Logical Methods in Computer Science
Volume 16, Issue 1, 2020, pp. 29:1–29:42
https://lmcs.episciences.org/

Submitted May 23, 2018
Published Mar. 02, 2020

TRANSLATION OF ALGORITHMIC DESCRIPTIONS OF DISCRETE FUNCTIONS TO SAT WITH APPLICATIONS TO CRYPTANALYSIS PROBLEMS

ALEXANDER SEMENOV, ILYA OTPUSCHENNIKOV, IRINA GRIBANOVA, OLEG ZAIKIN, AND STEPAN KOCHEMAZOV

Russia, Irkutsk, ISDCT SB RAS
e-mail address: biclop.rambler@yandex.ru

Russia, Irkutsk, ISDCT SB RAS
e-mail address: otilya@yandex.ru

Russia, Irkutsk, ISDCT SB RAS
e-mail address: the42dimension@gmail.com

Russia, Irkutsk, ISDCT SB RAS
e-mail address: zaikin.icc@gmail.com

Russia, Irkutsk, ISDCT SB RAS
e-mail address: veinamond@gmail.com

Abstract. In the present paper, we propose a technology for translating algorithmic descriptions of discrete functions to SAT. The proposed technology is aimed at applications in algebraic cryptanalysis. We describe how cryptanalysis problems are reduced to SAT in such a way that it should be perceived as natural by the cryptographic community. In the theoretical part of the paper we justify the main principles of general reduction to SAT for discrete functions from a class containing the majority of functions employed in cryptography. Then, we describe the Transalg software tool developed based on these principles with SAT-based cryptanalysis specifics in mind. We demonstrate the results of applications of Transalg to construction of a number of attacks on various cryptographic functions. Some of the corresponding attacks are state of the art. We compare the functional capabilities of the proposed tool with that of other domain-specific software tools which can be used to reduce cryptanalysis problems to SAT, and also with the CBMC system widely employed in symbolic verification. The paper also presents vast experimental data, obtained using the SAT solvers that took first places at the SAT competitions in the recent several years.

2012 ACM CCS: [**Theory of computation**]: Logic—Constraint and logic programming, [**Theory of computation**]: Logic—Automated reasoning, [**Software and its engineering**]: Software notations and tools—Context specific languages— Domain specific languages, [**Security and privacy**]: Cryptography—Symmetric cryptography and hash functions, [**Security and privacy**]: Cryptography—Cryptanalysis and other attacks.

Key words and phrases: SAT, SAT-based cryptanalysis, symbolic execution.

LOGICAL METHODS IN COMPUTER SCIENCE DOI:10.23638/LMCS-16(1:29)2020

Introduction

The state-of-the-art algorithms for solving the Boolean satisfiability problem (SAT) are successfully used in many practical areas including symbolic verification, scheduling and planning, bioinformatics, network science, etc. In the recent years, a growth of interest to applications of SAT in cryptanalysis is observed. The corresponding results are covered in, e.g., [Mas99, MM00, LM03, MZ06, DKV07, CB07, EM09, Bar09, SNC09, SZBP11, CGS12, Nos12, Cou13, Cou15, SZ16, Sto16, NNS+17, NLG+17, SZO+18].

SAT-based cryptanalysis implies two stages: on the first stage a SAT encoding of a considered cryptanalysis problem is constructed. On the second stage the obtained SAT instance is solved using some SAT solving algorithm. The success on the second stage is not guaranteed because SAT is an NP-hard problem, and also due to the fact that the hardness of cryptanalysis problems is usually preserved during their translation to SAT form [CM97]. Despite an impressive progress in the development of applied algorithms for solving SAT, the majority of cryptanalysis problems remain too hard even for cutting edge SAT solvers.

Meanwhile, the first stage is guaranteed to be effective for most cryptanalysis problems, at least in theory. It follows from the fact that cryptographic transformations are usually designed to be performed very fast. To reduce some cryptanalysis problem to SAT, one has to perform the so-called propositional encoding of a corresponding function. In practice, this task is quite nontrivial because modern cryptographic algorithms are constructed from a large number of basic primitives. Often, a researcher has to carry out a substantial amount of manual work to make a SAT encoding for a considered cipher due to some of its constructive features or because of specific requirements of an implemented attack.

There are two main classes of tools that can be used to automatically reduce a cryptanalysis problem to SAT. First, it is possible to use one of the several available domain-specific systems. Among them we would like to mention SAW [CFH+13] that can operate with the Cryptol language [LM03, EM09, ECW09] designed for specifying cryptographic algorithms. Another system that allows one to reduce cryptanalysis problems to SAT (albeit with some additional steps) is the URSA system [Jan12]. Finally, it is possible to use the Grain-of-Salt tool [Soo10] to construct SAT encodings of cryptographic functions from a limited class, formed by keystream generators based on feedback shift registers. Another approach to reducing cryptanalysis problems to SAT consists in using generic systems for symbolic verification in the form of Bounded Model Checking [BCCZ99, BCC+99, CKL04]. For example, one can employ the CBMC system [CKL04, Kro09] or LLBMC system [SMF12].

Both generic systems and domain-specific systems have their pros and cons. On the one hand, generic systems usually support widely employed programming languages, such as C, and therefore it is easy to adapt an existing implementation of a cryptographic function to such a tool. They also have a wide spectrum of applications, are well supported and have a good documentation. On the other hand, while domain-specific tools may lack in convenience of use, their languages are often purposefully enriched (compared to generic languages) by instructions and data types that improve their ability to deal with cryptographic functions. For example, such instructions and data types may allow them to work with bits directly, or to implement specific cryptographic attacks, such as guess-and-determine attacks [Bar09], attacks based on differential paths (e.g., the ones from [WLF+05, WY05]), etc.

In the present paper, we introduce a new software tool designed to encode algorithms that specify cryptographic functions to SAT. It is named Transalg (from TRANSlation of ALGorithms). Transalg uses a domain-specific language called TA language. The TA language is formed from a subset of the C language. It is designed to be much simpler and to avoid platform-dependent or undefined behavior. The language is extended by several specific data types and instructions that often allow Transalg to tackle common cryptanalysis tasks better than the competition. In particular, the TA language has a specific `bit` data type to represent a single bit of data and supports `bit` arrays of arbitrary size. This allows Transalg to reduce the redundancy of constructed propositional encodings and better preserve the structure of an original problem. The Transalg basic concept implies that a cryptographic function is interpreted starting from its input and ending with its output. Thus, the language has specific directives to declare variables and arrays as input and output ones. As a result, the first variables in the constructed SAT encoding always correspond to the function's input, and the last ones to its output. Transalg eliminates all the auxiliary variables that do not depend on input and do not influence the output, and uses minimization (in the form of Espresso logic minimization tool [BSVMH84]) during the construction of propositional encodings to reduce their size. The TA language also uses several specific constructions that make it easier for Transalg users to employ the constructed SAT encodings for implementing cryptographic attacks in the SAT context. In the present paper, we describe the Transalg tool and its theoretical foundations in detail, compare it with the competition and show its capabilities in application to cryptanalysis of several cryptographic systems which are currently used or have been used in recent past.

Let us give an outline of the paper. In Section 1, we briefly touch the basics of the Boolean satisfiability problem. In Section 2, we give the theoretical foundations of SAT-based cryptanalysis. Here we discuss several features of the procedures for translating programs defining discrete functions to SAT. As we show below, they are particularly important in the context of cryptographic applications. Also in the same section we discuss the main theoretical results that form the basis of the software tool that performs effective reductions of inversion problems of discrete functions to SAT. This software tool named Transalg is described in Section 3. In Section 4, we compare the functionality of Transalg with that of other software tools which can be used to encode cryptographic problems to SAT: CBMC; SAW; URSA; Grain-of-Salt. In Section 5, we describe SAT-based attacks on several relevant cryptographic functions. The corresponding SAT encodings were constructed using Transalg. It should be noted that some of the described attacks are currently the best known. Section 6 contains a brief review of related works.

The present paper is an extended version of the report [OSG+16] presented at the ECAI 2016 conference. The sources of Transalg are available at [OGS]. The examples of Transalg programs for various cryptographic primitives can be found at [OGZS]. All the instances considered in Section 4 are also available online at [OGZ+].

1. The Boolean satisfiability problem and algorithms for its solving used in cryptanalysis

The Boolean satisfiability problem (SAT) is a decision problem, in which for an arbitrary Boolean formula F it is necessary to decide whether there exists such truth assignment for its variables on which formula F takes the value of $True$. Hereinafter, let us denote values $True$ and $False$ by 1 and 0, respectively. It can be shown that SAT for an arbitrary Boolean formula F can be effectively (in polynomial time in the size of description of F) reduced to SAT for a formula in a Conjunctive Normal Form (CNF). Further we will consider SAT exactly in this sense. Also, below we view SAT not only as a decision problem but also as the corresponding search problem: if CNF C is satisfiable to find any truth assignment that satisfies C.

The decision variant of SAT is NP-complete [Coo71], while the search variant is NP-hard [GJ79]. Nevertheless, during the last three decades several highly effective practical SAT solving algorithms were developed, driven by numerous applications in various areas of science. The detailed information on SAT and the algorithms for its solving can be found in the book [BHvMW09].

In the present paper, we consider the applications of SAT solving algorithms to cryptanalysis problems, in particular to the problem of finding a preimage of a cryptographic function given its image. It can be viewed as a problem of finding solutions of a system of algebraic equations which interconnect ciphering algorithm's steps. Lately, the direction of research in which a cryptanalysis problem is viewed in the general context of the problem of solving algebraic equations is often referred to as *algebraic cryptanalysis* [Bar09]. As it will be shown below, from an algebraic system or even from an algorithm defining a cryptographic function one can effectively transition to SAT. A number of valuable results in algebraic cryptanalysis were obtained thanks to the use of SAT solvers: [MZ06, CGS12, CB07, Cou13, DKV07, Bar09, SNC09, SZBP11] and several others. The particular area of algebraic cryptanalysis which employs SAT solvers is known as *SAT-based cryptanalysis*.

We are not aware of theoretical results that would demonstrate advantage of some algorithms for solving SAT-based cryptanalysis instances over others. However, based on a large number of papers (both cited above and below) one can conclude that CDCL SAT solvers [MSLM09] suit best for solving such problems. The construction of first effective CDCL SAT solvers was the result of a deep modernization of the well-known DPLL algorithm [DLL62, DP60], which was undertaken in [MSS96, MS99, MMZ$^+$01, ZMMM01, ES04]. After this, CDCL-based SAT solvers became de facto algorithmic tools for solving computational problems in a number of areas, first and foremost in symbolic verification [BCCZ99, BCC$^+$99, BCC$^+$03, PBG05, MS08]. The computational potential of CDCL in application to cryptanalysis problems was realized approximately in the middle of 2000-s. As it was noted above, to the present day a lot of papers have been published in which CDCL SAT solvers were applied to cryptanalysis problems. A short review of the most prominent results in this direction will be given in Section 6.

2. Theoretical foundations of SAT-based cryptanalysis

As mentioned above, SAT-based cryptanalysis is an area of algebraic cryptanalysis (see [Bar09]) in which SAT solvers are used to solve equations that connect the input of a cryptographic algorithm with its output. A cryptanalysis problem usually involves searching for a preimage of a known image of a considered function (in this case, the term *preimage attack* is also used). In some cases, it is necessary to find several inputs whose outputs satisfy some additional constraints. Such constraints are used, for example, in the problem of finding collisions of cryptographic hash functions. Hereinafter, we refer to all these problems using the general term *inversion problems*.

In this section, we provide theoretical foundations of SAT-based cryptanalysis. In particular, we look into the construction of a Boolean formula that encodes a considered inversion problem. Below we follow the methodology of symbolic execution and in particular, Bounded Model Checking.

Symbolic Execution [Kin76] is a technique that associates with a program for a computer or for an abstract machine some symbolic expressions, usually Boolean formulas. *Bounded Model Checking* involves applying automated reasoning and combinatorial algorithms to a Boolean expression associated with a finite state system to prove some properties of such system [BCC+03]. From our point of view, this approach best fits the problem of constructing SAT encodings for inversion of discrete functions in general, and for cryptographic functions especially.

Hereinafter, denote by $\{0,1\}^n$ the set of all possible binary words of length n. By $\{0,1\}^+$ we denote the set of all binary words of length $n = 1, 2, \ldots$. Let us consider functions of the kind

$$f : \{0,1\}^+ \to \{0,1\}^+, \tag{2.1}$$

i.e. functions that map arbitrary binary words into binary words. Additionally, we assume that each function of the kind (2.1) is defined everywhere on $\{0,1\}^+$ (i.e. is total) and is specified by a Turing machine program $A(f)$, the complexity of which is bounded by a polynomial in the length of an input word. A program $A(f)$ specifies an infinite family of functions of the kind

$$f_n : \{0,1\}^n \to \{0,1\}^+, n = 1, 2, \ldots \tag{2.2}$$

It is clear that for an arbitrary $n = 1, 2, \ldots$ it follows that $Dom\, f_n = \{0,1\}^n$. Hereinafter, to functions (2.1) and (2.2) we refer as *discrete functions*.

Definition 2.1. For a discrete function f of the kind (2.1) the problem of its inversion consists in the following. Given $A(f)$, for an arbitrary $n = 1, 2, \ldots$ and arbitrary $y \in Range\, f_n$ to find such $x \in \{0,1\}^n$ that $f_n(x) = y$.

It is quite easy to give examples of cryptanalysis problems that can be naturally formulated in the context of inversion problems of corresponding functions. The main terms related to cryptography that we use below can be found, e.g., in [MVO96].

In our first example, suppose that given a secret key $x \in \{0,1\}^n$, f_n generates a pseudorandom sequence (generally speaking, of an arbitrary length) that is later used to cipher some plaintext via bit-wise XOR. Such a sequence is called a *keystream*. Knowing some fragment of plaintext lets us know the corresponding fragment of keystream, i.e. some word y for which we can consider the problem of finding such $x \in \{0,1\}^n$ that $f_n(x) = y$. Regarding cryptographic keystream generators, this corresponds to the so called *known plaintext attack*.

Let us give another example. Total functions of the kind $f : \{0,1\}^+ \to \{0,1\}^c$, where c is some constant, are called *hash functions*. If n is the length of an input message, and $n > c$, then there exist such x_1, x_2, $x_1 \neq x_2$ that $f_n(x_1) = f_n(x_2)$. Such a pair x_1, x_2 is called a *collision* of a hash function f. A cryptographic hash function is considered compromised if one is able to find collisions of that function in reasonable time.

For an arbitrary function of the kind (2.1) there exists an effective in theory procedure for reducing the problem of its inversion to SAT. Essentially, it follows from the Cook–Levin theorem, and to prove it one can use any known technique, e.g., from [Coo71] or [Gol08]. Below we briefly review the main techniques used to prove statements of such a kind. They play a crucial role in understanding basic principles that serve as a foundation of our software tool for translating algorithmic descriptions of discrete functions to SAT.

So, let us fix n and consider an arbitrary function of the kind (2.2). Since the runtime of a program defining the corresponding function is finite for any $x \in \{0,1\}^n$, we can consider this function in the following form:

$$f_n : \{0,1\}^n \to \{0,1\}^m. \tag{2.3}$$

It can be specified by a Boolean circuit $S(f_n)$ over an arbitrary complete basis. Hereinafter, we use the basis $\{\wedge, \neg\}$. On the current stage, assume that we are given a circuit $S(f_n)$. Note that it has n inputs and m outputs. Let us fix some order on the sets of inputs and outputs of $S(f_n)$. With each input of $S(f_n)$ we associate a Boolean variable. Denote the obtained ordered set of variables by $X = \{x_1, \ldots, x_n\}$. We will say that X encodes the input of function f_n. Similarly, let us encode the output of f_n via the Boolean variables forming the ordered set $Y = \{y_1, \ldots, y_m\}$. For a circuit $S(f_n)$ in linear time in the number of nodes in $S(f_n)$ we can construct a CNF denoted by $C(f_n)$. The corresponding algorithm traverses each inner node of a circuit exactly once. With each gate $G \in \{\wedge, \neg\}$ it associates an auxiliary variable $v(G)$ from the set $V : V \cap X = \emptyset$. For an arbitrary $v(G)$ a CNF $C(G)$ is then constructed which uses at most 3 Boolean variables. The exact representation of $C(G)$ depends on the gate G. The result of this process is the CNF:

$$C(f_n) = \bigwedge_{G \in S(f_n)} C(G). \tag{2.4}$$

The described technique of constructing a CNF for a circuit $S(f_n)$ is known as *Tseitin transformations* [Tse70].

Definition 2.2. To CNF $C(f_n)$ of the kind (2.4) we further refer as *template CNF* encoding the algorithm that specifies function f_n, or in short *template CNF for f_n*.

Let u be an arbitrary Boolean variable. Below we will use the following notation: by $l_\lambda(u)$, $\lambda \in \{0,1\}$ denote literal $\neg u$ if $\lambda = 0$, and literal u if $\lambda = 1$. Let $C(f_n)$ be a template CNF for f_n. Now let $x = (\alpha_1, \ldots, \alpha_n)$ and $y = (\gamma_1, \ldots, \gamma_m)$ be arbitrary truth assignments from $\{0,1\}^n$ and $\{0,1\}^m$, respectively. In other words, let us consider $x = (\alpha_1, \ldots, \alpha_n)$ as an assignment of variables from $X = \{x_1, \ldots, x_n\}$, and $y = (\gamma_1, \ldots, \gamma_m)$ as an assignment of variables from $Y = \{y_1, \ldots, y_m\}$. Consider the following two CNFs:

$$\begin{array}{rcl} C(x, f_n) & = & l_{\alpha_1}(x_1) \wedge \ldots \wedge l_{\alpha_n}(x_n) \wedge C(f_n), \\ C(f_n, y) & = & C(f_n) \wedge l_{\gamma_1}(y_1) \wedge \ldots \wedge l_{\gamma_m}(y_m). \end{array}$$

For many practical applications of SAT and, in particular, to describe many cryptographic attacks studied in algebraic cryptanalysis (see, e.g., [SZO$^+$18]) the following fact plays a very important role.

Remark 2.3. The application of only the *Unit Propagation* rule [DG84, MSLM09] to CNF $C(x, f_n)$ for a particular $x = (\alpha_1, \ldots, \alpha_n)$ results in the derivation of values for all remaining variables, including that of variables from Y: $y_1 = \gamma_1, \ldots, y_m = \gamma_m$, such that $f_n(x) = y$, $y = (\gamma_1, \ldots, \gamma_m)$.

This property was several times used in other papers (see, e.g., [JJ09, JBH12, SZ15]). Its proof in a very similar formulation can be found in [BKNW09]. Essentially, it follows from the fact that the set X in $C(f_n)$ is a Strong Unit Propagation Backdoor Set (SUPBS) [WGS03].

The following statement is a variant of the Cook–Levin theorem in the context of the problem of inverting functions of the kind (2.3). The basic steps of its proof are standard and can be found, for example, in [Gol08]. However, from our point of view there are several technical issues that should be clarified for better understanding of how the software tool described below works with data. That is why we present the short proof of this statement detailing only the features that play an important role in the context of this study.

Theorem 2.4. *Let f be an arbitrary function of the kind* (2.1). *Then there exists an algorithm A' such that given as an input a program $A(f)$, a number n (in unary form), and a word $y \in \{0,1\}^+$, in polynomial time it constructs a CNF $C(f_n, y)$ with the following properties:*

(1) *For $y \notin Range\, f_n$ the CNF $C(f_n, y)$ is unsatisfiable.*
(2) *For $y \in Range\, f_n$ the CNF $C(f_n, y)$ is satisfiable and from any of its satisfying assignments one can extract such a word $x \in \{0,1\}^n$ that $f_n(x) = y$.*

Sketch proof. Assume that the program $A(f)$ is executed on the Turing machine, described in [GJ79], which works only with binary data. By algorithm A' we mean the informal procedure that constructs a circuit $S(f_n)$ based on the text of the program $A(f)$ and a number n.

Note that the transition function in the machine from [GJ79] looks as follows:

$$\delta : (q, \theta) \to (q', \theta', s), \tag{2.5}$$

where $\theta \in \{0, 1, \emptyset\}$ is an input symbol, $q \in Q$ is an arbitrary state, and s is a variable that defines the direction in which the head of the Turing machine is going to shift, i.e. $s \in \{-1, 0, +1\}$. The function (2.5) describes the execution of one elementary command.

Let us consider the execution of program $A(f)$ as a sequence of time moments, such that during the transition from one time moment to another exactly one elementary command is executed. The moment $t = 0$ corresponds to a starting configuration. With each moment t we associate the set of Boolean variables X_t, $X_i \cap X_j = \emptyset$ if $i \neq j$. With the transition from t to $t+1$ we associate a formula

$$\Psi_{t\to t+1} = \bigvee_{(q,\theta)} \Phi^{(q,\theta)}_{t\to t+1}. \tag{2.6}$$

An arbitrary formula $\Phi^{(q,\theta)}_{t\to t+1}$ is a formula of the kind

$$\phi_t(q, \theta) \Rightarrow \phi_{t+1}(q', \theta'),$$

(here by $\Rightarrow$ we denote logical implication), which is constructed in the following manner. The formula $\phi_t(q, \theta)$ is a conjunction of literals over the set X_t that encodes a particular pair (q, θ) and also the state of the head of the Turing machine at the moment t. The formula $\phi_{t+1}(q', \theta')$ is the conjunction of literals over the set of Boolean variables X_{t+1} that encodes a pair (q', θ') and the state of the head corresponding to the triple (q', θ', s). The correspondence between (q, θ) and (q', θ', s) is defined by the transition function (2.5). In (2.6) the disjunction is performed over all possible pairs (q, θ) in the program $A(f)$.

It is very important to note that the cardinality of Q does not depend on n. Therefore, the size of the formula $\Psi_{t\to t+1}$ is a constant that does not depend on n. Omitting some details, let us note that from the above a formula $\Psi_{t\to t+1}$ defines a function

$$F_{t\to t+1} : \{0,1\}^{|X_t|} \to \{0,1\}^{|X_{t+1}|},$$

which can also be specified using Boolean circuit $S(F_{t\to t+1})$ over the basis $\{\wedge, \neg\}$, which has $|X_t|$ inputs and $|X_{t+1}|$ outputs. The sets X_t and X_{t+1} are the sets of input and output variables of a circuit $S(F_{t\to t+1})$, respectively, and X_{t+1} is the set of input variables of circuit $S(F_{t+1\to t+2})$.

Let $t(n)$ be the upper bound on the runtime of the program $A(f_n)$ over all inputs from $\{0,1\}^n$. By combining the circuits $S(F_{t_0 \to t_1}), \ldots, S(F_{t(n)-1 \to t(n)})$ according to the above we construct the circuit for which it is easy to see that it specifies the function f_n of the kind (2.3). This circuit is $S(f_n)$.

Let us construct the template CNF $C(f_n)$ for the circuit $S(f_n)$. Let y be an arbitrary assignment from $\{0,1\}^m$. Consider a CNF $C(f_n, y)$. Now we use the property mentioned above and conclude that X is a SUPBS in $C(f_n, y)$. Thus it follows that the points (1) and (2) from the Theorem formulation are valid. $\square$

We would like to give some additional comments regarding Theorem 2.4. Using the properties of the Tseitin transformations, it is easy to show that the reduction presented in Theorem 2.4 is parsimonious [GJ79], i.e. the number of assignments that satisfy CNF $C(f_n, y)$, $y \in \mathit{Range}\, f_n$, is equal to the number of preimages of y. For further purposes it will be enough for us to find at least one preimage of $y \in \mathit{Range}\, f_n$. It should be noted that the values of variables from X in an arbitrary satisfying assignment of $C(f_n, y)$, $y \in \mathit{Range}\, f_n$, specify a preimage of y. This follows directly from the fact that X is SUPBS for $C(f_n, y)$. Indeed, let α be a satisfying assignment of $C(f_n, y)$ and $x = (\alpha_1, ..., \alpha_n)$ be an assignment of variables from X extracted from α. Suppose that x is *not* a preimage of y. Then, since X is SUPBS, the application of UP to CNF

$$l_{\alpha_1}(x_1) \wedge \ldots \wedge l_{\alpha_n}(x_n) \wedge C(f_n, y)$$

should lead to a conflict. Thus, we have a contradiction with the fact that x is a part of the assignment that satisfies $C(f_n, y)$.

Based on Theorem 2.4, let us formulate the general concept of SAT-based cryptanalysis. Assume that we have a function f of the kind (2.1), and consider a problem of finding a preimage of a particular $y \in \mathit{Range}\, f_n$ for a fixed n. Then, using Theorem 2.4 we construct CNF $C(f_n, y)$. From any satisfying assignment of $C(f_n, y)$ it is easy to extract such $x \in \{0,1\}^n$ that $f_n(x) = y$.

As a concluding remark we would like to note that an input word from $\{0,1\}^n$, employed by the procedure used in the proof of Theorem 2.4 to transition from a program $A(f)$ to a template CNF $C(f_n)$, is not constrained in any way. In fact, this procedure takes Boolean variables $x_1, \ldots, x_n$ as an input and outputs $C(f_n)$, thus essentially performing symbolic execution.

3. Transalg: software tool for encoding algorithmic descriptions of discrete functions to SAT

In the present section, we describe the Transalg software tool that in essence implements the translation procedure for transforming algorithmic descriptions of functions of the kind (2.3) to SAT, which was outlined in Theorem 2.4. The only conceptual difference is that instead of the Turing machine Transalg uses an abstract machine with random access to memory cells.

Transalg takes as an input an algorithm that specifies a discrete function in a special TA language. Then it uses this description to construct a symbolic representation of the algorithm (in the sense of symbolic execution). We refer to the obtained representation as to *propositional encoding*. The propositional encoding is first built as a set of Boolean formulas, and then can be transformed to the DIMACS CNF format or the AIGER format [Bie07].

In TRANSALG the process of computing a value of a discrete function f_n is represented as a sequence of elementary operations with memory cells of an abstract machine. Each memory cell contains one bit of information. Any elementary operation o over data in memory cells is essentially a Boolean function of arity k, $k \geq 1$, where k is some constant that does not depend on the size of input (strictly speaking, it is possible to consider $k \in \{1, 2\}$). For example, if o has the arity of 2 then the result of applying o to data in memory cells c_1 and c_2 is one bit that is written to memory cell c_3. However, during the construction of a propositional encoding TRANSALG does not use the real data. Instead, it links with the cells c_1, c_2, c_3 the Boolean variables v_1, v_2 and w, respectively. Then it associates with operation o the Boolean formula

$$w \equiv \phi_o(v_1, v_2), \tag{3.1}$$

where $\phi_o(v_1, v_2)$ specifies a function o. For an arbitrary formula of the kind $\phi_o(v_1, v_2)$, we represent the corresponding function as a Boolean circuit. TRANSALG can work with different basis functions, but in the most simple case it can construct a circuit over $\{\wedge, \neg\}$.

3.1. **TA language.** To describe discrete functions, TRANSALG uses a domain specific language called TA language. The TA language has a C-like syntax and block structure. An arbitrary block (composite operator) is essentially a list of instructions, and has its own (local) scope. In the TA language, one can use nested blocks with no limit on depth. During the analysis of a program, TRANSALG constructs a scope tree with the global scope at its root. Every identifier in a TA program belongs to some scope. Variables and arrays declared outside of any block and also all functions belong to the global scope and therefore can be accessed in any point of a program.

A TA program is a list of functions. The `main` function is the entry point and, thus, must exist in every program. The TA language supports basic constructions used in procedural languages (variable declarations, assignment operators, conditional operators, loops, function calls, etc.), various integer operations and bit operations including bit shifting and comparison.

Similar to most symbolic execution systems, TRANSALG supports loops with fixed length and processes them via unwinding. It also supports conditional operators with any depth of nesting. On the level of ideas the corresponding solutions do not differ from those employed in symbolic verification systems, such as CBMC (see, e.g., [Kro09]). Briefly, the processing of a conditional operator is based on the following considerations. Each conditional operator of the kind `if then else` is associated with two arrays R_1 and R_2 in the memory of an abstract computing machine. The contents of these arrays represent two alternatives for data that will be in the memory of the machine after executing the conditional operator. With the cells of arrays R_1 and R_2 we first associate the encoding variables. Each encoding variable encodes the Boolean value that is the result of execution of this conditional operator.

The main data type in the TA language is the `bit` type, which can be used to specify arrays of bits of an arbitrary finite length. TRANSALG uses this data type to establish links between variables used in a TA program and Boolean variables included into a corresponding propositional encoding. It is important to note that TRANSALG does not establish such links for variables of other types, in particular `int` and `void`, which are used as service variables, e.g., as loop counters or to specify functions that do not return any value. We will refer to variables that appear in a TA program as *program variables*. All variables included in a propositional encoding are called *encoding variables*. Given a TA program A that specifies

f_n, TRANSALG constructs a propositional encoding of f_n. Below we will refer to this process as to the *translation* of the TA program A.

Declarations of global `bit` variables can have the `__in` or the `__out` attribute. The `__in` attribute marks variables that correspond to the algorithm's input. The `__out` attribute marks variables that correspond to the algorithm's output. Local `bit` variables cannot be declared with these attributes. Note, that the TA language strictly fixes the order in which it introduces Boolean variables at all steps of a considered algorithm. It means that the variables encoding the algorithm's input are always numbered from 1 to n, where n is the length of input in bits. The variables corresponding to output are always represented by the last m variables in an encoding, where m is the length of output in bits. Thus if necessary it is possible to exactly and explicitly associate input and output of an algorithm with the corresponding Boolean variables in a CNF (e.g., when manually invoking a SAT solver and processing its output).

3.2. **Techniques aimed at reducing the redundancy of propositional encodings.** In the process of symbolic execution of algorithms it is often the case that redundant variables and constraints are introduced. By calling them redundant we mean that they do not provide any additional information and can be safely removed. TRANSALG uses several techniques that often make it possible to significantly reduce the redundancy of a resulting SAT encoding.

The first technique exploits the fact that many algorithms can be represented in form of sequences of procedures which are simple and very similar to each other. Therefore, during the symbolic execution it is possible that the same Boolean formulas will be generated multiple times. Taking this fact into account, for each new formula TRANSALG first checks whether it is already present in the database. If the answer is "no", then the newly constructed formula is added to the database and associated with a new encoding variable. Otherwise, on the following steps the variable associated with the existing formula from the database is used. The approach is close to that introduced in [ABE00].

Another technique is related to the `__in` and `__out` attributes. Upon the generation of the resulting encoding, TRANSALG analyzes all functional dependencies of encoding variables on one another in order to define the minimally required set of variables that influence the construction of an output from an input. All the remaining variables and formulas defined using them are safely removed from an encoding without influencing its correctness. After this the variables are renumbered to exclude gaps.

The third technique aimed at reducing the number of auxiliary variables in a resulting propositional encoding works as follows. TRANSALG can use Boolean functions with arity $k > 2$ in the role of elementary operations over data in memory cells of its abstract machine. Therefore, as a result of each elementary step, a new encoding variable v is introduced and the following Boolean formula is constructed:

$$v \equiv \phi(\tilde{v}_1, \ldots, \tilde{v}_k). \tag{3.2}$$

Here, $\tilde{v}_1, \ldots, \tilde{v}_k$ are some encoding variables introduced at the previous steps. In other words, it is possible to represent f_n over any complete basis with arbitrarily complex basis functions. To transform (3.2) into CNF, TRANSALG uses the well-known ESPRESSO library [BSVMH84]. The arity of a function, which is given as an input to ESPRESSO, is often a very serious limitation: for functions with more than 20 inputs the performance of ESPRESSO is beginning to have a significant impact on the time of SAT encoding construction. To counter this issue, TRANSALG implements the ability to split formulas of this kind into several disjoint parts. Each such part is associated with a separate encoding variable. In the TA language these variables are declared using a special `__mem` attribute. The described technique gives the user more manual control: for example, using `__mem`, one can change the ratio between the number of variables and the number of clauses in the resulting SAT encoding.

3.3. **Cryptographic-specific features of TRANSALG.** TRANSALG also has several features that are specific for cryptographic algorithms and the use cases typical for algebraic cryptanalysis. In fact, one of the main features of TRANSALG that make it better for cryptographic problems is the `bit` type. The availability of this type is particularly useful when working with keystream generators that have shift registers of sizes which are not multiple of 8 (e.g., 19, 22, 23 bits). Note that in general purpose programming languages, such as C, memory is allocated in blocks of bits of a fixed size that is dependent both on the size of supported data types (8, 16, 32, 64 bits) and a particular compiler's implementation. Thus, a C program for a keystream generator with a register of size, say, 19 bits, inevitably processes excess data. For example, to represent such a shift register (of size 19) it would use a 32-bit integer variable. Consequently, this problem remains relevant for generic systems that employ the C language in that they require additional procedures to remove redundant Boolean variables from an encoding. Another important feature of TRANSALG is that it allows us to work with `bit` arrays simultaneously as arrays and as integer variables (represented in binary form). In particular, it can perform basic arithmetic operations (addition, subtraction, multiplication) with `bit` arrays without any additional data type transformations.

Cryptographic algorithms often use various bit shifting operators and also copy bits from one cell to another without changing their value. During the symbolic execution of such operators, there may appear elementary steps producing the formulas of the kind $v \equiv \tilde{v}$. However, we do not really need such formulas in a propositional encoding since it is evident that without the loss of correctness we can replace an arbitrary formula of the kind $v' \equiv \phi(v, \ldots)$ by a formula $v' \equiv \phi(\tilde{v}, \ldots)$. In such cases TRANSALG does not introduce new encoding variables.

When implementing cryptographic attacks in SAT form, it is sometimes desirable to manually track (and manipulate) the values of Boolean variables corresponding to specific program variables. The TA language allows us to add a special directive that outputs the numbers of encoding variables corresponding to specific program variables into the header of a propositional encoding. For this purpose it uses the `core_vars(<program variable>)` instruction. Here `program variable` can be a `bit` variable or a `bit` array. During the translation of a TA program, TRANSALG will put into the DIMACS file header the numbers of encoding variables associated with specified program variable at the moment when the `core_vars()` instruction is executed. Usually, the obtained variable numbers are used either to parameterize a SAT solver or in special heuristics, such as in [DKV07].

It is often demanded by attacks to impose specific constraints on the values of variables at certain steps of an algorithm. Since TRANSALG monitors the values of program variables inside TA program at any step of computing, it also allows us to impose any constraints on such variables. For this purpose the TA language uses the `assert(<expression>)` instruction. This instruction assumes that `expression` takes the value of $True$. The Boolean formula corresponding to `expression` is added to the resulting propositional encoding.

Note, that the functionality related to `core_vars` and `assert` instructions can only be used if TRANSALG outputs the propositional encoding in the DIMACS CNF format.

3.4. **Example of a TA program and its translation.** Let us consider the following example. Its goal is to demonstrate how new encoding variables and constraints involving them are introduced in the course of interpretation of a simple TA program. Note, that in this example we do not consider all technical nuances of the propositional encoding procedures implemented in TRANSALG. In particular, here we completely omit the post-processing stage, at which the tool removes all the variables and constraints that are redundant (because they do not influence the output in any way). The variable reindexing is also performed during the post-processing.

Example 3.1. Consider an encoding of a linear feedback shift register (LFSR) [MVO96] with TRANSALG. In Figure 1, we show the TA program for the LFSR with feedback polynomial $P(z) = z^{19} + z^{18} + z^{17} + z^{14} + 1$ over $GF(2)$ (here z is a formal variable). Let

```
define e 100;
__in bit reg[19];
__out bit output[e];

bit shift_reg(){
   bit u = reg[18];
   bit v = reg[18]^reg[17]
           ^reg[16]^reg[13];
   reg = reg >> 1;
   reg[0] = v;
   return u;
}
void main(){
   for(int i = 0; i < e; i = i+1)
      output[i] = shift_reg();
}
```

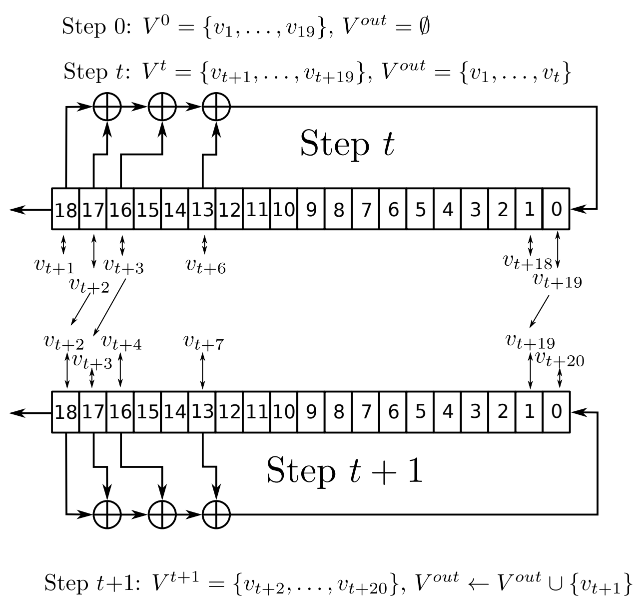


FIGURE 1. TA program for LFSR

us view the process of executing this TA program as a sequence of data modifications in a memory of an abstract computing machine at moments $\{1, \ldots, e\}$. At every moment $t \in \{0, 1, \ldots, e\}$, TRANSALG links a set V^t of encoding variables with program variables of the `bit` type. Denote $V = \bigcup_{t=0}^{e} V^t$. Let us separately denote by V^{in} and V^{out} the sets formed by encoding variables that correspond to input and output data, respectively. Note that during the translation of transition from moment t to moment $t+1$ it is not necessary to create new encoding variables for every cell of the register. If we copy data from one register cell to another, then we can use the same encoding variable to represent the corresponding data value at moments t and $t+1$. Therefore, at each moment TRANSALG creates only one new encoding variable and links it with program variable `reg[0]`. All the other program variables get linked with encoding variables created at previous moments.

In accordance with the above, the set of encoding variables corresponding to the initial values of the register is $V^{in} = V^0 = \{v_1, \ldots, v_{19}\}$. After each shift we encode values of register's cells with sets

$$V^1 = \{v_2, v_3, \ldots, v_{20}\}, V^2 = \{v_3, v_4, \ldots, v_{21}\}, \ldots, V^e = \{v_{e+1}, v_{e+2}, \ldots, v_{e+19}\}.$$

Note that $V^{out} = \{v_1, \ldots, v_e\}$. Thus the set of encoding variables for this program is $V = \{v_1, v_2, \ldots, v_{e+19}\}$, and the corresponding variables are connected between each other by the following Boolean formulas (here $\oplus$ stands for addition modulo 2):

$$\begin{array}{ccc} v_{20} & \equiv & v_1 \oplus v_2 \oplus v_3 \oplus v_6 \\ & \ldots & \\ v_{e+19} & \equiv & v_e \oplus v_{e+1} \oplus v_{e+2} \oplus v_{e+5}. \end{array}$$

Note that in the example the register's size is not a multiple of 8. As it was noted above, in the tools that employ general purpose programming languages, such as C, the description of such an algorithm would require either using a 32-bit integer variable to store register's state or using nineteen 8-bit variables each representing a single bit. Either way there would be quite a lot of unused bits in a constructed SAT encoding, which would need to be taken care of.

4. Comparison of tools for constructing SAT encodings of cryptanalysis problems

In the present section, we briefly describe major software tools that can translate cryptographic algorithms to SAT, compare their functional capabilities with that of Transalg, and study the performance of state-of-the-art SAT solvers on several classes of SAT encodings obtained by all discussed tools.

4.1. **Tools for translating cryptographic algorithms to SAT.**

We are aware of several domain-specific software tools (besides Transalg) that can be used to encode cryptographic algorithms to SAT. Below we provide their brief description.

The Grain-of-Salt tool [Soo10] is designed to produce SAT encodings only for cryptographic keystream generators based on the shift registers. It uses a special declarative language to describe each of keystream registers (by means of its feedback polynomials) and the general configuration of a generator. Unfortunately, Grain-of-Salt does not support many standard cryptographic operations and therefore can be applied only to a limited spectrum of cryptographic functions.

URSA (a system for Uniform Reduction to SAT [Jan12]) is a propositional encoding tool that is applicable to a wide class of combinatorial problems, varying from CSP (Constraint Satisfaction Problem) to cryptography. To describe these problems, URSA employs a declarative domain-specific language. The constructed SAT instances can be solved using two embedded solvers: ARGOSAT [Mar09] and CLASP [GKNS07].

Cryptol is a domain-specific Haskell-like programming language for specifying cryptographic algorithms [LM03, EM09, ECW09]. Software Analysis Workbench (SAW) [CFH+13] allows for producing SAT and SMT encodings of cryptographic problems described in Cryptol. Further we refer to SAW that takes as an input a program written in Cryptol as SAW+Cryptol.

Another major class of tools that can translate cryptographic algorithms to SAT is formed by the systems designed for software verification. Probably the most well known and powerful representative of this class is the CBMC tool (Bounded Model Checking for ANSI C [CKL04]). CBMC works with programs written in the C language. Note, that the primary application domain of CBMC is software verification. Therefore, for each program there should be a hypothesis that needs verifying or falsifying. Putting cryptanalysis attacks in this context requires some paradigm adjustment, but in the case of CBMC it can be done quite easily.

4.2. **Computational comparison of tools.** To compare the effectiveness of propositional encodings produced by the aforementioned tools, we chose several cryptographic keystream generators. Here they are, ordered by the resistance to SAT-based cryptanalysis (from the weakest to the strongest): Geffe [Gef73], Wolfram [Wol86], Bivium [Can06] and Grain [HJM07]. The Geffe generator is a simple generator, which is not resistant to many cryptographic attacks including the correlation attack proposed in [Sie84]. We considered the strengthened Geffe generator (to which we further refer as S_Geffe), which is a particular case of the threshold generator [Bru84]. It turned out that S_Geffe with a secret key length up to 160 bits is not resistant to SAT-based cryptanalysis (when implementing the known plaintext attack). We considered the variant of the S_Geffe generator that uses three LFSRs defined by the following primitive polynomials:

$$\begin{array}{l} z^{52} + z^{49} + 1; \\ z^{53} + z^{52} + z^{38} + z^{37} + 1; \\ z^{55} + z^{31} + 1. \end{array}$$

Thus, the considered generator has a secret key of 160 bits.

Unlike many other generators, the Wolfram generator does not use shift registers. It is based on a one-dimensional cellular automaton [von51]. This generator is not resistant to the attack proposed in [MS91] if its secret key length is less than 500 bits. Meanwhile, the cryptanalysis of the Wolfram generator with the secret key length of more than 150 bits is already a hard problem for state-of-the-art SAT solvers. Therefore, below we consider the version of the generator with 128-bit secret key. The Bivium generator [Can06] is a popular object for SAT-based and algebraic cryptanalysis. Nevertheless, SAT-based cryptanalysis of Bivium is quite a hard problem that, as the estimations show [SZ16], can be solved in reasonable time in a powerful distributed computing environment. Finally, we considered the Grain generator [HJM07], in particular, its v1 version. There are no known attacks on this version that are better than the brute force attack.

For each mentioned generator a SAT-based variant of the known plaintext attack was studied. It means that the following problem was considered: to invert a function of the form $g : \{0, 1\}^n \to \{0, 1\}^m$, where n is the amount of bits of registers' state that produces the analyzed keystream, and m is the length of the analyzed keystream. Using SAW+Cryptol, URSA, CBMC, and Transalg, we built propositional encodings of the following functions:

$$\begin{array}{lcl} g^{S\_Geffe} & : & \{0,1\}^{160} \to \{0,1\}^{250}, \\ g^{Wolfram} & : & \{0,1\}^{128} \to \{0,1\}^{128}, \\ g^{Bivium} & : & \{0,1\}^{177} \to \{0,1\}^{200}, \\ g^{Grain} & : & \{0,1\}^{160} \to \{0,1\}^{160}. \end{array}$$

It should be noted that since Grain-of-Salt operates only with shift registers, it was not possible to construct SAT encodings of the Wolfram generator via this tool. In Table 1 the obtained encodings are compared by the amount of variables, clauses, and literals.

Table 1. The characteristics of SAT encodings.

	Grain-of-Salt	URSA	SAW+Cryptol	Transalg	CBMC
S_Geffe					
Vars	1 910	2 394	1 883	1 000	2 668
Clauses	8 224	8 436	6 891	6 474	9 514
Literals	28 976	23 308	19 793	25 226	26 536
Wolfram					
Vars	-	24 704	24 620	12 544	32 904
Clauses	-	86 144	85 784	74 112	114 830
Literals	-	233 600	232 811	246 400	311 460
Bivium					
Vars	842	1 637	1 432	1 172	1 985
Clauses	6 635	5 975	5 308	7 405	7 044
Literals	29 455	16 995	15 060	29 745	19 866
Grain					
Vars	4 546	9 279	4 246	1 945	10 088
Clauses	74 269	37 317	16 522	34 165	40 596
Literals	461 069	105 925	46 402	190 388	115 178

At the second stage, we used the constructed encodings for implementing the known plaintext attack on the described generators. As it was mentioned above, the inversion problems for $g^{S\text{-}Geffe} : \{0,1\}^{160} \to \{0,1\}^{250}$ and $g^{Wolfram} : \{0,1\}^{128} \to \{0,1\}^{128}$ are quite easy even for sequential SAT solvers. Meanwhile, the inversion problem for $g^{Bivium} : \{0,1\}^{177} \to \{0,1\}^{200}$ is hard even for the best parallel SAT solvers. The inversion problem for $g^{Grain} : \{0,1\}^{160} \to \{0,1\}^{160}$ is extremely hard and cannot be solved in reasonable time in any modern distributed computing system that we are aware of. That is why we studied weakened variants of the last two problems. In particular, SAT solvers were given some information about unknown registers' state. In the case of Bivium, the last 30 bits of the second register were assumed to be known. In the case of Grain, 102 (out of 160) bits were known: the whole 80-bit linear register and the last 22 bits of the 80-bit nonlinear register. The constructed SAT instances are denoted by Bivium30 and Grain102, respectively.

For each considered generator, a set of 100 SAT instances was constructed by generating random values of corresponding registers states, which were used to produce keystreams. It means that for each generator we randomly constructed 100 variants of registers' initial states, then used the implementation of the generator to produce the corresponding 100 fragments of keystream. After this, we constructed SAT instances (by each of the considered tools) and SMT instances (by SAW+Cryptol and CBMC) for all generated pairs of initial states and keystream fragments. The goal of using the pre-generated pairs of initial register's states and keystream fragments is to perform the testing in equal conditions: indeed, it is often the case that for some keystream fragments the inversion problems are easier to tackle due to specific features of a corresponding algorithm. Thus it makes sense to compare over the exact same cryptanalysis problems encoded to SAT in different forms. All constructed SAT and SMT instances are available online [OGZ+].

Let us give a few more comments on the matter. It turned out that the Grain-of-Salt tool neither allows for constructing a template CNF nor has any instructions to assign the Boolean variables corresponding to keystream bits to some pre-specified values. It basically generates and uses a random keystream fragment every time it is run. Thus, we had to resort to extracting the randomly generated keystream fragment (and initial state) from the Grain-of-Salt encoding and using it to assign the necessary bits in the encodings constructed by all the other tools. To construct such encodings via Transalg and CBMC we used template CNFs. In SAW+Cryptol and URSA each encoding was constructed using individual program to which pre-specified data was added with special instructions.

In the experiments, we employed the SAT solvers that took prizes on the last SAT Competitions and also several SAT solvers which have shown good results on SAT-based cryptanalysis problems: MapleLcmDistChronoBt [NR18], Maple_LCM_Dist [XLL+17], MapleCOMSPS_LRB_VSIDS_2 [LOG+17], MapleSAT [LGPC16], Glucose [AS17], Maple_LCM_Scavel [XWCC18], COMiniSatPS Pulsar [Oh17], Cryptominisat5 [SNC09], Lingeling [Bie17], MiniSat [ES04], and Rokk [YO14].

We also used the SMT solvers that took prizes on the last SMT Competitions: CVC4 [BCD+11], Z3 [dMB08], and Yices [Dut14]. In all experiments described below, we employed the HPC-cluster "Academician V.M. Matrosov" [mat] as a computing platform. Each node of this cluster is equipped with two 18-core Intel Xeon E5-2695 CPUs. Each solver was run in the sequential mode (on 1 CPU core) with the time limit of 5000 seconds (following the rules of SAT competitions).

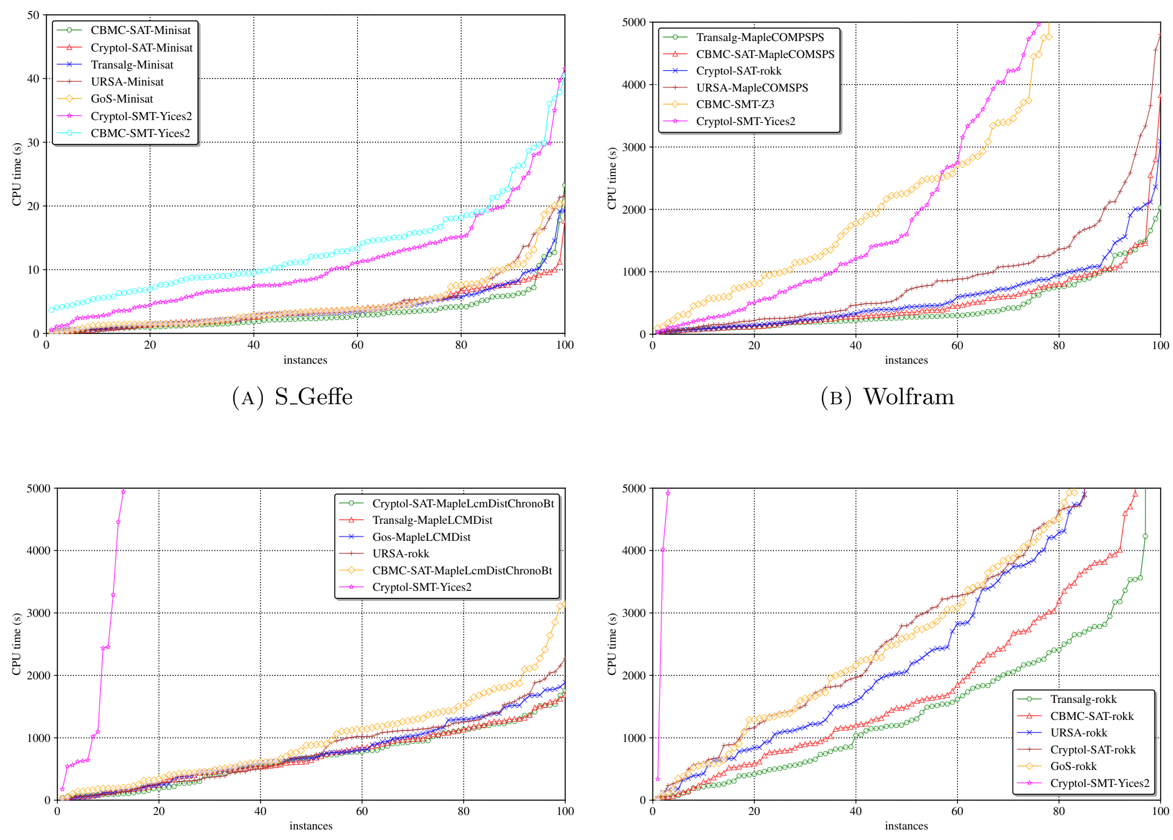
Table 2. Solving cryptanalysis problems for S_Geffe, Wolfram, Bivium, and Grain. For each considered pair (keystream generator, tool) only results obtained by the best solver are shown. Time is shown in seconds.

	TrAlg	GoS	URSA	Cr-SAT	Cb-SAT	Cr-SMT	Cb-SMT
S_Geffe							
Solver	MiniSat	MiniSat	MiniSat	MiniSat	**MiniSat**	yices	yices
Solved	100/100	100/100	100/100	100/100	**100/100**	100/100	100/100
Avg. time	4.05	4.61	4.64	3.98	**3.26**	11.17	13.61
Wolfram							
Solver	**MaCom**	-	MaCom	rokk	MaCom	yices	z3
Solved	**100/100**	-	100/100	100/100	100/100	76/100	78/100
Avg. time	**442**	-	931	614	536	1631	1844
Bivium30							
Solver	MaLcm	MaLcm	rokk	**MaChr**	MaChr	yices	-
Solved	100/100	100/100	100/100	**100/100**	100/100	13/100	0/100
Avg. time	725	781	788	**695**	995	1759	-
Grain102							
Solver	**rokk**	rokk	rokk	rokk	rokk	yices	-
Solved	**97/100**	83/100	85/100	85/100	95/100	3/100	0/100
Avg. time	**1407**	2290	2038	2364	1737	3090	-

In Table 2 for each considered pair (generator, tool) only the results obtained by the best solver are shown. In this table we use the following abbreviations:

- MaChr for MapleLcmDistChronoBt;
- MaCom for MapleCOMSPS_LRB_VSIDS_2;
- MaLcm for Maple_LCM_Dist;
- Cr-SAT for SAW+Cryptol_SAT;
- Cr-SMT for SAW+Cryptol_SMT;
- Cb-SAT for CBMC_SAT;
- Cb-SMT for CBMC_SMT;
- TrAlg for Transalg.

For each generator the best result is marked with bold. The corresponding cactus plots are shown in Figure 2. The detailed data for all considered solvers can be found in Appendix A. All cactus plots in the present paper were made by the mkplot script [Ign].



(A) S_Geffe (B) Wolfram

(C) Bivium30 (D) Grain102

Figure 2. Comparison of encodings on the best solvers

Let us comment on the obtained results. On the considered problems SAT solvers significantly outperform the SMT solvers. Among the latter, Yices showed the best results. As for SAT, the Transalg encodings showed the best results on Wolfram and the weakened

Grain, SAW+Cryptol outperformed the competitors on the weakened Bivium, while CBMC showed the best results on S_Geffe.

4.3. **Discussion.** The tests we considered above to compare different tools for constructing SAT encodings in fact represent one of the most simple cryptographic attacks in SAT form: each a single instance with values of variables (which weaken an instance) fixed in a straightforward manner by assigning values to variables directly. In practical cryptographic attacks it is often necessary to introduce more complex constraints. They may link together several Boolean variables, or require solving sequences of instances, where the next instance is constructed using the information obtained once the previous is solved, etc. Thus, in practice the specific abilities of the propositional encoding systems that make it possible to impose additional constraints and tune a SAT encoding become exceptionally important.

In this context, we would like to specifically remind the reader about several Transalg features described in Section 3 that prove to be quite useful when implementing various SAT-based cryptographic attacks. The distinctive feature is that Transalg can construct and explicitly output a template CNF $C(f_n)$ (see Definition 2.2) with a number of specific characteristics. A template CNF essentially encodes the algorithm in a pure symbolic execution sense. It has the clearly outlined input and output variables, and all the auxiliary and output variables depend on the input variables. As a result, in template CNFs constructed by Transalg, the variables are represented and numbered in the order in which they are introduced during the translation process. It means that the first variables correspond to a function's input and the last variables to its output. The Boolean variables not relevant to constructing the output are pruned out during the propositional encoding process. If necessary, Transalg performs re-numbering of variables to avoid gaps.

Thanks to the outlined sets of input and output variables, it is easy to use template CNFs to test the correctness of constructed SAT encodings, as well as to directly interpret satisfying assignments found by a SAT solver. The fact that in template CNFs the variables corresponding to the function input are outlined, makes it possible to use them for implementing the partitioning strategy [Hyv11] in a distributed computing environment (see for example [SZ15, SZ16]) without specific preparation. Another important feature of input variables is that they form a SUPBS [WGS03] in this template CNF. We will touch these questions in more detail in Section 5. Also, using template CNFs we can quickly generate families of instances that encode a cryptanalysis problem: to make a certain SAT instance for function inversion it is sufficient to add to a template CNF the unit clauses encoding the corresponding output.

When translating an algorithm to SAT, the data structures employed by Transalg preserve all the connections between the introduced Boolean variables and the corresponding cells of the abstract machine's memory. This fact allows one to effectively write auxiliary constraints on arbitrary subsets of program variables. For this purpose the corresponding constraints are introduced in a TA program using the `assert` instruction: they will then be correctly transferred to a resulting SAT encoding. This feature is very important when implementing the SAT variants of the so-called differential attacks on cryptographic hash functions [WLF+05, WY05]. In CBMC such constraints can be imposed by using the `assume` instruction.

In Table 3 we compare all considered systems with respect to several criteria. Let us give additional comments to it. Technically, CBMC and URSA can construct template CNFs. The CBMC system constructs template CNFs directly, but in order to do so one has to add to the corresponding program an empty hypothesis. For all practical purposes the obtained SAT encoding is a template CNF (despite having unit clauses induced by empty hypothesis), but this step does not look entirely natural. URSA adds the values of variables to a constructed SAT encoding by means of unit clauses. It also has an option of outputting the mapping between the variables from a program in its domain-specific language and the Boolean variables in a constructed SAT encoding. However, to make a template CNF one has to parse the URSA output and remove the unit clauses corresponding to function's outputs. To the best of our knowledge, both SAW+Cryptol and Grain-of-Salt have no option to output a template CNF with or without additional steps.

Table 3. Main functional abilities of Grain-of-Salt, URSA, SAW+Cryptol, CBMC, and Transalg.

System	GoS	URSA	SAW+Cryptol	CBMC	Transalg
Encodes conditional operators	-	+	+	+	+
Supports procedures and functions	-	-	+	+	+
Has embedded solvers	-	+	+	+	-
Constructs SMT encodings	-	-	+	+	-
Outlines sets of input and output variables	+	+	-	+	+
Constructs template CNFs	-	-	-	+/-	+
Adds auxiliary constraints on variables inside a program	-	-	-	+	+

Regarding the sets of input and output variables – almost all tools can output the corresponding information. However, in Transalg the input variables are always the first and the output variables are always the last, and always in the order specified in a TA program. The other tools usually provide the mapping of program variables to Boolean variables, but the ones corresponding to inputs and outputs are usually spread in the set of variables and rarely follow any specific order.

As for writing auxiliary constraints on arbitrary subsets of program variables, it appears that only Transalg and CBMC provide this functionality directly. The mapping of program variables to encoding ones (provided by URSA, for example) theoretically enables one to add them manually, but this is a very arduous process.

The remaining points in Table 3 reflect the richness of a language employed by a propositional encoding tool and its system capabilities such as whether it has embedded solvers and the ability to work with SMT encodings.

We believe that the lack of embedded solvers in Transalg and other tools should not be viewed as a drawback, at least in the cryptographic context. Indeed, the cryptanalysis instances are typically very hard, and to solve them one usually has to use the best available tools, such as cutting edge SAT solvers, including the parallel ones. Taking into account the fact that almost every year the SAT solvers become better and better (which can be tracked in annual SAT competitions), it is actually quite hard to maintain the list of embedded solvers up to date at all times. The existence of various API and Python interfaces for SAT solvers, e.g., [IMMS18] allows for using external libraries for fast prototyping and testing.

As for the ability to output SMT encodings, our empirical evaluation to date shows that SMT solvers typically work worse on cryptanalysis instances compared to cutting edge SAT solvers. We believe that adding this functionality to Transalg and investigating why SMT solvers often lose in comparisons similar to that summarized in, e.g., Table 2 is an interesting direction for future development and research.

5. Inversion of real world cryptographic functions using their translation to SAT

In this section, we present our results on SAT-based cryptanalysis of several ciphering systems. Some of these systems were used in practice in recent past, and some of them are still employed at the present moment. In all cases considered below we used the encodings produced by the Transalg system.

5.1. **Using Transalg to construct SAT-based guess-and-determine attacks on several ciphers.** Guess-and-determine is a general cryptanalysis strategy that can be used to evaluate the cryptographic weakness of various ciphers. The number of such attacks proposed in the recent two decades is very large. Here we would like to cite the book [Bar09], the major part of which studies guess-and-determine approach in the context of algebraic cryptanalysis.

The basic idea of the guess-and-determine strategy can be described as follows. Let F be an arbitrary cipher that works with binary data, and let $E(F)$ be a system of Boolean or algebraic equations that corresponds to some cryptanalysis problem for F. For example, for a given known pair (x, y), where x is a plaintext and y is a ciphertext, $E(F)$ can be a system from the solution of which one can extract the key z such that $F(x, z) = y$. Let us denote by X the set of all variables from $E(F)$. Let $\tilde{X}$, $\tilde{X} \subseteq X$ be such a set of l Boolean variables, that by assigning values to all variables from $\tilde{X}$ the problem of solving $E(F)$ becomes trivial. The simplest example in this context is when $\tilde{X}$ consists of variables corresponding to a secret key of a cipher F. Also $\tilde{X}$ can be formed by variables corresponding to the internal state of a cipher at some time moment, for example, to an internal state of keystream generator registers at some fixed step. In these cases by checking all possible assignments for variables from $\tilde{X}$, i.e. by performing exhaustive search over the set $\{0, 1\}^l$, we perform a *brute force* attack on a cipher F.

For some ciphers it is possible to find a set B, $B \subseteq X$, $|B| = s$, which has the following property. Let us consider all possible assignments of variables from B. For each assignment we set the corresponding values to variables from B into a system $E(F)$ and spend on solving each constructed weakened system by some fixed algorithm at most t seconds (or any other fitting complexity measure). Assume that if we search over the whole $\{0, 1\}^s$ in such a way, we find the solution for a considered cryptanalysis problem, and spend on this process at most $T = 2^s \cdot t$. In this case we can say that there is a *guess-and-determine attack* with complexity T. The set B is called a *set of guessed bits*. Guess-and-determine attacks with a complexity significantly less than that of brute force attack are of particular interest. Usually, for such attacks $s \ll l$.

In SAT-based cryptanalysis the problems of finding solutions for systems of equations of the type $E(F)$ are reduced to SAT. To construct a guess-and-determine attack in this case, one needs to know additional information about variables contained in the corresponding CNF. In particular, it is very useful to know which CNF variables correspond to secret key bits. The methodology of constructing template CNFs employed in TRANSALG allows one to naturally outline sets of guessed bits, and traverse the search space of such sets to construct guess-and-determine attacks with good runtime estimations (see, e.g., [SZ16, ZK17]). Below we will briefly describe guess-and-determine attacks on several ciphers for which SAT encodings were produced by TRANSALG.

In [SZBP11], a SAT-based guess-and-determine attack on the A5/1 keystream generator was constructed. In that attack the set of guessed bits of size 31 was used. The attack from [SZBP11] was later verified in the volunteer computing project SAT@home [PSZ12] by solving several dozens of the corresponding SAT instances using the technique from [SZBP11]. Later in [SZ15, SZ16] an automatic method for finding sets of guessed bits via optimization of a special function was proposed. A value of the function for a particular set of guessed bits is the runtime estimation of a corresponding guess-and-determine attack. Using this method, in [SZ16] a guess-and-determine attack on the Bivium cipher [Can06] was constructed. The corresponding runtime estimation makes this attack realistic for state-of-the-art distributed computing systems. Using the algorithms from [SZ16], in [ZK17] guess-and-determine attacks on several variants of the alternating step generator were constructed and implemented on a computing cluster.

In [SZO$^+$18], a new class of SAT-based guess-and-determine cryptographic attacks was proposed. It is based on the so-called *Inverse Backdoor Sets* (IBS). IBS is a modification of the notion of Strong Backdoor Set for Constraint Satisfaction Problems (including SAT), introduced in [WGS03]. IBS is oriented specifically on problems of SAT-based cryptanalysis. In more detail, a *Strong Backdoor Set* for a CNF C with respect to a polynomial algorithm A is such a subset B of a set of variables X of this CNF that setting values to variables from B in C in any possible way results in a CNF for which SAT is solved by A. This definition in its original form does not suit well to cryptanalysis problems. However, in [SZO$^+$18] we modified it in the context of discrete functions inversion problems. Conceptually, the modification consists in the following. Instead of demanding that A is polynomial, we limit its runtime by some value t. In the role of the set of guessed bits we use an arbitrary $B \subseteq X$ and demand that algorithm A can invert some portion of outputs of function f_n constructed for random inputs from $\{0,1\}^n$ in time $\leq t$ using the assignments of variables from B as hints. The portion of inverted outputs is the probability of a particular random event. If it is relatively large and the power of B is relatively small, then, as it was shown in [SZO$^+$18], B can be used to construct on its basis a nontrivial guess-and-determine attack on f_n. The set B defined in such a way is called *Inverse Backdoor Set (IBS)*. The effectiveness of guess-and-determine attacks based on IBS can be evaluated using the Monte Carlo method [MU49]. The problem of finding IBS with good runtime estimation of a corresponding guess-and-determine attack in [SZO$^+$18] was reduced to a problem of optimization of a black box function over the Boolean hypercube. Using IBSs, in [SZO$^+$18] the best or close to the best guess-and-determine attacks on several ciphers were constructed. In particular, the runtime estimation of the constructed attack using 2 known plaintext (2KP) on the block cipher AES-128 with 2.5 rounds is several dozen times better than that of the attack from [BDF11] and requires little to no memory, while the attack from [BDF11] needs colossal amounts of it. In [PSU19, PBU19] several evolutionary algorithms were used to minimize black box functions introduced in [SZO$^+$18] resulting in new guess-and-determine attacks on several keystream generators.

Let us once again emphasize the important features of translating algorithms to SAT, which are advantageous in the context of construction of guess-and-determine attacks. Here we first and foremost mean an ability to provide information about the interconnection between the variables in propositional encodings with corresponding elementary operations performed in an original cryptographic algorithm. For example, the ability to outline the variables that encode an input of a considered function for attacks described in [SZO+18] allows us to use template CNFs for effective generation of large random samples containing simplified CNFs. Each of them is formed by applying Unit Propagation to a template CNF augmented by the values corresponding to the known function input. When implementing a guess-and-determine attack with realistic runtime estimation, Transalg's features make it possible to naturally mount the attack using the incremental SAT technique [ES03]. In some cases it can lead to significant performance gains (see, e.g., [ZK17]).

5.2. **SAT-based cryptanalysis of hash functions from MD family.** In this subsection, we present examples of application of the Transalg system to cryptanalysis of cryptographic hash functions from the MD family. It should be noted that these functions are still considered to be interesting among cryptanalysts, and the first successful examples of application of SAT-based cryptanalysis to real world cryptosystems are related specifically to hash functions from the MD family [MZ06]. The present subsection is split into several parts: first we consider the problems of finding collisions for MD4 and MD5 functions. Then we construct preimage attacks on truncated variants of the MD4 function.

5.2.1. *Finding Collisions for MD4 and MD5.* Let $f : \{0,1\}^+ \to \{0,1\}^c$ be some cryptographic hash function that works with messages split into blocks of length n, $n > c$. It defines a function of the kind $f_n : \{0,1\}^n \to \{0,1\}^c$. To produce a SAT encoding for the problem of finding collisions of this function, we essentially translate the program describing f_n twice using disjoint sets of Boolean variables. Let C_1 and C_2 be the corresponding CNFs in which the sets of input and output variables are denoted by $X^1 = \{x_1^1, \ldots, x_n^1\}$, $X^2 = \{x_1^2, \ldots, x_n^2\}$ and $Y^1 = \{y_1^1, \ldots, y_c^1\}$, $Y^2 = \{y_1^2, \ldots, y_c^2\}$, respectively. Then finding collisions of f_n is reduced to finding an assignment that satisfies the following Boolean formula:

$$C_1 \wedge C_2 \wedge (y_1^1 \equiv y_1^2) \wedge \ldots \wedge (y_c^1 \equiv y_c^2) \wedge \left((x_1^1 \oplus x_1^2) \vee \ldots \vee (x_n^1 \oplus x_n^2)\right). \tag{5.1}$$

Below we consider the problems of constructing collisions for the MD4 and MD5 hash functions, which were actively used up until 2005. Let us first briefly remind the reader about features of the Merkle–Damgård construction [Mer89, Dam89], which serves as a basis of many cryptographic hash functions. In accordance with this construction, in MD4 and MD5 the process of computing a hash value is considered as a sequence of transformations of data stored in a special 128-bit register, to which we will refer as a *hash register*. The hash register is split into four 32-bit cells. At the initial stage a message called *Initial Value* (IV), which is specified in the algorithm's standard, is put into a hash register. Then the contents of the hash register are mixed with a 512-bit block of a hashed message by means of iterative transformations called *steps*. There are 48 and 64 steps in MD4 and MD5, respectively. At each step a 32-bit variable is associated with an arbitrary cell of a hash register. Such variables are called *chaining variables*. The transformations of data in a hash register, which were defined above, specify the so-called *compression function*. We will denote the compression functions used in MD4 and MD5 as f^{MD4} and f^{MD5}, respectively. The MD4 and MD5 algorithms can be used to construct hash values for messages of an arbitrary length. For this purpose an original message is first padded so that its length becomes a multiple of 512, and then is split into 512-bit blocks. Let $M = M_1, \ldots, M_k$ be a k-block message, where

M_i, $i \in \{1, \ldots, k\}$ are 512-bit blocks. A hash value for message M is constructed iteratively according to the following recurrence relations: $\chi_0 = IV$, $\chi_i = f^{MD}(\chi_{i-1}, M_i)$, $i = 1, \ldots, k$ (here MD is either MD4 or MD5). A word χ_k is the resulting hash value of a message M. If hash values of two different k-block messages coincide, then the corresponding messages form a *k-block collision* of the considered hash function.

The MD4 and MD5 algorithms were completely compromised with respect to finding collisions in [WLF+05, WY05]. The cryptanalysis methods used in the mentioned papers belong to a class of the so-called *differential attacks*. In the attacks from [WLF+05, WY05], the MD4 and MD5 hash functions were applied to two different messages. The processes of constructing hash values for these messages correspond to transformations of the contents of two hash registers. The main feature of differential attacks is that additional constraints are imposed on chaining variables associated with the corresponding cells of hash registers in the form of integer differences modulo 2^{32}. Also, special constraints on individual bits of these chaining variables can be used. These two types of constraints form the so-called *differential path*. In [WLF+05, WY05], differential paths were proposed that make it possible to effectively construct single-block collisions for MD4 and two-block collisions for MD5.

As we already mentioned, the first SAT variants of attacks from [WLF+05, WY05] were constructed in [MZ06]. To obtain a corresponding propositional encoding, it is first necessary to construct a formula of the kind (5.1) and then transform it to CNF using the Tseitin transformations. However, the resulting CNF turns out to be extremely hard even for state-of-the-art SAT solvers. A realistically feasible runtime of cryptanalysis is achievable only by adding to a constructed CNF the clauses which encode a differential path. In our experiments, the special `assert` instruction of the TA language made it possible to implement this step quite easily. We would like to additionally note that the constraints defining a non-zero differential path eliminate the need for constraints of the kind $\big((x_1^1 \oplus x_1^2) \vee \ldots \vee (x_n^1 \oplus x_n^2)\big)$ in (5.1), which indicate the difference between sets of values of the input variables (since only different inputs of a hash function can lead to a non-zero differential path).

In Table 4, we compare the SAT encodings of differential attacks for finding collisions for MD4 and MD5 used in [MZ06] with those constructed by Transalg.

Table 4. The parameters of SAT encodings for finding collisions of the MD4 and MD5 hash functions with differential paths from [WLF+05, WY05].

		SAT encodings from [MZ06]	Transalg encodings
MD4	variables	53228	18095
	clauses	221440	187033
MD5	variables	89748	34181
	clauses	375176	295773

For the problem of finding single block collisions of the MD4 hash function, we managed to find about 1000 MD4 collisions within 200 seconds on one core of Intel i7-3770K (16 Gb RAM) using the SAT encodings produced by Transalg and the Cryptominisat solver [SNC09]. Note that in [MZ06] it took about 500 seconds to construct one single block collision for MD4.

After this we studied the problem of finding two-block collisions of MD5. This process consists of two stages. At the first stage, we search for two 512-bit blocks M_1 and M_1'

that satisfy the differential path from [WY05]. We denote $\chi_1 = f^{MD5}(IV, M_1)$, $\chi'_1 = f^{MD5}(IV, M'_1)$. At the second stage, we look for second 512-bit blocks M_2 and M'_2 (which also satisfy the differential path from [WY05]) such that $f^{MD5}(\chi_1, M_2) = f^{MD5}(\chi'_1, M'_2)$.

For the problem of finding a first message blocks pair (M_1, M'_1), which turned out to be quite hard, we used the HPC-cluster "Academician V.M. Matrosov" [mat]. We ran state-of-the-art SAT solvers working in the multi-threaded mode (36 threads) on the cluster. In particular, we used PLINGELING, TREENGELING (versions from the SAT competition 2014 [Bie14]) and PLINGELING, TREENGELING [Bie17], PAINLESS [FBSK17], GLUCOSE-SYRUP [AS17] from the SAT competition 2017. Surprisingly, only TREENGELING 2014 managed to solve the corresponding SAT instances within a time limit (30 hours).

During these experiments, several message blocks with a lot of zeros in the beginning were found. A more detailed analysis showed that the maximum number of first message bytes that can be set to 0 simultaneously in M_1 and M'_1 is 10 bytes. Assignment of the 11^{th} byte to 0 in M_1 and M'_1 makes the corresponding SAT instance unsatisfiable (which can be proven quickly). Thus we outlined the class of message blocks pairs that satisfy the differential path from [WY05] and both blocks have first 10 zero bytes. The problem of finding a pair of such blocks is relatively simple and can be solved using a number of SAT solvers (compared to the situation when the first 10 bytes are not set to zero). On the corresponding SAT instance we ran four different SAT solvers (PAINLESS, GLUCOSE-SYRUP, TREENGELING 2014, TREENGELING 2017) each working in multi-threaded mode on one cluster node. In 24 hours each solver managed to find several message blocks pairs, except for TREENGELING 2014 SAT solver, which found only one. The corresponding results are given in Table 5.

Table 5. Finding a pair M_1, M'_1 that satisfies the differential path from [WY05], and both M_1 and M'_1 have first 10 zero bytes.

SAT solver	Solved instances	Avg. time (s)
PAINLESS	3	32327
GLUCOSE-SYRUP	3	38302
TREENGELING 2014	1	54335
TREENGELING 2017	3	33357

For the obtained pairs of first blocks, the problem of constructing such pairs (M_2, M'_2) that the messages $M_1|M_2$ (the concatenation of two 512-bit blocks M_1 and M_2) and $M'_1|M'_2$ form a two-block collision for MD5 turned out to be much simpler: on average one such pair (M_2, M'_2) can be found by any considered parallel solver in less than 500 seconds on one cluster node. An example of the two-block collision of the described kind is shown in Table 6.

In conclusion we would like to once more point out features of the Transalg system that made it possible to obtain the presented results. It is mainly thanks to the translation concept of Transalg that allows one to directly work with variables encoding each elementary step of a considered algorithm. That is why we can effectively reflect in SAT encoding any additional constraints, such as, for example, the ones that specify a differential path. As far as we know, only Transalg and CBMC allow for adding such constraints, whereas SAW+Cryptol, URSA and Grain-of-Salt do not have this capability.

Table 6. An example of two-block MD5 collision with first 10 zero bytes.

	00 00 00 00 00 00 00 00 00 00 20 74 67 a6 f5 48
	cb c1 6d a5 3e f7 b8 bc 67 a3 8d d9 3c 9b f5 b8
	55 ed 32 06 06 0a 74 a3 0f b6 84 87 47 cf 91 d0
	db 4c 6f 43 ef 64 f0 8d a4 1d 50 c6 26 df 95 fe
$M_1 \| M_2$	ff d1 2e c9 a0 90 aa b3 7d e7 e5 bc f2 3a 4e ab
	24 b8 d4 13 4c cc 7b 1b 00 29 eb f5 53 7a 0d d1
	5d 1f b7 79 af 36 ce 08 1e 44 a2 d0 51 ec 91 fb
	c5 4c a2 89 75 b3 a3 84 ac 97 7f f2 7e 50 d4 56
	00 00 00 00 00 00 00 00 00 00 20 74 67 a6 f5 48
	cb c1 6d 25 3e f7 b8 bc 67 a3 8d d9 3c 9b f5 b8
	55 ed 32 06 06 0a 74 a3 0f b6 84 87 47 4f 92 d0
	db 4c 6f 43 ef 64 f0 8d a4 1d 50 46 26 df 95 fe
$M'_1 \| M'_2$	ff d1 2e c9 a0 90 aa b3 7d e7 e5 bc f2 3a 4e ab
	24 b8 d4 93 4c cc 7b 1b 00 29 eb f5 53 7a 0d d1
	5d 1f b7 79 af 36 ce 08 1e 44 a2 d0 51 6c 91 fb
	c5 4c a2 89 75 b3 a3 84 ac 97 7f 72 7e 50 d4 56
Hash	c22664780a9766ceb57065eba36af06b

It should be noted that at the current stage SAT-based cryptanalysis is less effective in application to the collision search problems for cryptographic hash functions in comparison with specialized methods [SSA+09, SWOK07]. On the other hand, the use of new SAT encodings and state-of-the-art SAT solvers makes it possible to find collisions for MD4 hash function about 1000 times faster than it was done in [MZ06]. From our perspective, the potential for further improvements in this direction is far from being exhausted. It should be also mentioned that SAT-based cryptanalysis is, apparently, the most effective for preimage attacks on cryptographic hash functions. Below we build a new preimage attack on the 39-step version of the MD4 hash function using Transalg.

5.2.2. *Preimage attacks on truncated variants of MD4.* Despite the fact that MD4 is compromised with respect to collision finding, the problem of finding preimages for this function is still considered to be extremely hard. While it is believed that MD4 is not highly resistant to preimage attacks, all the arguments of this kind are mostly theoretical [Leu08]. We are not aware of papers in which the inversion problem of full-round MD4 would be solved in reasonable time. To the best of our knowledge, until recently, the paper [DKV07] was considered to be the best practical attack on MD4 since it made it possible to invert a truncated variant of MD4 with 39 (out of 48) steps. The attack from [DKV07] is a SAT-based variant of the attack proposed by H. Dobbertin in [Dob98]. Let us briefly review the results from these papers.

In fact in [Dob98] it was shown that the problem of inverting the first two rounds of MD4 (i.e. 32 steps) is not computationally hard. The main idea of that paper was to add some additional constraints on several chaining variables. These constraints significantly weaken the system of equations corresponding to the process of filling the hash register of MD4 during the first two rounds. In more detail, H. Dobbertin proposed to fix with constant

K the values of 12 chaining variables and showed that choosing the value of K at random with high probability leads to a consistent system which can be easily solved.

In [DKV07] a SAT-based attack on MD4 was proposed that used ideas from [Dob98]. More precisely, in [DKV07] the constant K was fixed to 0. Also, the authors of [DKV07] rejected one of the constraints from [Dob98]. Thus, in [DKV07] 11 constraints were used instead of 12. The constraints of "Dobbertin" type were added to the propositional encoding of the MD4 algorithm in the form of unit clauses. Hereinafter we refer to additional constraints of the "Dobbertin" type on chaining variables as *relaxation constraints*. In some cases the application of such constraints leads to propagation of the values of a large number of other variables. The variables that represent the unknown preimage of a known hash pose the main interest in this context.

In [DKV07], apart from the two-round variant of MD4, there were considered preimage attacks on truncated MD4 variants with k steps, up to and including $k = 39$. For an arbitrary $k < 48$ we will refer to a corresponding truncated variant of MD4 compression function as to MD4-k. The best result presented in [DKV07] was a successful inversion of MD4-39 for several hash values of a special kind. To solve each of such problems it took about 8 hours of the MiniSat SAT solver. It is surprising that the computational results achieved in [DKV07] remained state-of-the art for 10 years. In [GS18] we significantly improved them. It was the result of using a special technique which reduced the problem of finding *promising* relaxation constraints to the problem of optimization of a black box function over the Boolean hypercube. In all experiments in [GS18] we used the SAT encodings constructed by the Transalg system. Below let us briefly review results obtained in [GS18].

To automatically take into account information about relaxation constraints, special variables called *switching variables* [GZK+17] were added to the corresponding TA programs. The main idea of this approach consists in the following. Let C be a CNF that encodes the inversion of some function and X be a set of Boolean variables from C. Assume that we need to add to C new constraints that specify some predicate over variables from a set $\tilde{X}$, $\tilde{X} \subseteq X$. Let $R(\tilde{X})$ be a formula specifying this predicate. Now let us introduce new Boolean variable u, $u \notin X$. Consider the formula $C' = C \wedge (\neg u \vee R(\tilde{X}))$. It is clear that the constraint $R(\tilde{X})$ will be inactive when $u = 0$ and active when $u = 1$. Let us refer to variables similar to u as switching variables.

In application to preimage attacks on MD4-k, switching variables make it possible to reduce the problem of finding effective relaxation constraints (of the "Dobbertin" type) to the optimization problem over the Boolean hypercube. With that purpose in [GS18] we introduce a special measure μ that heuristically evaluates the effectiveness of a considered set of relaxation constraints. Each particular set of relaxation constraints is defined by an assignment of switching variables. The measure μ is a black box function. The relaxation constraints for which the value of μ lies in a particular range are considered to be promising. Thus, the arguments of the considered function are switching variables and its values are the values of μ on the corresponding sets of relaxation constraints. The function defined that way is maximized over the Boolean hypercube, each point of which represents an assignment of switching variables. Since the constructed function does not have an analytical representation, it is sensible to use metaheuristic methods for its maximization. In particular in [GS18] we used an algorithm from the *tabu search* [GL97] class. We view as promising such sets of relaxation constraints whose activation results in derivation by the Unit Propagation rule of a relatively large number of variables corresponding to the hashed message in a SAT encoding (the number of such variables gives the value of function μ). Similar to [DKV07] in the role of relaxation constraints we used the constraints meaning that the corresponding chaining variables should take the value $K = 0$. As a result, we managed to find new relaxation constraints that make it possible to invert the MD4-39 hash function much faster than in [DKV07]. Let us briefly mention the results of computational experiments from [GS18].

Let us note here that based on the features of the MD4 algorithm [Dob98] it is impossible to impose constraints on the first four and the last (preceding the calculation of the final hash value) four steps of the MD4-39 algorithm. According to this, the sets of new relaxation constraints were selected (using the values of the corresponding switching variables) from the set of cardinality 31. Thus, the problem of maximization of a function described above over the Boolean hypercube $\{0, 1\}^{31}$ was considered. The details of experiments can be found in [GS18]. As a result we found two new sets of relaxation constraints. We denote them as ρ_1 and ρ_2 and they represent the following assignments of corresponding switching variables:

$$\begin{array}{ll} \rho_1: & 0000000001101110111011101000000 \\ \rho_2: & 0000000000101110111011101100000 \end{array}$$

For example, vector ρ_1 specifies the set of 12 relaxation constraints: chaining variables on steps $14, 15, 17, 18, 19, 21, 22, 23, 25, 26, 27, 29$ are assigned with the value $K = 0$. The application of the relaxation constraints specified by ρ_1 and ρ_2 allows one to find preimages of the MD4-39 hash function for known hash values 0^{128} and 1^{128} within one minute of MiniSat 2.2 runtime. Note, that using constraints from [DKV07] the solution of the preimage finding problem for 1^{128} requires about 2 hours, and the preimage finding problem for 0^{128} cannot be solved in 8 hours. The corresponding results are presented in Table 7, where ρ_{De} denotes the set of relaxation constraints described in [DKV07] and $\rho_{Dobbertin}$ denotes the variant of Dobbertin's constraints from [Dob98] with constant $K = 0$. Below, these relaxation

Table 7. Finding the MD4-39 preimages for hash values 0^{128} and 1^{128}.

Relaxation constraints	Result / Solving time (s)	
	$\chi = 0^{128}$	$\chi = 1^{128}$
ρ_1	SAT / 20	SAT / 10
ρ_2	SAT / 60	UNSAT / < 1
$\rho_{Dobbertin}$	SAT / 20	Unknown / > 30000
ρ_{De}	Unknown / > 30000	SAT / 7000

constraints are specified by the vectors of values of switching variables from $\{0, 1\}^{31}$ (in the notation similar to that of ρ_1 and ρ_2):

$$\begin{array}{ll} \rho_{Dobbertin}: & 0000000011101110111011100000000 \\ \rho_{De}: & 0000000001101110111011100000000 \end{array}$$

What is particularly interesting is that the application of new sets of relaxation constraints ρ_1 and ρ_2 also allows one to find preimages of MD4-39 for randomly generated 128-bit Boolean vectors persistently. To obtain this result, we considered a test set of 500 randomly generated vectors from $\{0, 1\}^{128}$. Regarding each of these vectors we assumed that it is a hash value of MD4-39. After that the preimage finding problem for this value was solved using constraints specified by vectors ρ_1 and ρ_2. For the prevailing part of the tasks (65–75%) the solutions were successfully found using MiniSat 2.2. The average time of finding one preimage was less than 1 minute. The rest (25–35% of the tasks) corresponded to 128-bit vectors for which there were no MD4-39 preimages under constraints specified by ρ_1 and ρ_2 (this fact was proven by the SAT solver in under 1 minute on average). These results are presented in Table 8. Note that even in a few hours we did not manage to solve the preimage finding problem for any vector from the test set using constraints from [DKV07] or [Dob98].

Table 8. Finding the MD4-39 preimages for 500 randomly generated 128-bit Boolean vectors. Instances with preimages are satisfiable, while those with no preimages are unsatisfiable.

Relaxation constraints	Avg. solving time (s)	Max. solving time (s)	Solved instances (in % of total number of instances)	
			with preimages	with no preimages
ρ_1	12	80	65	35
ρ_2	46	250	75	25

Let us summarize the results of the present section. From our point of view, we convincingly demonstrated the power of SAT-based cryptanalysis methods. We believe that future ideas both in the area of reduction to SAT and in algorithms of state-of-the-art SAT solvers will make it possible to increase the effectiveness of SAT-based cryptanalysis and extend the spectrum of its applications.

6. Related works

The ideas of using general purpose combinatorial algorithms to solve cryptanalysis problems can be found in many papers starting from 90-th years of XX-th century. Apparently, S.A. Cook and D.G. Mitchell were the first to propose using SAT solving algorithms in cryptanalysis in [CM97]. The first example of propositional encoding of a cryptographic problem (in particular of DES cryptanalysis) was given in [Mas99, MM00]. In [JJ05] the problems of finding collisions of a number of cryptographic hash functions were reduced to SAT. In [MZ06] propositional encodings of hash functions from the MD family were also constructed. The authors of [MZ06] added to these encodings the constraints that encode the differential paths introduced in [WLF+05, WY05]. This addition made it possible to persistently construct single-block collisions for MD4 (it took about 10 minutes per collision). Thus, [MZ06] can be considered to be the first paper in which SAT-based cryptanalysis was successfully applied to relevant cryptographic algorithms. In the book [Bar09] SAT solvers are considered to be the primary tool for solving problems of algebraic cryptanalysis. It should be noted that in all the mentioned papers no automated system was used to construct propositional encodings of the considered functions.

In the present paper we described in detail the principles of constructing propositional encodings of discrete functions with the focus on functions employed in cryptography. We also compared several different systems that can produce such SAT encodings in an automatic mode. First it is the well-known CBMC system for symbolic verification [CKL04, Kro09], which have been developed for more than 15 years. CBMC is a generic system and it is not designed with cryptanalysis problems in mind. However, as we show in Section 4, it allows one to perform the majority of actions available to a limited number of considered domain-specific systems. Here we mean SAW+Cryptol, Grain-of-Salt, URSA and Transalg.

The first version of the Cryptol language was published in 2003 [LM03]. Its second version [EM09, ECW09] was later augmented by SAW [CFH+13] that allowed producing SAT encodings. In 2010, the Grain-of-Salt system was proposed [Soo10]. Approximately at this time we started the development of the Transalg software tool, which we describe in the present paper (Transalg was first mentioned in papers in Russian in 2011 [OS11]). In 2012, the URSA system, aimed at reducing to SAT various constraint programming problems, was published [Jan12]. It can be applied to construct encodings of cryptographic functions as well. Note that SAW+Cryptol, URSA and Transalg can encode to SAT algorithmic descriptions of a very wide class of functions working with binary data. Meanwhile, Grain-of-Salt is designed to work only with keystream generators based on shift registers. We considered the pros and cons of all mentioned systems in detail in Section 4.

CBMC, Transalg and other similar systems are based on symbolic execution of a program specifying a considered function. The idea to transform programs to Boolean formulas was first proposed by S.A. Cook in his paper [Coo71] which led to the creation and development of the theory of NP-completeness. The notion "Symbolic Execution" first appeared in the paper [Kin76] by J.C. King, where it is defined as a process of interpretation of a program in a special extended semantics, within the context of which it is allowed to take symbols as input and put formulas as output. Currently, symbolic execution combined with Bounded Model Checking is actively used in software verification (see, e.g., [Kro09]).

As we mentioned above, SAT-based cryptanalysis is still actively developing. The cryptographic attacks that employ SAT solvers show very good results for a number of keystream generators: Geffe, Wolfram (present paper), Crypto-1, Hitag2 (see [SNC09]). In [SZBP11], a successful SAT-based attack on the widely known A5/1 cryptographic keystream generator was described. Later several dozen SAT instances that encode the cryptanalysis problem for A5/1 were solved in the SAT@home volunteer computing project [PSZ12]. This result together with other attacks on A5/1 (see [BSZK18, GKN+08, Noh10]) provides an exhaustive argument towards not using A5/1 any more. The Bivium stream cipher [Can06] is a popular object of algebraic and SAT-based cryptanalysis [MCP07, EPV08, SNC09, EVP10]. In [SZ16] a SAT-based guess-and-determine attack on Bivium was proposed. The corresponding runtime estimation turned out to be realistic for modern distributed computing systems. The LTE stream cipher ZUC was analyzed by SAT in [LMH15]. In [ZK17], SAT-based guess-and-determine attacks on several variants of the alternating step generator were described. SAT-based cryptanalysis of stream ciphers from the CAESAR competition was described in [DKM+17].

In [SZO+18] a new class of SAT-based guess-and-determine attacks was described, in which the notion of Inverse Backdoor Set (IBS) is used. IBS is a modification of a well-known notion of Strong Backdoor Set [WGS03]. It made it possible to construct the best or close to the best guess-and-determine attacks on several ciphers. For example, the attack on 2.5-round AES-128 with 2 Known Plaintexts, presented in [SZO+18], is significantly better than the previously best known attack on this cipher proposed in [BDF11]. In [PSU19, PBU19], evolutionary and genetic algorithms were used to minimize the objective function from [SZO+18] in application to SAT-based guess-and-determine attacks on weakened variants of the Trivium stream cipher [Can06]. Note that the function introduced in [SZO+18] to associate with a particular IBS the estimation of effectiveness of a corresponding guess-and-determine attack is a concretization of the notion of SAT-immunity, introduced by N. Courtois in [Cou15, CGS12, Cou13].

As we already noted, [MZ06] was the first paper to demonstrate the applicability of SAT-based cryptanalysis to relevant cryptographic algorithms. In that paper, using the MiniSat solver [ES04] it was possible to quite effectively find single-block collisions for MD4. Using new propositional encoding methods (in particular, the Transalg system) and state-of-the-art SAT solvers one can find preimages for MD4 and MD5 several hundred times faster than it was done in [MZ06]. Nevertheless, on the current stage SAT-based cryptanalysis is less effective than specialized methods (see, for example [SWOK07, Ste12, SKP16]) on problems of finding collisions of cryptographic hash functions. However, as far as we know, it is the SAT-based approach that yields best known preimage attacks on truncated variants of hash functions [DKV07, Nos12, NNS$^+$17]. In application to MD4-39, for a long time the SAT-based preimage attack from [DKV07] was considered to be the best. In [GS18] we significantly improved the results from [DKV07]. It was possible for the large part thanks to functional capabilities of the Transalg system.

7. Conclusion and future work

In this paper, we study the principles of encoding the problems of inversion of discrete functions from a wide class to the Boolean satisfiability problem with a focus on cryptographic applications. We provide the theoretical basis of SAT-based cryptanalysis and use it to design the domain-specific system called Transalg for use in algebraic cryptanalysis.

In the comparison with relevant software tools for constructing propositional encodings, such as CBMC, SAW+Cryptol, URSA, and Grain-of-Salt, we showed that the Transalg encoding concepts often make it possible to build SAT encodings of cryptanalysis problems which are on par or better than that constructed by competitors. From the results of our study, it follows that in the cryptographic context the overall functional capabilities of Transalg match that of the CBMC system which is the recognized leader in SAT-based Bounded Model Checking. We also show how the distinctive features of Transalg can be useful in algebraic cryptanalysis on the example of the applications described in the final part of the paper.

In our opinion, Transalg often allows one to make better encodings than competition because it uses a number of techniques to reduce the redundancy of the encodings and also employs Boolean minimization (in form of the Espresso software tool) to make subformulas' representation in CNFs more compact. However, investigating this phenomenon in detail will take a large amount of computational experiments and time to comprehend their results. We are going to study these issues in the nearest future.

As a final comment, we would like to once more emphasize the theoretical and practical importance of SAT-based cryptanalysis and note that the corresponding problems can be viewed as interesting challenges for researchers. Therefore, they may stimulate the development of new algorithms and SAT solving techniques. We believe that the results presented in this paper will be useful in that context.

Acknowledgments

Authors express deep gratitude to professors Daniel Kroening, Predrag Janičić, and Aaron Tomb for feedback regarding several important details of CBMC, URSA and Cryptol, respectively. We are grateful to anonymous reviewers for their valuable comments that made it possible to significantly improve the quality of the present paper.

The research was funded by Russian Science Foundation (project No. 16-11-10046). Irina Gribanova and Stepan Kochemazov are additionally supported by the Council for Grants of the President of the Russian Federation (stipends SP-3545.2019.5 and SP-2017.2019.5, respectively).

References

[ABE00] Parosh Aziz Abdulla, Per Bjesse, and Niklas Eén. Symbolic reachability analysis based on SAT-solvers. In *Tools and Algorithms for the Construction and Analysis of Systems (TACAS)*, volume 1785 of *Lecture Notes in Computer Science*, pages 411–425, 2000.

[AS17] Gilles Audemard and Laurent Simon. Glucose and Syrup in the SAT'17. In Tomás Balyo, Marijn J. H. Heule, and Matti Järvisalo, editors, *SAT Competition 2017*, volume B-2017-1, pages 16–17, 2017.

[Bar09] Gregory V. Bard. *Algebraic Cryptanalysis*. Springer Publishing Company, Incorporated, 1st edition, 2009.

[BCC+99] Armin Biere, Alessandro Cimatti, Edmund M. Clarke, Masahiro Fujita, and Yunshan Zhu. Symbolic model checking using SAT procedures instead of BDDs. In *Proc. of the 36th Annual ACM/IEEE Design Automation Conference (DAC)*, pages 317–320, 1999.

[BCC+03] Armin Biere, Alessandro Cimatti, Edmund M. Clarke, Ofer Strichman, and Yunshan Zhu. Bounded Model Checking. *Advances in Computers*, 58:117–148, 2003.

[BCCZ99] Armin Biere, Alessandro Cimatti, Edmund M. Clarke, and Yunshan Zhu. Symbolic model checking without BDDs. In *Proc. of the 5th International Conference on Tools and Algorithms for Construction and Analysis of Systems (TACAS)*, volume 1579 of *Lecture Notes in Computer Science*, pages 193–207, 1999.

[BCD+11] Clark Barrett, Christopher L. Conway, Morgan Deters, Liana Hadarean, Dejan Jovanović, Tim King, Andrew Reynolds, and Cesare Tinelli. CVC4. In *Proc. of the 23rd International Conference on Computer Aided Verification (CAV)*, volume 6806 of *Lecture Notes in Computer Science*, pages 171–177, 2011. Snowbird, Utah.

[BDF11] Charles Bouillaguet, Patrick Derbez, and Pierre-Alain Fouque. Automatic search of attacks on round-reduced AES and applications. In *Proc. of Advances in Cryptology (CRYPTO)*, volume 6841 of *Lecture Notes in Computer Science*, pages 169–187, 2011.

[BHvMW09] Armin Biere, Marijn Heule, Hans van Maaren, and Toby Walsh, editors. *Handbook of Satisfiability*, volume 185 of *Frontiers in Artificial Intelligence and Applications*. IOS Press, 2009.

[Bie07] Armin Biere. The AIGER And-Inverter Graph (AIG) format version 20071012. Technical Report 07/1, Institute for Formal Models and Verification, Johannes Kepler University, Altenbergerstr. 69, 4040 Linz, Austria, 2007.

[Bie14] Armin Biere. Yet another local search solver and lingeling and friends entering the SAT Competition 2014. In Anton Belov, Daniel Diepold, Marijn Heule, and Matti Järvisalo, editors, *SAT Competition 2014*, volume B-2014-2, pages 39–40, 2014.

[Bie17] Armin Biere. CaDiCaL, Lingeling, Plingeling, Treengeling, YalSAT entering the SAT competition 2017. In Tomás Balyo, Marijn J. H. Heule, and Matti Järvisalo, editors, *SAT Competition 2017*, volume B-2017-1, pages 14–15, 2017.

[BKNW09] Christian Bessiere, George Katsirelos, Nina Narodytska, and Toby Walsh. Circuit complexity and decompositions of global constraints. In *Proc. of the 21st International Joint Conference on Artificial Intelligence (IJCAI)*, pages 412–418, 2009.

[Bru84] J. Bruer. On pseudo-random sequences as crypto generators. In *Proc. of the Int. Zurich Seminar on Digital Comm.*, volume 46, pages 157–161, 1984.

[BSVMH84] Robert King Brayton, Alberto L. Sangiovanni-Vincentelli, Curtis T. McMullen, and Gary D. Hachtel. *Logic Minimization Algorithms for VLSI Synthesis*. Kluwer Academic Publishers, Norwell, MA, USA, 1984.

[BSZK18] Vadim Bulavintsev, Alexander Semenov, Oleg Zaikin, and Stepan Kochemazov. A bitslice implementation of Anderson's attack on A5/1. *Open Engineering*, 8(1):7–16, 2018.

[Can06] Christophe De Cannière. Trivium: A stream cipher construction inspired by block cipher design principles. In *Proc. of the 9th International Conference on Information Security (ISC)*, volume 4176 of *Lecture Notes in Computer Science*, pages 171–186, 2006.

[CB07] Nicolas T. Courtois and Gregory V. Bard. Algebraic cryptanalysis of the data encryption standard. In *Proc. of Cryptography and Coding*, volume 4887 of *Lecture Notes in Computer Science*, pages 152–169, 2007.

[CFH+13] Kyle Carter, Adam Foltzer, Joe Hendrix, Brian Huffman, and Aaron Tomb. SAW: The software analysis workbench. In *Proc. of the 2013 ACM SIGAda Annual Conference on High Integrity Language Technology (HILT)*, pages 15–18, 2013.

[CGS12] Nicolas T. Courtois, Jerzy A. Gawinecki, and Guangyan Song. Contradiction immunity and guess-then-determine attacks on GOST. *Tatra Mountains Mathematical Publications*, 53(1):2–13, 2012.

[CKL04] Edmund Clarke, Daniel Kroening, and Flavio Lerda. A tool for checking ANSI-C programs. In *Proc. of Tools and Algorithms for the Construction and Analysis of Systems (TACAS)*, volume 2988 of *Lecture Notes in Computer Science*, pages 168–176, 2004.

[CM97] Stephen A. Cook and David G. Mitchell. Finding hard instances of the satisfiability problem: A survey. In *DIMACS Series in Discr. Math. and Theoretical Comp. Sci.*, volume 35, pages 1–17. American Mathematical Society, 1997.

[Coo71] Stephen A. Cook. The complexity of theorem-proving procedures. In *Proc. of the 3rd Annual ACM Symposium on Theory of Computing*, pages 151–158, 1971.

[Cou13] Nicolas T. Courtois. Low-complexity key recovery attacks on GOST block cipher. *Cryptologia*, 37(1):1–10, 2013.

[Cou15] Nicolas T. Courtois. Algebraic complexity reduction and cryptanalysis of GOST. Cryptology ePrint Archive, Report 2011/626, 2011–2015. `http://eprint.iacr.org/2011/626`.

[Dam89] Ivan Damgård. A design principle for hash functions. In *Proc. of Advances in Cryptology (CRYPTO)*, volume 435 of *Lecture Notes in Computer Science*, pages 416–427, 1989.

[DG84] William F. Dowling and Jean H. Gallier. Linear-time algorithms for testing the satisfiability of propositional horn formulae. *J. Log. Program.*, 1(3):267–284, 1984.

[DKM+17] Ashutosh Dhar Dwivedi, Milos Kloucek, Pawel Morawiecki, Ivica Nikolic, Josef Pieprzyk, and Sebastian Wójtowicz. SAT-based cryptanalysis of authenticated ciphers from the CAESAR competition. In Pierangela Samarati, Mohammad S. Obaidat, and Enrique Cabello, editors, *Proc. of the 14th International Joint Conference on e-Business and Telecommunications (ICETE 2017) - Volume 4: SECRYPT*, pages 237–246. SciTePress, 2017.

[DKV07] Debapratim De, Abishek Kumarasubramanian, and Ramarathnam Venkatesan. Inversion attacks on secure hash functions using SAT solvers. In *Proc. of Theory and Applications of Satisfiability Testing (SAT)*, volume 4501 of *Lecture Notes in Computer Science*, pages 377–382, 2007.

[DLL62] Martin Davis, George Logemann, and Donald Loveland. A machine program for theorem-proving. *Commun. ACM*, 5(7):394–397, 1962.

[dMB08] Leonardo de Moura and Nikolaj Bjørner. Z3: An efficient SMT solver. In *Proc. of Tools and Algorithms for the Construction and Analysis of Systems (TACAS)*, volume 4963 of *Lecture Notes in Computer Science*, pages 337–340, 2008.

[Dob98] Hans Dobbertin. The first two rounds of MD4 are not one-way. In *Proc. of Fast Software Encryption (FSE)*, volume 1372 of *Lecture Notes in Computer Science*, pages 284–292. Springer, 1998.

[DP60] Martin Davis and Hilary Putnam. A computing procedure for quantification theory. *J. ACM*, 7(3):201–215, 1960.

[Dut14] Bruno Dutertre. Yices 2.2. In *Proc. of Computer Aided Verification (CAV)*, volume 8559 of *Lecture Notes in Computer Science*, pages 737–744, 2014.

[ECW09] Levent Erkök, Magnus Carlsson, and Adam Wick. Hardware/software co-verification of cryptographic algorithms using Cryptol. In *Proc. of 9th International Conference on Formal Methods in Computer-Aided Design (FMCAD)*, pages 188–191. IEEE, 2009.

[EM09] Levent Erkök and John Matthews. Pragmatic equivalence and safety checking in Cryptol. In *Proc. of the 3rd Workshop on Programming Languages Meets Program Verification (PLPV)*, pages 73–82, New York, NY, USA, 2009. ACM.

[EPV08] Tobias Eibach, Enrico Pilz, and Gunnar Völkel. Attacking Bivium using SAT solvers. In *Theory and Applications of Satisfiability Testing (SAT)*, volume 4996 of *Lecture Notes in Computer Science*, pages 63–76, 2008.

[ES03] Niklas Eén and Niklas Sörensson. Temporal induction by incremental SAT solving. *Electr. Notes Theor. Comput. Sci.*, 89(4):543–560, 2003.

[ES04] Niklas Eén and Niklas Sörensson. An extensible SAT-solver. In *Proc. of the 6th International Conference on Theory and Applications of Satisfiability Testing (SAT)*, volume 2919 of *Lecture Notes in Computer Science*, pages 502–518, 2004.

[EVP10] Tobias Eibach, Gunnar Völkel, and Enrico Pilz. Optimising Gröbner bases on Bivium. *Mathematics in Computer Science*, 3(2):159–172, Apr 2010.

[FBSK17] Ludovic Le Frioux, Souheib Baarir, Julien Sopena, and Fabrice Kordon. painless-maplecomsps. In Tomás Balyo, Marijn J. H. Heule, and Matti Järvisalo, editors, *SAT Competition 2017*, volume B-2017-1, pages 26–27, 2017.

[Gef73] P. Geffe. How to protect data with ciphers that are really hard to break. *Electronics*, 46(1):99–101, 1973.

[GJ79] Michael R. Garey and David S. Johnson. *Computers and Intractability: A Guide to the Theory of NP-Completeness.* W. H. Freeman & Co., New York, NY, USA, 1979.

[GKN+08] Tim Güneysu, Timo Kasper, Martin Novotný, Christof Paar, and Andy Rupp. Cryptanalysis with COPACOBANA. *IEEE Trans. Comput.*, 57(11):1498–1513, 2008.

[GKNS07] Martin Gebser, Benjamin Kaufmann, André Neumann, and Torsten Schaub. clasp: A conflict-driven answer set solver. In *Proc. of Logic Programming and Nonmonotonic Reasoning*, volume 4483 of *Lecture Notes in Computer Science*, pages 260–265, 2007.

[GL97] Fred Glover and Manuel Laguna. *Tabu Search.* Kluwer Academic Publishers, Norwell, MA, USA, 1997.

[Gol08] Oded Goldreich. *Computational Complexity: A Conceptual Perspective.* Cambridge University Press, New York, NY, USA, 1 edition, 2008.

[GS18] Irina Gribanova and Alexander Semenov. Using automatic generation of relaxation constraints to improve the preimage attack on 39-step MD4. In *Proc. of the 41st International Convention on Information and Communication Technology, Electronics and Microelectronics (MIPRO)*, pages 1174–1179. IEEE, 2018.

[GZK+17] Irina Gribanova, Oleg Zaikin, Stepan Kochemazov, Ilya Otpuschennikov, and Alexander Semenov. The study of inversion problems of cryptographic hash functions from MD family using algorithms for solving Boolean satisfiability problem. In *Proc. of International Conference Mathematical and Information Technologies*, volume 1839, pages 98–113. CEUR-WS, 2017.

[HJM07] Martin Hell, Thomas Johansson, and Willi Meier. Grain: a stream cipher for constrained environments. *Int. J. Wire. Mob. Comput.*, 2(1):86–93, May 2007.

[Hyv11] Antti E. J. Hyvärinen. *Grid Based Propositional Satisfiability Solving.* PhD thesis, Aalto University, 2011.

[Ign] Alexey Ignatiev. mkplot: a Python script to create cactus and scatter plots based on matplotlib. URL: https://github.com/alexeyignatiev/mkplot.

[IMMS18] Alexey Ignatiev, Antonio Morgado, and João P. Marques-Silva. PySAT: A Python toolkit for prototyping with SAT oracles. In *Proc. of Theory and Applications of Satisfiability Testing (SAT)*, volume 10929 of *Lecture Notes in Computer Science*, pages 428–437, 2018.

[Jan12] Predrag Janicic. URSA: a system for uniform reduction to SAT. *Logical Methods in Computer Science*, 8(3):1–39, 2012.

[JBH12] Matti Järvisalo, Armin Biere, and Marijn Heule. Simulating circuit-level simplifications on CNF. *J. Autom. Reasoning*, 49(4):583–619, 2012.

[JJ05] Dejan Jovanovic and Predrag Janicic. Logical analysis of hash functions. In *Proc. of the 5th International Workshop on Frontiers of Combining Systems (FroCoS)*, volume 3717 of *Lecture Notes in Computer Science*, pages 200–215, 2005.

[JJ09] Matti Järvisalo and Tommi Junttila. Limitations of restricted branching in clause learning. *Constraints*, 14(3):325–356, Sep 2009.

[Kin76] James C. King. Symbolic execution and program testing. *Commun. ACM*, 19(7):385–394, July 1976.

[Kro09] Daniel Kroening. Software verification. In Biere et al. [BHvMW09], pages 505–532.

[Leu08] Gaëtan Leurent. MD4 is not one-way. In *Proc. of Fast Software Encryption (FSE)*, volume 5086 of *Lecture Notes in Computer Science*, pages 412–428, 2008.

[LGPC16] Jia Hui Liang, Vijay Ganesh, Pascal Poupart, and Krzysztof Czarnecki. Learning rate based branching heuristic for SAT solvers. In *Proc. of Theory and Applications of Satisfiability Testing (SAT)*, volume 9710 of *Lecture Notes in Computer Science*, pages 123–140. Springer, 2016.

[LM03] J. R. Lewis and B. Martin. Cryptol: high assurance, retargetable crypto development and validation. In *IEEE Military Communications Conference, 2003. MILCOM 2003.*, volume 2, pages 820–825, 2003.

[LMH15] Frédéric Lafitte, Olivier Markowitch, and Dirk Van Heule. SAT based analysis of LTE stream cipher ZUC. *Journal of Information Security and Applications*, 22:54 – 65, 2015. Special Issue on Security of Information and Networks.

[LOG+17] Jia Hui Liang, Chanseok Oh, Vijay Ganesh, Krzysztof Czarnecki, and Pascal Poupart. MapleCOMSPS_LRB_VSIDS and MapleCOMSPS_CHB_VSIDS. In Tomás Balyo, Marijn J. H. Heule, and Matti Järvisalo, editors, *SAT Competition 2017*, volume B-2017-1, pages 20–21, 2017.

[Mar09] Filip Marić. Formalization and implementation of modern SAT solvers. *J. Autom. Reason.*, 43(1):81–119, 2009.

[Mas99] Fabio Massacci. Using Walk-SAT and Rel-SAT for cryptographic key search. In *Proc. of the 16th International Joint Conference on Artifical Intelligence (IJCAI)*, pages 290–295. Morgan Kaufmann Publishers Inc., 1999.

[mat] Irkutsk Supercomputer Center of SB RAS. URL: `http://hpc.icc.ru`.

[MCP07] Cameron Mcdonald, Chris Charnes, and Josef Pieprzyk. Attacking Bivium with MiniSat. Technical Report 2007/040, ECRYPT Stream Cipher Project, 2007.

[Mer89] Ralph C. Merkle. A certified digital signature. In *Proc. of Advances in Cryptology (CRYPTO)*, volume 435 of *Lecture Notes in Computer Science*, pages 218–238, 1989.

[MM00] Fabio Massacci and Laura Marraro. Logical cryptanalysis as a SAT problem. *J. Autom. Reasoning*, 24(1/2):165–203, 2000.

[MMZ+01] Matthew W. Moskewicz, Conor F. Madigan, Ying Zhao, Lintao Zhang, and Sharad Malik. Chaff: Engineering an efficient SAT solver. In *Proc. of the 38th Annual Design Automation Conference (DAC)*, pages 530–535, 2001.

[MS91] Willi Meier and Othmar Staffelbach. Analysis of pseudo random sequences generated by cellular automata. In *Proc. of Advances in Cryptology (EUROCRYPT)*, volume 547 of *Lecture Notes in Computer Science*, pages 186–199, 1991.

[MS99] João P. Marques-Silva and Karem A. Sakallah. GRASP: a search algorithm for propositional satisfiability. *IEEE Trans. Computers*, 48(5):506–521, 1999.

[MS08] João P. Marques-Silva. Practical applications of Boolean satisfiability. In *Proc. of the 9th International Workshop on Discrete Event Systems*, pages 74–80, 2008.

[MSLM09] João P. Marques-Silva, Inês Lynce, and Sharad Malik. Conflict-driven clause learning SAT solvers. In Biere et al. [BHvMW09], pages 131–153.

[MSS96] João P. Marques-Silva and Karem A. Sakallah. GRASP: a new search algorithm for satisfiability. In *Proc. of the 1996 IEEE/ACM International Conference on Computer-aided Design (ICCAD)*, pages 220–227, 1996.

[MU49] N Metropolis and S Ulam. The Monte Carlo method. *J. Amer. statistical assoc.*, 44(247):335–341, 1949.

[MVO96] Alfred J. Menezes, Scott A. Vanstone, and Paul C. Van Oorschot. *Handbook of Applied Cryptography.* CRC Press, Inc., Boca Raton, FL, USA, 1st edition, 1996.

[MZ06] Ilya Mironov and Lintao Zhang. Applications of SAT solvers to cryptanalysis of hash functions. In *Proc. of Theory and Applications of Satisfiability Testing (SAT)*, volume 4121 of *Lecture Notes in Computer Science*, pages 102–115, 2006.

[NLG+17] Saeed Nejati, Jia Hui Liang, Catherine Gebotys, Krzysztof Czarnecki, and Vijay Ganesh. Adaptive restart and CEGAR-based solver for inverting cryptographic hash functions. In *Proc. of Verified Software. Theories, Tools, and Experiments*, volume 10712 of *Lecture Notes in Computer Science*, pages 120–131, 2017.

[NNS+17] Saeed Nejati, Zack Newsham, Joseph Scott, Jia Hui Liang, Catherine Gebotys, Pascal Poupart, and Vijay Ganesh. A propagation rate based splitting heuristic for divide-and-conquer solvers. In *Proc. of Theory and Applications of Satisfiability Testing (SAT)*, volume 10491 of *Lecture Notes in Computer Science*, pages 251–260, 2017.

[Noh10] Karsten Nohl. Attacking phone privacy. In *Proc. of BlackHat 2010 Lecture Notes, Las-Vegas, USA*, pages 1–6, 2010.

[Nos12] Vegard Nossum. SAT-based preimage attacks on SHA-1. Master's thesis, University of Oslo, Department of Informatics, 2012.

[NR18] Alexander Nadel and Vadim Ryvchin. Chronological backtracking. In *Proc. Theory and Applications of Satisfiability Testing (SAT)*, volume 10929 of *Lecture Notes in Computer Science*, pages 111–121, 2018.

[OGS] Ilya Otpuschennikov, Irina Gribanova, and Alexander Semenov. Transalg repository, URL: https://gitlab.com/transalg/transalg.

[OGZ+] Ilya Otpuschennikov, Irina Gribanova, Oleg Zaikin, Stepan Kochemazov, and Alexander Semenov. SAT and SMT encodings of cryptograhic problems made by the Transalg tool and its competitors, URL: https://github.com/Nauchnik/transalg-comparison.

[OGZS] Ilya Otpuschennikov, Irina Gribanova, Oleg Zaikin, and Alexander Semenov. The repository of SAT encodings for various cryptographic primitives constructed by Transalg, URL: https://gitlab.com/satencodings/satencodings.

[Oh17] Chanseok Oh. COMiniSatPS Pulsar and GHackCOMSPS. In Tomás Balyo, Marijn J. H. Heule, and Matti Järvisalo, editors, *SAT Competition 2017*, volume B-2017-1, pages 12–13, 2017.

[OS11] Ilya Otpuschennikov and Alexander Semenov. Technology for translating combinatorial problems into boolean equations. *Prikl. Diskr. Mat.*, 1(11):96–115, 2011.

[OSG+16] Ilya Otpuschennikov, Alexander Semenov, Irina Gribanova, Oleg Zaikin, and Stepan Kochemazov. Encoding cryptographic functions to SAT using TRANSALG system. In *Proc. of the 22nd European Conference on Artificial Intelligence (ECAI)*, volume 285 of *Frontiers in Artificial Intelligence and Applications*, pages 1594–1595. IOS Press, 2016.

[PBG05] Mukul R. Prasad, Armin Biere, and Aarti Gupta. A survey of recent advances in SAT-based formal verification. *International Journal on Software Tools for Technology Transfer*, 7(2):156–173, 2005.

[PBU19] Artem Pavlenko, Maxim Buzdalov, and Vladimir Ulyantsev. Fitness comparison by statistical testing in construction of SAT-based guess-and-determine cryptographic attacks. In *Proc. of the Genetic and Evolutionary Computation Conference (GECCO)*, pages 312–320. ACM, 2019.

[PSU19] Artem Pavlenko, Alexander Semenov, and Vladimir Ulyantsev. Evolutionary computation techniques for constructing SAT-based attacks in algebraic cryptanalysis. In *Proc. of the 22nd International Conference Applications of Evolutionary Computation*, volume 11454 of *Lecture Notes in Computer Science*, pages 237–253, 2019.

[PSZ12] Mikhail Posypkin, Alexander Semenov, and Oleg Zaikin. Using BOINC desktop grid to solve large scale SAT problems. *Computer Science*, 13(1):25, 2012.

[Sie84] Thomas Siegenthaler. Correlation-immunity of nonlinear combining functions for cryptographic applications. *IEEE Trans. Information Theory*, 30(5):776–780, 1984.

[SKP16] Marc Stevens, Pierre Karpman, and Thomas Peyrin. Freestart collision for full SHA-1. In *Proc. of Advances in Cryptology (EUROCRYPT)*, volume 9665 of *Lecture Notes in Computer Science*, pages 459–483. Springer Berlin Heidelberg, 2016.

[SMF12] Carsten Sinz, Florian Merz, and Stephan Falke. LLBMC: A bounded model checker for LLVM's intermediate representation. In *Proc. of Tools and Algorithms for the Construction and Analysis of Systems (TACAS)*, volume 7214 of *Lecture Notes in Computer Science*, pages 542–544, 2012.

[SNC09] Mate Soos, Karsten Nohl, and Claude Castelluccia. Extending SAT solvers to cryptographic problems. In *Proc. of Theory and Applications of Satisfiability Testing (SAT)*, volume 5584 of *Lecture Notes in Computer Science*, pages 244–257, 2009.

[Soo10] Mate Soos. Grain of Salt - an automated way to test stream ciphers through SAT solvers. In *Proc. of Workshop on Tools for Cryptanalysis (Tools)*, pages 131–144, 2010.

[SSA+09] Marc Stevens, Alexander Sotirov, Jacob Appelbaum, Arjen Lenstra, David Molnar, Dag Arne Osvik, and Benne de Weger. Short chosen-prefix collisions for MD5 and the creation of a rogue CA certificate. In *Proc. of Advances in Cryptology (CRYPTO)*, volume 5677 of *Lecture Notes in Computer Science*, pages 55–69, 2009.

[Ste12] Marc Stevens. Single-block collision attack on MD5. *IACR Cryptology ePrint Archive*, 2012:40, 2012.

[Sto16] Ko Stoffelen. Optimizing S-box implementations for several criteria using SAT solvers. In *Proc. of the 23rd International Conference on Fast Software Encryption (FSE)*, volume 9783 of *Lecture Notes in Computer Science*, pages 140–160. Springer Berlin Heidelberg, 2016.

[SWOK07] Yu Sasaki, Lei Wang, Kazuo Ohta, and Noboru Kunihiro. New message difference for MD4. In *Proc. of Fast Software Encryption (FSE)*, volume 4593 of *Lecture Notes in Computer Science*, pages 329–348, 2007.

[SZ15] Alexander Semenov and Oleg Zaikin. Using Monte Carlo method for searching partitionings of hard variants of Boolean satisfiability problem. In *Proc. of the 13th International Conference on Parallel Computing Technologies (PaCT)*, volume 9251 of *Lecture Notes in Computer Science*, pages 222–230, 2015.

[SZ16] Alexander Semenov and Oleg Zaikin. Algorithm for finding partitionings of hard variants of Boolean satisfiability problem with application to inversion of some cryptographic functions. *SpringerPlus*, 5(1):1–16, 2016.

[SZBP11] Alexander Semenov, Oleg Zaikin, Dmitry Bespalov, and Mikhail Posypkin. Parallel logical cryptanalysis of the generator A5/1 in BNB-grid system. In *Proc. of the 11th International Conference on Parallel Computing Technologies (PaCT)*, volume 6873 of *Lecture Notes in Computer Science*, pages 473–483, 2011.

[SZO+18] Alexander Semenov, Oleg Zaikin, Ilya Otpuschennikov, Stepan Kochemazov, and Alexey Ignatiev. On cryptographic attacks using backdoors for SAT. In *Proc. of the Thirty-Second AAAI Conference on Artificial Intelligence (AAAI)*, pages 6641–6648, 2018.

[Tse70] Grigori S Tseitin. On the complexity of derivation in propositional calculus. *Studies in constructive mathematics and mathematical logic, part II, Seminars in mathematics*, pages 115–125, 1970.

[von51] John von Neumann. The general and logical theory of automata. In L. A. Jeffress, editor, *Cerebral Mechanisms in Behavior – The Hixon Symposium*, pages 1–31. John Wiley & Sons, 1951.

[WGS03] Ryan Williams, Carla P. Gomes, and Bart Selman. Backdoors to typical case complexity. In *Proc. of the 18th International Joint Conference on Artificial Intelligence (IJCAI)*, pages 1173–1178, 2003.

[WLF+05] Xiaoyun Wang, Xuejia Lai, Dengguo Feng, Hui Chen, and Xiuyuan Yu. Cryptanalysis of the hash functions MD4 and RIPEMD. In *Proc. of Advances in Cryptology (EUROCRYPT)*, volume 3494 of *Lecture Notes in Computer Science*, pages 1–18, 2005.

[Wol86] Stephen Wolfram. Random sequence generation by cellular automata. *Advances in Applied Mathematics*, 7(2):123 – 169, 1986.

[WY05] Xiaoyun Wang and Hongbo Yu. How to break MD5 and other hash functions. In *Proc. of Advances in Cryptology (EUROCRYPT)*, volume 3494 of *Lecture Notes in Computer Science*, pages 19–35, 2005.

[XLL+17] Fan Xiao, Mao Luo, Chu-Min Li, Felip Manyà, and Zhipeng Lü. MapleLRB_LCM, Maple_LCM, Maple_Dist, MapleLRB_LCMoccRestart and Glucose-3.0+width in SAT Competition 2017. In Tomás Balyo, Marijn J. H. Heule, and Matti Järvisalo, editors, *SAT Competition 2017*, volume B-2017-1, pages 22–23, 2017.

[XWCC18] Yang Xu, Guanfeng Wu, Qingshan Chen, and Shuwei Chen. Maple_LCM_Scavel and Maple_LCM_Scavel_200. In Marijn Heule, Matti Järvisalo, and Martin Suda, editors, *Proc. of*

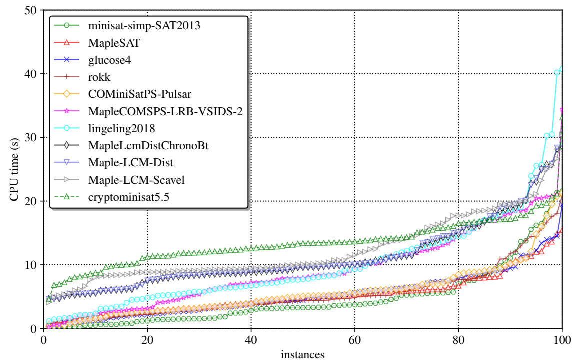
SAT Competition 2018 – Solver and Benchmark Descriptions, volume B-2018-1 of *Department of Computer Science Series of Publications B*, page 30. University of Helsinki, 2018.

[YO14] Takeru Yasumoto and Takumi Okuwaga. Rokk 1.0.1. In Anton Belov, Daniel Diepold, Marijn Heule, and Matti Järvisalo, editors, *SAT Competition 2014*, page 70, 2014.

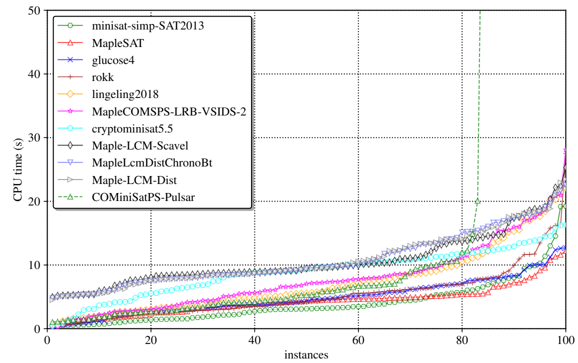
[ZK17] Oleg Zaikin and Stepan Kochemazov. An improved SAT-based guess-and-determine attack on the alternating step generator. In *Proc. of the 20th Information Security Conference (ISC)*, volume 10599 of *Lecture Notes in Computer Science*, pages 21–38, 2017.

[ZMMM01] Lintao Zhang, Conor F. Madigan, Matthew H. Moskewicz, and Sharad Malik. Efficient conflict driven learning in a Boolean satisfiability solver. In *Proc. of IEEE/ACM International Conference on Computer-aided Design (ICCAD)*, pages 279–285, 2001.

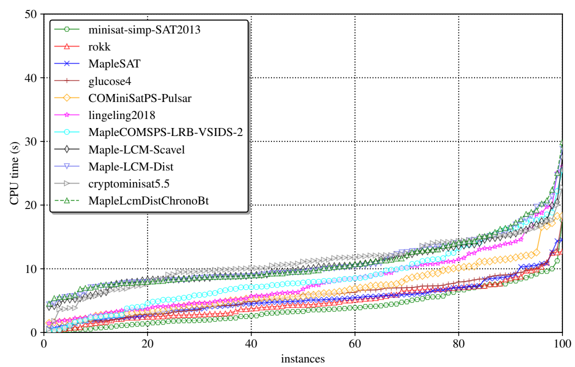
A. Detailed comparison of all considered solvers



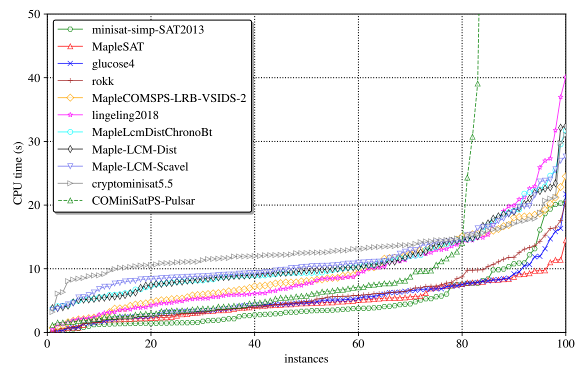
(a) URSA



(b) Transalg

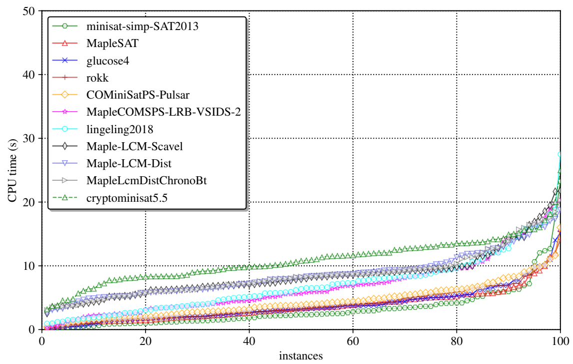


(c) Cryptol-SAT

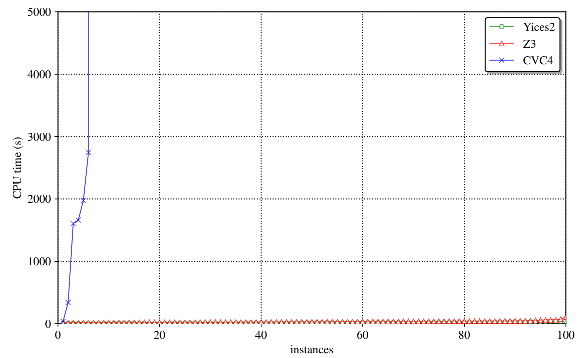


(d) Grain-of-Salt

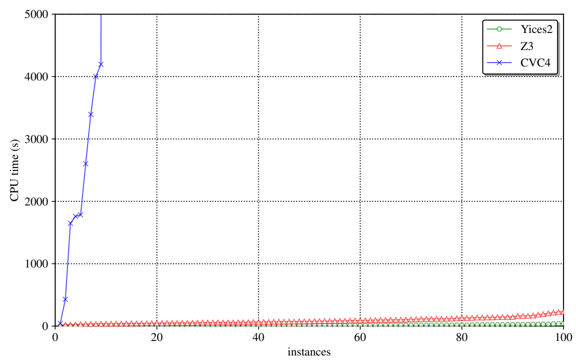
Figure 3. Solving cryptanalysis problem for S_Geffe via different SAT encodings (see Section 4). Part 1.



(A) CBMC-SAT

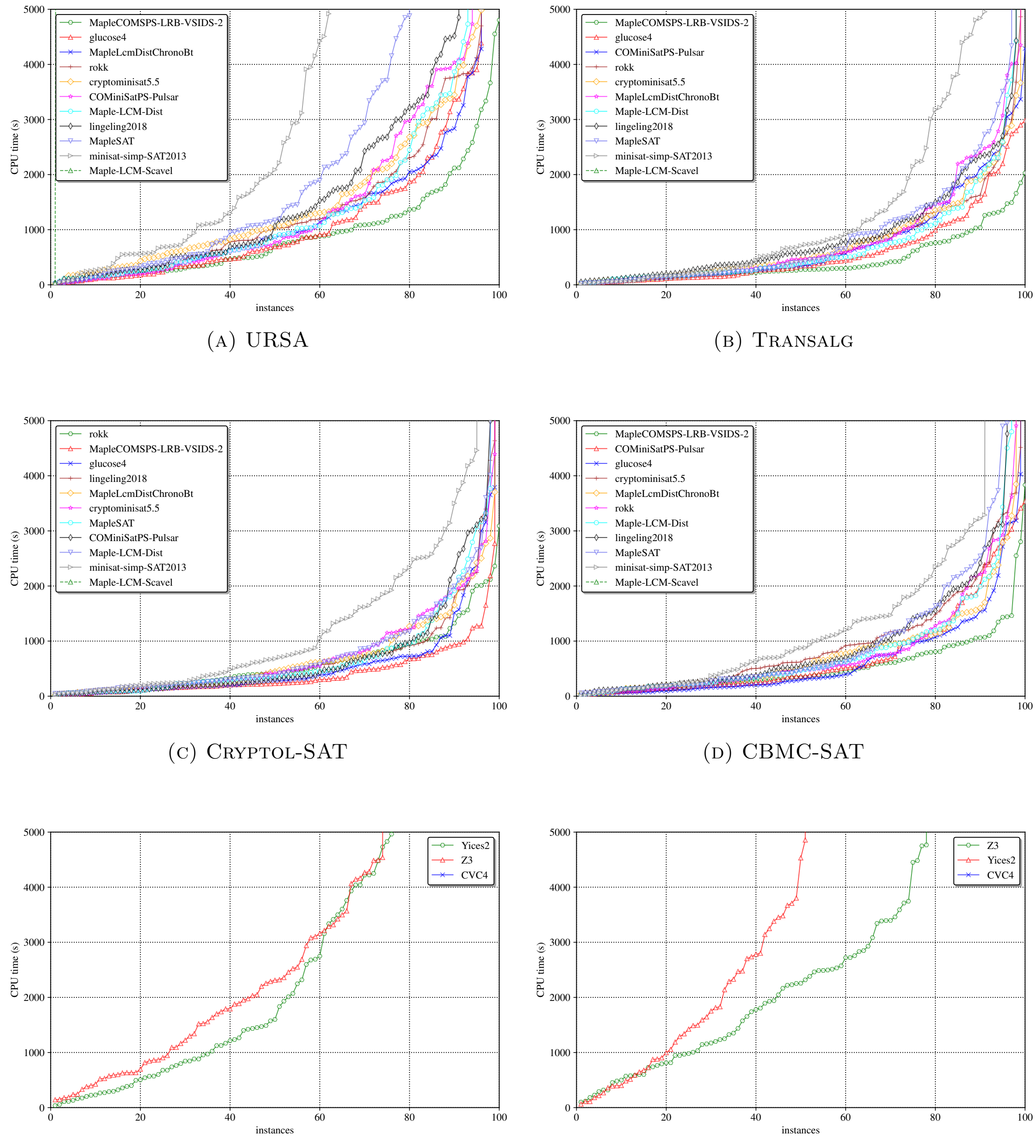


(B) CBMC-SMT



(C) CRYPTOL-SMT

FIGURE 4. Solving S_Geffe cryptanalysis instances using different encodings (see Section 4). Part 2.



(a) URSA

(b) Transalg

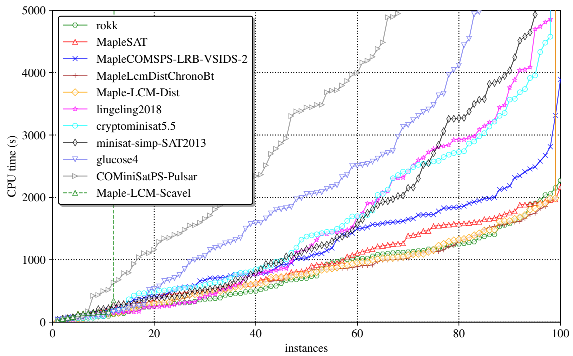
(c) Cryptol-SAT

(d) CBMC-SAT

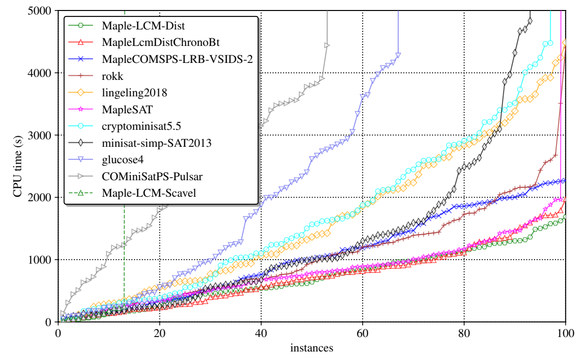
(e) Cryptol-SMT

(f) CBMC-SMT

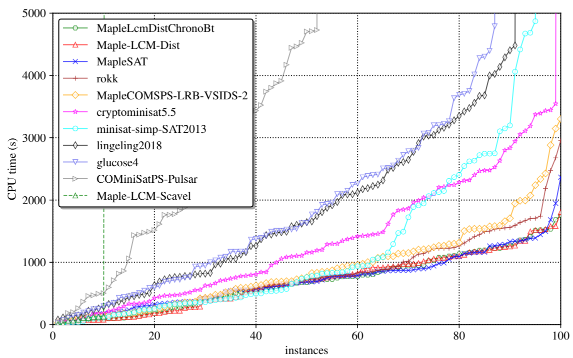
Figure 5. Solving Wolfram cryptanalysis problem via different SAT encodings (see Section 4)



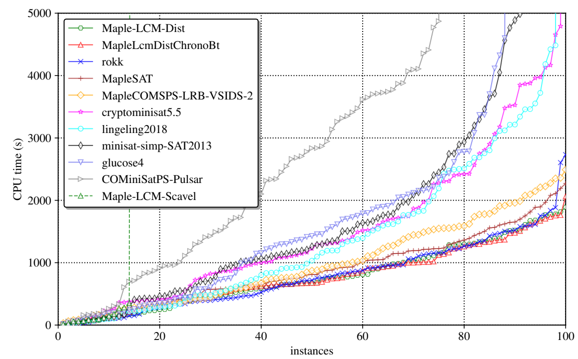
(A) URSA



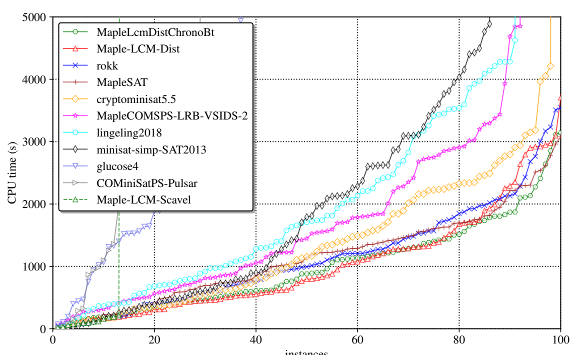
(B) Transalg



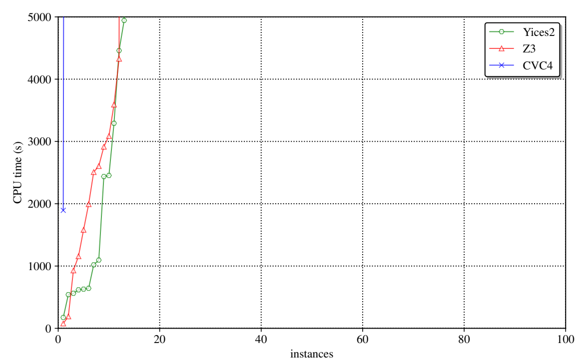
(C) Cryptol-SAT



(D) Grain-of-Salt

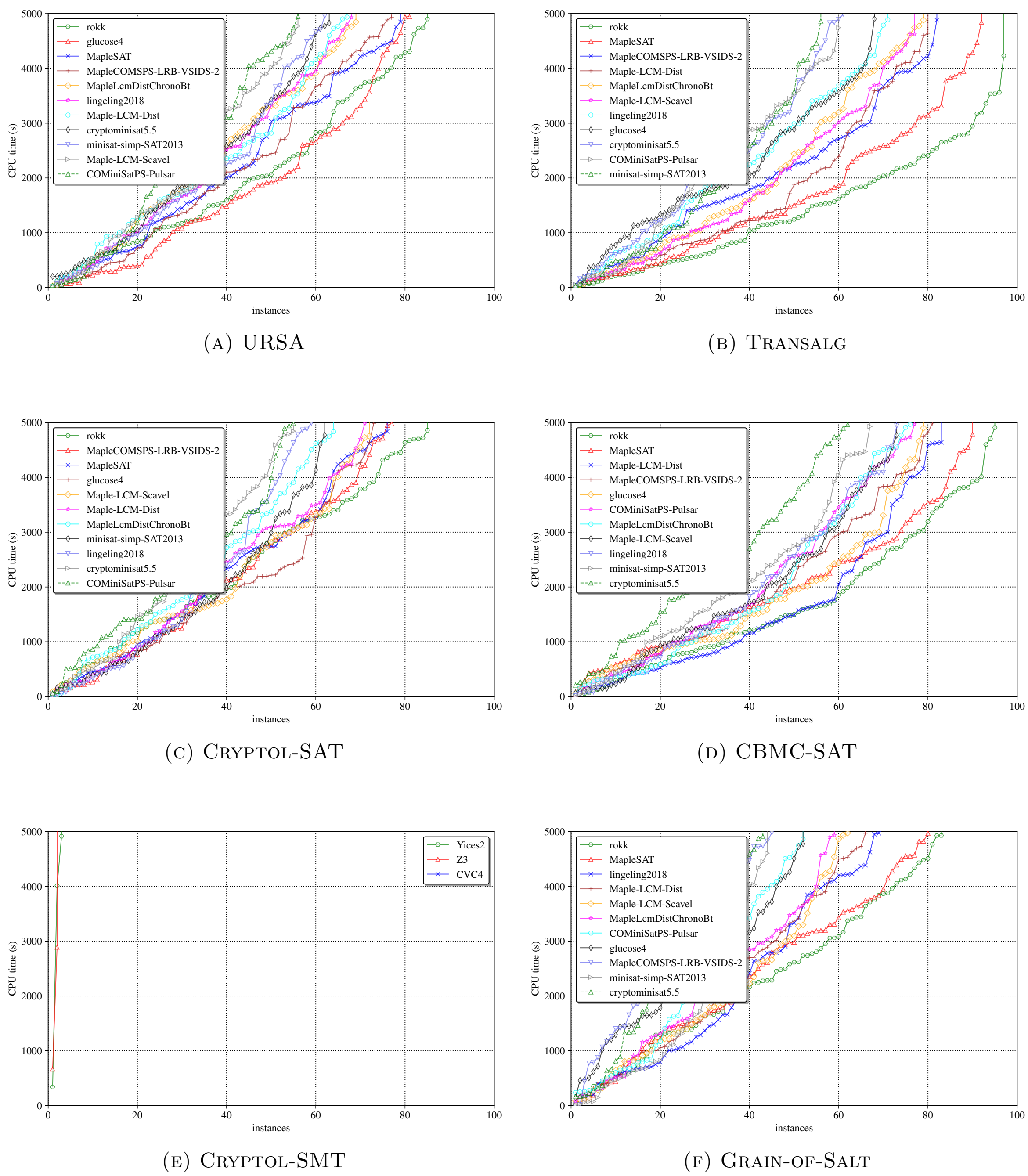


(E) CBMC-SAT



(F) Cryptol-SMT

Figure 6. Solving Bivium30 cryptanalysis problem via different SAT encodings (see Section 4).



(a) URSA

(b) Transalg

(c) Cryptol-SAT

(d) CBMC-SAT

(e) Cryptol-SMT

(f) Grain-of-Salt

Figure 7. Solving Grain102 cryptanalysis problem via different SAT encodings (see Section 4)