

AN EFFICIENT ALGORITHM TO DECIDE PERIODICITY OF b -RECOGNISABLE SETS USING LSDF CONVENTION*

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ABSTRACT. Let b be an integer strictly greater than 1. Each set of nonnegative integers is represented in base b by a language over $\{0, 1, \dots, b-1\}$. The set is said to be b -recognisable if it is represented by a regular language. It is known that ultimately periodic sets are b -recognisable, for every base b , and Cobham's theorem implies the converse: no other set is b -recognisable in every base b .

We consider the following decision problem: let S be a set of nonnegative integers that is b -recognisable, given as a finite automaton over $\{0, 1, \dots, b-1\}$, is S periodic? Honkala showed in 1986 that this problem is decidable. Later on, Leroux used in 2005 the convention to write number representations with the least significant digit first (LSDF), and designed a quadratic algorithm to solve a more general problem.

We use here LSDF convention as well and give a structural description of the minimal automata that accept periodic sets. Then, we show that it can be verified in linear time if a minimal automaton meets this description. In general, this yields a $O(bn \log(n))$ procedure to decide whether an automaton with n states accepts an ultimately periodic set of nonnegative integers.

Key words and phrases: integer-base systems; automata; recognisable sets; periodic sets; least significant digit first encodings.

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1. INTRODUCTION

Let b be a fixed integer strictly greater than 1, called the *base*. Every nonnegative integer n is *represented* (in base b) by a *word* u over the digit alphabet $A_b = \{0, 1, \dots, b-1\}$, and representation is unique up to leading 0's. Hence, *subsets* of \mathbb{N} are represented by *languages* of A_b^* . Depending on the base, a given subset of \mathbb{N} may be represented by a simple or complex language: the set of powers of 2 is represented in base 2 by the regular language 10^* ; whereas it is represented in base 3 by a language that is not context-free.

A subset of \mathbb{N} is said to be *b-recognisable* if it is represented by a regular (or rational, or recognisable) language over A_b . On the other hand, a subset of \mathbb{N} is said *recognisable* if it is, via the identification of \mathbb{N} with a^* ($n \leftrightarrow a^n$), a regular language of a^* . A subset of \mathbb{N} is recognisable if and only if it is *ultimately periodic* (u.p.) and we use the latter terminology in the sequel as it is both meaningful and more distinguishable from *b-recognisable*.

It is common knowledge that every u.p. set (of nonnegative integers) is *b-recognisable* for every b . However, a *b-recognisable* set for some b is not necessarily u.p., nor *c-recognisable* for some other c ; the set of all powers of 2, previously discussed, is an example of these two facts. It is a simple exercise to show that if b and c are *multiplicatively dependent* (that is, if there exist positive integers k and ℓ such that $b^k = c^\ell$), then every *b-recognisable* set is a *c-recognisable* set as well. A converse of these two properties is the theorem of Cobham [Cob69]: *a set of numbers that is both b- and c-recognisable, for multiplicatively independent b and c, is u.p.* It is a strong and deep result whose proof is difficult (see also [BHMV94, DR11]).

After Cobham's theorem, another natural question on *b-recognisable* sets is the decidability of periodicity. It was positively solved in 1986:

Theorem (Honkala [Hon86]). *It is decidable whether an automaton over A_b accepts an ultimately periodic set.*

The complexity of the decision procedure is not an issue in the original work. Neither are the properties or the structure of automata accepting u.p. sets. Given an automaton \mathcal{A} , Honkala shows that there are bounds on the parameters of the potential u.p. set accepted by \mathcal{A} . The property is then decidable as it is possible to enumerate all automata that accept sets with smaller parameters and check whether any of them is equivalent to \mathcal{A} .

As detailed below, subsequent works on automata and number representations brought some answers regarding the complexity of the decision procedure, explicitly or implicitly. In the present article, we follow the convention that number representations are written least significant digit first (LSDF convention) and show the following.

Theorem 1.1. *Let $b > 1$ be an integer. We assume that number representations are written in base b and with the least significant digit first. Given a minimal DFA \mathcal{A} with n states, it is decidable in time $\mathcal{O}(bn)$ whether \mathcal{A} accepts an ultimately periodic set.*

Corollary 1.2. *Given a DFA \mathcal{A} with n states, it is decidable in time $\mathcal{O}((bn) \log n)$ whether \mathcal{A} accepts an ultimately periodic set.*

On the order of digits. Honkala's problem gives birth to two different problems when one writes either the least or the most significant digit first (LSDF or MSDF, respectively). These two problems are not polynomially equivalent. In order to transform an instance \mathcal{A} of one of the problem into an instance of the other, one must run on \mathcal{A} a transposition and

then a determinisation. This potentially leads to an exponential blow-up of the number of states. This event occurs for the problem at hand for example with the language $L_n = 1(0+1)^n 1(0+1+\varepsilon)^n 0^*$ and its mirror K_n . The number of states in the minimal automaton accepting L_n (resp. K_n) grows linearly (resp. exponentially) with n . Evaluating L_n as LSDF encodings or K_n as MSDF encodings yields the same finite (thus u.p.) set.

A recent work by Boigelot et al. [BMMR17] gives a quasi-linear algorithm to solve Honkala's problem when number representations are written MSDF. As noted above, this result cannot be used to solve efficiently the problem using LSDF convention, which is the object of the present paper.

Related work in the multidimensional setting. New insights on Honkala's problem were obtained when stating it in a higher dimensional space. Let \mathbb{N}^d be the additive monoid of d -tuples of nonnegative integers. Every d -tuple in \mathbb{N}^d can be represented in base b by a d -tuple of words over A_b of the same length, as shorter words can be padded by 0's without changing the corresponding value. Such d -tuples can be read by (finite) automata over $(A_b^d)^*$ — automata reading on d synchronised tapes — and a subset of \mathbb{N}^d is b -recognisable if the set of the b -representations of its elements is accepted by such an automaton.

On the other hand, the recognisable and rational subsets of \mathbb{N}^d are defined in the classical way. A subset of \mathbb{N}^d is *recognisable* if it is saturated by a congruence of finite index, and is *rational* if it may be expressed by a rational expression. If $d = 1$, then $\mathbb{N}^d = \mathbb{N}$ is a free monoid and the family of rational sets is equal to the family of recognisable sets; in this case, they are typically called *regular languages* via the identification of \mathbb{N} with a^* . Otherwise, \mathbb{N}^d is not a free monoid and the two families do not coincide (cf. [Sak09]).

It is also common knowledge that every rational set of \mathbb{N}^d is b -recognisable for every b , and the example in dimension 1 is enough to show that a b -recognisable set is not necessarily rational. Semenov showed a generalisation of Cobham's theorem (cf. [Sem77, BHMV94, DR11]): *a subset of \mathbb{N}^d which is both b - and c -recognisable, for multiplicatively independent b and c , is rational.* The generalisation of Honkala's theorem went as smoothly.

Theorem (Muchnik [Muc03]). *It is decidable whether a b -recognisable subset of \mathbb{N}^d is rational.*

Theorem (Leroux [Ler05]). *Assuming that number representations are written LSDF, it is decidable in polynomial time whether a b -recognisable subset of \mathbb{N}^d is rational.*

Muchnik's algorithm is triply exponential while Leroux's is quadratic. This improvement is based on sophisticated geometric constructions that are detailed in [Ler06]. Note that Leroux's result, restricted to dimension $d = 1$, readily yields a quadratic procedure for Honkala's original problem. The improvement to quasilinear complexity that we present here (Corollary 1.2) is not due to a natural simplification of Leroux's construction for the case of dimension 1.

Rational sets of \mathbb{N}^d have been characterised by Ginsburg and Spanier [GS66] as sets definable in *Presburger arithmetic* (that is, definable by a formula of the first order logic with addition, denoted by $FO[\mathbb{N}, +]$). On the other hand, the Büchi-Bruyère theorem (cf. [Büc60, Bru85, BHMV94]) characterises b -recognisable subsets of \mathbb{N}^d : *A subset of \mathbb{N}^d is b -recognisable if and only if it is definable by a formula of $FO[\mathbb{N}, +, V_b]$.* (The function $V_b : \mathbb{N} \rightarrow \mathbb{N}$ maps each n to the greatest power of b that divides n .)

Using these two results, one may see that Muchnik’s problem can (and was indeed) stated in terms of logic: decide whether a formula of $FO[\mathbb{N}, +, V_b]$ has an equivalent formula in $FO[\mathbb{N}, +]$. However, the two statements are not equivalent for complexity issues. Using the Büchi-Bruyère Theorem to build an automaton from a formula may give rise to a multi-exponential blow-up of the size.

Related work in non-standard numeration systems. Generalisation of base p by nonstandard numeration systems gives an extension of Honkala’s problem, best expressed in terms of abstract numeration systems. Given a totally ordered alphabet A , any language $L \subseteq A^*$ defines an *abstract numeration system* (a.n.s.) S_L in which the integer $n \in \mathbb{N}$ is represented by the $(n+1)$ -th word of L in the radix order (*cf.* [LR10]). The a.n.s. is said to be regular if L is. A subset of \mathbb{N} is called S_L -recognisable if its representation in the a.n.s. S_L is a regular language. It is known that every u.p. subset of \mathbb{N} is S_L -recognisable for every regular a.n.s. S_L . The extended Honkala’s problem takes as input an S_L -recognisable set X and consists in deciding whether X is u.p.

It was observed in [ARS09, CRS12] that, for a subset of \mathbb{N} the *property of being u.p.* is definable by a formula of the Presburger arithmetic. Hence, if S_L is a regular a.n.s. in which addition is realised by a finite automaton, then the extended Honkala’s problem is decidable. In particular, this approach solves the case where the numeration system is a Pisot U-system (*cf.* [FS10]). On the other hand, with a proof similar to the one from the original Honkala’s paper, the problem was also shown to be decidable for a large class of U-systems [BCFR09, Cha09]. This class is incomparable with the class of Pisot U-systems. Finally, it is shown in [RM02, LR10] that the extended Honkala’s problem is equivalent to deciding whether an HD0L sequence is periodic (*cf.* [AS03]). Since then, this latter problem has been shown to be decidable [Dur13, Mit11]. Hence, the extended Honkala’s problem is also decidable in general.

These extensions were mentioned for the sake of completeness. The present article is focused on solving the original problem of Honkala when using LSDF convention.

Outline. As it is often the case, the linear complexity of our algorithm is obtained as the consequence of a structural characterisation. After preliminaries, Section 3 defines and study the class \mathbb{UP} of the minimal automata that accept u.p. sets. Then, we describe in Section 4 a set of structural properties about the shapes and positions of the strongly connected components (s.c.c.’s) and show that these properties characterise the class \mathbb{UP} (Theorem 4.3). Finally, Section 5 gives the linear algorithm underlying Theorem 1.1, which decides whether a given minimal automaton accepts a u.p. set. The delicate part is to obtain a linear complexity in the special case where the input automaton is strongly connected.

2. PRELIMINARIES

2.1. On automata. An *alphabet* A is a finite set of symbols, or *letters*; in our case, letters will always be digits and the term *digit* will be used as a synonym of *letter*. We call *word* over A a finite sequence of letters taken in A ; the empty word is denoted by ε and the length of a word $u = a_0a_1 \cdots a_{k-1}$ by $|u| = |a_0a_1 \cdots a_{k-1}| = k$. The set of words over A is denoted by A^* , and a subset of A^* is called a *language over* A .

In this article, we consider only automata that are deterministic and finite. Thus, an automaton is denoted by $\mathcal{A} = \langle A, Q, i, \delta, F \rangle$, where A is the *alphabet*, Q is the finite set of *states*, $i \in Q$ is the *initial state*, $F \subseteq Q$ is the set of *final states*, and $\delta : Q \times A \rightarrow Q$ is the *transition function*. As usual, δ is extended to a function $Q \times A^* \rightarrow Q$ by $\delta(q, \varepsilon) = q$ and $\delta(q, ua) = \delta(\delta(q, u), a)$. When the context is clear, $\delta(s, u)$ will also be denoted by $s \cdot u$. A *transition in \mathcal{A}* is an element (s, a, t) in $Q \times A_b \times Q$ such that $\delta(s, a) = t$; it is usually denoted by $s \xrightarrow[\mathcal{A}]{a} t$ or simply $s \xrightarrow{a} t$ when \mathcal{A} is clear from context. A *path in \mathcal{A}* is a sequence of transitions $s_0 \xrightarrow[\mathcal{A}]{a_0} s_1 \cdots \xrightarrow[\mathcal{A}]{a_k} s_{k+1}$ which is also denoted by $s_0 \xrightarrow[\mathcal{A}]{u} s_{k+1}$ where $u = a_0 \cdots a_k$, and we call s_0 the *origin*, u the *label* and s_{k+1} the *destination* of this path. Note that this path exists if $\delta(s_0, u) = s_{k+1}$.

We call *run* any path originating from the initial state, and *the run of a word u* refers to the run labelled by u if it exists; this path is well defined since our automata are deterministic. A word u in A^* is *accepted* by \mathcal{A} if its run ends in a final state, that is, if $(i \cdot u)$ exists and belongs to F . The language accepted by \mathcal{A} is denoted by $L(\mathcal{A})$. If every word has a run, \mathcal{A} is said to be *complete*. A state r is said *reachable from* another state s if there exists a path from r to s , and simply *reachable* if it is reachable from the initial state. An automaton is said *reachable* if all its states are *reachable*.

Drawing Convention. *In figures, most automata will be over two-letter alphabets ($\{0, 1\}$ or $\{0, g\}$). For the sake of clarity, we omit labels in such cases: transitions labelled by 1 will be drawn with a thick line, those labelled by 0 with a thin line, and those by g with a double line.*

Definition 2.1. Let \mathcal{A} and \mathcal{M} be two automata over the same alphabet A

- (i) An (*automaton*) *morphism* is a surjective function $\varphi : Q_{\mathcal{A}} \rightarrow Q_{\mathcal{M}}$ that meets the following three conditions.

$$\varphi(i_{\mathcal{A}}) = \varphi(i_{\mathcal{M}}) \quad (2.1a)$$

$$\varphi^{-1}(F_{\mathcal{M}}) = F_{\mathcal{A}} \quad (2.1b)$$

$$\forall a \in A, \forall s \in Q_{\mathcal{A}} \quad \varphi(s \cdot a) = \varphi(s) \cdot a \quad (2.1c)$$

- (ii) If φ denotes a morphism, we say that two states s and s' are *φ -equivalent* if they have the same image by φ .
- (iii) If there exists a morphism $\mathcal{A} \rightarrow \mathcal{M}$, we say that \mathcal{M} is a *quotient* of \mathcal{A} .

Given a regular language L , it is classical (*cf.* [Sak09], for instance) that there exists a *minimal automaton* \mathcal{M} that accepts L : it is the complete automaton accepting L with the minimal amount of states. Moreover, given an automaton \mathcal{A} that accept L , \mathcal{M} may be computed in quasi-linear time from \mathcal{A} and \mathcal{M} is a quotient of \mathcal{A} .

Definition 2.2. The transition monoid T of an automaton \mathcal{A} is the set of the functions induced by all words in A^* on the states of \mathcal{A} :

$$T = \left\{ \begin{array}{ccc} f_u : Q_{\mathcal{A}} & \longrightarrow & Q_{\mathcal{A}} \\ q & \longmapsto & q \cdot u \end{array} \mid u \in A^* \right\} .$$

Note that in the previous definition, since $Q_{\mathcal{A}}$ is finite, there is a finite number of functions $Q_{\mathcal{A}} \rightarrow Q_{\mathcal{A}}$ hence T is always finite.

Definition 2.3. An automaton \mathcal{A} over an alphabet A is called a *group automaton* if every state of \mathcal{A} has a unique incoming and a unique outgoing transition labelled by each letter of A .

It follows from Definition 2.3 that an automaton is a group automaton if and only if its transition monoid is a group. Moreover, that property is stable by quotient:

Property 2.4. Every quotient of a group automaton is a group automaton.

2.2. On strongly connected components. Two states s, s' of an automaton \mathcal{A} are *strongly connected* if \mathcal{A} contains a path from s to s' and a path from s' to s . This defines an equivalence relation whose classes are called the *strongly connected components* (s.c.c.'s) of \mathcal{A} . Every state s of \mathcal{A} then belongs to a unique s.c.c. Note that an s.c.c. does not necessarily contains a circuit. Indeed the s.c.c. of an *isolated state* s (that is, a state that do not belong to any circuit), is the singleton $\{s\}$ and is said *trivial*. Figures 1a and 1b show an automaton and its s.c.c.'s.

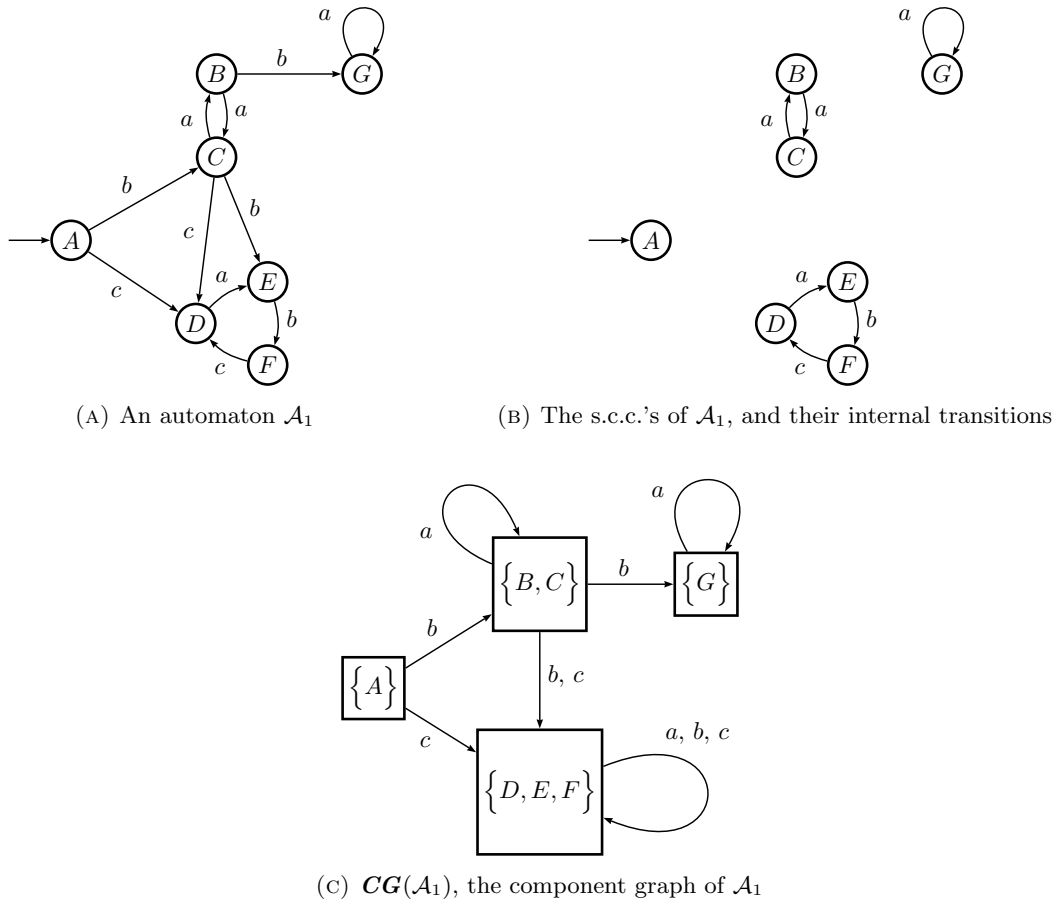


FIGURE 1. An automaton, its s.c.c.'s and its component graph.

The *component graph* $\mathbf{CG}(\mathcal{A})$ of a an automaton \mathcal{A} is the labelled d.a.g. (directed acyclic graph) that results from contracting each s.c.c. into a single vertex. For instance, Figure 1c shows the component graph of \mathcal{A}_2 . We say that an s.c.c. X is a *descendant* of another s.c.c. Y if X is a successor of Y in the component graph that is, if there is $x \in X$ and $y \in Y$ such that $x \xrightarrow{a} y$, for some letter a . It is classical that the component graph can be computed efficiently (*cf.* [CLRS09]), as stated below.

Theorem 2.5. *The component graph of an m -transitions automaton can be computed in time $O(m)$.*

2.3. On integer base numeration system. Let b be an integer strictly greater than 1 called the *base*. It will be fixed throughout the article. We briefly recall below the definition and elementary properties of base- b numeration systems. Note that we represent numbers with the Least Significant Digit First (LSDF) a convention used with some success in the past, for instance by Leroux [Ler05, Ler06].

Given two *positive integers* n and m , we denote by $n \div m$ and $n \% m$ respectively the quotient and the remainder of the Euclidean division of n by m , *i.e.* $n = (n \div m)m + (n \% m)$ and $0 \leq (n \% m) < m$. We index the letters of a word u from left to right: $u = a_0 a_1 \cdots a_n$.

Given a word $u = a_0 a_1 \cdots a_n$ over the alphabet $A_b = \{0, 1, \dots, b-1\}$, its *value* (in base b), denoted by \overline{u} , is given by the following expression.

$$\overline{u} = \overline{a_0 a_1 \cdots a_n} = \sum_{i=0}^n a_i b^i \quad (2.2)$$

Words whose values are equal to some integer k are called *b -expansions* of k . Exactly one among them does not **end** with the digit 0; it is called the *b -representation* of k , and is denoted by $\langle k \rangle$. We recall below formulas for evaluating concatenations of words; they follow from (2.2).

$$\forall a \in A_b, v \in A_b^* \quad \overline{av} = a + \overline{v}b \quad (2.3)$$

$$\forall a \in A_b, u \in A_b^* \quad \overline{ua} = \overline{u} + ab^{|u|} \quad (2.4)$$

$$\forall u, v \in A_b^* \quad \overline{uv} = \overline{u} + \overline{v}b^{|u|} \quad (2.5)$$

In this article, we are interested in the set of the values of the words accepted by automata over A_b . For the sake of consistency, we will only consider automata \mathcal{A} that *accept by value* that is, such that \mathcal{A} accepts either all words of value k , or none of them. This acceptance convention is generally more practical than the other one (accepting by representation): considered automata are usually smaller, proofs are more elegant, and in the multidimensional settings (which we do not consider here) it makes operations like projection much more efficient. In practice, it means that all automata we consider are such that the successor by 0 of a final state exists and is final while the successor by 0 of a non-final state is non-final, if it exists. Thus, we may say without ambiguity that an automaton *accepts* S , where S is a subset of \mathbb{N} ; indeed, it means that \mathcal{A} accepts the language $\langle S \rangle 0^*$.

3. AUTOMATON ACCEPTING AN ARBITRARY PERIODIC SET

The purpose of this section is to define and study the minimal automaton that accepts an arbitrary ultimately periodic set of nonnegative integers. Similar results and constructions were used in the literature in other contexts, for instance when considering automata for linear constraints (*cf.* [BC96]).

First, let us introduce some terminology and notation.

S	a	$\Delta(S, a)$
$\{0, 3, 4\}$	0	$\mapsto \{0, 2\}$
$\{0\} + 2\mathbb{N}$	0	$\mapsto \{0\} + \mathbb{N}$
$\{0\} + 3\mathbb{N}$	0	$\mapsto \emptyset + \mathbb{N}$
$\{0\} \oplus (\{0, 1, 2, 4\} + 5\mathbb{N})$	0	$\mapsto \{0\} \oplus (\{0, 1, 2, 3\} + 5\mathbb{N})$
$\{0\} \oplus (\{0, 1, 2, 4\} + 5\mathbb{N})$	1	$\mapsto \{0, 2, 3, 4\} + 5\mathbb{N}$

TABLE 2. A few values of function Δ in base 2**Definition 3.1.**

- (i) A set of integers $S \subseteq \mathbb{N}$ is said to be *purely periodic* if it may be written as $S = R + p\mathbb{N}$, for some positive integer p and $R \subseteq \{0, \dots, p-1\}$.
Moreover, we say that S is *canonically written as $R + p\mathbb{N}$* if there is no integer p' , $0 < p' < p$, and $R' \subseteq \{0, \dots, p'-1\}$ such that $S = (R' + p'\mathbb{N})$.
- (ii) Given two sets S and S' , we denote by $S \oplus S'$ their symmetric difference: an element belongs to $S \oplus S'$ if it is in S or in S' but not in both.
- (iii) A set $S \subseteq \mathbb{N}$ is said to be *ultimately periodic* (u.p.) if it may be written as $S = I \oplus S'$, where I is a finite subset of \mathbb{N} and S' is purely periodic.
Moreover we say that S is *canonically written as $I \oplus (R + p\mathbb{N})$* if S' is canonically written as $R + p\mathbb{N}$.
- (iv) If S denotes a u.p. subset of \mathbb{N} canonically written as $S = I \oplus (R + p\mathbb{N})$, then we call p the *period* of S ; R the *remainder set* of S ; I the *mismatch set* of S ; and m the *preperiod* of S , where $m = \max(I) + 1$ if $I \neq \emptyset$, and $m = 0$ otherwise.

For instance, let us consider the set $S = \{0, 6\} \cup (\{4, 5\} + 4\mathbb{N})$. It is canonically written as $S = \{1, 6\} \oplus (\{0, 1\} + 4\mathbb{N})$. Hence, the period of S is 4; its remainder set is $\{0, 1\}$; its mismatch set is $\{1, 6\}$; and its preperiod is 7. Similarly, the period of the empty set (resp. of \mathbb{N}) is 1 and its remainder set is \emptyset (resp. $\{0\}$).

3.1. The function Δ . In Section 3.1, we take interest in the function Δ that later on will be used as the common transition function of all minimal automata that accept u.p. subsets of \mathbb{N} . We denote by $\mathcal{P}(X)$ the set of the subsets of X .

Definition 3.2. Let Δ be the function $\Delta : (\mathcal{P}(\mathbb{N}) \times A_b) \rightarrow \mathcal{P}(\mathbb{N})$ defined by:

$$\forall S \subseteq \mathbb{N}, \forall a \in A_b \quad \Delta(S, a) = \{n \in \mathbb{N} \mid (nb + a) \in S\} . \quad (3.1)$$

As usual, Δ is extended as a function $(\mathcal{P}(\mathbb{N}) \times A_b^*) \rightarrow \mathcal{P}(\mathbb{N})$.

Table 2 gives a few instances of the function Δ in base 2. Given a letter a in A_b , the function $S \mapsto \Delta(S, a)$ corresponds to reading the letter a , as highlighted by the next equation (which follows from (2.3) and (3.1)).

$$\forall S \subseteq \mathbb{N}, \forall u \in A_b^*, \forall a \in A_b \quad \overline{au} \in S \iff \overline{u} \in \Delta(S, a) . \quad (3.2)$$

First, let us prove that the function Δ is stable over u.p. subsets of \mathbb{N} .

Lemma 3.3. *If S denotes a set of nonnegative integers, then the following are equivalent.*

- (i) S is u.p.
- (ii) For every a in A_b , $\Delta(S, a)$ is u.p.

Proof. (i) \implies (ii). We canonically write S as $S = I \oplus (R + p\mathbb{N})$ and we denote by m the preperiod of S . Moreover, we write

$$m' = \left\lceil \frac{m-a}{b} \right\rceil \quad \text{and} \quad p' = \frac{p}{\gcd(p, b)} \quad (3.3)$$

Let $n \geq m'$ be an integer. From (3.3), we have $(nb+a) \geq m$ and $p \mid (p'b)$. The proof of the forward direction is concluded by the following equivalences:

$$\begin{aligned} n \in \Delta(S, a) &\iff (nb+a) \in S \iff (nb+a+bp') \in S \\ &\iff (n+p')b+a \in S \iff (n+p') \in \Delta(S, a) \end{aligned}$$

(ii) \implies (i). From (3.1) and the properties of Euclidean divisions, it holds that

$$S = \bigcup_{a \in A_b} (b \times \Delta(S, a) + a) .$$

Since multiplication, addition and finite union preserves ultimate periodicity, S is u.p. \square

While showing the forward direction of Lemma 3.3, we also showed the following properties.

Properties 3.4. *Let S be a u.p. subset of \mathbb{N} and a be a letter. Let p be the period and m the preperiod of S . Let p' and m' be the period and preperiod of $\Delta(S, a)$.*

- (i) $p \geq p'$
- (ii) $m \geq m'$
- (iii) *If p is not coprime with b , then $p > p'$*
- (iv) *If $m > 1$, then $m > m'$*
- (v) *If $m > 0$ and $a \neq 0$, then $m > m'$*

We conclude our preliminary study of Δ by two technical statements that will be useful later on.

Property 3.5. *Let I and P be subsets of \mathbb{N} such that I is finite and P is purely periodic. For every letter a in A_b , $\Delta(I \oplus P, a) = \Delta(I, a) \oplus \Delta(P, a)$.*

Proof. Let u be a word in A_b^* .

$$\begin{aligned} \overline{u} \in \Delta(I \oplus P, a) &\iff \overline{au} \in I \oplus P \\ &\iff \overline{au} \in I \quad \text{or} \quad \overline{au} \in P \quad \text{but not both} \\ &\iff \overline{u} \in \Delta(I, a) \quad \text{or} \quad \overline{u} \in \Delta(P, a) \quad \text{but not both} \\ &\iff \overline{u} \in (\Delta(I, a) \oplus \Delta(P, a)) \end{aligned} \quad \square$$

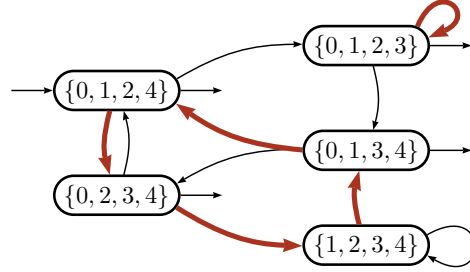
Lemma 3.6. *If S denotes a u.p. subset of \mathbb{N} , then the following hold.*

$$\forall u \in A_b^* \quad \overline{u} \in S \iff \Delta(S, u) \ni 0 \quad (3.4)$$

$$S = \{ u \in A_b^* \mid \Delta(S, u) \ni 0 \} \quad (3.5)$$

Sketch. Equation (3.4) is shown with an induction based on Equation (3.2) while (3.5) is a reformulation of (3.4). \square

e	$h_5(e, 0)$	$h_5(e, 1)$
0	0	2
1	3	0
2	1	3
3	4	1
4	2	4

TABLE 3. The function h_5 FIGURE 4. The automaton $\mathcal{U}_{(\{0,1,2,4\}+5\mathbb{N})}$

3.2. The class UP. In the following, we manipulate sets of sets of nonnegative integers (*i.e.*, subsets of $\mathcal{P}(\mathbb{N})$). For the sake of clarity, we denote such objects with a bold font.

If S and T denote two subsets of \mathbb{N} , we say that T is Δ -reachable from S if there is a word u in A_b^* such that $\Delta(S, u) = T$. If S is u.p., Lemma 3.3 yields that T is also u.p. Moreover, Properties 3.4(i) and 3.4(ii) ensure that the period and preperiod of T are smaller than the ones of S . Hence, finitely many sets are Δ -reachable from S .

Definition 3.7. Let S be a u.p. subset of \mathbb{N} and let $\mathbf{Q} \subseteq \mathcal{P}(\mathbb{N})$ be the set of all sets Δ -reachable from S . We denote by \mathcal{U}_S the automaton defined by:

$$\mathcal{U}_S = \langle A_b, \mathbf{Q}, S, \Delta|_{\mathbf{Q}}, \mathbf{F} \rangle,$$

where $\Delta|_{\mathbf{Q}}$ is the restriction of Δ to $\mathbf{Q} \times A_b$ and $\mathbf{F} = \{T \in \mathbf{Q} \mid T \ni 0\}$.

Then, the next proposition follows directly from Lemma 3.6.

Proposition 3.8. For every u.p. set $S \subseteq \mathbb{N}$, the automaton \mathcal{U}_S is the minimal automaton that accepts S .

Definition 3.9. We denote by \mathbb{UP} the class of all minimal automata that accept u.p. subsets of \mathbb{N} : $\mathbb{UP} = \{\mathcal{U}_S \mid S \text{ is a u.p. subset of } \mathbb{N}\}$.

Now, let us translate Lemma 3.3 in terms of automata for future reference.

Lemma 3.10. Let $\mathcal{A} = \langle A_b, Q, i, \delta, F \rangle$ be an automaton. The following are equivalent.

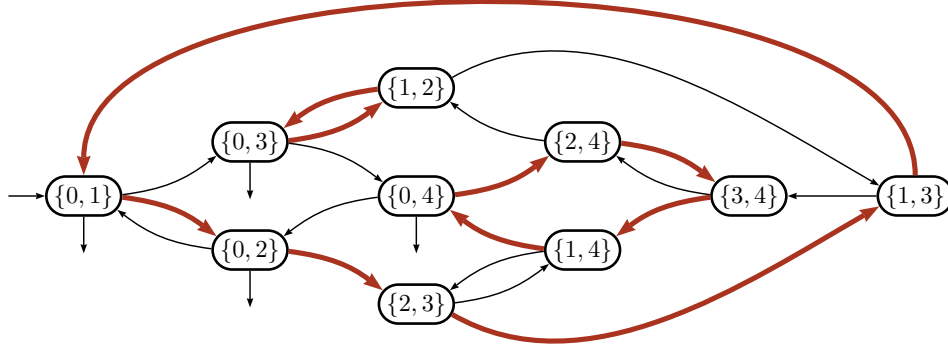
- (i) \mathcal{A} belongs to \mathbb{UP} ,
- (ii) For every letter $a \in A_b$, the automaton \mathcal{B}_a belongs to \mathbb{UP} , where \mathcal{B}_a is the reachable part of $\langle A_b, Q, \delta(i, a), \delta, F \rangle$.

3.3. Atomic automata. In this subsection, we take interest in some automata from \mathbb{UP} , that we call *atomic*.

Definition 3.11. An automaton \mathcal{U}_S in \mathbb{UP} is said *atomic* if S is a purely periodic and its period is coprime with b . For short, we say that an automaton is \mathbb{UP} -atomic if it belongs to \mathbb{UP} and is atomic.

Next, we work towards an explicit definition of \mathbb{UP} -atomic automata. For every positive integer p coprime with b , we denote by h_p the function $(\mathbb{Z}/p\mathbb{Z} \times A_b) \rightarrow \mathbb{Z}/p\mathbb{Z}$ defined by

$$\forall e \in \mathbb{Z}/p\mathbb{Z}, \forall a \in A_b \quad h_p(e, a) = (e - a)b^{-1}, \quad (3.6)$$

FIGURE 5. The automaton $\mathcal{U}_{(\{0,1\}+5\mathbb{N})}$.

where b^{-1} denotes the inverse of b in $\mathbb{Z}/p\mathbb{Z}$. For instance, Table 3 gives the explicit definition of h_5 . Let us denote by $\mathcal{P}_k(\mathbb{Z}/p\mathbb{Z})$ the set of the subsets of $\mathbb{Z}/p\mathbb{Z}$ that have cardinal k . Note that for each letter a , the function $e \mapsto h_p(e, a)$ is a permutation of $\mathbb{Z}/p\mathbb{Z}$. Hence, when k is fixed, we may lift h_p to a function $(\mathcal{P}_k(\mathbb{Z}/p\mathbb{Z}) \times A_b) \rightarrow \mathcal{P}_k(\mathbb{Z}/p\mathbb{Z})$ as usual: $h_p(E, a) = \{h_p(e, a) \mid e \in E\}$.

Proposition 3.12. *Let $R + p\mathbb{N}$ be a purely periodic subset of \mathbb{N} . The automaton $\mathcal{U}_{(R+p\mathbb{N})}$ is isomorphic to the reachable part of*

$$\langle A_b, \mathcal{P}_k(\mathbb{Z}/p\mathbb{Z}), R, h_p, \mathbf{F} \rangle,$$

where $k = \text{Card}(R)$, $\mathbf{F} = \{E \in \mathcal{P}_k(\mathbb{Z}/p\mathbb{Z}) \mid E \ni 0\}$, and R is naturally lifted to an element of $\mathcal{P}_k(\mathbb{Z}/p\mathbb{Z})$.

Proof. We denote by \mathcal{B} the reachable part of the automaton described in the statement. It is a routine to check that the function

$$\forall E \in \mathcal{P}_k(\mathbb{Z}/p\mathbb{Z}) \quad f(E) = E + p\mathbb{N}$$

is an isomorphism $\mathcal{B} \rightarrow \mathcal{U}_{(R+p\mathbb{N})}$. □

For instance, Figures 4 and 5 show respectively the automata $\mathcal{U}_{(\{0,1,2,4\}+5\mathbb{N})}$ and $\mathcal{U}_{(\{0,1\}+5\mathbb{N})}$, as defined in Proposition 3.12 (function h_5 is given in Table 3).

Lemma 3.13. *Let \mathcal{A} be an automaton in UP. The following are equivalent.*

- (i) \mathcal{A} is atomic.
- (ii) \mathcal{A} is a group automaton.
- (iii) \mathcal{A} is strongly connected.

Proof. (i) \implies (ii). Since \mathcal{A} is atomic, it may be defined as per Proposition 3.12. As noted before, for each letter a , the function $e \mapsto h_p(e, a)$ is a permutation of $\mathbb{Z}/p\mathbb{Z}$, hence the function $E \mapsto h_p(E, a)$ is a permutation of $\mathcal{P}_k(\mathbb{Z}/p\mathbb{Z})$.

(ii) \implies (iii). Reachable group automata are always strongly connected.

(iii) \implies (i). From Definition 3.7 and Lemma 3.3, each state of \mathcal{A} is a subset of \mathbb{N} that is u.p. Since by hypothesis, \mathcal{A} is strongly connected, properties 3.4(i) and 3.4(ii) yield that all the states of \mathcal{A} have the same period p and preperiod m . Then Property 3.4(iii) yields that p is coprime with b and, since moreover \mathcal{A} is complete, Property 3.4(v) yields that $m = 0$. □

From the Definition 3.7 of automata in \mathbb{UP} and the characterisation given in Lemma 3.13, it follows that the class of \mathbb{UP} -atomic automata is stable by *modification of the initial state*, as stated below.

Property 3.14. Let $\mathcal{A} = \langle A_b, Q, i, \delta, F \rangle$ be a \mathbb{UP} -atomic automaton. Then, for any q in Q , the automaton $\mathcal{B}_q = \langle A_b, Q, q, \delta, F \rangle$ is \mathbb{UP} -atomic.

Remark 3.15. Property 3.14 allows to say, by abuse of language, that some s.c.c. is \mathbb{UP} -atomic although it has no initial state.

4. STRUCTURAL CHARACTERISATION OF THE CLASS \mathbb{UP}

The purpose of this section is to show a structural characterisation of the class \mathbb{UP} (Theorem 4.3). Stating the characterisation first requires a few definitions.

Definition 4.1. We say that an s.c.c. C of an automaton \mathcal{A} is *embedded* in another s.c.c. D if there exists an *embedding function* $f : C \cup D \rightarrow D$, that is a function meeting the following.

- (i) For every s in D , $f(s) = s$.
- (ii) For every s in C and letter a , $(s \cdot a)$ exists if and only if $(f(s) \cdot a)$ does.
- (iii) For every s in $C \cup D$ and letter a such that $(s \cdot a) \in C \cup D$, it holds that $f(s \cdot a) = f(s) \cdot a$.

An embedding function $C \cup D \rightarrow D$ might be considered an automaton “pre-morphism”, in the sense that it satisfies (2.1c) but not necessarily (2.1a) or (2.1b).

Definition 4.2. We partition non-trivial s.c.c.’s in two types. The *type two* contains the simple circuits labelled only by the digit 0, or *0-circuits*. The *type one* contains the other s.c.c.’s, that is each s.c.c. with an internal transition labelled by a positive digit.

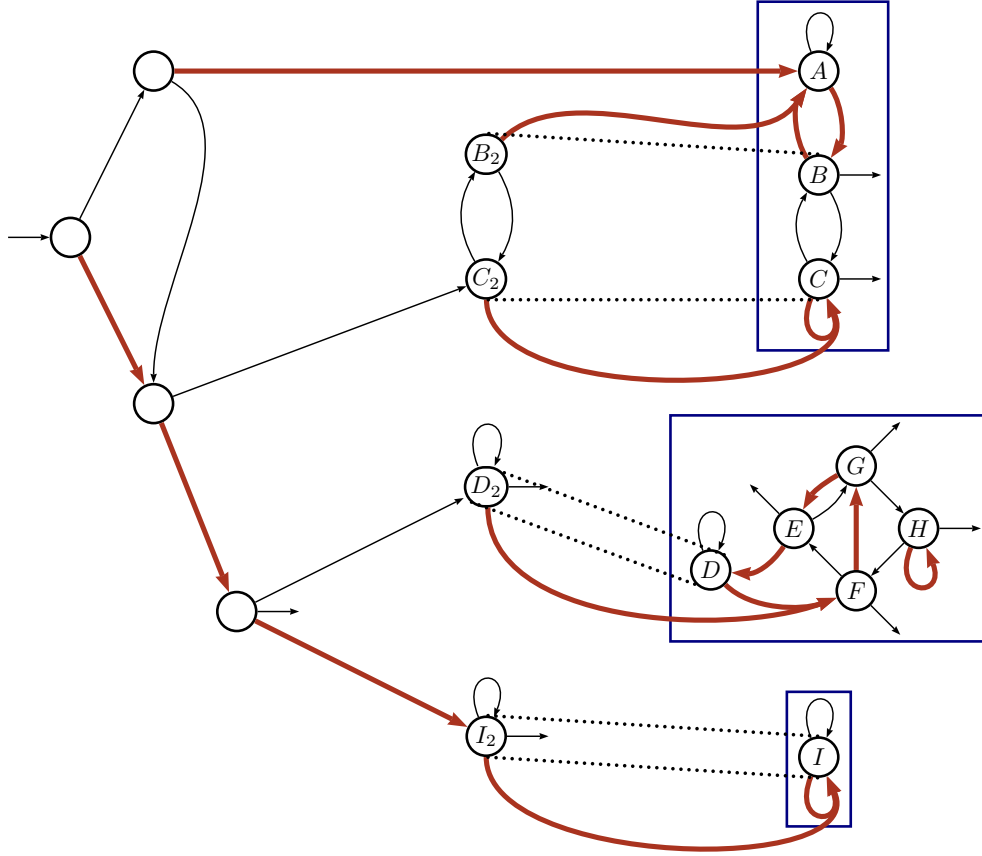
Theorem 4.3. *An automaton \mathcal{A} belongs to \mathbb{UP} if and only if the following holds, with $\mathbf{CG}(\mathcal{A})$ denoting the component graph of \mathcal{A} .*

- (UP0) *Each state and its successor by the digit 0 are both final or both non-final.*
- (UP1) *\mathcal{A} is minimal and complete.*
- (UP2) *Every type-one s.c.c. is \mathbb{UP} -atomic.*
- (UP3) *Every type-two s.c.c. has in $\mathbf{CG}(\mathcal{A})$ exactly one descendant, and that is a s.c.c. of type one.*
- (UP4) *Every type-two s.c.c. is embedded in its descendant in $\mathbf{CG}(\mathcal{A})$.*

Note that Condition (UP0) is not specific, it is more of a precondition (hence its number), which ensures that the automaton accepts by value. Proof of Theorem 4.3 takes the remainder of Section 4. Backward direction is shown in Section 4.1 and forward direction is shown in Section 4.2

In the following, we use (UP*) to refer to the conditions (UP0) to (UP4).

Example 4.4. Figure 6 shows an automaton \mathcal{A}_2 that satisfies Conditions (UP*). The framed s.c.c.’s are, from top to bottom, \mathcal{U}_{S_1} , \mathcal{U}_{S_2} and \mathcal{U}_{S_3} with $S_1 = (\{1, 2\} + 3\mathbb{N})$, $S_2 = \{0, 1, 2, 4\} + 5\mathbb{N}$ (cf. Figure 4) and $S_3 = \emptyset$. The three other non-trivial s.c.c.’s ($\{B_2, C_2\}$, $\{D_2\}$ and $\{I_2\}$) are simple 0-circuits. Embedding functions map each (relevant) node X_2 to X , with X in $\{B, C, D, I\}$.

FIGURE 6. \mathcal{A}_2 , an automaton that satisfies Conditions (UP^*)

In order to simplify the proof of both directions of Theorem 4.3, we will use Lemma 4.5, below. It follows directly from the definition of Conditions (UP^*) and states that the class of automata satisfying Conditions (UP^*) possess a property much like the class UP (cf. Lemma 3.10)

Lemma 4.5. *Let $\mathcal{A} = \langle A_b, Q, i, \delta, F \rangle$ be an automaton. The following are equivalent.*

- (i) \mathcal{A} satisfies Conditions (UP^*) .
- (ii) For every letter $a \in A_b$, the automaton \mathcal{B}_a satisfies Conditions (UP^*) , where \mathcal{B}_a is the reachable part of $\langle A_b, Q, \delta(i, a), \delta, F \rangle$.

4.1. Backward direction of Theorem 4.3.

Proposition 4.6. *An automaton that satisfies Conditions (UP^*) belongs to UP .*

Proof. Let $\mathcal{A} = \langle A_b, Q, i, \delta, F \rangle$ be an automaton that satisfies Conditions (UP^*) . Applying Lemmas 3.10 and 4.5 allows to reduce the general case to the case where the initial state of \mathcal{A} is part of a non-trivial s.c.c. If \mathcal{A} is strongly connected, it is a UP -atomic automaton and the statement obviously holds. Otherwise, Conditions (UP^*) imply that \mathcal{A} has exactly two s.c.c.'s such that:

- the s.c.c. containing the initial state, denoted by C , is a 0-circuit;

- the other s.c.c., denoted by D , is a UP-atomic automaton;
- C is embedded in D , and we denote by $f : (C \cup D) \rightarrow D$ the embedding function.

We write $j = f(i)$ and the automaton $\langle A_b, C, j, \delta|_C, F \cap C \rangle$ is thus $\mathcal{U}_{R+p\mathbb{N}}$, for some $p \in \mathbb{N}$ coprime with b , and $R \subseteq \{0, \dots, p-1\}$. Let u be a word of A_b^* that contains at least one non-0 digit. Since the initial s.c.c. is a 0-circuit, $\delta(i \cdot u)$ is a state in D . Since f is an embedding function, it holds that $\delta(i \cdot u) = \delta(j \cdot u)$. In other words, for every u such that $\bar{u} \neq 0$, \mathcal{A} accepts u if and only if $\bar{u} \in R + p\mathbb{N}$. On the other hand, since \mathcal{A} is minimal (from (UP1)), i and j must have a different final/non-final status. It follows that the set of numbers accepted by \mathcal{A} is $\{0\} \oplus (R + p\mathbb{N})$, hence that \mathcal{A} belongs to UP. \square

4.2. Forward direction of Theorem 4.3.

Proposition 4.7. *Every automaton in UP satisfies Conditions (UP*).*

Proof. Let \mathcal{U}_S be an automaton in UP which we write $\mathcal{U}_S = \langle A_b, \mathbf{Q}, S, \Delta|_{\mathbf{Q}}, \mathbf{F} \rangle$ (cf. Definitions 3.2 and 3.7). By definition, \mathcal{U}_S satisfies (UP0) and (UP1).

We write S canonically as $S = I \oplus (R + p\mathbb{N})$. Lemmas 3.10 and 4.5 allow to reduce the general case to the case where the initial state is part of a non-trivial s.c.c. In other words, there exists a non empty-word u such that $\Delta(S, u) = S$.

Claim 4.7.1. *p is coprime with b .*

Proof of Claim 4.7.1. For the sake of contradiction let us assume that p is not coprime with b . We factorise u as $u = av$ with a in A_b . From Property 3.4(iii), the smallest period of $\Delta(S, a)$ is strictly smaller than p . Hence, from Property 3.4(i) the smallest preperiod of $\Delta(\Delta(S, a), v) = S$ is strictly smaller than p , a contradiction.

Claim 4.7.2. *Either $I = \{0\}$ or $I = \emptyset$.*

Sketch. Claim 4.7.2 is proved just as Claim 4.7.1, but using Properties 3.4(iv) and 3.4(ii).

The case $I = \emptyset$ implies that \mathcal{U}_S is atomic; hence \mathcal{A} obviously satisfies Conditions (UP*). It remains to treat the case where $I = \{0\}$. In the following, we denote by \mathbf{K} the set of all purely periodic subsets of \mathbb{N} . We partition \mathbf{Q} as $\mathbf{C} \uplus \mathbf{D}$ where:

- \mathbf{C} contains every X (in \mathbf{Q}) that is not purely periodic;
- \mathbf{D} contains every X (in \mathbf{Q}) that is purely periodic.

Note that since $I = \{0\}$, every X in \mathbf{C} is of the form $\{0\} \oplus Y$, with $Y \in \mathbf{K}$.

Claim 4.7.3. *\mathbf{C} is a type-two s.c.c.*

Proof of Claim 4.7.3. From Property 3.4(v), reading any non-0 digit from any state in \mathbf{C} would reach a state in \mathbf{D} . On the other hand, from Property 3.4(ii), no state in \mathbf{C} is reachable from any state in \mathbf{D} . Since \mathcal{A} is reachable, all states in \mathbf{C} are reachable from S , hence it is necessary that \mathbf{C} is a 0-circuit.

Let f be the function $(\mathbf{C} \cup \mathbf{K}) \rightarrow \mathbf{K}$ defined as follows. For each X in \mathbf{C} , there is a $Y \in \mathbf{K}$ such that $X = \{0\} \oplus Y$ and we set $f(X) = Y$. For each X in \mathbf{K} , we set $f(X) = X$.

Claim 4.7.4. *$f(\mathbf{C}) \subseteq \mathbf{D}$*

Proof of Claim 4.7.4. Let X be an element in \mathbf{C} and we write $Y = f(X)$, hence $X = \{0\} \oplus Y$. Note that \mathcal{U}_Y is atomic, hence complete and strongly-connected. Thus, there exists a word w such that: (i) $\Delta(Y, w) = Y$; and (ii) w does not belong to 0^* . From (ii), $\Delta(\{0\}, w) = \emptyset$, hence Property 3.5 yields that $\Delta(X, w) = Y$. In other words, Y is reachable from X , hence also from S and by definition of \mathcal{U}_S , $Y \in \mathbf{Q}$.

Claim 4.7.5. *For every X in \mathbf{C} and every word u in A_b^* , $f(\Delta(X, u)) = \Delta(f(X), u)$.*

Proof of Claim 4.7.5. The whole statement reduces easily to the case where u is a letter a . The state X may be written has $\{0\} \oplus Y$ for some set Y in \mathbf{K} . Hence, the following concludes the proof of the claim.

$$\begin{aligned}
 f(\Delta(X, 0)) &= f(\Delta(\{0\} \oplus Y, a)) \\
 &= f(\Delta(\{0\}, a) \oplus \Delta(Y, a)) && \text{(From Property 3.5)} \\
 &= \Delta(Y, a) \\
 &= \Delta(f(\{0\} \oplus Y), a) \\
 &= \Delta(f(X), a)
 \end{aligned}$$

We denote by T the purely periodic set $T = f(S)$ (hence such that $S = \{0\} \oplus T$). Note that, from Claim 4.7.4, T belongs to \mathbf{D} .

Claim 4.7.6. *The automaton $\langle A_b, \mathbf{D}, T, \Delta|_{\mathbf{D}}, F \cap D \rangle$ is exactly \mathcal{U}_T .*

Proof of Claim 4.7.6. Note that the states of \mathcal{U}_S that are reachable from T are exactly the states of \mathcal{U}_T ; thus, it is enough to show that all states in \mathbf{D} are reachable from T . Let X be a state in \mathbf{D} . Since \mathcal{U}_S is reachable, there is a word w in A_b^* such that $\Delta(S, w) = X$. Claim 4.7.5 yields that $f(\Delta(S, w)) = \Delta(f(S), w)$. Since $f(S) = T$ and $f(\Delta(S, w)) = f(X) = X$, it follows that $\Delta(T, w) = X$.

Claim 4.7.6 implies that \mathbf{D} is an s.c.c. of type one and, since it is the only one, that \mathcal{U}_S satisfies (UP2). The only other s.c.c. of \mathcal{U}_S is \mathbf{C} and it is indeed of type two and has exactly one descendant (\mathbf{D}), hence \mathcal{U}_S satisfies (UP3). Moreover, Claim 4.7.4 ensures that we may restrict f to a function $(\mathbf{C} \cup \mathbf{D}) \rightarrow \mathbf{D}$. Finally, Claim 4.7.5 yields that f , thus restricted, indeed embeds \mathbf{C} in \mathbf{D} , hence that \mathcal{U}_S satisfies (UP4). \square

5. DECIDING MEMBERSHIP IN UP

The goal of Section 5 is to describe an algorithm that decides Problem 5.1 and that runs in time $\mathcal{O}(bn)$, where n is the number of states of the input automaton.

Problem 5.1. Given a minimal automaton \mathcal{A} , does \mathcal{A} satisfy Conditions (UP*)?

The hard part is to obtain a linear time-complexity in the special case where the input automaton is strongly connected. This algorithm is developed in details in Section 5.1. Then, the algorithm for the general case poses no particular difficulties and is given afterwards in Section 5.2.

5.1. The strongly connected case. From the definition of Conditions (UP^{*}), page 12, Problem 5.1 is the same as Problem 5.2, below, if the input automaton is strongly connected.

Problem 5.2 (UP-atomic). Given as input a minimal automaton \mathcal{A} , is \mathcal{A} UP-atomic?

We will see later on (Equation (5.9) and Proposition 5.21) that one can compute in linear time the only possible period p and remainder set R that could satisfy $\mathcal{A} = \mathcal{U}_{(R+p\mathbb{N})}$. In that light, Problem 5.2 reduces to Problem 5.3, below.

Problem 5.3 (Atomic Construction). Given a purely periodic set S , the period of which is coprime with b , build the automaton \mathcal{U}_S .

Proposition 3.12 gives an explicit construction of $\mathcal{U}_{(R+p\mathbb{N})}$. However, the time complexity of this construction is in $\mathcal{O}(bn \times \text{Card}(R))$, where n is the number of state in \mathcal{U}_S . Since $\text{Card}(R)$ may be up to linear in p , hence in n , this does not achieve the $\mathcal{O}(bn)$ time-complexity we require.

In the following, we use a different route to solve Problem 5.2 in linear time. In particular, we make great use of the fact that we are provided with the solution (the input automaton \mathcal{A}). Hence, the algorithm developed in the remainder of Section 5.1 does **not** solve Problem 5.3 in linear time.

The outline of Section 5.1 is as follows. Let $R + p\mathbb{N}$ be a purely periodic set such that p is coprime with b . In Section 5.1.1, we define a special automaton \mathcal{P}_p^R (called *Pascal automaton*) that accepts the purely periodic set $R + p\mathbb{N}$. Section 5.1.2 gives the precise structure of the transition monoid of \mathcal{P}_p^R (which, indeed is a group). In Section 5.1.3, we consider any strict quotient \mathcal{B} of \mathcal{P}_p^R and study the morphism φ that realises this quotient. In particular, we show that one can deduce from the structure of \mathcal{A} the values of p , of R and of a parameter (h, k) that characterises φ . Then, Section 5.1.4 gives a way to reconstruct \mathcal{B} in linear time when knowing p , R , and (h, k) . Finally, we describe in Section 5.1.5 an algorithm to decide whether a given automaton \mathcal{C} is the quotient of any Pascal automaton: (1) compute p , R , and (h, k) from the structure of \mathcal{C} ; (2) use these data to reconstruct the corresponding \mathcal{B} ; and (3) check whether \mathcal{C} and \mathcal{B} are isomorphic. This algorithm also solves Problem 5.2: the input automaton \mathcal{A} of Problem 5.2 is assumed to be minimal, hence \mathcal{A} is isomorphic to $\mathcal{U}_{(R+p\mathbb{N})}$ if and only if \mathcal{A} is a quotient of \mathcal{P}_p^R .

5.1.1. Pascal automaton: definition and elementary properties. Let us consider a purely periodic set, canonically written as $R + p\mathbb{N}$. We moreover assume that p is coprime with b . In the following, we define a special automaton that accepts $R + p\mathbb{N}$, called the *Pascal automaton of parameter (p, R)* and denoted by \mathcal{P}_p^R . Its principle indeed goes back to the work of the philosopher and mathematician Blaise Pascal (*cf.* [Sak09, preface]).

Since p and b are coprime, b is an invertible element of $\mathbb{Z}/p\mathbb{Z}$ and there exists some (smallest) positive integer ψ such that

$$b^\psi \equiv 1 [p] \quad \text{hence that} \quad \forall k \in \mathbb{N} \quad b^k \equiv b^{(k \% \psi)} [p]. \quad (5.1)$$

(In other words, ψ is the order of b in the multiplicative group of the invertible elements of $\mathbb{Z}/p\mathbb{Z}$.)

It follows from (2.4) and (5.1) that the value modulo p of a word ua can be computed using the length modulo ψ and the value modulo p of the word u :

$$\forall u \in A_b^*, \forall a \in A_b \quad \overline{ua} \% p \equiv (\overline{u} \% p) + ab^{|\overline{u}| \% \psi} [p]. \quad (5.2)$$

In the following, integers will often be used in the place of elements that should belong to $\mathbb{Z}/n\mathbb{Z}$, for some n ; in such a case, it is understood that the integer is lifted to its equivalence class modulo n . This typically occurs when the results of arithmetic operations are components of states, like in Equation (5.3) for instance.

Definition 5.4. The Pascal automaton of parameter (p, R) , denoted by \mathcal{P}_p^R , is the automaton:

$$\mathcal{P}_p^R = \langle A_b, \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/\psi\mathbb{Z}, (0, 0), \delta, R \times \mathbb{Z}/\psi\mathbb{Z} \rangle$$

where R is lifted as a subset of $\mathbb{Z}/p\mathbb{Z}$, and the transition function δ is defined by:

$$\forall (s, t) \in \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/\psi\mathbb{Z}, \forall a \in A_b \quad \delta((s, t), a) = (s, t) \cdot a = (s + ab^t, t + 1). \quad (5.3)$$

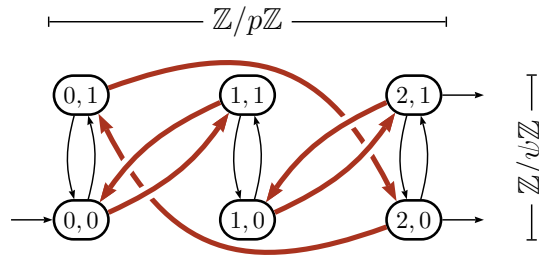


FIGURE 7. The Pascal automaton $\mathcal{P}_3^{\{2\}}$ in base 2

Example 5.5. Let $b = 2, p = 3, R = \{2\}$, hence $\psi = 2$. Figure 7 shows the Pascal automaton $\mathcal{P}_3^{\{2\}}$. Recall that transition labels are omitted in figures: transitions labelled by 1 are drawn with a thick line and transitions labelled by 0 with a thin line. Figure 8 shows $\mathcal{P}_7^{\{6\}}$; most transitions are dimmed for the sake of clarity.

Pascal automata have the expected behaviour, as stated below.

Proposition 5.6. *The Pascal automaton \mathcal{P}_p^R accepts $R + p\mathbb{N}$.*

Proposition 5.6 is a direct consequence of the Corollary 5.8 of the next lemma, which characterises the paths in \mathcal{P}_p^R .

Lemma 5.7. *Let u be a word in A_b^* . We write $h = \overline{u} \% p$ and $k = |u| \% \psi$. Then, for every state (s, t) of \mathcal{P}_p^R ,*

$$(s, t) \cdot u = (s + hb^t, t + k)$$

Proof. Induction over the length of u . The case $|u| = 0$ is trivial. Now we assume that $u \neq \varepsilon$. We write $u = va$ with $a \in A_b$ and $v \in A_b^*$. Moreover, we write $h' = \overline{v} \% p$ and $k' = |v| \% \psi (= k - 1)$. We apply below induction hypothesis and (5.3).

$$\begin{aligned} ((s, t) \cdot va) &= (s + h'b^t, t + k') \cdot a = (s + h'b^t + ab^{t+k'}, t + k' + 1) \\ &= (s + b^t(h' + ab^{k'}), t + k) \end{aligned}$$

Equation (5.2) yields the following and concludes the proof.

$$\overline{u} = \overline{va} \equiv \overline{v} \% p + ab^{|v| \% \psi} \equiv h' + ab^{k'} [p]. \quad \square$$

Corollary 5.8. *Let u be a word in A_b^* . The run of u in \mathcal{P}_p^R ends in the state $(\overline{u}, |u|)$.*

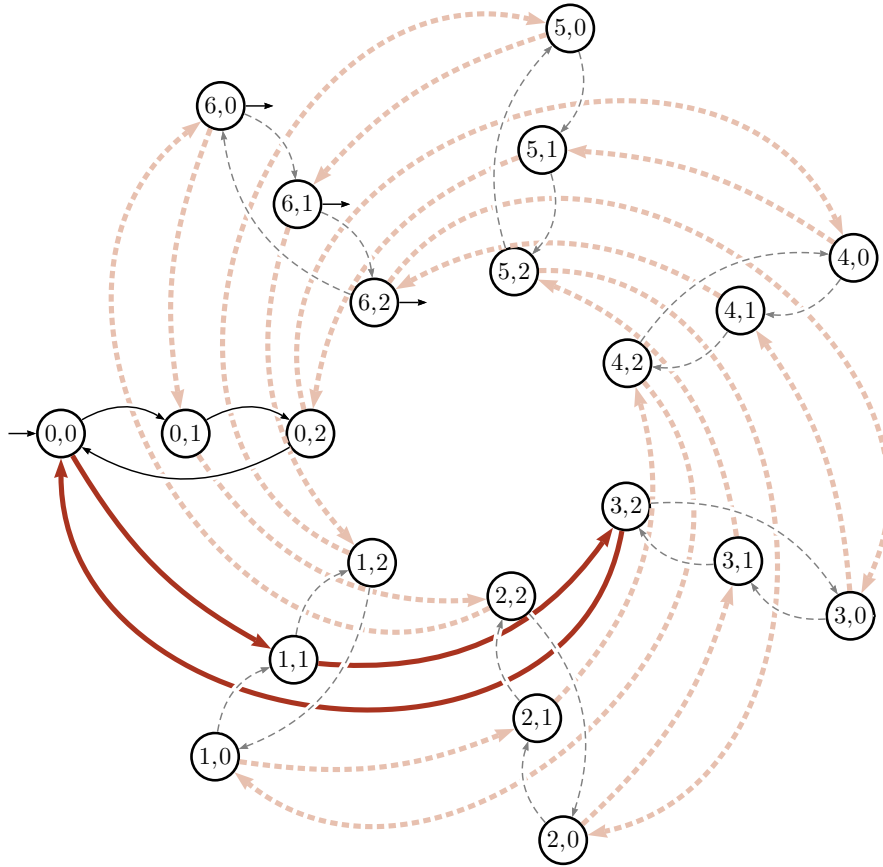


FIGURE 8. The Pascal automaton $\mathcal{P}_7^{\{6\}}$ in base 2

Much like $\mathcal{U}_{(R+p\mathbb{N})}$, every Pascal automaton \mathcal{P}_p^R (and indeed each quotient of \mathcal{P}_p^R) is a group automaton, as stated next.

Lemma 5.9. *Every Pascal automaton is a group automaton.*

Proof. Let \mathcal{P}_p^R be a Pascal automaton, (h, k) a state of \mathcal{P}_p^R , and a a letter in A_b . From (5.3), a state (s, t) is predecessor of (h, k) by a if and only if $s \equiv (h - ab^k) [p]$ and $t \equiv h - 1 [\psi]$; such a predecessor exists and is unique since p is coprime with b . \square

Corollary 5.10. *Every quotient of a Pascal automaton is a group automaton.*

The remainder of Section 5.1 is dedicated to devising an algorithm to decide the following problem.

Problem 5.11 (Quotient of a Pascal automaton). Given as input an automaton \mathcal{A} , is \mathcal{A} the quotient of some Pascal automaton?

Note that Problem 5.11 is more general than Problem 5.2. They become identical if we add in Problem 5.11 the extra assumption that \mathcal{A} is minimal

5.1.2. *Transition monoids of Pascal automata.* For a fixed period p , and a variable remainder set R , the Pascal automata \mathcal{P}_p^R are isomorphic, aside from the final-state set. In particular, their transition monoids are isomorphic as well. We denote this monoid by \mathbb{G}_p in the following; it is indeed a group from Lemma 5.9. Let us now study the structure of this group.

We recall that ψ denotes the smallest positive integer such that b^ψ is congruent to 1 modulo p , and that $\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/\psi\mathbb{Z}$ is the state set of \mathcal{P}_p^R .

Proposition 5.12. *The group \mathbb{G}_p is isomorphic to the semidirect product $\mathbb{Z}/p\mathbb{Z} \rtimes \mathbb{Z}/\psi\mathbb{Z}$.*

The proof of Proposition 5.12 requires additional definitions and properties. By definition of transition monoid, \mathbb{G}_p is the set of the permutations of $\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/\psi\mathbb{Z}$ (the state-set of \mathcal{P}_p^R) induced by words. For every u of A_b^* , the permutation induced by u , denoted by τ_u , is defined below.

$$\begin{aligned} \tau_u : \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/\psi\mathbb{Z} &\longrightarrow \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/\psi\mathbb{Z} \\ (s, t) &\longmapsto (s, t) \cdot u \end{aligned} \quad (5.4)$$

The next property follows directly from Lemma 5.7.

Property 5.13. For every words $u, v \in A_b^*$, the permutations τ_u and τ_v are equal if and only if both $\bar{u} \equiv \bar{v} [p]$ and $|u| \equiv |v| [\psi]$.

Hence, the group \mathbb{G}_p is isomorphic to the group $(\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/\psi\mathbb{Z}, \diamond)$; the operation \diamond is defined by

$$(s, t) \diamond (h, k) = (s + hb^t, t + k); \quad (5.5)$$

and the following function realises the isomorphism.

$$\begin{aligned} g : \mathbb{G}_p &\longrightarrow \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/\psi\mathbb{Z} \\ \tau_u &\longmapsto (\bar{u}, |u|) = \tau_u((0, 0)) \end{aligned} \quad (5.6)$$

We may rephrase the same fact by linking the transition function of \mathcal{P}_p^R (cf. Lemma 5.7) to the \diamond operation.

$$\forall u \in A_b^*, \forall (s, t) \in \mathbb{G}_p \quad ((s, t) \cdot u) = (s, t) \diamond (\bar{u}, |u|). \quad (5.7)$$

The next properties conclude the proof of Proposition 5.12. We recall that a subgroup H of a group G is *normal* if every x in G is such that $xHx^{-1} \subseteq H$.

Properties 5.14.

- (i) *The set $H = \mathbb{Z}/p\mathbb{Z} \times \{0\}$ is a normal subgroup of \mathbb{G}_p .*
- (ii) *The set $K = \{0\} \times \mathbb{Z}/\psi\mathbb{Z}$ is a subgroup of \mathbb{G}_p .*
- (iii) *The group \mathbb{G}_p is the internal semi-direct product $H \rtimes K$.*

Proof.

- (i) Let $(h, 0)$ and $(h', 0)$ be two elements of H . From (5.5), their product $(h, 0) \diamond (h', 0) = (h + h', 0)$ is indeed an element of H . Thus, H is a subgroup of $\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/\psi\mathbb{Z}$.

Let (s, t) be an element in $\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/\psi\mathbb{Z}$. It follows from (5.5) that the second component of its inverse, $(s, t)^{-1}$, is necessarily $-t$ modulo ψ . Hence, for every element (h, k) of $\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/\psi\mathbb{Z}$, the second component of $((s, t) \diamond (h, k) \diamond (s, t)^{-1})$ is equal to k . The case $k = 0$ yields that H is normal.

- (ii) Shown similarly from (5.5).

- (iii) Every element (h, k) of $\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/\psi\mathbb{Z}$ may be factorised as $(h, 0) \diamond (0, k)$, hence $H \diamond K = \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/\psi\mathbb{Z}$. Since moreover $H \cap K$ contains only the neutral element $(0, 0)$, $\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/\psi\mathbb{Z} = H \rtimes K$. □

In the following, we identify \mathbb{G}_p with $\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/\psi\mathbb{Z}$; we may then write the *permutation* $(s, t) \in \mathbb{G}_p$. (It is in fact the permutation τ_u , where u is any word that satisfies $\bar{u} \equiv s [p]$ and $|u| \equiv t [\psi]$.)

Since it is a transition monoid, \mathbb{G}_p is generated by the permutations induced by the letters of A_b . On the other hand, it is isomorphic to $\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/\psi\mathbb{Z}$ hence is obviously generated by the elements $(0, 1)$ and $(1, 0)$. The former is the permutation induced by the digit 0 while the latter is not induced by a letter, but rather by the word $10^{\psi-1}$. We define a new letter g whose action on \mathcal{P}_p^R is defined as the one of $10^{\psi-1}$:

$$\forall (s, t) \in \underbrace{\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/\psi\mathbb{Z}}_{=\mathbb{G}_p} \quad (s, t) \xrightarrow[A]{g} \underbrace{(s + b^t, t)}_{=(s,t) \diamond (1,0)} . \tag{5.8}$$

The next statement follows from Equation (5.5).

Property 5.15. For every letter a of A_b , the actions of a and of $g^a 0$ are equal.

Thus, the letter g allows to simplify \mathcal{P}_p^R into an automaton over the alphabet $\{0, g\}$ without losing information. This ‘equivalent’ automaton, denoted by \mathcal{P}'_p^R , is obtained by adding the letter g (which acts as the word $10^{\psi-1}$) and then deleting every letter $a \in A_b, a \neq 0$.

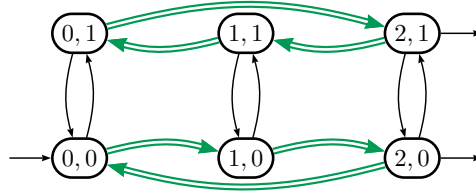


FIGURE 9. The simplified Pascal automaton $\mathcal{P}'_3^{\{2\}}$ in base 2

Example 5.16. Figures 9 and 10 show the automata $\mathcal{P}'_3^{\{2\}}$ and $\mathcal{P}'_7^{\{6\}}$ respectively. Once again, labels are omitted; transitions labelled by the digit 0 are drawn with a simple line while transitions labelled by g are drawn with a double line.

The structure of the transition monoid as a semidirect product is visible in Figure 9. First, 0 induces a permutation within each column and g induces a permutation within each row. Second, the action of 0 is the same in each column while the action of g depends on the line. A similar observation can be made about Figure 10 by replacing columns and rows by spokes and concentric circles.

Remark 5.17. The element $(0, \psi - 1)$ is, in \mathbb{G}_p , the inverse of $(0, 1)$. In Section 5.1.5, we will allow to take transitions backward; the action of g is then identical to the one of the word 10^{-1} . This word has the advantage to be shorter, and to be independent of ψ (hence independent of p)

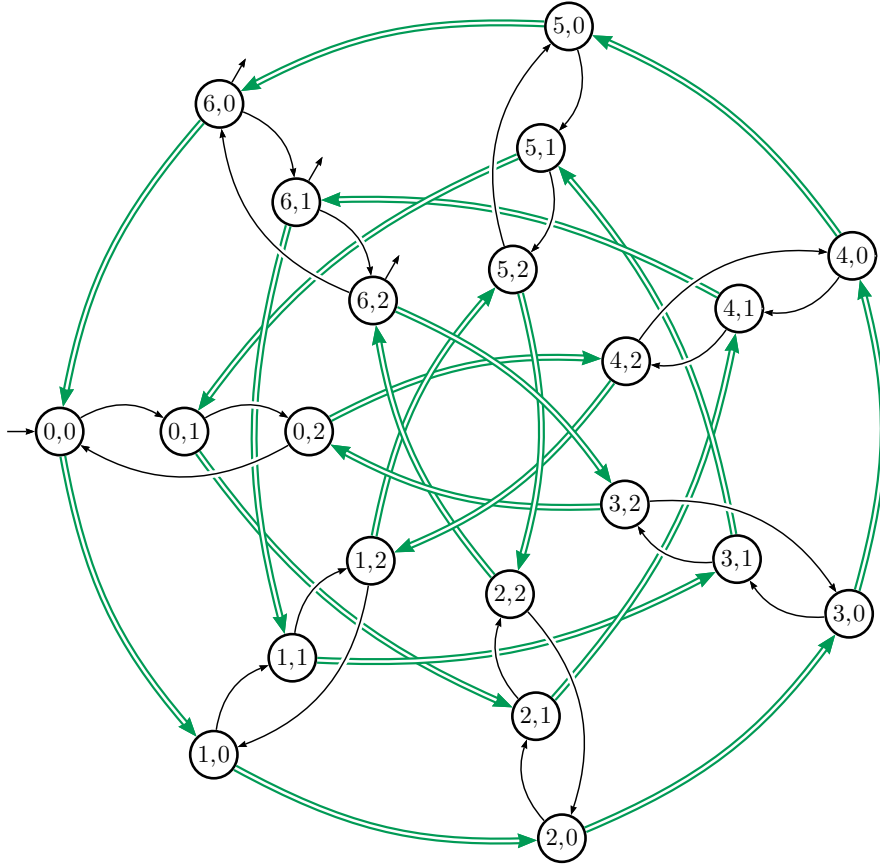


FIGURE 10. The simplified Pascal automaton $\mathcal{P}_7^{\{6\}}$ in base 2

5.1.3. *Properties of a quotient of Pascal automaton.* Here, we assume that \mathcal{A} denotes a strict quotient of \mathcal{P}_p^R , and that φ denotes the automaton morphism $\mathcal{P}_p^R \rightarrow \mathcal{A}$. Note that \mathcal{A} is a group automaton (Corollary 5.10).

As we did for \mathcal{P}_p^R in the previous Section 5.1.2, we add in \mathcal{A} transitions labelled by a new letter g whose action is the same as the one of 10^{-1} :

$$s \xrightarrow[\mathcal{A}]{g} s' \iff s \cdot 1 = s' \cdot 0 . \tag{5.9}$$

Since \mathcal{A} is a quotient of \mathcal{P}_p^R , the next property follows from Property 5.15.

Property 5.18. For every letter a of A_b , the action of a in \mathcal{A} is the same as the one of the word $g^a 0$.

In the following, we give a way to compute the parameter (p, R) of the Pascal automaton by observing the structure of \mathcal{A} . It is stated as Proposition 5.21 after two preliminary lemmas.

Lemma 5.19. For every $(s, 0)$ of \mathbb{G}_p distinct from $(0, 0)$, $\varphi((s, 0)) \neq \varphi((0, 0))$.

Proof. It follows from (5.8), which defines the transition labelled by g in \mathcal{P}_p^R , that

$$\forall h \in \mathbb{N} \quad (s, 0) \xrightarrow[\mathcal{P}_p^R]{g^h} (s + h, 0) \quad \text{and} \quad (0, 0) \xrightarrow[\mathcal{P}_p^R]{g^h} (h, 0) . \tag{5.10}$$

For the sake of contradiction, let us assume that $\varphi((s, 0)) = \varphi((0, 0))$; since φ is an automaton morphism it follows from (5.10) that

$$\forall h \in \mathbb{Z}/p\mathbb{Z} \quad \varphi((s + h, 0)) = \varphi((h, 0)) .$$

Since an automaton morphism preserves final states, for every $h \in \mathbb{Z}/p\mathbb{Z}$, $(h, 0)$ is final if and only if $(h + s, 0)$ is final. From the Definition 5.4 of Pascal automata (page 17), a state is final if and only if its first component belongs to R . Hence,

$$\forall h \in \mathbb{Z}/p\mathbb{Z} \quad h \in R \iff h + s \in R .$$

In other words, s is a period of $R + p\mathbb{N}$ strictly smaller than p , a contradiction. \square

Lemma 5.20. *Let (s, t) and (h, k) be two distinct elements of \mathbb{G}_p . If $t = k$, then $\varphi((s, t)) \neq \varphi((h, k))$.*

Proof. Let (s, t) and (h, k) be two distinct states of \mathcal{P}_p^R such that $t = k$. We write $u = 0^{\psi-t}g^s$. This word labels the two following paths:

$$(s, t) \xrightarrow{u} (0, 0) \quad \text{and} \quad (h, k) \xrightarrow{u} (h - s, 0) .$$

Since (s, t) and (h, k) are distinct, we necessarily have that $(h - s) \neq 0$.

It follows from the previous equation, that if $\varphi((s, t))$ and $\varphi((h, k))$ were equal, so would be $\varphi((0, 0))$ and $\varphi((h - s, 0))$, a contradiction to Lemma 5.19 above. \square

Proposition 5.21. *Let \mathcal{A} be the quotient of a Pascal automaton \mathcal{P}_p^R .*

- (i) *The circuits induced by the letter g in \mathcal{A} are all of length p .*
- (ii) *The word g^r is accepted by \mathcal{A} if and only if r belongs to R .*

Proof.

- (i) Let k be an element of $\mathbb{Z}/\psi\mathbb{Z}$. The g -circuit of \mathcal{P}_p^R that contains the state $(0, k)$ is

$$(0, k) \xrightarrow[\mathcal{P}_p^R]{g} (p^k, k) \xrightarrow[\mathcal{P}_p^R]{g} (2p^k, k) \xrightarrow[\mathcal{P}_p^R]{g} \dots \xrightarrow[\mathcal{P}_p^R]{g} ((n-1)p^k, k) \xrightarrow[\mathcal{P}_p^R]{g} (0, k) .$$

The image of this circuit by φ is:

$$\varphi((0, k)) \xrightarrow[\mathcal{P}_p^R]{g} \varphi((p^k, k)) \xrightarrow[\mathcal{P}_p^R]{g} \dots \xrightarrow[\mathcal{P}_p^R]{g} \varphi(((n-1)p^k, k)) \xrightarrow[\mathcal{P}_p^R]{g} \varphi((0, k)) .$$

Since φ is not necessarily injective, this last circuit might not be simple. In this case it would hold $\varphi((ip^k, k)) = \varphi((jp^k, k))$ for some distinct $i, j \in \mathbb{Z}/p\mathbb{Z}$, a contradiction to Lemma 5.20 above. Since every g -circuit of \mathcal{A} is necessarily the image of a g -circuit of \mathcal{P}'_p^R , item (i) holds.

- (ii) The run of the word g^r ends in \mathcal{P}'_p^R the state $(0, r)$ which by definition is a final state if and only if r belongs to R . Since \mathcal{A} is a quotient of \mathcal{P}'_p^R , they accept the same language.

Thus, g^r is accepted by \mathcal{A} if and only if it is accepted by \mathcal{P}'_p^R , concluding the proof. \square

Next, we give a method to characterise the automaton morphism $\varphi : \mathcal{P}_p^R \rightarrow \mathcal{A}$ with data observable in \mathcal{A} . Indeed, the morphism is entirely determined by the class of φ -equivalence of the state $(0, 0)$ of \mathcal{P}_p^R and in particular by the element (h, k) of this class such that k is the smallest but still positive.

This φ -equivalence class is characterised by the following lemma; it is a consequence of the definition of the letter g in \mathcal{P}_p^R .

Lemma 5.22. *Let (s, t) be in \mathbb{G}_p . The run of the word $g^s 0^t$ in \mathcal{A} reaches the initial states if and only if $\varphi((s, t)) = \varphi((0, 0))$.*

Let (h, k) be an element of \mathbb{G}_p . We denote by $\gamma_{(h,k)}$ the permutation of \mathbb{G}_p induced by the multiplication by (h, k) **on the left** (whereas τ_u corresponds to the multiplication by $(\bar{u}, |u|)$ on the right):

$$\forall (s, t) \in \mathbb{G}_p \quad \gamma_{(h,k)}((s, t)) = (h, k) \diamond (s, t) = (h + sb^k, k + t) . \quad (5.11)$$

We moreover write $\sigma_{(h,k)}$ the permutation resulting from the projection of $\gamma_{(h,k)}$ to its first component, $\mathbb{Z}/p\mathbb{Z}$:

$$\forall s \in \mathbb{Z}/p\mathbb{Z} \quad \sigma_{(h,k)}(s) = h + sb^k . \quad (5.12)$$

In the following we will always consider the permutations $\gamma_{(h,k)}$ and $\sigma_{(h,k)}$ parametrised by a special element (h, k) , called by abuse of language *the smallest state φ -equivalent to $(0, 0)$* , and defined as the unique¹ element satisfying the two following conditions:

- $\varphi((h, k)) = \varphi((0, 0))$;
- every element $(s, t) \in \mathbb{G}_p$ such that $(s, t) \neq (0, 0)$ and $\varphi((s, t)) = \varphi((0, 0))$ necessarily meets $k < t$.

The next lemma follows from definitions.

Lemma 5.23. *Every φ -equivalence class is stable by the permutation $\gamma_{(h,k)}$ (in \mathbb{G}_p).*

Proof. Let (s, t) be a state of \mathcal{P}_p^R and u a word such that $(\bar{u}, |u|) = (s, t)$.

$$\begin{aligned} \varphi(\gamma_{(h,k)}((s, t))) &= \varphi((h, k) \diamond (s, t)) \\ &= \varphi((h, k)) \cdot u \\ &= \varphi((0, 0)) \cdot u \\ &= \varphi((0, 0) \diamond (s, t)) \\ &= \varphi((s, t)) \end{aligned} \quad \square$$

Remark 5.24. In [Mar16], a statement stronger than Lemma 5.23 is shown: the φ -equivalence classes are in fact the orbits of $\gamma_{(h,k)}$.

5.1.4. *Construction of the quotient.* We keep here the settings of Section 5.1.3. Namely, p denotes a positive integer coprime with b , \mathcal{A} denotes a strict quotient of \mathcal{P}_p^R , φ denotes the automaton morphism $\mathcal{P}_p^R \rightarrow \mathcal{A}$ and (h, k) denotes the smallest state of \mathcal{P}_p^R that is φ -equivalent to $(0, 0)$. The purpose of this section is to show that, given as input p, R, h and k , one can build \mathcal{A} in linear time (with respect to the size of \mathcal{A}).

Definition 5.25. We denote by $\mathcal{A}_{(h,k)}$ the automaton

$$\mathcal{A}_{(h,k)} = \langle \{0, g\}, Q_{(h,k)}, \delta_{(h,k)}, (0, 0), F_{(h,k)} \rangle ,$$

where the state set is $Q_{(h,k)} = \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/k\mathbb{Z}$ (mind that the second operand of the Cartesian product is $\mathbb{Z}/k\mathbb{Z}$ and not $\mathbb{Z}/\psi\mathbb{Z}$); the final-state set is $Q_{(h,k)} = R \times \mathbb{Z}/k\mathbb{Z}$ (idem); the outgoing

¹ Uniqueness is a consequence of Lemma 5.20.

transitions of every state $(s, t) \in \mathbb{Q}$ are defined as follows.

$$(s, t) \cdot 0 = \begin{cases} (s, t + 1) & \text{if } t < (k - 1) \\ \gamma_{(h,k)}^{-1}((s, t + 1)) = \left(\frac{s - h}{b^k}, 0 \right) & \text{if } t = (k - 1) \end{cases}$$

$$(s, t) \cdot g = (s + b^t, t)$$

In the remainder of Section 5.1.4, we show that the automaton $\mathcal{A}_{(h,k)}$ is isomorphic to \mathcal{A} (Theorem 5.28). The proof of this statement needs preliminary results.

Lemma 5.26. *Let (s, t) be an element of $Q_{(h,k)}$ (hence both a state of $\mathcal{A}_{(h,k)}$ and of \mathcal{P}_p^R). Let x be a letter of $\{0, g\}$. We let (s', t') and (s'', t'') denote the successors of (s, t) by x , respectively in $\mathcal{A}_{(h,k)}$ and in \mathcal{P}_p^R . Then, as states of \mathcal{P}_p^R , (s', t') and (s'', t'') are φ -equivalent.*

Proof. From the definitions of $\mathcal{A}_{(h,k)}$ and \mathcal{P}_p^R , the only case where (s', t') and (s'', t'') are not equal happens when $a = 0$ and $t = k - 1$. In this case however, we have $\gamma_{(h,k)}((s'', t'')) = (s', t')$. Applying Lemma 5.23 concludes the proof. \square

Lemma 5.27. *Every φ -equivalence class contains exactly one state of $Q_{(h,k)}$.*

Proof. Existence. We denote by C any φ -equivalence class and (s, t) its smallest element (when ordered by second component); Lemma 5.20 ensures that (s, t) is well defined. If $t \geq k$, then $\gamma_{(h,k)}^{-1}((s, t))$ is equal to $(s', t - k)$ for some s' and it holds that $0 \leq t - k < t$. From Lemma 5.23, $(s', t - k)$ is moreover φ -equivalent to (s, t) , a contradiction to the choice of (s, t) . Hence $t < k$ and $(s, t) \in Q_{(h,k)}$.

Uniqueness. Ab Absurdo. Let (s, t) and (s', t') two distinct and φ -equivalent states of \mathcal{P}_p^R such that $0 \leq t, t' < k$. From Lemma 5.20, t and t' are not equal; we assume without loss of generality that $t < t'$, hence it holds that $0 < t' - t < k$. The state $(s', t') \diamond (s, t)^{-1}$ is φ -equivalent to $(0, 0)$ and equal to $(s'', t' - t)$ for some s'' , a contradiction to the definition of (h, k) as the smallest state φ -equivalent to $(0, 0)$. \square

Now, we establish that $\mathcal{A}_{(h,k)}$ is isomorphic to \mathcal{A} .

Theorem 5.28. *Let \mathcal{P}_p^R be a Pascal automaton and \mathcal{A} a non-trivial quotient of \mathcal{P}_p^R . We write φ the automaton morphism $\mathcal{P}_p^R \rightarrow \mathcal{A}$. Among the states φ -equivalent but not equal to $(0, 0)$, we denote by (h, k) the state with the smallest second component. Then, the automaton \mathcal{A} is isomorphic to $\mathcal{A}_{(h,k)}$.*

Proof. We define the function ξ .

$$\begin{aligned} \xi : Q_{\mathcal{A}} &\longrightarrow Q_{(h,k)} \\ q &\longmapsto \text{the unique state of } \varphi^{-1}(q) \cap Q_{(h,k)} \end{aligned}$$

Lemma 5.27 yields that ξ is well defined. Since the inverse images by φ of states of \mathcal{A} are disjoint, ξ is injective. It is also surjective since every state (s, t) of $Q_{(h,k)}$ is the image by ξ of $\varphi((s, t))$.

It remains to show that ξ is an automaton morphism $\mathcal{A} \rightarrow \mathcal{A}_{(h,k)}$. The state $(0, 0)$ is necessarily mapped by φ to $i_{\mathcal{A}}$, the initial state of \mathcal{A} , and belongs to $Q_{(h,k)}$ hence $\xi(i_{\mathcal{A}}) = (0, 0)$ which is the initial state of $\mathcal{A}_{(h,k)}$.

Similarly, φ preserves the final and non-final status of states hence so does ξ .

Finally, let $q \xrightarrow{a} q'$ be a transition of \mathcal{A} and let us show that $\xi(q) \xrightarrow{a} \xi(q')$ in $\mathcal{A}_{(h,k)}$. We denote by (s', t') and (s'', t'') the successors of $\xi(q)$ by x in $\mathcal{A}_{(h,k)}$ and \mathcal{P}_p^R , respectively. Since $\xi(q)$ belongs to $\varphi^{-1}(q)$ and since φ is a morphism (s'', t'') belongs to $\varphi^{-1}(q')$. Then, Lemma 5.26 implies that (s', t') belongs to $\varphi^{-1}(q')$ as well. Since (s', t') also belongs to $Q_{(h,k)}$, it holds that $\xi(q') = (s', t')$. \square

5.1.5. *Decision algorithm.* Let $\mathcal{A} = \langle Q, A_b, \delta, i, T \rangle$ be an automaton fixed in the following. We will describe here an algorithm to decide whether \mathcal{A} is the quotient of a Pascal automaton.

▷ Step 0 (Requirements). Every quotient of a Pascal automaton is necessarily a group automaton (Corollary 5.10) and necessarily accepts by value. It may be verified in linear time whether \mathcal{A} satisfies these two conditions. If it does not, reject \mathcal{A} . Moreover, we allow to take transitions (labelled by 0) backwards; computing these transitions may be done in one traversal of \mathcal{A} .

▷ Step 1 (Simplification). Let B be the alphabet $\{0, g\}$. Let us compute an automaton \mathcal{A}' over B . First, the automaton $\mathcal{A} = \langle Q, A_b, \delta, i, F \rangle$, whose alphabet is A_b , is transformed in the automaton $\mathcal{B} = \langle Q, A_b \cup B, \delta', i, F \rangle$, by adding transitions labelled by g : the transition $s \xrightarrow{g} s'$ is added in \mathcal{B} if and only if $s \xrightarrow{10^{-1}} s'$ exists in \mathcal{A} . Second, we ensure that no information is lost in the simplification process. From Property 5.18, if the automaton \mathcal{A} is the quotient of a Pascal automaton, the following equation necessarily holds (if it does not, reject \mathcal{A}):

$$\forall s \in Q, \forall a \in A_b \quad s \cdot a = s \cdot (g^a 0) \text{ in automaton } \mathcal{B} \quad (5.13)$$

Verifying that this equation is satisfied requires to run one test for every letter a and every state s , that is one test for each transition of \mathcal{A} . It is then sufficient that each test is executed in constant time in order for the general verification of (5.13) to be run in linear time. Keeping intermediary results allows to comply to this condition. Third, we delete from \mathcal{B} the transitions labelled by digits other than 0 or g and denote the result by \mathcal{A}' .

Running example. We consider an automaton \mathcal{A}_3 over the alphabet $A_3 = \{0, 1, 2\}$, hence the base in $b = 3$. Figure 11 shows the simplified automaton \mathcal{A}'_3 . (We did not include a representation of \mathcal{A}_3 because it has 30 transitions.)

▷ Step 2 (Analysis). For the whole step 2, we assume that \mathcal{A}' is the quotient of a Pascal automaton \mathcal{P}'_p^R , in order to compute p , R , φ and (h, k) . (If it is not the case, these parameters have no meaning and \mathcal{A} will be rejected during Step 3.) We first use Proposition 5.21 to compute p and R :

- p is the length of the g -circuit containing the initial state;
- R is the set of the exponents r such that g^r is accepted by \mathcal{A}' .

The order ψ of p in $(\mathbb{Z}/p\mathbb{Z}, \times)$ is computed in the usual way. The parameter (h, k) of the quotient is computed thanks to Lemma 5.22: we look for the *mixed circuit* $g^s 0^t$ with the smallest positive t ; then we write $(h, k) = (s, t)$.

Running example. Figure 12 highlights the g -circuit containing the initial state. It has length 5 (as have all other g -circuits), hence $p = 5$ and final states are at index 0 and 3, hence $R = \{0, 3\}$. Figure 13 shows the mixed circuit with the smallest number of 0's (and

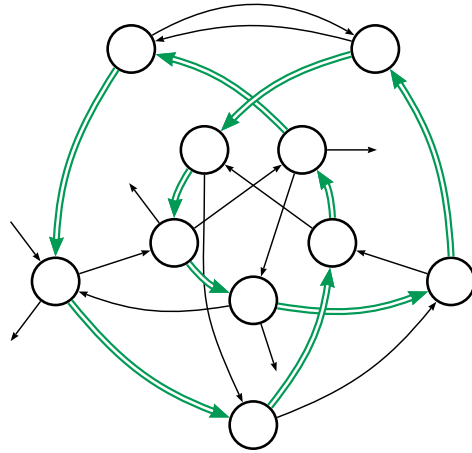


FIGURE 11. The simplified automaton \mathcal{A}'_3

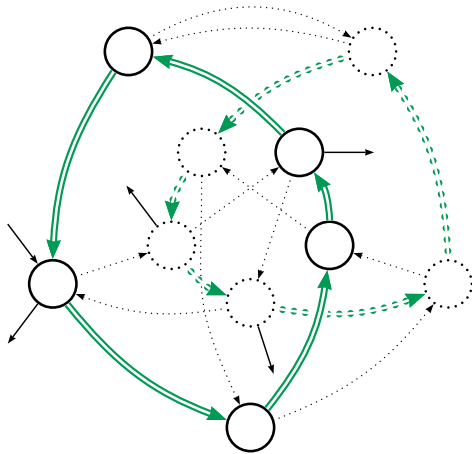


FIGURE 12. The g -circuit in \mathcal{A}'_3 containing the initial state

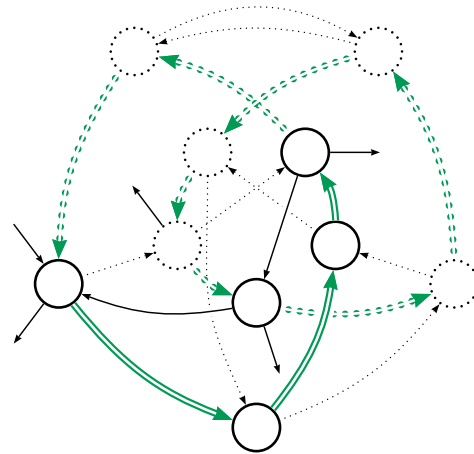


FIGURE 13. The smallest mixed circuit in \mathcal{A}'_3

Base	$b = 3$
Period	$p = 5$
Remainder set	$R = \{0, 3\}$
Parameter of the quotient	$(h, k) = (3, 2)$
Order of b in the group $(\mathbb{Z}/p\mathbb{Z}, \times)$	$\psi = 4$

TABLE 14. Summary of the parameters

(i)	$(s, 0) \cdot 0 = (s, 1)$	
(ii)	$(s, 1) \cdot 0 = (4s - 2, 1)$	$= (\frac{s-h}{p^k}, 0)$
(iii)	$(s, 0) \cdot g = (s + 1, 0)$	$= (s + p^0, 0)$
(iv)	$(s, 1) \cdot g = (s + 3, 1)$	$= (s + p^1, 1)$

TABLE 15. Transition function of $\mathcal{A}_{(h,k)}$

in this case it is the only one). Since it is labelled by the word $g^3 0^2$, the parameter of the quotient is $(h, k) = (3, 2)$. Table 14 sums up all relevant parameters.

▷ Step 3 (Verifications). From Theorem 5.28, if \mathcal{A} is the quotient of a Pascal automaton, it is isomorphic to $\mathcal{A}_{(h,k)}$. Hence, build $\mathcal{A}_{(h,k)}$ using Definition 5.25 and tests isomorphism to \mathcal{A}' with a simple traversal.

Running example. Table 15 gives the transition function of the automaton $\mathcal{A}_{(3,2)}$ (cf. Definition 5.25). Figure 16 shows the verification process, the isomorphism is built by visiting each transition of \mathcal{A}' and colouring the visited state by the corresponding state of $\mathcal{A}_{(3,2)}$.

5.2. Linear algorithm to solve Problem 5.1.

Theorem 5.29. *Let \mathcal{A} be a minimal automaton with n transitions. It can be decided in time $\mathcal{O}(bn)$ whether \mathcal{A} satisfies Conditions (UP*).*

Proof. A simple traversal is sufficient to check whether \mathcal{A} satisfies (UP0). Condition (UP1) is assumed to be satisfied by \mathcal{A} . The verification of the other conditions require to compute the component graph of \mathcal{A} ; this can be done in time $\mathcal{O}(bn)$ using classical algorithms (Theorem 2.5). Verifying (UP2) can be done in linear time thanks to the algorithm previously presented in Section 5.1. Verifying (UP3) requires a simple test for each of the affected s.c.c.'s.

Finally, condition (UP4) can be verified in the following way. Let C be an s.c.c. of type one and D the s.c.c. of type two that descends from it. We then define the function f as follows; it is the only function that may realise an embedding. Every state x in C is mapped to the unique state $f(x)$ in D such that

$$x \xrightarrow{1} y \quad \text{and} \quad f(x) \xrightarrow{1} y$$

(since D is a UP-atomic automaton, it is a group automaton, hence y and $f(x)$ are uniquely defined). Once f has been computed, checking whether f is an embedding function can be done in time $\mathcal{O}(bn)$. □

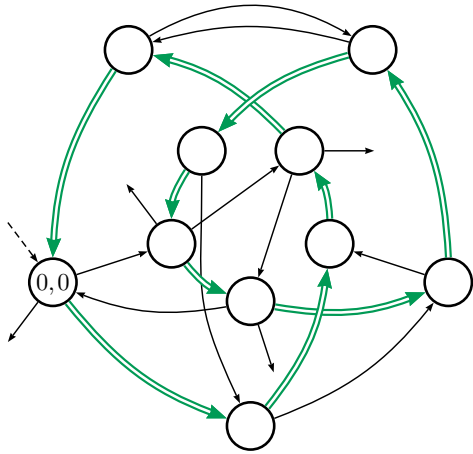
6. CONCLUSION AND FUTURE WORK

The main result of this article is stated again below. It follows directly from Theorems 4.3 and 5.29, shown in Sections 4 and 5.2 respectively.

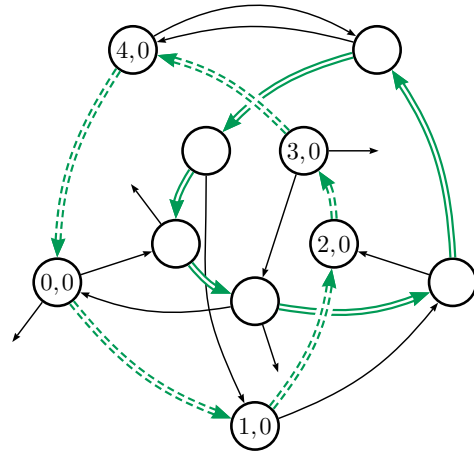
Theorem 1.1. *Let $b > 1$ be an integer. We assume that number representations are written in base b and with the least significant digit first. Given a minimal DFA \mathcal{A} with n states, it is decidable in time $\mathcal{O}(bn)$ whether \mathcal{A} accepts an ultimately periodic set.*

Corollary 1.2. *Given a DFA \mathcal{A} with n states, it is decidable in time $\mathcal{O}((bn) \log n)$ whether \mathcal{A} accepts an ultimately periodic set.*

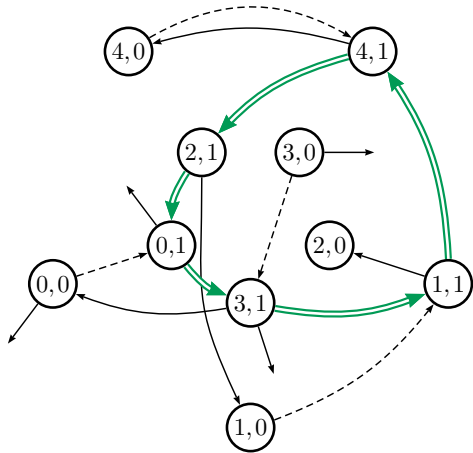
These results almost close the complexity question raised by Honkala's problem, when one writes representations LSDF. Two improvements are natural: getting rid, in Theorem 1.1, either of the condition of minimality, or of the condition of determinism. We are rather optimistic for a positive answer to the first one, by performing some kind of *partial* minimisation (which would run in linear time). For instance, the algorithm given in



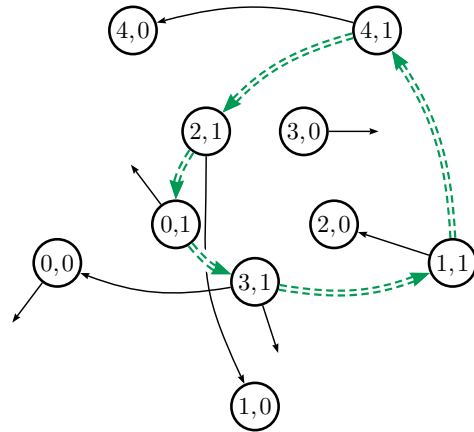
(A) The initial state is coloured by $(0, 0)$



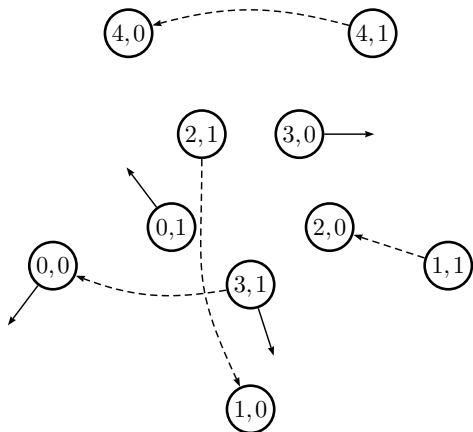
(B) Applying rule (iii) of Table 15:
 $(s, 0) \xrightarrow{a} (s + 1, 0)$



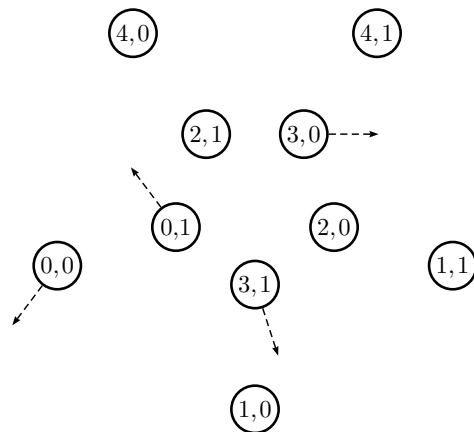
(C) Applying rule (i) of Table 15:
 $(s, 0) \xrightarrow{a} (s, 1)$



(D) Applying rule (iv) of Table 15:
 $(s, 1) \xrightarrow{a} (s + 3, 1)$



(E) Applying rule (ii) of Table 15:
 $(s, 1) \xrightarrow{0} (4s - 2, 0)$



(F) Verifying that (s, t) is final if and only if
 $s \in R = \{0, 3\}$

FIGURE 16. Verifications

Section 5.1 solves (a special case) even if the input automaton is not quite minimal. On the other hand, devising conditions similar to (UP^*) for non-deterministic automata seems to be much more difficult.

As for extensions, we are fairly confident that an approach similar to what we do here can be used for non-standard numeration systems, or at least for a family of U-systems to be identified. It would also be interesting to find an equivalent of Conditions (UP^*) for automata that accept rational subsets of \mathbb{N}^d .

The same questions arise in the case where number representations are written with the most significant digit first. We are hopeful that some of them can be addressed by building upon the recent work of Boigelot et al. [BMMR17].

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