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# UNDECIDABILITY OF A WEAK VERSION OF MSO+U

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Abstract. We prove the undecidability of MSO on $\omega$-words extended with the second-order predicate $\mathtt{U}_1(X)$ which says that the distance between consecutive positions in a set $X \subseteq \mathbb{N}$ is unbounded. This is achieved by showing that adding $\mathtt{U}_1$ to MSO gives a logic with the same expressive power as MSO+U, a logic on $\omega$-words with undecidable satisfiability. As a corollary, we prove that MSO on $\omega$-words becomes undecidable if allowing to quantify over sets of positions that are ultimately periodic, *i.e.*, sets $X$ such that for some positive integer $p$, ultimately either both or none of positions $x$ and $x+p$ belong to $X$.

## 1. Introduction

This paper is about monadic second-order logic (MSO) on $\omega$-words. Büchi's famous theorem says that given an MSO sentence describing a set of $\omega$-words over some alphabet, one can decide if the sentence is true in at least one $\omega$-word [8]. Büchi's theorem along with its proof using automata techniques have been the inspiration for a large number of decidability results for variants of MSO, including Rabin's theorem on the decidability of MSO on infinite trees [12].

By now it is quite well understood that MSO is a maximal decidable logic over finite words. One formalisation of this can be found in [10] (Theorem 9), which shows that the class of regular languages is the maximal class which: (a) has decidable satisfiability; and (b) is closed under Boolean operations and images under rational relations. The general idea behind the result in [10] is that a non-regular language has a Myhill-Nerode equivalence relation of infinite index, and this together with closure under Boolean operations and

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images under rational relations can be used to generate counters, Turing machines, and then arbitrary languages in the arithmetic hierarchy. However, for languages of infinite words, the situation is different as logic over omega-words can talk about asymptotic properties, and getting a counter from asymptotic properties can be much harder. One of the themes developed in the wake of Büchi's result is the question: what can be added to MSO on $\omega$-words so that the logic remains decidable? One direction, studied already in the sixties, has been to extend the logic with predicates such as *"position $x$ is a square number"* or *"position $x$ is a prime number"*. See [2, 9] and the references therein for a discussion on this line of research. Another direction, which is the one taken in this paper, is to study quantifiers which bind set variables. These new quantifiers may for instance talk about the number of sets satisfying a formula, hence being midway between the universal and the existential quantifiers, like the quantifier *"there exists uncountably many sets"* [1] (which *a posteriori* does not extend the expressivity of MSO on finitely branching trees). Another way is to consider quantifiers that talk about the asymptotic behaviors of infinite sets. This was already the direction followed in [3], where the authors define *Asymptotic* MSO, a logic which talks about the asymptotic behaviors of sequences of integers in a topological flavour. Another example is *the quantifier* $\mathtt{U}$ which was introduced in [5] and that says that some formula $\varphi(X)$ holds for arbitrarily large finite set $X$:

$$\mathtt{U}X\varphi(X) : \text{"for all } k \in \mathbb{N},\ \varphi(X) \text{ is true for some finite set } X \text{ of size at least } k\text{"}.$$

However, in [7] it was shown that MSO+$\mathtt{U}$, namely MSO extended with the quantifier $\mathtt{U}$, has undecidable satisfiability; see [4] for a discussion on this logic.

In this work, we study a, *a priori*, weaker version of MSO+$\mathtt{U}$, the logic MSO+$\mathtt{U}_1$, namely MSO extended with *the second-order predicate* $\mathtt{U}_1$ defined by:

$$\mathtt{U}_1(X) : \text{"for all } k \in \mathbb{N}, \text{ there exist two consecutive positions of } X \text{ at distance at least } k\text{"}.$$

**Example 1.1.**

- The set $X = \{10n \mid n \in \mathbb{N}\}$ does not satisfy $\mathtt{U}_1$, as for every $n$, $10(n+1) - 10n = 10$, so there are no two consecutive positions at distance at least 11.
- The set $X = \{2^n \mid n \in \mathbb{N}\}$ does satisfy $\mathtt{U}_1$, as for every $k$, $2^{k+1} - 2^k > k$, and $2^{k+1}$ and $2^k$ are consecutive in $X$.
- The set $X = \{10m+n \mid \text{n is the } m^{\text{th}} \text{ digit of } \pi\}$ does not satisfy $\mathtt{U}_1$, the difference between two consecutive elements being at most 19, as there is always an element between $10m$ and $10m+9$.

**Example 1.2** [6]**.** Consider the language of $\omega$-words of the form $a^{n_1}ba^{n_2}b\cdots$ such that $\{n_1, n_2, \ldots\}$ is unbounded. This language can be defined in MSO+$\mathtt{U}$ by a formula saying that there are factors of consecutive $a$'s of arbitrarily large size. It can also be defined in MSO+$\mathtt{U}_1$ saying that the set of the positions labeled by $b$ satisfies the predicate $\mathtt{U}_1$.

It is easy to see that the predicate $\mathtt{U}_1$ can be defined in the logic MSO+$\mathtt{U}$: a set $X$ satisfies $\mathtt{U}_1(X)$ if and only if there exist intervals (finite connected sets of positions) of arbitrarily large size which are disjoint with $X$. Therefore, the logic MSO+$\mathtt{U}_1$ can be seen as a fragment of MSO+$\mathtt{U}$. Is this fragment proper? The main contribution of this paper is showing that actually the two logics are the same:

**Theorem 1.3.** *The logics* MSO+$\mathtt{U}$ *and* MSO+$\mathtt{U}_1$ *define the same languages of $\omega$-words, and translations both ways are effective.*

We believe that MSO+$\mathtt{U}_1$ can be reduced to many extensions of MSO, in a simpler way than reducing MSO+$\mathtt{U}$.

As an example, we consider a quantifier which talks about ultimately periodic sets. A set of positions $X \subseteq \mathbb{N}$ is called *ultimately periodic* if there is some period $p \in \mathbb{N}$ such that for sufficiently large positions $x \in \mathbb{N}$, either both or none of $x$ and $x + p$ belong to $X$. For example, the set $\{10n \mid n \in \mathbb{N}\}$ is periodic (with period 10), but the set $\{10m + n \mid \text{n is the } m^{\text{th}} \text{ digit of } \pi\}$ is not periodic. We consider the logic MSO+$\mathtt{P}$, *i.e.*, MSO augmented with *the quantifier* $\mathtt{P}$ that ranges over ultimately periodic sets:

$$\mathtt{P}X\varphi(X) : \text{“the formula } \varphi(X) \text{ is true for all ultimately periodic sets } X\text{”}.$$

Though quantifier $\mathtt{P}$ extends the expressivity of MSO, it is *a priori* not clear whether it has decidable or undecidable satisfiability. However, using Theorem 1.3, we obtain the following result, which answers an open question raised in [6].

**Theorem 1.4.** *Satisfiability over $\omega$-words is undecidable for* MSO+$\mathtt{P}$.

Another example is the logic MSO +probability as defined in [11]. This line of thought is not developed in the present article, but we can make the following reasoning. A set X satisfies $\mathtt{U}_1(X)$ if and only if there exists an infinite set $Y$ of blocks in $X$ with the following property: there is nonzero probability of choosing a subset $Z \subseteq \mathbb{N}$ such that in every block from $Y$, there is at least one element of $Z$. Therefore, MSO +probability is more expressive than MSO+$\mathtt{U}_1$.

**Outline of the paper.** The rest of the paper is mainly devoted to the proof of Theorem 1.3. In Section 2 we introduce an intermediate logic MSO+$\mathtt{U}_2$. We first prove that MSO+$\mathtt{U}$ and MSO+$\mathtt{U}_2$ are effectively equivalent (Section 2.1, Lemma 2.2). Then, we prove that MSO+$\mathtt{U}_2$ and MSO+$\mathtt{U}_1$ are also effectively equivalent (Section 2.2, Lemma 2.3), assuming a certain property, namely Lemma 2.4. The proof of this latter lemma is the subject of Section 3. Finally, in Section 4, we discuss the expressive power of MSO+$\mathtt{P}$ with respect to MSO+$\mathtt{U}$. We prove Theorem 1.4 and we show that the property “ultimately periodic” can be expressed in MSO+$\mathtt{U}$ if allowing a certain encoding.

## 2. How MSO+$\mathtt{U}_1$ talks about vector sequences

In this section, we introduce a new predicate on pairs of sets of positions $\mathtt{U}_2(R, I)$ and consider the logic MSO extended with this predicate: MSO+$\mathtt{U}_2$. We first prove in Section 2.1, that MSO+$\mathtt{U}$ has the same expressive power as MSO+$\mathtt{U}_2$. Then, we show in Section 2.2, that MSO+$\mathtt{U}_2$ is itself as expressive as MSO+$\mathtt{U}_1$, assuming a certain property, namely Lemma 2.4. The proof of this lemma is given in Section 3.

### 2.1. From quantifier $\mathtt{U}$ to predicate $\mathtt{U}_2$.

We say that a sequence of natural numbers is *unbounded* if arbitrarily large numbers occur in it. For two disjoint sets of positions $R, I \subseteq \mathbb{N}$ with $R$ infinite, we define a sequence of numbers as in Figure 1: the $i$-th element of this sequence is the number of elements from $I$ between the $i$-th and the $(i+1)$-th elements from $R$. In particular, elements of $I$ in positions smaller than all the positions from $R$ are not relevant.

We now define the binary predicate $\mathtt{U}_2$:

$$\mathtt{U}_2(R, I) : \text{ “the sequence of numbers encoded by } R, I \text{ is defined and unbounded”.}$$

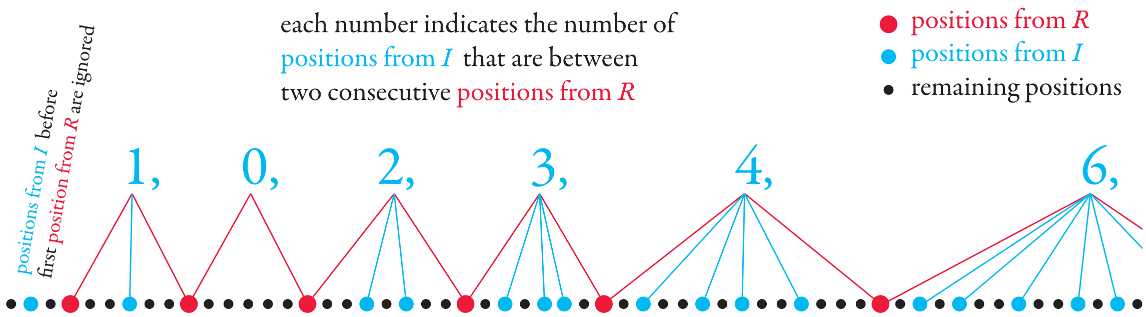



Figure 1: Two sets of positions $R, I \subseteq \mathbb{N}$ and the sequence in $\mathbb{N}^\omega$ that they define. The sequence is only defined when $R$ and $I$ are disjoint, and $R$ is infinite.

The difference between the predicate $\mathtt{U}_1$, which has one free set variable, and the predicate $\mathtt{U}_2$, which has two free set variables, is that the latter is able to ignore some positions. It is illustrated by the following example, which can be easily defined in MSO+$\mathtt{U}_2$.

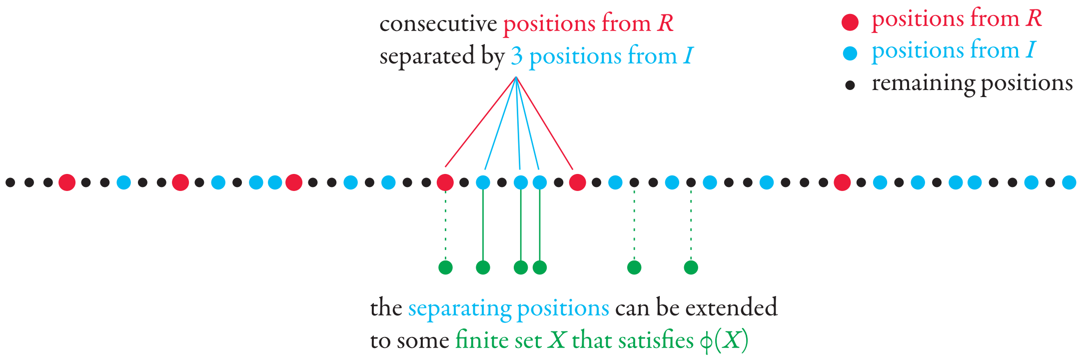
**Example 2.1.** Let $L$ be the union of the languages of $\omega$-words of the form $(ac^*)^{n_1}b(ac^*)^{n_2}b\cdots$ where $\{n_1, n_2, \ldots\}$ is unbounded. It can be defined in MSO+$\mathtt{U}_2$ in the following way: an $\omega$-word belongs to $L$ if and only if it belongs to $\left((ac^*)^*b\right)^\omega$ and the sequence of numbers encoded by $R = \{\text{set of positions labeled by } b\}$ and $I = \{\text{set of positions labeled by } a\}$ is unbounded. The language also admits a simple definition in MSO+$\mathtt{U}$: there exist arbitrarily large finite sets of positions labeled by $a$ such that no two positions from the set are separated by a $b$. It is however not straightforward to define it in the logic MSO+$\mathtt{U}_1$, since in every $\omega$-word, the distance between two successive $b$'s separated by at least one $a$ can be increased by adding occurrences of $c$'s, without changing the membership of the word to the language.

The following lemma follows essentially from the same argument as in [6, Lemma 5.5].

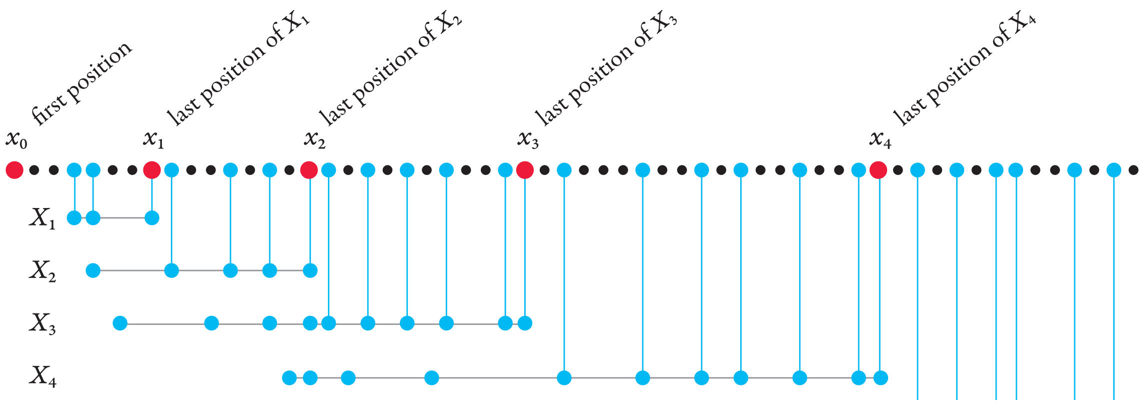
**Lemma 2.2.** *The logics* MSO+$\mathtt{U}$ *and* MSO+$\mathtt{U}_2$ *define the same languages of $\omega$-words, and translations both ways are effective.*

*Proof.* The predicate $\mathtt{U}_2$ is easily seen to be expressible in MSO+$\mathtt{U}$: a sequence of numbers encoded by $R$ and $I$ is unbounded if the subsets of elements of $I$ between two consecutive elements of $R$ are of arbitrarily large size, which is expressible in MSO+$\mathtt{U}$. For the converse implication, we use the following observation. A formula $\mathtt{U}X\varphi(X)$ is true if and only if the following condition holds:

($\star$)**:** There exist two sets $R, I \subseteq \mathbb{N}$ which satisfy $\mathtt{U}_2(R, I)$ such that for every two consecutive positions $r, s \in R$, there exists some finite $X$ satisfying $\varphi(X)$ which contains all positions of $I$ between $r$ and $s$. Here is a picture of property ($\star$):




Condition $(\star)$ is clearly expressible in MSO with the predicate $\mathtt{U}_2$. Hence the lemma will follow once we prove the equivalence: $\mathtt{U}X\varphi(X)$ if and only if $(\star)$. The right-to-left implication is easy to see. For the converse implication, we do the following construction. We define a sequence of positions $0 = x_0 < x_1 < \ldots$ as follows by induction. Define $x_0$ to be the first position, *i.e.*, the number 0. Suppose that $x_n$ has already been defined. By the assumption $\mathtt{U}X\varphi(X)$, there exists a set $X_{n+1}$ which satisfies $\varphi$ and which contains at least $n$ positions after $x_n$. Define $x_{n+1}$ to be the last position of $X_{n+1}$. This process is illustrated in the following picture:




Define $R$ to be all the positions $x_0, x_1, \ldots$ in the sequence thus obtained, and define $I$ to be the set of positions $x$ such that $x_n < x < x_{n+1}$ and $x \in X_{n+1}$ holds for some $n$. By construction, the sets $I$ and $R$ thus obtained will satisfy $\mathtt{U}_2$. □

2.2. **From predicate $\mathtt{U}_2$ to predicate $\mathtt{U}_1$.** Lemma 2.2 above states that MSO+U and MSO+$\mathtt{U}_2$ have the same expressive power, thus giving a first step towards the proof of Theorem 1.3. The second step, which is the key point of our result, is the following lemma, which states that MSO+$\mathtt{U}_2$ is as expressive as MSO+$\mathtt{U}_1$.

**Lemma 2.3.** *The logics* MSO+$\mathtt{U}_2$ *and* MSO+$\mathtt{U}_1$ *define the same languages of $\omega$-words, and translations both ways are effective.*

Note that one direction is straightforward, since $\mathtt{U}_1(X)$ holds if and only if $\mathtt{U}_2(X, \mathbb{N} \setminus X)$ holds. Hence, it remains to show that the predicate $\mathtt{U}_2$ can be defined by a formula of the logic MSO+$\mathtt{U}_1$ with two free set variables.

In our proof, we use terminology and techniques about sequences of vectors of natural numbers that were used in the undecidability proof for MSO+U [7]. A *vector sequence* is defined to be an element of $(\mathbb{N}^*)^\omega$, *i.e.*, a sequence of possibly empty tuples of natural numbers. For a vector sequence $\mathbf{f} \in (\mathbb{N}^*)^\omega$, we define its *dimension*, denoted by $\mathtt{dim}(\mathbf{f}) \in \mathbb{N}^\omega$, as being the number sequence of the dimensions of the vectors: the $i$-th element in $\mathtt{dim}(\mathbf{f})$ is *the dimension of the $i$-th vector* in $\mathbf{f}$, *i.e.*, the number of coordinates in the $i$-th tuple in $\mathbf{f}$. We say that a vector sequence $\mathbf{f}$ *tends towards infinity*, denoted by $\mathbf{f} \to \infty$, if every natural number appears in finitely many vectors from $\mathbf{f}$.

Recall Figure 1, which showed how to encode a number sequence using two sets of positions $R, I \subseteq \mathbb{N}$. We now show that the same two sets of positions can be used to define a vector sequence. As it was the case in Figure 1, we assume that $R$ is infinite and disjoint from $I$. Under these assumptions, we write $\mathbf{f}_{R,I} \in (\mathbb{N}^*)^\omega$ for the vector sequence defined according to the description from Figure 2, *i.e.*, the sequence of vectors whose coordinates are the lengths of intervals in $I$ between two consecutive elements of $R$. Remark that 0 is not encoded and thus that $\mathbf{f}_{R,I}$ contains possibly empty vectors with only positive coordinates. As for Figure 1, the positions of $I$ smaller than all the positions of $R$ are not relevant.

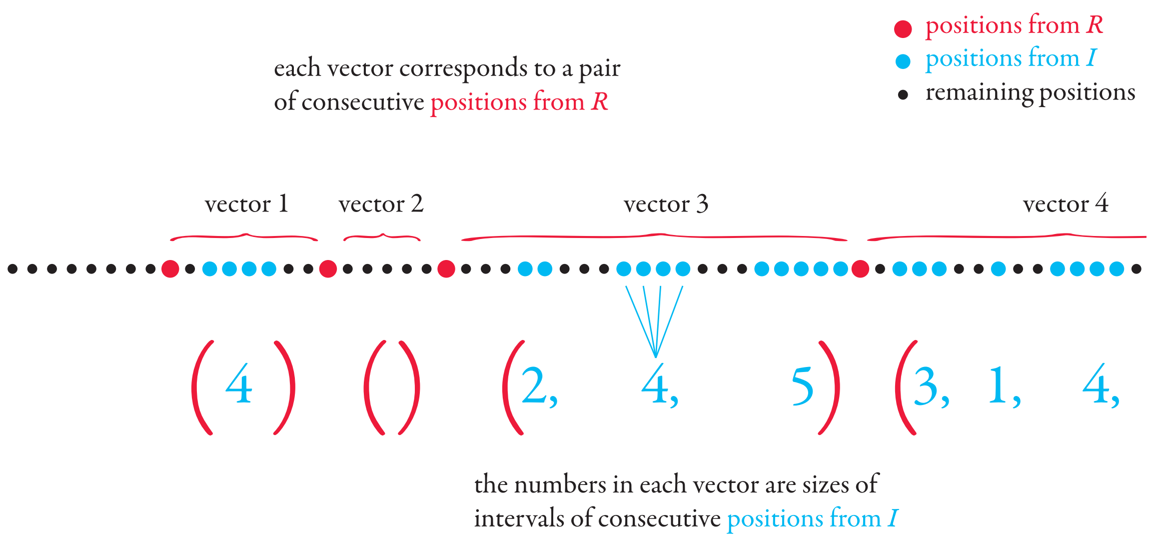



Figure 2: Two sets of positions $I, R \subseteq \mathbb{N}$ and the vector sequence that they encode. This vector sequence is defined only when $I$ and $R$ are disjoint and $R$ is infinite.

The following lemma states that the unboundedness of the dimensions of a vector sequence can be expressed in MSO+$\mathtt{U}_1$. Its proof will be the subject of Section 3.

**Lemma 2.4.** *There is a formula $\varphi(R, I)$ in* MSO+$\mathtt{U}_1$ *which is true if and only if the vector sequence $\mathbf{f}_{R,I}$ is defined and satisfies $\mathtt{dim}(\mathbf{f}_{R,I})$ is unbounded.*

*Proof of Lemma 2.3 assuming Lemma 2.4.* As the predicate $\mathtt{U}_1$ is easily definable in MSO+$\mathtt{U}_2$ ($\mathtt{U}_1(X)$ holds if and only if $\mathtt{U}_2(X, X \setminus \mathbb{N})$ holds), it remains to prove the converse, *i.e.*, to exhibit a formula $\varphi(X, Y)$ of MSO+$\mathtt{U}_1$ which holds if and only if $\mathtt{U}_2(X, Y)$ holds.

Since $\mathtt{U}_2(X, Y)$ holds if and only if $\mathtt{U}_2(X, Y \setminus X)$ holds and because $\mathtt{U}_2(X, Y)$ implies that $X$ is infinite, it is sufficient to write a formula $\varphi(R, I)$ which holds if and only if $\mathtt{U}_2(R, I)$

holds, assuming $R$ and $I$ are disjoint and $R$ is infinite. We fix such $R$ and $I$, which hence define a vector sequence $\mathbf{f}_{R,I}$ according to the encoding of Figure 2. Furthermore, $\mathtt{U}_2(R, I)$ holds if and only if summing all coordinates of each vector from the sequence yields a number sequence which is unbounded. We show that we can construct a $J \subseteq I$ such that $\mathtt{U}_2(R, I)$ holds if and only if the dimension of $\mathbf{f}_{R,J}$ is unbounded.

For every non negative integer $x$, if $2x$ belongs to $I$ as well as either $2x+1$ or $2x-1$, then we remove $2x$ from $I$. Let $J$ denote the thus obtained set. As at least one position from $I$ over two is kept in $J$, we have that $\mathtt{U}_2(R, I)$ holds if and only if $\mathtt{U}_2(R, J)$ holds. Furthermore, $J$ has the property that each of its point is isolated whence the sum of the coordinates of a vector from $\mathbf{f}_{R,J}$ is equal to its dimension. Therefore, $\mathtt{U}_2(R, J)$ holds if and only if the dimension of the vector sequence $\mathbf{f}_{R,J}$ is unbounded. Moreover, $J$ is definable in MSO in the sense that there is a MSO-formula with two free set variables $I$ and $J$ which holds if and only if $J$ is obtained from $I$ by the process given above.

We conclude the proof of Lemma 2.3 by applying Lemma 2.4. □

## 3. Expressing unboundedness of dimensions

This section is devoted to proving Lemma 2.4. Our proof has two steps. In Section 3.1, we show that in order to prove Lemma 2.4, it suffices to show the following lemma.

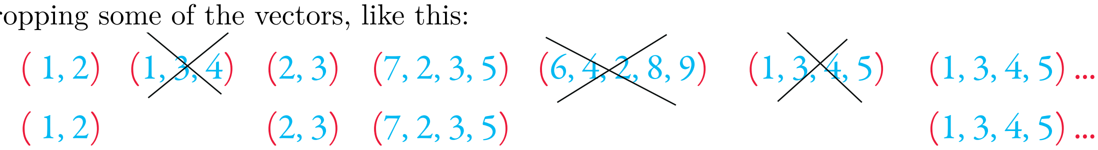
**Lemma 3.1.** *There is a formula $\varphi(R, I)$ in* MSO+$\mathtt{U}_1$ *which is true if and only if the vector sequence $\mathbf{f}_{R,I}$ is defined and satisfies $\mathbf{f}_{R,I} \to \infty$ and $\mathit{dim}(\mathbf{f}_{R,I})$ is unbounded.*

Lemma 3.1 itself will be proved in Section 3.2.

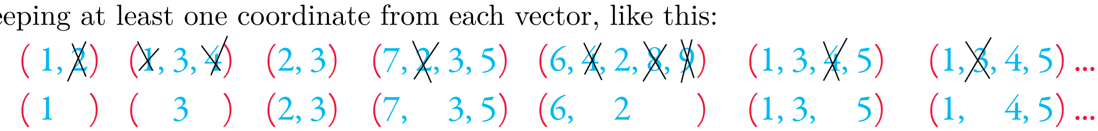
### 3.1. From Lemma 3.1 to Lemma 2.4.

Recall that the vector sequence $\mathbf{f}_{R,I}$ is defined if and only if $R$ is infinite and disjoint from a set $I$, which is clearly expressible in MSO. So from now on, we will always consider vector sequences $\mathbf{f}_{R,I}$ which are defined.

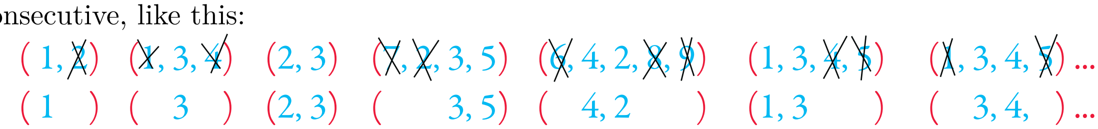
Given a vector sequence, a *sub-sequence* is an infinite vector sequence obtained by dropping some of the vectors, like this:



An *extraction* is obtained by removing some of the coordinates in some of the vectors, but keeping at least one coordinate from each vector, like this:



In particular a vector sequence containing at least one empty vector has no extraction. An extraction is called *interval-closed* if the coordinates which are kept in a given vector are consecutive, like this:



A 1-*extraction* is an extraction with only vectors of dimension 1. In particular, it is interval-closed. A *sub-extraction* is an extraction of a sub-sequence.

The following lemma gives two conditions equivalent to the unboundedness of the dimension of a vector sequence.

**Lemma 3.2.** *Let* $\mathbf{f}$ *be a vector sequence. Then* $\mathtt{dim}(\mathbf{f})$ *is unbounded if and only if one of the following conditions is true:*

**(1)** *There exists a sub-extraction* $\mathbf{g}$ *of* $\mathbf{f}$ *such that* $\mathbf{g} \to \infty$ *and* $\mathtt{dim}(\mathbf{g})$ *is unbounded.*

**(2)** *There exists an interval-closed sub-extraction* $\mathbf{g}$ *of* $\mathbf{f}$ *such that* $\mathtt{dim}(\mathbf{g})$ *is unbounded and* $\mathbf{g}$ *is bounded,* i.e.*, there exists a bound* $n \in \mathbb{N}$ *such that all the coordinates occurring in the sequence are less than* $n$*.*

*Proof.* The if implication is clear, since admitting a sub-extraction of unbounded dimension trivially implies the unboundedness of the dimension.

We now focus on the only-if implication. Assume that $\mathtt{dim}(\mathbf{f})$ is unbounded and condition (1) is false. Then, for all sub-extractions $\mathbf{g}$ of $\mathbf{f}$ we have that $\mathbf{g} \to \infty$ implies $\mathtt{dim}(\mathbf{g})$ is bounded. This implies that there exists a constant $m$ such that for all vectors $v$ in $\mathbf{f}$, the number of coordinates greater than $m$ is less than $m$. Indeed, if no such $m$ exists, we can exhibit a sub-extraction $\mathbf{g}$ of $\mathbf{f}$, which tends towards infinity and has unbounded dimension. Let us mark the coordinates smaller than $m$ in every vector of $\mathbf{f}$. For every integer $n$, we can find a vector of $\mathbf{f}$ with at least $n$ consecutive marked coordinates. Indeed suppose there is at most $n$ consecutive marked coordinates in every vector. Then, as there are at most $m$ unmarked coordinates, a vector cannot have dimension greater than $n(m+1)+m$ (*i.e.*, $m+1$ blocks of $n$ marked positions plus $m$ unmarked positions separating them). This contradicts that $\mathtt{dim}(\mathbf{f})$ is unbounded.

Thus by keeping for each vector one of the longest block of consecutive marked positions and removing vectors without any marked position, we can construct an interval-closed sub-extraction $\mathbf{g}$ of $\mathbf{f}$ which is bounded by $m$ and has unbounded dimension. □

We now proceed to show how Lemma 3.1 implies Lemma 2.4. Recall that Lemma 2.4 says that there is a formula $\varphi(R, I)$ in MSO+$\mathtt{U}_1$ which is true if and only if the vector sequence $\mathbf{f}_{R,I}$ is defined and satisfies $\mathtt{dim}(\mathbf{f}_{R,I})$ is unbounded.

*Proof of Lemma 2.4 assuming Lemma 3.1.* Consider $R$ and $I$ two sets of positions such that $\mathbf{f}_{R,I}$ is defined. We will use Lemma 3.2 to express in MSO+$\mathtt{U}_1$ that $\mathtt{dim}(\mathbf{f}_{R,I})$ is unbounded. The first condition of the lemma is expressible in MSO+$\mathtt{U}_1$ by Lemma 3.1, as shown in the next section. However, in order to express the second condition, we need that the gaps between consecutive intervals of $I$ have bounded length. This is ensured by extending the intervals as explained in the following. For any two disjoint sets $R$ and $I$, let us construct $J$ as follows: $J$ contains $I$, and for every two consecutive positions $x < y$ in $I$ such that there is no element of $R$ in between, we add to $J$ all the positions strictly between $x$ and $y$, except $y-1$, as in Figure 3. The set $J$ is expressible in MSO, in the sense that there is a MSO-formula with three free set variables $R$, $I$, $J$ which holds if and only if $J$ is obtained from $I$ by this process. Moreover, it is clear that $\mathtt{dim}(\mathbf{f}_{R,I})$ is unbounded if and only if $\mathtt{dim}(\mathbf{f}_{R,J})$ is unbounded since the dimension in both sequences are pairwise equal (see Figure 3). It suffices thus to prove Lemma 2.4 for sets $R$ and $I$ which satisfy:

**($\star$):** between two consecutive elements of $I$, there is either an element of $R$ or at most one position not in $I$.

Let $R$ and $I$ be two sets satisfying ($\star$). We want to express that $\mathtt{dim}(\mathbf{f}_{R,I})$ is unbounded. It suffices to show that for each condition in Lemma 3.2, there exists a formula of MSO+$\mathtt{U}_1$ which says that the condition is satisfied by $\mathbf{f}_{R,I}$. We treat the two cases separately.

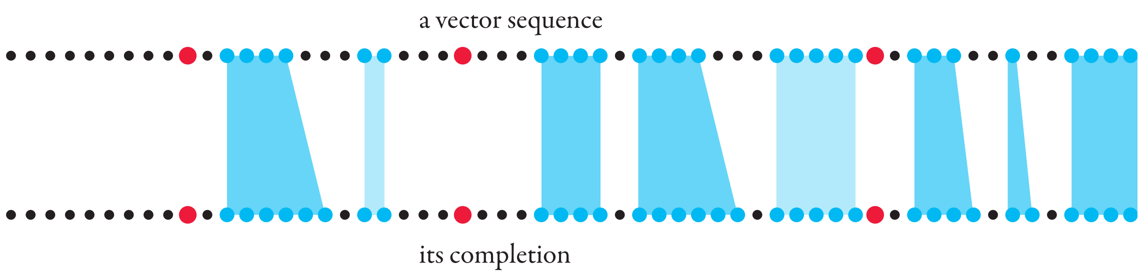



Figure 3: Completion of a vector sequence.

(1) We want to say that there exists a sub-extraction $\mathbf{g}$ of $\mathbf{f}_{R,I}$ such that $\mathbf{g} \to \infty$ and $\mathtt{dim}\,(\mathbf{g})$ is unbounded. Sub-extraction can be simulated in MSO according to the picture in Figure 4, and the condition "$\mathbf{g} \to \infty$ and $\mathtt{dim}\,(\mathbf{g})$ is unbounded" can be checked using Lemma 3.1.

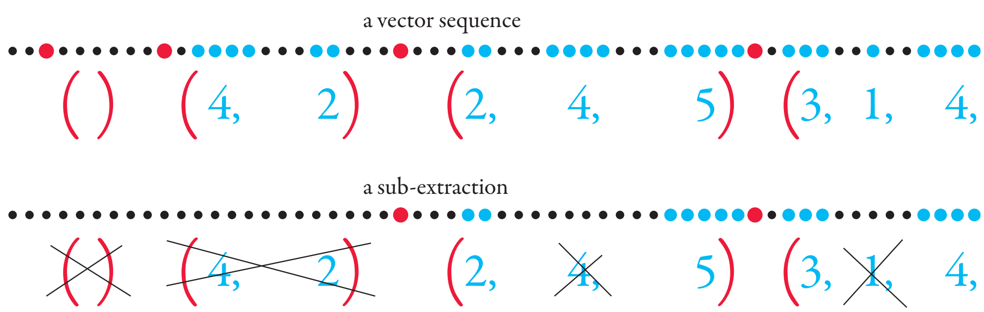



Figure 4: Sub-extraction of a vector sequence.

(2) We want to say that there exists an interval-closed sub-extraction $\mathbf{g}$ of $\mathbf{f}_{R,I}$ such that $\mathbf{g}$ is bounded and $\mathtt{dim}\,(\mathbf{g})$ is unbounded. We use the same approach as in the previous item, *i.e.*, we simulate sub-extraction in the logic using the encoding from Figure 4, thus obtaining $R', I'$. Additionally, we ensure that the selected sub-extraction is interval-closed. Importantly, taking an interval-closed sub-extraction preserves property $(\star)$. It remains therefore to write a formula in MSO+$\mathtt{U}_1$ which says that sets $R', I'$ satisfy $\mathbf{f}_{R',I'}$ is bounded and $\mathtt{dim}\,(\mathbf{f}_{R',I'})$ is unbounded.

- The boundedness of $\mathbf{f}_{R',I'}$ is expressed in MSO+$\mathtt{U}_1$ by the fact that the complement of $I'$ has to satisfy the negation of $\mathtt{U}_1$. Indeed, this exactly says that the intervals of $I'$ are bounded.
- Now, since $\mathbf{f}_{R',I'}$ is bounded then $\mathtt{dim}\,(\mathbf{f}_{R',I'})$ is unbounded if and only if the sequence of the number of occurrences of elements of $I'$ between two consecutive elements of $R'$ is unbounded. Consider the set $X$ of all the positions which are either in $R'$ or not adjacent to an element in $I'$ (which is expressible in MSO). Then, since $R'$ and $I'$ satisfy $(\star)$, $\mathtt{dim}\,(\mathbf{f}_{R',I'})$ is unbounded if and only if $\mathtt{U}_1(X)$ holds.

This completes the reduction of Lemma 2.4 to Lemma 3.1. □

3.2. **Unboundedness of dimensions for sequences tending to infinity.** We now prove Lemma 3.1 whence concluding the proof of Theorem 1.3. The lemma says that there is a formula $\varphi(R, I)$ in MSO+$\mathtt{U}_1$ which is true if and only if the vector sequence $\mathbf{f}_{R,I}$ is defined and satisfies $\mathbf{f}_{R,I} \to \infty$ and $\mathtt{dim}(\mathbf{f}_{R,I})$ is unbounded.

The main idea of the proof is to relate the unboundedness of the dimension with a property definable in MSO+$\mathtt{U}_1$, namely the *asymptotic mix* property, which was already used in [7] to prove undecidability of MSO+$\mathtt{U}$. This is done in Lemma 3.3 below, for which we need a couple of notions on vector sequences.

Given an infinite set $S \subseteq \mathbb{N}$ of positions and a number sequence $f \in \mathbb{N}^\omega$, the *restriction of $f$ to $S$* is the number sequence obtained by discarding elements of index not in $S$ and is denoted $f_{|S}$. Two number sequences $f$ and $g$ are called *asymptotically equivalent*, denoted by $f \sim g$, if they are bounded on the same sets of positions, *i.e.*, if for every set $S$, $f_{|S}$ is bounded if and only if $g_{|S}$ is bounded. For example, the number sequences $f = 1, 1, 3, 1, 5, 1, 7, 1, 9, \ldots$ and $g = 2, 1, 4, 1, 6, 1, 8, 1, 10, \ldots$ are asymptotically equivalent, but $f$ and $h = 1, 3, 1, 5, 1, 7, 1, 9, 1, \ldots$ are not.

We say that a vector sequence $\mathbf{f}$ is an *asymptotic mix* of a vector sequence $\mathbf{g}$ if for every 1-extraction $f$ of $\mathbf{f}$, there exists a 1-extraction $g$ of $\mathbf{g}$ such that $f \sim g$. In particular, the empty vector cannot occur in $\mathbf{f}$ or $\mathbf{g}$. Note that this is not a symmetric relation.

Given two vector sequences $\mathbf{f}$ and $\mathbf{g}$, we say that $\mathbf{g}$ *dominates* $\mathbf{f}$, denoted $\mathbf{f} \leq \mathbf{g}$, if for all $i$, the $i$-th vectors in both sequences have the same dimension, and the $i$-th vector of $\mathbf{f}$ is coordinatewise smaller than or equal to the $i$-th vector of $\mathbf{g}$. Furthermore, we suppose that all the coordinates in $\mathbf{f}$ and $\mathbf{g}$ are positive.

An extraction $\mathbf{g}$ of $\mathbf{f}$ is said to be *strict*, if at least one coordinate of each vector of $\mathbf{f}$ has been popped in $\mathbf{g}$, *i.e.*, the dimensions of the vectors of $\mathbf{g}$ are pointwise smaller than those of $\mathbf{f}$. In particular, a vector sequence with some vectors of dimension less than 2 admits no strict extraction, nevertheless, it may admit strict sub-extractions.

The following lemma relates the notion of asymptotic mix with the unboundedness of the dimension of a vector sequence, providing the vector sequence tends towards infinity.

**Lemma 3.3.** *Let $\mathbf{f}$ be a vector sequence such that $\mathbf{f} \to \infty$. Then, $\mathtt{dim}(\mathbf{f})$ is unbounded if and only if there exists a sub-sequence $\mathbf{h}$ and a strict extraction $\mathbf{g}$ of $\mathbf{h}$ satisfying:*

**($P_{\mathbf{g},\mathbf{h}}$):** *for every $\mathbf{h}' \leq \mathbf{h}$, there exists $\mathbf{g}' \leq \mathbf{g}$ such that $\mathbf{h}'$ is an asymptotic mix of $\mathbf{g}'$.*

*Proof.* The if direction follows from [7, Lemma 2.2], while the converse is new to this work. More precisely, it is proved in [7, Lemma 2.2] that given two vector sequences $\mathbf{h}$ and $\mathbf{g}$ tending towards infinity, if $\mathbf{h}$ and $\mathbf{g}$ have bounded dimension, then the two following properties are equivalent:

- **(1):** for infinitely many $i$'s, the $i$-th vector of $\mathbf{h}$ has higher dimension than the $i$-th vector of $\mathbf{g}$;
- **(2):** the negation of ($P_{\mathbf{g},\mathbf{h}}$): there exists $\mathbf{h}' \leq \mathbf{h}$ which is not an asymptotic mix of any $\mathbf{g}' \leq \mathbf{g}$.

Let $\mathbf{f}$ be a vector sequence which tends towards infinity and suppose that it has bounded dimension. Let $\mathbf{h}$ be a sub-sequence of $\mathbf{f}$ and $\mathbf{g}$ be a strict extraction of $\mathbf{h}$. By definition, $\mathbf{h}$ and $\mathbf{g}$ satisfy **(1)**, have bounded dimension and tend towards infinity. Therefore, by [7, Lemma 2.2], they satisfy **(2)**, *i.e.*, the negation of ($P_{\mathbf{g},\mathbf{h}}$).

We now prove the only if direction. We suppose that $\mathbf{f}$ tends towards infinity and that $\mathtt{dim}(\mathbf{f})$ is unbounded. There exists a sub-sequence $\mathbf{h}$ of $\mathbf{f}$ such that every vector occurring in $\mathbf{h}$ has dimension at least 2 and $\mathtt{dim}(\mathbf{h})$ tends towards infinity.

Define $\mathbf{g}$ as being the extraction obtained from $\mathbf{h}$ by dropping the last coordinate in each vector and observe that it is a strict extraction of $\mathbf{h}$. We prove now a result stronger than required, by exhibiting a universal $\mathbf{g}' \leq \mathbf{g}$ which makes $(P_{\mathbf{g},\mathbf{h}})$ true whatever the choice of $\mathbf{h}' \leq \mathbf{h}$, hence inverting the order of the universal and the existential quantifiers. Let $\mathbf{g}'$ be the vector sequence defined as follows: for each position $i$, denoting the $i$-th vector of $\mathbf{h}$ by $(v_1, v_2, \ldots, v_k)$, the $i$-th vector of $\mathbf{g}'$ is set to $(\min(v_1, 1), \min(v_2, 2), \ldots, \min(v_{k-1}, k-1))$. For instance:

$$\begin{array}{rllllll} \mathbf{h} = & (2,1), & (3,2,1), & (4,3,2,1), & (5,4,3,2,1), & (6,5,4,3,2,1), & \ldots \\ \mathbf{g} = & (2), & (3,2), & (4,3,2), & (5,4,3,2), & (6,5,4,3,2), & \ldots \\ \mathbf{g}' = & (1), & (1,2), & (1,2,2), & (1,2,3,2), & (1,2,3,3,2), & \ldots \end{array}$$

We now prove that every $\mathbf{h}' \leq \mathbf{h}$ is an asymptotic mix of $\mathbf{g}'$, that is, for every 1-extraction $h$ of $\mathbf{h}'$, there exists a 1-extraction $g$ of $\mathbf{g}'$ such that $h \sim g$. We fix an arbitrary vector sequence $\mathbf{h}' \leq \mathbf{h}$ and a 1-extraction $h$ of $\mathbf{h}'$. We define $g$ as the number sequence whose $i$-th number, for each position $i$, is the maximal coordinate in the $i$-th vector of $\mathbf{g}'$ which is less than or equal to the $i$-th number in $h$. This coordinate always exists since the first coordinate of every vector in the sequence $\mathbf{g}$' exists and is equal to 1. For instance, according to the previous example and given $\mathbf{h}'$ and $h$, the number sequence $g$ is defined as follows:

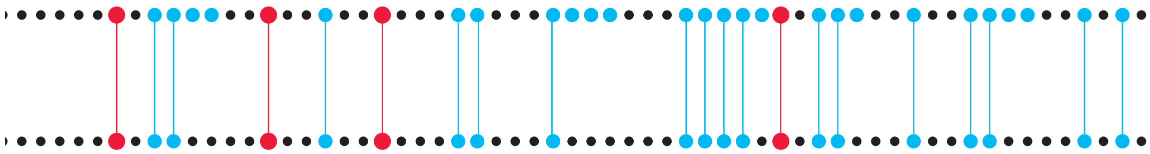
$$\begin{array}{rcccccc} \mathbf{h}' = & (1,1), & (2,1,1), & (3,2,1,1), & (4,3,2,1,1), & (5,4,3,2,1,1), & \ldots \\ h = & 1, & 2, & 1, & 4, & 1, & \ldots \\ \mathbf{g}' = & (1), & (1,2), & (1,2,2), & (1,2,3,2), & (1,2,3,3,2), & \ldots \\ \hline g = & 1, & 2, & 1, & 3, & 1, & \ldots \end{array}$$

Observe that, by definition, for every position $i$, the $i$-th number of $g$ is less than or equal to the $i$-th number of $h$. Hence, for every set $S$ of positions, if $h_{|S}$ is bounded then so is $g_{|S}$. Suppose now that for some set $S$ of positions, $h_{|S}$ is unbounded and fix an integer $n$. We want to prove that some number in $g_{|S}$ is greater than $n$. Since both $\mathbf{h}$ and $\mathtt{dim}(\mathbf{h})$ tend towards infinity, we can find a threshold $m \in \mathbb{N}$ such that for every position $i \geq m$, the $i$-th vector of $\mathbf{h}$ has dimension greater than $n$ and furthermore all its coordinates are greater than $n$. Thus, for all positions $i \geq m$, the $n$-th coordinate of the $i$-th vector of $\mathbf{g}'$ is defined and it is equal to $n$. Therefore, for all $i \in S$ with $i \geq m$ and such that the $i$-th number in $h$ is at least $n$, the $i$-th number in $g$ is at least $n$ as well. Hence, $g_{|S}$ is unbounded. □

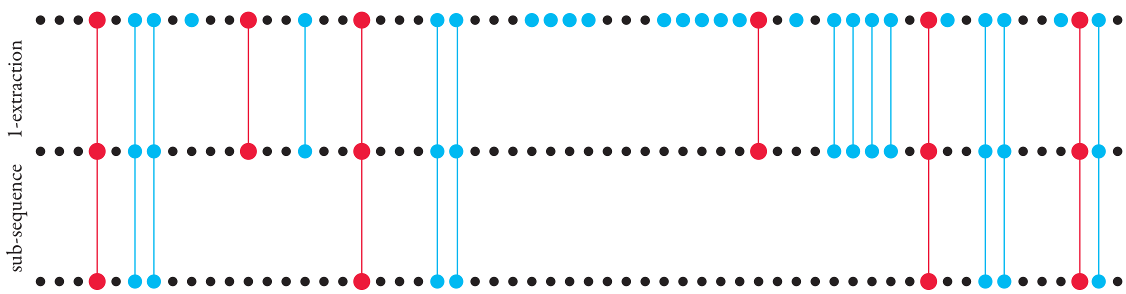
To conclude the proof of Lemma 3.1, it remains to express in MSO+$\mathtt{U}_1$ the property stated in Lemma 3.3: given two sets $R$ and $I$ encoding a vector sequence $\mathbf{f}_{R,I}$ which tends towards infinity, there exists a sub-sequence $\mathbf{h}$ of $\mathbf{f}_{R,I}$ and a strict extraction $\mathbf{g}$ of $\mathbf{h}$ satisfying $(P_{\mathbf{g},\mathbf{h}})$. Note that we can check in MSO+$\mathtt{U}_1$ that a sequence $\mathbf{f}_{R,I}$ (defined by two set variables $R$ and $I$) tends towards infinity. Indeed, this holds if and only if, for any bounded set $Y$ (*i.e.*, such that $\neg\mathtt{U}_1(Y)$), ultimately, the intervals of $I$ contain at least two elements of $Y$.

Given a vector sequence $\mathbf{f}$ and a set $S$, we say that an encoding of $\mathbf{f}$ is $S$-*synchronised* if it is of the form $S, J$ for some set $J$. Sub-sequences and strict extractions can be simulated in MSO by defining subsets of the original encoding, according to the picture in Figure 4. In particular, given $R$ and $I$, one can construct in MSO any sub-sequence $\mathbf{h}$ of $\mathbf{f}_{R,I}$ and any strict extraction $\mathbf{g}$ of $\mathbf{h}$, such that the encodings of $\mathbf{h}$ and $\mathbf{g}$ are $S$-synchronised for some $S \subseteq R$.

Now, we express $(P_{\mathbf{g},\mathbf{h}})$ on $\mathbf{h}$ and $\mathbf{g}$ encoded by $S, J$ and $S, J'$ respectively. Every dominated sequence $\mathbf{h}'$ (*resp.* $\mathbf{g}'$) of $\mathbf{h}$ (*resp.* $\mathbf{g}$) can be obtained by deleting some rightmost consecutive positions of intervals of $J$ (*resp.* $J'$), keeping at least one position for each interval. This can be done in MSO, preserving the $S$-synchronisation, as depicted below:



Finally, we prove that the property saying that $\mathbf{h}'$ is an asymptotic mix of $\mathbf{g}'$ can be expressed in MSO+$\mathtt{U}_1$ (using the $S$-synchronisation of the encodings of the two sequences). Indeed, 1-extractions can be simulated in MSO according to the picture in Figure 4, still preserving the $S$-synchronisation. Consider then $K, K'$ such that $S, K$ encodes a 1-extraction of $\mathbf{h}'$ and $S, K'$ encodes a 1-extraction of $\mathbf{g}'$, *i.e.*, such that between every two consecutive positions of $S$, the elements of $K$ (*resp.* $K'$) form an interval. It remains to prove that we can express in MSO+$\mathtt{U}_1$ that $\mathbf{f}_{S,K}$ and $\mathbf{f}_{S,K'}$, viewed as number sequences, are asymptotically equivalent. We select a subset of indices of the two number sequences $\mathbf{f}_{S,K}$ and $\mathbf{f}_{S,K'}$ by selecting elements of $S$ and keeping the maximal intervals in $K$ and $K'$ that directly follow the selected elements. Then, we check thanks to $\mathtt{U}_1$ that the corresponding sequences of numbers are both bounded or both unbounded. This process is depicted here:




## 4. How MSO+U may talk about ultimate periodicity

We consider now the quantifier $\mathtt{P}$ defined in the introduction. Recall that a set of positions $X \subseteq \mathbb{N}$ is called *ultimately periodic* if there is some period $p \in \mathbb{N}$ such that for sufficiently large positions $x \in \mathbb{N}$, either both or none of $x$ and $x + p$ belong to $X$. We consider the logic MSO+$\mathtt{P}$, *i.e.*, MSO augmented with *the quantifier* $\mathtt{P}$ that ranges over ultimately periodic sets:

$$\mathtt{P}X\varphi(X) : \text{“the formula } \varphi(X) \text{ is true for all ultimately periodic sets } X\text{”.}$$

**Example 4.1.** The language of $\omega$-words over $\{a, b\}$ such that the positions labeled by $a$ form an ultimately periodic set is definable in MSO+$\mathtt{P}$ by the following formula:

$$\exists X \quad \underbrace{\forall x \quad (x \in X \iff a(x))}_{X \text{ is the set of positions labeled by } a} \quad \wedge \quad \underbrace{\neg\Big(\mathtt{P}Y \quad (X \neq Y)\Big)}_{X \text{ is ultimately periodic}} .$$

From Example 4.1, one can see that our quantifier P extends strictly the expressivity of MSO, but it is *a priori* not clear whether it has decidable or undecidable satisfiability. However, using Theorem 1.3, we can prove Theorem 1.4 which states that satisfiability over $\omega$-words is undecidable for MSO+P.

*Proof of Theorem 1.4.* Since MSO+U has undecidable satisfiability [7], the same follows for MSO+$\mathtt{U}_1$ by Theorem 1.3. Moreover, the predicate $\mathtt{U}_1$ can be defined in terms of ultimate periodicity: a set $X$ of positions satisfies $\mathtt{U}_1(X)$ if and only if for all infinite ultimately periodic sets $Y$, there are at least two positions in $Y$ that are not separated by a position from $X$. It follows that every sentence of MSO+$\mathtt{U}_1$ can be effectively rewritten into a sentence of MSO+P which is true on the same $\omega$-words. Hence, Theorem 1.4 follows. □

The predicate $\mathtt{U}_1$ can thus be expressed in MSO augmented with the quantifier P. It is not clear that the converse is true, and we leave this as an open problem. However, in Theorem 4.2 below, we show that, up to a certain encoding, all languages expressible in MSO+P can be expressed in MSO+U.

Let $\Sigma$ be a finite alphabet, $\#$ be a symbol not belonging to $\Sigma$, and $\Sigma_\#$ denote $\Sigma \cup \{\#\}$. We define $\pi_\Sigma : {\Sigma_\#}^\omega \to \Sigma^\omega$ to be the function which erases all appearances of $\#$. This is a partial function, because it is only defined on $\omega$-words over $\Sigma_\#$ that contain infinitely many letters from $\Sigma$. We extend $\pi_\Sigma$ to languages of $\omega$-words in a natural way.

**Theorem 4.2.** *Every language definable in* MSO+P *over* $\Sigma$ *is equal to* $\pi_\Sigma(L)$ *for some language* $L \subseteq {\Sigma_\#}^\omega$ *definable in* MSO+U*.*

The main difficulty in expressing "ultimate periodicity" is to check the existence of a constant, namely the period, which is ultimately repeated. Such a property was already the crucial point in the proof of the undecidability of MSO+U [7]. Indeed the authors managed to express that an encoded vector sequence (as in Figure 2) tending towards infinity has "*ultimately constant dimension*", *i.e.*, all but finitely many vectors of the sequence have the same dimension (this dimension will represent the period). This is the content of the following lemma which is a direct consequence of [7, Lemma 3.1].

**Lemma 4.3.** *There exists a formula* $\varphi(R, I)$ *in* MSO+U *with two free set variables which holds if and only if* $\mathbf{f}_{R,I}$ *is defined, tends towards infinity, and* $\mathtt{dim}(\mathbf{f}_{R,I})$ *is ultimately constant.*

In order to use Lemma 4.3, we will encode a word $w$ over $\Sigma$ by adding factors of consecutive $\#$ between every two letters from $\Sigma$, such that the lengths of those factors tend towards infinity. Let $w = w_1 w_2 \cdots$ be an $\omega$-word over $\Sigma$ where the $w_i$'s are letters, and let $f = n_1, n_2, \ldots$ be a number sequence. We define the $\omega$-word $w_f$ over $\Sigma_\#$ as $w_1 \#^{n_1} w_2 \#^{n_2} \cdots$. In particular $\pi_\Sigma(w_f) = w$. Now, given a sentence $\varphi$ of MSO+P over $\Sigma$, we consider the language $L$ over $\Sigma_\#$ of all the $\omega$-words of the form $w_f$ for $w$ satisfying $\varphi$ and $f$ a number sequence tending towards infinity. As we can easily check in MSO+U that the sequence of lengths of the factors of $\#$ tends towards infinity, Theorem 4.2 follows immediately from the following lemma.

**Lemma 4.4.** *For every sentence* $\varphi$ *in* MSO+P*, one can effectively construct a formula* $\varphi_\#$ *in* MSO+U *such that for every* $\omega$*-word* $w \in \Sigma^\omega$*, the following conditions are equivalent:*
(1) $\varphi$ *is true in* $w$*;*
(2) $\varphi_\#$ *is true in* $w_f$ *for every* $f$ *which tends towards infinity.*

*Proof.* Let $\varphi$ be a sentence in MSO+P, and $f$ some number sequence tending towards infinity. Given a word $w$, we construct $w_f$ as explained above.

We construct $\varphi_\#$ by induction on the structure of $\varphi$, and show that at every step of the induction, $\varphi$ is true in $w$ if and only of $\varphi_\#$ is true in $w_f$. To do so, we use the natural mapping from positions of $w$ into positions of $w_f$ to handle free variables. In particular, every free set variable used in the induction hypothesis is supposed to contain only positions not labeled by #. Every formula not containing the quantifier P is translated straightforwardly, only ignoring all # positions when counting (a bounded number, of course). For example, the successor relation will be translated by "the first position to the right not labelled with #". It is routine to check that the satisfiability is the same in both $w$ and $w_f$, thanks to the natural mapping described above. The key point of the induction is the case of a formula $\mathtt{P}X\varphi(X)$. We translate it to a formula of the form: $\forall Y, \mathrm{UP}_\#(Y) \Rightarrow \varphi_\#(Y)$, where $\mathrm{UP}_\#(Y)$ holds for all sets $Y$ which contain only positions not labeled by # and which are "ultimately periodic when ignoring #'s".

Checking if $Y$ does not contain any position labeled by # is done by the formula $\forall y, y \in Y \Rightarrow \neg\#(y)$. It remains to express the property "$Y$ is ultimately periodic when ignoring #'s" as a formula of MSO+U with one free set variable $Y$ which contains only positions not labeled by #. It can be done in the following way: "$Y$ is ultimately periodic when ignoring #'s" if and only if there exist two sets $R$ and $I$ such that:

- $I$ is exactly the set of all the #, $R$ is infinite and disjoint from $I$ (hence $\mathbf{f}_{R,I}$ is defined);
- $\mathtt{dim}(\mathbf{f}_{R,I})$ is ultimately constant (this constant dimension will represent the period of $Y$);
- for every infinite set $R'$ disjoint from $I$, alternating with $R$ (*i.e.*, such that there is exactly one element of $R'$ between two consecutive elements of $R$) and such that $\mathtt{dim}(\mathbf{f}_{R',I})$ is ultimately constant, we have that all elements of $R'$ are either ultimately in $Y$ or ultimately not in $Y$.

These properties can be expressed in MSO+U thanks to Lemma 4.3, since, as $f$ tends towards infinity, the vector sequence $\mathbf{f}_{R,I}$, tends towards infinity. It is easy to check that the conjunction of these three properties is equivalent to "$Y$ is ultimately periodic when ignoring #'s". □

## 5. Conclusion

We have considered the extensions of MSO with the following features (two quantifiers and two second-order predicates):

**$\mathtt{U}X\varphi(X)$:** the formula $\varphi(X)$ is true for sets $X$ of arbitrarily large finite size.

**$\mathtt{U}_1(X)$:** for all $k \in \mathbb{N}$, there exist two consecutive positions of $X$ at distance at least $k$.

**$\mathtt{U}_2(R, I)$:** the sequence of numbers encoded by $R, I$ (Figure 1) is defined and unbounded.

**$\mathtt{P}X\varphi(X)$:** the formula $\varphi(X)$ is true for all ultimately periodic sets $X$.

We have proved that MSO+U, MSO+$\mathtt{U}_1$ and MSO+$\mathtt{U}_2$ have the same expressive power, and the translations between these logics are effective (Theorem 1.3, Lemmas 2.2 and 2.3). Furthermore, all three have undecidable satisfiability on $\omega$-words by [7]. Moreover we have seen that $\mathtt{U}_1$ is expressible in MSO+P, while the converse is also true up to some encoding (Theorem 4.2). As a consequence, MSO extended with the quantifier P has undecidable satisfiability (Theorem 1.4).

We believe that MSO+$\mathtt{U}_1$ can be reduced to many extensions of MSO, in a simpler way than reducing MSO+U. It is for instance the case for MSO augmented with the quantifier that ranges over periodic sets (and not necessarily ultimately periodic sets).

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