y:\mathbb{N}} \mathbf{K}_w(e, x, y, f)$ in $\mathbb{P}\Gamma(W \times \mathbb{N})$ has a Skolem arrow for \mathbb{N} .*

Proof. A Skolem arrow $\epsilon: W \rightarrow \mathbb{N}$ is determined by the function $(g, t) \mapsto t: W \rightarrow \mathbb{N}$. \square

Applying the results in previous sections we will obtain that

- (TCT) holds in $\Psi_{\mathcal{P}\mathcal{A}sm}$, $\text{Sts}_{\mathcal{A}sm}$ and $\text{Sub}_{\mathcal{E}ff}$;
- (CT) holds in $\Psi_{\mathcal{P}\mathcal{A}sm}$ and $\text{Sub}_{\mathcal{E}ff}$;

as these are all essentially inherited from the validity of (TCT) in $\mathbb{P}\Gamma$.

Accordingly with the previous sections, we shall write Π and Σ for the universal and the existential quantification in $\Psi_{\mathcal{P}\mathcal{A}sm}$, while we will write \forall and \exists for the universal and the existential quantification in $\mathbb{P}\Gamma$.

Proposition 4.8. *The doctrine $\Psi_{\mathcal{P}\mathcal{A}sm}$ satisfies (TCT).*

Proof. By Proposition 2.3 there is a right adjoint of doctrines $[\{-\}]: \mathbb{P}\Gamma \dashv \Psi_{\mathcal{P}\mathcal{A}sm}$. Let (W, V) be the weak exponential of $(\mathbb{N}, \text{id}_{\mathbb{N}})$ to its power with weak evaluation w . Since $[\{-\}]$ maps equality predicates to equality predicates and commutes with substitutions, the formula \mathbf{K}_w for $\Psi_{\mathcal{P}\mathcal{A}sm}$ is the image under $[\{-\}]$ of \mathbf{K}_w for $\mathbb{P}\Gamma$. Corollary 2.6, Lemma 4.6 and Lemma 4.7 ensure that, in $\Psi_{\mathcal{P}\mathcal{A}sm}$,

$$\vdash \Pi_{f:(W,V)} \Sigma_{e:\mathbb{N}} \Pi_{x:\mathbb{N}} \Sigma_{y:\mathbb{N}} \{\mathbf{K}_w\}(e, x, y, f).$$

if and only if

$$\vdash \{\forall_{f:(W,V)} \exists_{e:\mathbb{N}} \forall_{x:\mathbb{N}} \exists_{y:\mathbb{N}} \mathbf{K}_w(e, x, y, f)\}.$$

And this holds by Proposition 4.5. \square

Corollary 4.9. (i) *The doctrines $\text{Sts}_{\mathcal{A}sm}$ and $\text{Sub}_{\mathcal{E}ff}$ satisfy (TCT).*
(ii) *The doctrines $\Psi_{\mathcal{P}\mathcal{A}sm}$ and $\text{Sub}_{\mathcal{E}ff}$ satisfy (CT).*

Proof. (i) The doctrine $\mathbb{P}\Gamma$ satisfies (TCT) by Proposition 4.5. By Theorem 4.3, the doctrine $\text{Sts}_{\mathcal{A}sm}$ is (equivalent to) $\widehat{\mathbb{P}\Gamma}$, so Proposition 3.11 applies, and $\text{Sts}_{\mathcal{A}sm}$ satisfies (TCT). Besides, the doctrine $\Psi_{\mathcal{P}\mathcal{A}sm}$ satisfies (TCT) by Proposition 4.8. By the results in [RR01], the topos $\mathcal{E}ff$ is (equivalent to) $\mathcal{P}\mathcal{A}sm_{\text{ex}/\text{lex}}$; hence Corollary 3.12 applies, and the doctrine $\text{Sub}_{\mathcal{E}ff}$ satisfies (TCT).

(ii) By Proposition 3.15, it suffices to show that both $\Psi_{\mathcal{P}\mathcal{A}sm}$ and $\text{Sub}_{\mathcal{E}ff}$ satisfy (TCT) and $(\text{AC}_{\mathbb{N}})$. The doctrine $\Psi_{\mathcal{P}\mathcal{A}sm}$ satisfies (TCT) by Proposition 4.8; the doctrine $\text{Sub}_{\mathcal{E}ff}$ satisfies (TCT) by Corollary 3.12. As for $(\text{AC}_{\mathbb{N}})$, any variational doctrine satisfies (AC) so, in particular, it satisfies $(\text{AC}_{\mathbb{N}})$. Hence, by Corollary 3.14, $\text{Sub}_{\mathcal{E}ff}$ satisfies $(\text{AC}_{\mathbb{N}})$ as well. \square

Remark 4.10. As is well known, $\mathcal{P}\mathcal{A}sm$ is not cartesian closed, see *e.g.* [Hyl82]. One can see also that this is so because of the validity in $\Psi_{\mathcal{P}\mathcal{A}sm}$ of (CT) and of (AC). Indeed, if $\mathcal{P}\mathcal{A}sm$ were cartesian closed, since it has finite limits it would satisfy *extensionality of functions* in the following form: for all object X, Y in $\mathcal{P}\mathcal{A}sm$ and $f, g: X \rightarrow Y$

$$\vdash \forall_{x:A} (f(x) =_Y g(x)) \Rightarrow (\lambda x.f(x) =_{Y^X} \lambda x.g(x))$$

where Y^X indicates the exponential of Y over X and $\lambda x.f(x)$ is the usual λ -notation for the abstraction of f . But it is well known, see for example [Tv88], that (CT) and (AC) are inconsistent with the extensionality of function in a many-sorted first order theory including arithmetic and finite types.

Remark 4.11. The validity of (TCT) in $\text{Sts}_{\mathcal{A}sm}$ implies that neither (CT) nor (AUC) (hence (AC)) are valid in $\text{Sts}_{\mathcal{A}sm}$, as its underlying logic is boolean, see [MS05] for a logical argument.

Remark 4.12. Observe that, since the pnno in $\mathcal{A}sm$ coincides with that in $\mathcal{P}\mathcal{A}sm$, Lemma 4.6 and Lemma 4.7 can be proved also for $\text{Sts}_{\mathcal{A}sm}$. By Remark 2.5 there is an adjunction between $\text{Sts}_{\mathcal{A}sm}$ and $\text{Sub}_{\mathcal{A}sm}$ that satisfies the hypotheses of Corollary 2.6. Hence $\text{Sub}_{\mathcal{A}sm}$ satisfies (TCT) by Corollary 4.9-(i). We also know that $\text{Sub}_{\mathcal{A}sm}$ satisfies (CT), but an abstract proof of this requires an abstract treatment of the regular completion of a lex category, which we do not include here. We just stress that the regular completion of a lex category can be obtained as an instance of a more general construction introduced in [MPR17] that involves elementary doctrines and that produces $\mathcal{A}sm$ when such a construction is performed over $\Psi_{\mathcal{P}\mathcal{A}sm}$.

To compare $\mathbb{P}\Gamma$ and $\Psi_{\mathcal{P}\mathcal{A}sm}$ we can apply the reflection in Proposition 2.3. In this case it turn out that the object of a $\mathbb{P}\Gamma$ over A in $\mathcal{P}\mathcal{A}sm$ coincides with the double negated objects of $\Psi_{\mathcal{P}\mathcal{A}sm}$ over A . This fact can be deduced from a general result.

Proposition 4.13. *Suppose P is a weak hyperdoctrine with full weak comprehensions and comprehensive diagonals. Suppose that for every f in \mathcal{C} the left adjoint along f is stable under the double negation, i.e. $\mathcal{E}_f = \neg\neg\mathcal{E}_f$, and that $\llbracket - \rrbracket$ preserves bottom elements, i.e. $\llbracket \perp_A \rrbracket$ is the bottom element in $\Psi_{\mathcal{C}}(A)$. Then $\llbracket - \rrbracket \circ \mathcal{E}_-(\top): \Psi_{\mathcal{C}} \rightarrow \Psi_{\mathcal{C}}$ coincide with the double negation, i.e. it maps $\llbracket f \rrbracket$ to $\neg\neg\llbracket f \rrbracket$.*

Proof. Consider $f: X \rightarrow A$ and recall that $\llbracket - \rrbracket \circ \mathcal{E}_-(\top)(\llbracket f \rrbracket) = \llbracket \mathcal{E}_f \top_X \rrbracket$. Note that Corollary 2.6 implies that $\llbracket - \rrbracket$ commutes with the universal quantification and with the

implication. Then it is

$$[\{\mathcal{A}_f \top_X\}] = [\{\neg\neg\mathcal{A}_f \top_X\}] = [\{\neg V_f \neg \top_X\}] = [\neg \Pi_f \neg \{\top_X\}] = \neg\neg[\Sigma_f \{\top_X\}] = \neg\neg[\Sigma_f(\text{id}_X)]$$

and hence the claim as $[\Sigma_f(\text{id}_X)] = [f]$. \square

Hyland in [Hyl82] showed that assemblies are the $\neg\neg$ -separated objects of $\mathcal{E}ff$ for the Lawvere-Tierney topology of double negation, *i.e.* an object of $\mathcal{E}ff$ is in $\mathcal{A}sm$ if and only if its equality predicate is $\neg\neg$ -closed. This is also a corollary of our previous results.

Proposition 4.14 (Hyland). *The category $\mathcal{A}sm$ is the full reflective subcategory of $\mathcal{E}ff$ on $\neg\neg$ -separated objects.*

Proof. The 1-arrow $\{\{-\}\}: \mathbb{P}\Gamma \rightarrow \Psi_{\mathcal{P}\mathcal{A}sm}$ is full and has a left adjoint $\mathcal{E}_-(\top)$ by Corollary 2.3. Since the elementary quotient completion is a 2-functor, there is a full and faithful functor $G: \mathcal{Q}_{\mathbb{P}\Gamma} \rightarrow \mathcal{Q}_{\Psi_{\mathcal{P}\mathcal{A}sm}}$ which has a left adjoint F . Therefore (A, ρ) in $\mathcal{Q}_{\Psi_{\mathcal{P}\mathcal{A}sm}}$ is in $\mathcal{Q}_{\mathbb{P}\Gamma}$ if and only if $(A, \rho) \simeq GF(A, \rho)$. From the construction of F and G and from Proposition 4.13 this happens if and only if

$$(A, \rho) \simeq (A, \{\{-\}\}_{A \times A} \circ \mathcal{E}_-(\top)_{A \times A}(\rho)) = (A, \neg\neg\rho)$$

But ρ is the equality predicate over (A, ρ) for the doctrine $\widehat{\Psi_{\mathcal{P}\mathcal{A}sm}} \equiv \text{Sub}_{\mathcal{E}ff}$ by Corollary 2.8. The claim follows from $\mathcal{Q}_{\mathbb{P}\Gamma} \equiv \mathcal{A}sm$ by Theorem 4.3. \square

Remark 4.15. The category $\mathcal{E}qu$ of equilogical spaces introduced in [Sco96] is the domain of the elementary quotient completion of the doctrine of subspace inclusions on \mathcal{Top}_0 , see [MPR17], and also [Pas18a] where a more general situation is considered. In the same vein, one can show that $\mathcal{E}qu$ is the full and reflective subcategory of $(\mathcal{Top}_0)_{\text{ex/lex}}$ on those objects whose equality predicate is stable under the double negation [Ros00].

5. KLEENE'S REALIZABILITY INTERPRETATION IN $\Psi_{\mathcal{P}\mathcal{A}sm}$

It is well known that the interpretation of Intuitionistic Arithmetic (**HA**) in the internal logic of $\mathcal{E}ff$, *i.e.* the hyperdoctrine $\text{Sub}_{\mathcal{E}ff}: \mathcal{E}ff^{\text{op}} \rightarrow \mathbf{InfSL}$, extends Kleene's realizability interpretation, see [Hyl82, vO08]. However this is not evident in the tripos which produces $\mathcal{E}ff$ as explained in [HJP80] since the tripos does not validate Intuitionistic Arithmetic.

Here we show that $\Psi_{\mathcal{P}\mathcal{A}sm}$ is responsible for that result since $\mathcal{E}ff$ is the domain of the elementary quotient completion of $\Psi_{\mathcal{P}\mathcal{A}sm}$. Hence $\mathcal{E}ff$ inherits the interpretation of connectives and quantifiers from $\Psi_{\mathcal{P}\mathcal{A}sm}$, as explained in [MR13b].

The theory **HA** is interpreted in the arithmetic weak hyperdoctrine $\Psi_{\mathcal{P}\mathcal{A}sm}: \mathcal{P}\mathcal{A}sm^{\text{op}} \rightarrow \mathbf{InfSL}$ taking the pno $\mathbb{N} := (\mathbb{N}, \text{id}_{\mathbb{N}})$ in $\mathcal{P}\mathcal{A}sm$ as the domain of the interpretation and interpreting the operations with the standard operations on the pno.

For a formula ϕ in **HA** with at most n free variables x_1, \dots, x_n , let $[\phi^I: X \rightarrow \mathbb{N}^n]$ be interpretation of ϕ in $\Psi_{\mathcal{P}\mathcal{A}sm}$ as in [Jac99]. Write instead

$$R_\phi := \{(k_1, \dots, k_n, m) \in \mathbb{N}^{n+1} \mid m \Vdash_{\text{Kleene}} \phi[k_1/x_1, \dots, k_n/x_n]\},$$

where \Vdash_{Kleene} is Kleene realizability as presented in [Tv88], and let $\gamma_\phi: R_\phi \rightarrow \mathbb{N}$ be the function which maps an $(n+1)$ -ple to its encoding. If we let $\phi^\Vdash: R_\phi \rightarrow \mathbb{N}^n$ be the projection on the first n components, we obtain an arrow of partitioned assemblies $\phi^\Vdash: (R_\phi, \gamma_\phi) \rightarrow \mathbb{N}^n$. So $[\phi^\Vdash: (R_\phi, \gamma_\phi) \rightarrow \mathbb{N}^n]$ is an object of $\Psi_{\mathcal{P}\mathcal{A}sm}(\mathbb{N}^n)$.

Proposition 5.1. *For any \mathbf{HA} -formula ϕ with at most n free variables x_1, \dots, x_n , it is*

$$[\phi^I] = [\phi^{\text{tr}}].$$

The proof is an easy induction on the height of the formula ϕ and it is based on the constructions in $\mathcal{P}\mathcal{A}sm$.

Corollary 5.2. *A sentence ϕ in the language of \mathbf{HA} is true in the standard interpretation in $\Psi_{\mathcal{P}\mathcal{A}sm}$, if and only if ϕ has a realizer in the sense of Kleene realizability interpretation in [Kle45].*

Corollary 5.3 (Hyland). *A sentence ϕ in the language of \mathbf{HA} is true in the standard interpretation in $\text{Sub}_{\mathcal{E}ff}$, if and only if ϕ has a realizer in the sense of Kleene realizability interpretation in [Kle45].*

Proof. By Corollary 2.8 $\text{Sub}_{\mathcal{E}ff}$ is the elementary quotient completion of $\Psi_{\mathcal{P}\mathcal{A}sm}$, therefore the pno in $\mathcal{E}ff$ is of the form $(\mathbf{N}, \delta_{\mathbf{N}})$ where \mathbf{N} is a pno in $\mathcal{P}\mathcal{A}sm$, then not only $\text{Sub}_{\mathcal{E}ff}(\mathbf{N}, \delta_{\mathbf{N}}) = \Psi_{\mathcal{P}\mathcal{A}sm}(\mathbf{N})$, but quantifications, connectives and the equality predicate are the same. The claim follows by Corollary 5.2. \square

6. CONCLUSION

We have shown that $\mathcal{E}ff$ and $\mathcal{A}sm$ are elementary quotient completion of suitable doctrines. This fact is crucial to build models for the Minimalist Foundation (\mathbf{MF}), introduced in [MS05, Mai09], extended with the various forms of \mathbf{CT} . The reason is that \mathbf{MF} in [Mai09] has a two-level structure with an extensional level interpreted in the elementary quotient completion of its intensional level, as analyzed categorically in [MR13b]. Hence modeling \mathbf{MF} in $\mathcal{E}ff$ (or in $\mathcal{A}sm$) corresponds to build a morphism of doctrines from the elementary quotient completion of the syntactic doctrine of \mathbf{MF} to doctrines based on $\mathcal{E}ff$ (or $\mathcal{A}sm$).

In particular we would like to embed in $\mathcal{E}ff$ the already known models which provides the consistency of both levels of \mathbf{MF} with \mathbf{CT} in [MM15, MM16, IMMS18]. Since these models provide extraction of programs from constructive proofs in \mathbf{MF} as shown in [Mai17], we think that $\mathcal{E}ff$ should provide a framework to extend extraction of programs from proofs to extensions of \mathbf{MF} with general inductive definitions.

Finally we would also like to exploit the categorical structure of $\mathcal{A}sm$ to build models similar to that in [Str92] in order to show consistency of \mathbf{MF} (and of its extensions with inductive definitions) with classical logic and the weak form of \mathbf{CT} valid in the doctrine $\mathbb{P}\Gamma$.

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