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COMPANIONS, CODENSITY AND CAUSALITY

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Abstract. In the context of abstract coinduction in complete lattices, the notion of compatible function makes it possible to introduce enhancements of the coinduction proof principle. The largest compatible function, called the companion, subsumes most enhancements and has been proved to enjoy many good properties. Here we move to universal coalgebra, where the corresponding notion is that of a *final* distributive law. We show that when it exists, the final distributive law is a monad, and that it coincides with the codensity monad of the final sequence of the given functor. On sets, we moreover characterise this codensity monad using a new abstract notion of causality. In particular, we recover the fact that on streams, the functions definable by a distributive law or GSOS specification are precisely the causal functions. Going back to enhancements of the coinductive proof principle, we finally obtain that any causal function gives rise to a valid up-to-context technique.

1. Introduction

Coinduction has been widely studied in computer science since Milner's work on CCS [Mil89]. In concurrency theory, it is usually exploited to define behavioural equivalences or preorders on processes and to obtain powerful proof principles. Coinduction can also be used for programming languages, to define and manipulate infinite data-structures like streams or potentially infinite trees. For instance, streams can be defined using systems of differential equations [Rut05]. In particular, pointwise addition of two streams x, y can be defined by the following equations, where x_0 and x' respectively denote the head and the tail of a stream x.

$$\begin{aligned} (x \oplus y)_0 &= x_0 + y_0 \\ (x \oplus y)' &= x' \oplus y' \end{aligned} \tag{1.1}$$

Coinduction as a proof principle for concurrent systems can be nicely presented at the abstract level of complete lattices [Pou07, PS11]: bisimilarity is the greatest fixpoint of a

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monotone function on the complete lattice of binary relations. In contrast, coinduction as a tool to manipulate infinite data-structures requires one more step to be presented abstractly: moving to universal coalgebra [Jac16]. For instance, streams are the carrier of the final coalgebra of an endofunctor on the category Set of sets and functions, and simple systems of differential equations are just plain coalgebras. One usually distinguishes between coinduction as a tool to prove properties, and *corecursion* or *coiteration* as a tool to define functions. The theory we develop in the present work encompasses both in a uniform way, so that we do not emphasise their differences and just use the word *coinduction*.

In both cases one frequently needs enhancements of the coinduction principle [San98, SW01]. Indeed, rather than working with plain bisimulations, which can be rather large, one often uses "bisimulations up-to", which are not proper bisimulations but are nevertheless contained in bisimilarity [MPW92, Abr90, AG98, FLS00, JR04, Ler09, SVN+13]. The situation with infinite data-structures is similar. For instance, defining the shuffle product on streams is typically done using equations of the following shape,

$$\begin{aligned} (x \otimes y)_0 &= x_0 \times y_0 \\ (x \otimes y)' &= x \otimes y' \;\oplus\; x' \otimes y \end{aligned} \tag{1.2}$$

which fall out of the scope of plain coinduction due to the call to pointwise addition [Rut05, HKR17].

Enhancements of the bisimulation proof method have been introduced by Milner from the beginning [Mil89], and further studied by Sangiorgi [San98, SW01] and then by the first author [Pou07, PS11]. Let us recall the standard formulation of coinduction in complete lattices: by Knaster-Tarski's theorem [Kna28, Tar55], any monotone function b on a complete lattice admits a greatest fixpoint νb that satisfies the following *coinduction principle*:

$$\frac{x \leq y \leq b(y)}{x \leq \nu b}\ \textsc{coinduction} \tag{1.3}$$

In words, to prove that some point x is below the greatest fixpoint, it suffices to exhibit a point y above x which is an *invariant*, i.e., a post-fixpoint of b. Enhancements, or up-to techniques, make it possible to alleviate the second requirement: instead of working with post-fixpoints of b, one might use post-fixpoints of $b \circ f$, for carefully chosen functions f:

$$\frac{x \leq y \leq b(f(y))}{x \leq \nu b}\ \textsc{coinduction up to}\ f \tag{1.4}$$

Taking inspiration from the work of Hur et al. [HNDV13], the first author recently proposed to systematically use for f the largest *compatible* function [Pou16], i.e., the largest function t such that $t \circ b \leq b \circ t$. Such a function always exists and is called the *companion*. It enjoys many good properties, the most important one possibly being that it is a closure operator: $t \circ t = t$. Parrow and Weber characterised it extensionally in terms of the final sequence of the function b [PW16, Pou16]:

$$t : x \mapsto \bigwedge_{x \leq b_\alpha} b_\alpha \qquad \text{where} \begin{cases} b_\lambda \triangleq \bigwedge_{\alpha<\lambda} b_\alpha & \text{for limit ordinals} \\ b_{\alpha+1} \triangleq b(b_\alpha) & \text{for successor ordinals} \end{cases} \tag{1.5}$$

In the present paper, we give a categorical account of these ideas, generalising them from complete lattices to universal coalgebra, in order to encompass important instances of coinduction such as solving systems of equations on infinite data-structures.

Let us first be more precise about our example on streams. We consider there the Set functor $BX = \mathbb{R} \times X$, whose final coalgebra is the set $\mathbb{R}^\omega$ of streams over the reals. This means that any B-coalgebra (X, f) defines a function from X to streams. Take for instance the following coalgebra over the two-elements set $2 = \{0, 1\}$: $0 \mapsto (0.3, 1)$, $1 \mapsto (0.7, 0)$. This coalgebra can be seen as a system of two equations, whose unique solution is a function from 2 to $\mathbb{R}^\omega$, i.e, two streams, where the first has value 0.3 at all even positions and 0.7 at all odd positions.

In a similar manner, one can define binary operations on streams by considering coalgebras whose carrier consists of pairs of streams. For instance, the previous system of equations characterising pointwise addition (1.1) is faithfully represented by the following coalgebra:

$$\begin{aligned}(\mathbb{R}^\omega)^2 &\to B((\mathbb{R}^\omega)^2)\\ (x, y) &\mapsto \big(x_0 + y_0,\ (x', y')\big)\end{aligned} \tag{1.6}$$

Unfortunately, as explained above, systems of equations defining operations like shuffle product (1.2) cannot be represented easily in this way: we would need to call pointwise addition on streams that are not yet fully defined.

Instead, let S be the functor $SX = X^2$ so that pointwise stream addition can be seen as an S-algebra. Equations (1.2) can be represented by the following BS-coalgebra.

$$\begin{aligned}(\mathbb{R}^\omega)^2 &\to BS((\mathbb{R}^\omega)^2)\\ (x, y) &\mapsto \big(x_0 \times y_0,\ \big((x, y'),\ (x', y)\big)\big)\end{aligned} \tag{1.7}$$

The inner pairs (x, y') and (x', y) correspond to the corecursive calls, and thus to the shuffle products $x \otimes y'$ and $x' \otimes y$; in contrast, the intermediate pair $((x, y'), (x', y))$ corresponds to a call to the algebra on S, i.e., in this case, pointwise addition.

We should now explain in which sense and under which conditions such a BS-coalgebra gives rise to an operation on the final B-coalgebra. A preliminary step actually consists in proposing the following principle (Definition 3.1).

> *Let B be an endofunctor with a final B-coalgebra (Z, ζ), let F be a functor, and let $\alpha\colon FZ \to Z$ be an F-algebra on the final B-coalgebra.* Coinduction up to the algebra α is valid *if for every BF-coalgebra (X, g), there exists a unique morphism $g^\dagger\colon X \to Z$ making the following diagram commute.*

$$\begin{tikzcd} X \arrow[rr, "g^\dagger"] \arrow[d, "g"'] & & Z \arrow[d, "\zeta"] \\ BFX \arrow[r, "BFg^\dagger"'] & BFZ \arrow[r, "B\alpha"'] & BZ \end{tikzcd} \tag{1.8}$$

Intuitively, $g^\dagger$ gives the solution of the system of equations represented by g: it interprets the variables (X) into the denotational space of behaviours (Z); the above diagram ensures that the equations are satisfied when using the algebra α to interpret the F-part of the equations. For the previous example (1.7), one would take $F = S$ and pointwise addition for α. This notion of validity improves over the notion of soundness we proposed before [BPPR16] in that it fully specifies in which sense the equations are solved, by mentioning explicitly the algebra which is used on the final coalgebra.

In the literature on universal coalgebra, one would typically prove such an enhanced coinduction principle by using distributive laws. Typically, this principle holds if F is a monad and if there exists a distributive law $\lambda\colon FB \Rightarrow BF$ of a monad F over B (e.g., [Bar04, LPW00, UVP01, Jac06, MMS13]). The proof relies on the so-called generalised powerset construction [SBBR10] and the F-algebra α is the one canonically generated by λ. This precisely amounts to using an up-to technique. Such a use of distributive laws is actually rather standard in operational semantics [TP97, Bar04, Kli11]; they properly generalise the notion of compatible function. In order to follow [Pou16], we thus focus on the largest distributive law.

Our first contribution consists in showing that if a functor B admits a final distributive law (called the companion), then (1) this distributive law is that of a monad T over B, and thus (2) coinduction up to its associated algebra is valid (Section 4). In complete lattices, this corresponds to the facts that the companion is a closure operator and that it can be used as an up-to technique.

Then we move to conditions under which the companion exists. We start from the *final sequence* of the functor B, which is commonly used to obtain the existence of a final coalgebra [Adá74, Bar92], and we show that the companion actually coincides with the *codensity monad* of this sequence, provided that this codensity monad exists and is preserved by B (Theorem 6.3). Those conditions are satisfied by all polynomial functors. This link with the final sequence of the functor makes it possible to recover Parrow and Weber's characterisation (Equation (1.5)).

We can go even further for ω-continuous endofunctors on Set: the codensity monad of the final sequence can be characterised in terms of a new abstract notion of *causal algebra* (Definition 8.1). On streams, this notion coincides with the standard notion of causality [HKR17]: causal algebras (on streams) correspond to operations such that the n-th value of the result only depends on the n-th first values of the arguments. For instance, pointwise addition and shuffle product are causal algebras for the functor $SX = X^2$.

These two characterisations of the companion in terms of the codensity monad and in terms of causal algebras are the key theorems of the present paper. We study some of their consequences in Section 9. These apply to all polynomial functors.

First, coinduction up to, as presented in (1.8), is valid for every causal algebra for a functor F. Such a technique makes it possible to define shuffle product (1.2) in a streamlined way, without mentioning any distributive law. In the very same way, with the functor $BX = 2 \times X^A$ for deterministic automata, we immediately obtain the semantics of non-deterministic automata and context-free grammars using simple causal algebras on formal languages (Example 9.4)—distributive laws are now hidden from the end-user.

Second, we obtain that algebras on the final coalgebra are causal if and only if they can be defined by a distributive law. Similar results were known to hold for streams [HKR17] and languages [RBR16]. Our characterisation is more abstract and less syntactic; the precise relationship between those results remains to be studied.

Third, we can combine our results with some recent work [BPPR14] where we rely on (bi)fibrations to lift distributive laws on systems (e.g., automata) to obtain up-to techniques for coinductive predicates or relations on those systems (e.g., language equivalence). Doing so, we obtain that every causal algebra gives rise to a valid up-to context technique (Section 9.3). For instance, bisimulation up to pointwise addition and shuffle product is a valid technique for proving stream equalities coinductively.

In Section 10 we provide an expressivity result: while abstract GSOS specifications [TP97] seem more expressive than plain distributive laws, we show that this is actually not the case: any algebra obtained from an abstract GSOS specification can actually be defined from a plain distributive law.

Our main results on the construction of a companion through the final sequence apply to polynomial functors. In Section 11 we show a negative result: the finite powerset functor (which is not polynomial) does not satisfy the premises of our results, and hence falls outside its scope. We conclude the paper with related work (Section 12) and future work (Section 13).

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2. Preliminaries

Let $\mathcal{C}$ be a category. A *coalgebra* for a functor $B\colon \mathcal{C} \to \mathcal{C}$ is a pair (X, f) where X is an object in $\mathcal{C}$ and $f\colon X \to BX$ a morphism. A coalgebra homomorphism from (X, f) to (Y, g) is a $\mathcal{C}$-morphism $h\colon X \to Y$ such that $g \circ h = Bh \circ f$. A coalgebra (Z, ζ) is called *final* if it is final in the category of coalgebras, i.e., for every coalgebra (X, f) there exists a unique coalgebra morphism from (X, f) to (Z, ζ).

An *algebra* for a functor $F\colon \mathcal{C} \to \mathcal{C}$ is defined dually to a coalgebra, i.e., it is a pair (X, a) where $a\colon FX \to X$, and an algebra morphism from (X, a) to (Y, b) is a morphism $h\colon X \to Y$ such that $h \circ a = b \circ Fh$.

A *monad* is a triple (T, η, μ) where $T\colon \mathcal{C} \to \mathcal{C}$ is a functor, and $\eta\colon \mathsf{Id} \Rightarrow T$ and $\mu\colon TT \Rightarrow T$ are natural transformations called *unit* and *multiplication* respectively, such that $\mu \circ T\eta = \mathsf{id} = \mu \circ \eta T$ and $\mu \circ \mu T = \mu \circ T\mu$.

By $\mathcal{P}\colon \mathsf{Set} \to \mathsf{Set}$ we denote the (covariant) powerset functor, and by $\mathcal{P}_f\colon \mathsf{Set} \to \mathsf{Set}$ the finite powerset functor, defined by $\mathcal{P}_f(X) = \{S \subseteq X \mid S \text{ is finite}\}$.

2.1. Final sequence.

Let $B\colon \mathcal{C} \to \mathcal{C}$ be an endofunctor on a complete category $\mathcal{C}$. The *final sequence* is the unique ordinal-indexed sequence defined by $B_0 = 1$ (the final object of $\mathcal{C}$), $B_{i+1} = BB_i$ and $B_j = \lim_{i<j} B_i$ for a limit ordinal j, with connecting morphisms $B_{j,i}\colon B_j \to B_i$ for all $i \leq j$, satisfying $B_{i,i} = \mathsf{id}$, $B_{j+1,i+1} = BB_{j,i}$ and if j is a limit ordinal then $(B_{j,i})_{i<j}$ is a limit cone.

The final sequence is a standard tool for constructing final coalgebras: if there exists an ordinal k such that $B_{k+1,k}$ is an isomorphism, then $B^{-1}_{k+1,k}\colon B_k \to BB_k$ is a final B-coalgebra [Bar92, Theorem 1.3] (and dually for initial algebras [Adá74]). In the sequel, we shall sometimes present it as a functor $\bar{B}\colon \mathsf{Ord}^{\mathsf{op}} \to \mathcal{C}$, given by $\bar{B}(i) = B_i$ and $\bar{B}(j,i) = B_{j,i}$.

Example 2.1. Consider the functor $B\colon \mathsf{Set} \to \mathsf{Set}$ given by $BX = \mathbb{R} \times X$, whose coalgebras are stream systems over the real numbers. Then $B_0 = 1$ and $B_{i+1} = \mathbb{R} \times B_i$ for $0 < i < \omega$. Hence, for $i < \omega$, B_i is the set of all finite lists over $\mathbb{R}$ of length i. The limit B_ω consists of the set of all streams over $\mathbb{R}$. For each i, j with $i \leq j$, the connecting map $B_{j,i}$ maps a stream (if $j = \omega$) or a list (if $j < \omega$) to the prefix of length i. The set B_ω of streams is a final B-coalgebra.

Example 2.2. For the Set functor $BX = 2 \times X^A$ whose coalgebras are deterministic automata over A, B_i is (isomorphic to) the set of languages of words over A with length below i. In particular, $B_\omega = \mathcal{P}(A^*)$ is the set of all languages, and it is a final B-coalgebra.

Example 2.3. For the Set functor $BX = X + 1$, B_i is (isomorphic to) the set $\{0, 1, 2, \dots, i\}$. For $i < j$, the projection $B_{j,i}$ is given by $B_{j,i}(k) = \begin{cases} k & \text{if } k \leq i \\ i & \text{otherwise} \end{cases}$.

Example 2.4. Let $\mathcal{C}$ be a complete lattice, seen as a poset category. An endofunctor is just a monotone function b; the final coalgebra is its greatest fixed point νb; the final sequence corresponds to the sequence defined in Equation (1.5), and we recover Kleene's fixpoint theorem:

$$\nu b = \bigwedge_\alpha b_\alpha$$

A functor $B \colon \mathcal{C} \to \mathcal{C}$ is called *(ω)-continuous* if it preserves limits of ω^{op}-chains. For such a functor, B_ω is the carrier of a final B-coalgebra. The functors of stream systems and automata in the above examples are both ω-continuous.

3. Distributive laws and coinduction up-to

We define an abstract coinduction-up-to principle with respect to algebras on the final coalgebra, and show its validity under the assumption of a distributive law.

Definition 3.1. Let $B, F \colon \mathcal{C} \to \mathcal{C}$ be functors such that B has a final coalgebra (Z, ζ), and let $\alpha \colon FZ \to Z$ be an algebra on it. We say that *coinduction up to α is valid* if for every coalgebra $g \colon X \to BFX$ there exists a unique arrow $g^\dagger \colon X \to Z$ such that the following diagram commutes.

$$\begin{tikzcd} X \arrow[rr, "g^\dagger"] \arrow[d, "g"'] & & Z \arrow[d, "\zeta"] \\ BFX \arrow[r, "BFg^\dagger"'] & BFZ \arrow[r, "B\alpha"'] & BZ \end{tikzcd}$$

For $b, f \colon L \to L$ monotone functions on a complete lattice, the existence of an algebra α as in Definition 3.1 means that f preserves the greatest fixed point: $f(\nu b) \leq \nu b$. Validity in the sense of Definition 1.4 amounts to the validity of coinduction up to f (Equation 1.4, also called soundness) with the implicit requirement that $f(\nu b) \leq \nu b$. The latter is not required by soundness and does not follow from it [Pou08, page 53].

As mentioned in the Introduction, this definition also differs from the notion of soundness we proposed in [BPPR16, Section 3]: there we were focusing on coinduction up to a functor F rather than up to an algebra for a functor, and we were asking for the existence of an appropriate functor G from the category of BF-coalgebras to that of B-coalgebras. The problem with soundness there is that the sense in which the system of equations is solved is specified only through the functor G, and thus through the proof of soundness. In contrast, the present definition of validity is more fine-grained and fully characterises the way solutions behave.

Example 3.2. Recall, from Equation 1.7, how we presented the definition of the shuffle product $\otimes\colon (\mathbb{R}^\omega)^2 \to \mathbb{R}^\omega$ as a BS-coalgebra, for $SX = X^2$ and $BX = \mathbb{R} \times X$. Let $\oplus\colon (\mathbb{R}^\omega)^2 \to \mathbb{R}^\omega$ be pointwise addition of streams. Coinduction up to $\oplus$ is valid, and the shuffle product arises as the unique arrow from the above coalgebra to the final B-coalgebra. The validity of coinduction up to $\oplus$ can be derived through the use of distributive laws, explained below.

In the context of universal coalgebra, distributive laws are a useful concept to model interaction between algebra and coalgebra, with operational semantics as a prominent example [TP97, Kli11, Bar04]. We recall a few basic notions, and the application to coinduction up-to.

A *distributive law* of a functor $F\colon \mathcal{C} \to \mathcal{C}$ over a functor $B\colon \mathcal{C} \to \mathcal{C}$ is a natural transformation $\lambda\colon FB \Rightarrow BF$. If B has a final coalgebra (Z, ζ), then such a λ induces a unique algebra α making the following commute.

$$\begin{tikzcd} FZ \arrow[r, "F\zeta"] \arrow[d, "\alpha"'] & FBZ \arrow[r, "\lambda_Z"] & BFZ \arrow[d, "B\alpha"] \\ Z \arrow[rr, "\zeta"'] & & BZ \end{tikzcd} \tag{3.1}$$

We call α the *algebra induced by* λ (on the final coalgebra).

Let (T, η, μ) be a monad. A distributive law of (T, η, μ) over B is a natural transformation $\lambda\colon TB \Rightarrow BT$ such that $B\eta = \lambda \circ \eta B$ and $\lambda \circ \mu B = B\mu \circ \lambda T \circ T\lambda$.

Bartels [Bar04, Theorem 4.2.2, Corollary 4.3.6] (also see [Bar03]) makes the distributive law λ explicit in his notion of λ-coiteration, and proves (in our terminology) that coinduction up to the algebra induced by a distributive law is valid, under certain conditions:

Theorem 3.3. *Let $\lambda\colon TB \Rightarrow BT$ be a distributive law between functors $B, T\colon \mathcal{C} \to \mathcal{C}$, let (Z, ζ) be a final B-coalgebra and let $\alpha\colon TZ \to Z$ be the algebra induced by λ. If either*
- *$\mathcal{C}$ has countable coproducts, or*
- *T is a monad and λ a distributive law of that monad over B,*

then coinduction up to α is valid.

Example 3.4. Let $S, B\colon \mathsf{Set} \to \mathsf{Set}$ be as in the introduction (and Example 3.2), and define $\lambda\colon SB \Rightarrow BS$ by

$$\lambda_X((o_1, t_1), (o_2, t_2)) = (o_1 + o_2, (t_1, t_2)).$$

The algebra induced by λ on the final B-coalgebra $\mathbb{R}^\omega$ is given by pointwise addition $\oplus$, as in Equation 1.1. Theorem 3.3 asserts (since Set has countable coproducts) the validity of coinduction up to $\oplus$.

3.1. **Generalisation to morphisms of endofunctors.** Below we sometimes consider natural transformations of the more general form $\lambda\colon FA \Rightarrow BF$, for endofunctors $A\colon \mathcal{A} \to \mathcal{A}$, $B\colon \mathcal{C} \to \mathcal{C}$ and a functor $F\colon \mathcal{A} \to \mathcal{C}$. These form an instance of *morphisms of endofunctors*: 1-cells in a certain category of endofunctors in a 2-category [LPW00]. They appear in [CHL03] under the name of *generalized distributive laws*. If A has a final coalgebra (Z_A, ζ_A), and B a final coalgebra (Z_B, ζ_B), then such a λ induces a unique map $\alpha\colon FZ_A \to Z_B$ such that

$$\begin{tikzcd} FZ_A \arrow[r, "F\zeta_A"] \arrow[d, "\alpha"'] & FAZ_A \arrow[r, "\lambda_{Z_A}"] & BFZ_A \arrow[d, "B\alpha"] \\ Z_B \arrow[rr, "\zeta_B"'] & & BZ_B \end{tikzcd} \tag{3.2}$$

This is, of course, a generalisation of algebras induced by distributive laws.

Example 3.5. Define $F, A, B\colon \mathsf{Set} \to \mathsf{Set}$ by $FX = X$, $AX = \mathbb{R} \times \mathbb{R} \times X$ and $BX = \mathbb{R} \times X$. The set $\mathbb{R}^\omega$ of streams over $\mathbb{R}$ is a final coalgebra for A.

Consider $\lambda\colon FA \Rightarrow BF$ given by

$$\lambda_X(o_1, o_2, t) = (o_1, t)\,.$$

The unique map in (3.2) is given by $\mathsf{even}\colon \mathbb{R}^\omega \to \mathbb{R}^\omega$, $\mathsf{even}(\sigma) = (\sigma(0), \sigma(2), \sigma(4), \ldots)$.

It is straightforward to generalise Definition 3.1 to this setting, involving unique solutions of coalgebras $f\colon X \to AFX$. However, Theorem 3.3 does not generalise accordingly. For instance, for F, A, B as above, $f\colon 1 \to FA1$ given by $f(*) = (0, 1, *)$, the validity of the coinduction up to even principle would amount to a unique solution of

$$x_0 = 0 \quad x_1 = 1 \quad x'' = \mathsf{even}(x)\,,$$

which, however, has multiple solutions.

4. Properties of the companion

We define the notion of companion, and prove several of its abstract properties.

Definition 4.1. Let $B\colon \mathcal{C} \to \mathcal{C}$ be a functor. The category $\mathsf{DL}(B)$ of distributive laws is defined as follows. An object is a pair (F, λ) where $F\colon \mathcal{C} \to \mathcal{C}$ is a functor and $\lambda\colon FB \Rightarrow BF$ is a natural transformation. A morphism from (F, λ) to (G, ρ) is a natural transformation $\kappa\colon F \Rightarrow G$ such that $\rho \circ \kappa B = B\kappa \circ \lambda$.

$$\begin{tikzcd} FB \arrow[r, Rightarrow, "\kappa B"] \arrow[d, Rightarrow, "\lambda"'] & GB \arrow[d, Rightarrow, "\rho"] \\ BF \arrow[r, Rightarrow, "B\kappa"'] & BG \end{tikzcd}$$

The *companion* of B is the final object of $\mathsf{DL}(B)$, if it exists.

Morphisms in $\mathsf{DL}(B)$ are a special case of *morphisms of distributive laws*, see [PW02, Wat02, LPW00, KN15]. We first show that when it exists, the companion is a monad.

Theorem 4.2. *Let (T, τ) be the companion of an endofunctor B. There are unique $\eta\colon \mathsf{Id} \Rightarrow T$ and $\mu\colon TT \Rightarrow T$ such that (T, η, μ) is a monad and $\tau\colon TB \Rightarrow BT$ is a distributive law of this monad over B.*

Proof. Define η and μ as the unique morphisms from id_B and $\tau T \circ T\tau$ respectively to the companion:

$$
\begin{tikzcd}
B \arrow[r, equal] \arrow[d, Rightarrow, "\eta B"'] & B \arrow[d, Rightarrow, "B\eta"] \\
TB \arrow[r, Rightarrow, "\tau"] & BT
\end{tikzcd}
\qquad
\begin{tikzcd}
TTB \arrow[r, Rightarrow, "T\tau"] \arrow[d, Rightarrow, "\mu B"'] & TBT \arrow[r, Rightarrow, "\tau T"] & BTT \arrow[d, Rightarrow, "B\mu"] \\
TB \arrow[rr, Rightarrow, "\tau"] & & BT
\end{tikzcd}
$$

By definition, they satisfy the required axioms for τ to be a distributive law of monad over functor. The proof that (T, η, μ) is indeed a monad is routine, using finality of (T, τ); see [BPR17] for more details. □

Since the companion is a distributive law of a monad (Theorem 4.2), by Theorem 3.3 we obtain the following.

Corollary 4.3. *When a functor has a companion, coinduction up to the algebra induced by the companion is valid.*

Instantiated to the complete lattice case, this implies a soundness result (cf. Section 3): any invariant up to the companion (a post-fixpoint of $b \circ t$) is below the greatest fixpoint (νb).

If the underlying category has an intial object, then one can define the final coalgebra and the algebra induced by the companion explicitly:

Theorem 4.4. *Suppose $\mathcal{C}$ has an initial object 0. Let (T, τ) be the companion of a functor $B \colon \mathcal{C} \to \mathcal{C}$ and let (T, η, μ) be the corresponding monad. The B-coalgebra $(T0, \tau_0 \circ T!_{B0})$ is final, and the algebra induced on it by the companion is given by μ_0.*

Proof. Let (X, f) be a B-coalgebra. Write $\hat{X}$ for the constant-to-X functor, and $\hat{f}$ for the constant-to-f distributive law of $\hat{X}$ over B. By finality of the companion, there exists a unique natural transformation $\lambda \colon \hat{X} \Rightarrow T$ such that $B\lambda \circ \hat{f} = \tau \circ \lambda B$. One checks easily that λ_0 is the unique coalgebra homomorphism from (X, f) to $(T0, \tau_0 \circ T!_{B0})$.

To prove that μ_0 is the algebra induced by the companion, it suffices to prove that it is a coalgebra morphism of the correct type (3.1):

$$(\tau_0 \circ T!_{B0}) \circ \mu_0 = \tau_0 \circ \mu_{B0} \circ TT!_{B0} = B\mu_0 \circ \tau_{T0} \circ T\tau_0 \circ TT!_{B0} = B\mu_0 \circ \tau_{T0} \circ T(\tau_0 \circ T!_{B0})$$

which follows from naturality of μ, the fact that τ is a distributive law of the monad (T, η, μ) over B, and functoriality. □

More generally, the algebra induced by any distributive law factors through the algebra μ_0 induced by the companion.

Proposition 4.5. *Let (T, τ) be the companion of an endofunctor B and let (T, η, μ) be the corresponding monad. Let $\lambda \colon FB \Rightarrow BF$ be a distributive law, and $\alpha \colon FT0 \Rightarrow T0$ the algebra on the final coalgebra induced by it. Let $\bar{\lambda} \colon F \Rightarrow T$ be the unique natural transformation induced by finality of the companion. Then $\alpha = \mu_0 \circ \bar{\lambda}_{T0}$.*

Proof. By Theorem 4.4, $\tau_0 \circ T!_{B0} \colon T0 \to BT0$ is a final B-coalgebra. By definition of the algebra induced on the final coalgebra by λ, and uniqueness of morphisms into final coalgebras, it suffices to prove that the following diagram commutes.

$$\begin{tikzcd}
FT0 \arrow[r, "\bar\lambda_{T0}"] \arrow[d, "FT!_{B0}"'] & TT0 \arrow[r, "\mu_0"] \arrow[d, "TT!_{B0}"] & T0 \arrow[d, "T!_{B0}"] \\
FTB0 \arrow[r, "\bar\lambda_{TB0}"] \arrow[d, "F\tau_0"'] & TTB0 \arrow[r, "\mu_{B0}"] \arrow[d, "T\tau_0"] & TB0 \arrow[dd, "\tau_0"] \\
FBT0 \arrow[r, "\bar\lambda_{BT0}"] \arrow[d, "\lambda_{T0}"'] & TBT0 \arrow[d, "\tau_{T0}"] & \\
BFT0 \arrow[r, "B\bar\lambda_{T0}"'] & BTT0 \arrow[r, "B\mu_0"'] & BT0
\end{tikzcd}$$

Everything commutes, clockwise starting from the top right by naturality, the fact that τ is a distributive law of the monad (T, η, μ) over B, the fact that $\bar\lambda$ is a morphism from (F, λ) to (T, τ), and twice naturality. $\square$

5. Right Kan extensions and codensity monads

The notion of *codensity monad* is a special instance of a right Kan extension, which plays a central role in the following sections. We briefly define these notions here; see, e.g., [Lan98, nLab, Lei13] for a comprehensive study.

Let $F\colon \mathcal{C} \to \mathcal{D}$, $G\colon \mathcal{C} \to \mathcal{E}$ be two functors. Define the category $\mathcal{K}(F, G)$ whose objects are pairs (H, α) of a functor $H\colon \mathcal{D} \to \mathcal{E}$ and a natural transformation $\alpha\colon HF \Rightarrow G$. A morphism from (H, α) to (I, β) is a natural transformation $\kappa\colon H \Rightarrow I$ such that $\beta \circ \kappa F = \alpha$.

$$\begin{tikzcd}
HF \arrow[rr, Rightarrow, "\kappa F"] \arrow[dr, Rightarrow, "\alpha"'] & & IF \arrow[dl, Rightarrow, "\beta"] \\
& G &
\end{tikzcd}$$

The *right Kan extension* of G along F is a final object $(\mathsf{Ran}_F(G), \epsilon)$ in $\mathcal{K}(F, G)$; the natural transformation $\epsilon\colon \mathsf{Ran}_F(G)F \Rightarrow G$ is called its *counit*. A functor $K\colon \mathcal{E} \to \mathcal{F}$ is said to *preserve* $\mathsf{Ran}_F(G)$ if $K \circ \mathsf{Ran}_F(G)$ is a right Kan extension of KG along F, with counit $K\epsilon\colon K\mathsf{Ran}_F(G)F \Rightarrow KG$.

The codensity monad is a special case, with $F = G$. Explicitly, the *codensity monad* of a functor $F\colon \mathcal{C} \to \mathcal{D}$ consists of a functor $\mathsf{C}_F\colon \mathcal{D} \to \mathcal{D}$ and a natural transformation $\epsilon\colon \mathsf{C}_F F \Rightarrow F$ such that for every functor $H\colon \mathcal{D} \to \mathcal{D}$ and natural transformation $\alpha\colon HF \Rightarrow F$ there is a unique $\hat\alpha\colon H \Rightarrow \mathsf{C}_F$ such that $\epsilon \circ \hat\alpha F = \alpha$.

$$\begin{tikzcd}
HF \arrow[rr, Rightarrow, "\hat\alpha F"] \arrow[dr, Rightarrow, "\alpha"'] & & \mathsf{C}_F F \arrow[dl, Rightarrow, "\epsilon"] \\
& F &
\end{tikzcd}$$

As the name suggests, C_F is a monad: the unit η and the multiplication μ are the unique natural transformations such that the following diagrams commute.

$$\begin{tikzcd} F \arrow[r, Rightarrow, "\eta F"] \arrow[dr, equal] & \mathsf{C}_F F \arrow[d, Rightarrow, "\epsilon"] \\ & F \end{tikzcd} \qquad \begin{tikzcd} \mathsf{C}_F\mathsf{C}_F F \arrow[r, Rightarrow, "\mathsf{C}_F\epsilon"] \arrow[d, Rightarrow, "\mu F"'] & \mathsf{C}_F F \arrow[d, Rightarrow, "\epsilon"] \\ \mathsf{C}_F F \arrow[r, Rightarrow, "\epsilon"'] & F \end{tikzcd} \tag{5.1}$$

In the sequel we abbreviate the category $\mathcal{K}(F, F)$ as $\mathcal{K}(F)$.

Right Kan extensions (and in particular codensity monads) can be computed pointwise as a limit, if sufficient limits exist. For an object X in $\mathcal{D}$, denote by $\Delta_X : \mathcal{C} \to \mathcal{D}$ the functor that maps every object to X. By Δ_X/F we denote the comma category, where an object is a pair (Y, f) consisting of an object Y in $\mathcal{C}$ and an arrow $f : X \to FY$ in $\mathcal{D}$, and an arrow from (Y, f) to (Z, g) is a map $h: Y \to Z$ in $\mathcal{C}$ such that $Fh \circ f = g$. There is a forgetful functor $(\Delta_X/F) \to \mathcal{C}$, which remains unnamed below.

Lemma 5.1. *Let $F : \mathcal{C} \to \mathcal{D}$, $G : \mathcal{C} \to \mathcal{E}$ be functors. If, for every object X in $\mathcal{D}$, the limit $\lim\Big((\Delta_X/F) \to \mathcal{C} \xrightarrow{G} \mathcal{E}\Big)$ exists, then the right Kan extension $\mathsf{Ran}_F(G)$ exists, and is given on an object X by the corresponding limit.*

The hypotheses of Lemma 5.1 are met in particular if $\mathcal{C}$ is essentially small (equivalent to a category with a set of objects and a set of arrows), $\mathcal{D}$ is locally small and $\mathcal{E}$ is complete. The latter conditions hold for $\mathcal{D} = \mathcal{E} = \mathsf{Set}$. In that case, we have the following concrete presentation; see, e.g., [CP89, Section 2.5] for a proof.

Lemma 5.2. *Let $F, G : \mathcal{C} \to \mathsf{Set}$ be functors. Suppose that, for each set X, the collection $\{\alpha : (F-)^X \Rightarrow G\}$ is a set (rather than a proper class). Then the right Kan extension $\mathsf{Ran}_F(G)$ is given by*

$$\mathsf{Ran}_F(G)(X) = \{\alpha : (F-)^X \Rightarrow G\}$$

for each X. For $h: X \to Y$, $(\mathsf{Ran}_F(G)(h)(\alpha))_A : (FA)^Y \to GA$ is given by $f \mapsto \alpha_A(f \circ h)$.

The natural transformation $\epsilon : \mathsf{Ran}_F(G)F \Rightarrow G$ is given by $\epsilon_X(\alpha : F^{FX} \Rightarrow G) = \alpha_X(\mathsf{id}_{FX})$. Finally, given $H : \mathsf{Set} \to \mathsf{Set}$ and $\beta : HF \Rightarrow G$, the induced $\hat{\beta} : H \Rightarrow \mathsf{Ran}_F(G)$ is given by $(\hat{\beta}_X(S))_A : (FA)^X \to GA$, $f \mapsto \beta_A \circ Hf(S)$.

6. Constructing the companion via right Kan extensions

It is standard in the theory of coalgebras to compute the final coalgebra of a functor B as a limit of the final sequence $\bar{B}$, see Section 2. In this section, we show how the companion of a functor arises as the codensity monad of its final sequence.

We first adapt the definition of companion to more general natural transformations of the form $FA \Rightarrow BF$, fixing *two* functors; such natural transformations were discussed in Section 3.1. This generalisation is useful in the next sections, in the setting of causal functions. Moreover, the construction of the companion given in this section can be presented naturally at this level.

Definition 6.1. Let $A : \mathcal{A} \to \mathcal{A}$ and $B : \mathcal{C} \to \mathcal{C}$ be functors. The category $\mathsf{DL}(A, B)$ is defined as follows. An object is a pair (F, λ) where $F : \mathcal{A} \to \mathcal{C}$ is a functor and $\lambda : FA \Rightarrow BF$ is a natural transformation. A morphism from (F, λ) to (G, ρ) is a natural transformation

$\kappa\colon F \Rightarrow G$ such that $\rho \circ \kappa A = B\kappa \circ \lambda$. The *companion* of (A, B) is the final object (T, τ) of $\mathsf{DL}(A, B)$, if it exists.

Recall, from Section 2, that the final sequence of an endofunctor $A\colon \mathcal{A} \to \mathcal{A}$ in a complete category $\mathcal{A}$ can be presented as a functor $\bar{A}\colon \mathsf{Ord}^{\mathsf{op}} \to \mathcal{A}$. Given another functor $B\colon \mathcal{C} \to \mathcal{C}$, consider the right Kan extension $\mathsf{Ran}_{\bar{A}}(\bar{B})$ of the final sequence of $\bar{B}$ along the final sequence of $\bar{A}$. By definition, this is final in the category of natural transformations of the form $\alpha\colon F\bar{A} \Rightarrow \bar{B}$. The main result of this section is that, under certain conditions, the right Kan extension $\mathsf{Ran}_{\bar{A}}(\bar{B})$ is (the underlying functor of) the companion of (A, B), i.e., the final object in the category of distributive laws of the form $\lambda\colon FA \Rightarrow BF$ (Theorem 6.3). The following lemma is a first step: it associates, to every such distributive law $\lambda\colon FA \Rightarrow BF$, a natural transformation of the form $\alpha\colon F\bar{A} \Rightarrow \bar{B}$.

Lemma 6.2. *For every $\lambda\colon FA \Rightarrow BF$ there exists a unique $\alpha\colon F\bar{A} \Rightarrow \bar{B}$ such that for all $i \in \mathsf{Ord}$: $\alpha_{i+1} = B\alpha_i \circ \lambda_{A_i}$. This construction extends to a functor from $\mathsf{DL}(A, B)$ to $\mathcal{K}(\bar{A}, \bar{B})$.*

Moreover, if $A_{k+1,k}$ and $B_{k+1,k}$ are isomorphisms for some k, then $\alpha_k\colon FA_k \to B_k$ is the unique map induced by λ (as in (3.2)). In particular, if $A = B$ then α_k is the algebra induced by λ on the final coalgebra.

Proof. This natural transformation is completely determined by the successor case given in the definition; on a limit ordinal j, B_j is a limit, and naturality requires it to be defined as the unique arrow $\alpha_j\colon FA_j \to B_j$ such that

$$\begin{tikzcd} FA_j \arrow[r, dashed, "\alpha_j"] \arrow[d, "FA_{j,i}"'] & B_j \arrow[d, "B_{j,i}"] \\ FA_i \arrow[r, "\alpha_i"'] & B_i \end{tikzcd}$$

commutes, for all $i < j$.

For naturality, we have to prove that the relevant square (as above) commutes for all i, j with $i \leq j$. For $i = j$, this follows since $A_{j,j} = \mathsf{id}_{A_j}$ and $B_{j,j} = \mathsf{id}_{B_j}$ by definition of the final sequence. We prove that the square commutes for any i, j with $i < j$, by induction on j. The case that j is a limit ordinal follows immediately from the definition of α_j, without using the induction hypothesis.

Now suppose that, for any i with $i < j$, the square commutes for i, j. We need to prove that it commutes for all $i < j+1$. First observe that if $i < j$, then the square also commutes for $i + 1 < j + 1$:

$$\begin{tikzcd} FA_{j+1} = FAA_j \arrow[r, "\lambda_{A_j}"'] \arrow[rr, bend left, "\alpha_{j+1}"] \arrow[d, "FA_{j+1,i+1} = FAA_{j,i}"'] & BFA_j \arrow[r, "B\alpha_j"'] \arrow[d, "BFA_{j,i}"] & BB_j = B_{j+1} \arrow[d, "B_{j+1,i+1} = BB_{j,i}"] \\ FA_{i+1} = FAA_i \arrow[r, "\lambda_{A_i}"] \arrow[rr, bend right, "\alpha_{i+1}"'] & BFA_i \arrow[r, "B\alpha_i"] & BB_i = B_{i+1} \end{tikzcd}$$

by naturality (left square), assumption (right square) and definition of α on successor ordinals (crescents). Hence, the square commutes for any successor ordinal $i + 1$ strictly below $j + 1$.

For i a limit ordinal, consider the following diagram:

$$\begin{tikzcd} FA_j \arrow[r, "FA_{j,i}"] \arrow[d, "\alpha_j"'] & FA_i \arrow[r, "FA_{i,l}"] \arrow[d, "\alpha_i"] & FA_l \arrow[d, "\alpha_l"] \\ B_j \arrow[r, "B_{j,i}"'] & B_i \arrow[r, "B_{i,l}"'] & B_l \end{tikzcd}$$

For all $l < i$, the outer rectangle commutes by the induction hypothesis, and the right square by definition of α_i on the limit ordinal i. Since B_i is a limit with projections $B_{i,l}$ for $l \leq i$, it follows that the square on the left commutes, as desired.

To show that the construction extends to a functor, let $\kappa \colon F \Rightarrow G$ be a morphism in $\mathsf{DL}(A, B)$ from some (F, λ) to (G, ρ). The natural transformations (F, λ) and (G, ρ) respectively yield unique $\alpha^\lambda \colon F\bar{A} \Rightarrow \bar{B}$ and $\alpha^\rho \colon G\bar{A} \Rightarrow \bar{B}$ in the above way. We need to prove that κ is a morphism in $\mathcal{K}(\bar{A}, \bar{B})$, i.e., that

$$\begin{tikzcd} F\bar{A} \arrow[rr, Rightarrow, "\kappa\bar{A}"] \arrow[dr, Rightarrow, "\alpha^\lambda"'] & & G\bar{A} \arrow[dl, Rightarrow, "\alpha^\rho"] \\ & \bar{B} & \end{tikzcd}$$

commutes. This has a straightforward proof by induction; for the successor case one uses that κ is a morphism in $\mathsf{DL}(A, B)$, and for limit ordinals j the universal property of the limit B_j, the induction hypothesis and the definition of α^λ_j and α^ρ_j.

For the final point in the statement: if $B_{k+1,k} \colon B_{k+1} \to B_k$ is an isomorphism, then $B^{-1}_{k+1,k} \colon B_k \to B(B_{k+1})$ is a final B-coalgebra, and similarly for A and A_k. Hence, to show that α_k is the unique map induced by λ as in (3.2), it suffices to show that the following diagram commutes:

$$\begin{tikzcd} FA_k \arrow[r, "\alpha_k"] \arrow[d, "FA^{-1}_{k+1,k}"'] & B_k \arrow[dd, "B^{-1}_{k+1,k}"] \\ FAA_k \arrow[d, "\lambda_{A_k}"'] \arrow[dr, "\alpha_{k+1}"] & \\ BFA_k \arrow[r, "B\alpha_k"'] & BB_k \end{tikzcd}$$

The triangle commutes by definition of α, and the shape above it by naturality and the fact that $A_{k+1,k}$ and $B_{k+1,k}$ are isomorphisms. □

The natural transformation α arising from λ by the above lemma yields a natural transformation $\hat{\alpha} \colon F \Rightarrow \mathsf{Ran}_{\bar{A}}(\bar{B})$ due to the universal property of the right Kan extension. This will be shown to be the unique morphism in $\mathsf{DL}(A, B)$, turning $\mathsf{Ran}_{\bar{A}}(\bar{B})$ into the companion. However this requires a natural transformation $\mathsf{Ran}_{\bar{A}}(\bar{B})A \Rightarrow B\mathsf{Ran}_{\bar{A}}(\bar{B})$. For its existence, we assume that B preserves the right Kan extension $\mathsf{Ran}_{\bar{A}}(\bar{B})$. This condition, as well as the concrete form of the companion computed in this manner, becomes clearer when we instantiate this result to the case of lattices (Section 7) and to Set (Section 8).

Theorem 6.3. *Let $A \colon \mathcal{A} \to \mathcal{A}$ and $B \colon \mathcal{C} \to \mathcal{C}$ be endofunctors. Suppose the right Kan extension $\mathsf{Ran}_{\bar{A}}(\bar{B})$ exists and B preserves it. Then there is a natural transformation $\tau \colon \mathsf{Ran}_{\bar{A}}(\bar{B})A \Rightarrow B\mathsf{Ran}_{\bar{A}}(\bar{B})$ such that $(\mathsf{Ran}_{\bar{A}}(\bar{B}), \tau)$ is the companion of (A, B).*

Proof. By assumption, $(B\mathsf{Ran}_{\bar{A}}(\bar{B}), B\epsilon)$ is a right Kan extension of $B\bar{B}$ along $\bar{A}$. This means that for all $\alpha\colon H\bar{A} \Rightarrow B\bar{B}$, there exists a unique $\hat{\alpha}\colon H \Rightarrow B\mathsf{Ran}_{\bar{A}}(\bar{B})$ such that $\alpha = B\epsilon \circ \hat{\alpha}\bar{A}$. We use this universal property to define the natural transformation τ, choosing $H = \mathsf{Ran}_{\bar{A}}(\bar{B})A$.

To this end, consider the functor $S\colon \mathsf{Ord}^{\mathsf{op}} \to \mathsf{Ord}^{\mathsf{op}}$ defined by $S(i) = i+1$. For any $F\colon \mathcal{C} \to \mathcal{C}$, we have

$$\bar{F}S = F\bar{F} \tag{6.1}$$

simply expressing that $F_{i+1} = FF_i$ and $F_{j+1,i+1} = FF_{j,i}$ for all $i \leq j$, which both hold by definition of the final sequence. As a consequence, there is the natural transformation on the top row of the diagram below:

$$\begin{tikzcd}
\mathsf{Ran}_{\bar{A}}(\bar{B})A\bar{A} \arrow[r, equal] \arrow[d, Rightarrow, "\tau\bar{A}"'] & \mathsf{Ran}_{\bar{A}}(\bar{B})\bar{A}S \arrow[r, Rightarrow, "\epsilon S"] & \bar{B}S \arrow[r, equal] & B\bar{B} \\
B\mathsf{Ran}_{\bar{A}}(\bar{B})\bar{A} \arrow[rrru, Rightarrow, "B\epsilon"'] & & &
\end{tikzcd} \tag{6.2}$$

By the universal property of $(B\mathsf{Ran}_{\bar{A}}(\bar{B}), B\epsilon)$ we obtain $\tau\colon \mathsf{Ran}_{\bar{A}}(\bar{B})B \Rightarrow B\mathsf{Ran}_{\bar{A}}(\bar{B})$ as the unique natural transformation making the above diagram (6.2) commute.

We now show that $(\mathsf{Ran}_{\bar{A}}(\bar{B}), \tau)$ is the companion of (A, B), i.e., that it is final in the category $\mathsf{DL}(A, B)$. Let $\lambda\colon FA \Rightarrow BF$ be a natural transformation. We need to prove that there exists a unique $\hat{\alpha}\colon F \Rightarrow \mathsf{Ran}_{\bar{A}}(\bar{B})$ making the following diagram commute:

$$\begin{tikzcd}
FA \arrow[r, Rightarrow, "\hat{\alpha}A"] \arrow[d, Rightarrow, "\lambda"'] & \mathsf{Ran}_{\bar{A}}(\bar{B})A \arrow[d, Rightarrow, "\tau"] \\
BF \arrow[r, Rightarrow, "B\hat{\alpha}"'] & B\mathsf{Ran}_{\bar{A}}(\bar{B})
\end{tikzcd} \tag{6.3}$$

First, observe that for every natural transformation of the form $\alpha\colon F\bar{A} \Rightarrow \bar{B}$, there is a unique $\hat{\alpha}\colon F \Rightarrow \mathsf{Ran}_{\bar{A}}(\bar{B})$ such that $\epsilon \circ \hat{\alpha}\bar{A} = \alpha$, by the universal property of ϵ. We prove that $\hat{\alpha}$ satisfies (6.3) if and only if α makes the following diagram commute:

$$\begin{tikzcd}
FA\bar{A} \arrow[rr, Rightarrow, "\lambda\bar{A}"] \arrow[d, equal] & & BF\bar{A} \arrow[d, Rightarrow, "B\alpha"] \\
F\bar{A}S \arrow[r, Rightarrow, "\alpha S"'] & \bar{B}S \arrow[r, equal] & B\bar{B}
\end{tikzcd} \tag{6.4}$$

By Lemma 6.2, λ induces a unique α making the above diagram commute. Hence, it then follows that $\hat{\alpha}$ is the unique morphism to τ.

By the universal property of $B\epsilon$, (6.3) commutes if and only if the following equation holds:

$$B\epsilon \circ \tau\bar{A} \circ \hat{\alpha}A\bar{A} = B\epsilon \circ B\hat{\alpha}\bar{A} \circ \lambda\bar{A} \tag{6.5}$$

Hence, it suffices to prove that (6.4) is equivalent to (6.5).

Consider the following diagram:

$$
\begin{tikzcd}
FA\bar{A} \arrow[rrrr, Rightarrow, "\hat{\alpha}A\bar{A}"] \arrow[dr, equal] \arrow[dddd, Rightarrow, "\lambda\bar{A}"'] & & & & \mathsf{Ran}_{\bar{A}}(\bar{B})A\bar{A} \arrow[dl, equal] \arrow[dddd, Rightarrow, "\tau\bar{A}"] \\
& F\bar{A}S \arrow[rr, Rightarrow, "\hat{\alpha}\bar{A}S"] \arrow[dr, Rightarrow, "\alpha S"'] & & \mathsf{Ran}_{\bar{A}}(\bar{B})\bar{A}S \arrow[dl, Rightarrow, "\epsilon S"] & \\
& & \bar{B}S \arrow[d, equal] & & \\
& & B\bar{B} & & \\
BF\bar{A} \arrow[uurr, Rightarrow, "B\alpha"] \arrow[rrrr, Rightarrow, "B\hat{\alpha}\bar{A}"'] & & & & B\mathsf{Ran}_{\bar{A}}(\bar{B})\bar{A} \arrow[uull, Rightarrow, "B\epsilon"']
\end{tikzcd}
$$

The two triangles commute by definition of $\hat{\alpha}$, the upper trapezoid by the equality $A\bar{A} = \bar{A}S$, the right trapezoid by definition of τ. The left trapezoid is (6.4). The equivalence of (6.4) and (6.5) follows from a straightforward diagram chase. □

In case $A = B$, the right Kan extension $\mathsf{Ran}_{\bar{A}}(\bar{B})$ is the codensity monad $\mathsf{C}_{\bar{B}}$. It turns out that the unit and multiplication of this codensity monad coincide with the unique monad structure induced on the companion by Theorem 4.2. This follows from uniqueness of such a monad structure turning τ into a distributive law, together with the following theorem.

Theorem 6.4. *Let $B\colon \mathcal{C} \to \mathcal{C}$ be an endofunctor such that the right Kan extension $\mathsf{C}_{\bar{B}}$ exists and B preserves it. We have that τ is a distributive law of the codensity monad $(\mathsf{C}_{\bar{B}}, \eta, \mu)$ over B.*

Proof. For the unit axiom, we need to prove $\tau \circ \eta B = B\eta$, which we do by showing that $B\epsilon \circ \tau\bar{B} \circ \eta B\bar{B} = B\epsilon \circ B\eta\bar{B}$; the desired equality then follows by the universal property of the right Kan extension $(B\mathsf{C}_{\bar{B}}, B\epsilon)$. The equality follows from commutativity of:

$$
\begin{tikzcd}
& & \mathsf{C}_{\bar{B}}B\bar{B} \arrow[d, equal] \arrow[ddr, Rightarrow, "\tau\bar{B}"] & \\
B\bar{B} \arrow[urr, Rightarrow, "\eta B\bar{B}"] \arrow[r, equal] \arrow[dd, Rightarrow, "B\eta\bar{B}"'] \arrow[ddrr, equal] & \bar{B}S \arrow[r, Rightarrow, "\eta\bar{B}S"] \arrow[dr, equal] & \mathsf{C}_{\bar{B}}\bar{B}S \arrow[d, Rightarrow, "\epsilon S"] & \\
& & \bar{B}S \arrow[d, equal] & B\mathsf{C}_{\bar{B}}\bar{B} \arrow[dl, Rightarrow, "B\epsilon"] \\
B\mathsf{C}_{\bar{B}}\bar{B} \arrow[rr, Rightarrow, "B\epsilon"'] & & B\bar{B} &
\end{tikzcd}
$$

The two triangles within the big square commute by definition of η (5.1), the upper left triangle and the trapezoid in the square since $\bar{B}S = B\bar{B}$ (see (6.1)), and the right triangle by definition of τ (see (6.2)).

For the other axiom, we need to prove $\tau \circ \mu B = B\mu \circ \tau\mathsf{C}_{\bar{B}} \circ \mathsf{C}_{\bar{B}}\tau$ which, in a similar manner as above for the unit, follows from the universal property of $B\epsilon$ and commutativity of the following diagram.

$$
\begin{tikzcd}
\mathsf{C}_{\bar{B}}\mathsf{C}_{\bar{B}}B\bar{B} \arrow[rr, Rightarrow, "\mu B\bar{B}"] \arrow[dr, equal] \arrow[ddd, Rightarrow, "\mathsf{C}_{\bar{B}}\tau\bar{B}"'] & & \mathsf{C}_{\bar{B}}B\bar{B} \arrow[r, Rightarrow, "\tau\bar{B}"] \arrow[d, equal] & B\mathsf{C}_{\bar{B}}\bar{B} \arrow[dddd, Rightarrow, "B\epsilon"] \\
& \mathsf{C}_{\bar{B}}\mathsf{C}_{\bar{B}}\bar{B}S \arrow[r, Rightarrow, "\mu\bar{B}S"] \arrow[d, Rightarrow, "\mathsf{C}_{\bar{B}}\epsilon S"'] & \mathsf{C}_{\bar{B}}\bar{B}S \arrow[d, Rightarrow, "\epsilon S"] & \\
& \mathsf{C}_{\bar{B}}\bar{B}S \arrow[r, Rightarrow, "\epsilon S"'] \arrow[d, equal] & \bar{B}S \arrow[ddr, equal] & \\
\mathsf{C}_{\bar{B}}B\mathsf{C}_{\bar{B}}\bar{B} \arrow[r, Rightarrow, "\mathsf{C}_{\bar{B}}B\epsilon"] \arrow[dd, Rightarrow, "\tau\mathsf{C}_{\bar{B}}\bar{B}"'] & \mathsf{C}_{\bar{B}}B\bar{B} \arrow[d, Rightarrow, "\tau\bar{B}"'] & & \\
& B\mathsf{C}_{\bar{B}}\bar{B} \arrow[rr, Rightarrow, "B\epsilon"] & & B\bar{B} \\
B\mathsf{C}_{\bar{B}}\mathsf{C}_{\bar{B}}\bar{B} \arrow[ur, Rightarrow, "B\mathsf{C}_{\bar{B}}\epsilon"] \arrow[rrr, Rightarrow, "B\mu\bar{B}"'] & & & B\mathsf{C}_{\bar{B}}\bar{B} \arrow[u, Rightarrow, "B\epsilon"']
\end{tikzcd}
$$

The square in the middle commutes by definition of μ (see (5.1)). The rest commutes, clockwise starting from the north, by the equality $\bar{B}S = B\bar{B}$ (see (6.1)), twice definition of τ (see (6.2)), definition of μ (the south), naturality of τ and again definition of τ. ☐

The following result characterises the algebra induced on the final coalgebra by the distributive law of the companion, in terms of the counit ϵ of the codensity monad of $\bar{B}$. This plays an important role for the case $\mathcal{C} = \mathsf{Set}$ (Section 9).

Lemma 6.5. *Let A and B be endofunctors such that the right Kan extension $\mathsf{Ran}_{\bar{A}}(\bar{B})$ exists and B preserves it. If $A_{k+1,k}$ and $B_{k+1,k}$ are both isomorphisms for some k, then $\epsilon_k \colon \mathsf{Ran}_{\bar{A}}(\bar{B})A_k \to B_k$ is the unique map induced by τ as in the diagram below, where F is a shorthand for $\mathsf{Ran}_{\bar{A}}(\bar{B})$.*

$$
\begin{tikzcd}
FA_k \arrow[r, "\epsilon_k"] \arrow[d, "FA^{-1}_{k+1,k}"'] & B_k \arrow[dd, "B^{-1}_{k+1,k}"] \\
FAA_k \arrow[d, "\tau_{A_k}"'] & \\
BFA_k \arrow[r, "B\epsilon_k"'] & BB_k
\end{tikzcd}
$$

Proof. By definition of τ in the proof of Theorem 6.3, we have $B\epsilon_i \circ \tau_{A_i} = \epsilon_{i+1}$ for all i. The result follows by Lemma 6.2. ☐

Corollary 6.6. *Let $B \colon \mathcal{C} \to \mathcal{C}$ be an endofunctor such that the right Kan extension $\mathsf{C}_{\bar{B}}$ exists and B preserves it. Let $(\mathsf{C}_{\bar{B}}, \epsilon)$ be the codensity monad of $\bar{B}$, with distributive law τ and monad structure $(\mathsf{C}_{\bar{B}}, \eta, \mu)$. Further, suppose $\mathcal{C}$ has an initial object 0. If $B_{k+1,k}$ is an isomorphism for some k, then μ_0 is isomorphic (as a $\mathsf{C}_{\bar{B}}$-algebra) to ϵ_k, i.e., there is an* *isomorphism* $\iota\colon \mathsf{C}_{\bar{B}}0 \to B_k$ *(in* $\mathcal{C}$*) making the following diagram commute.*

$$\begin{tikzcd} \mathsf{C}_{\bar{B}}\mathsf{C}_{\bar{B}}0 \arrow[r, "\mathsf{C}_{\bar{B}}\iota"] \arrow[d, "\mu_0"'] & \mathsf{C}_{\bar{B}}B_k \arrow[d, "\epsilon_k"] \\ \mathsf{C}_{\bar{B}}0 \arrow[r, "\cong", "\iota"'] & B_k \end{tikzcd} \tag{6.6}$$

Proof. By Theorem 6.4, τ is a distributive law of the codensity monad $(\mathsf{C}_{\bar{B}}, \eta, \mu)$ over B. Since $\mathsf{C}_{\bar{B}}$ is the companion of B (Theorem 6.3), η and μ coincide with the natural transformations in Theorem 4.2. By Theorem 4.4, $\mathsf{C}_{\bar{B}}0$ is the carrier of a final coalgebra, and μ_0 is the algebra induced on it by τ.

Since $B_{k+1,k}$ is an isomorphism, B_k is a final coalgebra, and hence, since $\mathsf{C}_{\bar{B}}0$ is also final, there is an isomorphism (of coalgebras) $\iota\colon \mathsf{C}_{\bar{B}}0 \to B_k$. Further, by Lemma 6.5, ϵ_k is the algebra induced by τ on the final B-coalgebra B_k. It is now easy to establish (6.6). □

7. Codensity and the companion of a monotone function

Throughout this section, let $b\colon L \to L$ be a monotone function on a complete lattice. By Theorem 6.3, the companion of a monotone function b (viewed as a functor on a poset category) is given by the right Kan extension of the final sequence $\bar{b}\colon \mathsf{Ord}^{\mathsf{op}} \to L$ along itself. Using Lemma 5.1, we obtain the characterisation of the companion given in the Introduction (1.5).

Theorem 7.1. *The companion t of a monotone function b on a complete lattice is given by*

$$t : x \mapsto \bigwedge_{x \le b_i} b_i$$

Proof. By Lemma 5.1, the codensity monad $\mathsf{C}_{\bar{b}}$ can be computed by

$$\mathsf{C}_{\bar{b}}(x) = \mathsf{Ran}_{\bar{b}}(\bar{b})(x) = \bigwedge_{x \le b_i} b_i \,,$$

a limit that exists since L is a complete lattice. We apply Theorem 6.3 to show that $\mathsf{C}_{\bar{b}}$ is the companion of b. The preservation condition of the theorem amounts to the equality $b \circ \mathsf{Ran}_{\bar{b}}(\bar{b}) = \mathsf{Ran}_{\bar{b}}(b \circ \bar{b})$ which, by Lemma 5.1, in turn amounts to

$$b(\bigwedge_{x \le b_i} b_i) = \bigwedge_{x \le b_i} b(b_i)$$

for all $x \in L$. The sequence $(b_i)_{i \in \mathsf{Ord}}$ is decreasing and stagnates at some ordinal ϵ; therefore, the two intersections collapse into their last terms, say b_k and $b(b_k)$ (with k the greatest ordinal such that $x \not\le b_{k+1}$, or ϵ if such an ordinal does not exist). The equality follows. □

In fact, the category $\mathcal{K}(b)$ defined in Section 5 instantiates to the following: an object is a monotone function $f\colon L \to L$ such that $f(b_i) \le b_i$ for all $i \in \mathsf{Ord}$, and an arrow from f to g exists iff $f \le g$. The companion t is final in this category. This yields the following characterisation of functions below the companion.

Proposition 7.2. *Let t be the companion of a monotone function b on a complete lattice. For all monotone functions f we have $f \le t$ iff $\forall i \in \mathsf{Ord} : f(b_i) \le b_i$.*

A key intuition about up-to techniques is that they should at least preserve the greatest fixpoint (i.e., up-to context is valid only when bisimilarity is a congruence). It is however well-known that this is not a sufficient condition [San98, SW01]. The above proposition gives a stronger and better intuition: a technique should preserve all approximations of the greatest fixpoint (the elements of the final sequence) to be below the companion, and thus sound.

This intuition on complete lattices leads us to the abstract notion of causality we introduce in the following section.

8. Causality via right Kan extensions

We focus on the right Kan extension of the final sequence of A along the final sequence of B, where both A and B are ω-continuous Set endofunctors. For such functors, A_ω and B_ω are the carriers of the respective final coalgebras, and Lemma 5.2 provides us with a description of the codensity monad in terms of natural transformations of the form $(\bar{A}-)^X \Rightarrow \bar{B}$. We show that such natural transformations correspond to a new abstract notion which we call *causal operations*. Based on this correspondence and Theorem 6.3, we will get a concrete understanding of the companion of (A, B) in Section 9.

Definition 8.1. Let $A, B, F\colon \mathsf{Set} \to \mathsf{Set}$ be functors. An *(ω)-causal operation* (from A to B) is a map $\alpha\colon FA_\omega \to B_\omega$ such that for every set X, functions $f, g\colon X \to A_\omega$ and $i < \omega$, we have that $A_{\omega,i} \circ f = A_{\omega,i} \circ g$ implies $B_{\omega,i} \circ \alpha \circ Ff = B_{\omega,i} \circ \alpha \circ Fg$, i.e., the commutativity of the diagram on the left-hand side below implies commutativity of the right-hand side.

$$
\begin{tikzcd}
 & A_\omega \arrow[dr, "A_{\omega,i}"] & \\
X \arrow[ur, "f"] \arrow[dr, "g"'] & & A_i \\
 & A_\omega \arrow[ur, "A_{\omega,i}"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & FA_\omega \arrow[r, "\alpha"] & B_\omega \arrow[dr, "B_{\omega,i}"] & \\
FX \arrow[ur, "Ff"] \arrow[dr, "Fg"'] & & & B_i \\
 & FA_\omega \arrow[r, "\alpha"'] & B_\omega \arrow[ur, "B_{\omega,i}"'] &
\end{tikzcd}
$$

If $A = B$ then we refer to such α as a *causal algebra*. Further, an *(ω)-causal function* on $|V|$ arguments is a causal operation where $F = (-)^V$. Equivalently, $\alpha\colon (A_\omega)^V \to B_\omega$ is a causal function iff for every $h, k \in (A_\omega)^V$ and every $i < \omega$:

$$A_{\omega,i} \circ h = A_{\omega,i} \circ k \quad \text{implies} \quad B_{\omega,i}(\alpha(h)) = B_{\omega,i}(\alpha(k))\,.$$

Causal operations form a category $\mathsf{causal}(A, B)$: an object is a pair $(F, \alpha\colon FA_\omega \to B_\omega)$ where α is causal, and a morphism from (F, α) to (G, β) is a natural transformation $\kappa\colon F \Rightarrow G$ such that $\beta \circ \kappa_{A_\omega} = \alpha$.

Example 8.2. Recall from Example 2.1 that, for the functor $BX = \mathbb{R} \times X$, B_i is the set of lists of length i, and in particular B_ω is the set of streams over $\mathbb{R}$. We focus first on causal *functions*. To this end, for $\sigma, \tau \in B_\omega$, we write $\sigma \equiv_i \tau$ if σ and τ are equal up to i, i.e., $\sigma(k) = \tau(k)$ for all $k < i$. It is easy to verify that a function of the form $\alpha\colon (B_\omega)^n \to B_\omega$ is causal iff for all $\sigma_1, \ldots, \sigma_n, \tau_1, \ldots, \tau_n$ and all $i < \omega$: if $\sigma_j \equiv_i \tau_j$ for all $j \leq n$ then $\alpha(\sigma_1, \ldots, \sigma_n) \equiv_i \alpha(\tau_1, \ldots, \tau_n)$.

For instance, taking $n = 2$, $\mathsf{alt}(\sigma, \tau) = (\sigma(0), \tau(1), \sigma(2), \tau(3), \ldots)$ is causal, whereas $\mathsf{even}(\sigma) = (\sigma(0), \sigma(2), \ldots)$ (with $n = 1$) is not causal. Standard operations from the stream calculus [Rut05], such as pointwise stream addition and shuffle product, are causal.

The above notion of causal function (with a finite set of arguments V) agrees with the standard notion of causal stream function (e.g., [HKR17]). Our notion of causal *algebras* generalises it from single functions to algebras for arbitrary functors. This includes polynomial functors modelling a signature.

For $A = \mathbb{R}$, the algebra $\alpha\colon \mathcal{P}_f(B_\omega) \to B_\omega$ for the finite powerset functor $\mathcal{P}_f$, defined by $\alpha(S)(n) = \min\{\sigma(n) \mid \sigma \in S\}$, is a causal algebra which is not a causal function. The algebra $\beta\colon \mathcal{P}_f(B_\omega) \to B_\omega$ given by $\beta(S)(n) = \sum_{\sigma \in S} \sigma(n)$ is *not* causal according to Definition 8.1. Intuitively, $\beta(\{\sigma, \tau\})(i)$ depends on equality of σ and τ, since addition of real numbers is not idempotent.

Example 8.3. Let $BX = \mathbb{R} \times X$ as above, and $AX = \mathbb{R} \times \mathbb{R} \times X$. Then A_i is the set of lists over $\mathbb{R} \times \mathbb{R}$ of length i, isomorphic to the set of lists over $\mathbb{R}$ of length $2i$. In particular, A_ω is (isomorphic to) the set $\mathbb{R}^\omega$ of all lists over $\mathbb{R}$. The projections $A_{\omega,i}\colon A_\omega \to A_i$ map a stream to the first $2i$ elements.

An n-ary causal function from A to B is a map $\alpha\colon (\mathbb{R}^\omega)^n \to \mathbb{R}^\omega$ such that for all $\sigma_1, \ldots, \sigma_n, \tau_1, \ldots, \tau_n$ and all $i < \omega$: if $\sigma_j \equiv_{2i} \tau_j$ for all $j \leq n$ then $\alpha(\sigma_1, \ldots, \sigma_n) \equiv_i \alpha(\tau_1, \ldots, \tau_n)$. For instance, the function even described in the previous example is a causal function from A to B (but not from B to B).

The function $\mathsf{double}\colon \mathbb{R}^\omega \to \mathbb{R}^\omega$ given by $\mathsf{double}(\sigma) = (\sigma(0), \sigma(0), \sigma(1), \sigma(1), \ldots)$ is causal from B to A. It is easy to check that causal functions compose, so that $\mathsf{even} \circ \mathsf{double}$ is causal from B to B, and $\mathsf{double} \circ \mathsf{even}$ is causal from A to A. Such a behaviour is reminiscent of the sized-types approach to coinductive data-types in type theory [AP13].

Example 8.4. Let $BX = X + 1$. Recall the characterisation $B_i = \{0, 1, \ldots, i\}$ from Example 2.3; in particular $B_\omega = \mathbb{N} \cup \{\omega\}$, ordered as usual, with ω the top element. According to Definition 8.1, a function $f\colon B_\omega \to B_\omega$ is causal if for all $x, y \in B_\omega$, $i \in \mathbb{N}$:

$$(x = y \vee i \leq \min(x, y)) \to (f(x) = f(y) \vee i \leq \min(f(x), f(y))) \,.$$

It can be shown that this is equivalent to:

$$\forall x, y \in B_\omega. \, (x > f(x) \wedge y > f(x)) \to f(x) = f(y) \,.$$

Now let $AX = \mathbb{R} \times X$, and B as above. Using the notation $\equiv_i$ from Example 8.2, a function $f\colon \mathbb{R}^\omega \to \mathbb{N} \cup \{\omega\}$ is causal (from A to B) if for all $\sigma, \tau \in \mathbb{R}^\omega$ and all $i \in \mathbb{N}$:

$$\sigma \equiv_i \tau \to (f(\sigma) = f(\tau) \vee i \leq \min(f(\sigma), f(\tau))) \,.$$

For instance, the function $f(\sigma) = \min\{i \mid \sigma(i) = 0\}$ which computes the position of the first zero in σ, or ω if zero does not appear, is causal.

Example 8.5. For the functor $BX = 2 \times X^A$, $B_\omega = \mathcal{P}(A^*)$ is the set of languages over A (Example 2.2). Given languages L and K, we write $L \equiv_i K$ if L and K contain the same words of length below i. A function $\alpha\colon (\mathcal{P}(A^*))^n \to \mathcal{P}(A^*)$ is causal iff for all languages $L_1, \ldots, L_n, K_1, \ldots, K_n$: if $L_j \equiv_i K_j$ for all $j \leq n$ then $\alpha(L_1, \ldots, L_n) \equiv_i \alpha(K_1, \ldots, K_n)$. For instance, union, concatenation, Kleene star, and shuffle of languages are all causal. An example of a causal algebra which is not a causal function is $\alpha\colon \mathcal{P}(\mathcal{P}(A^*)) \to \mathcal{P}(A^*)$ defined by union.

The following result connects causal operations to natural transformations of the form $F\bar{A} \Rightarrow \bar{B}$ (which, from Section 5, form a category $\mathcal{K}(\bar{A}, \bar{B})$).

Theorem 8.6. *Let $A, B, F\colon \mathsf{Set} \to \mathsf{Set}$ be functors, and suppose A and B are ω-continuous. The category $\mathsf{causal}(A,B)$ of causal operations is isomorphic to the category $\mathcal{K}(\bar{A},\bar{B})$. Concretely, there is a one-to-one correspondence between natural transformations $\alpha\colon F\bar{A} \Rightarrow \bar{B}$ and causal algebras $\alpha_\omega\colon FA_\omega \to B_\omega$.*

$$\frac{\alpha\colon F\bar{A} \Rightarrow \bar{B}}{\alpha_\omega\colon FA_\omega \to B_\omega \qquad \text{causal}}$$

From top to bottom, this is given by evaluation at ω. Moreover, for any $\alpha\colon F\bar{A} \Rightarrow \bar{B}$, $\beta\colon G\bar{A} \Rightarrow \bar{B}$ and $\kappa\colon F \Rightarrow G$, we have $\beta \circ \kappa\bar{A} = \alpha$ (as on the left below) iff $\beta_\omega \circ \kappa A_\omega = \alpha_\omega$ (as on the right).

$$\begin{tikzcd} F\bar{A} \arrow[rr, "\kappa\bar{A}"] \arrow[dr, "\alpha"'] & & G\bar{A} \arrow[dl, "\beta"] \\ & \bar{B} & \end{tikzcd} \qquad \begin{tikzcd} FA_\omega \arrow[rr, "\kappa_{A_\omega}"] \arrow[dr, "\alpha_\omega"'] & & GA_\omega \arrow[dl, "\beta_\omega"] \\ & B_\omega & \end{tikzcd}$$

Proof. Let $\alpha\colon F\bar{A} \Rightarrow \bar{B}$. We need to prove that α_ω is causal; to this end, let $f, g\colon X \to FA_\omega$ be functions such that $A_{\omega,i} \circ f = A_{\omega,i} \circ g$ for some i. Then the following diagram commutes:

$$\begin{tikzcd} & FA_\omega \arrow[r, "\alpha_\omega"] \arrow[dr, "FA_{\omega,i}"'] & B_\omega \arrow[dr, "B_{\omega,i}"] & \\ FX \arrow[ur, "Ff"] \arrow[dr, "Fg"'] & & FA_i \arrow[r, "\alpha_i"] & B_i \\ & FA_\omega \arrow[r, "\alpha_\omega"'] \arrow[ur, "FA_{\omega,i}"] & B_\omega \arrow[ur, "B_{\omega,i}"'] & \end{tikzcd}$$

by assumption and naturality of α. Hence α_ω is causal.

Next, we show how to define a natural transformation $\alpha\colon F\bar{A} \Rightarrow \bar{B}$ from a given causal $\alpha_\omega\colon FA_\omega \to B_\omega$. Since A is ω-continuous (similarly for B) and any Set endofunctor preserves epimorphisms, one can prove by induction that for any $i < \omega$, the map $A_{\omega,i}$ is an epi. We will use that epis in Set split, i.e., every $A_{\omega,i}$ has a right inverse $A^{-1}_{\omega,i}$ with $A_{\omega,i} \circ A^{-1}_{\omega,i} = \mathsf{id}$.

Given $\alpha_\omega\colon FA_\omega \to B_\omega$, define $\alpha\colon F\bar{A} \Rightarrow \bar{B}$ on a component $i < \omega$ by

$$FA_i \xrightarrow{F(A^{-1}_{\omega,i})} FA_\omega \xrightarrow{\alpha_\omega} B_\omega \xrightarrow{B_{\omega,i}} B_i$$

where $A^{-1}_{\omega,i}$ is a right inverse of $A_{\omega,i}$. It follows from causality of α_ω that the choice of right inverse is irrelevant. On a component $i \geq \omega$, α is defined by

$$FA_i \xrightarrow{FA_{i,\omega}} FA_\omega \xrightarrow{\alpha_\omega} B_\omega \xrightarrow{B^{-1}_{i,\omega}} B_i$$

where $B^{-1}_{i,\omega}$ is the inverse of $B_{i,\omega}$ (which is an isomorphism, since B is ω-continuous).

We need to show that α is a natural transformation, and that the correspondence is bijective. For the bijective correspondence, first note that mapping α_ω to α and back trivially yields α_ω again. Conversely, given α, we need to prove that the following diagrams commute for $i < \omega$ (on the left) and $i \geq \omega$ (on the right):

$$\begin{tikzcd} FA_i \arrow[r, "\alpha_i"] \arrow[d, "F(A_{\omega,i}^{-1})"'] & B_i \\ FA_\omega \arrow[r, "\alpha_\omega"'] & B_\omega \arrow[u, "B_{\omega,i}"'] \end{tikzcd} \qquad \begin{tikzcd} FA_i \arrow[r, "\alpha_i"] \arrow[d, "FA_{i,\omega}"'] & B_i \\ FA_\omega \arrow[r, "\alpha_\omega"'] & B_\omega \arrow[u, "B_{i,\omega}^{-1}"'] \end{tikzcd}$$

The case $i < \omega$ follows by naturality of α and since $A_{\omega,i}^{-1}$ is a right inverse of $A_{\omega,i}$, the case $i \geq \omega$ by naturality of α and since $B_{i,\omega}^{-1}$ is a (left) inverse of $B_{i,\omega}$.

It remains to be shown that α, defined from a given α_ω as above, is natural, using that α_ω is causal. To this end, let $i \leq j$; to prove is that the following diagram commutes:

$$\begin{tikzcd} FA_i \arrow[r, "\alpha_i"] & B_i \\ FA_j \arrow[u, "FA_{j,i}"] \arrow[r, "\alpha_j"'] & B_j \arrow[u, "B_{i,j}"'] \end{tikzcd}$$

where α_i, α_j are defined from α_ω as above. We proceed with a case distinction.

If $i, j < \omega$, then the following diagram commutes:

$$\begin{tikzcd} A_j \arrow[r, "A_{j,i}"] \arrow[d, "A_{\omega,j}^{-1}"'] \arrow[dr, equal] & A_i \arrow[r, "A_{\omega,i}^{-1}"] \arrow[dr, equal] & A_\omega \arrow[d, "A_{\omega,i}"] \\ A_\omega \arrow[r, "A_{\omega,j}"] \arrow[rr, bend right, "A_{\omega,i}"'] & A_j \arrow[r, "A_{j,i}"] & A_i \end{tikzcd}$$

since $A_{\omega,i}^{-1}$ and $A_{\omega,j}^{-1}$ are right inverses (for the two triangles) and the final sequence $\bar{A}$ is a functor (for the crescent). By causality of α_ω (and functoriality of $\bar{B}$) we obtain commutativity of:

$$\begin{tikzcd} FA_i \arrow[r, "F(A_{\omega,i}^{-1})"] & FA_\omega \arrow[r, "\alpha_\omega"] & B_\omega \arrow[r, "B_{\omega,i}"] & B_i \\ FA_j \arrow[u, "FA_{j,i}"] \arrow[r, "F(A_{\omega,j}^{-1})"'] & FA_\omega \arrow[r, "\alpha_\omega"'] & B_\omega \arrow[r, "B_{\omega,j}"'] \arrow[ur, "B_{\omega,i}"] & B_j \arrow[u, "B_{j,i}"'] \end{tikzcd}$$

which is what we needed to prove, by definition of α_i and α_j.

If $i < \omega \leq j$, then the following diagram commutes:

$$\begin{tikzcd} A_i \arrow[r, "A_{\omega,i}^{-1}"] & A_\omega \arrow[r, "A_{\omega,i}"] & A_i \\ A_j \arrow[u, "A_{j,i}"] \arrow[rr, "A_{j,\omega}"'] & & A_\omega \arrow[u, "A_{\omega,i}"'] \end{tikzcd}$$

since $A_{\omega,i} \circ A_{\omega,i}^{-1} = \mathsf{id}$, and the final sequence $\bar{A}$ is a functor. Hence, by causality of α we obtain the commutativity of the large inner part in:

$$\begin{tikzcd}
FA_i \arrow[r, "F(A_{\omega,i}^{-1})"] & FA_\omega \arrow[r, "\alpha_\omega"] & B_\omega \arrow[r, "B_{\omega,i}"] & B_i \\
FA_j \arrow[u, "FA_{j,i}"] \arrow[r, "FA_{j,\omega}"'] & FA_\omega \arrow[r, "\alpha_\omega"'] & B_\omega \arrow[ur, "B_{\omega,i}"] \arrow[r, "B_{j,\omega}^{-1}"'] & B_j \arrow[u, "B_{j,i}"']
\end{tikzcd}$$

The triangle commutes since $\bar{B}$ is functorial and $B_{j,\omega}^{-1}$ is an inverse of $B_{j,\omega}$.

Finally, if $\omega \leq i \leq j$, then we immediately obtain commutativity of:

$$\begin{tikzcd}
FA_i \arrow[r, "FA_{i,\omega}"] & FA_\omega \arrow[r, "\alpha_\omega"] & B_\omega \arrow[r, "B_{i,\omega}^{-1}"] & B_i \\
FA_j \arrow[u, "FA_{j,i}"] \arrow[ur, "FA_{j,\omega}"'] \arrow[r, "FA_{j,\omega}"'] & FA_\omega \arrow[r, "\alpha_\omega"'] & B_\omega \arrow[ur, "B_{i,\omega}^{-1}"] \arrow[r, "B_{j,\omega}^{-1}"'] & B_j \arrow[u, "B_{j,i}"']
\end{tikzcd}$$

The triangles commute by functoriality of $\bar{A}, \bar{B}$ and the fact that $B_{i,\omega}^{-1}$ and $B_{j,\omega}^{-1}$ are inverses of $B_{i,\omega}$ and $B_{j,\omega}$ respectively.

This concludes the one-to-one correspondence between natural transformations α and causal algebras α_ω. We turn to the second correspondence in the statement: the equivalence

$$\beta \circ \kappa\bar{A} = \alpha \qquad \text{iff} \qquad \beta_\omega \circ \kappa_{A_\omega} = \alpha_\omega$$

for any $\alpha \colon F\bar{A} \Rightarrow \bar{B}$, $\beta \colon G\bar{A} \Rightarrow \bar{B}$ and $\kappa \colon F \Rightarrow G$. From left to right this is trivial. Conversely, suppose $\beta_\omega \circ \kappa_{A_\omega} = \alpha_\omega$. By the above, both α and β extend to natural transformations. First, suppose $i < \omega$. We need to prove that the outside of the following diagram commutes:

$$\begin{tikzcd}
FA_i \arrow[rrr, "\kappa_{A_i}"] \arrow[dr, "FA_{\omega,i}^{-1}"'] \arrow[ddd, "\alpha_i"'] & & & GA_i \arrow[ddd, "\beta_i"] \\
& FA_\omega \arrow[r, "\kappa_{A_\omega}"] \arrow[d, "\alpha_\omega"'] & GA_\omega \arrow[ur, "GA_{\omega,i}"'] \arrow[d, "\beta_\omega"] & \\
& B_\omega \arrow[r, equal] \arrow[dl, "B_{\omega,i}"'] & B_\omega \arrow[dr, "B_{\omega,i}"] & \\
B_i \arrow[rrr, equal] & & & B_i
\end{tikzcd}$$

The middle square commutes by assumption. The rest, clockwise starting at the north, by naturality of κ and $A_{\omega,i}^{-1}$ being a right inverse of $A_{\omega,i}$, naturality of β, trivially, and by naturality of α and $A_{\omega,i}^{-1}$ being a right inverse of $A_{\omega,i}$.

For $i \geq \omega$, we have $\alpha_i = B_{i,\omega}^{-1} \circ \alpha_\omega \circ FA_{i,\omega}$ and $\beta_i = B_{i,\omega}^{-1} \circ \beta_\omega \circ FA_{i,\omega}$, hence it suffices to prove commutativity of the diagram below.

$$\begin{tikzcd}
FA_i \arrow[rr, "\kappa_{A_i}"] \arrow[d, "FA_{i,\omega}"'] & & GA_i \arrow[d, "GA_{i,\omega}"] \\
FA_\omega \arrow[rr, "\kappa_{A_\omega}"] \arrow[d, "\alpha_\omega"'] & & GA_\omega \arrow[d, "\beta_\omega"] \\
B_\omega \arrow[rr, equal] \arrow[dr, "B_{i,\omega}^{-1}"'] & & B_\omega \arrow[dl, "B_{i,\omega}^{-1}"] \\
& B_i &
\end{tikzcd}$$

That follows, again, from naturality of κ and the assumption. □

By the above theorem, the universal property of the right Kan extension amounts to the following property of causal algebras.

Corollary 8.7. *Suppose $A, B\colon \mathsf{Set} \to \mathsf{Set}$ are ω-continuous. Let ϵ be the counit of $\mathsf{Ran}_{\bar{A}}(\bar{B})$. Then ϵ_ω is final in $\mathsf{causal}(A, B)$, i.e., ϵ_ω is causal and for every causal algebra $\alpha_\omega\colon FA_\omega \to B_\omega$, there is a unique natural transformation $\hat{\alpha}\colon F \Rightarrow \mathsf{Ran}_{\bar{A}}(\bar{B})$ such that $\epsilon_\omega \circ \hat{\alpha}_{A_\omega} = \alpha_\omega$.*

$$\begin{tikzcd}
FA_\omega \arrow[rr, "\hat{\alpha}_{A_\omega}"] \arrow[dr, "\alpha_\omega"'] & & \mathsf{Ran}_{\bar{A}}(\bar{B})A_\omega \arrow[dl, "\epsilon_\omega"] \\
& B_\omega &
\end{tikzcd}$$

By Lemma 5.2 and Theorem 8.6, we obtain the following concrete description of the relevant right Kan extension as a *functor of causal functions*.

Theorem 8.8. *Let $A, B\colon \mathsf{Set} \to \mathsf{Set}$ be ω-continuous functors. The right Kan extension $\mathsf{Ran}_{\bar{A}}(\bar{B})$ is given by*

$$\mathsf{Ran}_{\bar{A}}(\bar{B})(X) = \{\alpha\colon A_\omega^X \to B_\omega \mid \alpha \textit{ is a causal function}\}\,,$$
$$\mathsf{Ran}_{\bar{A}}(\bar{B})(h\colon X \to Y)(\alpha) = \lambda f.\, \alpha(f \circ h)\,,$$

and, for the counit $\epsilon\colon \mathsf{Ran}_{\bar{A}}(\bar{B})\bar{A} \Rightarrow \bar{B}$, we have $\epsilon_\omega(\alpha\colon A_\omega^{A_\omega} \to B_\omega) = \alpha(\mathsf{id}_{A_\omega})$.

Finally, given $\alpha\colon F\bar{A} \Rightarrow \bar{B}$, the unique natural transformation $\hat{\alpha}\colon F \Rightarrow \mathsf{Ran}_{\bar{A}}(\bar{B})$ such that $\epsilon \circ \hat{\alpha}\bar{A} = \alpha$ is given by $\hat{\alpha}_X(S)\colon A_\omega^X \to B_\omega$, $f \mapsto \alpha_\omega \circ Ff(S)$. Equivalently, $\hat{\alpha}$ is the unique natural transformation such that $\epsilon_\omega \circ \hat{\alpha}_{A_\omega} = \alpha_\omega$.

Taking $A = B = \mathbb{R} \times \mathsf{Id}$, the above right Kan extension (codensity monad) maps a set X to set of all causal stream functions with $|X|$ arguments. Similarly, for the functor $X \mapsto 2 \times X^A$ we obtain a functor of causal functions on languages.

Example 8.9. The right Kan extension $\mathsf{Ran}_{\bar{A}}(\bar{B})$ has a universal property in the category of causal algebras, via Corollary 8.7. We give an example for the case of streams, taking $A = B = \mathbb{R} \times \mathsf{Id}$, so the relevant right Kan extension $\mathsf{Ran}_{\bar{A}}(\bar{B})$ is the codensity monad $\mathsf{C}_{\bar{B}}$. Consider a causal algebra on streams, of the form $\alpha_\omega\colon \mathbb{R}^\omega \times \mathbb{R}^\omega \to \mathbb{R}^\omega$ for the functor

$SX = X^2$. It follows from Theorem 8.8 that the unique natural transformation $\hat{\alpha}\colon S \Rightarrow \mathsf{C}_{\bar{B}}$ such that $\epsilon_\omega \circ \hat{\alpha}_{\mathbb{R}^\omega} = \alpha_\omega$ is given on an X-component $\hat{\alpha}_X \colon X \times X \to \mathsf{C}_{\bar{B}}(X)$ by

$$\hat{\alpha}_X(x,y)(f\colon X \to \mathbb{R}^\omega) = \alpha_\omega(f(x), f(y))\,.$$

9. Companion of polynomial Set functors

The previous sections give us a concrete understanding of the codensity monad of the final sequence of a Set functor in terms of causal functions, and Theorem 6.3 provides us with a sufficient condition for this codensity monad to be the companion. We now focus on several applications of these results.

A rather general class of functors that satisfy the hypotheses of Theorem 6.3 is given by the *polynomial functors*. Automata, stream systems, Mealy and Moore machines, various kinds of trees, and many more are all examples of coalgebras for polynomial functors (e.g., [Jac16]). A functor $B\colon \mathsf{Set} \to \mathsf{Set}$ is called polynomial (in a single variable) if it is isomorphic to a functor of the form

$$X \mapsto \coprod_{a \in A} X^{B_a}$$

for some A-indexed collection $(B_a)_{a\in A}$ of sets. As explained in [GK13, 1.18], a Set functor B is polynomial if and only if it preserves connected limits. This implies existence and preservation by B of the codensity monad of $\bar{B}$, as required by Theorem 6.3.

Lemma 9.1. *If $B\colon \mathsf{Set} \to \mathsf{Set}$ is polynomial, then $\mathsf{C}_{\bar{B}}$ exists and B preserves it.*

For the proof, we first recall that a category is *connected* [Lan98] if it is inhabited and there is a zigzag of morphisms between any two objects X and Y: a finite collection of morphisms of the form

$$X = X_0 \longrightarrow X_1 \longleftarrow X_2 \longrightarrow X_3 \longleftarrow X_4 \longrightarrow \ldots \longleftarrow X_n = Y\,.$$

A *connected limit* is a limit over a connected category.

Proof. Since B is polynomial, $B_{k,\omega}$ is an isomorphism for each $k \geq \omega$, which implies that the category $\Delta_X/\bar{B}$ is essentially small for every set X. Hence, the limit

$$\lim\left((\Delta_X/\bar{B}) \to \mathsf{Ord}^{\mathsf{op}} \xrightarrow{\bar{B}} \mathsf{Set}\right) \tag{9.1}$$

exists for each X, which, by Lemma 5.1, defines the codensity monad $\mathsf{C}_{\bar{B}}$. As mentioned above, since B is polynomial, it preserves connected limits [GK13]. We show that $\Delta_X/\bar{B}$ is connected: $\Delta_X/\bar{B}$ is inhabited since there is the arrow $!_X\colon X \to B_0 = 1$; and for any $f\colon X \to B_i$, there is the arrow $B_{i,0}$ to $!_X\colon X \to B_0$, which is a morphism in $\Delta_X/\bar{B}$ by uniqueness. Hence, B preserves the limits in (9.1). This implies that B preserves $\mathsf{C}_{\bar{B}}$, which we describe in detail.

Denote, for a given set X, the limiting cone of (9.1) by

$$\{s^X_f \colon \mathsf{C}_{\bar{B}}X \to B_i\}_{f \in B_i^X, i \in \mathsf{Ord}}.$$

The counit of the codensity monad is defined by $\epsilon_i = s^{B_i}_{\mathsf{id}_{B_i}}$ (see, e.g., [nLab, Lan98]). Since B preserves these limits, for each X, we have that

$$\{Bs^X_f \colon B\mathsf{C}_{\bar{B}}X \to BB_i\}_{f \in B_i^X, i \in \mathsf{Ord}}$$

is the limit

$$\lim\left((\Delta_X/\bar{B}) \to \mathsf{Ord}^{\mathsf{op}} \xrightarrow{\bar{B}} \mathsf{Set} \xrightarrow{B} \mathsf{Set}\right).$$

Hence, by Lemma 5.1, $B\mathsf{C}_{\bar{B}}$ is a right Kan extension of $B\bar{B}$ along $\bar{B}$, with counit defined on $i \in \mathsf{Ord}$ by $Bs^{B_i}_{\mathsf{id}_{B_i}} = B\epsilon_i$ as desired. □

As a consequence, if B is polynomial, the functor of causal functions in Theorem 8.8 is the companion of B.

9.1. **Causal algebras and coinduction up-to.** As explained in Section 3, a distributive law of F over B allows one to strengthen the coinduction principle, formalised in terms of BF-coalgebras, leading to an expressive coinductive definition (and proof) technique. This approach is formally supported by Theorem 3.3. Based on the characterisation of the companion in terms of causal algebras, we obtain a new validity theorem, which does not mention distributive laws at all, but is stated purely in terms of causal algebras.

Theorem 9.2. *Let $B\colon \mathsf{Set} \to \mathsf{Set}$ be a polynomial functor, with final coalgebra (B_ω, ζ). Let $\alpha\colon FB_\omega \to B_\omega$ be a causal algebra. Then coinduction up to α is valid.*

Proof. Consider the codensity monad $\mathsf{C}_{\bar{B}}$, with counit ϵ. By Lemma 9.1, the functor B satisfies the hypotheses of Theorem 6.3 and hence $\mathsf{C}_{\bar{B}}$ is the underlying functor of the companion. By Lemma 6.5 (and since any polynomial functor is ω-continuous), ϵ_ω is the algebra induced by τ on the final coalgebra. Thus, by Corollary 4.3, coinduction up to ϵ_ω is valid.

Since α is causal, by Corollary 8.7 there is a unique natural transformation $\hat{\alpha}\colon F \Rightarrow \mathsf{C}_{\bar{B}}$ such that $\epsilon_\omega \circ \hat{\alpha}_{B_\omega} = \alpha$.

Let $f\colon X \to BFX$. Since coinduction up to ϵ_ω is valid, there is a unique $f^\dagger$ making the outside of the following diagram commute (note that we abuse notation, using $f^\dagger$ to refer to the unique map associated to the coalgebra $B\hat{\alpha}_X \circ f$ by the validity of coinduction up to ϵ_ω).

$$\begin{tikzcd}
X \arrow[rr, "f^\dagger"] \arrow[d, "f"'] & & B_\omega \arrow[d, "\zeta"] \\
BFX \arrow[r, "BFf^\dagger"] \arrow[d, "B\hat{\alpha}_X"'] & BFB_\omega \arrow[r, "B\alpha"] \arrow[d, "B\hat{\alpha}_{B_\omega}"'] & BB_\omega \arrow[d, equal] \\
B\mathsf{C}_{\bar{B}}X \arrow[r, "B\mathsf{C}_{\bar{B}}f^\dagger"'] & B\mathsf{C}_{\bar{B}}B_\omega \arrow[r, "B\epsilon_\omega"'] & BB_\omega
\end{tikzcd}$$

The lower left square commutes by naturality, and the lower right square by definition of $\hat{\alpha}$. Thus the outside of the diagram commutes if and only if the inner rectangle commutes. It follows that $f^\dagger$ is the unique map making the rectangle commute, which is what we needed to prove. □

Example 9.3. For the functor $BX = \mathbb{R} \times X$, B_ω is the set of streams. Let $SX = X^2$, and consider the coalgebra $f\colon 1 \to BS1$ with $1 = \{*\}$, defined by $* \mapsto (1, (*, *))$. Pointwise addition is a causal function on streams, modelled by an algebra on B_ω for the functor S. By Theorem 9.2 we obtain a unique solution $\sigma \in B_\omega$, satisfying $\sigma_0 = 1$ and $\sigma' = \sigma \oplus \sigma$.

Similarly, the shuffle product of streams is causal, so that by applying Theorem 9.2 with that algebra and the same coalgebra f we obtain a unique stream σ satisfying $\sigma_0 = 1$, $\sigma' = \sigma \otimes \sigma$.

As explained in the Introduction, this method also allows one to define functions on streams. For instance, for the shuffle product, define a BS-coalgebra $f\colon (B_\omega)^2 \to BS(B_\omega)^2$ by $f(\sigma, \tau) = (\sigma_0 \times \tau_0, ((\sigma', \tau), (\tau, \sigma')))$. Since addition of streams is causal, by Theorem 9.2 there is a unique $f^\dagger\colon B_\omega \times B_\omega \to B_\omega$ such that $f^\dagger(\sigma,\tau)(0) = \sigma(0) \times \tau(0)$ and $(f^\dagger(\sigma,\tau))' = (f^\dagger(\sigma',\tau) \oplus f^\dagger(\sigma,\tau'))$, matching the definition given in the Introduction (1.2). Notice that not every function defined in this way is causal; for instance, it is easy to define even (see Example 8.2), even with the standard coinduction principle (i.e., where $F = \mathsf{Id}$ and $\alpha = \mathsf{id}$).

Example 9.4. Consider the functor $BX = 2 \times X^A$, whose final coalgebra consists of the set $\mathcal{P}(A^*)$ of languages. A $B\mathcal{P}$-coalgebra $f\colon X \to 2 \times (\mathcal{P}(X))^A$ is a non-deterministic automaton. Taking the causal algebra $\alpha\colon \mathcal{P}(\mathcal{P}(A^*)) \to \mathcal{P}(A^*)$ defined by union, the unique map $f^\dagger\colon X \to \mathcal{P}(A^*)$ from Theorem 9.2 is the usual language semantics of non-deterministic automata.

In [WBR13], a context-free grammar (in Greibach normal form) is modelled as a $B\mathcal{P}^*$-coalgebra $f\colon X \to 2 \times (\mathcal{P}(X)^*)^A$, where X are the non-terminals, and its semantics is defined operationally by turning f into a deterministic automaton with state space $\mathcal{P}(X^*)$. In [RW13] this operational view is related to the semantics of CFGs in terms of language equations. Consider the causal algebra $\alpha\colon \mathcal{P}(\mathcal{P}(A^*)^*) \to \mathcal{P}(A^*)$ defined by union and language composition: $\alpha(S) = \bigcup_{L_1 \dots L_k \in S} L_1 L_2 \dots L_k$. By Theorem 9.2, any context-free grammar f has a unique solution $f^\dagger\colon X \to \mathcal{P}(A^*)$ assigning a language to every non-terminal; the commutativity of the diagram in 9.2 amounts to the fact that this is a solution of the grammar f viewed as a system of equations over the set $\mathcal{P}(A^*)$ of languages. As such, we obtain an elementary coalgebraic semantics of CFGs that does not require us to relate it to an operational semantics.

9.2. **Causal algebras and distributive laws.**

Another application of the fact that the codensity monad is the companion is that the final causal algebra in Corollary 8.7 is, by Lemma 6.5, the algebra induced by a distributive law. Hence, *any* causal algebra is "definable" by a distributive law, in the sense that it factors as a (component of a) natural transformation followed by the algebra induced by a distributive law. This is stated in Corollary 9.6 below, which follows from the following more general result.

Theorem 9.5. *Let $A, B\colon \mathsf{Set} \to \mathsf{Set}$ be functors, where A is ω-continuous and B is polynomial. An algebra $\alpha_\omega\colon FA_\omega \to B_\omega$ is causal if and only if there is a functor G, a natural transformation $\lambda\colon GA \Rightarrow BG$ and a natural transformation $\kappa\colon F \Rightarrow G$ such that the following diagram commutes:*

$$\begin{tikzcd} FA_\omega \arrow[rr, "\kappa_{A_\omega}"] \arrow[dr, "\alpha_\omega"'] & & GA_\omega \arrow[dl, "\beta_\omega"] \\ & B_\omega & \end{tikzcd}$$

where β_ω is the unique map induced by λ as in (3.2).

Proof. First note that the type of β_ω is correct: since A, B are both ω-continuous, A_ω and B_ω are final coalgebras. For the implication from right to left, by Lemma 6.2, a natural transformation $\lambda\colon GA \Rightarrow BG$ defines a natural transformation $\beta\colon G\bar{A} \Rightarrow \bar{B}$ such that β_ω

is the unique map induced by λ, and hence satisfies the above diagram. By Theorem 8.6, $\alpha_\omega = \beta_\omega \circ \kappa_{A_\omega}$ is causal, since $\beta \circ \kappa\bar{A} \colon F\bar{A} \Rightarrow \bar{B}$ is a natural transformation.

For the converse, let $\alpha \colon FA_\omega \to B_\omega$ be causal. We instantiate G and β_ω in the statement of the theorem respectively to $\mathsf{Ran}_{\bar{A}}(\bar{B})$ and ϵ_ω, for ϵ its counit. By Corollary 8.7, there is a natural transformation $\hat{\alpha} \colon F \Rightarrow \mathsf{Ran}_{\bar{A}}(\bar{B})$ such that $\alpha = \epsilon_\omega \circ \hat{\alpha}_{A_\omega}$. Further, by Lemma 9.1, B satisfies the hypotheses of Theorem 6.3. Hence, $\mathsf{Ran}_{\bar{A}}(\bar{B})$ is the companion with a natural transformation $\tau \colon \mathsf{Ran}_{\bar{A}}(\bar{B})A \Rightarrow B\mathsf{Ran}_{\bar{A}}(\bar{B})$, and by Lemma 6.5, ϵ_ω is the unique map induced by τ. □

In particular, taking $A = B$, the above theorem is an expressivity result for algebras defined by distributive laws. To make this more precise, suppose $B \colon \mathsf{Set} \to \mathsf{Set}$ has a final coalgebra (Z, ζ). We say an algebra $\alpha \colon FZ \to Z$ is *definable by a distributive law* if there exists a distributive law $\lambda \colon GB \Rightarrow BG$ with induced algebra $\beta \colon GZ \to Z$ and a natural transformation $\kappa \colon F \Rightarrow G$ such that the following commutes:

$$\begin{tikzcd} FZ \arrow[rr, "\kappa_Z"] \arrow[dr, "\alpha"'] & & GZ \arrow[dl, "\beta"] \\ & Z & \end{tikzcd}$$

Corollary 9.6. *Let $B \colon \mathsf{Set} \to \mathsf{Set}$ be polynomial. An algebra $\alpha_\omega \colon FB_\omega \to B_\omega$ is causal if and only if it is definable by a distributive law.*

Example 9.7. Since the functors for stream systems and automata are polynomial, by Corollary 9.6 we obtain that a function $f \colon (\mathbb{R}^\omega)^V \to \mathbb{R}^\omega$ on streams over $\mathbb{R}$, or a function $f \colon (\mathcal{P}(A^*))^V \to \mathcal{P}(A^*)$ on languages over A, is causal if and only if it is definable by a distributive law.

As explained in Example 8.3, the function even is not causal (from B to B, where $B = \mathbb{R} \times \mathsf{Id}$) but it is causal from A to B, where $A = \mathbb{R} \times \mathbb{R} \times \mathsf{Id}$. It follows from Corollary 9.6 that even is *not* definable by a distributive law of the form $GB \Rightarrow BG$. However, by Theorem 9.5, there is a distributive law $\lambda \colon GA \Rightarrow BG$ inducing an algebra $\beta_\omega \colon G\mathbb{R}^\omega \to \mathbb{R}^\omega$ (recall that $\mathbb{R}^\omega$ is the final coalgebra of both A and B) and a natural transformation $\kappa \colon \mathsf{Id} \Rightarrow G$ such that $\mathsf{even} = \beta_\omega \circ \kappa_{\mathbb{R}^\omega}$. Indeed, even arises directly as the unique operation induced by a natural transformation of the form $GA \Rightarrow BG$ where $G = \mathsf{Id}$, as shown in Example 3.5.

In [HKR17], a similar result to Corollary 9.6 is shown concretely for causal stream functions, and this is extended to languages in [RBR16]. In both cases, very specific presentations of distributive laws for the systems at hand are used to present the distributive law based on a "syntax", which however is not too clearly distinguished from the semantics: it consists of a single operation symbol for every causal function. In our case, in the proof of Theorem 9.5, we use the *companion*, which consists of the actual functions rather than a syntactic representation. Indeed, the setting of Theorem 9.5 applies more abstractly to *all* causal algebras, not just causal functions. However, it remains an intriguing question how to obtain a concrete syntactic characterisation of a distributive law for a given causal algebra.

9.3. **Soundness of up-to techniques.** The *contextual closure* of an algebra is one of the most powerful up-to techniques, which allows one to exploit algebraic structure in bisimulation proofs. In [BPPR16], it is shown that the contextual closure is sound (compatible) on any bialgebra for a distributive law. Here, we move away from the explicit requirement of a distributive law and give an elementary condition for soundness of the contextual closure on the final coalgebra: that the algebra under consideration is causal. In fact, we prove that this implies that the contextual closure lies below the companion, which not only gives soundness, but also allows to combine it with other up-to techniques.

Example 9.8. To motivate and illustrate the use of contextual closure as an up-to technique, we recall from [RBR16] a coinductive proof of *Arden's rule*.

First let $\mathsf{Rel}_{\mathcal{P}(A^*)} = \mathcal{P}(\mathcal{P}(A^*) \times \mathcal{P}(A^*))$, and consider the function $b\colon \mathsf{Rel}_{\mathcal{P}(A^*)} \to \mathsf{Rel}_{\mathcal{P}(A^*)}$ defined by

$$b(R) = \{(L,K) \mid \varepsilon \in L \text{ iff } \varepsilon \in K, \text{ and } \forall a \in A.\, (L_a, K_a)\}$$

where, given $L \in \mathcal{P}(A^*)$ and $a \in A$, $L_a = \{w \mid aw \in L\}$, called language derivative. A relation R is a *bisimulation* if $R \subseteq b(R)$; concretely, R is a bisimulation if for every pair $(L,K) \in R$: L contains the empty word iff K does, and $(L_a, K_a) \in R$ for all $a \in A$. The greatest fixpoint of b is language equality. Hence, the coinduction principle asserts that languages L, K are equal whenever they are contained in a bisimulation.

Arden's rule states that for every three languages $L, K, M \in \mathcal{P}(A^*)$, if $L = KL + M$ and K does not contain the empty word, then $L = K^*M$. To prove it, one may try to show that, for L, K, M satisfying the assumption, $R = \{(L, K^*M)\}$ is a bisimulation. However, this fails, since

$$L_a = (KL + M)_a = K_a L + M_a, \qquad \text{whereas} \qquad (K^*M)_a = K_a K^* M + M_a\,,$$

using so-called Brzozowski derivatives to compute the a-dervatives [Rut98]. The pair $(L_a, (K^*M)_a)$ is not related by R. However, it is related by the bigger relation $\mathsf{ctx}(\mathsf{rfl}(R))$, where $\mathsf{rfl}, \mathsf{ctx}\colon \mathsf{Rel}_{\mathcal{P}(A^*)} \to \mathsf{Rel}_{\mathcal{P}(A^*)}$ are the following functions on relations: $\mathsf{rfl}(R) = R \cup \{(L,L) \mid L \subseteq A^*\}$ is the reflexive closure and $\mathsf{ctx}(R)$ is the *contextual closure* of R (with respect to sum and composition). More precisely, $\mathsf{ctx}(R)$ is the least relation satisfying the following rules:

$$\frac{(L,K) \in R}{(L,K) \in \mathsf{ctx}(R)} \qquad \frac{(L_1,K_1) \in \mathsf{ctx}(R) \quad (L_2,K_2) \in \mathsf{ctx}(R)}{(L_1 + L_2, K_1 + K_2) \in \mathsf{ctx}(R)}$$

$$\frac{(L_1,K_1) \in \mathsf{ctx}(R) \quad (L_2,K_2) \in \mathsf{ctx}(R)}{(L_1 L_2, K_1 K_2) \in \mathsf{ctx}(R)}$$

Hence we have $R \subseteq b(\mathsf{ctx}(\mathsf{rfl}(R)))$, which means that R is a *bisimulation up to* $\mathsf{ctx} \circ \mathsf{rfl}$. To conclude that $L = K^*M$, it suffices to prove that ctx and rfl are both below the companion of b (cf. [Pou16]). Both rfl and ctx are, in fact, compatible, which follows from [BPPR16]. We focus on ctx. Showing compatibility using the techniques of [BPPR16] requires providing a distributive law. However, it also follows as an instance of Theorem 9.10 below that ctx is below the companion, relying on causality instead of having to provide an explicit distributive law.

We generalise the functions b and ctx from the above examples to speak more abstractly about soundness of the contextual closure for bisimilarity proofs, following the approach of [BPPR16] to up-to techniques. This approach is formulated at the abstract level of coinductive predicates in fibrations. However, we only recall a few necessary definitions, and refer to [BPPR16] for details.

For a set X, let $\mathsf{Rel}_X = \mathcal{P}(X \times X)$ be the lattice of relations, ordered by subset inclusion. For a functor $B\colon \mathsf{Set} \to \mathsf{Set}$, bisimulations on a B-coalgebra (X, f) are the post-fixed points of the monotone function $b_f\colon \mathsf{Rel}_X \to \mathsf{Rel}_X$, defined by

$$b_f(R) = (f \times f)^{-1} \circ \mathsf{Rel}(B)(R)\,.$$

Here $(f \times f)^{-1}$ is inverse image along $f \times f$, and $\mathsf{Rel}(B)$ is the *relation lifting* of B, defined for any relation R with projections π_1, π_2 by

$$\mathsf{Rel}(B)(R) = \{(x, y) \mid \exists z \in BR.\, x = B\pi_1(z),\, y = B\pi_2(z)\}\,,$$

see, e.g., [Jac16]. Contextual closure $\mathsf{ctx}_\alpha\colon \mathsf{Rel}_X \to \mathsf{Rel}_X$ with respect to an algebra $\alpha\colon FX \to X$ is defined dually by

$$\mathsf{ctx}_\alpha(R) = \textstyle\coprod_\alpha \circ \mathsf{Rel}(F)(R)$$

where $\coprod_\alpha$ is direct image along $\alpha \times \alpha$.

We first prove a general property of algebras and the contextual closure.

Lemma 9.9. *Let X be a set, $F, G\colon \mathsf{Set} \to \mathsf{Set}$ functors, $\alpha\colon FX \to X$ and $\beta\colon GX \to X$ algebras, and $\kappa\colon F \Rightarrow G$ a natural transformation, such that $\alpha = \beta \circ \kappa_X$. Then $\mathsf{ctx}_\alpha \le \mathsf{ctx}_\beta$.*

Proof. The natural transformation κ lifts to a natural transformation

$$\mathsf{Rel}(\kappa)\colon \mathsf{Rel}(F) \Rightarrow \mathsf{Rel}(G),$$

see [Jac16, Exercise 4.4.6]. It follows from a general property of fibrations (see [BPPR16, Lemma 14.5]) that there exists a natural transformation of the form

$$\textstyle\coprod_{\kappa_X} \circ \mathsf{Rel}(F) \Rightarrow \mathsf{Rel}(G)\colon \mathsf{Rel}_X \to \mathsf{Rel}_{GX}.$$

Hence, we obtain a natural transformation

$$\begin{aligned}
\mathsf{ctx}_\alpha &= \textstyle\coprod_\alpha \circ \mathsf{Rel}(F) \\
&= \textstyle\coprod_{\beta \circ \kappa_X} \circ \mathsf{Rel}(F) \\
&= \textstyle\coprod_\beta \circ \coprod_{\kappa_X} \circ \mathsf{Rel}(F) \\
&\Rightarrow \textstyle\coprod_\beta \circ \mathsf{Rel}(G) = \mathsf{ctx}_\beta\,.
\end{aligned}$$

This is a natural transformation in Rel_X, which just means that $\mathsf{ctx}_\alpha \le \mathsf{ctx}_\beta$. ☐

Theorem 9.10. *Let $B\colon \mathsf{Set} \to \mathsf{Set}$ be a polynomial functor, and (B_ω, ζ) a final B-coalgebra. Let t_ζ be the companion of b_ζ. For any causal algebra $\alpha\colon FB_\omega \to B_\omega$, we have $\mathsf{ctx}_\alpha \le t_\zeta$.*

Proof. By Lemma 9.1, B satisfies the hypotheses of Theorem 6.3, and hence by Lemma 6.5, ϵ_ω is the algebra induced by the distributive law τ of the companion. This means that $(B_\omega, \epsilon_\omega, \zeta)$ is a τ-bialgebra, and it follows from [BPPR16, Corollary 6.8] that $\mathsf{ctx}_{\epsilon_\omega}$ is b_ζ-compatible. Thus $\mathsf{ctx}_{\epsilon_\omega} \le t_\zeta$. Now, let $\alpha\colon FB_\omega \to B_\omega$ be causal. By Corollary 8.7, there exists a natural transformation $\hat\alpha\colon F \Rightarrow \mathsf{C}_{\bar B}$ such that $\alpha = \epsilon_\omega \circ \hat\alpha_{B_\omega}$. By Lemma 9.9 we obtain $\mathsf{ctx}_\alpha \le \mathsf{ctx}_{\epsilon_\omega}$, hence $\mathsf{ctx}_\alpha \le t_\zeta$. ☐

This implies that one can safely use the contextual closure for *any* causal algebra, such as union, concatenation and Kleene star of languages, or product and sum of streams. In particular, we recover the soundness of the contextual closure in Example 9.8 from the above theorem and the simple observation that language union and composition are causal.

Endrullis et al. [EHB13] prove the soundness of *causal contexts* in combination with other up-to techniques, for equality of streams. The soundness of causal algebras for streams is a special case of Theorem 9.10, but the latter provides more: being below the companion, it is possible to compose it to other such functions to obtain combined up-to techniques in a modular fashion, cf. [Pou16].

10. Abstract GSOS

To obtain expressive specification formats, Turi and Plotkin [TP97] use natural transformations of the form $\lambda\colon F(B \times \mathsf{Id}) \Rightarrow BF^*$, where F^* is the free monad for F. These are the so-called *abstract GSOS specifications*. In this section we show that they are actually equally expressive as plain distributive laws of a functor F over B, if the conditions of Theorem 6.3 apply (assuring that the codensity monad is the companion). This is in a similar spirit as Section 9.2, but we give a proof here that does not require results on causal algebras.

If B has a final coalgebra (Z, ζ), then any abstract GSOS specification $\lambda\colon F(B \times \mathsf{Id}) \Rightarrow BF^*$ defines an algebra $\alpha\colon FZ \to Z$ on it, which is the unique algebra making the following diagram commute.

$$\begin{tikzcd}
FZ \arrow[r, "F\langle\zeta,\mathsf{id}\rangle"] \arrow[d, "\alpha"'] & F(B\times\mathsf{Id})Z \arrow[r, "\lambda_Z"] & BF^*Z \arrow[d, "B\alpha^*"] \\
Z \arrow[rr, "\zeta"'] & & BZ
\end{tikzcd}$$

Here α^* is the Eilenberg-Moore algebra for the free monad corresponding to α. Intuitively, this algebra gives the interpretation of the operations defined by λ.

Like plain distributive laws (Lemma 6.2), abstract GSOS specifications induce natural transformations of the form $F\bar{B} \Rightarrow \bar{B}$.

Lemma 10.1. *For every $\lambda\colon F(B \times \mathsf{Id}) \Rightarrow BF^*$ there is a unique $\alpha\colon F\bar{B} \Rightarrow \bar{B}$ such that for all $i \in \mathsf{Ord}$: $\alpha_{i+1} = B\alpha_i^* \circ \lambda_{B_i} \circ F\langle \mathsf{id}, B_{i+1,i}\rangle$. Moreover, if $B_{k+1,k}$ is an isomorphism for some k, then α_k is the algebra induced by λ on the final coalgebra.*

Proof. The transformation α is determined by the successor case given in the definition. Naturality is proved in a similar way as in Lemma 6.2, with the relevant diagram in the successor case replaced by:

$$\begin{tikzcd}
FBB_j \arrow[r, "F\langle\mathsf{id},B_{j+1,j}\rangle"] \arrow[d, "FBB_{j,i}"'] & F(B\times\mathsf{Id})B_j \arrow[r, "\lambda_{B_j}"] \arrow[d, "F(B\times\mathsf{Id})B_{j,i}"] & BF^*B_j \arrow[r, "B\alpha_j^*"] \arrow[d, "BF^*B_{j,i}"] & BB_j \arrow[d, "BB_{j,i}"] \\
FBB_i \arrow[r, "F\langle\mathsf{id},B_{i+1,i}\rangle"'] & F(B\times\mathsf{Id})B_i \arrow[r, "\lambda_{B_i}"'] & BF^*B_i \arrow[r, "B\alpha_i^*"'] & BB_i
\end{tikzcd}$$

The left square commutes since $B_{i+1,i} \circ BB_{j,i} = B_{i+1,i} \circ B_{j+1,i+1} = B_{j+1,i} = B_{j,i} \circ B_{j+1,j}$ by functoriality and definition of the final sequence. The middle square commutes by naturality. The one on the right commutes, since $B_{j,i}$ is (by assumption in the inductive proof) an algebra morphism, i.e., $B_{j,i} \circ \alpha_j = \alpha_i \circ FB_{j,i}$, and hence $B_{j,i} \circ \alpha_j^* = \alpha_i^* \circ F^*B_{j,i}$ (it holds in general that the $(-)^*$ construction preserves algebra homomorphisms).

Suppose $B_{k+1,k}\colon B_{k+1} \to B_k$ is an isomorphism. Then $B_{k+1,k}^{-1}\colon B_k \to B(B_{k+1})$ is a final B-coalgebra. Consider the following diagram:

$$\begin{tikzcd}
FB_k \arrow[r, "F\langle B_{k+1,k}^{-1}{,}\mathsf{id}\rangle"] \arrow[dr, "FB_{k+1,k}^{-1}"'] \arrow[dd, "\alpha_k"'] & F(B \times \mathsf{Id})B_k \arrow[r, "\lambda_{B_k}"] & BF^*B_k \arrow[dd, "B\alpha_k^*"] \\
& FBB_k \arrow[u, "F\langle \mathsf{id}{,}B_{k+1,k}\rangle"'] \arrow[dr, "\alpha_{k+1}"] & \\
B_k \arrow[rr, "B_{k+1,k}^{-1}"'] & & BB_k
\end{tikzcd}$$

The big triangle commutes by naturality and the fact that $B_{k+1,k}$ is an isomorphism, the small triangle since $B_{k+1,k}$ is an isomorphism, and the remaining inner shape by definition of α. Hence, α_k is the algebra induced on the final coalgebra by λ. □

This places abstract GSOS specifications within the framework of the companion, constructed via the codensity monad of the final sequence $\bar{B}$. Whenever that construction applies (e.g., for polynomial functors), any algebra defined by an abstract GSOS is thus already definable by a plain distributive law over B.

Theorem 10.2. *Let $B\colon \mathcal{C} \to \mathcal{C}$ such that $\mathsf{C}_{\bar{B}}$ exists, B preserves $\mathsf{C}_{\bar{B}}$, and $B_{k+1,k}\colon B_{k+1} \to B_k$ is an iso for some k. Every algebra induced on the final coalgebra by an abstract GSOS specification $\lambda\colon F(B \times \mathsf{Id}) \Rightarrow BF^*$ is definable by a distributive law over B (cf. Section 9.2).*

Proof. By Lemma 10.1, the algebra induced by an abstract GSOS λ is given by α_k for some $\alpha\colon F\bar{B} \Rightarrow \bar{B}$. By the universal property of the codensity monad $(\mathsf{C}_{\bar{B}}, \epsilon)$, there exists a (unique) natural transformation $\hat{\alpha}\colon F \Rightarrow \mathsf{C}_{\bar{B}}$ such that $\alpha = \epsilon \circ \hat{\alpha}\bar{B}$. This means in particular that $\alpha_k = \epsilon_k \circ \hat{\alpha}_{B_k}$. By Lemma 6.5, ϵ_k is the algebra induced by a distributive law (i.e., the companion), so α_k is definable by a distributive law over B. □

In this sense, abstract GSOS is no more expressive than plain distributive laws. Note, however, that this does involve moving to a different (larger) syntax.

Remark 10.3. Every abstract GSOS specification $\lambda\colon F(B \times \mathsf{Id}) \Rightarrow BF^*$ corresponds to a unique distributive law $\lambda^\dagger\colon F^*(B \times \mathsf{Id}) \Rightarrow (B \times \mathsf{Id})F^*$ of the free monad F^* over the (cofree) copointed functor $B \times \mathsf{Id}$, see [LPW04]. The algebra induced by λ decomposes as the algebra induced by $\lambda^\dagger$ and the canonical natural transformation $F \Rightarrow F^*$. This implies that every algebra induced by an abstract GSOS is definable by a distributive law over the copointed functor $B \times \mathsf{Id}$. Theorem 10.2 strengthens this to definability by a distributive law over B.

11. Preserving the right Kan extension

Our main result for constructing the companion of functors (A, B) requires that B preserves the right Kan extension $\mathsf{Ran}_{\bar{A}}(\bar{B})$ (Theorem 6.3). In Section 9 we focused on polynomial functors on Set, which always satisfy this condition. However, polynomial functors exclude a particularly important example: the (finite) powerset functor $\mathcal{P}_f$. This functor and its variants are used to model, for instance, labelled transition systems as coalgebras. In the current section we distill a concrete condition on a functor $B\colon \mathsf{Set} \to \mathsf{Set}$ to preserve the Kan extension as above, taking $A = B$ to make the notation somewhat lighter. In particular, we show a negative result: the finite powerset functor does *not* preserve the relevant right Kan extension, and hence falls outside the scope of Theorem 6.3. Notice that $\mathcal{P}_f$ could still have a companion, and it could even be the codensity monad of the final sequence; however, it does not follow from our results, and remains an open question.

It will be useful to slightly reformulate preservation. Let $F\colon \mathcal{C} \to \mathcal{D}$, $G\colon \mathcal{C} \to \mathcal{E}$ and $K\colon \mathcal{E} \to \mathcal{F}$ be functors. Consider the right Kan extensions $(\mathsf{Ran}_F(G), \epsilon)$ and $(\mathsf{Ran}_F(KG), \epsilon')$. By the univeral property of the latter, there is a unique

$$\widehat{K\epsilon}\colon K\mathsf{Ran}_F(G) \Rightarrow \mathsf{Ran}_F(KG)$$

such that $\epsilon' \circ \widehat{K\epsilon}F = K\epsilon$. Now, K preserves $\mathsf{Ran}_F(G)$ iff this canonical map $\widehat{K\epsilon}$ is an isomorphism.

Lemma 11.1. *In the above, if $\mathcal{D} = \mathcal{E} = \mathcal{F} = \mathsf{Set}$ and the right Kan extensions are presented as in Lemma 5.2, then the canonical map $\widehat{K\epsilon}$ is given by*

$$\begin{array}{rcccl} ((\widehat{K\epsilon})_X(S))_A & : & (FA)^X & \to & KGA \\ & & f & \mapsto & K(\lambda\alpha.\alpha_A(f))(S) \end{array}$$

for all sets X, A and all $S \in K\mathsf{Ran}_F(G)(X)$.

Proof. By Lemma 5.2, we have that $((\widehat{K\epsilon})_X(S))_A\colon (FA)^X \to KGA$ is given by

$$f \mapsto (K\epsilon)_A \circ K\mathsf{Ran}_F(G)(f)(S) = K(\epsilon_A \circ \mathsf{Ran}_F(G)(f))(S)\,.$$

But, for all $\alpha\colon (F-)^X \Rightarrow G$, we have

$$\begin{aligned} \epsilon_A(\mathsf{Ran}_F(G)(f)(\alpha)) &= (\mathsf{Ran}_F(G)(f)(\alpha))_A(\mathsf{id}_{FA}) \\ &= (\lambda g.\alpha_A(g \circ f))(\mathsf{id}_{FA}) \\ &= \alpha_A(\mathsf{id}_{FA} \circ f) \\ &= \alpha_A(f) \end{aligned}$$

where the first two steps follow from Lemma 5.2. □

To simplify the notation, below we denote the finite powerset by P.

Proposition 11.2. *The finite powerset functor $P\colon \mathsf{Set} \to \mathsf{Set}$ does not preserve $\mathsf{Ran}_{\bar{P}}(\bar{P})$.*

Proof. We show that the canonical map $\widehat{P\epsilon}\colon P\mathsf{Ran}_{\bar{P}}(\bar{P}) \Rightarrow \mathsf{Ran}_{\bar{P}}(P\bar{P})$ is not an isomorphism, using the characterisation in Lemma 11.1. The latter gives

$$\begin{array}{rcccl} ((\widehat{P\epsilon})_X(S))_i & : & (P_i)^X & \to & PP_i \\ & & f & \mapsto & \{\alpha_i(f) \mid \alpha \in S\} \end{array} \tag{11.1}$$

for all $i \in \mathsf{Ord}$, since $P(\lambda\alpha.\alpha_i(f))(S) = \{\alpha_i(f) \mid \alpha \in S\}$.

Now, let $X = \{x, y\}$ and define $c^x\colon \bar{P}^X \Rightarrow \bar{P}$ by $c^x_i(f) = f(x)$, and similarly $c^y_i(f) = f(y)$. Further, define $d\colon \bar{P}^X \Rightarrow \bar{P}$ by

$$d_i(f) = \begin{cases} f(x) & \text{if } f(x) = \emptyset \\ f(y) & \text{otherwise} \end{cases}$$

for all $i \in \mathsf{Ord}$. It is easy to check that c^x and c^y are natural, and for d this follows since the direct image of a set is empty iff the set itself is empty.

By the concrete characterisation in (11.1) and the definition of c^x, c^y and d we have $(\widehat{P\epsilon})_X(\{c^x, c^y\}) = (\widehat{P\epsilon})_X(\{c^x, c^y, d\})$. Hence, $\widehat{P\epsilon}$ is not an isomorphism. □

12. Related Work

The central notion of *companion* proposed in the current paper is a categorical generalisation of the lattice-theoretic notion, which appeared first in [HNDV13] and was studied systematically in [Pou16]. The construction of the companion in terms of right Kan extensions generalises the lattice-theoretic results of Parrow and Weber [PW16, Pou16]. To the best of our knowledge, our categorical notion of companion, its structural properties, its construction as a right Kan extension and the implications for causality are orginal contributions.

The current paper fits in the tradition of distributive laws in universal coalgebra, started by Turi and Plotkin [TP97] and subsequently extended in numerous papers. The companion is characterised as the final object in a category of distributive laws, or, more generally, morphisms of endofunctors (e.g., [LPW00]). Morphisms between these distributive laws have been studied in various papers [PW02, Wat02, LPW00, KN15], but a final distributive law (the companion) does not appear there.

The current work is a thoroughly extended version of a conference paper [PR17]. The new material includes proofs of all results, examples, an introductory section on coinduction principles (Section 3), a new result on the powerset functor (Section 11) and a generalisation from the companion of a single functor to the companion of a pair (A, B) of functors, which allows to abstractly capture 'causal operations' between different final coalgebras (Section 8).

The recent [BPR17] takes a different and more abstract approach: there, the companion is constructed based on the theory of locally presentable categories and accessible functors. That paper also studies several other abstract properties of the companion, and features higher-order companions. The abstract constructions of [BPR17] do not mention causality or its implications, which we obtain here through the more explicit final sequence construction. A detailed comparison between the two constructions is left for future work.

The use of distributive laws in enhanced (co)induction and (co)iteration principles has been studied extensively, see, e.g., [Bar04, Bar03, LPW00, UVP01, Jac06, MMS13]. The notion of validity that we introduce in Section 3 comes essentially from the work of Bartels [Bar03] (where it is phrased in terms of the algebra induced by a distributive law). It slightly strengthens soundness in lattices, and also differs from the notion of soundness in [BPPR16]. In contrast to the above-mentioned approaches, our coinduction up-to principle for causal algebras (Section 9.1) does not explicitly refer to distributive laws.

Causality of stream functions is a well-established notion (e.g., [HKR17]). To the best of our knowledge, the generalisation to ω-continuous functors in Section 8, and the construction in terms of the final sequence, are new. The characterisation of causal operations in terms of distributive laws generalises a known construction for streams [HKR17] and automata [RBR16] (modulo a subtlety concerning syntax, see Section 9.2). The result that the contextual closure of any causal algebra lies below the companion (and hence is sound) generalises a soundness result of causal operations for streams [EHB13] (the latter also includes other up-to techniques, which we do not address here).

13. Future work

As explained in Section 11, whether the finite powerset functor has a companion remains open. This is important in practice as this functor is used, e.g., to handle labelled transition systems. This functor does not satisfy the main hypothesis of Theorem 6.3, but it could nevertheless be the case that its codensity monad is its companion. If it exists, one should understand its relationship with the family of *bounded companions* that one can obtain thanks to the accessibility of the finite powerset functor [BPR17]. (Note that the full powerset functor, which is not accessible, cannot have a companion: this would entail existence of a final coalgebra.)

Another research direction consists in studying semantics of open terms through the companion. Indeed, the present work shows that causal operations on the final coalgebra can be presented as (plain) distributive laws, to which the companion gives a canonical semantics by finality. One should thus understand under which conditions the usual notions of bisimulations on open terms coincide with this final semantics, or whether generic notions of bisimulations can be designed to capture it.

We discussed in Section 9.3 how to combine the present results with those from [BPPR16], where we use fibrations to relate coinduction at the level of systems (e.g., defining streams corecursively), and coinduction at the level of predicates and relations (e.g., proving equalities between streams coinductively, using bisimulations). We should still understand precisely the final sequence computation we use in the present work in that fibrational setting (the work of Hasuo et al. on final sequences in a fibration [HCKJ13] may be a good starting point). In particular, we would like to understand the connection between the companion of the lifted functor and the companions of the induced functors on the fibres. This would make it possible to handle other coinductive predicates than plain behavioural equivalence as in Theorem 9.10, and to understand the relationship between the lattice-theoretic and coalgebraic companions.

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