

MINIMIZATION OF VISIBLY PUSHDOWN AUTOMATA IS NP-COMPLETE

OLIVIER GAUWIN^a, ANCA MUSCHOLL^a, AND MICHAEL RASKIN^b

^a LaBRI, Univ. of Bordeaux, France

^b TU Munich, Germany

ABSTRACT. We show that the minimization of visibly pushdown automata is NP-complete. This result is obtained by introducing immersions, that recognize multiple languages (over a usual, non-visible alphabet) using a common deterministic transition graph, such that each language is associated with an initial state and a set of final states. We show that minimizing immersions is NP-complete, and reduce this problem to the minimization of visibly pushdown automata.

1. INTRODUCTION

Visibly pushdown automata (VPA) are a natural model for the control flow of recursive programs and have tight connections with tree automata and XML schemas. They were considered for parsing algorithms [11] under the name “input-driven pushdown automata”, and shown to have better space complexity than unrestricted pushdown automata. The name “visibly pushdown automata” is due to Alur and Madhusudan [2], who initiated their study from the perspective of program verification, and developed the theory in several directions (see <http://madhu.cs.illinois.edu/vpa/> for an exhaustive list of results). In particular, they showed that the class of visibly pushdown languages shares many desirable properties with the class of regular languages, like determinization, closure under boolean operations and the existence of a Myhill-Nerode congruence that defines canonical VPA [1]. However, the existence of a canonical VPA does not help for minimization, in contrast to regular languages. Similarly to many other more complex automata models, like automata over infinite words, two-way automata, etc, VPA do not have unique minimal automata. Even worse, the canonical VPA can be exponentially larger than a minimal VPA. Therefore, the minimization problem for VPA, besides being very relevant in practice, is also very challenging.

Minimization up to partitioning. Various minimization procedures have been proposed for some subclasses of deterministic VPA. Most of them use a partitioning of the state

Key words and phrases: visibly pushdown automata, minimization.

Work done while the third author was affiliated with the University of Bordeaux. This work was partially supported by the ANR project DeLTA (ANR-16-CE40-0007).

space into *modules*: when the VPA control is in a given module, and a call occurs, then the matching return brings the control back to the same module. The first model implementing this idea are *single-entry* VPA (SEVPA) [1], where each module has its own set of call symbols, and these sets are disjoint. Moreover, each module has a specific entry state: whenever a call of module m occurs, the VPA switches to the entry of m . For any fixed partition of call symbols [1] shows that there is a unique minimal deterministic SEVPA, and that it can be computed in polynomial time. *Multiple-entry* VPA (MEVPA) [8] allow several possible states when entering the module, but the symbol pushed on the stack by a call depends only on the state, not on the call symbol. MEVPA enjoy the same properties as SEVPA in terms of minimization: the minimal MEVPA is unique and computable in polynomial time. The two models SEVPA and MEVPA are subsumed by *call-driven automata* (CDA) [4], for which states are partitioned into modules, and a call leads to a state that depends only on the call symbol. A restricted version of CDA, called *expanded CDA* (ECDA) [4], further requires that only one call symbol can enter each module. Minimization of ECDA is easy, it resembles the Myhill-Nerode construction. A minimization procedure for CDA is obtained by adapting that of ECDA, and generalizes the ones for SEVPA and MEVPA, in the sense that these ones can be retrieved from the minimization of CDA.

The drawback of all the subclasses mentioned above (SEVPA, MEVPA, CDA and ECDA) is that there exist families of languages for which the minimal VPA within the respective class is exponentially larger than some minimal VPA. *Block* VPA (BVPA) [4] were proposed to overcome this problem: for every VPA, there exists an equivalent BVPA of quadratic size, so VPA can be minimized approximately via BVPA minimization. BVPA differ from SEVPA in that the entry state is determined by the call symbol, but may also depend on the current state. There is a unique minimal BVPA for a given visibly pushdown language, and this BVPA can be computed in cubic time, up to some partition of the language.

So all the approaches for VPA minimization rely on a fixed partition, either of the state space, or of the language, and the difficulty of minimization relies on finding a good partition. Given a BVPA and two integers k and s , knowing if there is an equivalent BVPA with k modules, each of size at most s , is NP-complete [5].

The main result of this paper is that VPA minimization is inherently difficult: we show that the problem is NP-complete. We obtain our result by showing NP-hardness for the following problem about deterministic finite state automata (DFA), that can be of independent interest: given n regular languages and a bound N , we ask if there exists some deterministic transition graph \mathcal{A} of size N such that for every given language we find a DFA accepting it by choosing an initial state and a set of final states of \mathcal{A} . We refer to this problem as *immersion minimization*.

Further related work. As for regular languages, finding a minimal *non-deterministic* automaton is computationally hard, namely EXPTIME-complete for non-deterministic VPA, [5] (hardness follows from the universality of non-deterministic VPA [2]). The paper [7] proposes an algorithm for computing *locally minimal* non-deterministic VPA, relying on a reduction to Partial Max-SAT. Results on the state complexity of VPA with respect to determinization, and various language operations are reported in the survey [12].

Some problems similar to the minimization of immersions also appear in the literature. However, to our best knowledge, no straightforward reduction exists from one of these problems to the minimization of immersions. The first problem is the minimization of non-deterministic finite automata with limited non-determinism. Whereas the minimization of

arbitrary non-deterministic finite automata is PSPACE-complete, it becomes NP-complete for automata that have a fixed number of initial states, and are otherwise deterministic [9]. Further NP-completeness results for minimization of automata with small degree of ambiguity are provided in [3]. A seemingly close problem from computational biology is the *shortest common superstring* problem, which asks for the shortest string containing each string from a given set as factor. This problem is known to be NP-complete [6].

Another related problem is the minimization of tree automata. Indeed, a word over a visibly pushdown alphabet can be viewed as the linearization of a tree, processed in a depth-first left-to-right traversal. This corresponds to an unranked tree, i.e, a finite ordered tree where the arity of each node is arbitrary. Several automata models exist for unranked trees, and the complexity of minimization ranges between PTIME and NP [10]. However, for each of these models, determinism does not correspond exactly to that of VPA, and minimization results do not transfer.

2. AUTOMATA

2.1. Visibly pushdown automata. A *visibly pushdown alphabet* $\widehat{\Sigma} = \Sigma_c \uplus \Sigma_r \uplus \Sigma_\ell$ is a finite set of *symbols* partitioned into *call symbols* in Σ_c , *return symbols* in Σ_r , and *internal symbols* in Σ_ℓ .

A *visibly pushdown automaton* (VPA for short) is a tuple $\mathcal{C} = \langle \widehat{\Sigma}, Q, I, F, \Gamma, \Delta \rangle$ where $\widehat{\Sigma}$ is a visibly pushdown alphabet, Q is a finite set of *states*, $I \subseteq Q$ and $F \subseteq Q$ are the sets of initial, resp. final states, and Γ is the (finite) stack alphabet. The set Δ has three types of transitions, depending on the type of the input symbol: *call* transitions $\Delta_c \subseteq Q \times \Sigma_c \times Q \times \Gamma$ that push a symbol on the stack, *return* transitions $\Delta_r \subseteq Q \times \Sigma_r \times \Gamma \times Q$ that pop a symbol from the stack, and *internal* transitions $\Delta_\ell \subseteq Q \times \Sigma_\ell \times Q$ that leave the stack unchanged.

A configuration of \mathcal{C} is a pair (q, σ) where $q \in Q$ is the current state and $\sigma \in \Gamma^*$ is the current stack content (the top of the stack is the rightmost symbol). A transition $(q, \sigma) \xrightarrow{a}_\mathcal{C} (q', \sigma')$ corresponds to one of the following cases:

- $a \in \Sigma_c$ and $\sigma' = \sigma A$ for some q, q', A with $(q, a, q', A) \in \Delta_c$,
- $a \in \Sigma_r$ and $\sigma = \sigma' A$ for some q, q', A with $(q, a, A, q') \in \Delta_r$,
- $a \in \Sigma_\ell$ and $\sigma = \sigma'$ for some q, q' with $(q, a, q') \in \Delta_\ell$.

Note that only return transitions can read the top stack symbol. The transition relation of \mathcal{C} extends to words from Σ^* as expected. The *language accepted by \mathcal{C}* is the set of words u such that $(q_0, \epsilon) \xrightarrow{u}_\mathcal{C} (q_f, \sigma)$ with $q_0 \in I$, $q_f \in F$ and $\sigma \in \Gamma^*$. In particular, acceptance does not require that the final configuration has an empty stack. A VPA is *deterministic* if it has a single initial state, Δ_c does not contain two rules (q, a, q_1, γ_1) and (q, a, q_2, γ_2) with $(q_1, \gamma_1) \neq (q_2, \gamma_2)$, Δ_r does not contain two rules (q, a, γ, q_1) and (q, a, γ, q_2) with $q_1 \neq q_2$, and Δ_ℓ do not contain two rules (q, a, q_1) and (q, a, q_2) with $q_1 \neq q_2$.

Minimization. We measure the size of a VPA by its number of states. This will be the parameter that we minimize. Another choice could be the size of the stack alphabet. The stack alphabet can be actually bounded by $|Q||\Sigma_c|$, as one can always choose it as $Q \times \Sigma_c$, [4]. The problem we consider here is the following:

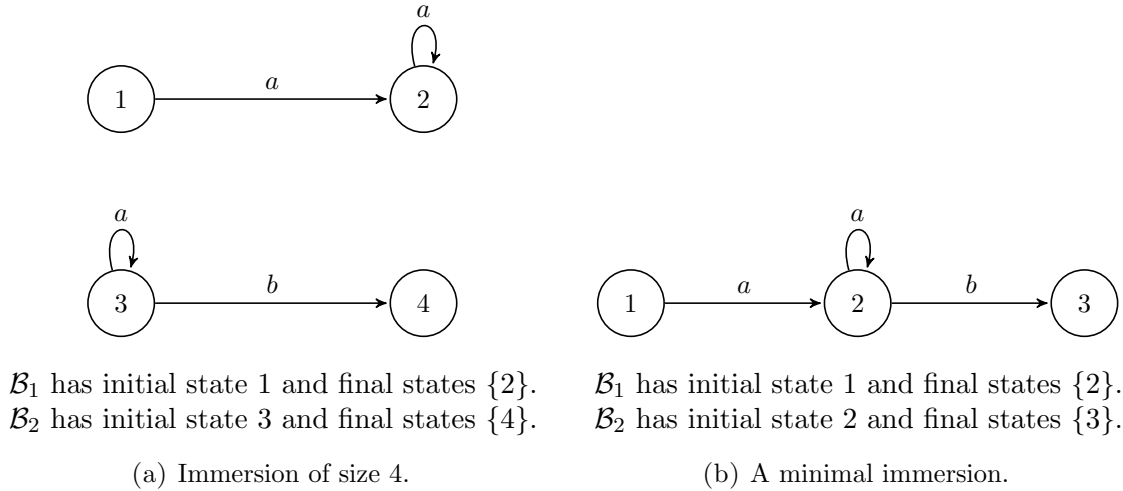


FIGURE 1. Two immersions for the languages $L_1 = a^+$ and $L_2 = a^*b$.

MINVPA

Input: deterministic VPA \mathcal{C} and integer N

Question: does a deterministic VPA \mathcal{C}' of size N exist that accepts the same language as \mathcal{C} ?

The main result of the paper is:

Theorem 2.1. *MINVPA is NP-complete.*

Proof. Since equivalence of deterministic VPA can be checked in polynomial time [2], it is clear that MINVPA belongs to NP. We show hardness through an intermediate problem called MINIMMERSION. We first prove that MINIMMERSION reduces to MINVPA (Proposition 2.2), and then show that MINIMMERSION is NP-hard, by reduction from 3-colorability of graphs (Section 3). \square

2.2. Immersions and VPA minimization. An immersion is a variant of a deterministic finite state automaton used to accept multiple regular languages. We show in this section that minimization of immersions reduces to minimization of VPA.

Sub-DFAs. Let $\mathcal{A} = \langle Q, \Sigma, \rightarrow \rangle$ be a finite, deterministic transition graph, so Q is a finite set of states, Σ is the alphabet, and \rightarrow is a partial function from $Q \times \Sigma$ to Q . A sub-automaton (*sub-DFA* for short) \mathcal{B} of \mathcal{A} is a tuple $\langle Q, \Sigma, \rightarrow, q_0, F \rangle$. So \mathcal{B} is a deterministic finite automaton (DFA for short) obtained by equipping \mathcal{A} with an initial state q_0 and a set $F \subseteq Q$ of final states. Note that all DFA in our constructions will have partially-defined transition functions. However, the results are not affected by introducing a dead-state.

Immersion. Given n languages $L_1, \dots, L_n \subseteq \Sigma^*$, an *immersion* for L_1, \dots, L_n consists of a finite *deterministic* transition graph \mathcal{A} , and n sub-DFA $\mathcal{B}_1, \dots, \mathcal{B}_n$ of \mathcal{A} such that $L(\mathcal{B}_i) = L_i$, for all $1 \leq i \leq n$. The size of the immersion is the number of states of its transition graph \mathcal{A} . For convenience we usually just write \mathcal{A} for the immersion, omitting the initial/final states.

Consider for instance the two languages $L_1 = a^+$ and $L_2 = a^*b$. A possible immersion for these two languages is obtained by taking the disjoint union of two DFA, one for each

language. This is illustrated in Figure 1a, and yields an immersion with 4 states. A smaller immersion is obtained by merging the states 2, 3 in Figure 1a, as depicted in Figure 1b. The resulting immersion has 3 states, and is minimal for L_1, L_2 . Another immersion with three states is obtained by merging the states 1, 4. The example shows that, in general, minimal immersions are not unique, as it is already the case for VPA.

MINIMMERSION

Input: DFA $\mathcal{A}_1, \dots, \mathcal{A}_n$, and integer N

Question: is there some immersion of size at most N for $L(\mathcal{A}_1), \dots, L(\mathcal{A}_n)$?

Proposition 2.2. MINIMMERSION reduces in polynomial time to MINVPA .

Proof. Let $\mathcal{A}_1, \dots, \mathcal{A}_n$ be DFA over the alphabet Σ_ℓ , and let $L_i = L(\mathcal{A}_i)$ for every i . We show that there exists an immersion of size k for L_1, \dots, L_n if and only if there exists a deterministic VPA of size $k + 2$ for the language $K = \bigcup_{i=1}^n c_i L_i r$, where $\Sigma_c = \{c_1, \dots, c_n\}$ and $\Sigma_r = \{r\}$.

Consider an immersion of size k for L_1, \dots, L_n with finite, deterministic transition graph $\mathcal{A} = \langle Q, \Sigma, \longrightarrow \rangle$, and sub-DFA $\mathcal{B}_1, \dots, \mathcal{B}_n$ of \mathcal{A} such that $L(\mathcal{B}_i) = L_i$, for all $1 \leq i \leq n$. Let q_i and F_i denote the initial state and the final states of \mathcal{B}_i , respectively. From \mathcal{A} we immediately get a deterministic VPA \mathcal{C} for K by letting $\mathcal{C} = (\widehat{\Sigma}, Q \uplus \{q_0, q_f\}, \{q_0\}, \{q_f\}, \Gamma, \Delta)$, with stack alphabet $\Gamma = \{1, \dots, n\}$, and Δ as follows:

- $\Delta_c = \{(q_0, c_i, q_i, i) \mid 1 \leq i \leq n\}$,
- $\Delta_\ell = \longrightarrow$,
- $\Delta_r = \{(q, r, i, q_f) \mid q \in F_i, 1 \leq i \leq n\}$.

Conversely, assume there is some deterministic VPA $\mathcal{C} = (\widehat{\Sigma}, Q, \{q_0\}, F, \Gamma, \Delta)$ of size $k + 2$ for $K = \bigcup_{i=1}^n c_i L_i r$. This language is included in $\Sigma_c \Sigma_\ell^* \Sigma_r$, so we can assume that \mathcal{C} has a single final state, that we call q_f , and which has no outgoing transitions. Let q_i denote the (unique) state of \mathcal{C} such that $(q_0, c_i, q_i, A_i) \in \Delta_c$, for some $A_i \in \Gamma$. We can also assume that i is used instead of A_i in these rules, as they are the only rules in Δ_c . We define an immersion for the languages L_1, \dots, L_n as the transition graph $\mathcal{A} = \langle Q_{\mathcal{A}}, \Sigma, \longrightarrow \rangle$, where $Q_{\mathcal{A}} = Q \setminus \{q_0, q_f\}$ and $\longrightarrow = \Delta_\ell$. The sub-DFA $\mathcal{B}_1, \dots, \mathcal{B}_n$ associated with this immersion are obtained by setting the initial state of \mathcal{B}_i to q_i , and setting $q \in F_i$ if $(q, r, i, q_f) \in \Delta_r$. It is clear that \mathcal{B}_i accepts precisely the words $w \in \Sigma_\ell^*$ such that $c_i w r \in L(\mathcal{C})$. So $L(\mathcal{B}_i) = L_i$ for every $1 \leq i \leq n$. \square

3. MINIMMERSION IS NP-COMPLETE

It is clear that MINIMMERSION is in NP. We show NP-hardness by a reduction from 3-colorability. Let $G = (V, E)$ be an undirected graph with vertex set $V = \{1, \dots, n\}$ and edge set $E \subseteq V^2 \setminus \{(i, i) \mid i \in V\}$. We ask whether there is a coloring $c : V \rightarrow \{0, 1, 2\}$ such that $c(i) \neq c(j)$, for every $(i, j) \in E$.

Before we define the DFA $\mathcal{A}_1, \dots, \mathcal{A}_n$ we need some notations. Let $m = 2n(n-1)+2$. We fix a set $P = \{p_1, p_2, p_3, q_1, q_2\}$ of five distinct prime numbers p such that $3n < p \leq c \cdot n$, for some suitable constant¹ c , such that no $p \in P$ divides m . Let also $N = 3m + p_1 + p_2 + p_3 + q_1 + q_2$. Note that $6n^2 < N < 9n^2$, for n sufficiently large.

¹Recall that Chebyshev's theorem says that there is always at least one prime between n and $2n$.

Notations. The alphabet used in the following for the DFA \mathcal{A}_i is $\Sigma = \{0, 1\}$. A path in some transition graph of the form $s_1 \xrightarrow{0} s_2 \xrightarrow{0} \cdots \xrightarrow{0} s_n$ will be called simply a *path*. Similarly, a *cycle* is a path as above, with $s_1 = s_n$. For any path $s_1 \xrightarrow{0} s_2 \xrightarrow{0} \cdots \xrightarrow{0} s_n$ we say that s_n is *0-reachable* from s_1 , and s_1 is *co-0-reachable* from s_n . A *k-cycle* denotes a cycle of length k . A *1-transition* is a transition labeled by 1.

We fix in the following a bijection between the set $\{2k + 1 : 1 \leq k \leq n(n - 1)\}$ and the set of ordered pairs of vertices $\{(i, j) : i, j \in V, i \neq j\}$. Hereby we denote by $\langle i, j \rangle$ the integer encoding the pair (i, j) w.r.t. this fixed bijection.

We are now ready to define the DFA \mathcal{A}_i , where $i \in V$ is a vertex of the given graph. The language L_i of the DFA \mathcal{A}_i will be a subset of 0^*10^* . Informally, \mathcal{A}_i consists of an m -cycle (called “dispatch” cycle), such that from some of the vertices of this cycle there is a 1-transition to some p -cycle (called “counting” cycle) with $p \in P$. Each p -cycle has a designated “entry” node, and all 1-transitions into the cycle point to this node. Assuming that the vertices of the m -cycle are numbered successively $1, \dots, m$, with 1 being the initial state, the DFA \mathcal{A}_i has the following transitions:

- (1) From vertex 1 there is a 1-transition to a p_1 -cycle, and from vertex 2 there is a 1-transition to a p_2 -cycle.
- (2) From every other *even* vertex there is a 1-transition to a p_3 -cycle.
- (3) Each of the remaining $n(n - 1)$ *odd* vertices is of the form $\langle j, k \rangle$, according to the bijection fixed above. The transitions out of these vertices are the following:
 - Each odd vertex $\langle i, j \rangle$ has a 1-transition to a q_1 -cycle.
 - Each odd vertex $\langle j, i \rangle$ with $\{i, j\} \notin E$, has a 1-transition to a q_1 -cycle.
 - Each odd vertex $\langle j, i \rangle$ with $\{i, j\} \in E$, has a 1-transition to a q_2 -cycle.

Note that there is no 1-transition outgoing from vertices $\langle j, k \rangle$ where $j \neq i$ and $k \neq i$. As already mentioned, the initial state of \mathcal{A}_i is the vertex 1 of the m -cycle. The final states are all the target states of the 1-transitions. Figure 2 shows an example \mathcal{A}_i .

Remark 3.1. Note that any DFA accepting $(0^p)^*$ must contain a cycle of length divisible by p , if $p > 1$ is a prime.

Let p be a prime from P . A vertex s of a transition graph \mathcal{A} over $\Sigma = \{0, 1\}$ is called a *p-vertex* if there is a sub-DFA of \mathcal{A} for the language $1(0^p)^*$ with initial state s .

Lemma 3.2. *Let \mathcal{A} be a minimal immersion for L_1, \dots, L_n of size at most N , and let C be a k -cycle of \mathcal{A} . Then exactly one of the two following cases holds:*

- (1) C contains at least one p_1 -vertex and k is divisible by m .
- (2) k is divisible by some unique prime $p \in P$.

Proof. By assumption there is some sub-DFA \mathcal{B}_i of \mathcal{A} accepting L_i , for every i . Note first that, by minimality of \mathcal{A} , every vertex of \mathcal{A} is either 0-reachable from the initial state of some \mathcal{B}_i , or co-0-reachable from a final state of some \mathcal{B}_i . This will ensure that one of the two cases in the statement of the lemma holds for any cycle.

Recall that immersions were defined as *deterministic* transition graphs. Using the assumption $|\mathcal{A}| \leq N$, note that a vertex of \mathcal{A} cannot be both a p -vertex and a p' -vertex, for two different primes p, p' from P . Then otherwise the size of \mathcal{A} would be at least $9n^2 > N$, which is a contradiction.

Let us denote a vertex s of \mathcal{A} as *special* if s is p_1 -vertex and $s \xrightarrow{0} s'$, with s' being p_2 -vertex.

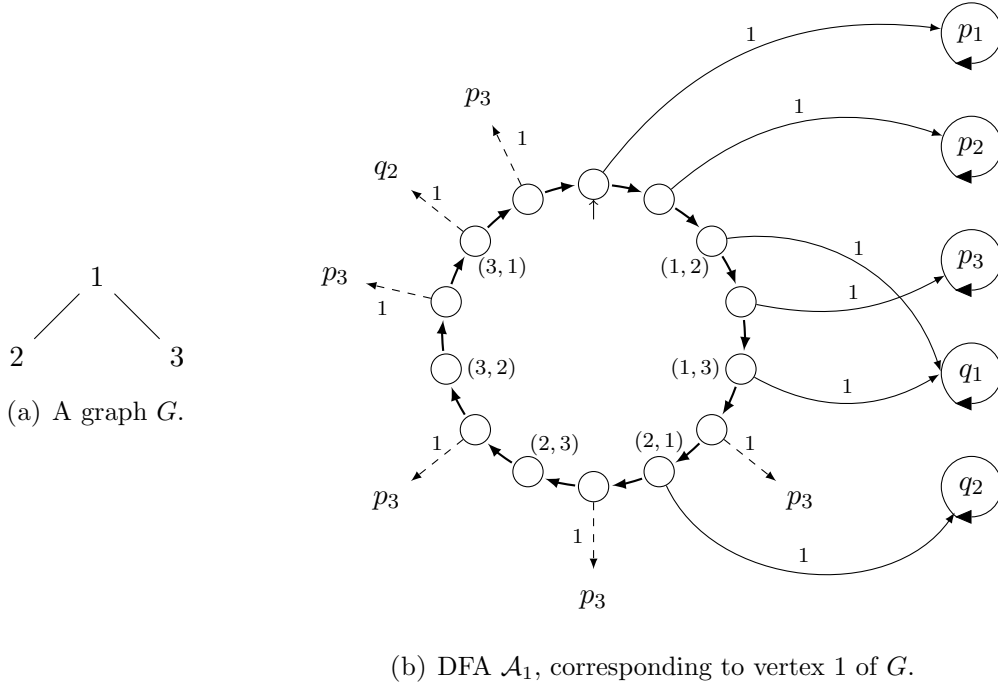


FIGURE 2. The dispatch cycle (thick) and the counting cycles of \mathcal{A}_1 (p_1, p_2, p_3, q_1, q_2). Dashed edges point to one of counting cycles. Unlabeled edges are 0-transitions. The final states are all target states of the 1-transitions.

The first case in the statement corresponds to C being 0-reachable from the initial state of some \mathcal{B}_i . Clearly, C needs to have at least one special vertex s . Note also that any vertex t such that $s \xrightarrow{0^{m \cdot j}} t$, where $j \geq 0$, must be special, because the length of the dispatch cycle is m . Assume by contradiction that k is not divisible by m , and let $d = k \pmod m$. If d is odd, then it follows from the previous remark that C must contain a p_1 -vertex that is at the same time a p_2 -vertex or a p_3 -vertex. If d is even and not zero, then similarly, C must contain a vertex that is at the same time a p_2 -vertex and a p_3 -vertex. So in both cases we obtain a contradiction to $|\mathcal{A}| \leq N$, as already noted.

The second case is where C is co-0-reachable from a final state of some \mathcal{B}_i . Here, k must be divisible by some $p \in P$. This prime is unique, as already observed.

We argue finally that the two cases are mutually exclusive. If k were both divisible by m and by $p \in P$ then $k > pm > N$, since p does not divide m . But this is again a contradiction to $|\mathcal{A}| \leq N$. \square

Assume that \mathcal{A} is a minimal immersion for L_1, \dots, L_n of size at most N . From Lemma 3.2 we deduce that the vertex set of \mathcal{A} is the disjoint union of two sets V_1, V_2 , such that:

- Transitions within each V_i ($i \in \{1, 2\}$) are labeled only by 0.
- Transitions from V_1 to V_2 are labeled only by 1.
- There are no transitions from V_2 to V_1 .

To see this, let us ignore the 1-labeled transitions of \mathcal{A} . Then we obtain a disjoint union of graphs (transitions are labeled only by 0s). Each such graph consists of a cycle, plus some

simple paths reaching the cycle. From Lemma 3.2 we know that each cycle is used either to “dispatch” (case 1) or to “count” modulo some prime (case 2), and that the two cases are mutually exclusive. So by the minimality of \mathcal{A} we can conclude that 1-labeled transitions of \mathcal{A} go only from “dispatch” cycles to “counting” cycles. Again by minimality we can bound the size and number of cycles in V_1 and V_2 :

Lemma 3.3. *Assume that \mathcal{A} is a minimal immersion for L_1, \dots, L_n of size at most N . Then the vertex set of \mathcal{A} is the disjoint union of two sets V_1, V_2 as above, such that:*

- (1) V_1 consists of at most three m -cycles.
- (2) V_2 consists of p -cycles, one for each $p \in P$.

Proof. By Lemma 3.2 we know that V_2 contains at least $|P|$ cycles, one for each $p \in P$. By minimality of \mathcal{A} , V_2 has exactly one p -cycle, for each $p \in P$.

Now we consider V_1 . By Lemma 3.2 the cycles of V_1 have length divisible by m . As before, by minimality V_1 is a disjoint union of cycles. Each cycle C has the property that it accepts some language $\{u \in 0^* : u1v \in L_i \text{ for some } v \in 0^*\}$ from one of the p_1 -vertices of C . In particular, C is equal to C'^j for some dispatch cycle C' of one of the L_i and some $j \geq 1$. By minimality of \mathcal{A} we obtain that $C = C'$ and $j = 1$. Finally, by the choice of N , we conclude that V_1 consists of at most three m -cycles. \square

From Lemma 3.3 we see that each of the sub-DFA for any of the L_i consists of one of the m -cycles in V_1 , with the p_1 -vertex as initial state, together with transitions labeled by 1 to the required p -cycles in V_2 .

Lemma 3.4. *The graph G is 3-colorable if and only if there is some minimal immersion for L_1, \dots, L_n of size at most N .*

Proof. Let us first assume that G is 3-colorable. Then we argue that \mathcal{A} can be built from at most three dispatch cycles C_0, C_1, C_2 , one for each color 0,1 and 2 (together with p -cycles, one for each $p \in P$). Cycle C_α can be used for all vertices $i \neq j$ of color α , since they are pairwise unconnected. To see this, note that vertex $\langle i, j \rangle$ of C_α is a q_1 -vertex according to the definition of \mathcal{A}_i ; and $\langle i, j \rangle$ is also q_1 -vertex according to L_j , since $\{i, j\} \notin E$.

Conversely, if \mathcal{A} has size at most N then by Lemma 3.3 there are at most three m -cycles C_0, C_1, C_2 in \mathcal{A} . We color vertex i by α if the sub-DFA for L_i uses C_α . This coloring is proper, because if the sub-DFA for L_i, L_j both use the same dispatch cycle, then $\{i, j\} \notin E$ since otherwise vertex $\langle i, j \rangle$ would be both a q_1 - and a q_2 -vertex, contradicting Lemma 3.3. \square

Lemma 3.4 yields finally the claimed result, and also the proof of Theorem 2.1:

Theorem 3.5. *MINIMMERSION is NP-complete.*

CONCLUSIONS

We have shown that the VPA minimization is intrinsically difficult, by exhibiting an NP-lower bound. A minor modification of the construction reduces approximation of the chromatic number to approximating the minimal size of an equivalent VPA, thus any constant-factor approximation of VPA minimisation is NP-hard. Our result raises the quest for efficient implementations of SAT-based minimization algorithms for VPA.

REFERENCES

- [1] R. Alur, V. Kumar, P. Madhusudan, and M. Viswanathan. Congruences for visibly pushdown languages. In *Automata, Languages and Programming, 32nd International Colloquium, ICALP 2005*, volume 3580 of *Lecture Notes in Computer Science*, pages 1102–1114. Springer, 2005.
- [2] R. Alur and P. Madhusudan. Visibly pushdown languages. In *Proceedings of the 36th Annual ACM Symposium on Theory of Computing*, pages 202–211. ACM, 2004.
- [3] H. Björklund and W. Martens. The tractability frontier for NFA minimization. *J. of Comp. and Syst. Sci.*, 78(1):198–210, 2012.
- [4] P. Chervet and I. Walukiewicz. Minimizing variants of visibly pushdown automata. In *Mathematical Foundations of Computer Science 2007, 32nd International Symposium, MFCS 2007*, volume 4708 of *Lecture Notes in Computer Science*, pages 135–146. Springer, 2007.
- [5] R. Duvignau. On minimization of visibly pushdown automata. Master’s thesis, University of Bordeaux, 2012.
- [6] M. R. Garey and D. S. Johnson. *Computers and Intractability: A Guide to the Theory of NP-Completeness*. W. H. Freeman & Co., New York, NY, USA, 1990.
- [7] M. Heizmann, C. Schilling, and D. Tischner. Minimization of visibly pushdown automata using partial max-sat. In *Tools and Algorithms for the Construction and Analysis of Systems - 23rd International Conference, TACAS 2017*, volume 10205 of *Lecture Notes in Computer Science*, pages 461–478, 2017.
- [8] V. Kumar, P. Madhusudan, and M. Viswanathan. Minimization, learning, and conformance testing of boolean programs. In *Concurrency Theory, 17th International Conference, CONCUR 2006*, volume 4137 of *Lecture Notes in Computer Science*, pages 203–217. Springer, 2006.
- [9] A. Malcher. Minimizing finite automata is computationally hard. *Theoretical Computer Science*, 327(3):375–390, 2004.
- [10] W. Martens and J. Niehren. On the minimization of XML schemas and tree automata for unranked trees. *Journal of Computer and System Sciences*, 73(4):550–583, 2007.
- [11] K. Mehlhorn. Pebbling mountain ranges and its application to DCFL-recognition. In *Automata, Languages and Programming, 7th International Colloquium, ICALP 1980*, pages 422–435. Springer Berlin Heidelberg, 1980.
- [12] A. Okhotin and K. Salomaa. Complexity of input-driven pushdown automata. *SIGACT News*, 45(2):47–67, 2014.