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GAME CHARACTERIZATIONS AND LOWER CONES IN THE WEIHRAUCH DEGREES

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Abstract. We introduce a parametrized version of the Wadge game for functions and show that each lower cone in the Weihrauch degrees is characterized by such a game. These parametrized Wadge games subsume the original Wadge game, the eraser and backtrack games as well as Semmes's tree games. In particular, we propose that the lower cones in the Weihrauch degrees are the answer to Andretta's question on which classes of functions admit game characterizations. We then discuss some applications of such parametrized Wadge games. Using machinery from Weihrauch reducibility theory, we introduce games characterizing every (transfinite) level of the Baire hierarchy via an iteration of a pruning derivative on countably branching trees.

1. Introduction

The interest in characterizations of classes of functions in descriptive set theory via infinite games began with a re-reading of the seminal work of Wadge [Wad83], who introduced a game in order to analyze a notion of reducibility—*Wadge reducibility*—between subsets of the Baire space. In the variant—which by a slight abuse we call the *Wadge game*—two players, I and II, are given a partial function $f :\subseteq \mathbb{N}^\mathbb{N} \to \mathbb{N}^\mathbb{N}$ and play with perfect information for ω rounds. In each run of this game, at each round player I first picks a natural number and player II responds by either picking a natural number or passing, although she must pick natural numbers at infinitely many rounds. Thus, after ω rounds I and II build elements $x \in \mathbb{N}^\mathbb{N}$ and $y \in \mathbb{N}^\mathbb{N}$, respectively, and II wins the run if and only if $x \notin \mathrm{dom}(f)$ or $f(x) = y$.

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It is an easy consequence of the original work of Wadge that this game *characterizes* the continuous functions in the following sense.

Theorem 1.1. *A partial function $f : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ is relatively continuous iff player* II *has a winning strategy in the Wadge game for f.*

By giving player II more freedom in how she builds her sequence $y \in \mathbb{N}^{\mathbb{N}}$, one can obtain games characterizing larger classes of functions. For example, in the *eraser game* (implicit in [Dup01]) characterizing the Baire class 1 functions, player II is allowed to erase past moves, the rule being that she is only allowed to erase each position of her output sequence finitely often. In the *backtrack game* (implicit in [Van78]) characterizing the functions which preserve the class of $\mathbf{\Sigma}^0_2$ sets under preimages, player II is allowed to erase *all* of her past moves at any given round, the rule in this case being that she only do this finitely many times.

In his PhD thesis [Sem09], Semmes introduced the *tree game* characterizing the class of Borel functions in the Baire space. Player I plays as in the Wadge game and therefore builds some $x \in \mathbb{N}^{\mathbb{N}}$ after ω rounds, but at each round n player II now plays a finite *labeled tree*, i.e., a pair (T_n, ϕ_n) consisting of a finite tree $T_n \subseteq \mathbb{N}^{<\mathbb{N}}$ and a function $\phi_n : T_n \setminus \{\langle\rangle\} \to \mathbb{N}$, where $\langle\rangle$ denotes the empty sequence. The rules are that $T_n \subseteq T_{n+1}$ and $\phi_n \subseteq \phi_{n+1}$ must hold for each n, and that the *final* labeled tree $(T, \phi) = (\bigcup_{n\in\mathbb{N}} T_n, \bigcup_{n\in\mathbb{N}} \phi_n)$ must be an infinite tree with a unique infinite branch. Player II then wins if the sequence of labels along this infinite branch is exactly $f(x)$. By providing suitable extra requirements on the structure of the final tree, Semmes was able to obtain a game characterizing the Baire class 2 functions, and although this is not done explicitly in [Sem09], it is not difficult to see that restrictions of the tree game also give his *multitape* and *multitape eraser* games from [Sem07], which respectively characterize the class of functions which preserve $\mathbf{\Sigma}^0_3$ under preimages and the class of functions for which the preimage of any $\mathbf{\Sigma}^0_2$ set is a $\mathbf{\Sigma}^0_3$ set.

As examples of applications of these games, Semmes found a new proof of a theorem of Jayne and Rogers characterizing the class of functions which preserve $\mathbf{\Sigma}^0_2$ under preimages and extended this theorem to the classes characterized by the multitape and multitape eraser games, by performing a detailed analysis of the corresponding game in each case.

Given the success of such game characterizations, in [And07] Andretta raised the question of which classes of functions admit a characterization by a suitable game. Significant progress towards an answer was made by Motto Ros in [Mot11]: Starting from a general definition of a reduction game, he shows how to construct new games from existing ones in ways that mirror the typical constructions of classes of functions (e.g., piecewise definitions, composition, pointwise limits). In particular, Motto Ros's results show that all the usual subclasses of the Borel functions studied in descriptive set theory admit game characterizations.

In order to study the classes of functions characterizable by games, we will use the language of Weihrauch reducibility theory. This reducibility (in its modern form) was introduced by Gherardi and Marcone [GM09] and Brattka and Gherardi [BG11b, BG11a] based on earlier work by Weihrauch on a reducibility between sets of functions analogous to Wadge reducibility [Wei92a, Wei92b].

We will show that game characterizations and Weihrauch degrees correspond closely to each other. We can thus employ the results and techniques developed for Weihrauch reducibility to study function classes in descriptive set theory, and vice versa. In particular, we can use the algebraic structure available for Weihrauch degrees [HP13, BP18] to obtain game characterizations for derived classes of functions from game characterizations for the original classes, similar to the constructions found by Motto Ros [Mot11]. As a further feature of our work, we should point out that our results apply to the effective setting firsthand, and are then lifted to the continuous setting via relativization. They thus follow the recipe laid out by Moschovakis, e.g., in [Mos09, Section 3I].

While the traditional scope of descriptive set theory is restricted to Polish spaces, their subsets, and functions between them, these restrictions are immaterial for the approach presented here. Our results naturally hold for multi-valued functions between represented spaces. As such, this work is part of a larger development to extend descriptive set theory to a more general setting, cf., e.g., [dB13, PdB15, Peq15, KP14, Pau14].

After recalling and preparing some notions and results related to Weihrauch reducibility in Section 2, we introduce our parametrized version of the Wadge game in Section 3 and discuss applications. In Section 4, we introduce a notion of pruning derivative for countably-branching trees, and show how this gives rise to games characterizing each (transfinite) level of the Baire hierarchy.

We will freely use standard concepts and notation from descriptive set theory, referring to [Kec95] for an introduction.

2. Represented spaces and Weihrauch reducibility

Represented spaces and continuous or computable maps between them form the setting for computable analysis. The classical reference for computable analysis is the textbook by Weihrauch [Wei00]; for a comprehensive introduction more in line with the style of this paper we refer to [Pau16].

A *represented* space $\mathbb{X} = (X, \delta_\mathbb{X})$ is given by a set X and a partial surjection $\delta_\mathbb{X} :\subseteq \mathbb{N}^\mathbb{N} \to X$. We will always consider $\mathbb{N}^\mathbb{N}$ as represented by id, and $\mathbb{N}$ as represented by the function $\delta_\mathbb{N}(p) = n$ iff $p = 0^n 1^\mathbb{N}$. Given a represented space $\mathbb{X}$ and $n \in \mathbb{N}$, $\mathbb{X}^n$ is the represented space of n -tuples represented in the natural way since $\mathbb{N}^\mathbb{N}$ inherits particularly nice tupling functions $\ulcorner \cdot \urcorner$ of all finite arities from $\mathbb{N}$. The *coproduct* of a family of represented spaces $\{\mathbb{Y}_x\,;\, x \in \mathbb{X}\}$ indexed by $\mathbb{X}$ is the represented space $\coprod_{x\in\mathbb{X}} \mathbb{Y}_x$ composed of pairs (x, y) such that $y \in \mathbb{Y}_x$, with the representation given in the natural way, letting a name for (x, y) be a $\mathbb{N}^\mathbb{N}$ -pair of a name for x and one for y. We denote by $\mathbb{X}^{<\mathbb{N}}$ the represented space $\coprod_{n\in\mathbb{N}} \mathbb{X}^n$; thus $\mathbb{N}^{<\mathbb{N}}$ can intuitively be seen as a represented space such that σ is named by p iff p encodes the length $|\sigma|$ of σ as well as the $|\sigma|$ elements of σ Finally, $\mathbb{X}^\mathbb{N}$ is the represented space in which tuples $\langle x_n \rangle_{n\in\mathbb{N}}$ are named by infinite tuples composed of a name for each x_n —recall that $\mathbb{N}^\mathbb{N}$ has a countable tupling function $\ulcorner \cdot \urcorner : (\mathbb{N}^\mathbb{N})^\mathbb{N} \to \mathbb{N}^\mathbb{N}$ given by $\ulcorner p_n \urcorner_{n\in\mathbb{N}} = p$ iff $p(\ulcorner n, k \urcorner) = p_n(k)$ for each $n, k \in \mathbb{N}$. This tupling function is naturally associated to ω-many corresponding projections: for each $n \in \mathbb{N}$ and $p \in \mathbb{N}^\mathbb{N}$, we define $(p)_n \in \mathbb{N}^\mathbb{N}$ by $(p)_n(k) = p(\ulcorner n, k \urcorner)$.

A (multi-valued) function between represented spaces is just a (multi-valued) function on the underlying sets. We say that a partial function $F :\subseteq \mathbb{N}^\mathbb{N} \to \mathbb{N}^\mathbb{N}$ is a *realizer* for a multi-valued function $f :\subseteq \mathbb{X} \rightrightarrows \mathbb{Y}$, denoted by $F \vdash f$, if $\delta_\mathbb{Y}(F(p)) \in f(\delta_\mathbb{X}(p))$ for all $p \in \mathrm{dom}(f\delta_\mathbb{X})$. Then, given a class Λ of partial functions in $\mathbb{N}^\mathbb{N}$, we say $f :\subseteq \mathbb{X} \rightrightarrows \mathbb{Y}$ is in $(\delta_\mathbb{X}, \delta_\mathbb{Y})$-$\Lambda$ if it has a realizer in Λ. When $\delta_\mathbb{X}$ and $\mathbb{Y}$ are clear from the context, we will just say f is in Λ; thus we have computable, continuous, etc., functions between represented spaces.

Represented spaces and continuous functions (in the sense just defined) generalize Polish spaces and continuous functions (in the usual sense). Indeed, let (X, τ) be some Polish space, and fix a countable dense sequence $\langle a_i\,;\, i \in \mathbb{N}\rangle$ and a compatible metric d. Now define $\delta_{\mathbb{X}}$ by $\delta_{\mathbb{X}}(p) = x$ iff $d(a_{p(i)}, x) < 2^{-i}$ holds for all $i \in \mathbb{N}$. In other words, we represent a point by a sequence of basic points converging to it with a prescribed speed. It is a foundational result in computable analysis that the notion of continuity for the represented space $(X, \delta_{\mathbb{X}})$ coincides with that for the Polish space (X, τ).

Definition 2.1. Let f and g be partial multi-valued functions between represented spaces. We say that f is *Weihrauch-reducible* to g, in symbols $f \leq_{\mathrm{W}} g$, if there are computable functions $K :\subseteq \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ and $H :\subseteq \mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ such that whenever $G \vdash g$, the function $F := (p \mapsto H(p, G(K(p))))$ is a realizer for f. If there are computable functions $K, H :\subseteq \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ such that whenever $G \vdash g$ then $HGK \vdash f$, then we say that f is *strongly Weihrauch-reducible* to g ($f \leq_{\mathrm{sW}} f$). We write $f \leq^{\mathrm{t}}_{\mathrm{W}} g$ and $f \leq^{\mathrm{t}}_{\mathrm{sW}} g$ for the variations where “computable” is replaced with “continuous”.

In this paper, we almost always denote (multi- or single-valued) function composition by juxtaposition (e.g., as we did for HGK in Definition 2.1). However, because some of the functions we use have English words for names, for the sake of clarity, when talking about compositions involving such functions we will use the symbol $\circ$ to denote composition.

We refer the reader to [BGP17] for a recent comprehensive survey on Weihrauch reducibility.

A multi-valued function f *tightens* g or is a *tightening* of g, denoted by $f \preceq g$, if $\mathrm{dom}(g) \subseteq \mathrm{dom}(f)$ and $f(x) \subseteq g(x)$ whenever $x \in \mathrm{dom}(g)$, cf. [Wei08, Definition 7]. The following result illustrates how the notion of tightening is a good tool for expressing concepts in Weihrauch reducibility theory. First, for any represented space $\mathbb{X}$, let $\Delta_{\mathbb{X}} : \mathbb{X} \to \mathbb{X} \times \mathbb{X}$ be the total computable function given by $\Delta_{\mathbb{X}}(x) = (x, x)$

Proposition 2.2 (Folklore; cf., e.g., [Pau12, Chapter 4])**.** *Let $f :\subseteq \mathbb{X} \rightrightarrows \mathbb{Y}$ and $g :\subseteq \mathbb{Z} \rightrightarrows \mathbb{W}$.*

(1) *The following are equivalent:*
  (a) *$f \leq_{\mathrm{sW}} g$ ($f \leq^{\mathrm{t}}_{\mathrm{sW}} g$)*
  (b) *there exist computable (continuous) $k :\subseteq \mathbb{X} \rightrightarrows \mathbb{Z}$ and $h :\subseteq \mathbb{W} \rightrightarrows \mathbb{Y}$ such that $f \succeq hgk$.*

(2) *The following are equivalent:*
  (a) *$f \leq_{\mathrm{W}} g$ ($f \leq^{\mathrm{t}}_{\mathrm{W}} g$);*
  (b) *there exist computable (continuous) $k :\subseteq \mathbb{X} \rightrightarrows \mathbb{Z}$ and $h :\subseteq \mathbb{X} \times \mathbb{W} \rightrightarrows \mathbb{Y}$ such that $f \succeq h(\mathrm{id}_{\mathbb{X}} \times gk)\Delta_{\mathbb{X}}$;*
  (c) *there exist computable (continuous) $k :\subseteq \mathbb{X} \rightrightarrows \mathbb{Z}$ and $h :\subseteq \mathbb{X} \times \mathbb{W} \rightrightarrows \mathbb{Y}$ such that $f = h(\mathrm{id}_{\mathbb{X}} \times gk)\Delta_{\mathbb{X}}$.*

Although the Weihrauch degrees form a very rich algebraic structure (cf., e.g., [BGH15, BP18, HP13] for surveys covering this aspect of the Weihrauch lattice), we only need two operations on the Weihrauch degrees, parallelization and sequential composition. Given a map $f :\subseteq \mathbb{X} \rightrightarrows \mathbb{Y}$ between represented spaces, its *parallelization* is the map $\widehat{f} :\subseteq \mathbb{X}^{\mathbb{N}} \rightrightarrows \mathbb{Y}^{\mathbb{N}}$ given by $\langle y_n\rangle_{n\in\mathbb{N}} \in \widehat{f}(\langle x_n\rangle_{n\in\mathbb{N}})$ iff $y_n \in f(x_n)$ for each $n \in \mathbb{N}$. We say that f is *parallelizable* if $f \equiv_{\mathrm{W}} \widehat{f}$. It is not hard to see that parallelization is a closure operator in the Weihrauch degrees. Rather than defining the sequential composition operator $\star$ explicitly as in [BP18], we will make use of the following characterization:

Theorem 2.3 (Brattka & Pauly [BP18])**.** $f \star g \equiv_{\mathrm{W}} \max_{\leq_{\mathrm{W}}}\{f'g' \, ; \, f' \leq_{\mathrm{W}} f \;\wedge\; g' \leq_{\mathrm{W}} g\}$.

2.1. **Transparent cylinders.** In this section we study properties of *transparent cylinders*, which will play a central role in our parametrized version of the Wadge game.

Definition 2.4 (Brattka & Gherardi [BG11b])**.** Let $f : \subseteq \mathbb{X} \rightrightarrows \mathbb{Y}$. We call f

(1) a *cylinder* if $\mathrm{id}_{\mathbb{N}^\mathbb{N}} \times f \leq_{\mathrm{sW}} f$;
(2) *transparent* iff for any computable or continuous $g : \subseteq \mathbb{Y} \rightrightarrows \mathbb{Y}$ there is a computable or continuous, respectively, $f_g : \subseteq \mathbb{X} \rightrightarrows \mathbb{X}$ such that $f f_g \preceq g f$.

The transparent (single-valued) functions on the Baire space were studied by de Brecht under the name *jump operator* in [dB14]. One of the reasons for their relevance is that they induce endofunctors on the category of represented spaces, which in turn can characterize function classes in DST [PdB13]. The term *transparent* was coined in [BGM12]. Our extension of the concept to multi-valued functions between represented spaces is rather straightforward, but requires the use of the notion of tightening. Note that if $f : \subseteq \mathbb{X} \rightrightarrows \mathbb{Y}$ is transparent, then for every $y \in \mathbb{Y}$ there is some $x \in \mathrm{dom}(f)$ with $f(x) = \{y\}$, i.e., f is *slim* in the terminology of [BGM12, Definition 2.7].

Two examples of transparent cylinders which will be relevant in what follows are the functions $\lim$ and $\lim_\Delta : \subseteq \mathbb{N}^\mathbb{N} \to \mathbb{N}^\mathbb{N}$ defined by letting $\lim(p) = \lim_{n \in \mathbb{N}} (p)_n$ and letting $\lim_\Delta(p)$ be the restriction of $\lim$ to the domain $\{p \in \mathbb{N}^\mathbb{N} \, ; \, \exists n \in \mathbb{N} \forall k \geq n ((p)_k = (p)_n)\}$. To see a further example, related to Semmes's tree game characterizing the Borel functions, one first needs to define the appropriate represented space of labeled trees. For this, it is best to work in a quotient space of labeled trees under bisimilarity. The resulting quotient space can be thought of as the space of labeled trees in which the order of the subtrees rooted at the children of a node, and possible repetitions among these subtrees, are abstracted away. Then the map Prune, which removes from (any representative of the equivalence class of) a labeled tree which has one infinite branch all of the nodes which are not part of that infinite branch, is a transparent cylinder. This idea will be developed in full in Section 4 below.

Proposition 2.5. *Let $f : \subseteq \mathbb{X} \rightrightarrows \mathbb{Y}$ and $g : \subseteq \mathbb{Y} \rightrightarrows \mathbb{Z}$ be cylinders. If f is transparent then gf is a cylinder and $gf \equiv_{\mathrm{W}} g \star f$. Furthermore, if g is also transparent then so is gf.*

Proof. (gf is a cylinder) As g is a cylinder, there are computable $h : \subseteq \mathbb{Z} \rightrightarrows \mathbb{N}^\mathbb{N} \times \mathbb{Z}$ and $k : \subseteq \mathbb{N}^\mathbb{N} \times \mathbb{Y} \rightrightarrows \mathbb{Y}$ such that $\mathrm{id}_{\mathbb{N}^\mathbb{N}} \times g \succeq hgk$. Likewise, there are computable $h' : \subseteq \mathbb{Y} \rightrightarrows \mathbb{N}^\mathbb{N} \times \mathbb{Y}$ and $k' : \subseteq \mathbb{N}^\mathbb{N} \times \mathbb{X} \rightrightarrows \mathbb{X}$ such that $\mathrm{id}_{\mathbb{N}^\mathbb{N}} \times f \succeq h'fk'$. As composition respects tightening [PZ13, Lemma 2.4(1b)], we conclude that $(\mathrm{id}_{\mathbb{N}^\mathbb{N}} \times g)(\mathrm{id}_{\mathbb{N}^\mathbb{N}} \times f) = \mathrm{id}_{\mathbb{N}^\mathbb{N}} \times (gf) \succeq hgkh'fk'$. Note that $kh' : \subseteq \mathbb{Y} \rightrightarrows \mathbb{Y}$ is computable, and as f is transparent, there is some computable $f_{kh'} : \subseteq \mathbb{X} \rightrightarrows \mathbb{X}$ with $kh'f \succeq f f_{kh'}$. But then $\mathrm{id}_{\mathbb{N}^\mathbb{N}} \times (gf) \succeq hgkh'fk' \succeq hgff_{kh'}k'$, thus h and $f_{kh'}k'$ witness that $\mathrm{id}_{\mathbb{N}^\mathbb{N}} \times (gf) \leq_{\mathrm{sW}} gf$, i.e., gf is a cylinder.

($gf \equiv_{\mathrm{W}} g \star f$) The direction $gf \leq_{\mathrm{W}} g \star f$ is immediate. Let f' and g' be such that $f' \leq_{\mathrm{W}} f$, $g' \leq_{\mathrm{W}} g$, and $g'f'$ is defined. We need to show that $g'f' \leq_{\mathrm{W}} gf$. As g and f are cylinders, we find that already $g' \leq_{\mathrm{sW}} g$ and $f' \leq_{\mathrm{sW}} f$. Let h, k witness the former and h', k' the latter. We conclude $hgkh'fk' \preceq g'f'$. As above, there then is some computable $f_{kh'}$ with $kh'f \succeq ff_{kh'}$. Then h and $f_{kh'}k'$ witness that $g'f' \leq_{\mathrm{sW}} gf$.

Now suppose that g is also transparent.

(gf is transparent) Let $h : \subseteq \mathbb{Z} \rightrightarrows \mathbb{Z}$ be computable. By assumption that g is transparent, there is some computable $g_h : \subseteq \mathbb{Y} \rightrightarrows \mathbb{Y}$ such that $gg_h \preceq hg$. Then there is some computable

$f_h :\subseteq \mathbb{X} \rightrightarrows \mathbb{X}$ with $ff_h \preceq g_h f$. As composition respects tightening [PZ13, Lemma 2.4.1.b], we find that $hgf \succeq gg_h f \succeq gff_h$, which is what we need. □

Definition 2.6. Given a function $f :\subseteq A \rightrightarrows B$ and $C \subseteq B$, the *corestriction* of f to C, denoted $f\lfloor C$, is the restriction of f to domain $\{x \in \mathrm{dom}(f)\,;\, f(x) \subseteq C\}$. This notion extends to functions between represented spaces in a natural way. A represented space $(X, \delta_{\mathbb{X}})$ is a *subspace* of (Y, δ_Y), denoted $(X, \delta_{\mathbb{X}}) \subseteq (Y, \delta_Y)$, if $X \subseteq Y$ and $\delta_{\mathbb{X}} = \delta_Y\lfloor X$.

Proposition 2.7. *If $f :\subseteq \mathbb{X} \rightrightarrows \mathbb{Y}$ and $\mathbb{Z} \subseteq \mathbb{W} \subseteq \mathbb{Y}$, then $f\lfloor\mathbb{Z} \leq_{\mathrm{sW}} f\lfloor\mathbb{W}$.*

Proposition 2.8. *Any corestriction of a transparent map is transparent.*

Proof. Let $f :\subseteq \mathbb{X} \rightrightarrows \mathbb{Y}$ be transparent, and let $\mathbb{Z}$ be a subspace of $\mathbb{Y}$. Let $g :\subseteq \mathbb{Z} \rightrightarrows \mathbb{Z}$ be computable. Then $g :\subseteq \mathbb{Y} \rightrightarrows \mathbb{Y}$, and therefore there exists a computable $f_g :\subseteq \mathbb{X} \rightrightarrows \mathbb{X}$ such that $ff_g \preceq gf$. Note that $\mathrm{dom}(gf) \subseteq \mathrm{dom}((f\lfloor\mathbb{Z})f_g)$. Indeed, if $x \in \mathrm{dom}(gf)$, then $ff_g(x) \subseteq gf(x) \subseteq \mathbb{Z}$, so $f_g(x) \subseteq \mathrm{dom}(f\lfloor\mathbb{Z})$ and therefore $x \in \mathrm{dom}((f\lfloor\mathbb{Z})f_g)$ as desired. From this it immediately follows that $((f\lfloor\mathbb{Z})f_g)\restriction\mathrm{dom}(gf) = (ff_g)\restriction\mathrm{dom}(gf)$, from which we conclude $(f\lfloor\mathbb{Z})f_g \preceq g(f\lfloor\mathbb{Z})$. □

Theorem 2.9. *Any multi-valued function between represented spaces is strongly-Weihrauch-equivalent to a multi-valued function on $\mathbb{N}^{\mathbb{N}}$.*

Proof. Let $f :\subseteq \mathbb{X} \rightrightarrows \mathbb{Y}$ be given. Define $f' :\subseteq \mathbb{N}^{\mathbb{N}} \rightrightarrows \mathbb{N}^{\mathbb{N}}$ by $\mathrm{dom}(f') = \mathrm{dom}(f\delta_{\mathbb{X}})$ and $q \in f'(p)$ iff $\delta_{\mathbb{Y}}(q) \in f\delta_{\mathbb{X}}(p)$. To see that $f \equiv_{\mathrm{sW}} f'$, first suppose $F \vdash f$, i.e., for any $p \in \mathrm{dom}(f\delta_{\mathbb{X}})$ we have $\delta_{\mathbb{Y}}F(p) \in f\delta_{\mathbb{X}}(p)$. Then $F(p) \in f'(p)$, so $F \vdash f'$. Conversely, suppose for any $p \in \mathrm{dom}(f') = \mathrm{dom}(f\delta_{\mathbb{X}})$ we have $F(p) \in f'(p)$. But this happens iff $\delta_{\mathbb{Y}}F(p) \in f\delta_{\mathbb{X}}(p)$, i.e., $F \vdash f$. □

Definition 2.10. The space $\mathcal{M}(\mathbb{X}, \mathbb{Y})$ of the *strongly continuous functions* between represented spaces $\mathbb{X}$ and $\mathbb{Y}$ is defined by letting p be a name for f iff $p = 0^n 1 q$ and, letting M be the n^{th} Turing machine, we have

(1) for every $r \in \mathrm{dom}(f\delta_{\mathbb{X}})$ and every $r' \in \mathbb{N}^{\mathbb{N}}$ we have that M computes a $\delta_{\mathbb{Y}}$-name for an element of $f\delta_{\mathbb{X}}(r)$ when given $\ulcorner r, r' \urcorner$ as input and q as oracle;
(2) for every $x \in \mathrm{dom}(f)$ and every $y \in f(x)$ there exist a $\delta_{\mathbb{X}}$-name r for x and an $r' \in \mathbb{N}^{\mathbb{N}}$ such that M computes a $\delta_{\mathbb{Y}}$-name for y when given $\ulcorner r, r' \urcorner$ as input and q as oracle;
(3) for every $x \in \mathbb{X} \setminus \mathrm{dom}(f)$, every $\delta_{\mathbb{X}}$-name r for x, and every $r' \in \mathbb{N}^{\mathbb{N}}$, we have that M does not compute an element in $\mathrm{dom}(\delta_{\mathbb{Y}})$ when given $\ulcorner r, r' \urcorner$ as input and q as oracle.

In this case we also say that f is *strongly continuous*, and that M *strongly computes* f with oracle p. As expected, if the oracle $q \in \mathbb{N}^{\mathbb{N}}$ in the definition above is computable then f is called *strongly computable*.

Theorem 2.11 (Brattka & Pauly [BP18, Lemma 13])**.** *Every computable or continuous $f :\subseteq \mathbb{X} \rightrightarrows \mathbb{Y}$ has a strongly computable or strongly continuous, respectively, tightening $g :\subseteq \mathbb{X} \rightrightarrows \mathbb{Y}$.*

Proof. We can assign to each Turing machine M and oracle q a function $g_{M,q} :\subseteq \mathbb{X} \rightrightarrows \mathbb{Y}$ given by $\mathrm{dom}(g_{M,q}) = \{\delta_{\mathbb{X}}(r)\,;\, M$ produces an element of $\mathrm{dom}(\delta_{\mathbb{Y}})$ when run on input r with oracle $q\}$ and $g_{M,q}(x) = \{\delta_{\mathbb{Y}}(q')\,;$ there exists a $\delta_{\mathbb{X}}$-name r for x such that q' is the output of M when run with input r and oracle $q\}$. Now, if M with oracle q computes a realizer for $f :\subseteq \mathbb{X} \rightrightarrows \mathbb{Y}$, then it immediately follows that $g_{M,q} \preceq f$. Finally, to see that $g_{M,q}$ is strongly continuous or strongly computable (in case q is computable), let M' be the Turing machine which, on input $\ulcorner r, r' \urcorner$ and with oracle q, simply runs the Turing machine M on input r and oracle q. We now have that M' strongly computes $g_{M,q}$ with oracle q. $\square$

Theorem 2.12 (Brattka & Pauly, implicit in [BP18, Section 3.2])**.** *Every multi-valued function f is strongly Weihrauch-equivalent to some transparent cylinder f^{tc}, which can furthermore be taken to have codomain $\mathbb{N}^{\mathbb{N}}$.*

Proof. By Theorem 2.9, it is enough to prove the result for $f :\subseteq \mathbb{N}^{\mathbb{N}} \rightrightarrows \mathbb{N}^{\mathbb{N}}$. Let $f^{\mathrm{tc}} :\subseteq \mathcal{M}(\mathbb{N}^{\mathbb{N}}, \mathbb{N}^{\mathbb{N}}) \times \mathbb{N}^{\mathbb{N}} \rightrightarrows \mathbb{N}^{\mathbb{N}}$ be given by $f^{\mathrm{tc}}(h, x) = hf(x)$. That $f^{\mathrm{tc}} \leq_{\mathrm{sW}} f$ holds is of course immediate, and conversely we have $f \leq_{\mathrm{sW}} f^{\mathrm{tc}}$ since $\mathrm{id}_{\mathbb{N}^{\mathbb{N}}}$ has a computable name in $\mathcal{M}(\mathbb{N}^{\mathbb{N}}, \mathbb{N}^{\mathbb{N}})$, so the function $K(x) = (\mathrm{id}_{\mathbb{N}^{\mathbb{N}}}, x)$ is computable and $f = f^{\mathrm{tc}}K$. To see that f^{tc} is a cylinder, define computable $K :\subseteq \mathbb{N}^{\mathbb{N}} \times (\mathcal{M}(\mathbb{N}^{\mathbb{N}}, \mathbb{N}^{\mathbb{N}}) \times \mathbb{N}^{\mathbb{N}}) \to \mathbb{N}^{\mathbb{N}}$ and $H :\subseteq \mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}}$ by $K(p, (h, x)) = (h_p, x)$ where $h_p(y) = \ulcorner p, h(y) \urcorner$ and $H(\ulcorner p, y \urcorner) = (p, y)$. Then $Hf^{\mathrm{tc}}K(p, (h, x)) = H(h_p f(x)) = H(\ulcorner p, f(x) \urcorner) = (p, f(x))$, so $\mathrm{id}_{\mathbb{N}^{\mathbb{N}}} \times f \leq_{\mathrm{sW}} f^{\mathrm{tc}}$. Since $f \equiv_{\mathrm{sW}} f^{\mathrm{tc}}$, this suffices. Finally, to see that f^{tc} is transparent, let $g :\subseteq \mathbb{N}^{\mathbb{N}} \rightrightarrows \mathbb{N}^{\mathbb{N}}$ be continuous or computable. Define $g' :\subseteq \mathcal{M}(\mathbb{N}^{\mathbb{N}}, \mathbb{N}^{\mathbb{N}}) \times \mathbb{N}^{\mathbb{N}} \rightrightarrows \mathcal{M}(\mathbb{N}^{\mathbb{N}}, \mathbb{N}^{\mathbb{N}}) \times \mathbb{N}^{\mathbb{N}}$ by $g'(h, x) = (gh, x)$. Note that g' is continuous or computable, respectively, since g is. Furthermore, we have $f^{\mathrm{tc}}g'(h, x) = f^{\mathrm{tc}}(gh, x) = ghf(x) = gf^{\mathrm{tc}}(h, x)$, i.e., $f^{\mathrm{tc}}g' = gf^{\mathrm{tc}}$ as desired. $\square$

Definition 2.13. We say that a represented space $\mathbb{X}$ *(strongly) encodes* $\mathbb{N}^{\mathbb{N}}$ if any $f :\subseteq \mathbb{N}^{\mathbb{N}} \rightrightarrows \mathbb{N}^{\mathbb{N}}$ is (strongly) Weihrauch-equivalent to some $f' :\subseteq \mathbb{N}^{\mathbb{N}} \rightrightarrows \mathbb{X}$.

Note that if $\mathbb{X}$ has a subspace which is computably isomorphic to $\mathbb{N}^{\mathbb{N}}$, then $\mathbb{X}$ strongly encodes $\mathbb{N}^{\mathbb{N}}$.

Theorem 2.14. *Let $f :\subseteq \mathbb{X} \rightrightarrows \mathbb{Y}$ be a transparent cylinder. If $\mathbb{Z} \subseteq \mathbb{Y}$ (strongly) encodes $\mathbb{N}^{\mathbb{N}}$, then $f\lfloor\mathbb{Z}$ is transparent and (strongly) Weihrauch-equivalent to f. In the strong case, $f\lfloor\mathbb{Z}$ is also a cylinder.*

Proof. Note that $f\lfloor\mathbb{Z} \leq_{\mathrm{sW}} f$ holds for any f and $\mathbb{Z}$, and if f is transparent then so is $f\lfloor\mathbb{Z}$. Now, by Theorem 2.9, there is some $g :\subseteq \mathbb{N}^{\mathbb{N}} \rightrightarrows \mathbb{N}^{\mathbb{N}}$ which is strongly Weihrauch-equivalent to f. Therefore, by assumption, there exists $g' :\subseteq \mathbb{N}^{\mathbb{N}} \rightrightarrows \mathbb{Z}$ such that g' is (strongly) Weihrauch-equivalent to f. Since f is a transparent cylinder, there exists a computable $h :\subseteq \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ such that $g' \succeq fh$. Hence, since the codomain of g' is $\mathbb{Z}$, it follows that $g' \succeq (f\lfloor\mathbb{Z})h$, i.e., $g' \leq_{\mathrm{sW}} f\lfloor\mathbb{Z}$ and therefore f is (strongly) Weihrauch-reducible to $f\lfloor\mathbb{Z}$. Finally, if $\mathbb{Z}$ strongly encodes $\mathbb{N}^{\mathbb{N}}$, then we have $\mathrm{id}_{\mathbb{N}^{\mathbb{N}}} \times f\lfloor\mathbb{Z} \leq_{\mathrm{sW}} \mathrm{id}_{\mathbb{N}^{\mathbb{N}}} \times f \leq_{\mathrm{sW}} f \equiv_{\mathrm{sW}} f\lfloor\mathbb{Z}$, so $f\lfloor\mathbb{Z}$ is a cylinder. $\square$

3. Parametrized Wadge games

3.1. **The definition.** In order to define our parametrization of the Wadge game, first we need the following notion, which is just the dual notion to being an admissible representation as in [Sch02].

Definition 3.1. A *probe* for $\mathbb{Y}$ is a computable partial function $\pi :\subseteq \mathbb{Y} \to \mathbb{N}^{\mathbb{N}}$ such that for every computable or continuous $f :\subseteq \mathbb{Y} \rightrightarrows \mathbb{N}^{\mathbb{N}}$ there is a computable or continuous, respectively, $e :\subseteq \mathbb{Y} \rightrightarrows \mathbb{Y}$ such that $\pi e \preceq f$.

Note that a probe is always transparent, and that the partial inverse of a computable embedding from $\mathbb{N}^\mathbb{N}$ into $\mathbb{Y}$ is always a probe. The following definition generalizes the definition of a reduction game from [Mot11, Subsection 3.1], which is recovered as the special case in which all involved spaces are $\mathbb{N}^\mathbb{N}$, the map π is the identity on $\mathbb{N}^\mathbb{N}$, and Ξ is a single-valued function.

Definition 3.2. Let $\pi :\subseteq \mathbb{Y} \to \mathbb{N}^\mathbb{N}$ be a probe and $\Xi :\subseteq \mathbb{X} \rightrightarrows \mathbb{Y}$. The *Wadge game parametrized by* Ξ *and* π, in short the (Ξ, π)-*Wadge game*, is played by two players, I and II, who take turns in infinitely many rounds. At each round of a run of the game for a given function $f :\subseteq \mathbb{Z} \rightrightarrows \mathbb{W}$, player I first plays a natural number and player II then either plays a natural number or passes, as long as she plays natural numbers infinitely often. Therefore, after ω rounds player I builds $x \in \mathbb{N}^\mathbb{N}$ and II builds $y \in \mathbb{N}^\mathbb{N}$, and player II *wins* the run of the game if $x \notin \mathrm{dom}(f\delta_\mathbb{Z})$, or $y \in \mathrm{dom}(\delta_\mathbb{W}\pi\Xi\delta_\mathbb{X})$ and $\delta_\mathbb{W}\pi\Xi\delta_\mathbb{X}(y) \subseteq f\delta_\mathbb{Z}(x)$.

Thus, the (Ξ, π)-Wadge game is like the Wadge game but, instead of player I building an element $x \in \mathrm{dom}(f)$ and player II trying to build $f(x)$, now player I builds a name for some element $x \in \mathrm{dom}(f)$ and player II tries to build a name for some element $y \in \mathbb{Y}$ which is transformed by $\pi\Xi$ into a name for an element in $f(x)$. Intuitively, the idea is that the main transformation is done by Ξ, but because fixing a parametrized game entails fixing Ξ, in order for a fixed game to be able to deal with functions between different represented spaces there needs to be some map which will work as an intermediary between the target space of Ξ and the target space, say $\mathbb{W}$, of the function in question. This role will be played by the computable map $\delta_\mathbb{W}\pi$.

It is easy to see that, restricted to single-valued functions on $\mathbb{N}^\mathbb{N}$, the original Wadge game is the $(\mathrm{id}_{\mathbb{N}^\mathbb{N}}, \mathrm{id}_{\mathbb{N}^\mathbb{N}})$-Wadge game, the eraser game is the $(\lim, \mathrm{id}_{\mathbb{N}^\mathbb{N}})$-Wadge game, and the backtrack game is the $(\lim_\Delta, \mathrm{id}_{\mathbb{N}^\mathbb{N}})$-Wadge game. Semmes's tree game for the Borel functions is the $(\mathrm{Prune}, \mathrm{Label})$-Wadge game, where Label is the function extracting the infinite running label from (any representative of the equivalence class of) a pruned labeled tree consisting of exactly one infinite branch. The details of this last example, including the definitions of the represented spaces involved, will be given in Section 4 below.

Theorem 3.3. *Let* Ξ*,* π*, and* f *be as in Definition* 3.2*, and furthermore suppose* Ξ *is a transparent cylinder. Then player* II *has a (computable) winning strategy in the* (Ξ, π)*-Wadge game for* f *iff* $f \leq^\mathrm{t}_\mathrm{W} \Xi$ *(*$f \leq_\mathrm{W} \Xi$*).*

Proof. ($\Rightarrow$) Any (computable) strategy for player II gives rise to a continuous (computable) function $k :\subseteq \mathbb{N}^\mathbb{N} \to \mathbb{N}^\mathbb{N}$. If the strategy is winning, then $\delta_\mathbb{W}\pi\Xi\delta_\mathbb{X}k \preceq f\delta_\mathbb{Z}$, which implies $\delta_\mathbb{W}\pi\Xi\delta_\mathbb{X}k\delta_\mathbb{Z}^{-1} \preceq f\delta_\mathbb{Z}\delta_\mathbb{Z}^{-1} = f$. Thus the continuous (computable) maps $\delta_\mathbb{W}\pi$ and $\delta_\mathbb{X}k\delta_\mathbb{Z}^{-1}$ witness that $f \leq^\mathrm{t}_\mathrm{sW} \Xi$ ($f \leq_\mathrm{sW} \Xi$).

($\Leftarrow$) As Ξ is a cylinder, if $f \leq^\mathrm{t}_\mathrm{W} \Xi$ ($f \leq_\mathrm{W} \Xi$), then already $f \leq^\mathrm{t}_\mathrm{sW} \Xi$ ($f \leq_\mathrm{sW} \Xi$). Thus, there are continuous (computable) h, k with $h\Xi k \preceq f$. As $\delta_\mathbb{W}\delta_\mathbb{W}^{-1} = \mathrm{id}_\mathbb{W}$, we find that $\delta_\mathbb{W}\delta_\mathbb{W}^{-1}h\Xi k \preceq f$. Now $\delta_\mathbb{W}^{-1}h :\subseteq \mathbb{Y} \rightrightarrows \mathbb{N}^\mathbb{N}$ is continuous (computable), so by definition of a probe, there is some continuous (computable) $e :\subseteq \mathbb{Y} \rightrightarrows \mathbb{Y}$ with $\delta_\mathbb{W}\pi e\Xi k \preceq f$. As Ξ is transparent, there is some continuous (computable) g with $e\Xi \succeq \Xi g$, thus $\delta_\mathbb{W}\pi\Xi gk \preceq f$. As $gk :\subseteq \mathbb{Z} \rightrightarrows \mathbb{X}$ is continuous (computable), it has some (continuous) computable realizer $K :\subseteq \mathbb{N}^\mathbb{N} \to \mathbb{N}^\mathbb{N}$. By Theorem 1.1, player II has a winning strategy in the Wadge game for K, and it is easy to see that this strategy also wins the (Ξ, π)-Wadge game for f for her. $\square$

Corollary 3.4. *Let Ξ and Ξ' be transparent cylinders. If the (Ξ, π)-Wadge game characterizes the class Λ and the (Ξ', π')-Wadge game characterizes the class Λ', then the $(\Xi'\Xi, \pi')$-Wadge game characterizes the class $\Lambda'\Lambda := \{fg\,;\, f \in \Lambda' \wedge g \in \Lambda\}$.*

Proof. If player II has a (computable) winning strategy in the $(\Xi'\Xi, \pi')$-Wadge game for $f :\subseteq \mathbb{A} \rightrightarrows \mathbb{B}$, then by Theorem 3.3 we have $f \leq^{\mathrm{t}}_{\mathrm{W}} \Xi'\Xi$ ($f \leq_{\mathrm{W}} \Xi'\Xi$). Thus, by Proposition 2.2, there exist continuous (computable) $k :\subseteq \mathbb{A} \rightrightarrows \mathbb{X}$ and $h :\subseteq \mathbb{A} \times \mathbb{Z} \rightrightarrows \mathbb{B}$ such that $h(\mathrm{id}_{\mathbb{A}} \times \Xi'\Xi k)\Delta_{\mathbb{A}} = f$. Now let $g' = h(\mathrm{id}_{\mathbb{A}} \times \Xi')$ and $g = (\mathrm{id}_{\mathbb{A}} \times \Xi k)\Delta_{\mathbb{A}}$. Then $f = g'g$, and since $g' \leq^{\mathrm{t}}_{\mathrm{W}} \Xi'$ ($g' \leq_{\mathrm{W}} \Xi'$) and $g \leq^{\mathrm{t}}_{\mathrm{W}} \Xi$ ($g \leq_{\mathrm{W}} \Xi$), we have $g' \in \Lambda'$ and $g \in \Lambda$, as desired. Conversely, if $g = f'f$ with $f' \in \Lambda'$ and $f \in \Lambda$, then by Theorem 3.3 we have $f' \leq_{\mathrm{W}} \Xi'$ and $f \leq_{\mathrm{W}} \Xi$. Now, by Proposition 2.5, it follows that $f'f \leq_{\mathrm{W}} \Xi'\Xi$. Finally, since by Proposition 2.5 we have that $\Xi'\Xi$ is a transparent cylinder, again by Theorem 3.3 it follows that II has a winning strategy in the $(\Xi'\Xi, \pi')$-Wadge game for g. □

We thus get game characterizations of many classes of functions, including, e.g., ones not covered by Motto Ros's constructions in [Mot11]. For example, consider the function $\mathrm{Sort} : 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ given by $\mathrm{Sort}(p) = 0^n 1^{\mathbb{N}}$ if p contains exactly n occurrences of 0 and $\mathrm{Sort}(p) = 0^{\mathbb{N}}$ otherwise. This map was introduced by Carroy in [Car14] (where it was called count_0) and studied by Neumann and Pauly in [NP18]. From the results in [NP18] it follows that the class Λ of total functions on $\mathbb{N}^{\mathbb{N}}$ which are Weihrauch-reducible to Sort is neither the class of pointwise limits of functions in some other class, nor the class of $\boldsymbol{\Gamma}$-measurable functions for any boldface pointclass $\boldsymbol{\Gamma}$ of subsets of $\mathbb{N}^{\mathbb{N}}$ closed under countable unions and finite intersections. By Theorem 2.12, Sort is Weihrauch-equivalent to some transparent cylinder $\mathrm{Sort}^{\mathrm{tc}}$ with codomain $\mathbb{N}^{\mathbb{N}}$. Thus, by Theorem 3.3, Λ is characterized by the $(\mathrm{Sort}^{\mathrm{tc}}, \mathrm{id}_{\mathbb{N}^{\mathbb{N}}})$-Wadge game.

The converse of Theorem 3.3 is almost true, as well:

Proposition 3.5. *If the (Ξ, π)-Wadge game characterizes a lower cone in the Weihrauch degrees, then it is the lower cone of $\pi\Xi$, and $\pi\Xi$ is a transparent cylinder.*

Proof. Similar to the corresponding observation in Theorem 3.3, note that whenever player II has a (computable) winning strategy in the (Ξ, π)-Wadge game for f, this induces a (strong) Weihrauch-reduction $f \leq^{\mathrm{t}}_{\mathrm{sW}} \pi\Xi$ ($f \leq_{\mathrm{sW}} \pi\Xi$). Conversely, by simply copying player I 's moves, player II wins the (Ξ, π)-Wadge game for $\pi\Xi$. This establishes the first claim. Now, as $\mathrm{id}_{\mathbb{N}^{\mathbb{N}}} \times (\pi\Xi) \leq_{\mathrm{W}} \pi\Xi$, the assumption that the (Ξ, π)-Wadge game characterize a lower cone in the Weihrauch degrees implies that player II wins the (Ξ, π)-Wadge game for $\mathrm{id}_{\mathbb{N}^{\mathbb{N}}} \times (\pi\Xi)$. Thus, $\mathrm{id}_{\mathbb{N}^{\mathbb{N}}} \times (\pi\Xi) \leq_{\mathrm{sW}} \pi\Xi$ follows, and we find $\pi\Xi$ to be a cylinder. For the remaining claim that $\pi\Xi$ is transparent, let $g :\subseteq \mathbb{N}^{\mathbb{N}} \rightrightarrows \mathbb{N}^{\mathbb{N}}$ be continuous (computable). Then $g\pi\Xi \leq^{\mathrm{t}}_{\mathrm{W}} \pi\Xi$ ($g\pi\Xi \leq_{\mathrm{W}} \pi\Xi$), hence player II has a (computable) winning strategy in the (Ξ, π)-Wadge game for $g\pi\Xi$. This strategy induces some continuous (computable) $H :\subseteq \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ with $g\pi\Xi\delta_{\mathbb{X}} \succeq \pi\Xi\delta_{\mathbb{X}} H$. Thus, $\delta_{\mathbb{X}} H \delta_{\mathbb{X}}^{-1}$ is the desired witness. □

3.2. **Using game characterizations.** One main advantage of having game characterizations of some properties is realized together with determinacy: either by choosing our set-theoretic axioms accordingly, or by restricting to simple cases and invoking, e.g., Borel determinacy, we can conclude that if the property is false, i.e., player II has no winning strategy, then player I has a winning strategy. Thus, player I 's winning strategies serve as explicit witnesses of the failure of a property. Applying this line of reasoning to our parametrized Wadge games, we obtain the following corollaries of Theorem 3.3:

Corollary 3.6 (ZFC)**.** *Let Ξ be a transparent cylinder and π a probe such that $\pi\Xi$ is single-valued and* $\mathrm{dom}(\pi\Xi)$ *is Borel. Then for any $f : \mathbb{X} \rightrightarrows \mathbb{Y}$ such that* $\mathrm{dom}(\delta_{\mathbb{X}})$ *and $f(x)$ are Borel for any $x \in \mathbb{X}$, we find that $f \not\leq^{\mathrm{t}}_{\mathrm{W}} \Xi$ iff player* I *has a winning strategy in the (Ξ, π)-Wadge game for f.*

Corollary 3.7 (ZF + DC + AD)**.** *Let Ξ be a transparent cylinder and π a probe. Then $f \not\leq^{\mathrm{t}}_{\mathrm{W}} \Xi$ iff player* I *has a winning strategy in the (Ξ, π)-Wadge game for f.*

Unfortunately, as determinacy fails in a computable setting (cf., e.g., [CR92, LRP15]), we do not retain the computable counterparts. More generally, we lack a clear grasp on the connections between winning strategies of player I in the (Ξ, π)-Wadge game for a function f and *positive* witnesses of the fact that f is not in the class characterized by the game. As pointed out by Carroy and Louveau in private communication, this is true even for the original Wadge game for functions, i.e., the $(\mathrm{id}_{\mathbb{N}^\mathbb{N}}, \mathrm{id}_{\mathbb{N}^\mathbb{N}})$-Wadge game. Here we already have a notion of positive witnesses for discontinuity, viz. *points of discontinuity*, and can therefore make this discussion mathematically precise:

Question 3.8. Let a point of discontinuity of a function $f : \mathbb{N}^\mathbb{N} \to \mathbb{N}^\mathbb{N}$ be given as a sequence $(x_n)_{n\in\mathbb{N}}$, a point $x \in \mathbb{N}^\mathbb{N}$, and $\sigma \in \mathbb{N}^{<\mathbb{N}}$ with $\sigma \subseteq f(x)$ such that $\forall n(d(x_n, x) < 2^{-n} \wedge \sigma \not\subseteq f(x_n))$. Let DiscPoint be the multi-valued map that takes as input a winning strategy for player I in the $(\mathrm{id}_{\mathbb{N}^\mathbb{N}}, \mathrm{id}_{\mathbb{N}^\mathbb{N}})$-Wadge game for some function $f : \mathbb{N}^\mathbb{N} \to \mathbb{N}^\mathbb{N}$, and outputs a point of discontinuity for that function. Is DiscPoint computable? More generally, what is the Weihrauch degree of DiscPoint?

We can somewhat restrict the range of potential answers for the preceding question:

Theorem 3.9. *Let player* I *have a computable winning strategy in the $(\mathrm{id}_{\mathbb{N}^\mathbb{N}}, \mathrm{id}_{\mathbb{N}^\mathbb{N}})$-Wadge game for $f : \mathbb{N}^\mathbb{N} \to \mathbb{N}^\mathbb{N}$. Then f has a computable point of discontinuity.*

The proof of the theorem will require some recursion theoretic preparations. Given $p, q \in \mathbb{N}^\mathbb{N}$, let $[p \mid q] \in \mathbb{N}^\mathbb{N}$ be defined as $[p \mid q] = 0^{q(0)}(p(0) + 1)0^{q(1)}(p(1) + 1)0^{q(2)} \ldots$, i.e., $[p \mid q]$ increases each number in p by 1, and then intersperses zeros between the entries, with the number of repetitions being provided by q. Now, given $r \in \mathbb{N}^\mathbb{N}$ and some $A \subseteq \mathbb{N}^\mathbb{N}$, let $A^{+r} := \{[p \mid q]\,;\, p \in A \wedge \forall n \in \mathbb{N}(q(n) \geq r(n))\}$.

The proof of the following lemma is based on helpful comments by Takayuki Kihara in personal communication.

Lemma 3.10. *Let $F : \mathbb{N}^\mathbb{N} \to \mathbb{N}^\mathbb{N}$ be computable, $r \in \mathbb{N}^\mathbb{N}$, $A, B \subseteq \mathbb{N}^\mathbb{N}$, $B \neq \varnothing$ be such that $F[B^{+r}] \subseteq A$ and A is Σ^0_2. Then A contains a computable point.*

Proof. Let $A = \bigcup_{n\in\mathbb{N}} Q_n$ with Π^0_1 -sets Q_n. For the sake of a contradiction, assume that A and thus all Q_n contain no computable points. Pick some $p \in B$.

As $F(0^\mathbb{N})$ is computable, we find $F(0^\mathbb{N}) \notin Q_0$. As Q_0 is Π^0_1 and F computable, there is some $m_0 \geq r(0)$ such that $F[0^{m_0}\mathbb{N}^\mathbb{N}] \cap Q_0 = \varnothing$. Next, consider $F(0^{m_0}p(0)0^\mathbb{N})$. Again, this is a computable point, hence there is some $m_1 \geq r(1)$ such that $F[0^{m_0}p(0)0^{m_1}\mathbb{N}^\mathbb{N}] \cap Q_1 = \varnothing$. We proceed in this manner to choose all m_i, and then define $q \in \mathbb{N}^\mathbb{N}$ by $q(i) = m_i$. Note that $q \geq r$. Then $[p \mid q] \in B^{+r}$, but $F([p \mid q]) \notin A$ by construction, hence we derive the desired contradiction and conclude that A contains a computable point. □

Proof of Theorem 3.9. Let us assume that player I has a winning strategy in the $(\mathrm{id}, \mathrm{id})$-Wadge game for $f : \mathbb{N}^\mathbb{N} \to \mathbb{N}^\mathbb{N}$. We describe how player II can coax player I into playing a point of discontinuity of f. Player II starts passing, causing player I to produce longer and longer prefixes of some $p \in \mathbb{N}^\mathbb{N}$. If player I ever produces a prefix $p_{\leq n_0}$ such that $\exists k_0 \ f[p_{\leq n}\mathbb{N}^\mathbb{N}] \subseteq k_0\mathbb{N}^\mathbb{N}$, then player II will play k_0, and then goes back to passing. If subsequently, there is some n_1, such that $\exists k_1 \ f[p_{\leq n_1}\mathbb{N}^\mathbb{N}] \subseteq k_0k_1\mathbb{N}^\mathbb{N}$, then player II plays k_1, and starts passing again, etc. If f is continuous at p, then player II will play a correct response to f, hence contradict the assumption that player I is following a winning strategy. Thus, p has to be a point of discontinuity of f.

Note that if player II passes even more than necessary, this does not change the argument at all. Thus, we find that there is some non-empty set B and $r \in \mathbb{N}^\mathbb{N}$ such that the computable response function $S : \mathbb{N}^\mathbb{N} \to \mathbb{N}^\mathbb{N}$ maps B^{+r} into the set of points of discontinuity of f. The latter is a Σ^0_2 -set, hence Lemma 3.10 implies that it contains a computable point. □

A more convenient way of exploiting determinacy of the (Ξ, π)-Wadge games could perhaps be achieved if a more symmetric version were found. In this, we could hope for a dual principle S, where for any f either $f \leq^c_\mathrm{W} \Xi$ or $S \leq^c_\mathrm{W} f$ holds. More generally, we hope that a better understanding of the (Ξ, π)-Wadge games would lead to structural results about the Weihrauch lattice, similar to the results obtained by Carroy on the strong Weihrauch reducibility [Car13].

3.3. **Generalized Wadge reducibility.** As mentioned in the introduction, the Wadge game was introduced not to characterize continuous functions, but in order to reason about a reducibility between sets. Given $A, B \subseteq \mathbb{N}^\mathbb{N}$, we say that A is *Wadge-reducible* to B, in symbols $A \leq_\mathcal{W} B$, if there exists a continuous $F : \mathbb{N}^\mathbb{N} \to \mathbb{N}^\mathbb{N}$ such that $F^{-1}[B] = A$ (we use the notation $\leq_\mathcal{W}$ instead of the more established $\leq_W$ in order to help avoid confusion with Weihrauch reducibility, $\leq_\mathrm{W}$). Equivalently, we could consider the multi-valued total function $\frac{B}{A} : \mathbb{N}^\mathbb{N} \rightrightarrows \mathbb{N}^\mathbb{N}$ defined by $\frac{B}{A}(x) = B$ if $x \in A$ and $\frac{B}{A}(x) = (\mathbb{N}^\mathbb{N} \setminus B)$ if $x \notin A$. It is easy to see that we have $A \leq_\mathcal{W} B$ iff $\frac{B}{A}$ is continuous. It is a famous structural result due to Wadge (using Borel determinacy) that for any Borel $A, B \subseteq \mathbb{N}^\mathbb{N}$, either $A \leq_\mathcal{W} B$ or $\mathbb{N}^\mathbb{N} \setminus B \leq_\mathcal{W} A$. In particular, the Wadge hierarchy on the Borel sets is a strict weak order of width 2.[1]

Both definitions generalize in a natural way to the case where $A \subseteq \mathbb{X}$ and $B \subseteq \mathbb{Y}$ for represented spaces $\mathbb{X}$, $\mathbb{Y}$: $A \leq'_\mathcal{W} B$ iff there exists a continuous $f : \mathbb{X} \to \mathbb{Y}$ such that $A = f^{-1}[B]$, and $\frac{B}{A} : \mathbb{X} \rightrightarrows \mathbb{Y}$ is defined by letting $\frac{B}{A}(x) = B$, if $x \in A$, and $\frac{B}{A}(x) = \mathbb{Y} \setminus B$, otherwise. It is easy to see that if $A \leq'_\mathcal{W}$, then $\frac{B}{A}$ is continuous, since if $f : \mathbb{X} \to \mathbb{Y}$ is such that $A = f^{-1}[B]$, then any realizer of f also realizes $\frac{B}{A}$. However, since not every continuous multi-valued function has a continuous uniformization, the converse does not hold in general. As noted, e.g., by Hertling [Her96], the relation $\leq'_\mathcal{W}$ restricted to $\mathbb{X} = \mathbb{Y} = \mathbb{R}$ already introduces infinite antichains in the resulting degree structure, and Ikegami showed that in fact the partial order $(\wp(\mathbb{N}), \subseteq_\mathrm{fin})$ can be embedded into that degree structure [Ike10, Theorem 5.1.2]. The generalization of $\frac{B}{A}$ was proposed by Pequignot [Peq15] as an alternative[2].

It is a natural variation to replace *continuous* in the definition of Wadge reducibility by some other class of functions (ideally one closed under composition). Motto Ros has shown that for the typical candidates of more restrictive classes of functions, the resulting

[1] A relation R is a *strict weak order* on a set X if there exists some ordinal number α and a partition $\langle X_\beta \,;\, \beta < \alpha \rangle$ of X such that $x \mathrel{R} y$ holds iff $x \in X_\beta$, $y \in X_\gamma$, and $\beta < \gamma$. The *width* of R is the supremum of the cardinalities of the parts in the partition.

[2] While Pequignot only introduces the notion for second countable T_0 spaces, the extension to all represented spaces is immediate. Note that one needs to take into account that for general represented spaces, the Borel sets can show unfamiliar properties, e.g., even singletons can fail to be Borel (cf. also [SS15, SS14, Hoy17]).

degree structures will not share the nice properties of the standard Wadge degrees (they are *bad*) [MR14]. Larger classes of functions as reduction witnesses have been explored by Motto Ros, Schlicht, and Selivanov [MRSS15] in the setting of quasi-Polish spaces—using the generalization of the first definition of the reduction. Here, we explore the second generalization; thus, we define $A \leq_{\mathcal{W}} B$ iff $\frac{B}{A}$ is continuous.

Definition 3.11. Given a multi-valued function Ξ and $A \subseteq \mathbb{X}$, $B \subseteq \mathbb{Y}$ for represented spaces $\mathbb{X}$ and $\mathbb{Y}$, let $A \leq_\Xi B$ iff $\frac{B}{A} \leq^{\mathrm{t}}_{\mathrm{W}} \Xi$.

Observation 3.12. If $\Xi \star \Xi \equiv_{\mathrm{W}} \Xi$, then $\leq_\Xi$ is a quasiorder.

The following partially generalizes [Mot11, Theorem 6.10]:

Theorem 3.13. *Let $A \subseteq \mathbb{X}$ and $B \subseteq \mathbb{Y}$, let $\Xi : \mathbb{U} \rightrightarrows \mathbb{V}$ be a transparent cylinder, and let $\pi : \subseteq \mathbb{Y} \to \mathbb{N}^\mathbb{N}$ be a probe such that the (Ξ, π)-Wadge game for $\frac{B}{A}$ is determined. Then either $A \leq_\Xi B$ or $B \leq_{\mathcal{W}} \mathbb{N}^\mathbb{N} \setminus A$.*

Proof. If player II has a winning strategy in the (Ξ, π)-Wadge game for $\frac{B}{A}$, then by Theorem 3.3, we find that $\frac{B}{A} \leq^{\mathrm{t}}_{\mathrm{W}} \Xi$, hence $A \leq_\Xi B$. Otherwise, player I has a winning strategy in that game. This winning strategy induces a continuous function $s : \mathbb{N}^\mathbb{N} \to \mathbb{N}^\mathbb{N}$, such that if player II plays $y \in \mathbb{N}^\mathbb{N}$, then player I plays $s(y) \in \mathbb{N}^\mathbb{N}$. As Ξ is a transparent cylinder and π a probe, since $\mathrm{id}_{\mathbb{N}^\mathbb{N}} \leq_{\mathrm{W}} \Xi$, by Theorem 3.3 player II has a winning strategy in the (Ξ, π)-Wadge game for $\mathrm{id}_{\mathbb{N}^\mathbb{N}}$. This strategy induces a continuous function $t : \subseteq \mathbb{N}^\mathbb{N} \to \mathbb{N}^\mathbb{N}$ such that $\pi \Xi \delta_\mathbb{X} t \preceq \mathrm{id}_{\mathbb{N}^\mathbb{N}}$, and since $\mathrm{id}_{\mathbb{N}^\mathbb{N}}$ is total and single-valued, we have that t is total and $\pi \Xi \delta_\mathbb{X} t = \mathrm{id}_{\mathbb{N}^\mathbb{N}}$ Now we consider $st : \mathbb{N}^\mathbb{N} \to \mathbb{N}^\mathbb{N}$. If $\delta_\mathbb{Z}(x) \in A$, then if player II plays $t(x)$, player I needs to play some $s(t(x))$ such that $\delta_\mathbb{W}(s(t(x))) \notin B$. Likewise, if $\delta_\mathbb{Z}(x) \notin A$, then for player I to win, it needs to be the case that $\delta_\mathbb{W}(s(t(x))) \in B$. Thus, st is a continuous realizer of $\frac{B}{\mathbb{N}^\mathbb{N} \setminus A}$, and $B \leq_{\mathcal{W}} \mathbb{N}^\mathbb{N} \setminus A$ follows. □

Corollary 3.14 (ZF + DC + AD)**.** *Suppose $\Xi \star \Xi \equiv_{\mathrm{W}} \Xi$. Then $<_\Xi$ is strict weak order of width at most* 2.

In [Mot09], in a different formalism, Motto Ros has identified sufficient conditions on a general reduction to ensure that its degree structure is equivalent to the Wadge degrees. We leave for future work the task of determining precisely for which Ξ the degree structure of $<_\Xi$ (restricted to subsets of $\mathbb{N}^\mathbb{N}$) is equivalent to the Wadge degrees, and which other structure types are realizable.

4. Games for functions of a fixed Baire class

4.1. **Spaces of trees.** An *unlabeled tree*, or simply *tree*, is a subset of $\mathbb{N}^{<\mathbb{N}}$ closed under the operation of taking initial segments. We will typically denote trees by the letters T, S, U with or without sub- or superscripts. Given a tree T and $\sigma \in \mathbb{N}^{<\mathbb{N}}$, we denote

(1) $\mathrm{Conc}(T, \sigma) := \{\tau \in \mathbb{N}^{<\mathbb{N}} \,;\, \sigma^\frown\tau \in T\}$
(2) $\mathrm{Ext}(T, \sigma) := \{\tau \in T \,;\, \sigma \subseteq \tau\}$

We call a tree *linear* if each of its nodes has at most one child, *finitely branching* if each of its nodes has only finitely many children, and *pruned* if each of its nodes has at least one child. Given a tree T and $\sigma \in T$ we define the *rank* of σ in T, denoted by $\mathrm{rk}_T(\sigma)$, by the recursion $\mathrm{rk}_T(\sigma) := \sup\{\mathrm{rk}_T(\tau) + 1 \,;\, \sigma \subset \tau \in T\}$, if σ is in the wellfounded part of

T, and $\mathrm{rk}_T(\sigma) := \infty$ otherwise. By letting $\infty > \alpha$ whenever α is a countable ordinal and $\infty + n = \infty$ for all $n \in \mathbb{N}$, we get $\mathrm{rk}_T(\sigma) = \sup\{\mathrm{rk}_T(\tau) + 1\,;\, \sigma \subset \tau \in T\}$ in either case. Furthermore, if $\sigma \subset \tau \in T$ and $|\tau| \geq |\sigma| + n$, then $\mathrm{rk}_T(\sigma) \geq \mathrm{rk}_T(\tau) + n$. The *rank* of T is $\mathrm{rk}_T(\langle\rangle)$, if $T \neq \varnothing$, or 0 if $T = \varnothing$.

Given $s \in \mathbb{N}^{\leq\mathbb{N}}$ with $|s| > 0$, let the *left shift* of s, denoted by $\mathrm{shift}(s)$, be the unique $t \in \mathbb{N}^{\leq\mathbb{N}}$ such that $s = \langle s(0)\rangle^\frown t$. Given $p \in \mathbb{N}^{\mathbb{N}}$ and $\sigma \in \mathbb{N}^{<\mathbb{N}}$, we say that σ is a *path through* p if $p(0) \neq 0$ and recursively $\mathrm{shift}(\sigma)$ is a path through $(\mathrm{shift}(p))_{\sigma(0)}$ in case $\sigma \neq \langle\rangle$. Let $\mathbb{UT}$ be the space of unlabeled trees represented by the total function $\delta_{\mathbb{UT}}$ given by

$$\delta_{\mathbb{UT}}(p) := \{\sigma \in \mathbb{N}^{<\mathbb{N}}\,;\, \sigma \text{ is a path through } p\}.$$

A *labeled tree* is a pair (T, φ) where T is a tree, called the *domain* of (T, φ), and $\varphi : T \setminus \{\langle\rangle\} \to \mathbb{N}$ is called the *labeling function* of (T, φ). We typically denote labeled trees by the letter Υ, with or without sub- or superscripts. A labeled tree Υ is a *subtree* of a labeled tree Υ', denoted $\Upsilon \subseteq \Upsilon'$, if the domain of Υ is a subset of the domain of Υ' and the labeling function of Υ is a restriction of that of Υ'. It is not hard to see that there exist a computable enumeration $\mathrm{e}_{\mathbb{LT}} : \mathbb{N} \to \mathbb{LT}$ of all finite labeled trees and a computable function $\mathrm{size} : \mathbb{N} \to \mathbb{N}$ such that $\mathrm{e}_{\mathbb{LT}}(m) \subset \mathrm{e}_{\mathbb{LT}}(n)$ implies $m < n$, and such that $\mathrm{e}_{\mathbb{LT}}(n)$ has exactly $\mathrm{size}(n)$ nodes. We will in general overload notation from unlabeled to labeled trees; whenever some such notation is used without prior introduction, the intended meaning will be intuitive. For example, for $\Upsilon = (T, \varphi)$ we will write $\sigma \in \Upsilon$ to mean $\sigma \in T$, or $\mathrm{rk}_\Upsilon(\sigma)$ instead of $\mathrm{rk}_T(\sigma)$, etc.

If $\sigma \neq \langle\rangle$ is a path through p, then its *label according to* p is $p(\sigma(0)) - 1$, if $|\sigma| = 1$, or the label of $\mathrm{shift}(\sigma)$ according to $(\mathrm{shift}(p))_{\sigma(0)}$, otherwise. Let $\mathbb{LT}$ be the space of labeled trees represented by the total function $\delta_{\mathbb{LT}}$ given by $\delta_{\mathbb{LT}}(p) = (\delta_{\mathbb{UT}}(p), \varphi)$, where $\varphi(\sigma)$ is the label of σ according to p.

Given labeled trees $\Upsilon = (T, \varphi)$ and $\Upsilon' = (T', \varphi')$, a relation $B \subseteq T \times T'$ is called a *bisimulation* between Υ and Υ' in case $\sigma\ B\ \tau$ implies $|\sigma| = |\tau|$ and:

$$\varphi(\sigma) = \varphi'(\tau) \tag{label}$$

$$\forall \sigma' \in T\ (\sigma \subset \sigma' \ \Rightarrow\ \exists \tau' \in T'(\tau \subset \tau' \wedge \sigma'\ B\ \tau')) \tag{forth}$$

$$\forall \tau' \in T'\ (\tau \subset \tau' \ \Rightarrow\ \exists \sigma' \in T(\sigma \subset \sigma' \wedge \sigma'\ B\ \tau')) \tag{back}$$

$$|\sigma| = \ell + 1 \ \Rightarrow\ \sigma{\restriction}\ell\ B\ \tau{\restriction}\ell \tag{parent}$$

It is easily seen that the union of any family of bisimulations between given labeled trees is also a bisimulation between those trees. Therefore, between any pair of labeled trees Υ and Υ' there always exists a largest bisimulation, denoted $\rightleftarrows_{\Upsilon,\Upsilon'}$. We say Υ and Υ' are *bisimilar*, denoted $\Upsilon \rightleftarrows \Upsilon'$, in case $\rightleftarrows_{\Upsilon,\Upsilon'}$ is nonempty. A particular case of a bisimulation between $\Upsilon = (T, \varphi)$ and $\Upsilon' = (T', \varphi')$ is an *isomorphism* between those trees, i.e., a bijection $\iota : T \to T'$ satisfying, for any $\sigma, \tau \in T$:

(1) $\sigma \subseteq \tau$ iff $\iota(\sigma) \subseteq \iota(\tau)$,
(2) $|\sigma| = |\iota(\sigma)|$, and
(3) $\varphi(\sigma) = \varphi'(\iota(\sigma))$.

The trees Υ and Υ' are *isomorphic*, denoted $\Upsilon \simeq \Upsilon'$, if there exists an isomorphism between them.

Lemma 4.1. *If $B \subseteq \Upsilon \times \Upsilon'$ is a bisimulation and $\sigma\ B\ \tau$ holds, then* $\mathrm{rk}_\Upsilon(\sigma) = \mathrm{rk}_{\Upsilon'}(\tau)$.

Proof. If $\mathrm{rk}_\Upsilon(\sigma) = \infty$, i.e., if σ is on an infinite branch of Υ, then it is easy to see that τ is on an infinite branch of Υ' and therefore $\mathrm{rk}_{\Upsilon'}(\tau) = \infty$ as well. By the same argument, we have that if $\mathrm{rk}_{\Upsilon'}(\tau) = \infty$ then $\mathrm{rk}_\Upsilon(\sigma) = \infty$. If $\sigma \in \mathrm{WF}(\Upsilon)$, then we proceed by induction on $\mathrm{rk}_\Upsilon(\sigma)$. For the base case, note that $\mathrm{rk}_\Upsilon(\sigma) = 0$ iff σ is a leaf of Υ, and in this case $\sigma\ B\ \tau$ implies that τ is also a leaf of Υ' and therefore also has rank 0. Now suppose the result holds for every node of rank $< \mathrm{rk}_\Upsilon(\sigma)$. For each $\beta < \mathrm{rk}_\Upsilon(\sigma)$ there exists some descendant σ' of σ in Υ such that $\mathrm{rk}_\Upsilon(\sigma') = \beta$. Since B is a bisimulation, there exists a descendant τ' of τ in Υ' such that $\sigma'\ B\ \tau'$. By induction hypothesis we have $\mathrm{rk}_{\Upsilon'}(\tau') = \beta$, and since $\beta < \mathrm{rk}_\Upsilon(\sigma)$ was arbitrary we have $\mathrm{rk}_{\Upsilon'}(\tau) \geq \mathrm{rk}_\Upsilon(\sigma)$. Analogously we can prove $\mathrm{rk}_{\Upsilon'}(\tau) \leq \mathrm{rk}_\Upsilon(\sigma)$, so the result follows. □

An *abstract tree* is an equivalence class of labeled trees under the relation of bisimilarity. Let $\mathbb{AT}$ be the space of abstract trees represented by the total function $\delta_{\mathbb{AT}}$ given by $\delta_{\mathbb{AT}}(p) = \delta_{\mathbb{LT}}(p)/\rightleftarrows$. We typically denote abstract trees by $\mathcal{A}$, with or without sub- or superscripts. As usual with quotient constructions, any property of labeled trees can be extended to abstract trees by stipulating that an abstract tree has the property in question if one of its representatives does. Note that for some properties this extension behaves better than for some others. For example, the property of *having rank* α behaves well, since by Lemma 4.1 any two bisimilar labeled trees have the same rank. On the other hand, the property of *being finitely branching* does not behave as well, since every finitely branching labeled tree is bisimilar to an infinitely branching one.

Note that, according to our definition, formally speaking an abstract tree is not itself a tree but only a certain type of set of labeled trees. However, for the sake of intuition it can be helpful to think of an abstract tree as an unordered tree without any concrete underlying set of vertices, as follows. We call an *informal tree* a (possibly empty) countable set I of objects of the form (n, J), where n is a natural number and J is again an informal tree. The intuition is that such a tree I is the tree for which each such object (n, J) represents a child of the root of I with label n and whose subtree is exactly J. See Figure 1 for the depiction of a simple informal tree.

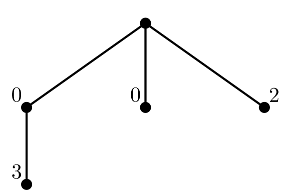

Figure 1. Depiction of the informal tree $\{(0, \{(3, \varnothing)\}), (0, \varnothing), (2, \varnothing)\}$.

To see how informal trees correspond to abstract trees, let $\delta_{\mathbb{IT}}$ be the partial function defined by corecursion with

$$\delta_{\mathbb{IT}}(p) = \{(n, \delta_{\mathbb{IT}}(q))\,;\, \exists k((p)_k = \langle n+1\rangle^\frown q)\}.$$

Then we say an informal tree I *corresponds* to an abstract tree $\mathcal{A}$ if there exists $p \in \mathrm{dom}(\delta_{\mathbb{IT}})$ with $\delta_{\mathbb{IT}}(p) = I$ and $\delta_{\mathbb{AT}}(p) = \mathcal{A}$.

Proposition 4.2. *In ZFC, the domain of $\delta_{\mathbb{IT}}$ is the set of $p \in \mathbb{N}^{\mathbb{N}}$ for which $\delta_{\mathbb{AT}}(p)$ is wellfounded. Therefore, in ZFC no informal tree corresponds to an illfounded abstract tree.*

This is, of course, because if p is such that $\delta_{\mathbb{AT}}(p)$ is illfounded, then in order for $p \in \mathrm{dom}(\delta_{\mathbb{IT}})$ to hold there would have to exist an infinite $\in$-descending chain of sets starting at $\delta_{\mathbb{IT}}(p)$, contradicting the axiom of foundation.

However—as is often the case with definitions by corecursion [MD97]—, this definition and the correspondence would also work for illfounded trees if one were to work in a system of non-wellfounded set theory such as $\mathrm{ZFC}^- + \mathrm{AFA}$, where AFA is the axiom of anti-foundation first formulated by Forti and Honsell [FH83] and later popularized by Aczel [Acz88]—in the style of Aczel [Acz88, Chapter 6], in $\mathrm{ZFC}^- + \mathrm{AFA}$ the set of informal trees can be defined as the greatest fixed point of the class operator Φ defined by letting $\Phi(X)$ be the class of all countable sets of elements of the form (n, T), with $n \in \mathbb{N}$ and T a countable subset of X. Thus in $\mathrm{ZFC}^- + \mathrm{AFA}$ the set of informal trees is exactly

$$\bigcup\{x \in V \,;\, x \subseteq \Phi(x)\}.$$

Proposition 4.3 ($\mathrm{ZFC}^- + \mathrm{AFA}$)**.** *The correspondence between abstract and informal trees is a bijection.*

We will not pursue this line of investigation any further; we thus now move back to our setting of ZFC for the remainder of the paper.

Computable functions between abstract trees. We denote by $\mathcal{O}(\mathbb{N})$ the represented space of subsets of $\mathbb{N}$ given by *enumeration*, i.e., so that p is a name for $X \subseteq \mathbb{N}$ iff $X = \{n \in \mathbb{N} \,;\, \exists k \in \mathbb{N}(p(k) = n+1)\}$. Note that any computable function of type $\subseteq \mathcal{O}(\mathbb{N}) \rightrightarrows \mathcal{O}(\mathbb{N})$ has a computable realizer which uses only *positive information*, by which we mean that it works by only following rules of the form "enumerate a certain natural number into the output set only after having seen some finite set of natural numbers enumerated into the input set", i.e., via enumeration operators (cf., e.g., [Odi99, Chapter XIV]). Let $\mathrm{SubTrees} : \mathbb{LT} \to \mathcal{O}(\mathbb{N})$ be defined by letting $\mathrm{SubTrees}(\Upsilon) = \{n \in \mathbb{N} \,;\, \mathrm{e}_{\mathbb{LT}}(n) \text{ is a labeled subtree of } \Upsilon\}$. It is easy to see that SubTrees is computable.

Lemma 4.4. *There exists a computable map* $\mathrm{ConsTree} : \subseteq \mathcal{O}(\mathbb{N}) \rightrightarrows \mathbb{LT}$ *such that the composition* $\mathrm{ConsTree} \circ \mathrm{SubTrees}$ *is total and* $\Upsilon' \in \mathrm{ConsTree} \circ \mathrm{SubTrees}(\Upsilon)$ *implies* $\Upsilon' \simeq \Upsilon$.

Proof. ConsTree can be defined as follows. Suppose we are at stage k of the construction, when some $n \in \mathbb{N}$ is enumerated into the input. If some m has been enumerated at some earlier stage such that $\mathrm{e}_{\mathbb{LT}}(m) \supset \mathrm{e}_{\mathbb{LT}}(n)$, then we proceed to the next stage. Otherwise let X be the set of $m \in \mathbb{N}$ such that $\mathrm{e}_{\mathbb{LT}}(m)$ is a maximal subtree of $\mathrm{e}_{\mathbb{LT}}(n)$ among those m which have been enumerated at earlier stages. By construction, for each $m \in X$ we have defined an associated $a(m) \in \mathbb{N}$ and an isomorphism $\iota_m : \mathrm{e}_{\mathbb{LT}}(m) \to \mathrm{e}_{\mathbb{LT}}(a(m))$, in such a way that $\mathrm{e}_{\mathbb{LT}}(a(m))$ is guaranteed to be a subtree of the output tree we are constructing. Let $N \geq n$ be least such that there exists an isomorphism $\iota_n : \mathrm{e}_{\mathbb{LT}}(n) \to \mathrm{e}_{\mathbb{LT}}(N)$ extending ι_m for each $m \in X$ (in particular $\mathrm{e}_{\mathbb{LT}}(a(m)) \subset \mathrm{e}_{\mathbb{LT}}(N)$ for every $m \in X$) and such that no node of $\mathrm{e}_{\mathbb{LT}}(N)$ which is not in $\bigcup_{m \in X} \mathrm{e}_{\mathbb{LT}}(a(m))$ has been promised to be part of our current partial output. Then let $a(n) := N$ and guarantee that $\mathrm{e}_{\mathbb{LT}}(N)$ will be a subtree of our output tree.

It is now straightforward to check that running the algorithm above on a name for $\mathrm{SubTrees}(\Upsilon)$ we have $\Upsilon = \bigcup_{n \in \mathrm{dom}(a)} \mathrm{e}_{\mathbb{LT}}(n)$ and that $\iota := \bigcup_{n \in \mathrm{dom}(a)} \iota_n$ is an isomorphism between Υ and $\Upsilon' := \bigcup_{n \in \mathrm{dom}(a)} \mathrm{e}_{\mathbb{LT}}(a(n))$. □

Lemma 4.5. *Let $G \vdash g :\subseteq \mathbb{AT} \rightrightarrows \mathbb{AT}$. Suppose $F :\subseteq \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ and $H :\subseteq \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ are such that $\delta_{\mathbb{LT}} F(p) \rightleftarrows \delta_{\mathbb{LT}}(p)$ and $\delta_{\mathbb{LT}} H(q) \rightleftarrows \delta_{\mathbb{LT}}(q)$ for any $p \in \mathrm{dom}(F)$ and $q \in \mathrm{dom}(H)$, and $\mathrm{dom}(G) \subseteq \mathrm{dom}(HGF)$. Then $HGF \vdash g$.*

Proof. We have $\delta_{\mathbb{AT}} HGF(p) = \delta_{\mathbb{AT}} GF(p)$ and $\delta_{\mathbb{AT}} F(p) = \delta_{\mathbb{AT}}(p)$, therefore $\delta_{\mathbb{AT}} HGF(p) = \delta_{\mathbb{AT}} G(p)$. □

Corollary 4.6. *Let F, H be computable realizers of* ConsTree *and* SubTrees*, respectively. If G is a computable realizer of some $g :\subseteq \mathbb{AT} \rightrightarrows \mathbb{AT}$, then so is $FHGFH$.*

Proof. Indeed, we have $\delta_{\mathbb{LT}} FH(p) \in \mathrm{ConsTree} \circ \mathrm{SubTrees} \delta_{\mathbb{LT}}(p)$, so $\delta_{\mathbb{LT}} FH(p) \rightleftarrows \delta_{\mathbb{LT}}(p)$. □

Note that HGF is a computable realizer of some function $g' :\subseteq \mathcal{O}(\mathbb{N}) \rightrightarrows \mathcal{O}(\mathbb{N})$; thus we can assume that $FHGFH$ works by only following rules of the form "make a certain finite labeled tree a subtree of the output only after having seen some finite set of finite labeled trees as subtrees of the input", which is to say, "make a certain finite labeled tree a subtree of the output only after having seen a certain finite labeled tree as a subtree of the input".

4.2. **The pruning derivative.** We now define the main operation which will be used in the game characterization of the class of functions of any fixed Baire class. First, let us recall the definition from [Pau1X] of the space $\mathbb{CO}$ of countable ordinals represented by the function δ_{nK} defined recursively by

(1) $\delta_{\mathrm{nK}}(0p) = 0$
(2) $\delta_{\mathrm{nK}}(1p) = \delta_{\mathrm{nK}}(p) + 1$
(3) $\delta_{\mathrm{nK}}(2\ulcorner p_n \urcorner_{n\in\mathbb{N}}) = \sup_{n\in\mathbb{N}} \delta_{\mathrm{nK}}(p_n)$.

Definition 4.7. We call *pruning derivative* the operation $\mathrm{PD} : \mathbb{UT} \to \mathbb{UT}$ which assigns to each unlabeled tree T its subtree $\mathrm{PD}(T) := \{\sigma \in T\,;\, \sigma$ has descendants of arbitrary lengths in $T\}$. We overload notation and also denote by $\mathrm{PD} : \mathbb{LT} \to \mathbb{LT}$ the operation which assigns to each labeled tree $\Upsilon = (T, \varphi)$ its subtree $\mathrm{PD}(\Upsilon)$ whose domain is $\mathrm{PD}(T)$. As usual, these definitions can be iterated transfinitely in a natural way by letting

(1) $\mathrm{PD}^\star(\,\cdot\,, 0) = \mathrm{id}$
(2) $\mathrm{PD}^\star(\,\cdot\,, \alpha + 1) = \mathrm{PD}(\mathrm{PD}^\star(\,\cdot\,, \alpha))$
(3) $\mathrm{PD}^\star(\,\cdot\,, \lambda) = \bigcap_{\alpha<\lambda} \mathrm{PD}^\star(\,\cdot\,, \alpha)$, for limit $\lambda > 0$.

Since trees are countable, for any given tree the iteration described above will stabilize at some countable stage for each tree. Thus, as functions between represented spaces, we can consider them as having types $\mathrm{PD}^\star : \mathbb{UT} \times \mathbb{CO} \to \mathbb{UT}$ and $\mathrm{PD}^\star : \mathbb{LT} \times \mathbb{CO} \to \mathbb{LT}$, respectively.

Lemma 4.8. *For $\sigma \in T$, we have $\sigma \in \mathrm{PD}^\star(T, \alpha)$ iff $\mathrm{rk}_T(\sigma) \geq \omega \cdot \alpha$.*

Lemma 4.9. *If $B \subseteq \Upsilon \times \Upsilon'$ is a bisimulation and $\sigma\ B\ \tau$ holds, then $\mathrm{rk}_\Upsilon(\sigma) = \mathrm{rk}_{\Upsilon'}(\tau)$.*

Proof. By induction on $\mathrm{rk}_\Upsilon(\sigma)$. For the base case, note that $\mathrm{rk}_\Upsilon(\sigma) = 0$ iff σ is a leaf of Υ, and in this case $\sigma\ B\ \tau$ implies that τ is also a leaf of Υ' and therefore also has rank 0. Now suppose the result holds for every node of rank $< \mathrm{rk}_\Upsilon(\sigma)$. For each $\beta < \mathrm{rk}_\Upsilon(\sigma)$ there exists some descendant σ' of σ in Υ such that $\mathrm{rk}_\Upsilon(\sigma') = \beta$. Since B is a bisimulation, there exists a descendant τ' of τ in Υ' such that $\sigma'\ B\ \tau'$. By induction hypothesis we have $\mathrm{rk}_{\Upsilon'}(\tau') = \beta$, and since $\beta < \mathrm{rk}_\Upsilon(\sigma)$ was arbitrary we have $\mathrm{rk}_{\Upsilon'}(\tau) \geq \mathrm{rk}_\Upsilon(\sigma)$. Analogously we can prove $\mathrm{rk}_{\Upsilon'}(\tau) \leq \mathrm{rk}_\Upsilon(\sigma)$, so the result follows. □

Corollary 4.10. *If* $\Upsilon \rightleftarrows \Upsilon'$ *then* $\mathrm{PD}^\star(\Upsilon, \alpha) \rightleftarrows \mathrm{PD}^\star(\Upsilon', \alpha)$ *for any* $\alpha < \omega_1$.

We overload notation yet again and denote by $\mathrm{PD} : \mathbb{AT} \to \mathbb{AT}$ and $\mathrm{PD}^\star : \mathbb{AT} \times \mathbb{CO} \to \mathbb{AT}$ the operations assigning to each abstract tree $\mathcal{A}$ with representative Υ and each countable ordinal α the subtrees $\mathrm{PD}(\mathcal{A})$ and $\mathrm{PD}^\star(\mathcal{A}, \alpha)$ with representatives $\mathrm{PD}(\Upsilon)$ and $\mathrm{PD}^\star(\Upsilon, \alpha)$, respectively; Corollary 4.10 guarantees that these are well-defined operations. Whenever not specified otherwise by the context, in what follows PD and $\mathrm{PD}^\star$ will refer to the operations on abstract trees.

4.3. **The Weihrauch degree of the pruning derivative.** In order to analyze the Weihrauch degree of $\mathrm{PD}^\star$, we will first introduce and analyze several operations on trees. We introduce and analyze them as modularly as possible, in the hope that this will increase the clarity of the presentation and the potential for applicability of the operations in other situations.

Given $\sigma_0, \dots, \sigma_{n-1} \in \mathbb{N}^{<\mathbb{N}}$ such that $|\sigma_i| = |\sigma_j| = \ell$ for each $i, j < n$, let $\ulcorner \sigma_0, \dots, \sigma_{n-1} \urcorner \in \mathbb{N}^\ell$ be defined by $\ulcorner \sigma_0, \dots, \sigma_{n-1} \urcorner(m) = \ulcorner \sigma_0(m), \dots, \sigma_{n-1}(m) \urcorner$ for each $m < \ell$. Note that $\ulcorner \sigma_0, \dots, \sigma_{n-1} \urcorner \subseteq \ulcorner \tau_0, \dots, \tau_{n-1} \urcorner$ iff $\sigma_i \subseteq \tau_i$ for every $i < n$. Now, given trees $T_0, \dots, T_{n-1}$, let their *product* be the tree $\bigotimes_{i<n} T_i := \{\ulcorner \sigma_0, \dots, \sigma_{n-1} \urcorner ; \forall i < n(\sigma_i \in T_i \text{ and } |\sigma_i| = |\sigma_0|)\}$. If $n = 2$ then we use the smaller infix notation $T_0 \otimes T_1$ to denote the product.

Lemma 4.11. *The operation* $\bigotimes : \mathbb{UT}^{<\mathbb{N}} \to \mathbb{UT}$ *is computable and*
(1) $\bigotimes_{i<n} T_i = \varnothing$ *iff* $T_i = \varnothing$ *for some* $i < n$.
(2) $\mathrm{PD}(\bigotimes_{i<n} T_i) = \bigotimes_{i<n} \mathrm{PD}(T_i)$.
(3) $\bigcap_{\beta<\alpha} \bigotimes_{i<n} T_i^\beta = \bigotimes_{i<n} \bigcap_{\beta<\alpha} T_i^\beta$ *for any ordinal* α.

In particular, $\mathrm{PD}^\star(\bigotimes_{i<n} T_i, \alpha) = \bigotimes_{i<n} \mathrm{PD}^\star(T_i, \alpha)$ for any ordinal α.

We extend the binary product $\otimes$ to type $\mathbb{LT} \times \mathbb{UT} \to \mathbb{LT}$ by letting $(T, \varphi) \otimes S = (T \otimes S, \varphi')$, where $\varphi'(\ulcorner \sigma, \tau \urcorner) = \varphi(\sigma)$.

Lemma 4.12. *If* S *is pruned and nonempty, then* $(T, \varphi) \rightleftarrows (T, \varphi) \otimes S$.

Proof. Let $B \subseteq T \times (T \otimes S)$ be given by $\sigma \mathrel{B} \tau$ iff $\tau = \ulcorner \sigma, \xi \urcorner$ for some $\xi \in S$. It is easy to see that B satisfies conditions (label) and (parent). Suppose $\sigma \mathrel{B} \tau$, and let $\xi \in S$ be such that $\tau = \ulcorner \sigma, \xi \urcorner$. For (forth), let σ' be a child of σ in (T, φ). Since S is pruned, ξ has a child ξ' in S, and therefore $\tau' := \ulcorner \sigma', \xi' \urcorner$ is a child of τ in $(T, \phi) \otimes S$. Now $\sigma' \mathrel{B} \tau'$ follows. For (back), let τ' be a child of τ in $(T, \phi) \otimes S$. Thus σ' is a child of σ in (T, φ), from which $\sigma' \mathrel{B} \tau'$ follows. $\square$

For our next operation on trees, let us first define some auxiliary notation $[\sigma_0, \dots, \sigma_{n-1}]$, for $\sigma_0, \dots, \sigma_{n-1} \in \mathbb{N}^{<\mathbb{N}}$.

Definition 4.13. We define $[\,] := \langle\rangle$. Then, given $\sigma_0, \dots, \sigma_{n-1} \in \mathbb{N}^{<\mathbb{N}}$ such that $n = |\sigma_0| > 0$ and $|\sigma_i| = n - i$ for each $i < n$, let $[\sigma_0, \dots, \sigma_{n-1}]$ be defined by

$$[\sigma_0, \dots, \sigma_{n-1}](m) = \ulcorner \sigma_0(m), \sigma_1(m-1), \sigma_2(m-2), \dots, \sigma_m(0) \urcorner$$

for each $m < n$. Note that $[\sigma_0, \dots, \sigma_{n-1}] \subseteq [\tau_0, \dots, \tau_{m-1}]$ iff $n \leq m$ and $\sigma_i \subseteq \tau_i$ for each $i < n$. Now, given trees $\langle T_n \rangle_{n \in \mathbb{N}}$, let their *countable product* be the tree

$$\mathop{\boxtimes}_{n \in \mathbb{N}} T_n := \{[\sigma_0, \dots, \sigma_{k-1}] \,;\, \forall m < k(|\sigma_m| = k - m \text{ and } \sigma_m \in T_m)\}.$$

Note that $\langle\rangle \in \boxtimes_{n\in\mathbb{N}} T_n$ always holds. In particular, it is *not* always the case that $\mathrm{PD}^\star(\boxtimes_{n\in\mathbb{N}} T_n, \alpha) = \boxtimes_{n\in\mathbb{N}} \mathrm{PD}^\star(T_n, \alpha)$ holds for all α, contrary to the situation for finite products of trees.

Lemma 4.14. *The operation $\boxtimes : \mathbb{UT}^{\mathbb{N}} \to \mathbb{UT}$ is computable and*

(1) *For $m > 0$ we have $\mathrm{rk}_S([\sigma_0, \dots, \sigma_{m-1}]) \le \min_{i<m} \mathrm{rk}_{T_i}(\sigma_i)$, with equality in case $\mathrm{rk}(T_j) \ge \min_{i<m} \mathrm{rk}_{T_i}(\sigma_i)$ holds for each $j \ge m$. As a consequence, we have $\mathrm{rk}(S) \le \min_{n\in\mathbb{N}}(\mathrm{rk}(T_n) + n)$.*

(2) *For every α we have $\mathrm{PD}^\star(\boxtimes_{n\in\mathbb{N}} T_n, \alpha) \subseteq \boxtimes_{n\in\mathbb{N}} \mathrm{PD}^\star(T_n, \alpha)$, with equality in case $\alpha = 0$ or $\mathrm{PD}^\star(T_n, \alpha) \neq \varnothing$ for all $n \in \mathbb{N}$.*

(3) *If all T_n are pruned and nonempty then so is $\boxtimes_{n\in\mathbb{N}} T_n$.*

Proof. The computability of $\boxtimes$ is straightforward.

(1) By induction on $\mathrm{rk}_{T_i}(\sigma_i)$, we show that $\mathrm{rk}_S([\sigma_0, \dots, \sigma_{m-1}]) \le \mathrm{rk}_{T_i}(\sigma_i)$. If $\mathrm{rk}_{T_i}(\sigma_i) = 0$, this is easy to see. For $\mathrm{rk}_{T_i}(\sigma_i) > 0$ we have that every descendant of σ_i in T_i has rank less than $\mathrm{rk}_{T_i}(\sigma_i)$, so by inductive hypothesis every descendant of $[\sigma_0, \dots, \sigma_{m-1}]$ in S has rank less than $\mathrm{rk}_{T_i}(\sigma_i)$, and therefore $\mathrm{rk}_S([\sigma_0, \dots, \sigma_{m-1}]) \le \mathrm{rk}_{T_i}(\sigma_i)$. Conversely, by induction on α we show that if $\mathrm{rk}_{T_i}(\sigma_i), \mathrm{rk}(T_j) \ge \alpha$ holds for all $i < m$ and $j \ge m$, then $\mathrm{rk}_S([\sigma_0, \dots, \sigma_{m-1}]) \ge \alpha$ as well. The case $\alpha = 0$ is clear. Now suppose $\alpha > 0$. Given $\beta < \alpha$, for each $i < m$ let σ'_i be an immediate child of σ_i in T_i of rank at least β, and let $\sigma'_m \in T_m$ have length 1 and rank at least β. Then $[\sigma'_0, \dots, \sigma'_m]$ is an immediate child of $[\sigma_0, \dots, \sigma_{m-1}]$ in S. Since $\mathrm{rk}(T_j) \ge \beta$ for each $j \ge m+1$, by induction hypothesis we get $\mathrm{rk}_S([\sigma'_0, \dots, \sigma'_m]) \ge \beta$. Therefore $\mathrm{rk}_S([\sigma_0, \dots, \sigma_{m-1}]) > \beta$, and since $\beta < \alpha$ was arbitrary we get $\mathrm{rk}_S([\sigma_0, \dots, \sigma_{m-1}]) \ge \alpha$, as desired. Finally $\tau \in S$ has length $n+1$, then $\tau = [\sigma_0, \dots, \sigma_n]$ where $\sigma_i \in T_i$ for every $i \le n$. In particular, $\mathrm{rk}_S(\tau) \le \mathrm{rk}_{T_n}(\sigma_n) < \mathrm{rk}(T_n)$, so $\mathrm{rk}(S) = \mathrm{rk}_S(\langle\rangle) \le \mathrm{rk}(T_n) + n$.

(2) Follows by combining (1) with Lemma 4.8.

(3) Follows from (1) since a tree is pruned and nonempty iff all its nodes have rank ∞. □

Definition 4.15. Given trees $T_0, \dots, T_{n-1}$, let their *mix* be the tree $\bigoplus_{i<n} T_i$ which satisfies $\bigoplus_{i<n} T_i = \varnothing$ iff $T_i = \varnothing$ for some $i < n$, and otherwise $\langle\rangle \in \bigoplus_{i<n} T_i$ and $\mathrm{Conc}(\bigoplus_{i<n} T_i, \langle\ulcorner m, k\urcorner\rangle) = \mathrm{Conc}(T_m, \langle k\rangle)$ for each $m < n$ and $k \in \mathbb{N}$. Intuitively, the mix of $T_0, \dots, T_{n-1}$ is the tree obtained by merging the roots of those trees into a single root. If $n = 2$ then we use the smaller infix notation $T_0 \oplus T_1$ to denote the mix.

Lemma 4.16. *The operation $\bigoplus : \mathbb{UT}^{<\mathbb{N}} \to \mathbb{UT}$ is computable and*

(1) $\mathrm{PD}(\bigoplus_{i<n} T_i) = \bigoplus_{i<n} \mathrm{PD}(T_i)$.

(2) $\bigcap_{\beta<\alpha} \bigoplus_{i<n} T_i^\beta = \bigoplus_{i<n} \bigcap_{\beta<\alpha} T_i^\beta$ *for any ordinal α.*

In particular, $\mathrm{PD}^\star(\bigoplus_{i<n} T_i, \alpha) = \bigoplus_{i<n} \mathrm{PD}^\star(T_i, \alpha)$ for any ordinal α

Definition 4.17. Given trees $\langle T_n\rangle_{n\in\mathbb{N}}$, let their *countable mix* be the tree $\boxplus_{n\in\mathbb{N}} T_n$ such that $\langle\rangle \in \boxplus_{n\in\mathbb{N}} T_n$ and $\mathrm{Conc}(\boxplus_{n\in\mathbb{N}} T_n, \langle\ulcorner m, k\urcorner\rangle) = \mathrm{Conc}(T_m, \langle k\rangle)$ for each $m, k \in \mathbb{N}$. As before with countable products, $\boxplus$ will not commute with $\mathrm{PD}^\star(\,\cdot\,, \alpha)$ for all α in general.

Lemma 4.18. *The operation $\boxplus : \mathbb{UT}^{\mathbb{N}} \to \mathbb{UT}$ is computable and satisfies $\mathrm{PD}^\star(\boxplus_{n\in\mathbb{N}} T_n, \alpha) \subseteq \boxplus_{n\in\mathbb{N}} \mathrm{PD}^\star(T_n, \alpha)$, with equality in case $\alpha = 0$ or $\mathrm{PD}^\star(T_n, \alpha) \neq \varnothing$ for some $n \in \mathbb{N}$.*

To proceed, we need the notion of a Borel truth value. This represented space was introduced in [GKP17] (built on ideas from [Mos09]), and further investigated in [Pau1X]. Our definition differs slightly from the one given in the literature, but is easily seen to be equivalent.

Definition 4.19. A *Borel truth value* is a pair $b = (T, \mu)$ such that T is a wellfounded tree and μ is a function, called a *tagging* function, assigning to each node of T one of the *tags* $\bot, \top, \forall, \exists$, in such a way that each leaf is tagged $\top$ or $\bot$, and each non-leaf node is tagged $\forall$ or $\exists$ (in alternating fashion, i.e., so that if a node tagged $\forall$ has a parent, then the parent is tagged $\exists$ and vice versa). A name for a Borel truth value (T, μ) is an element $p \in 5^{\mathbb{N}}$ which is a $\delta_{\mathbb{UT}}$ -code for T and such that if $\sigma \in T$, i.e., if σ is a path through p, then

$$\mu(\sigma) = \begin{cases} \bot, & \text{if } \sigma = \langle\rangle \text{ and } p(0) = 1, \text{ or } |\sigma| = 1 \text{ and } p(\sigma(0)) = 1 \\ \top, & \text{if } \sigma = \langle\rangle \text{ and } p(0) = 2, \text{ or } |\sigma| = 1 \text{ and } p(\sigma(0)) = 2 \\ \forall, & \text{if } \sigma = \langle\rangle \text{ and } p(0) = 3, \text{ or } |\sigma| = 1 \text{ and } p(\sigma(0)) = 3 \\ \exists, & \text{if } \sigma = \langle\rangle \text{ and } p(0) = 4, \text{ or } |\sigma| = 1 \text{ and } p(\sigma(0)) = 4 \\ \mu'(\mathrm{shift}(\sigma)), & \text{if } |\sigma| > 1, \end{cases}$$

where (T', μ') is the Borel truth value named by $(\mathrm{shift}(p))_{\sigma(0)}$. In other words, intuitively in a name for a Borel truth value, zeroes indicate absence of the corresponding node, and nonzero values indicate both presence of the corresponding node and its tag. The *value* $\mathrm{Val}(b) \in \{\top, \bot\}$ of a Borel truth value b is defined by recursion on the rank of b in a straightforward way. The space of Borel truth values is denoted by $\mathbb{S}(\mathcal{B})$. The Σ^0_α *-truth values* (denoted $\mathbb{S}(\Sigma^0_\alpha)$) are those with rank $\leq \alpha$ and root tagged $\exists$, and the Π^0_α *-truth values* (denoted $\mathbb{S}(\Pi^0_\alpha)$) are those with rank $\leq \alpha$ and root tagged $\forall$.

Given an ordinal α, with $\alpha = \lambda + n$ for some limit ordinal λ and $n \in \mathbb{N}$, let $\hat{\alpha} = \lambda + 2n$ and $\check{\alpha} = \lambda + \lceil \frac{n}{2} \rceil$.

Proposition 4.20. *The map* $\mathrm{isPresent} :\subseteq \mathbb{LT} \times \mathbb{CO} \times \mathbb{N} \rightrightarrows \coprod_{\alpha \in \mathbb{CO}} \mathbb{S}(\Pi^0_\alpha)$*, mapping* (Υ, α, ℓ) *such that* $\mathrm{e}_{\mathbb{LT}}(\ell)$ *is linear to* $(\max\{1, \hat{\alpha}\}, b)$ *where* $\mathrm{Val}(b) = \top$ *iff* $\mathrm{e}_{\mathbb{LT}}(\ell) \subseteq \mathrm{PD}^\star(\Upsilon, \alpha)$*, is computable.*

Proof. It is straightforward to see that $\alpha \mapsto \max\{1, \hat{\alpha}\} : \mathbb{CO} \to \mathbb{CO}$ is computable. Computability of the second component is shown by induction over the δ_{nK} -name q of α provided. If $q = 0q'$, then we check whether $\mathrm{e}_{\mathbb{LT}}(\ell) \subseteq \Upsilon$ and return either the tree of rank 1 with root tagged $\forall$ and children tagged $\top$ (if yes), or with root tagged $\forall$ and children tagged $\bot$ (if no). If $q = 1q'$, then $\alpha = \beta + 1$ and q' is a name for β. Let h be the height of $\mathrm{e}_{\mathbb{LT}}(\ell)$. We start searching for confirmation that $\beta > 0$. Until we find it, we add children with tag $\exists$ to the root tagged $\forall$, and then for each $\ell' \in \mathbb{N}$ such that $\mathrm{e}_{\mathbb{LT}}(\ell')$ is a linear tree of height $h' > h$ extending $\mathrm{e}_{\mathbb{LT}}(\ell)$, we add a grandchild tagged $\top$ or $\bot$ to the $(h' - h)$ $^{\text{th}}$ child, depending on whether or not $\mathrm{e}_{\mathbb{LT}}(\ell') \subseteq \Upsilon$. If we do receive confirmation that $\beta > 0$, we add a grandchild tagged $\top$ to each $\exists$ -child produced so far, and then ignore these children. Then, for each $\ell' \in \mathbb{N}$ such that $\mathrm{e}_{\mathbb{LT}}(\ell')$ is a linear tree of height $h' > h$ extending $\mathrm{e}_{\mathbb{LT}}(\ell)$, we compute $b_{\ell'} \in \mathbb{S}(\Pi^0_{\max\{1,\hat{\beta}\}})$ denoting whether or not $\mathrm{e}_{\mathbb{LT}}(\ell') \subseteq \mathrm{PD}^\star(\Upsilon, \beta)$. Then we add each $b_{\ell'}$ as a grandchild of the root of b via the $(h' - h)$ $^{\text{th}}$ new child tagged $\exists$. If $q = 2^\ulcorner q_i \urcorner_{i \in \mathbb{N}}$, then $\alpha = \sup_{i \in \mathbb{N}} \alpha_i$ and each q_i is a name for α_i. For each $i \in \mathbb{N}$, we compute whether $\mathrm{e}_{\mathbb{LT}}(\ell) \subseteq \mathrm{PD}^\star(\Upsilon, \alpha_i)$ as $b_i \in \mathbb{S}(\Pi^0_{\max\{1,\hat{\alpha_i}\}})$. By induction, each b_i has root tagged $\forall$, and we now obtain the answer b as the mix of the b_i.

Claim 4.21. $\mathrm{Val}(b) = \top$ iff $\mathrm{e}_{\mathbb{LT}}(\ell) \subseteq \mathrm{PD}^\star(\Upsilon, \alpha)$.

By induction on the name q of α. If $q = 0q'$ then it is immediate to see that the claim holds. Suppose the claim holds for q' and let $q = 1q'$. Let $\beta = \delta_{\mathrm{nK}}(q')$ and suppose $\mathrm{e}_{\mathbb{LT}}(\ell)$

has height h. Then $\mathrm{e}_{\mathbb{LT}}(\ell) \subseteq \mathrm{PD}^\star(\Upsilon, \alpha)$ iff for every $n \in \mathbb{N}$ there exists some $\ell' \in \mathbb{N}$ such that $\mathrm{e}_{\mathbb{LT}}(\ell')$ is a linear tree of height $h + n$ extending $\mathrm{e}_{\mathbb{LT}}(\ell)$, with $\mathrm{e}_{\mathbb{LT}}(\ell') \subseteq \mathrm{PD}^\star(\Upsilon, \beta)$. By induction, the result of iterating the algorithm for (Υ, β, ℓ') gives the correct output. Therefore $\mathrm{Val}(b) = \top$ iff $\mathrm{e}_{\mathbb{LT}}(\ell) \subseteq \mathrm{PD}^\star(\Upsilon, \alpha)$. Finally, suppose the claim holds for each q_n and let $q = 2\ulcorner q_n \urcorner_{n\in\mathbb{N}}$. Let $\alpha_n = \delta_{\mathrm{nK}}(q_n)$. Again, by induction the result of iterating the algorithm for $(\Upsilon, \alpha_n, \ell)$ gives the correct output. Therefore $\mathrm{Val}(b) = \top$ iff all children of the roots of all b_n have value $\top$ iff $\mathrm{e}_{\mathbb{LT}}(\ell) \subseteq \mathrm{PD}^\star(\Upsilon, \alpha_n)$ for all $n \in \mathbb{N}$ iff $\mathrm{e}_{\mathbb{LT}}(\ell) \subseteq \mathrm{PD}^\star(\Upsilon, \alpha)$, as desired.

Claim 4.22. The Borel truth value b has rank $\leq \max\{1, \hat{\alpha}\}$.

We again proceed by induction on the name q of α. If $q = 0q'$ then by construction b has rank 1. If $q = 1q'$ and $\beta := \delta_{\mathrm{nK}}(q') = 0$, then again by construction b has rank $\leq 2 = \hat{1}$. If $q = 1q'$ and $\beta := \delta_{\mathrm{nK}}(q') > 0$, then by induction each b_τ as defined in the algorithm has rank $\leq \hat{\beta}$, and therefore b has rank $\leq \hat{\beta} + 2 = \hat{\alpha}$. Finally, if $q = 2\ulcorner q_i \urcorner_{i\in\mathbb{N}}$, for each i let $\alpha_i = \delta_{\mathrm{nK}}(q_i)$. By induction, each b_i as defined in the algorithm has rank $\leq \max\{1, \hat{\alpha}_i\}$, and by construction b has rank $\leq \max\{1, \sup_{i\in\mathbb{N}} \hat{\alpha}_i\} = \max\{1, \hat{\alpha}\}$. $\square$

Proposition 4.23. *The map* $\mathrm{Witness} :\subseteq \mathbb{S}(\mathcal{B}) \rightrightarrows \mathbb{UT}$*, mapping* b *of rank* $\alpha > 0$ *to some* T *such that if* $\mathrm{Val}(b) = \top$ *then* $\mathrm{PD}^\star(T, \check{\alpha})$ *is a nonempty pruned tree, and if* $\mathrm{Val}(b) = \bot$ *then* $\mathrm{PD}^\star(T, \check{\alpha}) = \varnothing$*, is computable.*

Proof. If b is composed of a single node, then we output $\mathbb{N}^{<\mathbb{N}}$ or $\varnothing$ according to whether $\mathrm{Val}(b) = \top$ or $\mathrm{Val}(b) = \bot$. Otherwise, we iteratively compute trees $(T_n)_{n\in\mathbb{N}}$ for all the subtrees rooted at the children of the root of b, and output $\boxtimes_{n\in\mathbb{N}} T_n$ if the root of b is tagged $\forall$, or output $\boxplus_{n\in\mathbb{N}} T_n$ if the root of b is tagged $\exists$.

Claim 4.24. Suppose $\beta > 0$ is such that $\mathrm{PD}^\star(T_n, \beta)$ is pruned for each $n \in \mathbb{N}$. Then $\mathrm{PD}^\star(T, \beta)$ is pruned, if the root of b is tagged $\forall$, and $\mathrm{PD}^\star(T, \beta + 1)$ is pruned, if the root of b is tagged $\exists$. Furthermore, if $\mathrm{PD}^\star(T, \delta)$ is pruned, then it is nonempty in case $\mathrm{Val}(b) = \top$ and empty in case $\mathrm{Val}(b) = \bot$.

If b is composed of a single node then the claim follows easily. Otherwise, suppose the root of b is tagged $\forall$, so that $T = \boxtimes_{n\in\mathbb{N}} T_n$. If $\mathrm{Val}(b) = \top$ then each $\mathrm{PD}^\star(T_n, \beta)$ is pruned and nonempty, and therefore the same holds for $\mathrm{PD}^\star(T, \beta)$. Conversely, if $\mathrm{Val}(b) = \bot$ then $\mathrm{PD}^\star(T_{n_0}, \beta) = \varnothing$ for some $n_0 \in \mathbb{N}$. Thus there is some $\gamma < \beta$ and $H \in \mathbb{N}$ such that $\mathrm{PD}^\star(T_{n_0}, \gamma)$ has height $\leq H < \omega$. Then $\mathrm{PD}^\star(T, \gamma)$ has height $\leq H' < \omega$ for some H' depending on H and n_0, and therefore $\mathrm{PD}^\star(T, \beta) = \varnothing$. Now suppose the root of b is tagged $\exists$, so that $T = \boxplus_{n\in\mathbb{N}} T_n$. If $\mathrm{Val}(b) = \top$ then some $\mathrm{PD}^\star(T_n, \beta)$ is pruned and nonempty. Therefore the same holds for $\mathrm{PD}^\star(T, \beta)$. Otherwise, if $\mathrm{Val}(b) = \bot$ then each $\mathrm{PD}^\star(T_n, \beta)$ is empty. Therefore $\mathrm{PD}^\star(T, \beta) \subseteq \{\langle\rangle\}$, and thus $\mathrm{PD}^\star(T, \beta + 1)$ is empty.

Claim 4.25. Let b' be a Borel truth value and let T' be the result of applying the algorithm above to b'. For $\beta = \mathrm{rk}(b')$, we have that if the root of b' has tag $\forall$ then $\mathrm{PD}^\star(T', \check{\beta})$ is pruned, and if the root of b' has tag $\exists$ then $\mathrm{PD}^\star(T', \check{\beta} + 1)$ is pruned.

By induction on β. If $\beta = 0$, i.e., if b' is a single node, then by construction T' is pruned. Now suppose $\beta > 0$, and let the n $^{\text{th}}$ child σ_n of the root of b' have rank $\beta_n < \beta$. Suppose the root of b' has tag $\exists$, so that each σ_n is either a leaf or has tag $\forall$. By induction, the result T_n of applying the algorithm to the subtree of b' rooted at σ_n is such that $\mathrm{PD}^\star(T_n, \check{\beta}_n)$ is pruned. Since $\sup_{n\in\mathbb{N}} \check{\beta}_n \leq \check{\beta}$, by the preceding claim it follows that $\mathrm{PD}^\star(T, \check{\beta} + 1)$ is pruned.

Finally, suppose the root of b' has tag $\forall$, so that each σ_n is either a leaf or has tag $\exists$. By induction, the result T_n of applying the algorithm to the subtree of b' rooted at σ_n is such that $\mathrm{PD}^\star(T_n, \check{\beta}_n + 1)$ is pruned. If $\sup_{n\in\mathbb{N}}(\check{\beta}_n + 1) \le \check{\beta}$ for each n, then by the preceding claim we are done. Otherwise, say $\check{\beta}_n = \check{\beta}$ for some n. Then β_n is odd and $\beta = \beta_n + 1$. In particular $\beta_n = \gamma_n + 1$ for some γ_n, and therefore $\check{\delta} \le \check{\gamma}_n < \check{\beta}_n$ for each $\delta < \beta_n$. Hence, since by induction the result S of applying the algorithm to any subtree of β' rooted at some child of σ_n is such that $\mathrm{PD}^\star(S, \check{\delta})$ is pruned for some $\delta < \beta_n$, by the preceding claim it follows that $\mathrm{PD}^\star(T_n, \check{\gamma}_n + 1) = \mathrm{PD}^\star(T_n, \check{\beta}_n)$ is pruned. Thus, again by the preceding claim, $\mathrm{PD}^\star(T, \check{\beta})$ is pruned. $\square$

Lemma 4.26. *For each $\alpha \in \mathbb{CO}$ we have that $\mathrm{PD}^\star(\,\cdot\,, \alpha)$ is parallelizable.*

Proof. Given abstract trees $\langle \mathcal{A}_n \rangle_{n\in\mathbb{N}}$ with respective representatives $\langle \Upsilon_n \rangle_{n\in\mathbb{N}}$, let $\mathcal{A}$ be the abstract tree represented by the labeled tree Υ in which the root has a child σ_n labeled n for each $n \in \mathbb{N}$ such that $\Upsilon_n \neq \varnothing$, and such that $\mathrm{Conc}(\Upsilon, \sigma_n) = \Upsilon_n$ in the positive case. It is now straightforward to see that each $\mathrm{PD}^\star(\mathcal{A}_n, \alpha)$ can be reconstructed from $\mathrm{PD}^\star(\mathcal{A}, \alpha)$. $\square$

Let $\underline{2}$ be the represented space composed of two elements, $\top$ and $\bot$, the first represented by $1^{\mathbb{N}}$ and the latter by $0^{\mathbb{N}}$.

Corollary 4.27. *For each $\alpha > 0$ we have that $\mathrm{PD}^\star(\,\cdot\,, \alpha)$ is Weihrauch-equivalent to the parallelization of $\mathrm{id}_\alpha : \mathbb{S}(\Pi^0_{\hat{\alpha}}) \to \underline{2}$. Furthermore, the reductions in both directions can be taken to be uniform in α.*

Proof. To reduce $\mathrm{PD}^\star(\,\cdot\,, \alpha)$ to the parallelization of id_α, note that we can use isPresent from Proposition 4.20 to compute for each linear labeled tree whether or not to include it in $\mathrm{PD}^\star(\Upsilon, \alpha)$ as a $\mathbb{S}(\Pi^0_{\hat{\alpha}})$-truth value. We then use the parallelization of $\mathrm{id}_\alpha : \mathbb{S}(\Pi^0_{\hat{\alpha}}) \to \underline{2}$ to convert all of these into booleans, and can thus construct $\mathrm{PD}^\star(\Upsilon, \alpha)$. For the converse, we use $\mathrm{Witness}(\hat{\alpha}, b)$ from Proposition 4.23 to obtain some Υ such that $\mathrm{PD}^\star(\Upsilon, \alpha) = \varnothing$ if $\mathrm{Val}(b) = \bot$ and $\mathrm{PD}^\star(\Upsilon, \alpha) \neq \varnothing$ if $\mathrm{Val}(b) = \top$. As $\{\varnothing\}$ is a decidable subset of $\mathbb{LT}$, we can recover $b \in \underline{2}$ after obtaining $\mathrm{PD}^\star(\Upsilon, \alpha)$ from the oracle. $\square$

Theorem 4.28 (Folklore)**.** *If $A \subseteq \mathbb{N}^{\mathbb{N}}$ is Wadge-complete for $\mathbf{\Pi}^0_\alpha$ (respectively effectively Wadge-complete for Π^0_α), then $\widehat{\chi_A}$ is continuously Weihrauch-complete for Baire class α (respectively Weihrauch-complete for effective Baire class α), where $\chi_A : \mathbb{N}^{\mathbb{N}} \to \underline{2}$ is given by $\chi_A(x) = \top$ iff $x \in A$.*

Proof. That $\widehat{\chi_A}$ is Baire class α follows from noticing that

$$\widehat{\chi_A}^{-1}[\sigma] = \bigcap_{\substack{n<|\sigma| \\ \sigma(n)=0}} \{x \in \mathbb{N}^{\mathbb{N}}\,;\, (x)_n \in A\} \cap \bigcap_{\substack{n<|\sigma| \\ \sigma(n)=1}} \{x \in \mathbb{N}^{\mathbb{N}}\,;\, (x)_n \notin A\},$$

which is the intersection of a $\mathbf{\Pi}^0_\alpha$ set with a $\mathbf{\Sigma}^0_\alpha$ set.

Now let $F :\subseteq \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ be a Baire class α realizer of $f :\subseteq \mathbb{X} \rightrightarrows \mathbb{Y}$. Let $\langle \sigma_n\,;\, n \in \mathbb{N} \rangle$ be some enumeration of $\mathbb{N}^{<\mathbb{N}}$. Since F is Baire class α, there exists some countable collection $\langle X_{n,m}\,;\, n, m \in \mathbb{N} \rangle$ of $\mathbf{\Sigma}^0_\alpha$ sets such that $F^{-1}[\sigma_n] = \bigcup_{m\in\mathbb{N}} X_{n,m}$. Since A is Wadge-complete for $\mathbf{\Pi}^0_\alpha$, for each $n, m \in \mathbb{N}$ there exists a continuous $f_{n,m} : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ such that $X_{n,m} = f^{-1}_{n,m}[\mathbb{N}^{\mathbb{N}} \smallsetminus A]$. Now, defining a continuous $K : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ by $(K(x))_{\ulcorner n,m \urcorner} = f_{n,m}(x)$, we have $\sigma_n \subseteq F(x)$ iff $x \in X_{n,m}$ for some m iff $\widehat{\chi_A}(K(x))(\ulcorner n, m \urcorner) = 1$ for some m. Finally, defining a continuous $H :\subseteq \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ by $H(x) = \bigcup\{\sigma_n\,;\, \exists m (x(\ulcorner n, m \urcorner) = 1\}$ with its natural domain, we have $H\widehat{\chi_A}K \preceq F$. Therefore $F \le_{\mathrm{sW}} \widehat{\chi_A}$, and $f \le_{\mathrm{sW}} \widehat{\chi_A}$ as well. $\square$

Corollary 4.29. *For each $\alpha > 0$ the parallelization of the map* $\mathrm{id}_\alpha : \mathbb{S}(\Pi^0_{\hat\alpha}) \to \underline{2}$ *is (continuously) Weihrauch-complete for (effective) Baire class α.*

Proof. It is enough to show that for each $\alpha > 0$ the characteristic function of any $\mathbf{\Pi}^0_\alpha$ set is continuously Weihrauch-reducible to id_α, and that id_α is Weihrauch-reducible to the characteristic function of some $\mathbf{\Pi}^0_\alpha$ set. Both of these claims can be easily proved by induction. □

Corollary 4.30. $\mathrm{PD}^\star(\,\cdot\,,\alpha)$ *is Weihrauch-complete for Baire class $\hat\alpha$.*

Corollary 4.31. $\mathrm{PD}^\star(\,\cdot\,,\,\cdot\,) \equiv_{\mathrm{W}} \mathrm{UC}_{\mathbb{N}^\mathbb{N}}$.

Proof. It was shown in [Pau1X] that $\mathrm{id} : \widehat{\mathbb{S}(\mathcal{B})} \to \underline{2} \equiv_{\mathrm{W}} \mathrm{UC}_{\mathbb{N}^\mathbb{N}}$. □

4.4. **The transparency of the pruning derivative.**

Proposition 4.32. *The map* $\mathrm{isAbsent} : \subseteq \mathbb{LT} \times \mathbb{CO} \times \mathbb{N} \rightrightarrows \coprod_{\alpha \in \mathbb{CO}} \mathbb{S}(\Pi^0_\alpha)$, *mapping* (Υ, α, ℓ) *such that* $\mathrm{e}_{\mathbb{LT}}(\ell)$ *is linear to* $(\max\{1, \alpha_\uparrow\}, b)$ *where* $\mathrm{Val}(b) = \top$ *iff* $\mathrm{e}_{\mathbb{LT}}(\ell) \not\subseteq \mathrm{PD}^\star(\Upsilon, \alpha)$, *is computable.*

Proof. We just run isPresent on (Υ, α, ℓ) then dualize the output by exchanging tags $\forall$ with $\exists$ and $\top$ with $\bot$. □

Corollary 4.33. *The operation* $\mathrm{Neg} : \mathbb{LT} \times \mathbb{CO} \times \mathbb{N} \rightrightarrows \mathbb{UT}$, *given by* $S \in \mathrm{Neg}(\Upsilon, \alpha, m)$ *iff* $\mathrm{PD}^\star(S, \alpha)$ *is a pruned tree and* $\mathrm{PD}^\star(S, \alpha) \neq \varnothing$ *iff* $\mathrm{e}_{\mathbb{LT}}(m) \subseteq \mathrm{PD}^\star(\Upsilon, \alpha)$, *is computable.*

Proof. Let $\Upsilon_0, \ldots, \Upsilon_n$ be the linear subtrees of $\mathrm{e}_{\mathbb{LT}}(m)$, and for each $i \leq n$ let $\ell_i \in \mathbb{N}$ be such that $\mathrm{e}_{\mathbb{LT}}(\ell_i) = \Upsilon_i$. By Propositions 4.20 and 4.23, letting $S_i \in \mathrm{Witness} \circ \mathrm{isPresent}(\Upsilon, \alpha, \ell_i)$, we have that $\mathrm{PD}^\star(S_i, \alpha)$ is pruned and $\mathrm{PD}^\star(S_i, \alpha) \neq \varnothing$ iff $\Upsilon_i \subseteq \mathrm{PD}^\star(\Upsilon, \alpha)$. Now letting $S = \bigotimes_{i \leq n} S_i$ we have that $\mathrm{PD}^\star(S, \alpha)$ is pruned and that $\mathrm{PD}^\star(S, \alpha) \neq \varnothing$ iff $\mathrm{e}_{\mathbb{LT}}(m) \subseteq \mathrm{PD}^\star(\Upsilon, \alpha)$, as desired. □

Corollary 4.34. *The operation* $\mathrm{WitnessAbsence} : \mathbb{LT} \times \mathbb{CO} \times \mathbb{N}^\mathbb{N} \rightrightarrows \mathbb{UT}$, *given by* $S \in \mathrm{WitnessAbsence}(\Upsilon, \alpha, x)$ *iff in case* $\alpha > 0$ *then* $\mathrm{PD}^\star(S, \alpha)$ *is a pruned tree and* $\mathrm{PD}^\star(S, \alpha) = \varnothing$ *iff* $\mathrm{e}_{\mathbb{LT}}(x(m)) \subseteq \mathrm{PD}^\star(\Upsilon, \alpha)$ *for some* $m \in \mathbb{N}$, *is computable.*

Proof. For each m and each linear subtree $\Upsilon^m_0, \ldots, \Upsilon^m_{n_m}$ of $\mathrm{e}_{\mathbb{LT}}(x(m))$, let $\ell^m_i \in \mathbb{N}$ be such that $\mathrm{e}_{\mathbb{LT}}(\ell^m_i) = \Upsilon^m_i$. By Propositions 4.32 and 4.23, letting $S^m_i \in \mathrm{Witness} \circ \mathrm{isAbsent}(\Upsilon, \alpha, \ell^m_i)$, we have that $\mathrm{PD}^\star(S^m_i, \alpha)$ is a pruned tree and $\mathrm{PD}^\star(S^m_i, \alpha) = \varnothing$ iff $\Upsilon^m_i \subseteq \mathrm{PD}^\star(\Upsilon, \alpha)$. Now letting $S^m = \bigotimes_{i \leq n_m} S^m_i$ we have that $\mathrm{PD}^\star(S^m, \alpha)$ is pruned and that $\mathrm{PD}^\star(S^m, \alpha) = \varnothing$ iff $\mathrm{e}_{\mathbb{LT}}(x(m)) \subseteq \mathrm{PD}^\star(\Upsilon, \alpha)$. Now let $S = \boxtimes_{m \in \mathbb{N}} S^m$. Suppose $\alpha > 0$, and first suppose that $\mathrm{e}_{\mathbb{LT}}(x(m)) \subseteq \mathrm{PD}^\star(\Upsilon, \alpha)$ for some $m \in \mathbb{N}$. Then $\mathrm{PD}^\star(S^m, \alpha) = \varnothing$, so for some $\beta < \alpha$ we have that $\mathrm{PD}^\star(S^m, \beta)$ has some finite height H. Hence $\mathrm{PD}^\star(S, \beta)$ also has some finite height H' (which depends on H and m), and therefore $\mathrm{PD}^\star(S, \alpha) = \varnothing$, as desired. Now suppose $\mathrm{e}_{\mathbb{LT}}(x(m)) \not\subseteq \mathrm{PD}^\star(\Upsilon, \alpha)$ for all $m \in \mathbb{N}$. Then each $\mathrm{PD}^\star(S^m, \alpha)$ is pruned and nonempty, and therefore the same holds for $\mathrm{PD}^\star(S, \alpha)$. □

Proposition 4.35 (Pauly [Pau1X, Theorem 31])**.** *The function* $\min : \mathbb{CO} \times \mathbb{CO} \to \mathbb{CO}$ *is computable.*

Proposition 4.36. *The function* $\mathrm{TreeWithRank} : \mathbb{CO} \rightrightarrows \mathbb{UT}$, *given by*

$$T \in \mathrm{TreeWithRank}(\alpha) \quad \text{iff} \quad T \text{ is a wellfounded tree and } \mathrm{rk}(T) = \alpha,$$

is computable.

Proof. We will define a computable $F \vdash \mathrm{TreeWithRank}$. Given $p \in \mathrm{dom}(\delta_{\mathrm{nK}})$, if $p(0) = 0$ we let $F(p) = 10^{\mathbb{N}}$, i.e., a code for the tree $\{\langle\rangle\}$. If $p = 1q$, we let $F(p)$ be a code for the tree $T := \{\langle\rangle\} \cup \{\langle 0\rangle^\frown\sigma\,;\, \sigma \in \delta_{\mathbb{UT}}F(p)\}$. Finally, if $p = 2q_0q_1\ldots$, we let $F(p)$ be a code for the mix of the trees coded by the $F(q_n)$. It is now routine to check that $F \vdash \mathrm{TreeWithRank}$. □

Corollary 4.37. *The operation* $\mathrm{Pos} : \mathbb{LT} \times \mathbb{CO} \times \mathbb{N} \rightrightarrows \mathbb{UT}$*, given by* $S \in \mathrm{Pos}(\Upsilon, \alpha, n)$ *iff*

$$\mathrm{PD}^\star(S,\alpha) = \begin{cases} \{\langle\rangle\}, & \text{if } \mathrm{e}_{\mathbb{LT}}(n) \subseteq \mathrm{PD}^\star(\Upsilon,\alpha) \\ \varnothing, & \text{otherwise,} \end{cases}$$

is computable.

Proof. Given a labeled tree $\Upsilon = (T, \varphi)$, a countable ordinal α, and a natural number n, we output a tree S of rank $\beta := \min(\{\omega \cdot \alpha\} \cup \{\mathrm{rk}_T(\sigma)\,;\, \sigma \in \mathrm{e}_{\mathbb{LT}}(n)\})$.

We have $\mathrm{PD}^\star(S,\alpha) = \{\langle\rangle\}$ iff $\beta = \omega\cdot\alpha$ iff $\mathrm{rk}_T(\sigma) \geq \omega \cdot \alpha$ for each $\sigma \in \mathrm{e}_{\mathbb{LT}}(n)$ iff $\sigma \in \mathrm{PD}^\star(\Upsilon,\alpha)$ for each $\sigma \in \mathrm{e}_{\mathbb{LT}}(n)$ iff $\mathrm{e}_{\mathbb{LT}}(n) \subseteq \mathrm{PD}^\star(\Upsilon,\alpha)$, and $\mathrm{PD}^\star(S,\alpha) = \varnothing$ otherwise. □

Definition 4.38. We define $\mathrm{Graft} : \mathbb{LT} \times \mathbb{UT} \times \mathbb{UT} \times \mathbb{N}^{<\mathbb{N}} \to \mathbb{LT}$ by

(1) $\mathrm{Graft}(\Upsilon, S, U, \sigma) \setminus \mathrm{Ext}(\mathbb{N}^{<\mathbb{N}}, \sigma) = \Upsilon \setminus \mathrm{Ext}(\mathbb{N}^{<\mathbb{N}}, \sigma)$
(2) $\mathrm{Conc}(\mathrm{Graft}(\Upsilon, S, U, \sigma), \sigma) = (\mathrm{Conc}(\Upsilon, \sigma) \otimes S) \oplus U$

Definition 4.39. We define $\mathrm{Aux} : \mathbb{LT} \times \mathbb{LT} \times \mathbb{UT} \times \mathbb{CO} \times \mathbb{N}^{<\mathbb{N}} \times \mathbb{N}^{<\mathbb{N}} \rightrightarrows \mathbb{LT}$ as follows. Given $\Upsilon, \Upsilon_{\mathrm{aux}} \in \mathbb{LT}$, $\alpha \in \mathbb{CO}$, and $\sigma, \tau \in \mathbb{N}^{<\mathbb{N}}$ such that $|\sigma| = |\tau| > 0$, let $\Upsilon' \in \mathrm{Aux}(\Upsilon, \Upsilon_{\mathrm{aux}}, U, \alpha, \sigma, \tau)$ iff $\Upsilon' = \mathrm{Graft}(\Upsilon, S_{\mathrm{N}}, S_{\mathrm{P}} \otimes U, \sigma)$ for some

$$\begin{aligned} S_{\mathrm{N}} &\in \mathrm{Neg}(\Upsilon_{\mathrm{aux}}, \alpha, \perp(\tau)) \\ S_{\mathrm{P}} &\in [\textstyle\bigotimes_{n<|\tau|-1}(\mathrm{Neg}(\Upsilon_{\mathrm{aux}}, \alpha, \tau(n)))] \otimes \mathrm{Pos}(\Upsilon_{\mathrm{aux}}, \alpha, \perp(\tau)). \end{aligned}$$

Recall from Theorem 2.11 that every computable or continuous multi-valued function between represented spaces is tightened by a strongly computable or strongly continuous, respectively, multi-valued function between the same spaces. Therefore, in order to conclude that $\mathrm{PD}^\star(\,\cdot\,,\alpha)$ is transparent for each α, it is enough to prove the following stronger result.

Theorem 4.40. *There is a computable operation* $\mathrm{Trans} : \mathcal{M}(\mathbb{AT}, \mathbb{AT}) \times \mathbb{CO} \rightrightarrows \mathcal{M}(\mathbb{AT}, \mathbb{AT})$ *such that* $g \in \mathrm{Trans}(f, \alpha)$ *iff* $\mathrm{dom}(f\mathrm{PD}^\star(\,\cdot\,,\alpha)) \subseteq \mathrm{dom}(g)$ *and* $\mathrm{PD}^\star(\mathcal{A}_g, \alpha) \in f(\mathrm{PD}^\star(\mathcal{A}, \alpha))$ *for any* $\mathcal{A} \in \mathrm{dom}(f\mathrm{PD}^\star(\,\cdot\,,\alpha))$ *and* $\mathcal{A}_g \in g(\mathcal{A})$.

Proof. Let $f \in \mathcal{M}(\mathbb{AT}, \mathbb{AT})$ be given in the form of a Turing machine M which strongly computes f with some given oracle q. Let $F : \subseteq \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ be defined with $\mathrm{dom}(F) = \mathrm{dom}(f\delta_{\mathbb{AT}})$ by letting $F(p)$ be the output of M on input $\ulcorner p, 0^{\mathbb{N}} \urcorner$ and oracle q. Thus F is a computable realizer of f, so by Corollary 4.6 we can assume that for each m there exists a computable subset $X_m \subseteq \mathbb{N}$ such that $\mathrm{e}_{\mathbb{LT}}(m) \subseteq \delta_{\mathbb{LT}}F(p)$ iff $\mathrm{e}_{\mathbb{LT}}(n) \subseteq \delta_{\mathbb{LT}}(p)$ for some $n \in X_m$. Thus we can construct a computable labeled tree Υ_F which *represents* F, as follows. The nodes of length 1 of Υ_F are bijectively associated to the pairs (n, ℓ) such that $\mathrm{e}_{\mathbb{LT}}(\ell)$ is a linear tree of height 1 and $n \in X_\ell$. If $\sigma \in \Upsilon_F$ is associated to (n, ℓ), then

(1) the label of σ in Υ_F is the label of the node of $\mathrm{e}_{\mathbb{LT}}(\ell)$ at height $|\sigma|$, and
(2) the children of σ in Υ_F are bijectively associated to the pairs (n', ℓ') such that $\mathrm{e}_{\mathbb{LT}}(\ell')$ is a linear tree of height $|\sigma| + 1$, $n' \in X_{\ell'}$, and $\mathrm{e}_{\mathbb{LT}}(\ell) \subseteq \mathrm{e}_{\mathbb{LT}}(\ell')$.

It is now straightforward to check that if $\delta_{\mathbb{LT}}F(p)$ is not empty then it is bisimilar to the subtree Υ_p of Υ_F composed of the root plus those σ which are associated to (n, ℓ) with $\mathrm{e}_{\mathbb{LT}}(n) \subseteq \delta_{\mathbb{LT}}(p)$.

Formally, our goal now is to computably define a Turing machine M' from M, q, and α, such that the function $g := g_{M',q}$ from the proof of Theorem 2.11 has the desired properties. To simplify the presentation, we will define g directly and leave the definition of M' implicit. Thus, we want to define a computable $g :\subseteq \mathbb{LT} \rightrightarrows \mathbb{LT}$ such that for any $p \in \mathrm{dom}(F)$ and any $\Upsilon' \in g(\delta_{\mathbb{LT}}(p))$, letting $\delta_{\mathbb{LT}}(p') = \mathrm{PD}^\star(\delta_{\mathbb{LT}}(p), \alpha)$, we have:

(1) if $\delta_{\mathbb{LT}}F(p) \neq \varnothing$ then $\mathrm{PD}^\star(\Upsilon', \alpha) \rightleftarrows \Upsilon_{p'}$;
(2) if $\delta_{\mathbb{LT}}F(p) = \varnothing$ then $\mathrm{PD}^\star(\Upsilon', \alpha) = \varnothing$.

Again, since F is computable, there exists a computable $z \in \mathbb{N}^\mathbb{N}$ such that $\delta_{\mathbb{LT}}F(p) = \varnothing$ iff $\mathrm{e}_{\mathbb{LT}}(z(n)) \subseteq \delta_{\mathbb{LT}}(p)$ holds for some $n \in \mathbb{N}$. Given $p \in \mathrm{dom}(f\mathrm{PD}^\star(\,\cdot\,, \alpha)\delta_{\mathbb{AT}})$, let $\Upsilon := \delta_{\mathbb{LT}}(p)$ and $U \in \mathrm{WitnessAbsence}(\Upsilon, \alpha, z)$. Therefore, if $\delta_{\mathbb{LT}}F(p) \neq \varnothing$ then $\mathrm{e}_{\mathbb{LT}}(z(m)) \not\subseteq \delta_{\mathbb{LT}}(p)$ for all $m \in \mathbb{N}$ and thus $\mathrm{PD}^\star(U, \alpha)$ is pruned and nonempty, and if $\delta_{\mathbb{LT}}F(p) = \varnothing$ then $\mathrm{e}_{\mathbb{LT}}(z(m)) \subseteq \delta_{\mathbb{LT}}(p)$ for some $m \in \mathbb{N}$ and thus $\mathrm{PD}^\star(U, \alpha) = \varnothing$. Let $V \in \mathrm{TreeWithRank}(\omega \cdot \alpha)$, so that $\mathrm{PD}^\star(V, \alpha) = \{\langle\rangle\}$. Let $\Upsilon_0 = (\Upsilon_F \oplus V) \otimes U$, so that if $\delta_{\mathbb{LT}}F(p) \neq \varnothing$ then $\mathrm{PD}^\star(\Upsilon_0, \alpha) \rightleftarrows \mathrm{PD}^\star(\Upsilon_F, \alpha)$, and if $\delta_{\mathbb{LT}}F(p) = \varnothing$ then $\mathrm{PD}^\star(\Upsilon_0, \alpha) = \varnothing$. We let any node in Υ_0 coming from Υ_F be associated to the same pair (n, ℓ) as the corresponding node in Υ_F.

Now suppose we are at stage $s > 0$ of the construction, so that we have already built a tree Υ_{s-1}. Let $\sigma := \mathrm{e}_\mathbb{N}(s)$, where $\mathrm{e}_\mathbb{N} : \mathbb{N} \to \mathbb{N}^{<\mathbb{N}}$ is any computable bijection such that $\mathrm{e}_\mathbb{N}(s) \subseteq \mathrm{e}_\mathbb{N}(s')$ implies $s \leq s'$. If $\sigma \notin \Upsilon_{s-1}$ or $\sigma \in \Upsilon_{s-1}$ but is not associated to any (n, ℓ), then let $\Upsilon_s = \Upsilon_{s-1}$. Otherwise suppose $\sigma{\upharpoonright}(m+1)$ is associated to some (n_m, ℓ_m) for each $m < |\sigma|$. Let $*\sigma := \langle n_0, \ldots, n_{|\sigma|-1}\rangle$ and define $\Upsilon_s := \mathrm{Aux}(\Upsilon_{s-1}, \Upsilon, U, \alpha, \sigma, *\sigma)$. Recall that in this case we have

$$\mathrm{Conc}(\Upsilon_s, \sigma) = (\mathrm{Conc}(\Upsilon_{s-1}, \sigma) \otimes S_\mathrm{N}) \oplus (S_\mathrm{P} \otimes U)$$

for some $S_\mathrm{N}, S_\mathrm{P}$ as in the definition of Aux. Hence we let each descendant $\sigma^\frown\tau$ of σ in Υ_s in which τ comes from the $\mathrm{Conc}(\Upsilon_{s-1}, \sigma) \otimes S_\mathrm{N}$ component of $\oplus$ above be associated to the same (n, ℓ) as the corresponding node in Υ_{s-1}.

We then define Υ' by letting $\sigma \in \Upsilon'$ iff $\sigma \in \Upsilon_s$ for $s = \mathrm{e}_\mathbb{N}^{-1}(\sigma)$, with the label for σ being its label in Υ_s in the positive case.

Claim 4.41. Every node of $\mathrm{PD}^\star(\Upsilon', \alpha)$ other than the root is associated to some pair (n, ℓ).

Indeed, it is easy to see that this is true of Υ_0. Thus if a node $\xi \in \mathrm{PD}^\star(\Upsilon', \alpha)$ is not associated to some such pair, this means that ξ was added to Υ' at some stage $s > 0$ of the construction. Let $\sigma = \mathrm{e}_\mathbb{N}(s)$. In this case we have $\Upsilon_s := \mathrm{Aux}(\Upsilon_{s-1}, \Upsilon, U, \alpha, \sigma, *\sigma)$, i.e., $\Upsilon_s = \mathrm{Graft}(\Upsilon_{s-1}, S_\mathrm{N}, S_\mathrm{P} \otimes U, \sigma)$ for some

$$\begin{aligned} S_\mathrm{N} &\in \mathrm{Neg}(\Upsilon, \alpha, \bot(*\sigma)) \text{ and} \\ S_\mathrm{P} &\in [\textstyle\bigotimes_{m<|\sigma|-1}(\mathrm{Neg}(\Upsilon, \alpha, *\sigma(m)))] \otimes \mathrm{Pos}(\Upsilon, \alpha, \bot(*\sigma)). \end{aligned}$$

The fact that ξ is not associated to any pair (n, ℓ) implies that $\xi = \sigma^\frown\eta$ for some $\eta \neq \langle\rangle$ coming from $S_\mathrm{P} \otimes U$. By construction the subtree of ξ in Υ' is the same as in Υ_s, since for any $s' > s$ such that $\mathrm{e}_\mathbb{N}(s') \supseteq \xi$ we have $\Upsilon_{s'} = \Upsilon_{s'-1}$, and for any $s' > s$ such that $\sigma' = \mathrm{e}_\mathbb{N}(s') \not\supseteq \xi$ we have $\Upsilon_{s'} \smallsetminus \mathrm{Ext}(\mathbb{N}^{<\mathbb{N}}, \sigma') = \Upsilon_{s'-1} \smallsetminus \mathrm{Ext}(\mathbb{N}^{<\mathbb{N}}, \sigma')$. Hence $\mathrm{Conc}(\mathrm{PD}^\star(\Upsilon', \alpha), \xi) \subseteq \mathrm{Conc}(\mathrm{PD}^\star(S_\mathrm{P} \otimes U, \alpha), \eta) = \varnothing$, i.e., $\xi \notin \mathrm{PD}^\star(\Upsilon', \alpha)$.

Claim 4.42. If $\delta_{\mathbb{LT}}F(p) = \varnothing$ then $\mathrm{PD}^\star(\Upsilon', \alpha) = \varnothing$.

Indeed, if $\delta_{\mathbb{LT}}F(p) = \varnothing$ then $\mathrm{PD}^\star(U, \alpha) = \varnothing$. Hence $\mathrm{PD}^\star(\Upsilon_0, \alpha) = \varnothing$, and at each stage $s > 0$ we either keep $\Upsilon_s = \Upsilon_{s-1}$, or else Υ_s differs from Υ_{s-1} only in that

$$\mathrm{Conc}(\Upsilon_s, \sigma) = (\mathrm{Conc}(\Upsilon_{s-1}, \sigma) \otimes S_\mathrm{N}) \oplus (S_\mathrm{P} \otimes U)$$

for some $S_{\mathrm{N}}, S_{\mathrm{P}}$ as in the definition of $\mathrm{Aux}(\Upsilon_{s-1}, \Upsilon, U, \alpha, \sigma, *\sigma)$, where $\sigma = \mathrm{e}_{\mathbb{N}}(s)$. But then we have that

$$\begin{aligned}&\mathrm{Conc}(\mathrm{PD}^\star(\Upsilon_s, \alpha), \sigma)\\ &\qquad = (\mathrm{Conc}(\mathrm{PD}^\star(\Upsilon_{s-1}, \alpha), \sigma) \otimes \mathrm{PD}^\star(S_{\mathrm{N}}, \alpha)) \oplus \mathrm{PD}^\star(S_{\mathrm{P}} \otimes U, \alpha)\\ &\qquad = \mathrm{Conc}(\mathrm{PD}^\star(\Upsilon_{s-1}, \alpha), \sigma) \otimes \mathrm{PD}^\star(S_{\mathrm{N}}, \alpha),\end{aligned}$$

so assuming by induction that $\mathrm{PD}^\star(\Upsilon_{s-1}, \alpha) = \varnothing$ holds, it follows that $\mathrm{Conc}(\mathrm{PD}^\star(\Upsilon_s, \alpha), \sigma) = \varnothing$ as well. But then $\mathrm{PD}^\star(\Upsilon_s, \alpha) = \varnothing$, as desired. Therefore we have $\mathrm{PD}^\star(\Upsilon', \alpha) = \varnothing$.

For the rest of the proof we assume that $\delta_{\mathbb{LT}} F(p) \neq \varnothing$, which implies that $\mathrm{PD}^\star(U, \alpha)$ is a pruned and nonempty tree. Furthermore, since $\mathrm{PD}^\star(V, \alpha) = \{\langle\rangle\}$, we have $\langle\rangle \in \mathrm{PD}^\star(\Upsilon', \alpha)$.

Claim 4.43. Suppose $\sigma \in \Upsilon' \smallsetminus \{\langle\rangle\}$. Then $\sigma \in \mathrm{PD}^\star(\Upsilon', \alpha)$ iff $\mathrm{e}_{\mathbb{LT}}(*\sigma(m)) \subseteq \mathrm{PD}^\star(\Upsilon, \alpha)$ for each $m < |\sigma|$.

Let $s = \mathrm{e}_{\mathbb{N}}^{-1}(\sigma)$. Suppose $\mathrm{e}_{\mathbb{LT}}(*\sigma(m)) \nsubseteq \mathrm{PD}^\star(\Upsilon, \alpha)$ for some $m < |\sigma|$. Let $s' = \mathrm{e}_{\mathbb{N}}^{-1}(\sigma \restriction (m{+}1))$. Note that $\mathrm{PD}^\star(S_{\mathrm{N}}, \alpha) = \mathrm{PD}^\star(S_{\mathrm{P}} \otimes U, \alpha) = \varnothing$ for any $S_{\mathrm{N}} \in \mathrm{Neg}(\Upsilon, \alpha, *\sigma(m))$ and $S_{\mathrm{P}} \in \mathrm{Pos}(\Upsilon, \alpha, *\sigma(m))$. Thus we also have $\mathrm{Conc}(\mathrm{PD}^\star(\Upsilon_s, \alpha), \sigma \restriction (m+1)) = \varnothing$. Put together, and also considering the preceding claim, the two last statements imply $\tau \notin \mathrm{PD}^\star(\Upsilon_{s''}, \alpha)$ for any $\tau \supseteq \sigma \restriction (m{+}1)$. Thus $\sigma \notin \mathrm{PD}^\star(\Upsilon', \alpha)$. Conversely, suppose $\mathrm{e}_{\mathbb{LT}}(*\sigma(m)) \subseteq \mathrm{PD}^\star(\Upsilon, \alpha)$ for every $m < |\sigma|$. Then for any

$$S_{\mathrm{P}} \in [\bigotimes_{m<|\sigma|-1} (\mathrm{Neg}(\Upsilon, \alpha, *\sigma(m)))] \otimes \mathrm{Pos}(\Upsilon, \alpha, \bot(*\sigma))$$

we have $\mathrm{PD}^\star(S_{\mathrm{P}} \otimes U, \alpha) = \mathrm{PD}^\star(S_{\mathrm{P}}, \alpha) = \{\langle\rangle\}$. In particular it follows that the descendants of σ in Υ_s which are not associated to any (n, ℓ) already guarantee that $\sigma \in \mathrm{PD}^\star(\Upsilon', \alpha)$, as desired.

Let p' be such that $\delta_{\mathbb{LT}}(p') = \mathrm{PD}^\star(\Upsilon, \alpha)$.

Claim 4.44. The trees $\mathrm{PD}^\star(\Upsilon', \alpha)$ and $\Upsilon_{p'}$ are bisimilar.

Define $B \subseteq \mathrm{PD}^\star(\Upsilon', \alpha) \times \Upsilon_{p'}$ by letting $\sigma\ B\ \tau$ iff $\sigma = \tau = \langle\rangle$ or $|\sigma| = |\tau|$, $\sigma \restriction n\ B\ \tau \restriction n$ for each $n < |\sigma|$, and σ and τ are associated to the same pair (n, ℓ). In order to verify that B is a bisimulation, the only nontrivial properties to check are (back) and (forth). So suppose $\sigma\ B\ \tau$ and for (back) let τ' be a child of τ in $\Upsilon_{p'}$. Then τ' is associated to some (n', ℓ') where $\mathrm{e}_{\mathbb{LT}}(n') \subseteq \delta_{\mathbb{LT}}(p') = \mathrm{PD}^\star(\Upsilon, \alpha)$. But then by construction σ has some child σ' in Υ' associated to (n', ℓ'). By Claim 4.43 we have $\sigma' \in \mathrm{PD}^\star(\Upsilon', \alpha)$, and $\sigma'\ B\ \tau'$ follows. Finally, for (forth) let σ' be a child of σ in $\mathrm{PD}^\star(\Upsilon', \alpha)$. Again by Claim 4.43 we get that σ' is associated to some (n', ℓ') such that $\mathrm{e}_{\mathbb{LT}}(n') \subseteq \mathrm{PD}^\star(\Upsilon, \alpha)$. But then τ must have a child τ' in $\Upsilon_{p'}$ which is also associated to (n', ℓ'), and therefore $\sigma'\ B\ \tau'$.

Our assumption that $\delta_{\mathbb{LT}} F(p) \neq \varnothing$ implies that both $\mathrm{PD}^\star(\Upsilon', \alpha)$ and $\Upsilon_{p'}$ are nonempty trees, and $B \neq \varnothing$. Hence we have $\mathrm{PD}^\star(\Upsilon', \alpha) \rightleftarrows \Upsilon_{p'}$ as desired. ☐

Theorem 4.45. *The operation* $\mathrm{PD}^\star(\,\cdot\,, \alpha)$ *is a transparent cylinder.*

Proof. Transparency follows directly from Theorem 4.40. To see that $\mathrm{PD}^\star(\,\cdot\,, \alpha)$ is a cylinder, given a code p of an abstract tree $\mathcal{A}$, let $\mathcal{A}_p$ be the abstract tree obtained from $\mathcal{A}$ by changing each of its labels ℓ to $\ulcorner 1, \ell \urcorner$ plus adding an infinite path with induced label $\langle \ulcorner 0, p(n) \urcorner \rangle_{n\in\mathbb{N}}$. Then $\mathrm{PD}^\star(\mathcal{A}_p, \alpha)$ is obtained from $\mathrm{PD}^\star(\mathcal{A}, \alpha)$ by the same change of labels as above plus the addition of the same infinite path. Now both p and $\mathrm{PD}^\star(\mathcal{A}, \alpha)$ can easily be reconstructed from $\mathrm{PD}^\star(\mathcal{A}_p, \alpha)$ without needing direct access to p; in other words, $\mathrm{id}_{\mathbb{N}^\mathbb{N}} \times \mathrm{PD}^\star(\,\cdot\,, \alpha) \leq_{\mathrm{sW}} \mathrm{PD}^\star(\,\cdot\,, \alpha)$. ☐

Let $\mathbb{AT}_{\text{lin}}$ be the subspace of $\mathbb{AT}$ composed of the linear abstract trees, and let $\mathbb{AT}^*_{\text{lin}}$ be the subspace of $\mathbb{AT}_{\text{lin}}$ composed of the linear abstract trees which have a unique infinite induced label. The spaces $\mathbb{AT}_{\text{fb}}$ and $\mathbb{AT}^*_{\text{fb}}$ are defined analogously for finitely branching trees. Note that $\mathbb{AT}^*$lin is composed exactly of the nonempty pruned linear trees. Let Prune_{fb} be the restriction of PD to $\mathbb{AT}^*_{\text{fb}}$, and note that $\text{Prune}_{\text{fb}} : \mathbb{AT}^*_{\text{fb}} \to \mathbb{AT}^*_{\text{lin}}$.

Lemma 4.46. *The operation* Prune_{fb} *is Weihrauch-equivalent to* lim.

Proof. ($\lim \leq_{\text{W}} \text{Prune}_{\text{fb}}$) Given $p \in \text{dom}(\lim)$, we can build an abstract finitely branching tree whose induced labels are exactly the sequences of the form $(p)_n{\restriction}n$. Since $\lim(p)$ is well defined, this tree is in the domain of Prune_{fb}; applying this map to this tree results in a linear tree with an infinite branch labeled $\lim(p)$.

($\text{Prune}_{\text{fb}} \leq_{\text{W}} \lim$) Given a name p of an abstract tree $\mathcal{A}$ in the domain of Prune_{fb}, let $\Upsilon = \delta_{\mathbb{LT}}(p)$ be one of its representatives. Since Υ is bisimilar to a finitely branching tree, for each $\sigma \in \Upsilon$ by Kőnig's lemma we have that $\sigma \in \text{PD}(\Upsilon)$ iff $\text{Conc}(\Upsilon, \sigma)$ has infinite height. Therefore deciding whether $\sigma \in \text{PD}(\Upsilon)$ holds can be done with a single use of lim, and since lim is parallelizable, one application of lim suffices to decide this for all $\sigma \in \Upsilon$ at once. With this information we can construct $\text{Prune}_{\text{fb}}(\mathcal{A})$. □

Theorem 4.47. *The operation* Prune_{fb} *is transparent.*

Proof. The proof is a simplified version of the proof of Theorem 4.40.

Let $f :\subseteq \mathbb{AT}^*_{\text{lin}} \rightrightarrows \mathbb{AT}^*_{\text{lin}}$ be computable. Then f has a realizer F such that for each $\tau \in \mathbb{N}^{<\mathbb{N}}$ there exists a computable $X_\tau \subseteq \mathbb{N}^{<\mathbb{N}}$ such that τ is an induced label of the tree $\delta_{\mathbb{AT}}F(p)$ iff ξ is an induced label of $\delta_{\mathbb{AT}}(p)$ for some $\xi \in X_\tau$. Let p be given and $\Upsilon := \delta_{\mathbb{LT}}(p)$. We can computably define a labeled tree Υ_G with the following properties. The nodes at level 1 of Υ_G are bijectively associated to the pairs (ξ, τ) such that $|\tau| = 1$ and $\xi \in X_\tau$ is an induced label of Υ. Recursively, if $\sigma \neq \langle\rangle$ is in Υ_G and is associated to a pair (ξ, τ), then we have:

(1) The induced label of σ in Υ_G is τ.
(2) If some node of Υ with induced label ξ has rank at least $|\tau| + 1$, then the children of σ in Υ_G are bijectively associated to the pairs (ξ', τ') such that $|\tau'| = |\tau| + 1$, $\tau' \supset \tau$, and $\xi' \in X_{\tau'}$ is an induced label of Υ; otherwise σ is a leaf of Υ_G.

Claim 4.48. The trees $\text{PD}(\Upsilon_G)$ and $\Upsilon_{FH} := \delta_{\mathbb{LT}}FH(p)$ are bisimilar.

Let $H \vdash \text{PD} : \mathbb{LT} \to \mathbb{LT}$. To see that $\text{PD}(\Upsilon_G) \rightleftarrows \Upsilon_{FH}$, let $\sigma\ B\ \tau$ iff $\sigma = \tau = \langle\rangle$ or σ and τ have the same induced labels in $\text{PD}(\Upsilon_G)$ and Υ_{FH}, respectively. Now suppose $\sigma\ B\ \tau$, and let σ be associated to (ξ_0, τ_1). Let σ' be a child of σ in $\text{PD}(\Upsilon_G)$. It follows that σ' is associated to some pair (ξ_1, τ_1) such that $\tau_0 \subseteq \tau_1$. Since σ' is in the pruning derivative of $\delta_{\mathbb{LT}}G(p)$, by condition 2 of the construction it follows that there are nodes of Υ of arbitrary length whose labels extend ξ_1. Since Υ is bisimilar to a finitely branching tree, this implies that some node ν of Υ with induced label ξ_1 is the root of a subtree of Υ of infinite height. Thus ν is in $\delta_{\mathbb{LT}}H(p)$, and since $\xi_1 \in X_{\tau_1}$ it follows that some node with induced label τ_1 is in Υ_{FH}. Finally, since Υ_{FH} is linear, it follows that τ has a child τ' with induced label τ_1, and thus $\sigma'\ B\ \tau'$. Conversely, let τ' be a child of τ in Υ_{FH}, and let τ_1 be its induced label. Therefore, some $\xi_1 \in X_{\tau_1}$ is an induced label in $\text{PD}(\Upsilon)$, and thus some node ν of Υ has τ_1 as its induced label and is the root of a subtree of Υ of infinite height. This implies that some child σ' of σ in Υ_G is associated to (ξ_1, τ_1), and that such σ' is also in $\text{PD}(\Upsilon_G)$. Therefore $\sigma'\ B\ \tau'$. Finally, note that Υ_{FH} contains an infinite path since $f :\subseteq \mathbb{AT}^*_{\text{lin}} \rightrightarrows \mathbb{AT}^*_{\text{lin}}$, which implies that Υ_G and $\mathrm{PD}(\Upsilon_G)$ also contain an infinite path. Therefore $B \neq \varnothing$ and $\mathrm{PD}(\Upsilon_G)$ is bisimilar to Υ_{FH}.

Claim 4.49. The tree Υ_G is bisimilar to a finitely branching tree.

By construction, nodes of Υ_G which have the same induced label have bisimilar (indeed, isomorphic) subtrees. Thus if some node σ of Υ_G has infinitely many children σ_n which are roots of non-bisimilar subtrees, then the labels of the σ_n are pairwise distinct. Therefore the σ_n must be associated to elements (ξ_n, τ_n) such that the τ_n are pairwise $\subseteq$-incomparable. But Υ is bisimilar to a finitely branching tree; thus in particular only finitely many different labels occur on each of its levels. This implies that $\lim_{n\in\mathbb{N}} |\tau_n| = \infty$, and therefore arbitrarily long prefixes of the infinite induced label of Υ occur among the prefixes of the τ_n. But then we cannot have that all ξ_n have the same length $|\sigma|+1$, a contradiction. □

Lemma 4.50. *The operation* $\mathrm{Prune}_{\mathrm{fb}}$ *is a cylinder.*

Proof. Given a name p of an abstract tree $\mathcal{A} \in \mathrm{dom}(\mathrm{Prune}_{\mathrm{fb}})$, let $\mathcal{A}_p$ be the tree obtained from $\mathcal{A}$ by changing the label ℓ of any node $\sigma \neq \langle\rangle$ to $\ulcorner \ell, p(|\sigma|)\urcorner$. Then $\mathrm{Prune}_{\mathrm{fb}}(\mathcal{A}_p)$ is obtained from $\mathrm{Prune}_{\mathrm{fb}}(\mathcal{A})$ via the same transformation, and since $\mathrm{Prune}_{\mathrm{fb}}(\mathcal{A})$ has an infinite branch, it is easy to reconstruct both p and $\mathrm{Prune}_{\mathrm{fb}}(\mathcal{A})$ from $\mathrm{Prune}_{\mathrm{fb}}(\mathcal{A}_p)$. In other words, $\mathrm{id}_{\mathbb{N}^\mathbb{N}} \times \mathrm{Prune}_{\mathrm{fb}} \leq_{\mathrm{sW}} \mathrm{Prune}_{\mathrm{fb}}$. □

4.5. **Games for functions of a fixed Baire class.** For an ordinal $\alpha = \lambda + 2n$, with λ a limit ordinal and n a natural number, let Prune^α be the corestriction of $\mathrm{PD}^\star(\,\cdot\,, \lambda+n)$ to $\mathbb{AT}^*_{\mathrm{lin}}$, let $\mathrm{Prune}^\alpha_{\mathrm{fb}}$ be the corestriction of $\mathrm{PD}^\star(\,\cdot\,, \lambda+n)$ to $\mathbb{AT}^*_{\mathrm{fb}}$, and finally let $\mathrm{Prune}^{\alpha+1} = \mathrm{Prune}_{\mathrm{fb}} \circ \mathrm{Prune}^\alpha_{\mathrm{fb}}$.

Corollary 4.51. *Let* $\alpha < \omega_1$. *We have that* Prune^α *is a transparent cylinder which is Weihrauch-complete for the Baire class* α *functions. Therefore the* $(\mathrm{Prune}^\alpha, \mathrm{Label})$*-Wadge game characterizes the Baire class* α *functions.*

Proof. Suppose $\alpha = \lambda + 2n$. We have that $\mathrm{PD}^\star(\,\cdot\,, \lambda+n)$ is a transparent cylinder which is Weihrauch-complete for the Baire class $\lambda + 2n$ functions, so to see that the same holds for $\mathrm{Prune}^{\lambda+2n}$, by Theorem 2.14 it is enough to show that $\mathbb{AT}^*_{\mathrm{lin}}$ strongly encodes $\mathbb{N}^\mathbb{N}$. But any $F :\subseteq \mathbb{N}^\mathbb{N} \rightrightarrows \mathbb{N}^\mathbb{N}$ is easily seen to be strongly Weihrauch-equivalent to the map $F' :\subseteq \mathbb{N}^\mathbb{N} \rightrightarrows \mathbb{AT}^*_{\mathrm{lin}}$ which assigns $x \in \mathrm{dom}(F)$ to any linear abstract tree whose unique infinite label is in $F(x)$. Now suppose $\alpha = \lambda + 2n + 1$. Since $\mathbb{AT}^*_{\mathrm{lin}} \subseteq \mathbb{AT}^*_{\mathrm{fb}} \subseteq \mathbb{AT}$, by Proposition 2.7 and the fact that $\mathrm{Prune}^{\lambda+2n}$ is Weihrauch-complete for Baire class $\lambda+2n$ it follows that $\mathrm{Prune}^{\lambda+2n}_{\mathrm{fb}}$ also has this property. Now, since $\mathrm{Prune}_{\mathrm{fb}}$ is a transparent cylinder which is Weihrauch-complete for the Baire class 1 functions, the result follows. □

In other words, for $\alpha = \lambda + 2n$ with λ a limit ordinal and n a natural number, the restriction of the tree game in which the final tree built by player II must have $(\lambda+n)^{\mathrm{th}}$ pruning derivative bisimilar to a linear tree characterizes the Baire class α functions, and for $\alpha = \lambda + 2n + 1$, the restriction of the tree game in which the final tree built by player II must have $(\lambda+n)^{\mathrm{th}}$ pruning derivative bisimilar to a finitely branching tree characterizes the Baire class α functions.

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