

RULE FORMATS FOR NOMINAL PROCESS CALCULI*

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ABSTRACT. The nominal transition systems (NTSs) of Parrow et al. describe the operational semantics of nominal process calculi. We study NTSs in terms of the nominal residual transition systems (NRTSs) that we introduce. We provide rule formats for the specifications of NRTSs that ensure that the associated NRTS is an NTS and apply them to the operational specifications of the early and late pi-calculus. We also explore alternative specifications of the NTSs in which we allow residuals of abstraction sort, and introduce translations between the systems with and without residuals of abstraction sort. Our study stems from the Nominal SOS of Cimini et al. and from earlier works in nominal sets and nominal logic by Gabbay, Pitts and their collaborators.

Key words and phrases: nominal sets, nominal structural operational semantics, process algebra, nominal transition systems, scope opening, rule formats.

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1. INTRODUCTION

The goal of this paper is to develop the foundations of a framework for studying the meta-theory of structural operational semantics (SOS) [Plo04] for process calculi with names and name-binding operations, such as the π -calculi [MPW92, SW01]. To this end, we build on the large body of work on rule formats for SOS, as surveyed in [AFV01, MRG07], and on the nominal techniques of Gabbay, Pitts and their co-workers [UPG04, CP07, GM09, Pit13].

Rule formats provide syntactic templates guaranteeing that the models of the calculi, whose semantics they specify, enjoy some desirable properties. A first design decision that has to be taken in developing a theory of rule formats for a class of languages is therefore the choice of the semantic objects specified by the rules. The target semantic model we adopt in our study is that of *nominal transition systems* (NTSs), which have been introduced by Parrow *et al.* in [PBE⁺15, PWBE17] as a uniform model to describe the operational semantics of a variety of calculi with names and name-binding operations. Based on this choice, a basic sanity criterion for a collection of rules describing the operational semantics of a nominal calculus is that they specify an NTS, and we present a rule format guaranteeing this property (Thm. 5.13).

As a first stepping stone in our study, we introduce *nominal residual transition systems* (NRTSs), and study NTSs in terms of NRTSs (Section 2). More specifically, the only requirement of an NRTS is that its transition relation is equivariant, which means that it treats names uniformly. This is a desirable property of models of nominal calculi, such as NTSs. Moreover, NTSs are NRTSs that, in addition to having an equivariant transition relation, satisfy a property Parrow *et al.* call *alpha-conversion of residuals* (see Def. 2.6 for the details). The latter property formalises a key aspect of calculi in which names can be scoped to represent local resources. To wit, one crucial feature of the π -calculus is scope opening [MPW92]. Consider a transition $p \xrightarrow{\bar{a}(vb)} p'$ in which a process p exports a private/local channel name b along channel a . Since the name b is local, it ‘can be subject to alpha-conversion’ [PBE⁺15] and the transitions $p \xrightarrow{\bar{a}(vc)} p\{c/b\}$ should also be present for each ‘fresh name’ c .

In contrast to related work [CMRG12, FG07], our approach uses *nominal terms* [Pit13] to connect the specification system with the semantic model. This has the advantage of capturing the requirement that transitions be ‘up to alpha-equivalence’ (typical in nominal calculi) without instrumenting alpha-conversion explicitly in the specification system.

We specify an NRTS by means of a nominal residual transition system specification (NRTSS), which describes the syntax of a nominal calculus in terms of a nominal signature (Section 3) and its semantics by means of a set of inference rules (Section 4). We develop the basic theory of the NRTS/NRTSS framework, building on the nominal algebraic datatypes of Pitts [Pit13] and the nominal rewriting framework of Fernández and Gabbay [FG07]. Based on this framework, we provide rule formats [AFV01, MRG07] for NRTSSs (Section 5) that ensure that the induced transition relation is equivariant (Thm. 5.3) and enjoys alpha-conversion of residuals (Thm. 5.13), and is therefore an NTS. Section 6 presents an example of application of these rule formats to the setting of the π -calculus. Section 7 explores alternative specifications of the NTSs in which we allow a residual to be an atom abstraction (hereafter referred to as *residual with abstraction sort*). We introduce translations between the systems with and without residuals of abstraction sort (Defs. 7.1 and 7.4). We develop a rule format that guarantees that these translations are the inverse of each other (Thms. 7.8 and 7.9). Section 8 presents an example of application of this rule format to the early

π -calculus and to a slightly modified version of the late π -calculus. We also show that both the specification with and without residuals of abstraction sort induce the same model of computation. Finally, Section 9 discusses related work, as well as avenues for future work, and concludes.

The appendix accompanying the paper collects some proofs that are omitted in the main text.

This paper is an extended version of a paper with the same title presented at CONCUR 2017 [AFGP⁺17]. The novel content in this extended version is summarised below:

- In Section 2 we recall the notion of finite renamings, which play a prominent role throughout this paper since they replace the permutations in the moderated terms of the CONCUR 2017 paper.
- In Section 6.1 we introduce an NRTSS that faithfully captures the original semantics of the early π -calculus [San96]. The NRTSS of the CONCUR 2017 paper induced a semantics that failed to capture some transitions in the original early π -calculus.
- In Section 6.2 we introduce an NRTSS whose induced semantics differs minimally from the original semantics of the late π -calculus [San96] (see Remark 6.1 for further discussion). We also apply the rule format for alpha-conversion of residuals to this version of the late π -calculus. This section is entirely novel.
- Section 7, where we study alternative formulations of the NTSs in which we allow residuals of abstraction sorts, is entirely novel.
- In Section 8 we apply the rule formats from Section 7 to the early π -calculus and to our version of the late π -calculus. This section is entirely novel too.
- We have included the detailed proofs of all lemmas and theorems in the paper, some of which were missing in the conference version.

2. PRELIMINARIES

This section collects some earlier foundational work by Gabbay and Pitts on nominal sets and finitary renamings [GH08, GP02, Pit13, Pit16] on which our work builds, and recalls the nominal transition systems of Parrow *et al.* [PBE⁺15].

Nominal Sets. We assume a countably infinite set \mathbb{A} of *atoms* and consider $\text{Perm } \mathbb{A}$ as the group of *finite permutations of atoms* (hereafter *permutations*) ranged over by π , where we write ι for the *identity*, \circ for *composition* and π^{-1} for the *inverse* of permutation π . We are particularly interested in *transpositions* of two atoms: (ab) stands for the permutation that swaps a with b and leaves all other atoms fixed. Every permutation π is equal to the composition of a finite number of transpositions, i.e. $\pi = (a_1 b_1) \circ \dots \circ (a_n b_n)$ with $n \geq 0$.

An *action* of the group $\text{Perm } \mathbb{A}$ on a set S is a binary operation mapping each $\pi \in \text{Perm } \mathbb{A}$ and $s \in S$ to an element $\pi \cdot s \in S$, and satisfying the identity law $\iota \cdot s = s$ and the composition law $(\pi_1 \circ \pi_2) \cdot s = \pi_1 \cdot (\pi_2 \cdot s)$. A $\text{Perm } \mathbb{A}$ -*set* is a set equipped with an action of $\text{Perm } \mathbb{A}$.

We say that a set of atoms A *supports* an object s iff $\pi \cdot s = s$ for every permutation π that leaves each element $a \in A$ invariant. In particular, we are interested in sets all of whose elements have finite support (Def. 2.2 of [Pit13]).

Definition 2.1 (Nominal sets). A *nominal set* is a $\text{Perm } \mathbb{A}$ -set all of whose elements are finitely supported.

For each element s of a nominal set, we write $\text{supp}(s)$ for the least set that supports s , called the *support* of s . (Intuitively, the action of permutations on a set S determines that a finitely supported $s \in S$ only depends on atoms in $\text{supp}(s)$, and no others.) The set \mathbb{A} of atoms is a nominal set when $\pi \cdot a = \pi a$ since $\text{supp}(a) = \{a\}$ for each atom $a \in \mathbb{A}$. The set $\text{Perm } \mathbb{A}$ of finite permutations is also a nominal set where the permutation action on permutations is given by conjugation, i.e. $\pi \cdot \pi' = \pi \circ \pi' \circ \pi^{-1}$, and the support of a permutation π is $\text{supp}(\pi) = \{a \mid \pi a \neq a\}$.

Given two $\text{Perm } \mathbb{A}$ -sets S and T and a function $f : S \rightarrow T$, the action of permutation π on function f is given by conjugation, i.e. $(\pi \cdot f)(s) = \pi \cdot f(\pi^{-1} \cdot s)$ for each $s \in S$. We say that a function $f : S \rightarrow T$ is *equivariant* iff $\pi \cdot f(s) = f(\pi \cdot s)$ for every $\pi \in \text{Perm } \mathbb{A}$ and every $s \in S$. The intuition is that an equivariant function f is atom-blind, in that f does not treat any atom preferentially. It turns out that a function f is equivariant iff $\text{supp}(f) = \emptyset$ (Rem. 2.13 of [Pit13]). The function supp is equivariant (Prop. 2.11 of [Pit13]).

Let S be a $\text{Perm } \mathbb{A}$ -set, we write S_{fs} for the nominal set that contains the elements in S that are finitely supported. Let S_1 and S_2 be nominal sets. The product $S_1 \times S_2$ is a nominal set (Prop. 2.14 of [Pit13]). The permutation action for products is given componentwise (Eq (1.12) of [Pit13]).

Conjugation yields that, for every $\text{Perm } \mathbb{A}$ -set S , the action of π on $s \in S$ is equivariant. Indeed,

$$\pi \cdot (\pi_1 \cdot s) = (\pi \circ \pi_1) \cdot s = (\pi \circ \pi_1 \circ \pi^{-1} \circ \pi) \cdot s = ((\pi \cdot \pi_1) \circ \pi) \cdot s = (\pi \cdot \pi_1) \cdot (\pi \cdot s).$$

It is also straightforward to show that composition of permutations is equivariant. In fact,

$$\pi \cdot (\pi_1 \circ \pi_2) = \pi \circ (\pi_1 \circ \pi_2) \circ \pi^{-1} = (\pi \circ \pi_1 \circ \pi^{-1}) \circ (\pi \circ \pi_2 \circ \pi^{-1}) = (\pi \cdot \pi_1) \circ (\pi \cdot \pi_2).$$

An element $s_1 \in S_1$ is *fresh* in $s_2 \in S_2$, written $s_1 \# s_2$, iff $\text{supp}(s_1) \cap \text{supp}(s_2) = \emptyset$. The freshness relation is equivariant (Eq. (3.2) of [Pit13]).

We consider *atom abstractions* (Sec. 4 of [Pit13]), which represent alpha-equivalence classes of elements.

Definition 2.2 (Atom abstraction). Given a nominal set S , the *atom abstraction* of atom a in element $s \in S$, written $\langle a \rangle s$, is the $\text{Perm } \mathbb{A}$ -set $\langle a \rangle s = \{(b, (ba) \cdot s) \mid b = a \vee b \# s\}$, whose permutation action is $\pi \cdot \langle a \rangle s = \{(\pi \cdot b, \pi \cdot ((ba) \cdot s)) \mid \pi \cdot b = \pi \cdot a \vee \pi \cdot b \# \pi \cdot s\}$.

We write $[\mathbb{A}]S$ for the set of *atom abstractions* in elements of S , which is a nominal set (Def. 4.4 of [Pit13]), since $\text{supp}(\langle a \rangle s) = \text{supp}(s) \setminus \{a\}$ for each atom a and element $s \in S$.

Remark 2.3. Notice that, by Lemma 4.3 in [Pit11], $s = s'$ whenever $\langle a \rangle s = \langle a \rangle s'$. ■

Nominal sets are the objects of a category **Nom** whose morphisms are the equivariant functions. The category **Nom** is closed under finite products and both finite and infinite coproducts.¹ We write $s = \text{inj}_i s'$ with $i \in I$ and $s' \in S_i$ for an element s in a coproduct $\sum_{i \in I} (S_i)$. (For a finite coproduct $S_1 + \dots + S_n$ we let $I = \{1, \dots, n\}$.) For other set-theoretical operations (i.e. infinite products, functions, partial functions, power sets) the following caveat applies. The category of nominal sets is closed under the variant of each operation that restricts any universal quantification that is involved in the operation to quantify only over finitely supported elements (see Sections 2.2 to 2.5 of [Pit13]).

The *nominal function set* between nominal sets S and T (Definition 2.18 of [Pit13]) is the nominal set $(T^S)_{\text{fs}}$ of finitely supported functions from S to T —be they equivariant or not; recall that an equivariant function has empty support. (We may write $S \rightarrow_{\text{fs}} T$ in lieu of

¹In **Nom**, coproducts correspond to disjoint unions.

$(T^S)_{\text{fs}}$.) The application and currying functions can be respectively restricted to equivariant functions $\text{app} : (X \rightarrow_{\text{fs}} Y) \times X \rightarrow Y$ and $\text{curry} : (Z \times X \rightarrow_{\text{fs}} Y) \rightarrow Z \rightarrow (X \rightarrow_{\text{fs}} Y)$ such that the nominal function set coincides with the *exponential object* in \mathbf{Nom} , i.e. there is a bijection between hom-sets $\mathbf{Nom}(Z \times X, Y)$ and $\mathbf{Nom}(Z, X \rightarrow_{\text{fs}} Y)$ given by sending $f \in \mathbf{Nom}(Z \times X, Y)$ to $\text{curry}(f) \in \mathbf{Nom}(Z, X \rightarrow_{\text{fs}} Y)$. (Section 2.4 in [Pit13] spells out all the details on this isomorphism.)

Finally, the category \mathbf{Nom} is Cartesian closed (Theorem 2.19 in [Pit13]), i.e., \mathbf{Nom} admits all the finite products (including the empty product $\mathbf{1}$ which is the terminal object) and all the exponentials.

Renamings. We consider the *finitely supported renamings* (hereafter *renamings*) ranged over by ρ , which are finitely supported functions $\rho : \mathbb{A} \rightarrow_{\text{fs}} \mathbb{A}$, that is, functions that act like the identity on all but finitely many atoms. We write ι for the *identity function* and ‘;’ for *diagrammatical composition*, that is, $f;g$ denotes the function $g \circ f$. We are particularly interested in *replacements* of an atom by another: b/a stands for the replacement that substitutes a with b and leaves all other atoms fixed. Every renaming ρ is equal to the composition of a finite number of replacements [GH08], i.e. $\rho = b_1/a_1; \dots; b_n/a_n$ with $n \geq 0$. Notice that $\mathbb{A} \rightarrow_{\text{fs}} \mathbb{A}$ with ‘;’ as composition operator and ι as identity element is a monoid [GH08].

An *action* of the monoid $\mathbb{A} \rightarrow_{\text{fs}} \mathbb{A}$ on a set S is a binary operation mapping each $\rho \in \mathbb{A} \rightarrow_{\text{fs}} \mathbb{A}$ and $s \in S$ to an element $s\{\rho\} \in S$, and satisfying the identity law $s\{\iota\} = s$ and the composition law $(s\{\rho_1\})\{\rho_2\} = s\{\rho_1; \rho_2\}$. We will provide an action of renaming for the raw terms to be defined in Section 3. An action of renaming could be defined for every object in \mathbf{Nom} , which ultimately gives rise to the category \mathbf{Ren} of renamings as described in [GH08], which is a generalisation of \mathbf{Nom} . We are interested in interpreting our terms as the nominal algebraic datatypes of [Pit13], which live in \mathbf{Nom} , and therefore we refrain ourselves from interpreting our terms in the category \mathbf{Ren} , and we treat renamings as the exponential objects $\mathbb{A} \rightarrow_{\text{fs}} \mathbb{A}$ in the former category.

Notice that every permutation is an instance of a renaming. For every permutation π , we may write $s\{\pi\} = \pi \cdot s$ for the action of renaming π on s , and for every renaming ρ , we may write $\pi; \rho$ for the diagrammatical composition of ρ after π . As we have mentioned above, the renamings are the exponential object $\mathbb{A} \rightarrow_{\text{fs}} \mathbb{A}$ in the category \mathbf{Nom} , and therefore they are equipped with a permutation action given by $\pi \cdot \rho = \pi^{-1}; \rho; \pi$. As for any other element of an object in \mathbf{Nom} , the support of a renaming ρ is the least set A such that $\pi \cdot \rho = \rho$ for every permutation π that leaves each element of A invariant.

Example 2.4. Consider the replacement b/a . Its support is $\text{supp}(b/a) = \{a, b\}$, as we show next. Let π be a permutation such that $\pi \cdot a = a$ and $\pi \cdot b = b$. We show that $\pi^{-1}; b/a; \pi = b/a$. For atom a ,

$$a\{\pi^{-1}; b/a; \pi\} = (\pi^{-1} \cdot a)\{b/a; \pi\} = a\{b/a; \pi\} = b\{\pi\} = \pi \cdot b = b = a\{b/a\}.$$

For any other atom $c \neq a$,

$$c\{\pi^{-1}; b/a; \pi\} = (\pi^{-1} \cdot c)\{b/a; \pi\} = (\pi^{-1} \cdot c)\{\pi\} = \pi \cdot (\pi^{-1} \cdot c) = c = c\{b/a\},$$

since $\pi^{-1} \cdot c \neq a$ by the assumptions on π . Therefore $\{a, b\}$ supports b/a , while it is not hard to see that no subset of $\{a, b\}$ does so. \blacksquare

Since every renaming ρ is finitary, its support can be defined alternatively as in the proposition below.

Proposition 2.5. *Let ρ be a renaming. The support $\text{supp}(\rho) = \{a, (\rho a) \mid \rho a \neq a\}$.*

The proof of Proposition 2.5 is in Appendix A.

Nominal Transition Systems. Nominal transition systems adopt the state/residual presentation for transitions of [BP09], where a residual is a pair consisting of an action and a state. In [PBE⁺15], Parrow *et al.* develop modal logics à la Hennessy-Milner for process nominal calculi. Here we are mainly interested in the transition relation and we adapt Definition 1 in [PBE⁺15] by removing the predicates. We write $\mathcal{P}_\omega(\mathbb{A})$ for the *finite power set* of \mathbb{A} .

Definition 2.6 (Nominal transition system). A *nominal transition system* (NTS) is a quadruple $(S, \text{Act}, \text{bn}, \longrightarrow)$ where S and Act are nominal sets of *states* and *actions* respectively, $\text{bn} : \text{Act} \rightarrow \mathcal{P}_\omega(\mathbb{A})$ is an equivariant function that delivers the *binding names* in an action, and $\longrightarrow \subseteq S \times (\text{Act} \times S)$ is an equivariant binary transition relation from states to *residuals* (we let $\text{Act} \times S$ be the set of residuals). The function bn is such that $\text{bn}(\ell) \subseteq \text{supp}(\ell)$ for each $\ell \in \text{Act}$. We often write $p \longrightarrow (\ell, p')$ in lieu of $(p, (\ell, p')) \in \longrightarrow$.

Finally, the transition relation \longrightarrow must satisfy *alpha-conversion of residuals*, that is, if $a \in \text{bn}(\ell)$, $b \# (\ell, p')$ and $p \longrightarrow (\ell, p')$ then also $p \longrightarrow ((ab) \cdot \ell, (ab) \cdot p')$, or equivalently $p \longrightarrow (ab) \cdot (\ell, p')$.

We will consider an NTS (without its associated binding-names function bn) as a particular case of a nominal residual transition system, which we introduce next.

Definition 2.7 (Nominal residual transition system). A *nominal residual transition system* (NRTS) is a triple (S, R, \longrightarrow) where S and R are nominal sets, and where $\longrightarrow \subseteq S \times R$ is an equivariant binary transition relation. We say S is the set of *states* and R is the set of *residuals*.

The connection between NTSs and NRTSs will be studied in more detail in Section 5.

3. NOMINAL TERMS

This section is devoted to the notion of nominal terms, which are syntactic objects that make use of the atom abstractions of Definition 2.2 and represent terms up to alpha-equivalence. As a first step, we introduce raw terms, devoid of any notion of alpha-equivalence. Our raw terms resemble those from the literature, mainly from [UPG04, FG07, CP07, Pit13], but with some important differences. In particular, our terms include both variables (i.e. unknowns) and moderated terms (i.e. explicit renamings over raw terms), and we consider atom and abstraction sorts. (The raw terms of [Pit13] do not include moderated terms, and the ones in [UPG04, FG07] only consider moderated variables where the delayed renaming is a permutation. In [CP07] the authors consider neither atom nor abstraction sorts.) We also adopt the classic presentation of free algebras and term algebras in [GTWW77, BS00] in a different way from that in [CP07, Pit13]. The raw terms correspond to the standard notion of free algebra over a signature generated by a set of variables. We then adapt the Σ -structures of [CP07] to our sorting schema. Finally, the nominal terms are the interpretations of the ground terms in the initial Σ -structure; we show that they coincide with the nominal algebraic terms of [Pit13].

Definition 3.1 (Nominal signature and nominal sort). A *nominal signature* (or simply a *signature*) Σ is a triple (Δ, A, F) where $\Delta = \{\delta_1, \dots, \delta_n\}$ is a finite set of *base sorts*, A is a countable set of *atom sorts*, and F is a finite set of *function symbols*. The *nominal sorts* over Δ and A are given by the grammar

$$\sigma ::= \delta \mid \alpha \mid [\alpha]\sigma \mid \sigma_1 \times \dots \times \sigma_k,$$

with $k \geq 0$, $\delta \in \Delta$ and $\alpha \in A$. The sort $[\alpha]\sigma$ is the *abstraction sort*. Symbol \times denotes the *product sort*, which is associative; $\sigma_1 \times \dots \times \sigma_k$ stands for the sort of the empty product when $k = 0$, which we may write as $\mathbf{1}$. We write \mathbf{S} for the set of nominal sorts. We arrange the function symbols in F based on the sort of the data (base sort) that they produce. We write $f_{ij} \in F$ with $1 \leq i \leq n$ and $1 \leq j \leq m_i$ such that f_{ij} has arity $\sigma_{ij} \rightarrow \delta_i$, where δ_i is a base sort.

The theory of nominal sets extends to the case of many-sorted atoms (see Sec. 4.7 in [Pit13]). We assume that \mathbb{A} contains a countably infinite collection of atoms $a_\alpha, b_\alpha, c_\alpha, \dots$ for each atom sort α such that the sets of atoms \mathbb{A}_α of each sort are mutually disjoint. We write $\text{Perm}_s \mathbb{A} = \{\pi \in \text{Perm } \mathbb{A} \mid \forall \alpha \in A. \forall a \in \mathbb{A}_\alpha. \pi a \in \mathbb{A}_\alpha\}$ for the subgroup of finite permutations that respect the sorting. The sorted nominal sets are the $\text{Perm}_s \mathbb{A}$ -sets whose elements are finitely supported. We also consider renamings that respect the sorting, which we write $(\mathbb{A} \rightarrow_{\text{fs}} \mathbb{A})_s = \{\rho \in \mathbb{A} \rightarrow_{\text{fs}} \mathbb{A} \mid \forall \alpha \in A. \forall a \in \mathbb{A}_\alpha. \rho a \in \mathbb{A}_\alpha\}$. (Notice that every permutation in $\text{Perm}_s \mathbb{A}$ is a renaming that respects the sorting.) In the sequel we may drop the s subscript in $\text{Perm}_s \mathbb{A}$ and in $(\mathbb{A} \rightarrow_{\text{fs}} \mathbb{A})_s$, and omit the ‘sorted’ epithet from ‘sorted nominal sets’.

We let \mathcal{V} be a set that contains a countably infinite collection of *variable names* (variables for short) $x_\sigma, y_\sigma, z_\sigma, \dots$ for each sort σ , such that the sets of variables \mathcal{V}_σ of each sort are mutually disjoint. We also assume that \mathcal{V} is disjoint from \mathbb{A} .

Definition 3.2 (Raw terms). Let $\Sigma = (\Delta, A, F)$ be a signature. The set of *raw terms over signature Σ and set of variables \mathcal{V}* (raw terms for short) is given by the grammar

$$t_\sigma ::= x_\sigma \mid a_\alpha \mid (t_\sigma \{\rho\})_\sigma \mid ([a_\alpha]t_\sigma)_{[\alpha]\sigma} \mid (t_{\sigma_1}, \dots, t_{\sigma_k})_{\sigma_1 \times \dots \times \sigma_k} \mid (f_{ij}(t_{\sigma_{ij}}))_{\delta_i},$$

where term x_σ is a *variable* of sort σ , term a_α is an *atom* of sort α , term $(t_\sigma \{\rho\})_\sigma$ is a *moderated term* (i.e. the explicit, or delayed, renaming ρ over term t_σ), term $([a_\alpha]t_\sigma)_{[\alpha]\sigma}$ is the *abstraction of atom a_α in term t_σ* , term $(t_{\sigma_1}, \dots, t_{\sigma_k})_{\sigma_1 \times \dots \times \sigma_k}$ is the *product of terms* $t_{\sigma_1}, \dots, t_{\sigma_k}$, and term $(f_{ij}(t_{\sigma_{ij}}))_{\delta_i}$ is the *datum of base sort δ_i constructed from term $t_{\sigma_{ij}}$ and function symbol $f_{ij} : \sigma_{ij} \rightarrow \delta_i$* . When they are clear from the context or immaterial, we leave the arities and sorts implicit and write $x, a, t\{\rho\}, [a]t, (t_1, \dots, t_k), f(t)$, etc.

Given a raw term t , the *size of t* is the number of nodes of t ’s abstract syntax tree.

The raw terms are the inhabitants of the carrier of the free algebra over the set of variables \mathcal{V} and over the \mathbf{S} -sorted conventional signature that consists of the function symbols in F , together with a constant symbol for each atom a_α , a unary symbol that produces moderated terms for each renaming ρ and each sort σ , a unary symbol that produces abstractions for each atom a_α and sort σ , and a k -ary symbol that produces a product of sort $\sigma_1 \times \dots \times \sigma_k$ for each sequence of sorts $\sigma_1, \dots, \sigma_k$. (See [GTWW77] for a classic presentation of term algebras, initial algebra semantics and free algebras.)

We write $\mathbb{T}(\Sigma, \mathcal{V})_\sigma$ for the set of raw terms of sort σ . A raw term t is *ground* iff no variables occur in t . We write $\mathbb{T}(\Sigma)_\sigma$ for the set of ground terms of sort σ . The sets of raw terms (resp. ground terms) of each sort are mutually disjoint as terms carry sort information.

Therefore we sometimes identify the family $(\mathbb{T}(\Sigma, \mathcal{V})_\sigma)_{\sigma \in \mathcal{S}}$ of \mathcal{S} -indexed raw terms and the family $(\mathbb{T}(\Sigma)_\sigma)_{\sigma \in \mathcal{S}}$ of \mathcal{S} -indexed ground terms with their respective ranges $\bigcup_{\sigma \in \mathcal{S}} \mathbb{T}(\Sigma, \mathcal{V})_\sigma$ and $\bigcup_{\sigma \in \mathcal{S}} \mathbb{T}(\Sigma)_\sigma$, which we abbreviate as $\mathbb{T}(\Sigma, \mathcal{V})$ and $\mathbb{T}(\Sigma)$ respectively.

The set $\mathbb{T}(\Sigma, \mathcal{V})$ of raw terms is a nominal set, with the Perm \mathbb{A} -action and the support of a raw term given by:

$$\begin{array}{ll} \pi \cdot x &= x & \text{supp}(x) &= \emptyset \\ \pi \cdot a &= \pi a & \text{supp}(a) &= \{a\} \\ \pi \cdot (t\{\rho\}) &= (\pi \cdot t)\{\pi \cdot \rho\} & \text{supp}(t\{\rho\}) &= \text{supp}(t) \cup \text{supp}(\rho) \\ \pi \cdot [a]t &= [\pi a](\pi \cdot t) & \text{supp}([a](t)) &= \{a\} \cup \text{supp}(t) \\ \pi \cdot (t_1, \dots, t_k) &= (\pi \cdot t_1, \dots, \pi \cdot t_k) & \text{supp}((t_1, \dots, t_k)) &= \text{supp}(t_1) \cup \dots \cup \text{supp}(t_k) \\ \pi \cdot (f(t)) &= f(\pi \cdot t), & \text{supp}(f(t)) &= \text{supp}(t). \end{array}$$

It is straightforward to check that the permutation action for raw terms is sort-preserving (remember that permutations are also sort-preserving). The set $\mathbb{T}(\Sigma)$ of ground terms is also a nominal set since it is closed with respect to the Perm \mathbb{A} -action given above.

Below on the left we introduce the action of renaming for a raw term t , which replaces each occurrence of a free atom a in t by $a\{\rho\}$. On the right we present the function $\text{fa} : \mathbb{T}(\Sigma, \mathcal{V}) \rightarrow \mathcal{P}_\omega(\mathbb{A})$, which delivers the set of free atoms in a raw term:

$$\begin{array}{ll} x\{\rho\} &= x & \text{fa}(x) &= \emptyset \\ a\{\rho\} &= \rho a & \text{fa}(a) &= \{a\} \\ (t\{\rho_1\})\{\rho_2\} &= t\{\rho_1; \rho_2\} & \text{fa}(t\{\rho\}) &= \text{fa}(t\{\rho\}) \\ ([a]t)\{\rho\} &= [\rho a](t\{\rho\}) & \text{fa}([a]t) &= \text{fa}(t) \setminus \{a\} \\ (t_1, \dots, t_k)\{\rho\} &= (t_1\{\rho\}, \dots, t_k\{\rho\}) & \text{fa}(t_1, \dots, t_k) &= \text{fa}(t_1) \cup \dots \cup \text{fa}(t_k) \\ (f(t))\{\rho\} &= f(t\{\rho\}), & \text{fa}(f(t)) &= \text{fa}(t). \end{array}$$

Notice that the set of free atoms in a raw term differs from the support of the term. For instance, $\text{fa}([a](a, b)) = \{b\}$, but $\text{supp}([a](a, b)) = \{a, b\}$.

Remark 3.3. Let t be a raw term and ρ a renaming. Then the size of $t\{\rho\}$ equals the size of t , which can be checked in a straightforward way by the definition above. \blacksquare

Observe that the action of renaming is equivariant.

Lemma 3.4. *Let t be a term, ρ be a renaming and π be a permutation. Then, $\pi \cdot (t\{\rho\}) = (\pi \cdot t)\{\pi \cdot \rho\}$.*

As expected, the free atoms of a raw term are contained in its support.

Lemma 3.5. *Let t be a raw term. Then $\text{fa}(t) \subseteq \text{supp}(t)$.*

The proof of Lemmas 3.4 and 3.5 are in Appendix B.

Example 3.6 (π -calculus). Consider a signature Σ for the π -calculus [SW01, CMRG12] given by a single atom sort ch of channel names, and base sorts pr and ac for processes and actions respectively. The function symbols (adapted from [SW01]) are the following:

$$F = \left\{ \begin{array}{lll} \text{null} : \mathbf{1} \rightarrow \text{pr}, & \text{par} : (\text{pr} \times \text{pr}) \rightarrow \text{pr}, & \text{tau}A : \mathbf{1} \rightarrow \text{ac}, \\ \text{tau} : \text{pr} \rightarrow \text{pr}, & \text{sum} : (\text{pr} \times \text{pr}) \rightarrow \text{pr}, & \text{in}A : (\text{ch} \times \text{ch}) \rightarrow \text{ac}, \\ \text{in} : (\text{ch} \times [\text{ch}]\text{pr}) \rightarrow \text{pr}, & \text{rep} : \text{pr} \rightarrow \text{pr}, & \text{out}A : (\text{ch} \times \text{ch}) \rightarrow \text{ac}, \\ \text{out} : (\text{ch} \times \text{ch} \times \text{pr}) \rightarrow \text{pr}, & \text{new} : [\text{ch}]\text{pr} \rightarrow \text{pr}, & \text{bout}A : (\text{ch} \times \text{ch}) \rightarrow \text{ac} \end{array} \right\}.$$

Recalling terminology from [SW01, CMRG12], *null* stands for inaction, *tau*(p) for the internal action after which process p follows, *in*($a, [b]p$) for the input at channel a where the input name is bound to b in the process p that follows, *out*(a, b, p) for the output of name b through channel a after which process p follows, *par*(p, q) for parallel composition, *sum*(p, q) for nondeterministic choice, *rep*(p) for parallel replication, and *new*($[a]p$) for the restriction of channel a in process p (a is private in p). Actions and processes belong to different sorts. We use *tauA*, *outA*(a, b), *inA*(a, b) and *boutA*(a, b) respectively for the internal action, the output action, the input action and the bound output action.

The set of terms of the π -calculus corresponds to the subset of ground terms over Σ of sort **pr** and **ac** in which no moderated (sub-)terms occur. For instance, the process $(\nu b)(\bar{a}b.0)$ corresponds to the ground term $\text{new}([b](\text{out}(a, b, \text{null})))$, whose support is $\{a, b\}$. Both free and bound channel names (such as the a and b respectively in the example process) are represented by atoms. The set of ground terms also contains generalised processes and actions with moderated (sub-)terms $p\{\rho\}$, which stand for a delayed renaming ρ that ought to be applied to a term p , e.g. $\text{new}([b](\text{out}(a, b, \text{null})))\{\rho\}$. ■

Raw terms allow variables to occur in the place of any ground subterm. The variables represent *unknowns*, and should be mistaken with neither free nor bound channel names. For instance, the raw term $\text{new}([b](\text{out}(a, b, x)))$ represents a π -calculus process $(\nu b)(\bar{a}b.P)$ where the x is akin to the meta-variable P , which stands for some unknown process. The process $(\nu b)(\bar{a}b.P)$ unifies with $(\nu b)(\bar{a}b.0)$ by replacing P with 0 . In the nominal setting, the raw term $\text{new}([b](\text{out}(a, b, x)))$ unifies with ground term $\text{new}([b](\text{out}(a, b, \text{null})))$, by means of a *substitution* φ such that $\varphi(x) = \text{null}$. Formally, substitutions are defined below.

Definition 3.7 (Substitution). A *substitution* $\varphi : \mathcal{V} \rightarrow_{\text{fs}} \mathbb{T}(\Sigma, \mathcal{V})$ is a sort-preserving, finitely supported function from variables to raw terms. The *domain* $\text{dom}(\varphi)$ of a substitution φ is the set $\{x \mid \varphi(x) \neq x\}$. A substitution φ is *ground* iff $\varphi(x) \in \mathbb{T}(\Sigma)$ for every variable $x \in \text{dom}(\varphi)$.

The set of substitutions is a nominal set. The *extension to raw terms* $\bar{\varphi}$ of substitution φ is the unique homomorphism induced by φ from the free algebra $\mathbb{T}(\Sigma, \mathcal{V})$ to itself, which coincides with the function given by:

$$\begin{aligned} \bar{\varphi}(x) &= \varphi(x) \\ \bar{\varphi}(a) &= a \\ \bar{\varphi}(t\{\rho\}) &= \bar{\varphi}(t)\{\rho\} \\ \bar{\varphi}([a]t) &= [a](\bar{\varphi}(t)) \\ \bar{\varphi}(t_1, \dots, t_k) &= (\bar{\varphi}(t_1), \dots, \bar{\varphi}(t_k)) \\ \bar{\varphi}(f(t)) &= f(\bar{\varphi}(t)). \end{aligned}$$

Given substitutions φ and γ we write $\varphi \circ \gamma$ for their composition, which is defined as follows: For every variable x , $(\varphi \circ \gamma)(x) = \bar{\varphi}(t)$ where $\gamma(x) = t$. It is straightforward to check that $(\bar{\varphi} \circ \bar{\gamma})(t) = \bar{\varphi}(\bar{\gamma}(t))$. We note that our definition of substitution is different from those in both [UPG04, CP07], where the authors consider delayed permutations instead of delayed renamings, and where their substitution function performs the delayed permutations of the moderated terms *on-the-fly*.

Lemma 3.8 (Extension to raw terms is equivariant). *Let φ be a substitution and π a permutation. Then, $\pi \cdot \bar{\varphi} = \bar{\pi} \cdot \bar{\varphi}$.*

Proof. We prove $(\pi \cdot \bar{\varphi})(t) = \overline{\pi \cdot \varphi}(t)$ by induction on the structure of raw term t . By conjugation,

$$\begin{aligned} (\pi \cdot \bar{\varphi})(x) &= \pi \cdot \bar{\varphi}(\pi^{-1} \cdot x) = \pi \cdot \bar{\varphi}(x) = \pi \cdot \varphi(x) \\ &= \pi \cdot \varphi(\pi^{-1} \cdot x) = (\pi \cdot \varphi)(x) = \overline{\pi \cdot \varphi}(x) \end{aligned}$$

and the lemma holds for the base case $t = x$. Similarly,

$$(\pi \cdot \bar{\varphi})(a) = \pi \cdot \bar{\varphi}(\pi^{-1} \cdot a) = \pi \cdot (\pi^{-1} \cdot a) = a = \overline{\pi \cdot \varphi}(a)$$

and the lemma holds for the base case $t = a$. The rest of the cases are straightforward by induction. \square

It is easy to check that the support of $\bar{\varphi}$ coincides with the support of φ . By the above lemma, the set of extended substitutions is also a nominal set, since it is closed with respect to the Perm \mathbb{A} -action. Hereafter we sometimes write $\varphi(t)$, where t is a raw term, instead of $\bar{\varphi}(t)$. We may also write φ^π instead of $\pi \cdot \bar{\varphi}$ or $\overline{\pi \cdot \varphi}$ for short.

The following result highlights the relation between substitution and the permutation action.

Lemma 3.9 (Substitution and permutation action). *Let φ be a substitution, π a permutation and t a raw term. Then, $\pi \cdot \varphi(t) = \varphi^\pi(\pi \cdot t)$.*

Proof. By definition of φ^π , we have that $\varphi^\pi(\pi \cdot t) = \pi \cdot \varphi(\pi^{-1} \cdot (\pi \cdot t)) = \pi \cdot \varphi(t)$ and we are done. \square

Our goal is to give meaning to ground terms in nominal sets. To this end, we need a suitable class of algebraic structures that can be used to give an *interpretation* of those ground terms.

Definition 3.10 (Σ -structure). Let $\Sigma = (\Delta, A, F)$ be a signature. A Σ -structure M consists of a nominal set $M[[\sigma]]$ for each sort σ defined as follows

$$\begin{aligned} M[[\alpha]] &= \mathbb{A}_\alpha \\ M[[[\alpha]\sigma]] &= [\mathbb{A}_\alpha](M[[\sigma]]) \\ M[[\sigma_1 \times \cdots \times \sigma_k]] &= M[[\sigma_1]] \times \cdots \times M[[\sigma_k]], \end{aligned}$$

where the $M[[\delta_i]]$ with $\delta_i \in \Delta$ are given, as well as an equivariant function $M[[f_{ij}]] : M[[\sigma_{ij}]] \rightarrow M[[\delta_i]]$ for each symbol $(f_{ij})_{\sigma_{ij} \rightarrow \delta_i} \in F$.

The notion of Σ -structure adapts that of Σ -structure in [CP07] to our sorting convention with atom and abstraction sorts. The Σ -structures characterise a range of interpretations of ground terms into elements of nominal sets, such that any sort σ gives rise to the expected nominal set, i.e. atom sorts give rise to sets of atoms, abstraction sorts give rise to sets of atom abstractions, and product sorts give rise to finite products of nominal sets.

Next we define the *interpretation of a ground term in a Σ -structure*, which resembles the *value of a term* in [CP07].

Definition 3.11 (Interpretation of ground terms in a Σ -structure). Let Σ be a signature and M be a Σ -structure. The *interpretation* $M[[p]]$ of a ground term p in M is given by:

$$\begin{aligned} M[[a]] &= a \\ M[[p\{\rho\}]] &= M[[p\{\rho\}]] \\ M[[[a]p]] &= \langle a \rangle(M[[p]]) \\ M[[p_1, \dots, p_k]] &= (M[[p_1]], \dots, M[[p_k]]) \\ M[[f(p)]] &= M[[f]](M[[p]]). \end{aligned}$$

Notice that the moderated ground term $p\{\iota\}$ is syntactically different from the ground term p , although both terms have the same interpretation in any Σ -structure since $p\{\iota\} = p$.

The next lemma states that interpretation in a Σ -structure is equivariant and highlights the relation between interpretation and moderated terms.

Lemma 3.12 (Interpretation and moderated terms). *Let M be a Σ -structure. Interpretation in M is equivariant, that is, $\pi \cdot M\llbracket p \rrbracket = M\llbracket \pi \cdot p \rrbracket$ for every ground term p and permutation π .*

Proof. Recall that the size of a ground term is the number of nodes of its abstract syntax tree. We proceed by induction on the size of p . The base case $p = a$ is trivial.

If $p = q\{\rho\}$, then $\pi \cdot M\llbracket q\{\rho\} \rrbracket = \pi \cdot M\llbracket q\{\rho\} \rrbracket$, which by the induction hypothesis is equal to

$$\begin{aligned} M\llbracket \pi \cdot (q\{\rho\}) \rrbracket &= M\llbracket q\{\rho; \pi\} \rrbracket = M\llbracket q\{\pi; \pi^{-1}; \rho; \pi\} \rrbracket = M\llbracket q\{\pi; \pi \cdot \rho\} \rrbracket = M\llbracket (q\{\pi\})\{\pi \cdot \rho\} \rrbracket \\ &= M\llbracket (\pi \cdot q)\{\pi \cdot \rho\} \rrbracket = M\llbracket (\pi \cdot q)\{\pi \cdot \rho\} \rrbracket = M\llbracket \pi \cdot (q\{\rho\}) \rrbracket. \end{aligned}$$

If $p = [a]q$, then $\pi \cdot M\llbracket [a]q \rrbracket = \pi \cdot (\langle a \rangle(M\llbracket q \rrbracket)) = \langle \pi \cdot a \rangle(\pi \cdot (M\llbracket q \rrbracket))$, which, by the induction hypothesis, is equal to $\langle \pi \cdot a \rangle(M\llbracket \pi \cdot q \rrbracket) = M\llbracket [\pi \cdot a](\pi \cdot q) \rrbracket = M\llbracket \pi \cdot ([a]q) \rrbracket$.

The remaining cases are straightforward by the induction hypothesis. \square

Moreover, the interpretation $M\llbracket p\{\pi\} \rrbracket$ of a suspension whose delayed renaming is a permutation π , is equal to the permutation $\pi \cdot M\llbracket p \rrbracket$. Indeed, by Definition 3.11 and Lemma 3.12 we have $M\llbracket p\{\pi\} \rrbracket = M\llbracket \pi \cdot p \rrbracket = \pi \cdot M\llbracket p \rrbracket$

Finally, we introduce the Σ -structure NT , which formalises the set of *nominal terms*.

Definition 3.13 (Σ -structure for nominal terms). Let Σ be a signature. The Σ -structure NT for nominal terms is given by the least tuple $(NT\llbracket \delta_1 \rrbracket, \dots, NT\llbracket \delta_n \rrbracket)$ satisfying

$$NT\llbracket \delta_i \rrbracket = NT\llbracket \sigma_{i1} \rrbracket + \dots + NT\llbracket \sigma_{im_i} \rrbracket \quad \text{for each base sort } \delta_i \in \Delta, \text{ and}$$

$$NT\llbracket f_{ij} \rrbracket = \text{inj}_j : NT\llbracket \sigma_{ij} \rrbracket \rightarrow NT\llbracket \delta_i \rrbracket, \text{ for each function symbol } f_{ij} \in F.$$

In the conditions above, the ‘less than or equal to’ relation for tuples is pointwise set inclusion. The $NT\llbracket f_{ij} \rrbracket$ is the j th injection of the i th component in $(NT\llbracket \delta_1 \rrbracket, \dots, NT\llbracket \delta_n \rrbracket)$.

Nominal terms represent alpha-equivalence classes of raw terms by using the atom abstractions of Definition 2.2.

Definition 3.14 (Nominal terms). Let Σ be a signature. The set $\mathbb{N}(\Sigma)_\sigma$ of *nominal terms over Σ of sort σ* is the domain of interpretation of the ground terms of sort σ in the Σ -structure NT , that is, $\mathbb{N}(\Sigma)_\sigma = NT\llbracket \sigma \rrbracket$.

We sometimes write p, ℓ instead of $NT\llbracket p \rrbracket, NT\llbracket \ell \rrbracket$ when it is clear from the context that we are referring to the interpretation into nominal terms of ground terms p and ℓ .

Nominal Terms and Nominal Algebraic Datatypes. We check that the nominal sets $\mathbb{N}(\Sigma)_\sigma$ coincide (up to isomorphism) with the nominal algebraic datatypes of Definition 8.9 in [Pit13]. We first illustrate the nominal terms by means of the signature Σ for the π -calculus in Example 3.6.

Example 3.15. The Σ -structure NT is given by the least pair $(NT[\mathbf{pr}], NT[\mathbf{ac}])$ of nominal sets satisfying the following set equations

$$\begin{aligned}
NT[\mathbf{pr}] &= NT[\mathbf{1}] + NT[\mathbf{pr}] + NT[\mathbf{ch} \times [\mathbf{ch}]\mathbf{pr}] + NT[\mathbf{ch} \times \mathbf{ch} \times \mathbf{pr}] \\
&\quad + NT[\mathbf{pr} \times \mathbf{pr}] + NT[\mathbf{pr} \times \mathbf{pr}] + NT[\mathbf{pr}] + NT[[\mathbf{ch}]\mathbf{pr}] \\
&= \{()\} + NT[\mathbf{pr}] + (\mathbb{A}_{\mathbf{ch}} \times [\mathbb{A}_{\mathbf{ch}}](NT[\mathbf{pr}])) + (\mathbb{A}_{\mathbf{ch}} \times \mathbb{A}_{\mathbf{ch}} \times NT[\mathbf{pr}]) \\
&\quad + (NT[\mathbf{pr}] \times NT[\mathbf{pr}]) + (NT[\mathbf{pr}] \times NT[\mathbf{pr}]) + NT[\mathbf{pr}] + [\mathbb{A}_{\mathbf{ch}}](NT[\mathbf{pr}]), \\
NT[\mathbf{ac}] &= NT[\mathbf{1}] + NT[\mathbf{ch} \times \mathbf{ch}] + NT[\mathbf{ch} \times \mathbf{ch}] + NT[\mathbf{ch} \times \mathbf{ch}] \\
&= \{()\} + (\mathbb{A}_{\mathbf{ch}} \times \mathbb{A}_{\mathbf{ch}}) + (\mathbb{A}_{\mathbf{ch}} \times \mathbb{A}_{\mathbf{ch}}) + (\mathbb{A}_{\mathbf{ch}} \times \mathbb{A}_{\mathbf{ch}}),
\end{aligned}$$

together with an equivariant function for each function symbol in F (we only show a few)

$$\begin{aligned}
NT[\mathit{null}] &= \mathit{inj}_1 : \{()\} \rightarrow NT[\mathbf{pr}] \\
NT[\mathit{tau}] &= \mathit{inj}_2 : NT[\mathbf{pr}] \rightarrow NT[\mathbf{pr}] \\
NT[\mathit{out}] &= \mathit{inj}_4 : \mathbb{A}_{\mathbf{ch}} \times [\mathbb{A}_{\mathbf{ch}}](NT[\mathbf{pr}]) \rightarrow NT[\mathbf{pr}] \\
NT[\mathit{new}] &= \mathit{inj}_8 : [\mathbb{A}_{\mathbf{ch}}](NT[\mathbf{pr}]) \rightarrow NT[\mathbf{pr}] \\
NT[\mathit{tauA}] &= \mathit{inj}_1 : \{()\} \rightarrow NT[\mathbf{ac}] \\
NT[\mathit{boutA}] &= \mathit{inj}_4 : \mathbb{A}_{\mathbf{ch}} \times \mathbb{A}_{\mathbf{ch}} \rightarrow NT[\mathbf{ac}].
\end{aligned}$$

For example, the process $(\nu b)(\bar{a}b.0)$ is encoded as the ground term $\mathit{new}([b](\mathit{out}(a, b, \mathit{null})))$, whose interpretation in NT is $\mathit{inj}_8(\langle b \rangle(\mathit{inj}_4(a, b, \mathit{inj}_1())))$. ■

Remark 3.16. Recall that the constructor inj_j for disjoint union has the polymorphic type

$$\mathit{inj}_j : \forall(S_1 + \dots + S_m).S_j \rightarrow S_1 + \dots + S_m, \quad \text{where } j \leq m.$$

Therefore, a nominal term may have ‘polymorphic sort’ and the sets of nominal terms of each sort may not be mutually disjoint. For instance, both ground terms null and tauA have the same interpretation $\mathit{inj}_1()$ in NT . However, each of the $NT[\mathit{null}]$ and $NT[\mathit{tauA}]$ live in different components of the carrier $(NT[\mathbf{pr}], NT[\mathbf{ac}])$ of the T -algebra induced by the Σ -structure NT and, by all means, the sort information is never lost. Here we are not concerned with this technical subtlety and, at any rate, we can always determine the ‘monomorphic sort’ of a given nominal term by using implicit type parameters (within curly braces) that fix the set $S_1 + \dots + S_n$ over which each constructor inj_j is universally quantified, i.e. $NT[\mathit{null}] = \mathit{inj}_1\{NT[\mathbf{pr}]\}()$. ■

The nominal term with implicit type parameters that corresponds to process $(\nu b)(\bar{a}b.0)$ is $\mathit{inj}_8\{NT[\mathbf{pr}]\}(\langle b \rangle(\mathit{inj}_4\{NT[\mathbf{pr}]\}(a, b, \mathit{inj}_1\{NT[\mathbf{pr}]\}())))$.

The remainder of this section shows that the nominal terms are connected to the elements of the nominal algebraic data types of Definition 8.9 in [Pit13]. We follow closely the exposition on initial algebraic semantics for nominal algebraic data types in [Pit13]. The reader is advised to read Sections 8.3 and 8.4 of [Pit13] alongside.

Let $\mathbf{Nom}^n = \mathbf{Nom} \times \dots_n \text{ times} \dots \times \mathbf{Nom}$ be the n -product category and let $T : \mathbf{Nom}^n \rightarrow \mathbf{Nom}^n$ be the nominal algebraic functor induced by a signature Σ (see Section 8.3 of [Pit13]), which we describe next. Given an n -tuple $S = (S_1, \dots, S_n)$ of nominal sets, each sort σ gives rise to a nominal set $[[\sigma]S]$ defined by:

$$\begin{aligned}
[[\delta_i]S] &= S_i \\
[[\alpha]S] &= \mathbb{A}_\alpha \\
[[[\alpha]\sigma]S] &= [\mathbb{A}_\alpha]([[\sigma]S]) \\
[[\sigma_1 \times \dots \times \sigma_k]S] &= [[\sigma_1]S] \times \dots \times [[\sigma_k]S].
\end{aligned}$$

Let the sorts σ_{ij} be such that $f_{ij} : \sigma_{ij} \rightarrow \delta_i$ are the function symbols of signature Σ . The nominal algebraic functor T has components $T_i : \mathbf{Nom}^n \rightarrow \mathbf{Nom}$ mapping each $S = (S_1, \dots, S_m) \in \mathbf{Nom}^n$ to $T_i S = \llbracket \sigma_{i1} \rrbracket S + \dots + \llbracket \sigma_{im_i} \rrbracket S$, and similarly for n -tuples of equivariant functions.

A Σ -structure M gives rise to a T -algebra whose carrier is the n -tuple of nominal sets $S = (M\llbracket \delta_1 \rrbracket, \dots, M\llbracket \delta_n \rrbracket)$ and whose morphism is the n -tuple of equivariant functions $F = (F_1, \dots, F_n)$ where $F_i(\text{inj}_j s) = M\llbracket f_{ij} \rrbracket(s)$ for each $s \in S_j$.

Theorem 3.17. *The nominal sets $\mathbb{N}(\Sigma)_\sigma$ coincide (up to isomorphism) with the nominal algebraic datatypes of Definition 8.9 in [Pit13].*

Proof. Let $D = (NT\llbracket \delta_1 \rrbracket, \dots, NT\llbracket \delta_n \rrbracket)$ together with morphism $I = (I_1, \dots, I_n)$ such that $I_i(\text{inj}_j s) = NT\llbracket f_{ij} \rrbracket(s)$ be the T -algebra induced by Σ -structure NT . It is routine to check that I maps $T(D)$ to D , where T is the nominal algebraic functor induced by signature Σ , and that the morphism I coincides with the identity. Since D is the least tuple satisfying this condition, the tuple coincides with the least fixed point of functor T . By a well known result by Lambek [Lam68], (D, I) constitutes the initial T -algebra. The theorem follows by Theorem 8.15 in [Pit13]. \square

4. SPECIFICATIONS OF NRTSS

The NRTSSs of Definition 2.7 are meant to be a model of computation for calculi with name-binding operators and state/residual presentation. In this section we present syntactic specifications for NRTSSs. We start by defining nominal residual signatures.

Definition 4.1 (Nominal residual signature). A *nominal residual signature* (a residual signature for short) is a quintuple $\Sigma = (\Delta, A, \sigma, \varrho, F)$ such that (Δ, A, F) is a nominal signature and σ and ϱ are distinguished nominal sorts over Δ and A , which we call *state sort* and *residual sort* respectively. We say that $\mathbb{N}(\Sigma)_\sigma$ is the set of *states* and $\mathbb{N}(\Sigma)_\varrho$ is the set of *residuals*.

Let $\mathcal{T} = (S, R, \longrightarrow)$ be an NRTS and $\Sigma = (\Delta, A, \sigma, \varrho, F)$ be a residual signature. We say that \mathcal{T} is an NRTS *over signature* Σ iff the sets of states S and residuals R coincide with the sets of nominal terms of state sort $\mathbb{N}(\Sigma)_\sigma$ and residual sort $\mathbb{N}(\Sigma)_\varrho$ respectively.

Our next goal is to introduce syntactic specifications of NRTSSs, which we call nominal residual transition system specifications adapting a terminology introduced by Groote and Vaandrager [GV92]. To this end, we will make use of residual formulas and freshness assertions over raw terms, which are defined below.

Definition 4.2 (Residual formula and freshness assertion). A *residual formula* (a *formula* for short) over a residual signature Σ is a pair (s, r) , where $s \in \mathbb{T}(\Sigma, \mathcal{V})_\sigma$ and $r \in \mathbb{T}(\Sigma, \mathcal{V})_\varrho$. We use the more suggestive $s \longrightarrow r$ in lieu of (s, r) . A formula $s \longrightarrow r$ is *ground* iff s and r are ground terms.

A *freshness assertion* (an *assertion* for short) over a signature Σ is a pair (a, t) where $a \in \mathbb{A}$ and $t \in \mathbb{T}(\Sigma, \mathcal{V})$. We will write $a \not\# t$ in lieu of (a, t) . An assertion is *ground* iff t is a ground term.

Remark 4.3. Formulas and assertions are raw syntactic objects, similar to raw terms, which will occur in the rules of the nominal residual transition system specifications to be defined, and whose purpose is to represent respectively transitions and freshness relations involving

nominal terms. A formula $s \longrightarrow r$ (resp. an assertion $a \not\# t$) unifies with a ground formula $\varphi(s) \longrightarrow \varphi(r)$ (resp. a ground assertion $a \not\# \varphi(t)$), which in turn represents a transition $NT[\varphi(s)] \longrightarrow NT[\varphi(r)]$ (resp. a freshness relation $a \# NT[\varphi(t)]$). For the assertions, notice how the symbols $\not\#$, $\#$ and $NT[\]$ interact. The ground assertion $a \not\# [a]a$ represents the freshness relation $a \# NT[[a]a]$, which is true. On the other hand, the freshness relation $a \#[a]a$ is false because $a \in \text{supp}([a]a)$. \blacksquare

Permutation action, substitution and the function fa extend to residual formulas and freshness assertions in the expected way, i.e.

$$\begin{aligned} \pi \cdot (s \longrightarrow r) &= \pi \cdot s \longrightarrow \pi \cdot r \\ \pi \cdot (a \not\# t) &= \pi \cdot a \not\# \pi \cdot t \\ \varphi(s \longrightarrow r) &= \varphi(s) \longrightarrow \varphi(r) \\ \varphi(a \not\# t) &= a \not\# \varphi(t) \\ \text{fa}(r \longrightarrow s) &= \text{fa}(r) \cup \text{fa}(s) \\ \text{fa}(a \not\# t) &= \{a\} \cup \text{fa}(t). \end{aligned}$$

Residual formulas and freshness assertions are elements of nominal sets. The support of a residual formula (respectively a freshness assertion) is the union of the supports of the raw terms in it. We write $\text{supp}(t \longrightarrow t')$ and $\text{supp}(a \not\# t)$ for the supports of residual formula $t \longrightarrow t'$ and of freshness assertion $a \not\# t$ respectively. We write $b \# (t \longrightarrow t')$ and $b \# (a \not\# t)$ for the freshness relations that involve atom b and residual formula $t \longrightarrow t'$ and freshness assertion $a \not\# t$ respectively.

Definition 4.4 (Nominal residual transition system specification). Let Σ be a residual signature $(\Delta, A, \sigma, \rho, F)$. A *transition rule* over Σ (a *rule*, for short) is of the form

$$\frac{\{u_i \longrightarrow u'_i \mid i \in I\} \quad \{a_j \not\# v_j \mid j \in J\}}{t \longrightarrow t'}$$

abbreviated as $H, \nabla / t \longrightarrow t'$, where $H = \{u_i \longrightarrow u'_i \mid i \in I\}$ is a finitely supported set of formulas over Σ (we call H the set of *premisses*) and where $\nabla = \{a_j \not\# v_j \mid j \in J\}$ is a finite set of assertions over Σ (we call ∇ the *freshness environment*). We say formula $t \longrightarrow t'$ over Σ is the *conclusion*, where t is the *source* and t' is the *target*. A rule is an *axiom* iff it has an empty set of premisses. Note that axioms might have a non-empty freshness environment.

A *nominal residual transition system specification* over Σ (abbreviated to NRTSS) is a set of transition rules over Σ .

Permutation action and substitution extend to rules in the expected way; they are applied to each of the formulas and freshness assertions in the rule.

Notice that the rules of an NRTSS are elements of a nominal set. The support of a rule $H, \nabla / t \longrightarrow t'$ is the union of the support of H , the support of ∇ and the support of $t \longrightarrow t'$. In the sequel we write $\text{supp}(\text{RU})$ for the support of rule RU , and $a \# \text{RU}$ for a freshness relation involving atom a and rule RU . Observe that the set H of premisses of a rule may be infinite, but its support must be finite. However, the freshness environment ∇ must be finite in order to make the simplification rules of Definition 5.5 to follow terminating. These simplification rules will be used in Section 5 to define the rule format in Definition 5.12.

Let \mathcal{R} be an NRTSS. We say that the formula $s \longrightarrow r$ *unifies* with rule RU in \mathcal{R} iff RU has conclusion $t \longrightarrow t'$ and $s \longrightarrow r$ is a substitution instance of $t \longrightarrow t'$. If s and r are ground terms, we also say that transition $NT[[s]] \longrightarrow NT[[r]]$ unifies with RU .

Definition 4.5. Let $a \not\# t$ be a freshness assertion and φ a ground substitution. We say that $\varphi(a \not\# t)$ *holds* iff the freshness relation $a \# NT[\varphi(t)]$ holds.

Let $\nabla = \{a_j \not\# t_j \mid j \in J\}$ be a freshness environment. We say that $\varphi(\nabla)$ *holds* iff the conjunction $\bigwedge_{j \in J} (a_j \# NT[\varphi(t_j)])$ holds.

Definition 4.6 (Proof tree). Let Σ be a residual signature and \mathcal{R} be an NRTSS over Σ . A *proof tree* in \mathcal{R} of a transition $NT[s] \longrightarrow NT[r]$ is an upwardly branching rooted tree without paths of infinite length whose nodes are labelled by transitions such that

- (i) the root is labelled by $NT[s] \longrightarrow NT[r]$, and
- (ii) if $K = \{NT[q_i] \longrightarrow NT[q'_i] \mid i \in I\}$ is the set of labels of the nodes directly above a node with label $NT[p] \longrightarrow NT[p']$, then there exist a rule

$$\frac{\{u_i \longrightarrow u'_i \mid i \in I\} \quad \{a_j \not\# v_j \mid j \in J\}}{t \longrightarrow t'}$$

in \mathcal{R} and a ground substitution φ such that $\varphi(t \longrightarrow t') = p \longrightarrow p'$, for each $i \in I$ $\varphi(u_i \longrightarrow u'_i) = q_i \longrightarrow q'_i$, and $\varphi(\{a_j \not\# t_j \mid j \in J\})$ holds.

We say that $NT[s] \longrightarrow NT[r]$ is *provable* in \mathcal{R} iff it has a proof tree in \mathcal{R} . The transition relation specified by \mathcal{R} consists of all the transitions that are provable in \mathcal{R} .

The nodes of a proof tree are labelled by transitions, which contain nominal terms (i.e. syntactic objects that use the atom abstractions of Definition 2.2). The use of nominal terms in a proof tree captures the convention in typical nominal calculi of considering terms ‘up to alpha-equivalence’.

Example 4.7. Consider the residual signature with base sort \mathbf{b} , atom sort \mathbf{a} , two function symbols f, g with arity $[a]a \rightarrow \mathbf{b}$ and state and residual sorts equal to \mathbf{b} . Let \mathcal{R} be the NRTSS defined by the rules:

$$\frac{}{g(x) \longrightarrow g(x)} \text{Ax} \quad \frac{g([a]a) \longrightarrow g([b]b) \quad a \not\# b}{f([a]a) \longrightarrow f([b]b)} \text{RU}, \quad \text{where } a, b \in \mathbb{A}_a.$$

The nominal term $NT[f([a]a)]$ is equal to $NT[f([b]b)]$, and $NT[g([a]a)]$ is equal to $NT[g([b]b)]$, so the transition $NT[f([a]a)] \longrightarrow NT[f([a]a)]$ is provable with the following proof tree, where rule AX is instantiated using a ground substitution φ such that $\varphi(x) = [a]a$.²

$$\frac{\frac{NT[g([a]a)] \longrightarrow NT[g([b]b)]}{NT[f([a]a)] \longrightarrow NT[f([b]b)]} \text{Ax} \quad a \# b}{\text{RU}}.$$

Intuitively, the freshness assertion $a \not\# b$ in rule RU is superfluous because it references atoms a and b , which do not occur free in the rule (i.e. $a, b \notin \text{fa}(f([a]a) \longrightarrow f([b]b))$ and $a, b \notin \text{fa}(g([a]a) \longrightarrow g([b]b))$). ■

The fact that the nodes of a proof tree are labelled by nominal terms is the main difference between our approach and previous work in nominal structural operational semantics [CMRG12, ACG⁺], nominal rewriting [UPG04, FG07] and nominal algebra [GM09]. In all these works, the ‘up-to-alpha-equivalence’ transitions are explicitly instrumented

²Extending the existing convention to our notion of proof tree, we depict proof trees as trees of inference rules where the conclusion and premisses in each rule are replaced by *the transitions denoted by* their substitution instances, and where the freshness assertions in each rule are replaced by *the freshness relations denoted by* their substitution instances.

within the model of computation by adding inference rules that perform alpha-conversion of raw terms to the specification system.

5. RULE FORMATS FOR NRTSSs

This section defines two rule formats for NRTSSs that ensure that:

- (i) an NRTSS induces an equivariant transition relation, and thus an NRTS in the sense of Definition 2.7;
- (ii) an NRTSS induces a transition relation which, together with an equivariant function bn , corresponds to an NTS of Definition 2.6 [PBE⁺15]. For the latter, we need to ensure that the induced transition relation is equivariant and satisfies *alpha-conversion of residuals* (recall, if $p \longrightarrow (\ell, p')$ is provable in \mathcal{R} and a is in the set of binding names of ℓ , then for every atom b that is fresh in (ℓ, p') the transition $p \longrightarrow (ab) \cdot (\ell, p')$ is also provable).

As a first step, we introduce a rule format ensuring equivariance of the induced transition relation.

Definition 5.1 (Equivariant format). Let \mathcal{R} be an NRTSS. \mathcal{R} is in *equivariant format* iff the rule $(ab) \cdot \text{RU}$ is in \mathcal{R} , for every rule RU in \mathcal{R} and for each $a, b \in \mathbb{A}$.

Lemma 5.2. *Let \mathcal{R} be an NRTSS in equivariant format. For every rule RU in \mathcal{R} and for every permutation π , the rule $\pi \cdot \text{RU}$ is in \mathcal{R} .*

Proof. The claim follows straightforwardly by Definition 5.1, since each permutation π can be expressed as a composition of transpositions $(a_1 b_1) \circ \dots \circ (a_n b_n)$ with $n \geq 0$. \square

Theorem 5.3 (Rule format for NRTSSs). *Let \mathcal{R} be an NRTSS. If \mathcal{R} is in equivariant format then \mathcal{R} induces an NRTS.*

Proof. We prove that the transition relation induced by \mathcal{R} is equivariant. That is, if $NT[p] \longrightarrow NT[p']$ then $\pi \cdot NT[p] \longrightarrow \pi \cdot NT[p']$ for every permutation π . We proceed by induction on the height of the proof tree of $NT[p] \longrightarrow NT[p']$. Assume that the last rule used in this proof is

$$\frac{\{u_i \longrightarrow u'_i \mid i \in I\} \quad \{a_j \# v_j \mid j \in J\}}{t \longrightarrow t'} \text{RU}$$

and that, for some ground substitution φ ,

- (i) the premisses $NT[\varphi(u_i)] \longrightarrow NT[\varphi(u'_i)]$ with $i \in I$ are provable in \mathcal{R} ,
- (ii) the freshness relations $a_j \# NT[\varphi(v_j)]$ with $j \in J$ hold, and
- (iii) $\varphi(t) \longrightarrow \varphi(t') = p \longrightarrow p'$.

Since \mathcal{R} is in equivariant format, by Lemma 5.2 \mathcal{R} contains the rule

$$\frac{\{\pi \cdot u_i \longrightarrow \pi \cdot u'_i \mid i \in I\} \quad \{\pi \cdot a_j \# \pi \cdot v_j \mid j \in J\}}{\pi \cdot t \longrightarrow \pi \cdot t'} \text{RU}_\pi.$$

Our goal now is to show that the transition $\pi \cdot NT[p] \longrightarrow \pi \cdot NT[p']$ is provable using rule RU_π and substitution φ^π defined on page 10. Let $j \in J$. By Lemma 3.9 we know that $\pi \cdot \varphi(v_j) = \varphi^\pi(\pi \cdot v_j)$. Moreover, since $\#$ is equivariant, by Lemma 3.12, the freshness relation $\pi \cdot a_j \# NT[\varphi^\pi(\pi \cdot v_j)]$ holds. Assume now that $i \in I$. We know that the premiss $\pi \cdot NT[\varphi(u_i)] \longrightarrow \pi \cdot NT[\varphi(u'_i)]$ is provable in \mathcal{R} by the induction hypothesis ($I = \emptyset$

corresponds to the base case, i.e. a rule without premisses). By Lemmas 3.9 and 3.12, this premiss is equal to $NT[\varphi^\pi(\pi \cdot u_i)] \longrightarrow NT[\varphi^\pi(\pi \cdot u'_i)]$. Therefore, the transition $\pi \cdot NT[p] \longrightarrow \pi \cdot NT[p']$ is provable using rule RU_π and substitution φ^π because it is equal to $NT[\varphi^\pi(\pi \cdot t)] \longrightarrow NT[\varphi^\pi(\pi \cdot t')]$ by Lemmas 3.9 and 3.12. \square

Remark 5.4. It is straightforward to check that the proof tree of transition $NT[(ab) \cdot p] \longrightarrow NT[(ab) \cdot p']$ obtained in the proof above coincides with the proof tree of $(ab) \cdot (NT[p]) \longrightarrow (ab) \cdot (NT[p'])$, where atoms a and b have been transposed. Both proof trees have the same height. \blacksquare

Before introducing a rule format ensuring alpha-conversion of residuals, we adapt to our freshness environments the simplification rules and the entailment relation of Definition 10 and Lemma 15 in [FG07].

Definition 5.5 (Simplification of freshness environments). Consider a signature Σ . The following rules, where ∇ is a freshness environment over Σ , define *simplification of freshness environments*:

$$\begin{aligned}
\{a \not\# b\} \cup \nabla &\Longrightarrow \nabla \quad \text{if } a \neq b \\
\{a \not\# b\{\rho\}\} \cup \nabla &\Longrightarrow \{a \not\# \rho b\} \cup \nabla \\
\{a \not\# (t\{\rho_1\})\{\rho\}\} \cup \nabla &\Longrightarrow \{a \not\# t\{\rho_1; \rho\}\} \cup \nabla \\
\{a \not\# ([b]t)\{\rho\}\} \cup \nabla &\Longrightarrow \{a \not\# [\rho b](t\{\rho\})\} \cup \nabla \\
\{a \not\# (t_1, \dots, t_k)\{\rho\}\} \cup \nabla &\Longrightarrow \{a \not\# t_1\{\rho\}, \dots, a \not\# t_k\{\rho\}\} \cup \nabla \\
\{a \not\# (f(t))\{\rho\}\} \cup \nabla &\Longrightarrow \{a \not\# t\{\rho\}\} \cup \nabla \\
\{a \not\# [b]t\} \cup \nabla &\Longrightarrow \begin{cases} \{a \not\# t\} \cup \nabla & \text{if } a \neq b \\ \nabla & \text{otherwise} \end{cases} \\
\{a \not\# (t_1, \dots, t_k)\} \cup \nabla &\Longrightarrow \{a \not\# t_1, \dots, a \not\# t_k\} \cup \nabla \\
\{a \not\# f(t)\} \cup \nabla &\Longrightarrow \{a \not\# t\} \cup \nabla.
\end{aligned}$$

The rules define a reduction relation on freshness environments. We write $\nabla \Longrightarrow \nabla'$ when ∇' is obtained from ∇ by applying one simplification rule, and \Longrightarrow^* for the reflexive and transitive closure of \Longrightarrow .

Lemma 5.6. *The relation \Longrightarrow is confluent and terminating.*

A freshness assertion is *reduced* iff it is of the form $a \not\# a$, $a \not\# x$ or $a \not\# x\{\rho\}$. We say that $a \not\# a$ is *inconsistent* and $a \not\# x$ and $a \not\# x\{\rho\}$ are *consistent*. (Notice that assertions $a \not\# x$ and $a \not\# x\{\iota\}$ are syntactically different, although both represent the same freshness relation.) An environment ∇ is *reduced* iff it consists only of reduced assertions. An environment containing a freshness assertion that is not reduced can always be simplified using one of the rules in Definition 5.5. Therefore, by Lemma 5.6, an environment ∇ reduces by \Longrightarrow^* to a unique reduced environment, which we call the *normal form* of ∇ , written $\langle \nabla \rangle nf$. An environment ∇ is *inconsistent* iff $\langle \nabla \rangle nf$ contains some inconsistent assertion.

We write $\langle \tilde{\nabla} \rangle nf$ for the environment obtained by replacing every assertion $a \not\# x$ in $\langle \nabla \rangle nf$ by the assertion $a \not\# x\{\iota\}$. Both $\langle \nabla \rangle nf$ and $\langle \tilde{\nabla} \rangle nf$ denote the same set of freshness relations. Adding the identity renaming ι to variables that are not moderated simplifies the definition of the entailment relation below.

Lemma 5.7. *Let ∇ be an environment over Σ and let φ be a ground substitution. Then $\varphi(\nabla)$ holds iff $\varphi(\langle \nabla \rangle nf)$ holds. Moreover, $\varphi(\langle \nabla \rangle nf)$ holds iff $\varphi(\langle \tilde{\nabla} \rangle nf)$ holds.*

The proof of Lemma 5.7 is in Appendix C.

Notice that if ∇ is inconsistent, then for every ground substitution φ none of $\varphi(\nabla)$, $\varphi(\langle \nabla \rangle nf)$ and $\varphi(\langle \widetilde{\nabla} \rangle nf)$ holds.

Our notion of entailment $\nabla \vdash \nabla'$ to be defined below represents that the freshness relations in $\varphi(\nabla)$ imply the freshness relations in $\varphi(\nabla')$. In the presence of assertions of the shape $a \not\# x\{\rho\}$, checking that one environment entails another requires some care. Take the entailment $\{a \not\# x\{a/b\}\} \vdash \{b \not\# x\{b/a\}\}$. We have

$$\begin{aligned} (ab) \cdot NT[\varphi(x\{a/b\})] &= (ab) \cdot NT[\varphi(x)\{a/b\}] = (ab) \cdot NT[\varphi(x)\{a/b\}] \\ &= NT[(ab) \cdot (\varphi(x)\{a/b\})] = NT[((ab) \cdot \varphi(x))\{(ab) \cdot a/b\}] \\ &= NT[((ab) \cdot \varphi(x))\{(ab); a/b; (ab)\}] = NT[\varphi(x)\{(ab); (ab); a/b; (ab)\}] \\ &= NT[\varphi(x)\{a/b; (ab)\}] = NT[\varphi(x)\{b/a\}] = NT[\varphi(x)\{b/a\}] = NT[\varphi(x)\{b/a\}], \end{aligned}$$

for every ground substitution φ . By equivariance of $\#$, $a \# NT[\varphi(x\{a/b\})]$ holds iff $b \# NT[\varphi(x\{b/a\})]$ holds. The permutation (ab) mediates between the atoms a and b and between the renamings a/b and b/a . Definition 5.8 below considers such a mediating permutation.

Definition 5.8. We say ∇ entails ∇' (written $\nabla \vdash \nabla'$) iff either ∇ is inconsistent, or otherwise for every assertion $a_1 \not\# x\{\rho_1\}$ in $\langle \widetilde{\nabla} \rangle nf$ there exists a permutation π and a freshness assertion $a_2 \not\# x\{\rho_2\}$ in $\langle \widetilde{\nabla}' \rangle nf$ such that $\pi a_1 = a_2$ and $\rho_1; \pi = \rho_2$.

Lemma 5.9. Let ∇ and ∇' be environments over Σ such that $\nabla \vdash \nabla'$. Then, for every ground substitution φ , if $\varphi(\nabla)$ holds then $\varphi(\nabla')$ holds.

Corollary 5.10. In particular, if $\emptyset \vdash \nabla$ then $\varphi(\nabla)$ holds for every ground substitution φ .

The proof of Lemma 5.9 is in Appendix C.

We are interested in NTSs [PBE⁺15], which consider signatures with base sorts **ac** (for actions) and **pr** (for processes), with a single atom sort **ch** and with source and residual sorts **pr** and **ac** \times **pr** respectively. We let Σ_{NTS} be any such signature parametric on a set F of function symbols that we keep implicit. We let $\text{bn} : \mathbb{N}(\Sigma)_{\text{ac}} \rightarrow \mathcal{P}_\omega(\mathbb{A}_{\text{ch}})$ be the binding-names function of a given NTS. From now on we restrict our attention to the NTS of [PBE⁺15] (without predicates), and the definitions and results to come apply to NRTS/NRTSS over a signature Σ_{NTS} . We require that the rules of an NRTSS only contain ground actions ℓ and therefore function bn is always defined over $NT[\ell]$. (Recall that we write $\text{bn}(\ell)$ instead of $\text{bn}(NT[\ell])$ since it is clear in this context that the ℓ stands for a nominal term.) The rule format that we introduce in Definition 5.12 relies on identifying the rules that give rise to transitions with actions ℓ such that $\text{bn}(\ell)$ is non-empty, which are the transitions that meet the conditions of the property of alpha-conversion of residuals. To this end, we adapt the notion of strict stratification from [FV03, AFGI17].

Definition 5.11 (Partial strict stratification). Let \mathcal{R} be an NRTSS over a signature Σ_{NTS} and bn be a binding-names function. Let S be a partial map from pairs of ground processes and actions to ordinal numbers. S is a *partial strict stratification of \mathcal{R} with respect to bn* iff

- (i) $S(\varphi(t), \ell) \neq \perp$, for every rule in \mathcal{R} with conclusion $t \longrightarrow (\ell, t')$ such that $\text{bn}(\ell)$ is non-empty and for every ground substitution φ , and
- (ii) $S(\varphi(u_i), \ell_i) < S(\varphi(t), \ell)$ and $S(\varphi(u_i), \ell_i) \neq \perp$, for every rule RU in \mathcal{R} with conclusion $t \longrightarrow (\ell, t')$ such that $S(\varphi(t), \ell) \neq \perp$, for every premiss $u_i \longrightarrow (\ell_i, u'_i)$ of RU and for every ground substitution φ .

We say a pair (p, ℓ) of ground process and action *has order* $S(p, \ell)$.

The choice of S determines which rules will be considered by the rule format for NRTSSs of Definition 5.12 below, which guarantees that the induced transition relation satisfies alpha-conversion of residuals and, therefore, the associated transition relation together with function bn are indeed an NTS. We will intend the map S to be such that the only rules whose source and label of the conclusion have defined order are those that may take part in proof trees of transitions with some binding atom in the action.

Definition 5.12 (Alpha-conversion-of-residuals format). Let \mathcal{R} be an NRTSS over a signature Σ_{NTS} , bn be a binding-names function and S be a partial strict stratification of \mathcal{R} with respect to bn . Assume that all the actions occurring in the rules of \mathcal{R} are ground. Let

$$\frac{\{u_i \longrightarrow (\ell_i, u'_i) \mid i \in I\} \quad \nabla}{t \longrightarrow (\ell, t')} \text{RU}$$

be a rule in \mathcal{R} . Let D be the set of variables that occur in the source t of RU but do not occur in the premisses $u_i \longrightarrow (\ell_i, u'_i)$ with $i \in I$, the environment ∇ or the target t' of the rule. The rule RU is in *alpha-conversion-of-residuals format with respect to S* (ACR format with respect to S for short) iff for each ground substitution φ such that $S(\varphi(t), \ell) \neq \perp$, there exists a ground substitution γ such that $\text{dom}(\gamma) \subseteq D$, and for every atom a in the set $\{c \mid \langle \{c \not\# t\} \rangle \text{nf} \neq \emptyset\}$ and for every atom $b \in \text{bn}(\ell)$, the following hold:

- (i) $\{a \not\# t'\} \cup \nabla \vdash \{a \not\# u'_i \mid i \in I\}$,
- (ii) $\{a \not\# t'\} \cup \nabla \cup \{a \not\# u_i \mid i \in I\} \vdash \{a \not\# \gamma(t)\}$, and
- (iii) $\nabla \cup \{b \not\# u_i \mid i \in I \wedge b \in \text{bn}(\ell_i)\} \vdash \{b \not\# \gamma(t)\}$.

An NRTSS \mathcal{R} , together with a binding-names function bn , is in *ACR format with respect to a partial strict stratification S* iff \mathcal{R} is in equivariant format and all the rules in \mathcal{R} are in ACR format with respect to S .

Given a transition $p \longrightarrow (\ell, q)$ that unifies with the conclusion of RU, the rule format ensures that any atom a that is fresh in (ℓ, q) is also fresh in p , and also that the binding atom b is fresh in p . We have obtained the constraints of the rule format by considering the variable flow in each node of a proof tree and the freshness relations that we want to ensure. Constraints (i) and (ii) cover the case for the freshness relation $a \# p$ and Constraint (iii) covers the case for the freshness relation $b \# p$. The purpose of substitution γ is to ignore the variables that occur in the source of a rule but are dropped everywhere else in the rule. Constraints (i) and (ii) are not required for atoms a that for sure are fresh in p , and this explains why the a in the rule format ranges over $\{c \mid \langle \{c \not\# t\} \rangle \text{nf} \neq \emptyset\}$. For example, take the instance of rule RES in Figure 1 from Section 6.1 with $\ell = \text{bout}A(a, b)$. Condition (i)

$$\{c \not\# (\text{bout}A(a, b), \text{new}([c]y)), c \not\# \text{bout}A(a, b)\} \vdash \{c \not\# (\text{bout}A(a, b), y)\}$$

does not hold because $c \not\# [c]y$ does not entail that $c \not\# y$. However, c is fresh in $NT[\text{new}([c]p)]$ even if it is not fresh in $NT[p]$.

Theorem 5.13 (Rule format for NTSs). *Let \mathcal{R} be an NRTSS over a signature Σ_{NTS} , bn be a binding-names function and S be a partial strict stratification of \mathcal{R} with respect to bn . If \mathcal{R} is in ACR format with respect to S then the NRTS induced by \mathcal{R} and bn constitute an NTS—that is, the transition relation induced by \mathcal{R} is equivariant and satisfies alpha-conversion of residuals.*

Sketch of the proof. Given a transition $NT[\varphi(t)] \longrightarrow NT[\varphi(\ell, t')]$, we first prove the freshness relations $a\#NT[\varphi(\gamma(t))]$ and $b\#NT[\varphi(\gamma(t))]$, for each $a \in \mathbb{A} \setminus \{c \in \text{supp}(t) \mid \langle \{c \neq t\} \rangle nf = \emptyset\}$ and for every atom $b \in \text{bn}(\ell)$. Both relations are proven by induction on $S(\varphi(\gamma(t)), \ell)$, and by analysing the variable flow in the rule unifying with $\varphi(t) \longrightarrow \varphi(\ell, t')$. For the first relation, we assume $a\#NT[\varphi(t')]$, use Constraint (i) to prove that $a\#NT[\varphi(u'_i)]$ for each target u'_i of a premiss, apply the induction hypothesis to obtain $a\#NT[\varphi(\gamma(u_i))]$ for each source of a premiss u_i , and use Constraint (ii) to conclude that $a\#NT[\varphi(\gamma(t))]$. For the second relation, the induction hypothesis ensures that $b\#NT[\varphi(\gamma(u_i))]$ for each source u_i of a premiss having b as a binding name, and we use Constraint (iii) to conclude that $b\#NT[\varphi(\gamma(t))]$. From these two freshness relations it is straightforward to prove that $NT[\varphi(t)] \longrightarrow (ab) \cdot NT[\varphi(\ell, t')]$ and we are done. \square

The full proof of Theorem 5.13 is in Appendix C.

6. EXAMPLE OF APPLICATION OF THE ACR-FORMAT TO THE π -CALCULUS

In this section we consider two different semantics of the π -calculus. These semantics differ in the moment at which substitution is performed at input processes. In the *early* semantics, substitution is performed whenever a process makes an input transition. To wit, an input process $in(a, [c]p)$ can perform a transition to a process $p\{b/c\}$ that is obtained from p by renaming the channel name c with a channel name b received through channel a .

In the *late* semantics, substitution is postponed to the moment when an input process and an output process synchronise. For instance, a parallel composition $par(in(a, [c]p), out(a, b, q))$ can perform a transition to $par(p\{b/c\}, q)$ whose left component is obtained from p by renaming the channel name c with a channel name b received through channel a .

6.1. Early Semantics of the π -Calculus. Consider the NRTSS \mathcal{R}_E in Figure 1 for the early semantics of the π -calculus [MPW92] over the residual signature Σ_{NTS} as defined on page 18 of Section 5, where F is the set of function symbols from Example 3.6. Omitted rules EPARR, EPARRSR, ECOMMR, ECLOSER and SUMR are respectively the symmetric version of rules EPARL, EPARRSL, ECOMML, ECLOSEL and SUML.

In the rule EIN, the moderated term $x\{b/c\}$ is used in order to indicate that the renaming b/c will be performed over the term substituted for variable x .

The rule ECLOSEL specifies the interaction of a process like $NT[\text{new}([b](out(a, b, p)))]$, which exports a private channel name b through channel a , composed in parallel with an input process such as $NT[in(a, [c]q)]$ that reads through channel a . The private name b is exported and the resulting process $NT[\text{new}([b](par(p, (cb) \cdot q)))]$ is the parallel composition of processes p and q where atom b is restricted. For illustration, consider the raw terms $t \equiv \text{new}([b](out(a, b, p)))$ and $t' \equiv (boutA(a, b), p)$. The transition $NT[t] \longrightarrow NT[t']$ is provable in \mathcal{R}_E by the following proof tree:

$$\frac{NT[out(a, b, p)] \longrightarrow NT[(outA(a, b), p)] \quad \text{OUT}}{NT[\text{new}([b](out(a, b, p)))] \longrightarrow NT[(boutA(a, b), p)] \quad \text{OPEN.}} \quad b\#a$$

Notice that the nodes of the proof tree above are labelled by transitions involving nominal terms. Therefore, if we were to start with the raw term $q \equiv \text{new}([c](out(a, c, p)))$

$$\begin{array}{c}
\frac{}{in(a, [b]x) \longrightarrow (inA(a, c), x\{\mathbf{b}/c\})} \text{EIN} \\
\\
\frac{}{out(a, b, x) \longrightarrow (outA(a, b), x)} \text{OUT} \qquad \frac{}{tau(x) \longrightarrow (\tau, x)} \text{TAU} \\
\\
\ell \notin \{boutA(a, b), \mid a, b \in \mathbb{A}_{\text{ch}}\} \frac{x_1 \longrightarrow (\ell, y_1)}{par(x_1, x_2) \longrightarrow (\ell, (par(y_1, x_2)))} \text{EPARL} \\
\\
\frac{x_1 \longrightarrow (boutA(a, b), y_1) \quad b \not\# x_2}{par(x_1, x_2) \longrightarrow (boutA(a, b), (par(y_1, x_2)))} \text{EPARRESL} \\
\\
\frac{x_1 \longrightarrow (outA(a, b), y_1) \quad x_2 \longrightarrow (inA(a, b), y_2)}{par(x_1, x_2) \longrightarrow (tauA, (par(y_1, y_2)))} \text{ECOMML} \\
\\
\frac{x_1 \longrightarrow (boutA(a, b), y_1) \quad x_2 \longrightarrow (inA(a, b), y_2) \quad b \not\# x_2}{par(x_1, x_2) \longrightarrow (tauA, new([b](par(y_1, y_2))))} \text{ECLOSEL} \\
\\
\frac{x_1 \longrightarrow (\ell, y_1)}{sum(x_1, x_2) \longrightarrow (\ell, y_1)} \text{SUML} \qquad \frac{x \longrightarrow (\ell, y)}{rep(x) \longrightarrow (\ell, (par(y, rep(x))))} \text{REP} \\
\\
\frac{x \longrightarrow (outA(a, b), y_1) \quad x \longrightarrow (inA(a, b), y_2)}{rep(x) \longrightarrow (tauA, par(par(y_1, y_2), rep(x)))} \text{EREPCOMM} \\
\\
\frac{x \longrightarrow (boutA(a, b), y_1) \quad x \longrightarrow (inA(a, b), y_2) \quad b \not\# x}{rep(x) \longrightarrow (tauA, par(new([b](par(y_1, y_2))), rep(x)))} \text{EREPCLOSE} \\
\\
\frac{x \longrightarrow (outA(a, b), y) \quad b \not\# a}{new([b]x) \longrightarrow (boutA(a, b), y)} \text{OPEN} \qquad \frac{x \longrightarrow (\ell, y) \quad b \not\# \ell}{new([b]x) \longrightarrow (\ell, new([b]y))} \text{RES}
\end{array}$$

where $a, b, c \in \mathbb{A}_{\text{ch}}$ and ℓ is a ground action.

Figure 1: NRTSS \mathcal{R}_E for the early π -calculus.

where $c\#(a, p)$ —which is alpha-equivalent to t —then the transition $NT[[q]] \longrightarrow NT[[t']]$ would have the same proof tree as above, since $NT[[t]]$ and $NT[[q]]$ are the same nominal term.

We use the rule format of Definition 5.12 to show that \mathcal{R}_E , together with the equivariant function bn_E such that $\text{bn}_E(boutA(a, b)) = \{b\}$, and $\text{bn}_E(\ell) = \emptyset$ otherwise, specifies an NTS. We consider the following partial strict stratification

$$\begin{array}{rcl}
S(out(a, b, p), outA(a, b)) & = & 0 \\
S(par(p, q), \ell) & = & 1 + \max\{S(p, \ell), S(q, \ell)\} \\
S(sum(p, q), \ell) & = & 1 + \max\{S(p, \ell), S(q, \ell)\} \\
S(rep(p), \ell) & = & 1 + S(p, \ell) \\
S(new([c]p), \ell) & = & 1 + S(p, \ell) \quad \text{if } c\#\ell \\
S(new([b]p), boutA(a, b)) & = & 1 + S(p, outA(a, b)) \\
S(p, \ell') & = & \perp \quad \text{otherwise}
\end{array}$$

where $a, b \in \mathbb{A}_{\text{ch}}$ and $\ell \in \{boutA(a, b), outA(a, b) \mid a, b \in \mathbb{A}_{\text{ch}}\}$. Operators \max and $+$ above are extended with \perp in the following way:

$$\begin{aligned} \max(\{s_1, \dots, s_n\} \cup \{\perp\}) &= \max\{s_1, \dots, s_n\} \\ \max\{\perp\} &= \perp \\ \perp + s &= \perp \\ s + \perp &= \perp. \end{aligned}$$

We check that \mathcal{R}_E , together with the binding-names function bn_E , is in ACR format with respect to S as follows. First of all, notice that, from the definition of S , we have that $S(p, \text{tau}A) = S(p, \text{in}A(a, b)) = \perp$, for each p and $a, b \in \mathbb{A}_{\text{ch}}$. Observe that S meets Definition 5.11(i) because a formula with either action $\text{tau}A$ or $\text{in}A(a, b)$ does not take part in any proof tree that proves a transition whose action has binding names. Therefore, the only rules in \mathcal{R}_E whose sources and actions unify with pairs of processes and actions that have defined order are OUT , OPEN and EPARRESL , the instance of rule EPARL where $\ell = outA(a, b)$, and the instances of rules SUML , REP and RES where $\ell \in \{boutA(a, b), outA(a, b)\}$ (and the corresponding instances of the symmetric versions EPARRESR , EPARR and SUMR , which are omitted in the excerpt and will not be checked). Observe that S meets Definition 5.11(ii) because for each rule whose conclusion has either action $boutA(a, b)$ or $outA(a, b)$, the order of the ground transition that unifies with its conclusion is always bigger than the order of the ground transitions that unify with its premisses.

For rule OUT , we have an empty set of premisses and the set D of variables that are in $\text{supp}(out(a, b, x))$ but are not in $\text{supp}(outA(a, b), x)$ is empty. Therefore we can do away with substitution γ . Every atom c is such that $\langle \{c \not\# out(a, b, x)\} \rangle \text{nf} \neq \emptyset$, and the set $\text{bn}_E(outA(a, b))$ is empty. We only need to check that for every atom c , the obligation $\{c \not\# (outA(a, b), x)\} \vdash \{c \not\# out(a, b, x)\}$ holds. For atoms $c \in \text{supp}(outA(a, b), x)$ this obligation vacuously holds, and therefore it suffices to pick an atom c fresh in the rule and check that $\{c \not\# (outA(a, b), x)\} \vdash \{c \not\# out(a, b, x)\}$, which simplifies to $\{c \not\# x\{\iota\}\} \vdash \{c \not\# x\{\iota\}\}$. The permutation ι witnesses that this entailment trivially holds as in Definition 5.8(i).

For rule OPEN the set D is empty and every atom $c \# b$ is such that $\langle \{c \not\# new([b]x)\} \rangle \text{nf} \neq \emptyset$. It suffices to pick atom c fresh in the rule (and therefore different from b) and check that

$$\begin{aligned} \{c \not\# (boutA(a, b), y), b \not\# a\} \vdash \{c \not\# (boutA(a, b), y)\} &\quad \text{and} \\ \{c \not\# (boutA(a, b), y), b \not\# a, c \not\# x\} \vdash \{c \not\# new([b]x)\} &\quad \text{and} \\ \{b \not\# x, b \not\# a\} \vdash \{b \not\# new([b]x)\}, & \end{aligned}$$

which holds because $b \not\# new([b]x)$ reduces to the empty set.

For rule EPARRESL we have premiss $x_1 \longrightarrow (boutA(a, b), y_1)$ and the set D is empty. Every atom c is such that $\langle \{c \not\# par(x_1, x_2)\} \rangle \text{nf} \neq \emptyset$ and the set $\text{bn}_E(boutA(a, b))$ contains atom b . We check that

$$\begin{aligned} \{c \not\# (boutA(a, b), par(y_1, x_2)), b \not\# x_2\} \vdash \{c \not\# (boutA(a, b), y_1)\} &\quad \text{and} \\ \{c \not\# (boutA(a, b), par(y_1, x_2)), b \not\# x_2, c \not\# x_1\} \vdash \{c \not\# par(x_1, x_2)\} &\quad \text{and} \\ \{b \not\# x_1, b \not\# x_2\} \vdash \{b \not\# par(x_1, x_2)\}. & \end{aligned}$$

Atom c is either fresh in the rule, or otherwise $c = a$ or $c = b$. In all three cases, checking the obligations above is straightforward.

Consider the instance of rule EPARL where $\ell = outA(a, b)$. That rule instance has premiss $x_1 \longrightarrow (outA(a, b), y_1)$ and the set D is empty. Every atom c is such that $\langle \{c \not\# par(x_1, x_2)\} \rangle \text{nf} \neq \emptyset$ and the set $\text{bn}_E(outA(a, b))$ is empty. We consider the three cases over

c as before and check that

$$\begin{aligned} \{c \# (outA(a, b), par(y_1, x_2))\} \vdash \{c \# (outA(a, b), y_1)\} \quad \text{and} \\ \{c \# (outA(a, b), par(y_1, x_2)), c \# x_1\} \vdash \{c \# par(x_1, x_2)\}, \end{aligned}$$

which is straightforward.

Consider now the instance of rule SUML where $\ell = boutA(a, b)$. We have premiss $x_1 \longrightarrow (boutA(a, b), y_1)$ and the set D contains only x_2 . We pick γ such that $\gamma(x_2) = null$. Every atom c is such that $\langle \{c \# sum(x_1, x_2)\} \rangle nf \neq \emptyset$ and the set $bn_E(boutA(a, b))$ contains only atom b . Again, we check that

$$\begin{aligned} \{c \# (boutA(a, b), y_1)\} \vdash \{c \# (boutA(a, b), y_1)\} \quad \text{and} \\ \{c \# (boutA(a, b), y_1), c \# x_1\} \vdash \{c \# \gamma(sum(x_1, x_2))\} \quad \text{and} \\ \{b \# x_1\} \vdash \{b \# \gamma(sum(x_1, x_2))\}, \end{aligned}$$

which holds since $\gamma(sum(x_1, x_2)) = sum(x_1, null)$ and $b \# null$ reduces to the empty set.

The instance of rule SUML, where $\ell = outA(a, b)$, has premiss $x_1 \longrightarrow (outA(a, b), y_1)$, and the set D and the substitution γ are the same as for the previous instance of SUML. Every atom c is such that $\langle \{c \# sum(x_1, x_2)\} \rangle nf \neq \emptyset$ and the set $bn_E(outA(a, b))$ is empty. We check that

$$\begin{aligned} \{c \# (outA(a, b), y_1)\} \vdash \{c \# (outA(a, b), y_1)\} \quad \text{and} \\ \{c \# (outA(a, b), y_1), c \# x_1\} \vdash \{c \# \gamma(sum(x_1, x_2))\}, \end{aligned}$$

which hold as before.

For the instance of rule REP, where $\ell = boutA(a, b)$, the set D is empty and every atom c is such that $\langle \{c \# rep(x)\} \rangle nf \neq \emptyset$. We need to check that

$$\begin{aligned} \{c \# (boutA(a, b), par(y, rep(x)))\} \vdash \{c \# (boutA(a, b), y)\} \quad \text{and} \\ \{c \# (boutA(a, b), par(y, rep(x))), c \# x\} \vdash \{c \# rep(x)\} \quad \text{and} \\ \{b \# x\} \vdash \{b \# rep(x)\}, \end{aligned}$$

which is straightforward.

For the instance of rule REP, where $\ell = outA(a, b)$, the set D is empty and every atom c is such that $\langle \{c \# rep(x)\} \rangle nf \neq \emptyset$. It suffices to check that

$$\begin{aligned} \{c \# (outA(a, b), par(y, rep(x)))\} \vdash \{c \# (outA(a, b), y)\} \quad \text{and} \\ \{c \# (outA(a, b), par(y, rep(x))), c \# x\} \vdash \{c \# rep(x)\}, \end{aligned}$$

which is straightforward.

For the instance of the rule RES, where $\ell = boutA(a, b)$, the set D is empty and every atom $d \# c$ is such that $\langle \{d \# new([c]x)\} \rangle nf \neq \emptyset$. We check that

$$\begin{aligned} \{d \# (boutA(a, b), new([c]y)), c \# boutA(a, b)\} \vdash \{d \# (boutA(a, b), y)\} \quad \text{and} \\ \{d \# (boutA(a, b), new([c]y)), c \# boutA(a, b), d \# x\} \vdash \{d \# new([c]x)\} \quad \text{and} \\ \{b \# x, c \# boutA(a, b)\} \vdash \{b \# new([c]x)\}. \end{aligned}$$

Atom d is either fresh in the rule, or otherwise $d = a$ or $d = b$. In all three cases, checking the obligations above is straightforward. For instance, in the second and third obligations, $d \# x$ and $b \# x$ entail $d \# new([c]x)$ and $b \# new([c]x)$ respectively.

For the instance of the rule RES where $\ell = outA(a, b)$ the set D is empty and every atom $d \# c$ is such that $\langle \{d \# new([c]x)\} \rangle nf \neq \emptyset$. We consider the three cases over d as before and check that

$$\begin{aligned} \{d \# (outA(a, b), new([c]y)), c \# outA(a, b)\} \vdash \{d \# (outA(a, b), y)\} \quad \text{and} \\ \{d \# (outA(a, b), new([c]y)), c \# outA(a, b), d \# x\} \vdash \{d \# new([c]x)\}, \end{aligned}$$

which holds because $d \not\# x$ entails $d \not\# \text{new}([c]x)$.

Atoms a , b and c in \mathcal{R}_E range over \mathbb{A}_{ch} , and thus \mathcal{R}_E is in equivariant format. Therefore \mathcal{R}_E is in ACR format with respect to S . By Theorem 5.13 the NRTS induced by \mathcal{R}_E , together with function bn_E , constitute an NTS of Definition 2.6.

6.2. Late Semantics of the π -Calculus. The NRTSS \mathcal{R}_L over the residual signature Σ_{NTS} models the late semantics of the π -calculus [MPW92] in our target semantic model, which is an NTS. \mathcal{R}_L consists of the rules in Figure 2 together with rules OUT,TAU, SUML, REP, OPEN and RES from Figure 1 in Section 6.1, and the omitted symmetric versions LPARR, LPARRESR, LCOMMR, LCLOSER and SUMR.

\mathcal{R}_L is an NRTSS over signature Σ_{NTS} , where the free-input actions are replaced by *bound-input actions* (page 159 of [SW01]), which we write $\text{bin}A(a, b)$. We let the binding-names function bn_L be such that the binding name of both the bound-output action $\text{bout}A(a, b)$ and the bound-input action $\text{bin}A(a, b)$ be b , that is, $\text{bn}_L(\text{bout}A(a, b)) = \text{bn}_L(\text{bin}A(a, b)) = \{b\}$ and $\text{bn}_L(\ell) = \emptyset$ otherwise.

$$\frac{b \not\# a}{\text{in}(a, [b]x) \longrightarrow (\text{bin}A(a, b), x)} \text{LIN}$$

$$\ell \notin \{\text{bout}A(a, b), \text{bin}A(a, b) \mid a, b \in \mathbb{A}_{\text{ch}}\} \frac{x_1 \longrightarrow (\ell, y_1)}{\text{par}(x_1, x_2) \longrightarrow (\ell, (\text{par}(y_1, x_2)))} \text{LPARL}$$

$$\ell \in \{\text{bout}A(a, b), \text{bin}A(a, b) \mid a, b \in \mathbb{A}_{\text{ch}}\} \frac{x_1 \longrightarrow (\ell, y_1) \quad b \not\# x_2}{\text{par}(x_1, x_2) \longrightarrow (\ell, (\text{par}(y_1, x_2)))} \text{LPARRESL}$$

$$\frac{x_1 \longrightarrow (\text{out}A(a, b), y_1) \quad x_2 \longrightarrow (\text{bin}A(a, c), y_2)}{\text{par}(x_1, x_2) \longrightarrow (\text{tau}A, (\text{par}(y_1, y_2 \{b/c\})))} \text{LCOMML}$$

$$\frac{x_1 \longrightarrow (\text{bout}A(a, b), y_1) \quad x_2 \longrightarrow (\text{bin}A(a, b), y_2)}{\text{par}(x_1, x_2) \longrightarrow (\text{tau}A, \text{new}([b](\text{par}(y_1, y_2))))} \text{LCLOSEL}$$

$$\frac{x \longrightarrow (\text{out}A(a, b), y_1) \quad x \longrightarrow (\text{bin}A(a, c), y_2)}{\text{rep}(x) \longrightarrow (\text{tau}A, \text{par}(\text{par}(y_1, y_2 \{b/c\}), \text{rep}(x)))} \text{LREPCOMM}$$

$$\frac{x \longrightarrow (\text{bout}A(a, b), y_1) \quad x \longrightarrow (\text{bin}A(a, b), y_2)}{\text{rep}(x) \longrightarrow (\text{tau}A, \text{par}(\text{new}([b](\text{par}(y_1, y_2))), \text{rep}(x)))} \text{LREPCLOSE}$$

where $a, b, c \in \mathbb{A}_{\text{ch}}$ and ℓ is a ground action.

Figure 2: NRTSS \mathcal{R}_L for the late π -calculus.

In rule LIN, the binding input action $\text{bin}A(a, b)$ binds atom b in the term substituted for variable x on the right side of the residual. In rules LCOMML and LREPCOMM, the moderated term $y_2 \{b/c\}$ is used in order to indicate that the renaming b/c will be performed over the term substituted for variable y_2 .

Remark 6.1. In order to represent the binding input action of the late π -calculus in an NTS, rule LIN ensures that the binding atom b is different from the communication channel a by requiring $b \not\approx a$. This is similar to the requirement $b \not\approx a$ in rule OPEN. As a result, the obtained semantics minimally differs from the original one in [San96]. Consider the original late π -calculus and take the transitions $a(b).((ab) \cdot p) \xrightarrow{a(b)} (ab) \cdot p$ where b is either a or fresh in p . Our rule LIN prevents the transition $NT\llbracket in[a]p \rrbracket \longrightarrow NT\llbracket (binA(a, a), p) \rrbracket$, and our semantics fails to faithfully represent the above-mentioned transition in the original late π -calculus when $b = a$. By alpha-conversion of residuals, if the state $NT\llbracket in(a, [a]p) \rrbracket$ has derivative $NT\llbracket (binA(a, a), p) \rrbracket$, then the same state has to have all the derivatives $\{NT\llbracket (binA(c, c), (ac) \cdot p) \rrbracket \mid c \# (binA(a, a), p)\}$, but these derivatives do not represent valid transitions in the original late π -calculus.

However, the discrepancy between the original and our semantics has very limited consequences, since the binding name of an input process vanishes when communication is performed. Our semantics allows for the transition

$$\begin{aligned} & NT\llbracket par(out(a, a, null), in(a, [b](out(c, b, null)))) \rrbracket \longrightarrow \\ & NT\llbracket (tauA, par(null, (out(c, b, null))\{a/b\})) \rrbracket = NT\llbracket (tauA, par(null, out(c, a, null))) \rrbracket, \end{aligned} \quad (6.1)$$

where the name a is transmitted over the channel with the same name and where $b \# (a, c)$. The transition in (6.1) faithfully represents $(\bar{a}a.0 \parallel a(b).\bar{c}b.0) \xrightarrow{\tau} (0 \parallel \bar{c}a.0)$ in the original late π -calculus. By the nominal interpretations of terms, the process $NT\llbracket in(a, [b](out(c, b, null))) \rrbracket$ with binding atom b is equal to the process $NT\llbracket in(a, [a](out(c, a, null))) \rrbracket$ with binding atom a , and thus the transition in (6.1) also represents $(\bar{a}a.0 \parallel a(a).\bar{c}a.0) \xrightarrow{\tau} (0 \parallel \bar{c}a.0)$ in the original late π -calculus. \blacksquare

As we did in Section 6.1, we use the rule format of Definition 5.12 to show that \mathcal{R}_L , together with equivariant function bn_L specifies an NTS. We consider the following partial strict stratification

$$\begin{aligned} S(out(a, b, p), outA(a, b)) &= 0 \\ S(in(a, [b]p), binA(a, b)) &= 0 \\ S(par(p, q), \ell) &= 1 + \max\{S(p, \ell), S(q, \ell)\} \\ S(sum(p, q), \ell) &= 1 + \max\{S(p, \ell), S(q, \ell)\} \\ S(rep(p), \ell) &= 1 + S(p, \ell) \\ S(new([c]p), \ell) &= 1 + S(p, \ell) \quad \text{if } c \# \ell \\ S(new([b]p), boutA(a, b)) &= 1 + S(p, outA(a, b)) \\ S(p, \ell') &= \perp \quad \text{otherwise,} \end{aligned}$$

where $\ell \in \{boutA(a, b), outA(a, b), binA(a, b) \mid a, b \in \mathbb{A}_{ch}\}$.

Notice that the differences between the S above and the partial strict stratification from Section 6.1 are the inclusion of the second clause above, which defines an order for the pair of input process and bound-input action, and the addition of the bound-input action to the set over which the ℓ above ranges.

We check that \mathcal{R}_L , together with the binding-names function bn_L , is in ACR format with respect to S as follows. First of all, the definition of S yields that $S(p, tauA) = \perp$, for each p . Observe that S meets Definition 5.11(i) because a formula with action $tauA$ does not take part in any proof tree that proves a transition whose action has binding names. Therefore, the only rules in \mathcal{R}_L whose sources and actions unify with pairs of processes and actions that have defined order are LIN, OUT, OPEN and LPARRESL, the instance of rule LPARL where $\ell = outA(a, b)$, and the instances of rules SUML, REP and RES where

$\ell \in \{boutA(a, b), outA(a, b), binA(a, b)\}$ (and the corresponding instances of the symmetric versions LPARRESR, LPARR and SUMR, which are omitted in the excerpt and will not be checked). Observe that S meets Definition 5.11(ii) because for each rule whose conclusion has any of the actions $boutA(a, b)$, $outA(a, b)$ or $binA(a, b)$, the order of the transition that unifies with its conclusion is always bigger than the order of the transitions that unify with its premisses.

We have already checked the ACR-format for rules OUT and OPEN in Section 6.1. We have also checked the ACR-format for the instances of rules SUML, REP and RES where $\ell \in \{boutA(a, b), outA(a, b)\}$, and we will not check the ACR-format for the instances where $\ell = binA(a, b)$ because the checking proceeds exactly as in the case where $\ell = boutA(a, b)$. We will limit ourselves to checking that rule LIN is in the ACR-format with respect to S , as the checking for the other rules are similar to those presented earlier.

For rule LIN we have an empty set of premisses, the set D is empty, and every atom $c\#b$ is such that $\langle \{b\# in(a, [b]x)\} \rangle nf \neq \emptyset$. We check that

$$\{c\# (binA(a, b), x)\} \vdash \{c\# in(a, [b]x)\} \quad \text{and} \quad \{b\# a\} \vdash \{b\# in(a, [b]x)\}.$$

Let us consider the obligation on the left first. If $c = a$, that obligation vacuously holds since its left-hand-side is inconsistent. If $c \neq a$, the obligation simplifies to

$$\{c\# x\{\iota\}\} \vdash \{c\# x\{\iota\}\},$$

which holds straightforwardly. Checking the obligation on the right is also straightforward.

Atoms a , b and c in \mathcal{R}_L range over \mathbb{A}_{ch} , and thus \mathcal{R}_L is in equivariant format. Therefore \mathcal{R}_L is in ACR format with respect to S . By Theorem 5.13 the NRTS induced by \mathcal{R}_L , together with function bn_L , constitute an NTS of Definition 2.6.

7. NTSS WITH RESIDUALS OF ABSTRACTION SORT

In this section we explore alternative specifications of the NTSs à la Parrow in which we allow for the use of residuals of abstraction sort. Intuitively, by the requirement of alpha-conversion of residuals, the NTSs à la Parrow treat the actions with binding names as binding operators. In the systems with residuals of abstraction sorts, we let the binding name in an action to be the binding atom of the residual in which the action occurs. Our aim is to provide translations between the systems with and without residuals of abstraction sort, and to give conditions under which the translations are inverse to each other.

We have already defined the signature Σ_{NTS} on page 18, which is parametric on a set F of function symbols that we keep implicit. For the alternative specifications with residuals of abstraction sort, we consider signatures with base and residual sorts pr and $[ch](ac \times pr)$, and we let $\Sigma_{NTS}^{[ch]}$ be any such signature parametric on the set F of function symbols.

We let \mathcal{T} and $\mathcal{T}^{[ch]}$ range over NRTSs over signatures Σ_{NTS} and $\Sigma_{NTS}^{[ch]}$, respectively, where we write \longrightarrow and $\longrightarrow_{[ch]}$ for the transition relations of \mathcal{T} and $\mathcal{T}^{[ch]}$ respectively. We let bn range over equivariant functions that deliver the binding names in an action. A tuple (\mathcal{T}, bn) where \longrightarrow enjoys alpha-conversion of residuals constitutes an NTS. In what follows, we assume that $|bn(\ell)| \leq 1$ for every action $\ell \in ac$.³

³It is straightforward to generalise the results in this section to the case where $|bn(\ell)| \leq n$ by iterating n abstractions in the residuals, i.e., by adopting a sort $[ch_1] \dots [ch_n](ac \times pr)$ for the residuals and fixing function bn so that it returns an ordered list $(a_{ch_1}, \dots, a_{ch_n})$ of names instead of a set. We omit this generalisation here in order not to clutter notation.

The translation from an NTS (\mathcal{T}, bn) to an NRTS $\mathcal{T}^{\text{[ch]}}$ is given in the definition below.

Definition 7.1. Let (\mathcal{T}, bn) be an NTS with equivariant transition relation \longrightarrow and with equivariant function bn where $|\text{bn}(\ell)| \leq 1$ such that \longrightarrow enjoys alpha-conversion of residuals. The NTS (\mathcal{T}, bn) *translates to* an NRTS $\mathcal{T}^{\text{[ch]}}$ with transition relation $\longrightarrow_{\text{[ch]}}$, which is the least relation satisfying that for all p, ℓ and p' ,

$$p \longrightarrow (\ell, p') \implies p \longrightarrow_{\text{[ch]}} [a](\ell, p'),$$

where either $a\#(\ell, p')$ and $\text{bn}(\ell) = \emptyset$, or $\text{bn}(\ell) = \{a\}$.

We write $\mathfrak{T}^{\text{[ch]}}$ for the translation function, i.e., $\mathcal{T}^{\text{[ch]}} = \mathfrak{T}^{\text{[ch]}}(\mathcal{T}, \text{bn})$.

We prove that the transition relation $\longrightarrow_{\text{[ch]}}$ obtained by Definition 7.1 is equivariant and thus the translation produces an NRTS.

Lemma 7.2 (Equivariance of $\longrightarrow_{\text{[ch]}}$). *The relation $\longrightarrow_{\text{[ch]}}$ obtained by Definition 7.1 is equivariant. More formally, $p \longrightarrow_{\text{[ch]}} [a](\ell, p')$ implies $\pi \cdot p \longrightarrow_{\text{[ch]}} [\pi \cdot a](\pi \cdot \ell, \pi \cdot p')$ for every permutation π .*

Proof. Let a be an atom, p and p' be processes and ℓ be an action. We assume that $p \longrightarrow_{\text{[ch]}} [a](\ell, p')$, which has been obtained from $p \longrightarrow (\ell, p')$ by Definition 7.1. Now we prove that $\pi \cdot p \longrightarrow_{\text{[ch]}} [\pi \cdot a](\ell, p')$ for every permutation π . We distinguish the following cases.

Case $\text{bn}(\ell) = \emptyset$: Then $a\#(\ell, p')$. Since $\#$ is equivariant, we have that $(\pi \cdot a)\#(\pi \cdot \ell, \pi \cdot p')$.

Since \longrightarrow is equivariant, it follows that $\pi \cdot p \longrightarrow (\pi \cdot \ell, \pi \cdot p')$. By the translation function, and since $(\pi \cdot a)\#(\pi \cdot \ell, \pi \cdot p')$, we have that $\pi \cdot p \longrightarrow_{\text{[ch]}} [\pi \cdot a](\pi \cdot \ell, \pi \cdot p')$. Since bn is equivariant, $\text{bn}(\pi \cdot \ell)$ is empty and we are done.

Case $\text{bn}(\ell) = \{a\}$: Since bn is equivariant, $\text{bn}(\pi \cdot \ell) = \{\pi \cdot a\}$. Since \longrightarrow is equivariant, $\pi \cdot p \longrightarrow (\pi \cdot \ell, \pi \cdot p')$. By the translation function, and since $\text{bn}(\pi \cdot \ell) = \{\pi \cdot a\}$, we have that $\pi \cdot p \longrightarrow_{\text{[ch]}} [\pi \cdot a](\pi \cdot \ell, \pi \cdot p')$. \square

Remark 7.3. Note that, as expected, the fact the transition relation \longrightarrow in an NTS enjoys alpha-conversion of residuals does not play a role in the proof of the above result. \blacksquare

The translation from an NRTS $\mathcal{T}^{\text{[ch]}}$ into an NTS (\mathcal{T}, bn) is given in the definition below.

Definition 7.4. Let $\mathcal{T}^{\text{[ch]}}$ be an NRTS with equivariant transition relation $\longrightarrow_{\text{[ch]}}$. The NRTS $\mathcal{T}^{\text{[ch]}}$ *translates to* an NTS (\mathcal{T}, bn) with transition relation \longrightarrow , which is the least relation satisfying that for all p, a, ℓ and p' ,

$$p \longrightarrow_{\text{[ch]}} [a](\ell, p') \implies p \longrightarrow ((ba) \cdot \ell, (ba) \cdot p')$$

for $b = a$ and for each $b\#(\ell, p')$, and with binding-names function

$$\text{bn}(\ell) = \{a \mid p \longrightarrow_{\text{[ch]}} [a](\ell, p') \wedge a \in \text{supp}(\ell)\}.$$

We write \mathfrak{T} for the translation function, i.e., $(\mathcal{T}, \text{bn}) = \mathfrak{T}(\mathcal{T}^{\text{[ch]}})$.

Notice that if $a \in \text{supp}(\ell, p')$, then \mathfrak{T} maps transition $p \longrightarrow_{\text{[ch]}} [a](\ell, p')$ into every transition in $\{p \longrightarrow ((ba) \cdot \ell, (ba) \cdot p') \mid b = a \vee b\#(\ell, p')\}$, which encompasses the alpha-equivalence class of the target $[a](\ell, p')$.

We prove that the \longrightarrow and bn obtained by Definition 7.4 are equivariant, and that \longrightarrow enjoys alpha-conversion of residuals. Thus, the translation is sound.

Lemma 7.5 (Equivariance of \longrightarrow). *The relation \longrightarrow obtained by Definition 7.4 is equivariant. More formally, $p \longrightarrow (\ell, p')$ implies $\pi \cdot p \longrightarrow (\pi \cdot \ell, \pi \cdot p')$ for every permutation π .*

Proof. Let a be an atom, p and p' be processes and ℓ be an action. We assume $p \longrightarrow (\ell, p')$ which is one of the transitions in the set $\{p \longrightarrow ((ba) \cdot \ell, (ba) \cdot p') \mid b = a \vee b\#(\ell, p')\}$ obtained from $p \longrightarrow_{[\text{ch}]} [a](\ell, p')$ by Definition 7.4. Now we prove that for every permutation π , if $p \longrightarrow ((ba) \cdot \ell, (ba) \cdot p')$ where $b = a$ or $b\#(\ell, p')$, then $\pi \cdot p \longrightarrow (\pi \cdot (ba) \cdot \ell, \pi \cdot (ba) \cdot p')$. Since $\longrightarrow_{[\text{ch}]}$ is equivariant, we have that $p \longrightarrow_{[\text{ch}]} [a](\ell, p')$ implies $\pi \cdot p \longrightarrow_{[\text{ch}]} \pi \cdot [a](\ell, p')$. By Definition 7.4, for every atom $c = \pi \cdot a$ or $c\#(\pi \cdot \ell, \pi \cdot p')$ we have that $\pi \cdot p \longrightarrow ((c(\pi \cdot a)) \cdot \pi \cdot \ell, (c(\pi \cdot a)) \cdot \pi \cdot p')$. Therefore it suffices to find such an atom c that entails that $(c(\pi \cdot a)) \cdot \pi \cdot (\ell, p') = \pi \cdot (ba) \cdot (\ell, p')$. The latter equation holds by choosing $c = \pi \cdot b$. If $b = a$, then the transpositions $(c(\pi \cdot a))$ and (ba) are equal to ι and the equation above trivially follows. Otherwise, $b\#(\ell, p')$ and the equation above follows since $\pi \cdot b\#\pi \cdot (\ell, p')$ by equivariance of $\#$, and since $((\pi \cdot a)(\pi \cdot b)) \cdot \pi \cdot (\ell, p') = (\pi \cdot (ab)) \cdot (\pi \cdot (\ell, p')) = \pi \cdot (ab) \cdot (\ell, p')$ by equivariance of the permutation action. \square

Lemma 7.6 (Equivariance of bn). *The function bn obtained by Definition 7.4 is equivariant. More formally, for every permutation π and every action ℓ we have that $\text{bn}(\pi \cdot \ell) = \pi \cdot \text{bn}(\ell)$.*

Proof. By calculating

$$\begin{aligned}
\text{bn}(\pi \cdot \ell) &= \{a \mid p \longrightarrow_{[\text{ch}]} [a](\pi \cdot \ell, p') \in \mathcal{T}^{[\text{ch}]} \wedge a \in \text{supp}(\pi \cdot \ell)\} \\
&= \text{by considering } a = \pi \cdot b, p = \pi \cdot q \text{ and } p' = \pi \cdot q' \\
&\quad \{\pi \cdot b \mid \pi \cdot q \longrightarrow_{[\text{ch}]} [\pi \cdot b](\pi \cdot \ell, \pi \cdot q') \in \mathcal{T}^{[\text{ch}]} \wedge \pi \cdot b \in \text{supp}(\pi \cdot \ell)\} \\
&= \text{by equivariance of } \longrightarrow_{[\text{ch}]} \text{ and } \text{supp} \\
&\quad \pi \cdot \{b \mid q \longrightarrow_{[\text{ch}]} [b](\ell, q') \in \mathcal{T}^{[\text{ch}]} \wedge b \in \text{supp}(\ell)\} \\
&= \text{by Definition 7.4} \\
&\quad \pi \cdot \text{bn}(\ell). \quad \square
\end{aligned}$$

Lemma 7.7 (Alpha-conversion of residuals). *Given \longrightarrow and bn obtained by Definition 7.1, if $p \longrightarrow (\ell, p')$, $a \in \text{bn}(\ell)$ and $b\#(\ell, p')$, then $p \longrightarrow ((ba) \cdot \ell, (ba) \cdot p')$.*

Proof. Transition $p \longrightarrow (\ell, p')$ stems from a transition $p \longrightarrow_{[\text{ch}]} [a](\ell, p')$ and, since $a \in \text{bn}(\ell)$, by Definition 7.4 we know that $a \in \text{supp}(\ell)$. Thus, a is not fresh in (ℓ, p') and since $b\#(\ell, p')$ we have that $b \neq a$. Transition $p \longrightarrow ((ba) \cdot \ell, (ba) \cdot p')$ follows by Definition 7.4 and we are done. \square

Although both the translations in Definitions 7.1 and 7.4 are sound, they are not the inverse of each other. Consider an NRTS $\mathcal{T}^{[\text{ch}]}$ that contains a transition $p \longrightarrow_{[\text{ch}]} [a](\ell, p')$ where $a \in \text{supp}(p')$ and $a\#\ell$. (Atom a is abstracted over p' but is fresh in ℓ .) By Definition 7.4, $\mathcal{T}^{[\text{ch}]}$ translates to an NTS $(\mathcal{T}, \text{bn}) = \mathfrak{T}(\mathcal{T}^{[\text{ch}]})$ that contains a transition $p \longrightarrow (\ell, (ba) \cdot p')$ for each atom $b = a \vee b\#(\ell, p')$ and where $a \notin \text{bn}(\ell)$. (The exported name a does not occur as a binding name of ℓ .) Taking the translation in Definition 7.1 back, we obtain an NRTS $\mathcal{T}^{[\text{ch}]'} = \mathfrak{T}^{[\text{ch}]}(\mathcal{T}, \text{bn})$ that has a distinct transition $p \longrightarrow [c](\ell, (ba) \cdot p')$ for each atom $b = a \vee b\#(\ell, p')$ and where $c\#(\ell, (ba) \cdot p')$, but which does not contain the original transition $p \longrightarrow [a](\ell, p')$ —equal to any of its alpha-equivalent representations $p \longrightarrow [b](\ell, (ba) \cdot p')$ with $b\#(\ell, p')$ —because $b \in \text{supp}((ba) \cdot p')$ and thus $c \neq b$. (The original transition with name a abstracted in the residual's body cannot be obtained back.) Therefore $\mathcal{T}^{[\text{ch}]'} \neq \mathcal{T}^{[\text{ch}]}$.

However, given an NTS (\mathcal{T}, bn) , translating it to an NRTS with atom-abstractions in the residuals and then back, delivers the same NTS (\mathcal{T}, bn) . The following lemma states this fact.

Theorem 7.8. *Let (\mathcal{T}, bn) be an NTS such that $|\text{bn}(\ell)| \leq 1$ for every action ℓ . Then, $\mathfrak{T}(\mathfrak{T}^{\text{ch}}(\mathcal{T}, \text{bn})) = (\mathcal{T}, \text{bn})$.*

Proof. Let $(\mathcal{T}', \text{bn}') = \mathfrak{T}(\mathfrak{T}^{\text{ch}}(\mathcal{T}, \text{bn}))$. We prove that $\mathcal{T}' = \mathcal{T}$ and that $\text{bn}' = \text{bn}$. Let $p \longrightarrow (\ell, p')$ be a transition in \mathcal{T} . It suffices to prove that

- transition $p \longrightarrow (\ell, p')$ maps through the composition of \mathfrak{T}^{ch} and \mathfrak{T} to a set of transitions that contains itself, and such that every other transition in the set is already in \mathcal{T} , and
- $a \in \text{bn}(\ell)$ iff $a \in \text{bn}'(\ell)$.

We consider the following cases.

Case $\text{bn}(\ell) = \emptyset$: By Definition 7.1, transition $p \longrightarrow (\ell, p')$ maps to transition $p \longrightarrow_{\text{[ch]}} [a](\ell, p')$ with $a\#\ell, p'$ in $\mathfrak{T}^{\text{ch}}(\mathcal{T}, \text{bn})$. By Definition 7.4, transition $p \longrightarrow_{\text{[ch]}} [a](\ell, p')$ maps to transition $p \longrightarrow (\ell, p')$ in \mathcal{T}' because $a\#\ell, p'$. Furthermore, by Definition 7.4, $a \notin \text{bn}'(\ell)$ because $a\#\ell$.

Case $\text{bn}(\ell) = \{a\}$: By Definition 7.1, transition $p \longrightarrow (\ell, p')$ maps to transition $p \longrightarrow_{\text{[ch]}} [a](\ell, p')$ in $\mathfrak{T}^{\text{ch}}(\mathcal{T}, \text{bn})$, where a is abstracted in the residual (ℓ, p') . By Definition 7.4, transition $p \longrightarrow_{\text{[ch]}} [a](\ell, p')$ maps to the set $T = \{p \longrightarrow ((ab) \cdot \ell, (ab) \cdot p') \mid b = a \vee b\#\ell, p'\}$ in \mathcal{T}' . The set T contains the original transition $p \longrightarrow (\ell, p')$ and, by alpha-conversion of residuals, every other transition in T is in \mathcal{T} . Furthermore, by Definition 7.4, $a \in \text{bn}'(\ell)$ because $a \in \text{supp}(\ell)$. \square

In order to prove that the composition of the translations in the inverse order is the identity—i.e., $\mathfrak{T}^{\text{ch}}(\mathfrak{T}(\mathcal{T}^{\text{ch}})) = \mathcal{T}^{\text{ch}}$ —it suffices to prevent that the abstracted atom in a residual occurs in the process but not in the action.

Theorem 7.9. *Let \mathcal{T}^{ch} be an NRTS such that for every transition $p \longrightarrow_{\text{[ch]}} [a](\ell, p')$ in \mathcal{T}^{ch} , $a\#\ell$ implies that $a\#p'$. Then, $\mathfrak{T}^{\text{ch}}(\mathfrak{T}(\mathcal{T}^{\text{ch}})) = \mathcal{T}^{\text{ch}}$.*

Proof. Let $\mathcal{T}^{\text{ch}'} = \mathfrak{T}^{\text{ch}}(\mathfrak{T}(\mathcal{T}^{\text{ch}}))$. We prove that $\mathcal{T}^{\text{ch}'} = \mathcal{T}^{\text{ch}}$. Let $p \longrightarrow_{\text{[ch]}} [a](\ell, p')$ be a transition in \mathcal{T}^{ch} . It suffices to prove that $p \longrightarrow_{\text{[ch]}} [a](\ell, p')$ maps through the composition of \mathfrak{T} and \mathfrak{T}^{ch} to itself.

We consider the following cases.

Case $a\#\ell$: By assumption, $a\#p'$. By Definition 7.4, transition $p \longrightarrow_{\text{[ch]}} [a](\ell, p')$ maps to transition $p \longrightarrow (\ell, p')$ in \mathcal{T} , and $a \notin \text{bn}(\ell)$, where $\mathfrak{T}(\mathcal{T}^{\text{ch}}) = (\mathcal{T}, \text{bn})$. By Definition 7.1, since $a\#\ell$, transition $p \longrightarrow (\ell, p')$ maps to transition $p \longrightarrow_{\text{[ch]}} [a](\ell, p')$ in $\mathcal{T}^{\text{ch}'}$.

Case $a \in \text{supp}(\ell)$: By Definition 7.4, transition $p \longrightarrow_{\text{[ch]}} [a](\ell, p')$ maps to every transition in the set $T = \{p \longrightarrow ((ba) \cdot \ell, (ba) \cdot p') \mid b = a \vee b\#\ell, p'\}$ in \mathcal{T} , and $a \in \text{bn}(\ell)$, where $\mathfrak{T}(\mathcal{T}^{\text{ch}}) = (\mathcal{T}, \text{bn})$. For each b such that $b = a$ or $b\#\ell, p'$, by Definition 7.4, transition $p \longrightarrow ((ba) \cdot \ell, (ba) \cdot p')$ maps to transition $p \longrightarrow_{\text{[ch]}} [b]((ba) \cdot \ell, (ba) \cdot p')$ in $\mathcal{T}^{\text{ch}'}$ because $b \in \text{bn}((ba) \cdot \ell)$. By definition of atom-abstraction, $[b]((ba) \cdot \ell, (ba) \cdot p') = [a](\ell, p')$ for every b such that $b = a$ or $b\#\ell, p'$. Therefore, every transition in T maps to $p \longrightarrow_{\text{[ch]}} [a](\ell, p')$ in $\mathcal{T}^{\text{ch}'}$ and we are done. \square

Below we introduce a rule format for NRTSSs over signature $\Sigma_{\text{NRTS}}^{[\text{ch}]}$ that ensures that the composition $\mathfrak{T}^{[\text{ch}]} \circ \mathfrak{T}$ is the identity over the associated NRTS. To this end we adapt the notion of partial strict stratification from Definition 5.11.

Definition 7.10 (Partial strict stratification with atom-abstractions). Let $\mathcal{R}^{[\text{ch}]}$ be an NRTSS over a signature $\Sigma_{\text{NRTS}}^{[\text{ch}]}$. Let $S^{[\text{ch}]}$ be a partial map from ground nominal terms of sort $\text{pr} \times [\text{ch}]\text{ac}$ to ordinal numbers. $S^{[\text{ch}]}$ is a *partial strict stratification with atom-abstractions* of $\mathcal{R}^{[\text{ch}]}$ iff

- (i) $S^{[\text{ch}]}(\varphi(t), [a]\ell) \neq \perp$, for every rule in $\mathcal{R}^{[\text{ch}]}$ with conclusion $t \longrightarrow [a](\ell, t')$ such that $a \# \ell$ and for every ground substitution φ , and
- (ii) $S^{[\text{ch}]}(\varphi(u_i), [a_i]\ell_i) < S^{[\text{ch}]}(\varphi(t), [a]\ell)$ and $S^{[\text{ch}]}(\varphi(u_i), [a_i]\ell_i) \neq \perp$, for every rule RU in $\mathcal{R}^{[\text{ch}]}$ with conclusion $t \longrightarrow [a](\ell, t')$ such that $S^{[\text{ch}]}(\varphi(t), [a]\ell) \neq \perp$, for every premiss $u_i \longrightarrow [a_i](\ell_i, u'_i)$ of RU such that $a_i \# \ell_i$ and for every ground substitution φ .

We say a ground nominal term $(p, [a]\ell)$ of sort $\text{pr} \times [\text{ch}]\text{ac}$ has order $S^{[\text{ch}]}(p, [a]\ell)$.

The choice of $S^{[\text{ch}]}$ determines which rules will be considered by the rule format for NRTSSs defined below, which guarantees that for every transition $p \longrightarrow_{[\text{ch}]} [a](\ell, p')$ in the induced transition relation, $a \# \ell$ implies $a \# p'$. We will intend the map $S^{[\text{ch}]}$ to be such that the only rules whose source, abstracted atom and label of the conclusion have defined order are those that may take part in proof trees of transitions where the abstracted atom in its residual is fresh in its action.

Definition 7.11 (Binding-actions format). Let $\mathcal{R}^{[\text{ch}]}$ be an NRTSS over a signature $\Sigma_{\text{NRTS}}^{[\text{ch}]}$ and $S^{[\text{ch}]}$ be a partial strict stratification with atom-abstractions of $\mathcal{R}^{[\text{ch}]}$. Assume that all the actions occurring in the rules of $\mathcal{R}^{[\text{ch}]}$ are ground. Let

$$\frac{\{u_i \longrightarrow [a_i](\ell_i, u'_i) \mid i \in I\} \quad \nabla}{t \longrightarrow [a](\ell, t')} \text{RU}$$

be a rule in $\mathcal{R}^{[\text{ch}]}$. The rule RU is in *binding-actions format with respect to $S^{[\text{ch}]}$* (BA format with respect to $S^{[\text{ch}]}$ for short) iff either $a \in \text{supp}(\ell)$, or otherwise the following holds:

$$\nabla \cup \{a_i \# u'_i \mid i \in I \wedge a_i \# \ell_i\} \vdash \{a \# t'\}.$$

An NRTSS $\mathcal{R}^{[\text{ch}]}$ is in *BA format with respect to $S^{[\text{ch}]}$* iff all the rules in $\mathcal{R}^{[\text{ch}]}$ are in BA format with respect to $S^{[\text{ch}]}$.

Theorem 7.12. *Let $\mathcal{R}^{[\text{ch}]}$ be an NRTSS over a signature $\Sigma_{\text{NRTS}}^{[\text{ch}]}$ and $S^{[\text{ch}]}$ be a partial strict stratification with atom-abstractions of $\mathcal{R}^{[\text{ch}]}$. Assume that $\mathcal{R}^{[\text{ch}]}$ is in BA format with respect to $S^{[\text{ch}]}$ and let $\mathcal{T}^{[\text{ch}]}$ be the NRTS induced by $\mathcal{R}^{[\text{ch}]}$. Then, for every transition $p \longrightarrow_{[\text{ch}]} [a](\ell, p')$ in $\mathcal{T}^{[\text{ch}]}$, $a \# \ell$ implies that $a \# p'$.*

Proof. Let $NT[p] \longrightarrow_{[\text{ch}]} NT[[a](\ell, p')]$ be provable in $\mathcal{R}^{[\text{ch}]}$ and assume that the last rule used in the proof of $NT[p] \longrightarrow_{[\text{ch}]} NT[[a](\ell, p')]$ is

$$\frac{\{u_i \longrightarrow [a_i](\ell_i, u'_i) \mid i \in I\} \quad \{a_j \# v_j \mid j \in J\}}{t \longrightarrow [a](\ell, t')} \text{RU}$$

where I and J are disjoint. Therefore, for some ground substitution φ ,

- $NT[p] = NT[\varphi(t)]$ and $NT[[a](\ell, p')] = NT[[a](\ell, \varphi(t'))]$,
- the premisses $NT[\varphi(u_i)] \longrightarrow NT[[a_i](\ell_i, \varphi(u'_i))]$ with $i \in I$ are provable in \mathcal{R} , and

- the freshness relations $a_j \# NT[\varphi(v_j)]$ with $j \in J$ hold.

Recall that the actions ℓ and ℓ_i where $i \in I$ are ground, and thus φ is not applied to the actions in the items above. Since $NT[[a](\ell, \varphi(t'))] = NT[[a](\ell, p')]$, by Definition 3.11 and Remark 2.3, $NT[\varphi(t')] = NT[p']$.

We need to prove that $a \# \ell$ implies $a \# NT[p']$. If $a \in \text{supp}(\ell)$ then a is not fresh in ℓ and we are done. Otherwise, we know that $a \# \ell$ and we show that $a \# NT[\varphi(t')]$. Observe that $S^{[\text{ch}]}(\varphi(t), [a]\ell)$ is defined because $a \# \ell$. We proceed by induction on $S^{[\text{ch}]}(\varphi(t), [a]\ell)$.

Since rule RU is in BA format with respect to $S^{[\text{ch}]}$,

$$\{a_j \# v_j \mid j \in J\} \cup \{a_i \# u'_i \mid i \in I \wedge a_i \# \ell_i\} \vdash \{a \# t'\}.$$

We use Lemma 5.7 to obtain the implication

$$(1) \bigwedge_{j \in J} (a_j \# NT[\varphi(v_j)]) \wedge \bigwedge_{i \in I \wedge a_i \# \ell_i} (a_i \# NT[\varphi(u'_i)]) \implies a \# NT[\varphi(t')].$$

By the existence of the proof tree, all the $a_j \# NT[\varphi(v_j)]$ with $j \in J$ hold, and it suffices to prove $\bigwedge_{i \in I \wedge a_i \# \ell_i} (a_i \# NT[\varphi(u'_i)])$. The base case is when $S^{[\text{ch}]}(\varphi(t), [a]\ell)$ is minimal. By Definition 7.10 the rule RU has no premisses and the set I is empty, which makes $\bigwedge_{i \in I \wedge a_i \# \ell_i} (a_i \# NT[\varphi(u'_i)])$ trivially true and we are done. Now we assume that $S^{[\text{ch}]}(\varphi(t), [a]\ell)$ is not minimal. Condition (ii) in Definition 7.10 ensures that $S^{[\text{ch}]}(\varphi(t), [a]\ell) \neq \perp$ and $S^{[\text{ch}]}(\varphi(u_i), [a_i]\ell_i) < S^{[\text{ch}]}(\varphi(t), [a]\ell)$ for every $i \in I$ such that $a_i \# \ell_i$. Thus, we can apply the induction hypothesis to obtain $a_i \# NT[\varphi(u'_i)]$ for every $i \in I$ such that $a_i \# \ell_i$ and the theorem holds. \square

8. EXAMPLE OF APPLICATION OF THE BA-FORMAT TO THE π -CALCULUS

In this section we introduce the NRTSSs with residuals of abstraction sort $\mathcal{R}_E^{[\text{ch}]}$ and $\mathcal{R}_L^{[\text{ch}]}$, which respectively define our versions of the early and the late semantics of the π -calculus. For each of these semantics, we aim at showing that the induced NTRSs with and without residuals of abstraction sort represent the same model of computation, in the sense that $\mathcal{T}_E^{[\text{ch}]} = \mathfrak{T}^{[\text{ch}]}(\mathcal{T}_E, \text{bn}_E)$ and $(\mathcal{T}_E, \text{bn}_E) = \mathfrak{T}(\mathcal{T}_E^{[\text{ch}]})$ (and respectively for $(\mathcal{T}_L, \text{bn}_L)$ and $\mathcal{T}_L^{[\text{ch}]}$). Since we have already checked that both \mathcal{R}_E and \mathcal{R}_L are in ACR-format in Section 6, in order to establish that the models of computation are the same we need to check that

- (i) both $\mathcal{R}_E^{[\text{ch}]}$ and $\mathcal{R}_L^{[\text{ch}]}$ are in BA-format, and
- (ii) $\mathcal{T}_E^{[\text{ch}]} = \mathfrak{T}^{[\text{ch}]}(\mathcal{T}_E, \text{bn}_E)$, where $\mathcal{T}_E^{[\text{ch}]}$ is induced by $\mathcal{R}_E^{[\text{ch}]}$ (and respectively for $\mathcal{T}_L^{[\text{ch}]}$ and $(\mathcal{T}_L, \text{bn}_L)$).

The translations between these systems with and without residuals of abstraction sort are inverse to each other, and thus the two-way correspondence holds.

8.1. Early Semantics of the π -Calculus. Consider the NRTSS $\mathcal{R}_E^{[\text{ch}]}$ in Figure 3 for our version of the early semantics π -calculus [MPW92] over the residual signature $\Sigma_{\text{NTS}}^{[\text{ch}]}$ as defined on page 26 in Section 7.1, where F is the set of function symbols from Example 3.6. Omitted rules APARR, AECOMMR, AECLOSER and ASUMR are, respectively, the symmetric version of rules APARL, AECOMML, AECLOSEL and ASUML.

$$\begin{array}{c}
\frac{d \not\# (a, c, x\{b/c\})}{in(a, [b]x) \longrightarrow [d](inA(a, c), x\{b/c\})} \text{AEIN} \qquad \frac{c \not\# (a, b, x)}{out(a, b, x) \longrightarrow [c](outA(a, b), x)} \text{AOUT} \\
\\
\frac{a \not\# x}{tau(x) \longrightarrow [a](tauA, x)} \text{ATAU} \qquad \frac{x_1 \longrightarrow [a](\ell, y_1) \quad a \not\# x_2}{par(x_1, x_2) \longrightarrow [a](\ell, (par(y_1, x_2)))} \text{APARL} \\
\\
\frac{x_1 \longrightarrow [c](outA(a, b), y_1) \quad x_2 \longrightarrow [c](inA(a, b), y_2)}{par(x_1, x_2) \longrightarrow [c](tauA, (par(y_1, y_2)))} \text{AECOMML} \\
\\
\frac{x_1 \longrightarrow [b](boutA(a, b), y_1) \quad x_2 \longrightarrow [c](inA(a, b), y_2) \quad b \not\# x_2 \quad c \not\# y_1}{par(x_1, x_2) \longrightarrow [c](tauA, new([b](par(y_1, y_2))))} \text{AECLOSEL} \\
\\
\frac{x_1 \longrightarrow [a](\ell, y_1)}{sum(x_1, x_2) \longrightarrow [a](\ell, y_1)} \text{ASUML} \qquad \frac{x \longrightarrow [a](\ell, y) \quad a \not\# x}{rep(x) \longrightarrow [a](\ell, (par(y, rep(x))))} \text{AREP} \\
\\
\frac{x \longrightarrow [c](outA(a, b), y_1) \quad x \longrightarrow [c](inA(a, b), y_2) \quad c \not\# x}{rep(x) \longrightarrow [c](tauA, par(par(y_1, y_2), rep(x)))} \text{AEREP COMM} \\
\\
\frac{x \longrightarrow [b](boutA(a, b), y_1) \quad x \longrightarrow [c](inA(a, b), y_2) \quad b \not\# x \quad c \not\# (x, y_1)}{rep(x) \longrightarrow [c](tauA, par(new([b](par(y_1, y_2))), rep(x)))} \text{AEREP CLOSE} \\
\\
\frac{x \longrightarrow [c](outA(a, b), y) \quad b \not\# a}{new([b]x) \longrightarrow [b](boutA(a, b), y)} \text{AOPEN} \qquad \frac{x \longrightarrow [a](\ell, y) \quad b \not\# \ell}{new([b]x) \longrightarrow [a](\ell, new([b]y))} \text{ARES}
\end{array}$$

where $a, b, c, d \in \mathbb{A}_{\text{ch}}$ and ℓ is a ground action.

Figure 3: NRTSS for the early π -calculus with atom-abstractions in the residuals.

We use the rule format of Definition 7.11 to show that for every transition $p \longrightarrow_{[\text{ch}]} [a](\ell, p')$ in $\mathcal{T}_{\text{E}}^{[\text{ch}]}$, $a \# \ell$ implies $a \# p'$. We consider the following partial strict stratification with atom abstractions

$$\begin{aligned}
S^{[\text{ch}]}(in(a, [b]p), [d]inA(a, c)) &= 0 \\
S^{[\text{ch}]}(out(a, b, p), [c]outA(a, b)) &= 0 \\
S^{[\text{ch}]}(tau(p), [a]tauA) &= 0 \\
S^{[\text{ch}]}(par(p, q), [a]\ell) &= 1 + \max\{S^{[\text{ch}]}(p, [a]\ell), S^{[\text{ch}]}(q, [a]\ell), \\
&\quad S^{[\text{ch}]}(p, [c](outA(a, b))), \\
&\quad S^{[\text{ch}]}(q, [c](inA(a, b)))\} \\
S^{[\text{ch}]}(sum(p, q), [a]\ell) &= 1 + \max\{S^{[\text{ch}]}(p, [a]\ell), S^{[\text{ch}]}(q, [a]\ell)\} \\
S^{[\text{ch}]}(rep(p), [a]\ell) &= 1 + \max\{S^{[\text{ch}]}(p, [a]\ell), \\
&\quad S^{[\text{ch}]}(p, [c](outA(a, b))), \\
&\quad S^{[\text{ch}]}(p, [c](inA(a, b)))\} \\
S^{[\text{ch}]}(new([b]p), [a]\ell) &= 1 + S^{[\text{ch}]}(p, [a]\ell) \\
S^{[\text{ch}]}(p, t) &= \perp \quad \text{otherwise,}
\end{aligned}$$

where $a, b \in \mathbb{A}_{\text{ch}}$ and $\ell \in \{inA(a, b), outA(a, b), tauA \mid a, b \in \mathbb{A}_{\text{ch}}\}$.

$$\begin{array}{c}
\frac{b \not\# a}{in(a, [b]x) \longrightarrow [b](binA(a, b), x)} \text{ALIN} \\
\\
\frac{x_1 \longrightarrow [d](outA(a, b), y_1) \quad x_2 \longrightarrow [c](binA(a, c), y_2) \quad d \not\# y_2 \{b/c\}}{par(x_1, x_2) \longrightarrow [d](tauA, (par(y_1, y_2 \{b/c\})))} \text{ALCOMML} \\
\\
\frac{x_1 \longrightarrow [b](boutA(a, b), y_1) \quad x_2 \longrightarrow [b](binA(a, b), y_2) \quad c \not\# [b](y_1, y_2)}{par(x_1, x_2) \longrightarrow [c](tauA, new([b](par(y_1, y_2))))} \text{ALCLOSEL} \\
\\
\frac{x \longrightarrow [d](outA(a, b), y_1) \quad x \longrightarrow [c](binA(a, c), y_2) \quad d \not\# (x, y_2 \{b/c\})}{rep(x) \longrightarrow [d](tauA, par(par(y_1, y_2 \{b/c\}), rep(x)))} \text{ALREPCOMM} \\
\\
\frac{x \longrightarrow [b](boutA(a, b), y_1) \quad x \longrightarrow [b](binA(a, b), y_2) \quad c \not\# (x, [b](y_1, y_2))}{rep(x) \longrightarrow [c](tauA, par(new([b](par(y_1, y_2))), rep(x)))} \text{ALREPCLOSE} \\
\\
\text{where } a, b, c, d \in \mathbb{A}_{\text{ch}}.
\end{array}$$

Figure 4: NRTSS for the late π -calculus with atom-abstractions in the residuals.

not require renaming of channel names, since the channel through which communication takes place is abstracted in the residual of the input process.

Remark 8.2. Similar to rule LIN in Section 6.2, and as commented in Remark 6.1, rule ALIN ensures that the binding atom b is different from the communication channel. Our semantics allows for the transition

$$\begin{array}{l}
NT[\![par(out(a, a, null), in(a, [b](out(c, b, null))))]\!] \longrightarrow \\
NT[\![b](tauA, par(null, (out(c, b, null))\{a/b\}))]\!] = NT[\![b](tauA, par(null, out(c, a, null))]\!],
\end{array}$$

where $b\#(a, c)$, which models both $(\bar{a}a.0 \parallel a(b).\bar{c}b.0) \xrightarrow{\tau} (0 \parallel \bar{c}a.0)$ and $(\bar{a}a.0 \parallel a(a).\bar{c}a.0) \xrightarrow{\tau} (0 \parallel \bar{c}a.0)$ in the original late π -calculus. \blacksquare

We use the rule format of Definition 7.11 to show that for every transition $p \longrightarrow_{[\text{ch}]}$ $[a](\ell, p')$ in $\mathcal{T}_L^{[\text{ch}]}$, $a\#\ell$ implies $a\#p'$. We consider the following partial strict stratification with atom abstractions

$$\begin{array}{l}
S^{[\text{ch}]}(out(a, b, p), [c]outA(a, b)) = 0 \\
S^{[\text{ch}]}(tau(p), [a]tauA) = 0 \\
S^{[\text{ch}]}(par(p, q), [a]\ell) = 1 + \max\{S^{[\text{ch}]}(p, [a]\ell), S^{[\text{ch}]}(q, [a]\ell), \\
\quad S^{[\text{ch}]}(p, [c](outA(a, b))), \\
\quad S^{[\text{ch}]}(q, [c](in(a, b)))\} \\
S^{[\text{ch}]}(sum(p, q), [a]\ell) = 1 + \max\{S^{[\text{ch}]}(p, [a]\ell), S^{[\text{ch}]}(q, [a]\ell)\} \\
S^{[\text{ch}]}(rep(p), [a]\ell) = 1 + \max\{S^{[\text{ch}]}(p, [a]\ell), \\
\quad S^{[\text{ch}]}(p, [c](outA(a, b))), \\
\quad S^{[\text{ch}]}(p, [c](inA(a, b)))\} \\
S^{[\text{ch}]}(new([b]p), [a]\ell) = 1 + S^{[\text{ch}]}(p, [a]\ell) \\
S^{[\text{ch}]}(p, t) = \perp \text{ otherwise,}
\end{array}$$

where $a, b \in \mathbb{A}_{\text{ch}}$ and $\ell \in \{outA(a, b), tauA \mid a, b \in \mathbb{A}_{\text{ch}}\}$.

We check that $\mathcal{R}_L^{[\text{ch}]}$ is in BA-format with respect to $S^{[\text{ch}]}$ as follows. Consider a transition $p \longrightarrow_{[\text{ch}]} [b](\ell, p')$. The abstracted atom b is in the support of ℓ iff $\ell \in \{\text{bout}A(a, b), \text{bin}A(a, b) \mid a \in \mathbb{A}_{\text{ch}}\}$. By the definition of $S^{[\text{ch}]}$, we have that

$$S^{[\text{ch}]}(p, [b]\text{bout}A(a, b)) = S^{[\text{ch}]}(p, [b]\text{bin}A(a, b)) = \perp,$$

for each p and $a, b \in \mathbb{A}_{\text{ch}}$. Observe that $S^{[\text{ch}]}$ meets Definition 7.10(i) because a formula with either a residual $[b](\text{bout}A(a, b), p')$ or $[b](\text{bin}A(a, b), p')$ does not take part in any proof tree that proves a transition with a residual $[b](\ell, p'')$ such that $b \# \ell$. Therefore, the only rules in $\mathcal{R}_L^{[\text{ch}]}$ whose sources, abstracted atoms, and actions have defined order are AOUT, ATAU, ALREPCOMM, ALREPCLOSE, rules ALCOMML, ALCLOSEL and their symmetric versions, and the instance of rules APARL, ASUML, AREP, ARES where $t = [a]\ell$ and $a \# \ell$ (and the corresponding instance of the symmetric versions APARR and ASUMR). Observe that $S^{[\text{ch}]}$ meets Definition 7.10(ii) because for each rule whose conclusion has a residual $[b](\ell, p')$ such that $b \# \ell$, the order of the transition that unifies with its conclusion is always bigger than the order of the transitions that unify with those premisses $u_i \longrightarrow [a_i](\ell_i, p'_i)$ with $a_i \# \ell_i$.

We will limit ourselves to checking that rules ALCOMML and ALREPCOMM are in BA-format with respect to $S^{[\text{ch}]}$, since the checks for the other rules are similar to those presented earlier.

For rule ALCOMML, we need to check that

$$\{d \# y_2 \{b/c\}\} \cup \{d \# y_1\} \vdash \{d \# \text{par}(y_1, y_2 \{b/c\})\},$$

which trivially holds.

For rule ALREPCOMM, we need to check that

$$\{d \# (x, y_2 \{b/c\})\} \cup \{d \# y_1\} \vdash \{d \# \text{par}(\text{par}(y_1, y_2 \{b/c\}), \text{rep}(x))\},$$

which trivially holds.

Atoms a, b, c and d in $\mathcal{R}_L^{[\text{ch}]}$ range over \mathbb{A}_{ch} , and thus $\mathcal{R}_L^{[\text{ch}]}$ is in equivariant format. Since $\mathcal{R}_L^{[\text{ch}]}$ is in the BA-format with respect to $S^{[\text{ch}]}$, by Theorems 7.12 and 7.9, $\mathfrak{T}^{[\text{ch}]}(\mathfrak{T}(\mathcal{T}_L^{[\text{ch}]}) = \mathcal{T}_L^{[\text{ch}]}$. Since \mathcal{R}_L is in ACR-format and by Theorem 7.8, $\mathfrak{T}(\mathfrak{T}^{[\text{ch}]}(\mathcal{T}_L, \text{bn}_L)) = (\mathcal{T}_L, \text{bn}_L)$. Thus, in order to show that \mathcal{T}_L and $\mathcal{T}_L^{[\text{ch}]}$ represent the same model of computation, it suffices to check that $\mathfrak{T}^{[\text{ch}]}(\mathcal{T}_L, \text{bn}_L) = \mathcal{T}_L^{[\text{ch}]}$.

Lemma 8.3. *Let \mathcal{T}_L be the NRTS induced by \mathcal{R}_L in Figure 2 of Section 6.2, bn_L be its associated binding-names function, and $\mathcal{T}_L^{[\text{ch}]}$ be the NRTS induced by $\mathcal{R}_L^{[\text{ch}]}$. Then, $\mathfrak{T}^{[\text{ch}]}(\mathcal{T}_L, \text{bn}_L) = \mathcal{T}_L^{[\text{ch}]}$.*

The proof of Lemma 8.3 is in Appendix D. By Lemma 8.3, the NTS $(\mathcal{T}_L, \text{bn}_L)$ and the NRTS $\mathcal{T}_L^{[\text{ch}]}$ represent the same model of computation.

9. RELATED AND FUTURE WORK

The work we have presented in this paper stems from the Nominal SOS (NoSOS) framework [CMRG12] and from earlier proposals for nominal logic in [UPG04, CP07, GM09]. It is by no means the only approach studied so far in the literature that aims at a uniform treatment of binders and names in programming and specification languages. Other existing approaches that accommodate variables and binders within the SOS framework are those proposed by Fokkink and Verhoef in [FV98], by Middelburg in [Mid01, Mid03], by Bernstein

in [Ber98], by Ziegler, Miller and Palamidessi in [ZMP06] and by Fiore and Staton in [FS09] (originally, by Fiore and Turi in [FT01]). The aim of all of the above-mentioned frameworks is to establish sufficient syntactic conditions guaranteeing the validity of a semantic result (congruence in the case of [Ber98, Mid01, ZMP06, FS09] and conservativity in the case of [FV98, Mid03]). In addition, Gabbay and Mathijssen present a nominal axiomatisation of the λ -calculus in [GM10]. None of these approaches addresses equivariance nor the property of alpha-conversion of residuals in [PBE⁺15]. The proposal that is closest to ours is the one in [FS09]. In that paper, Fiore and Staton presented a GSOS-like rule format for name-passing process calculi, where operational specifications corresponds to theories in nominal logic, and show that a natural notion of bisimilarity is preserved by operations specified in that format.

Nominal techniques have been implemented also in programming languages. This is the case of FreshML [SPG03] where Shinwell, Pitts and Gabbay extend ML with constructs for defining and working with data involving binding operations. In particular, FreshML adds the keyword *fresh* to ML in order to generate a fresh new name in an expression inside the code.

In [GH08], Gabbay and Hofmann study the category **Ren**, which is a generalisation of **Nom** in which the renamings play the role of the permutations in **Nom**. In this paper we use the action of renaming to define the interpretation of moderated terms into nominal terms, which are akin to the nominal algebraic datatypes of Pitts [Pit13] and live in **Nom**.

Our moderated terms $t\{\rho\}$ are reminiscent of the terms in calculi with explicit substitutions in the tradition of the $\lambda\sigma$ calculus of Abadi *et al.* [ACCL91]. In fact, a replacement a/b is an instance of an explicit substitution that substitutes atom b with atom a , and both the renamings and the explicit substitutions are composable, as the equality

$$NT\llbracket t\{\rho_1\}\{\rho_2\} \rrbracket = NT\llbracket (t\{\rho_1\})\{\rho_2\} \rrbracket = NT\llbracket t\{\rho_1; \rho_2\} \rrbracket$$

in this paper and the equality $a[s][t] = a[s \circ t]$ in [ACCL91] witness. However, renamings and explicit substitutions are very different objects: a renaming is a semantic object in $\mathbb{A} \rightarrow_{\text{fs}} \mathbb{A}$ that maps atoms to atoms, while an explicit substitution is a syntactic object that specifies how to substitute placeholders with arbitrary terms.

In [MT05], Miller and Tiu use an approach to higher-order abstract syntax that is called *λ -tree syntax*, which allows one to encode both the static and dynamic structure of abstractions. Their logic $FO\lambda^{\Delta\nabla}$ uses the *new quantifier* $\nabla a.\phi$, whose meaning is that atom a is fresh in the formula ϕ that lies within the scope of the quantifier. The logic $FO\lambda^{\Delta\nabla}$ is equipped with a sequent calculus that deals with the issues concerning name-binding operations. This sequent calculus uses renamings as a primitive operation.

In the NTSs of Parrow *et al.* [PBE⁺15], scope opening is modelled by the property of alpha-conversion of residuals. We have explored an alternative in which scope opening is encoded by a *residual abstraction* of sort $[\text{ch}](\text{ac} \times \text{pr})$. Similarly, Parrow has recently proposed an alternative definition of his nominal transition systems in which scope opening is represented as an alpha-equivalence condition encoded by explicit name abstraction [Par18]. We have developed mutual, one-to-one translations between the NTSs and the NRTSs with residual abstractions. The generality of our NRTSs also allows for neat specifications of our versions of the early and the late semantics of the π -calculus.

Our current proposal aims at following closely the spirit of the seminal work on nominal techniques by Gabbay, Pitts and their co-workers, and paves the way for the development of results on rule formats akin to those presented in the aforementioned references. Amongst

those, we consider the development of a congruence format for the notion of bisimilarity presented in [PBE⁺15, Def. 2] to be of particular interest. The logical characterisation of bisimilarity given in [PBE⁺15] opens the intriguing possibility of employing the divide-and-congruence approach from [FvGdW06] to obtain an elegant congruence format and a compositional proof system for the logic.

We also plan to lift the congruence formats guaranteeing various *bounded nondeterminism* properties (including determinism) to the setting of NRTSS [ABI⁺12, AFGI17, FV03]. In order to increase the applicability of those results it would also be useful to extend the results in this paper to a setting with state predicates. Such predicates are an important component in the theory and application of NTSs to some advanced calculi that include them, *e.g.*, active substitutions and fusions.

Developing rule formats for SOS is always the result of a trade-off between ease of application and generality. Our rule format for alpha-conversion of residuals in Definition 5.12 is no exception and might be generalised in various ways. Together with substitution γ in conditions (ii) and (iii), a substitution γ_i could be used in condition (i) for each premiss, in order to discard variables that are used in the target of the premisses but are dropped in the target of the rule. Moreover, the restrictions on atom a in conditions (i) and (ii) could be relaxed by considering a subset of premisses in the conditions.

Finally, we are developing rule formats for properties other than alpha-conversion of residuals. One such rule format ensures a property for NRTSs to the effect that, in each transition, the support of a state is a subset of the support of its derivative. Another such format would ensure the converse property. That is, in each transition, the support of the derivative is a subset of the support of the state. In [Par18], Parrow considers properties analogous to the previous one in the setting of NTSs.

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REFERENCES

- [ABI⁺12] L. Aceto, A. Birgisson, A. Ingólfssdóttir, M. R. Mousavi, and M. A. Reniers. Rule formats for determinism and idempotence. *Science of Computer Programming*, 77(7–8):889–907, 2012.
- [ACCL91] M. Abadi, L. Cardelli, P.-L. Curien, and J.-J. Lévy. Explicit substitutions. *Journal of Functional Programming*, 1(4):375–416, 1991.
- [ACG⁺] L. Aceto, M. Cimini, M. J. Gabbay, A. Ingólfssdóttir, M. R. Mousavi, and M. A. Reniers. Nominal structural operational semantics. In preparation.
- [AFGI17] L. Aceto, I. Fábregas, A. García-Pérez, and A. Ingólfssdóttir. A unified rule format for bounded nondeterminism in SOS with terms as labels. *Journal of Logical and Algebraic Methods in Programming*, 92:64–86, 2017.
- [AFGP⁺17] L. Aceto, I. Fábregas, A. García-Pérez, A. Ingólfssdóttir, and Y. Ortega-Mallén. Rule formats for nominal process calculi. In *28th International Conference on Concurrency Theory*, volume 85 of *LIPICs*, pages 10:1–10:16. Schloss Dagstuhl, 2017.
- [AFV01] L. Aceto, W. Fokkink, and C. Verhoef. Structural operational semantics. In *Handbook of Process Algebra*, chapter 3, pages 197–292. Elsevier, 2001.

- [Ber98] K. L. Bernstein. A congruence theorem for structured operational semantics of higher-order languages. In *13th Annual IEEE Symposium on Logic in Computer Science*, pages 153–164. IEEE Computer Society, 1998.
- [BP09] J. Bengtson and J. Parrow. Formalising the pi-calculus using nominal logic. *Logical Methods in Computer Science*, 5(2), 2009.
- [BS00] S. Burris and H. P. Sankappanavar. *A Course in Universal Algebra: The Millennium Edition*. Springer Verlag, 2000.
- [CMRG12] M. Cimini, M. R. Mousavi, M. A. Reniers, and M. J. Gabbay. Nominal SOS. *Electronic Notes in Theoretical Computer Science*, 286:103–116, 2012.
- [CP07] R. Clouston and A. Pitts. Nominal equational logic. *Electronic Notes in Theoretical Computer Science*, 172:223–257, 2007.
- [FG07] M. Fernández and M. J. Gabbay. Nominal rewriting. *Information and Computation*, 205(6):917–965, 2007.
- [FS09] M. P. Fiore and S. Staton. A congruence rule format for name-passing process calculi. *Information and Computation*, 207(2):209–236, 2009.
- [FT01] M. P. Fiore and D. Turi. Semantics of name and value passing. In *16th Annual IEEE Symposium on Logic in Computer Science*, pages 93–104. IEEE Computer Society, 2001.
- [FV98] W. Fokkink and C. Verhoef. A conservative look at operational semantics with variable binding. *Information and Computation*, 146(1):24–54, 1998.
- [FV03] W. Fokkink and T. D. Vu. Structural operational semantics and bounded nondeterminism. *Acta Informatica*, 39(6-7):501–516, 2003.
- [FvGdW06] W. Fokkink, R. J. van Glabbeek, and P. de Wind. Compositionality of Hennessy-Milner logic by structural operational semantics. *Theoretical Computer Science*, 354(3):421–440, 2006.
- [GH08] M. J. Gabbay and M. Hofmann. Nominal renaming sets. In *Logic for Programming, Artificial Intelligence, and Reasoning, 15th International Conference*, volume 5330 of *Lecture Notes in Computer Science*, pages 158–173. Springer, 2008.
- [GM09] M. J. Gabbay and A. Mathijssen. Nominal (universal) algebra: Equational logic with names and binding. *Journal of Logic and Computation*, 19(6):1455–1508, 2009.
- [GM10] M. J. Gabbay and A. Mathijssen. A nominal axiomatization of the lambda calculus. *Journal of Logic and Computation*, 20(2):501–531, 2010.
- [GP02] M. J. Gabbay and A. Pitts. A new approach to abstract syntax with variable binding. *Formal Aspects of Computing*, 13(3–5):341–363, 2002.
- [GTWW77] J. A. Goguen, J. W. Thatcher, E. G. Wagner, and J. B. Wright. Initial algebra semantics and continuous algebras. *Journal of the ACM*, 24(1):68–95, 1977.
- [GV92] J. F. Groote and F. W. Vaandrager. Structured operational semantics and bisimulation as a congruence. *Information and Computation*, 100(2):202–260, 1992.
- [Lam68] J. Lambek. A fixed point theorem for complete categories. *Mathematische Zeitung*, 103:151–161, 1968.
- [Mid01] C. A. Middelburg. Variable binding operators in transition system specifications. *Journal of Logic and Algebraic Programming*, 47(1):15–45, 2001.
- [Mid03] C. A. Middelburg. An alternative formulation of operational conservativity with binding terms. *Journal of Logic and Algebraic Programming*, 55(1-2):1–19, 2003.
- [MPW92] R. Milner, J. Parrow, and D. Walker. A calculus of mobile processes (Parts I and II). *Information and Computation*, 100(1):1–77, 1992.
- [MRG07] M. R. Mousavi, M. A. Reniers, and J. F. Groote. SOS formats and meta-theory: 20 years after. *Theoretical Computer Science*, 373(3):238–272, 2007.
- [MT05] D. Miller and A. Tiu. A proof theory for generic judgments. *ACM Transactions on Computational Logic*, 6(4):749–783, 2005.
- [Par18] J. Parrow. Nominal transition systems defined by name abstraction, 2018. Personal communication.
- [PBE⁺15] J. Parrow, J. Borgström, L.-H. Eriksson, R. Gutkovas, and T. Weber. Modal logics for nominal transition systems. In *26th International Conference on Concurrency Theory*, volume 42 of *LIPICs*, pages 198–211. Schloss Dagstuhl, 2015.
- [Pit11] A. Pitts. Nominal sets, 2011. Invited course on Nominal sets and their applications at the Midlands Graduate School (http://www.cs.nott.ac.uk/~pszvc/mgs/MGS2011_nominal_sets.pdf).

- [Pit13] A. Pitts. *Nominal Sets: Names and Symmetry in Computer Science*. Cambridge University Press, 2013.
- [Pit16] A. Pitts. Nominal techniques. *ACM SIGLOG News*, 3(1):57–72, 2016.
- [Plo04] G. D. Plotkin. A structural approach to operational semantics. *Journal of Logic and Algebraic Programming*, 60-61:17–139, 2004.
- [PWBE17] J. Parrow, T. Weber, J. Borgström, and L.-H. Eriksson. Weak nominal modal logic. In *Formal Techniques for Distributed Objects, Components, and Systems - 37th IFIP WG 6.1 International Conference, FORTE 2017*, volume 10321 of *Lecture Notes in Computer Science*, pages 179–193. Springer, 2017.
- [San96] D. Sangiorgi. pi-Calculus, internal mobility, and agent-passing calculi. *Theoretical Computer Science*, 167(1&2):235–274, 1996.
- [SPG03] M. R. Shinwell, A. Pitts, and M. J. Gabbay. FreshML: programming with binders made simple. *SIGPLAN Notices*, 38(9):263–274, 2003.
- [SW01] D. Sangiorgi and D. Walker. *The π -calculus — A Theory of Mobile Processes*. Cambridge University Press, 2001.
- [UPG04] C. Urban, A. Pitts, and M. J. Gabbay. Nominal unification. *Theoretical Computer Science*, 323(1–3):473–497, 2004.
- [ZMP06] A. Ziegler, D. Miller, and C. Palamidessi. A congruence format for name-passing calculi. *Electronic Notes in Theoretical Computer Science*, 156(1):169–189, 2006.

APPENDIX A. PRELIMINARIES

Proof of Proposition 2.5. First, we show that the set $A = \{a, (\rho a) \mid \rho a \neq a\}$ supports ρ . This requires us to show that for all permutation π that leaves each element in A invariant, and every atom b , $b\{\pi^{-1}; \rho; \pi\} = b\{\rho\}$. We distinguish the following cases. If $b \in A$ the result holds since $b\{\pi^{-1}; \rho; \pi\} = b\{\rho; \pi\} = (\rho b)\{\pi\} = \rho c = b\{\rho\}$. If $b \notin A$ we have, $b\{\pi^{-1}; \rho; \pi\} = (\pi^{-1} \cdot b)\{\rho; \pi\}$. Now, by definition of the set A and the permutation π , it must be the case that $(\pi^{-1} \cdot b) \notin A$. Otherwise, we would have that there exists $c \in A$ such that $\pi \cdot c = b \neq c$, which results in a contradiction. Hence, $(\pi^{-1} \cdot b)\{\rho; \pi\} = (\pi^{-1} \cdot b)\{\pi\} = b = b\{\rho\}$ since $b \notin A$, and we are done.

Finally, we prove that A is the smallest set supporting ρ . Assume towards a contradiction that there exists $A' \subset A$ that supports ρ . Without loss of generality, assume that there exists an atom $a \in A$ which is not in A' . Let π be the permutation that leaves each element in A' invariant and such that $\pi \cdot a = b$ and $\pi \cdot b = a$ for some $b \notin A$. We have,

$$a\{\pi^{-1}; \rho; \pi\} = (\pi^{-1} \cdot a)\{\rho; \pi\} = b\{\rho; \pi\} = (\rho b)\{\pi\} = \pi \cdot b = a,$$

which results in a contradiction because $a\{\rho\} = \rho a \neq a$ since $a \in A$. Therefore, the set A is the smallest set that supports ρ , which ends the proof. \square

APPENDIX B. NOMINAL TERMS

Proof of Lemma 3.4. By induction on the size of t . If $t = a$, then

$$\pi \cdot (a\{\rho\}) = a\{\rho; \pi\} = a\{\pi; \pi^{-1}; \rho; \pi\} = (\pi a)\{\pi^{-1}; \rho; \pi\} = (\pi a)\{\pi \cdot \rho\}.$$

If $t = t'\{\rho'\}$, then

$$\begin{aligned} \pi \cdot ((t'\{\rho'\})\{\rho\}) &= \pi \cdot (t'\{\rho'; \rho\}) = (\pi \cdot t')\{\pi \cdot (\rho'; \rho)\} \\ &= (\pi \cdot t')\{\pi^{-1}; \rho'; \rho; \pi\} = (\pi \cdot t')\{\pi^{-1}; \rho'; \pi; \pi^{-1}; \rho; \pi\} \\ &= (\pi \cdot t')\{\pi \cdot \rho'; \pi \cdot \rho\} = ((\pi \cdot t')\{\pi \cdot \rho'\})\{\pi \cdot \rho\} = (\pi \cdot (t'\{\rho'\}))\{\pi \cdot \rho\}. \end{aligned}$$

If $t = [a]t'$, then

$$\pi \cdot (([a]t')\{\rho\}) = \pi \cdot ([\rho a](t'\{\rho\})) = [\pi \cdot (\rho a)](\pi \cdot (t'\{\rho\})) = [\pi \cdot (a\{\rho\})](\pi \cdot (t'\{\rho\})).$$

By the induction hypothesis,

$$\begin{aligned} [(\pi a)\{\pi \cdot \rho\}](\pi \cdot (t'\{\rho\})) &= [(\pi \cdot \rho)(\pi a)](\pi \cdot (t'\{\rho\})) \\ &= ([\pi a](\pi \cdot t'))\{\pi \cdot \rho\} = (\pi \cdot ([a]t'))\{\pi \cdot \rho\}. \end{aligned}$$

The remaining cases are straightforward by the induction hypothesis. \square

We lift the action of renaming $A\{\rho\}$ to sets of atoms A in the obvious way. Let t be a raw term. Lemma 11.1 in [GH08] states that the support of $t\{\rho\}$ is a subset of $(\text{supp}(t))\{\rho\}$.

Lemma B.1. *Let A be a set of atoms. Then, $A\{\rho\} \subseteq A \cup \text{supp}(\rho)$.*

Proof. Consider atom $a \in A\{\rho\}$. If $a \in A$, then the result trivially follows. Otherwise, $a \in A\{\rho\} \setminus A$. We claim that $a \in \text{supp}(\rho)$. Indeed, $a = \rho b$ for some $b \in A$. Since a is not in A , it follows that $a \neq b$, and therefore $a \in \text{supp}(\rho)$ by Proposition 2.5. \square

Proof of Lemma 3.5. By induction on the size of t . The only non-trivial case is $t = t'\{\rho\}$. By definition, $\text{fa}(t'\{\rho\}) = \text{fa}(t'\{\rho\})$. By the induction hypothesis, $\text{fa}(t'\{\rho\}) \subseteq \text{supp}(t'\{\rho\})$. By Lemma B.1, $(\text{supp}(t'))\{\rho\} \subseteq \text{supp}(t') \cup \text{supp}(\rho) = \text{supp}(t'\{\rho\})$ and the claim follows by Lemma 11.1 in [GH08]. \square

APPENDIX C. RULE FORMATS FOR NRTSSs

The proofs of some of the lemmas to come use induction on the size of a freshness environment. We let the *size of a freshness environment* ∇ be the sum of the sizes of the raw terms in its assertions.

Proof of Lemma 5.6. The proof goes along the same lines as the proof of Lemma 11 in [FG07]. Since the simplification rules do not overlap, there are no critical pairs and confluence holds trivially. Each simplification rule decreases the size of some assertion in the environment, except for the rule $\{a \# ([b]t)\{\rho\}\} \cup \nabla \Longrightarrow \{a \# [\rho b](t\{\rho\})\} \cup \nabla$. However, the environments that pattern-match with that rule simplify as follows

$$\{a \# ([b]t)\{\rho\}\} \cup \nabla \Longrightarrow \{a \# [\rho b](t\{\rho\})\} \cup \nabla \Longrightarrow \begin{cases} \{a \# t\{\rho\}\} \cup \nabla & \text{if } a \neq \rho b \\ \nabla & \text{otherwise} \end{cases}$$

and thus the assertion $a \# ([b]t)\{\rho\}$ either decreases its size or vanishes after the two consecutive simplification steps above. Since the reduction relation is confluent, the environments of the shape above can always be reduced in this fashion. Therefore the reduction relation is terminating. \square

Proof of Lemma 5.7. We first prove that $\varphi(\nabla)$ holds iff $\varphi(\langle \nabla \rangle nf)$ holds. We proceed by induction on the size of ∇ . If $\nabla = \langle \nabla \rangle nf$ then the result follows trivially. Without loss of generality we let $\nabla = \{a \# t\} \cup \nabla'$ and consider the cases where some simplification rule is applicable. If the assertion $a \# t$ vanishes after the simplification step, then t is either an atom $b \# a$, or $t = [a]t'$, and in both cases $a \# NT[\varphi(t)]$ for every ground substitution φ . If the assertion $a \# t$ simplifies to a set of assertions $\{a \# t_i \mid i \in I\}$, then we show that $a \# NT[\varphi(t)]$ iff $\bigwedge_{i \in I} (a \# NT[\varphi(t_i)])$, for every ground substitution φ .

For illustration, we provide the proof for the cases

$$\nabla = \{a \# (t' \{\rho_1\})\{\rho\}\} \cup \nabla' \text{ and } \nabla = \{a \# ([b]t')\{\rho\}\} \cup \nabla'.$$

The rest of the cases are straightforward by the induction hypothesis.

If $\nabla = \{a \# (t' \{\rho_1\})\{\rho\}\} \cup \nabla'$, then

$$\begin{aligned} NT[\varphi((t' \{\rho_1\})\{\rho\})] &= NT[(\varphi(t') \{\rho_1\})\{\rho\}] = NT[(\varphi(t') \{\rho_1\})\{\rho\}] \\ &= NT[\varphi(t') \{\rho_1; \rho\}] = NT[\varphi(t' \{\rho_1; \rho\})], \end{aligned}$$

and therefore $a \# NT[\varphi((t' \{\rho_1\})\{\rho\})]$ iff $a \# NT[\varphi(t' \{\rho_1; \rho\})]$, and the lemma follows by the induction hypothesis since the assertion $a \# (t' \{\rho_1\})\{\rho\}$ simplifies to $a \# t' \{\rho_1; \rho\}$.

If $\nabla = \{a \# ([b]t')\{\rho\}\} \cup \nabla'$, then we consider the following cases. If $a = \rho b$ then the environment ∇ simplifies to ∇' in two steps and the lemma follows by the induction hypothesis. If $a \neq \rho b$, then

$$\begin{aligned} NT[\varphi(([b]t')\{\rho\})] &= NT[(\varphi([b]t'))\{\rho\}] = NT[(\varphi([b]t'))\{\rho\}] \\ &= NT[\langle \rho b \rangle (\varphi(t')\{\rho\})] = \langle \rho b \rangle (NT[\varphi(t')\{\rho\}]) = \langle \rho b \rangle (NT[\varphi(t')\{\rho\}]) \\ &= NT[\langle \rho b \rangle (\varphi(t')\{\rho\})] = NT[\varphi(\langle \rho b \rangle (t' \{\rho\}))], \end{aligned}$$

and therefore $a \# NT[\varphi(([b]t')\{\rho\})]$ iff $a \# NT[\varphi(\langle \rho b \rangle (t' \{\rho\}))]$, and the lemma follows by the induction hypothesis since the assertion $a \# ([b]t')\{\rho\}$ simplifies to $a \# \langle \rho b \rangle (t' \{\rho\})$.

Now we show that $\varphi(\langle \nabla \rangle nf)$ holds iff $\varphi(\langle \tilde{\nabla} \rangle nf)$ holds. If $\langle \tilde{\nabla} \rangle nf$ contains assertion $a \# x\{\iota\}$, then $\langle \nabla \rangle nf$ contains either $a \# x$ or $a \# x\{\iota\}$, or both. The result follows trivially since $NT[\varphi(x\{\iota\})] = NT[(\varphi(x))\{\iota\}] = NT[\varphi(x)]\{\iota\} = NT[\varphi(x)]$. \square

Remark C.1. Notice that if ∇ in the lemma above is inconsistent, then the lemma follows trivially since no substitution φ exists such that $\varphi(\nabla)$ holds. This is so because $\langle \nabla \rangle nf$ contains some freshness assertion of the form $a \not\# a$, and neither the conjunction of the freshness relations denoted by $\varphi(\nabla)$ holds, nor the conjunction of the ones denoted by $\varphi(\langle \nabla \rangle nf)$ does. \blacksquare

Proof of Lemma 5.9. Assume $\varphi(\nabla)$ holds. By Lemma 5.7, $\langle \widetilde{\nabla} \rangle nf$ holds. Without loss of generality we assume $\langle \widetilde{\nabla}' \rangle nf = \{a_i \not\# b_i \mid i \in I\} \cup \{a_j \not\# x_j \{\rho_j\} \mid j \in J\}$. We prove that the conjunction

$$\bigwedge_{i \in I} (a_i \# b_i) \wedge \bigwedge_{j \in J} (a_j \# NT[(\varphi(x_j)) \{\rho_j\}])$$

holds. Since $\nabla \vdash \nabla'$, each of the assertions $a_i \not\# b_i$ is contained in $\langle \widetilde{\nabla} \rangle nf$ and $\bigwedge_{i \in I} (a_i \# b_i)$ holds by assumptions. For each assertion $a_j \not\# x_j \{\rho_j\}$, we know that there exist a permutation π and an assertion $b_j \not\# x_j \{\rho'_j\}$ in $\langle \widetilde{\nabla} \rangle nf$ such that $\pi a_j = b_j$ and $\rho_j; \pi = \rho'_j$. We have

$$\begin{aligned} \pi^{-1} \cdot NT[(\varphi(x_j)) \{\rho'_j\}] &= \pi^{-1} \cdot (NT[\varphi(x_j)] \{\rho'_j\}) = NT[\varphi(x_j)] \{\rho'_j; \pi^{-1}\} \\ &= NT[\varphi(x_j)] \{\rho_j; \pi; \pi^{-1}\} = NT[\varphi(x_j)] \{\rho_j\} = NT[\varphi(x_j) \{\rho_j\}], \end{aligned}$$

and thus by equivariance of the freshness relation

$$b_j \# NT[\varphi(x_j) \{\rho'_j\}] \quad \text{iff} \quad a_j \# NT[(\varphi(x_j)) \{\rho_j\}].$$

Therefore, the conjunction

$$\bigwedge_{j \in J} (a_j \# NT[(\varphi(x_j)) \{\rho_j\}])$$

holds and we are done. \square

Remark C.2. Notice that if ∇ in the lemma above is inconsistent, then the lemma follows trivially since no substitution φ exists such that the antecedent $\varphi(\nabla)$ of the implication holds. \blacksquare

Lemma C.3. Let \mathcal{R} be an NRTSS and RU be a rule

$$\frac{\{u_i \longrightarrow (\ell_i, u'_i) \mid i \in I\} \quad \{a_j \not\# v_j \mid j \in J\}}{t \longrightarrow (\ell, t')} \text{RU}$$

in \mathcal{R} . Let D be the set of variables that occur in the source t of RU but do not occur in the premisses $u_i \longrightarrow (\ell_i, u'_i)$ with $i \in I$, the environment ∇ or the target t' of the rule. For every $\gamma : D \rightarrow \mathbb{T}(\Sigma_{\text{NTS}})$ and every substitution φ , a proof tree for transition $NT[\varphi(\gamma(t))] \longrightarrow NT[\varphi(\ell, t')]$ that uses RU as last rule exists iff a proof tree for transition $NT[\varphi(t)] \longrightarrow NT[\varphi(\ell, t')]$ that uses RU as last rule exists.

Proof. Since the domain D of γ contains variables neither in the premisses nor in the target of rule RU, $NT[\varphi(\gamma(t))] \longrightarrow NT[\varphi(\ell, t')]$ is equal to $NT[\varphi(\gamma(t))] \longrightarrow NT[\varphi(\gamma(\ell, t'))]$. Consider a proof tree of transition $NT[\varphi(\gamma(t))] \longrightarrow NT[\varphi(\gamma(\ell, t'))]$ that uses RU as last rule, if it exists. Since none of the variables occurring in the premisses and in the environment are in the domain of γ , $\varphi(w) = \varphi(\gamma(w))$ for each w in $\{u_i, u'_i \mid i \in I\} \cup \{v_j \mid j \in J\}$. Hence, the sub-trees that prove the premisses $NT[\varphi(\gamma(u_i))] \longrightarrow NT[\varphi(\gamma(\ell_i, u'_i))]$, with $i \in I$, also prove transitions $NT[\varphi(u_i)] \longrightarrow NT[\varphi(\ell_i, u'_i)]$; and also all $a_j \# \varphi(\gamma(v_j))$ and $a_j \# \varphi(v_j)$ with $j \in J$ hold. Therefore, in the case they exist, the proof trees for transitions $NT[\varphi(\gamma(t))] \longrightarrow NT[\varphi(\ell, t')]$ and $NT[\varphi(t)] \longrightarrow NT[\varphi(\ell, t')]$ that use RU as last rule share

the same sub-trees for the premisses, the freshness assertions hold, and the only difference between the proof trees is the root node. \square

Proof of Theorem 5.13. Since \mathcal{R} is in equivariant format, \mathcal{R} induces an NRTS with an equivariant transition relation (Theorem 5.3). We prove that this transition relation also enjoys alpha-conversion of residuals. That is, if $NT[p] \longrightarrow NT[(\ell, p')]$ is provable in \mathcal{R} and $b \in \text{bn}(\ell)$, then $NT[p] \longrightarrow (ab) \cdot NT[(\ell, p')]$ for every atom a that is fresh in $NT[(\ell, p')]$.

Assume that the last rule used in the proof of $NT[p] \longrightarrow NT[(\ell, p')]$ is

$$\frac{\{u_i \longrightarrow (\ell_i, u'_i) \mid i \in I\} \quad \{a_j \not\# v_j \mid j \in J\}}{t \longrightarrow (\ell, t')} \text{RU}$$

and therefore that for some ground substitution φ

- $NT[p] = NT[\varphi(t)]$ and $NT[(\ell, p')] = NT[\varphi(\ell, t')]$,
- the premisses $NT[\varphi(u_i)] \longrightarrow NT[\varphi(\ell_i, u'_i)]$ with $i \in I$ are provable in \mathcal{R} , and
- the freshness relations $a_j \# NT[\varphi(v_j)]$ with $j \in J$ hold.

Observe that $S(\varphi(t), \ell)$ is defined because $\text{bn}(\ell)$ is non-empty. Thus, as rule RU is in ACR format, there is a ground substitution γ whose domain is contained in the set of variables D occurring in t but nowhere else in the rule, meeting conditions (i)-(iii) in Definition 5.12 for each atom a in the set $\mathbb{A} \setminus \{c \in \text{supp}(t) \mid \langle \{c \not\# t\} \rangle \text{nf} = \emptyset\}$ and each atom b in $\text{bn}(\ell)$.

Let us fix any $b \in \text{bn}(\ell)$ and any atom a that is fresh in $NT[(\ell, p')]$, we first show that transition $NT[\varphi(t)] \longrightarrow (ab) \cdot NT[\varphi(\ell, t')]$ is provable under the assumption that $a \# NT[\varphi(\gamma(t))]$ and $b \# NT[\varphi(\gamma(t))]$. We will then show that those assumptions hold.

By Lemma C.3 we know that a proof tree of $NT[\varphi(\gamma(t))] \longrightarrow NT[\varphi(\ell, t')]$ that uses RU as last rule exists, and since \mathcal{R} is in equivariant format, a proof tree of $(ab) \cdot NT[\varphi(\gamma(t))] \longrightarrow (ab) \cdot NT[\varphi(\ell, t')]$ that uses $(ab) \cdot \text{RU}$ as last rule exists. By our assumptions, we have that $(ab) \cdot NT[\varphi(\gamma(t))] = NT[\varphi(\gamma(t))]$ and therefore $NT[\varphi(\gamma(t))] \longrightarrow (ab) \cdot NT[\varphi(\ell, t')]$. Again by Lemma C.3, a proof tree of $NT[\varphi(t)] \longrightarrow (ab) \cdot NT[\varphi(\ell, t')]$ that uses $(ab) \cdot \text{RU}$ as last rule exists, and the theorem holds.

In the remainder we prove the assumptions $a \# NT[\varphi(\gamma(t))]$ and $b \# NT[\varphi(\gamma(t))]$.

We prove first $a \# NT[\varphi(\gamma(t))]$. We distinguish two cases:

- If $a \in \{c \in \text{supp}(t) \mid \langle \{c \not\# t\} \rangle \text{nf} = \emptyset\}$ then $\vdash \{a \not\# t\}$ and by Lemmas 5.7 and 5.9, $a \# NT[(\overline{\varphi \circ \gamma})(t)] = NT[\varphi(\gamma(t))]$ holds.
- Otherwise, since \mathcal{R} is in ACR format with respect to S ,
 - $\{a \not\# t'\} \cup \nabla \vdash \{a \not\# u'_i \mid i \in I\}$ and
 - $\{a \not\# t'\} \cup \nabla \cup \{a \not\# u_i \mid i \in I\} \vdash \{a \not\# \gamma(t)\}$.

We use Lemmas 5.7 and 5.9 to obtain the implications

- (1) $(a \# NT[\varphi(t')] \wedge \bigwedge_{j \in J} (a_j \# NT[\varphi(v_j)])) \implies \bigwedge_{i \in I} (a \# NT[\varphi(u'_i)])$ and
- (2) $(a \# NT[\varphi(t')] \wedge \bigwedge_{j \in J} (a_j \# NT[\varphi(v_j)]) \wedge \bigwedge_{i \in I} (a \# NT[\varphi(u_i)])) \implies a \# NT[\varphi(\gamma(t))]$.

Since the set D does not contain any variable occurring in t' it follows that $\varphi(\gamma(t')) = \varphi(t')$. Now we prove the statement $a \# NT[\varphi(t')] \implies a \# NT[\varphi(\gamma(t))]$ by induction on $S(\varphi(\gamma(t)), \ell)$ (this suffices to show the claim since $a \# NT[\varphi(t')]$ holds by assumption). The base case is when $S(\varphi(\gamma(t)), \ell)$ is minimal. By Definition 5.11 the rule RU has no premisses and the set I is empty, which makes $\bigwedge_{i \in I} (a \# NT[\varphi(u_i)])$ trivially true and what we were proving holds by (2). Now assume that $S(\varphi(\gamma(t)), \ell)$ is not minimal. Since all $a_j \# NT[\varphi(v_j)]$ with $j \in J$ hold, all $a \# NT[\varphi(u'_i)]$ with $i \in I$ hold by (1). Condition (ii) in Definition 5.11 ensures that $S(\varphi(u_i), \ell_i) \neq \perp$ and $S(\varphi(\gamma(u_i)), \ell_i) < S(\varphi(\gamma(t)), \ell)$. Thus, we can apply the induction hypothesis to obtain the implications $a \# NT[\varphi(u'_i)] \implies$

$a\#NT[\varphi(\gamma(u_i))]$, with $i \in I$. For each $i \in I$, since the variables that occur in u_i are not in $\text{dom}(\gamma)$, we have that $a\#NT[\varphi(u'_i)] \implies a\#NT[\varphi(u_i)]$. And now by (2) we know that $a\#NT[\varphi(\gamma(t))]$ which is what was to be shown.

To finish the proof we prove the statement $b\#NT[\varphi(\gamma(t))]$ by induction on $S(\varphi(\gamma(t)), \ell)$. Since \mathcal{R} is in ACR format with respect to S we have that $\nabla \cup \{b \not\# u_i \mid i \in I \wedge b \in \text{bn}(\ell_i)\} \vdash \{b \not\# \gamma(t)\}$. We use Lemmas 5.7 and 5.9 to obtain the implication

$$(3) \left(\bigwedge_{j \in J} (a_j \# NT[\varphi(v_j)]) \wedge \bigwedge_{i \in I \wedge b \in \text{bn}(\ell_i)} (b \# NT[\varphi(u_i)]) \right) \implies b \# NT[\varphi(\gamma(t))].$$

The base case for the induction is when $S(\varphi(\gamma(t)), \ell)$ is minimal. By Definition 5.11 the rule RU has no premisses and the set I is empty, so that $\{i \mid i \in I \wedge b \in \text{bn}(\ell_i)\}$ is empty as well, in which case $\bigwedge_{i \in I \wedge b \in \text{bn}(\ell_i)} (b \# NT[\varphi(u_i)])$ is trivially true and $b \# NT[\varphi(\gamma(t))]$ holds. Now assume that $S(\varphi(\gamma(t)), \ell)$ is not minimal. Condition (ii) in Definition 5.11 ensures that $S(\varphi(u_i), \ell_i) \neq \perp$ and $S(\varphi(\gamma(u_i)), \ell_i) < S(\varphi(\gamma(t)), \ell)$ for every $i \in I$. Thus, we can apply the induction hypothesis to obtain $b \# NT[\varphi(\gamma(u_i))]$ for each $i \in I$ such that $b \in \text{bn}(\ell_i)$. For each $i \in I$, since the set D does not contain any variable occurring in u_i we know that $\varphi(\gamma(u_i)) = \varphi(u_i)$. In particular this holds for $i \in I$ such that $b \in \text{bn}(\ell_i)$. By (3) we know that $b \# NT[\varphi(\gamma(t))]$ and we are done. \square

APPENDIX D. EXAMPLE OF APPLICATION OF THE BA-FORMAT TO THE π -CALCULUS

Proof of Lemma 8.1. For every transition $p \longrightarrow (\ell, p')$, we have to show that $p \longrightarrow (\ell, p')$ has a proof tree in \mathcal{R}_E iff $p \longrightarrow_{[\text{ch}]} [a](\ell, p')$ has a proof tree in $\mathcal{R}_E^{[\text{ch}]}$, where either $\text{bn}_E(\ell) = \{a\}$, or $\text{bn}_E(\ell) = \emptyset$ and $a \# (\ell, p')$. We proceed by induction on the height of the proof tree of $p \longrightarrow (\ell, p')$. We prove the “if” direction first.

The base case is when $p \longrightarrow (\ell, p')$ is provable by any of the axioms EIN, OUT or TAU. In all of these cases, $\text{bn}_E(\ell) = \emptyset$. The transition $p \longrightarrow_{[\text{ch}]} [a](\ell, p')$ is provable by axioms AEIN, AOUT or ATAU respectively, where we let $a \# (\ell, p')$.

For the inductive step, we distinguish the following sub-cases depending on the last rule used in the proof of $p \longrightarrow (\ell, p')$:

- The last rule used is EPARL or EPARR. Without loss of generality, we assume that the last rule used is EPARL and thus $p = \text{par}(p_1, p_2)$ and $p' = \text{par}(p'_1, p_2)$ where $p_1 \longrightarrow (\ell, p'_1)$ is provable in \mathcal{R}_E where $\ell \notin \{\text{bout}A(a, b) \mid a, b \in \mathbb{A}_{\text{ch}}\}$ and therefore $\text{bn}_E(\ell) = \emptyset$. By the induction hypothesis, $p_1 \longrightarrow_{[\text{ch}]} [a](\ell, p'_1)$ is provable in $\mathcal{R}_E^{[\text{ch}]}$, where $a \# (\ell, p'_1)$. Without loss of generality, we let $a \# p_2$. The transition $p \longrightarrow_{[\text{ch}]} [a](\ell, \text{par}(p'_1, p_2))$ is provable by rule APARL.
- The last rule used is EPARRESL or EPARRESR. Without loss of generality, we assume that the last rule used is EPARRESL and thus $p = \text{par}(p_1, p_2)$, where $\text{par}(p_1, p_2) \longrightarrow (\text{bout}A(a, b), \text{par}(p'_1, p_2))$ is provable in \mathcal{R}_E and $b \# p_2$. By the induction hypothesis, $p_1 \longrightarrow_{[\text{ch}]} [b](\text{bout}A(a, b), p'_1)$ is provable in $\mathcal{R}_E^{[\text{ch}]}$, and

$$\text{par}(p_1, p_2) \longrightarrow_{[\text{ch}]} [b](\text{bout}A(a, b), \text{par}(p'_1, p_2))$$

is provable by rule APARL since $b \# p_2$.

- The last rule used is ECOMML or ECOMMR, and thus $\ell = \text{tau}A$. Without loss of generality, we assume that the last rule used is ECOMML and thus $p = \text{par}(p_1, p_2)$ and $p' = \text{par}(p'_1, p'_2)$, where $p_1 \longrightarrow (\text{out}A(a, b), p'_1)$ and $p_2 \longrightarrow (\text{in}A(a, b), p'_2)$ are provable in \mathcal{R}_E . By the induction hypothesis, $p_1 \longrightarrow_{[\text{ch}]} [c](\text{out}A(a, b), p'_1)$ and $p_2 \longrightarrow_{[\text{ch}]} [d](\text{in}A(a, b), p'_2)$ are

provable in $\mathcal{R}_E^{[ch]}$, and since $\text{bn}_E(\text{out}A(a, b)) = \text{bn}_E(\text{in}A(a, b)) = \emptyset$, therefore $c\#p'_1$ and $d\#p'_2$. Without loss of generality we assume that $c = d$. Therefore, $p \rightarrow_{[ch]} [c](\ell, p')$ is provable in $\mathcal{R}_E^{[ch]}$ by rule AECOMML where $\text{bn}_E(\text{tau}A) = \emptyset$ and $c\#(\text{tau}A, \text{par}(p'_1, p'_2))$.

- The last rule used is any of ECLOSEL, ECLOSER, EREPCOMM or EREPCLOSE. Consider that the last rule used is ECLOSEL, and thus $\ell = \text{tau}A$, $p = \text{par}(p_1, p_2)$ and $p' = \text{new}([b](\text{par}(p'_1, p'_2)))$, where $p_1 \rightarrow (\text{bout}A(a, b), p'_1)$ and $p_2 \rightarrow (\text{in}A(a, b), p'_2)$ are provable in \mathcal{R}_E and $b\#p_2$. By the induction hypothesis, $p_1 \rightarrow_{[ch]} [b](\text{bout}A(a, b), p'_1)$ and $p_2 \rightarrow_{[ch]} [c](\text{in}A(a, b), p'_2)$ are provable in $\mathcal{R}_E^{[ch]}$ where $c\#p'_2$. Without loss of generality, we let $c\#p'_1$. Therefore $p \rightarrow_{[ch]} [c](\text{tau}A, p')$ is provable in $\mathcal{R}_E^{[ch]}$ by rule AECLOSEL where $\text{bn}_E(\text{tau}A) = \emptyset$ and $c\#\text{new}([b](\text{par}(p'_1, p'_2)))$. The cases where the last rule used is any of ECLOSER, EREPCOMM or EREPCLOSE are analogous.
- The last rule used is REP, thus $p = \text{rep}(p_1) \rightarrow (\ell, \text{par}(p'_1, \text{rep}(p_1)))$ where $p_1 \rightarrow (\ell, p'_1)$ is provable in \mathcal{R}_E . By the induction hypothesis, $p_1 \rightarrow [a](\ell, p'_1)$ is provable in $\mathcal{R}_E^{[ch]}$. If $\text{bn}_E(\ell) = \emptyset$, then $a\#(\ell, p'_1)$. Without loss of generality, we let $a\#p_1$, and therefore $\text{rep}(p_1) \rightarrow [a](\ell, \text{par}(p'_1, \text{rep}(p_1)))$ is provable in $\mathcal{R}_E^{[ch]}$ by rule AREP and $\text{bn}_E(\ell) = \emptyset$ and $a\#(\ell, \text{par}(p'_1, \text{rep}(p_1)))$.

If $\text{bn}_E(\ell) = \{a\}$, then $a\#p_1$ since \mathcal{R}_E is in ACR-format (see page 19), which guarantees that the binding name a in transition $p_1 \rightarrow (\ell, p'_1)$ is fresh in its source p_1 . Therefore, transition $p = \text{rep}(p_1) \rightarrow_{[ch]} [a](\ell, \text{par}(p'_1, \text{rep}(p_1)))$ is provable in $\mathcal{R}_E^{[ch]}$ by rule AREP.

- The last rule used is OPEN. We have that $\ell = \text{bout}A(a, b)$, $\text{bn}_E(\text{bout}A(a, b)) = \{b\}$ and $b\#a$. Transition $p \rightarrow_{[ch]} [b](\text{bout}A(a, b), p')$ is provable by applying the induction hypothesis and by rule AOPEN.
- The last rule used is any of SUML, SUMR or RES. These cases are analogous to the case EPARL.

The “only if” direction can be checked similarly, except for the observation that for a transition $p \rightarrow_{[ch]} [a](\ell, p')$ provable in $\mathcal{R}_E^{[ch]}$ and where the last rule used is APARL, we distinguish the cases where $a\#\ell$ and where $a \in \text{supp}(\ell)$, and use the induction hypothesis together with rule EPARL or rule EPARRSL, respectively, to prove that $p \rightarrow (\ell, p')$ is provable in \mathcal{R}_E . \square

Proof of Lemma 8.3. For every transition $p \rightarrow (\ell, p')$, we have to show that $p \rightarrow (\ell, p')$ has a proof tree in \mathcal{R}_L iff $p \rightarrow_{[ch]} [a](\ell, p')$ has a proof tree in $\mathcal{R}_L^{[ch]}$, where either $\text{bn}_L(\ell) = \{a\}$, or $\text{bn}_L(\ell) = \emptyset$ and $a\#(\ell, p')$. We proceed by induction on the height of the proof tree of $p \rightarrow (\ell, p')$. We prove the “if” direction first.

The base case is when $p \rightarrow (\ell, p')$ is provable by any of the axioms LIN, OUT or TAU. In the case where $p \rightarrow (\ell, p')$ is provable by axiom LIN, we know that $\ell = \text{bin}A(a, b)$, $\text{bn}_L(\text{bin}A(a, b)) = \{b\}$ and $b\#a$. Therefore $p \rightarrow_{[ch]} [b](\text{bin}A(a, b), p')$ is provable by axiom ALIN. In the other two cases are already proven in Proof of Lemma 8.1.

For the inductive step, we distinguish the following sub-cases depending on the last rule used in the proof of $p \rightarrow (\ell, p')$:

- The last rule used is LPARL or LPARR. Without loss of generality, we assume that the last rule used is LPARL and thus $p = \text{par}(p_1, p_2)$ and $p' = \text{par}(p'_1, p_2)$ where $p_1 \rightarrow (\ell, p'_1)$ is provable in \mathcal{R}_L where $\ell \notin \{\text{bout}A(a, b), \text{bin}A(a, b) \mid a, b \in \mathbb{A}_{ch}\}$ and therefore $\text{bn}_L(\ell) = \emptyset$. By the induction hypothesis, $p_1 \rightarrow_{[ch]} [a](\ell, p'_1)$ is provable in $\mathcal{R}_L^{[ch]}$, where $a\#(\ell, p'_1)$.

Without loss of generality, we let $a \# p_2$. The transition $p \rightarrow_{[\text{ch}]} [a](\ell, \text{par}(p'_1, p_2))$ is provable by rule APARL.

- The last rule used is LPARRESL or LPARRESR. Without loss of generality, we assume that the last rule used is LPARRESL and thus $p = \text{par}(p_1, p_2)$, where $\text{par}(p_1, p_2) \rightarrow (\ell, \text{par}(p'_1, p_2))$ with $\text{bn}_L(\ell) = \{b\}$ is provable in \mathcal{R}_L and $b \# p_2$. By the induction hypothesis, $p_1 \rightarrow_{[\text{ch}]} [b](\ell, p'_1)$ is provable in $\mathcal{R}_L^{[\text{ch}]}$, and

$$\text{par}(p_1, p_2) \rightarrow_{[\text{ch}]} [b](\ell, \text{par}(p'_1, p_2))$$

is provable by rule APARL since $b \# p_2$.

- The last rule used is LCOMML or LCOMMR, and thus $\ell = \text{tau}A$. Without loss of generality, we assume that the last rule used is LCOMML and thus $p = \text{par}(p_1, p_2)$ and $p' = \text{par}(p'_1, p'_2)$, where $p_1 \rightarrow (\text{out}A(a, b), p'_1)$ and $p_2 \rightarrow (\text{bin}A(a, c), p'_2)$ are provable in \mathcal{R}_L . By the induction hypothesis, $p_1 \rightarrow_{[\text{ch}]} [d](\text{out}A(a, b), p'_1)$ and $p_2 \rightarrow_{[\text{ch}]} [c](\text{bin}A(a, c), p'_2)$ are provable in $\mathcal{R}_L^{[\text{ch}]}$, and since $\text{bn}_L(\text{out}A(a, b)) = \emptyset$, therefore $d \# p'_1$. Without loss of generality we assume that $d \# p'_2 \{b/c\}$. Therefore, $p \rightarrow_{[\text{ch}]} [d](\text{tau}A, p')$ is provable in $\mathcal{R}_L^{[\text{ch}]}$ by rule ALCOMML where $\text{bn}_L(\text{tau}A) = \emptyset$ and $d \# (\text{tau}A, \text{par}(p'_1, p'_2 \{b/c\}))$.
- The last rule used is any of LCLOSEL, LCLOSER, LREPCOMM or LREPCLOSE. Consider that the last rule used is LCLOSEL, and thus $\ell = \text{tau}A$, $p = \text{par}(p_1, p_2)$ and $p' = \text{new}([b](\text{par}(p'_1, p'_2)))$, where $p_1 \rightarrow (\text{bout}A(a, b), p'_1)$ and $p_2 \rightarrow (\text{bin}A(a, b), p'_2)$ are provable in \mathcal{R}_L . By the induction hypothesis, $p_1 \rightarrow_{[\text{ch}]} [b](\text{bout}A(a, b), p'_1)$ and $p_2 \rightarrow_{[\text{ch}]} [b](\text{bin}A(a, b), p'_2)$ are provable in $\mathcal{R}_L^{[\text{ch}]}$. Without loss of generality, we let $c \# [b](p'_1, p'_2)$. Therefore $p \rightarrow_{[\text{ch}]} [c](\text{tau}A, p')$ is provable in $\mathcal{R}_L^{[\text{ch}]}$ by rule AECLOSEL where $\text{bn}_L(\text{tau}A) = \emptyset$ and $c \# \text{new}([b](\text{par}(p'_1, p'_2)))$. The cases where the last rule used is any of LCLOSER, LREPCOMM or LREPCLOSE are analogous.

The cases where the last rule used is any of REP, OPEN, SUML, SUMR and RES are already proven in Proof of Lemma 8.1.

The “only if” direction can be checked similarly, except for the observation that for a transition $p \rightarrow_{[\text{ch}]} [a](\ell, p')$ provable in $\mathcal{R}_L^{[\text{ch}]}$ and where the last rule used is APARL, we distinguish the cases where $a \# \ell$ and where $a \in \text{supp}(\ell)$, and use the induction hypothesis together with rule LPARL or rule LPARRESL, respectively, to prove that $p \rightarrow (\ell, p')$ is provable in \mathcal{R}_L . \square