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# A NEW COINDUCTIVE CONFLUENCE PROOF FOR INFINITARY LAMBDA CALCULUS

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Abstract. We present a new and formal coinductive proof of confluence and normalisation of Böhm reduction in infinitary lambda calculus. The proof is simpler than previous proofs of this result. The technique of the proof is new, i.e., it is not merely a coinductive reformulation of any earlier proofs. We formalised the proof in the Coq proof assistant.

## 1. Introduction

Infinitary lambda calculus is a generalisation of lambda calculus that allows infinite lambda terms and transfinite reductions. This enables the consideration of "limits" of terms under infinite reduction sequences. For instance, for a term $M \equiv (\lambda mx.mm)(\lambda mx.mm)$ we have

$$M \to_\beta \lambda x.M \to_\beta \lambda x.\lambda x.M \to_\beta \lambda x.\lambda x.\lambda x.M \to_\beta \ldots$$

Intuitively, the "value" of $M$ is an infinite term $L$ satisfying $L \equiv \lambda x.L$, where by $\equiv$ we denote identity of terms. In fact, $L$ is the normal form of $M$ in infinitary lambda calculus.

In [19, 17] it is shown that infinitary reductions may be defined coinductively. The standard non-coinductive definition makes explicit mention of ordinals and limits in a certain metric space [24, 22, 4]. A coinductive approach is better suited to formalisation in a proof-assistant. Indeed, with relatively little effort we have formalised our results in Coq (see Section 7).

In this paper we show confluence of infinitary lambda calculus, modulo equivalence of so-called meaningless terms [26]. We also show confluence and normalisation of infinitary Böhm reduction over any set of strongly meaningless terms. All these results have already been obtained in [24, 26] by a different and more complex proof method.

In a related conference paper [10] we have shown confluence of infinitary reduction modulo equivalence of root-active subterms, and confluence of infinitary Böhm reduction over root-active terms. The present paper is quite different from [10]. A new and simpler method is used. The proof in [10] follows the general strategy of [24]. There first confluence modulo equivalence of root-active terms is shown, proving confluence of an auxiliary $\epsilon$-calculus

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as an intermediate step. Then confluence of Böhm reduction is derived from confluence modulo equivalence of root-active terms. Here we first show that every term has a unique normal form reachable by a special standard infinitary $N_{\mathcal{U}}$-reduction. Then we use this result to derive confluence of Böhm reduction, and from that confluence modulo equivalence of meaningless terms. We do not use any $\epsilon$-calculus at all. See the beginning of Section 5 for a more detailed discussion of our proof method.

1.1. **Related work.** Infinitary lambda calculus was introduced in [24, 23]. Meaningless terms were defined in [26]. The confluence and normalisation results of this paper were already obtained in [24, 26], by a different proof method. See also [22, 4, 18] for an overview of various results in infinitary lambda calculus and infinitary rewriting.

Joachimski in [21] gives a coinductive confluence proof for infinitary lambda calculus, but Joachimski's notion of reduction does not correspond to the standard notion of a strongly convergent reduction. Essentially, it allows for infinitely many parallel contractions in one step, but only finitely many reduction steps. The coinductive definition of infinitary reductions capturing strongly convergent reductions was introduced in [19]. Later [16, 17] generalised this to infinitary term rewriting systems. In [10] using the definition from [19], confluence of infinitary lambda calculus modulo equivalence of root-active subterms was shown coinductively. The proof in [10] follows the general strategy of [24, 23]. The proof in the present paper bears some similarity to the proof of the unique normal forms property of orthogonal iTRSs in [30]. It is also similar to the coinductive confluence proof for nearly orthogonal infinitary term rewriting systems in [12], but there the "standard" reduction employed is not unique and need not be normalising.

Confluence and normalisation results in infinitary rewriting and infinitary lambda calculus have been generalised to the framework of infinitary combinatory reduction systems [27, 28, 29].

There are three well-known variants of infinitary lambda calculus: the $\Lambda^{111}$, $\Lambda^{001}$ and $\Lambda^{101}$ calculi [4, 18, 24, 23]. The superscripts 111, 001, 101 indicate the depth measure used: *abc* means that we shall add $a/b/c$ to the depth when going down/left/right in the tree of the lambda term [24, Definition 6]. In this paper we are concerned only with a coinductive presentation of the $\Lambda^{111}$-calculus.

In the $\Lambda^{001}$-calculus, after addition of appropriate $\bot$-rules, every finite term has its Böhm tree [23] as the normal form. In $\Lambda^{111}$ and $\Lambda^{101}$, the normal forms are, respectively, Berarducci trees and Levy-Longo trees [24, 23, 6, 35, 36]. With the addition of infinite $\eta$- or $\eta!$-reductions it is possible to also capture, respectively, $\eta$-Böhm or $\infty\eta$-Böhm trees as normal forms [40, 44].

The addition of $\bot$-rules may be avoided by basing the definition of infinitary terms on ideal completion. This line of work is pursued in [1, 2, 3]. Confluence of the resulting calculi is shown, but the proof depends on the confluence results of [24].

## 2. Infinite terms and corecursion

In this section we define many-sorted infinitary terms. We also explain and justify guarded corecursion using elementary notions. The results of this section are well-known.

**Definition 2.1.** A *many-sorted algebraic signature* $\Sigma = \langle \Sigma_s, \Sigma_c \rangle$ consists of a collection of *sort symbols* $\Sigma_s = \{s_i\}_{i \in I}$ and a collection of *constructors* $\Sigma_c = \{c_j\}_{j \in J}$. Each constructor $c$

has an associated *type* $\tau(c) = (s_1, \ldots, s_n; s)$ where $s_1, \ldots, s_n, s \in \Sigma_s$. If $\tau(c) = (; s)$ then $c$ is a *constant* of sort $s$. In what follows we use $\Sigma, \Sigma'$, etc., for many-sorted algebraic signatures, $s, s'$, etc., for sort symbols, and $f, g, c, d$, etc., for constructors.

The set $\mathcal{T}^\infty(\Sigma)$, or just $\mathcal{T}(\Sigma)$, of *infinitary terms over* $\Sigma$ is the set of all finite and infinite terms over $\Sigma$, i.e., all finite and infinite labelled trees with labels of nodes specified by the constructors of $\Sigma$ such that the types of labels of nodes agree. More precisely, a term $t$ over $\Sigma$ is a partial function from $\mathbb{N}^*$ to $\Sigma_c$ satisfying:

- $t(\epsilon)\downarrow$, and
- if $t(p) = c \in \Sigma_c$ with $\tau(c) = (s_1, \ldots, s_n; s)$ then
  - $t(pi) = d \in \Sigma_c$ with $\tau(d) = (s'_1, \ldots, s'_{m_i}; s_i)$ for $i < n$,
  - $t(pi)\uparrow$ for $i \geq n$,
- if $t(p)\uparrow$ then $t(pi)\uparrow$ for every $i \in \mathbb{N}$,

where $t(p)\uparrow$ means that $t(p)$ is undefined, $t(p)\downarrow$ means that $t(p)$ is defined, and $\epsilon \in \mathbb{N}^*$ is the empty string. We use obvious notations for infinitary terms, e.g., $f(g(t, s), c)$ when $c, f, g \in \Sigma_c$ and $t, s \in \mathcal{T}(\Sigma)$, and the types agree. We say that a term $t$ *is of sort* $s$ if $t(\epsilon)$ is a constructor of type $(s_1, \ldots, s_n; s)$ for some $s_1, \ldots, s_n \in \Sigma_s$. By $\mathcal{T}_s(\Sigma)$ we denote the set of all terms of sort $s$ from $\mathcal{T}(\Sigma)$.

**Example 2.2.** Let $A$ be a set. Let $\Sigma$ consist of two sorts $\mathfrak{s}$ and $\mathfrak{d}$, one constructor $\mathtt{cons}$ of type $(\mathfrak{d}, \mathfrak{s}; \mathfrak{s})$ and a distinct constant $a \in A$ of sort $\mathfrak{d}$ for each element of $A$. Then $\mathcal{T}_\mathfrak{s}(\Sigma)$ is the set of streams over $A$. We also write $\mathcal{T}_\mathfrak{s}(\Sigma) = A^\omega$ and $\mathcal{T}_\mathfrak{d}(\Sigma) = A$. Instead of $\mathtt{cons}(a, t)$ we usually write $a : t$, and we assume that $:$ associates to the right, e.g., $x : y : t$ is $x : (y : t)$. We also use the notation $x : t$ to denote the application of the constructor for $\mathtt{cons}$ to $x$ and $t$. We define the functions $\mathtt{hd} : A^\omega \to A$ and $\mathtt{tl} : A^\omega \to A^\omega$ by

$$\begin{aligned} \mathtt{hd}(a : t) &= a \\ \mathtt{tl}(a : t) &= t \end{aligned}$$

Specifications of many-sorted signatures may be conveniently given by coinductively interpreted grammars. For instance, the set $A^\omega$ of streams over a set $A$ could be specified by writing

$$A^\omega ::= \mathtt{cons}(A, A^\omega).$$

A more interesting example is that of finite and infinite binary trees with nodes labelled either with $a$ or $b$, and leaves labelled with one of the elements of a set $V$:

$$T ::= V \parallel a(T, T) \parallel b(T, T).$$

As such specifications are not intended to be formal entities but only convenient notation for describing sets of infinitary terms, we will not define them precisely. It is always clear what many-sorted signature is meant.

For the sake of brevity we often use $\mathcal{T} = \mathcal{T}(\Sigma)$ and $\mathcal{T}_s = \mathcal{T}_s(\Sigma)$, i.e., we omit the signature $\Sigma$ when clear from the context or irrelevant.

**Definition 2.3.** The class of *constructor-guarded* functions is defined inductively as the class of all functions $h : \mathcal{T}_s^m \to \mathcal{T}_{s'}$ (for arbitrary $m \in \mathbb{N}$, $s, s' \in \Sigma_s$) such that there are a constructor $c$ of type $(s_1, \ldots, s_k; s')$ and functions $u_i : \mathcal{T}_s^m \to \mathcal{T}_{s_i}$ $(i = 1, \ldots, k)$ such that

$$h(y_1, \ldots, y_m) = c(u_1(y_1, \ldots, y_m), \ldots, u_k(y_1, \ldots, y_m))$$

for all $y_1, \ldots, y_m \in \mathcal{T}_s$, and for each $i = 1, \ldots, k$ one of the following holds:

- $u_i$ is constructor-guarded, or

- $u_i$ is a constant function, or
- $u_i$ is a projection function, i.e., $s_i = s$ and there is $1 \le j \le m$ with $u_i(y_1, \ldots, y_m) = y_j$ for all $y_1, \ldots, y_m \in \mathcal{T}_s$.

Let $S$ be a set. A function $h : S \times \mathcal{T}_s^m \to \mathcal{T}_{s'}$ is constructor-guarded if for every $x \in S$ the function $h_x : \mathcal{T}_s^m \to \mathcal{T}_{s'}$ defined by $h_x(y_1, \ldots, y_m) = h(x, y_1, \ldots, y_m)$ is constructor-guarded. A function $f : S \to \mathcal{T}_s$ is defined by *guarded corecursion* from $h : S \times \mathcal{T}_s^m \to \mathcal{T}_s$ and $g_i : S \to S$ ($i = 1, \ldots, m$) if $h$ is constructor-guarded and $f$ satisfies

$$f(x) = h(x, f(g_1(x)), \ldots, f(g_m(x)))$$

for all $x \in S$.

The following theorem is folklore in the coalgebra community. We sketch an elementary proof. In fact, each set of many-sorted infinitary terms is a final coalgebra of an appropriate set-functor. Then Theorem 2.4 follows from more general principles. We prefer to avoid coalgebraic terminology, as it is simply not necessary for the purposes of the present paper. See e.g. [20, 38] for a more general coalgebraic explanation of corecursion.

**Theorem 2.4.** *For any constructor-guarded function $h : S \times \mathcal{T}_s^m \to \mathcal{T}_s$ and any $g_i : S \to S$ ($i = 1, \ldots, m$), there exists a unique function $f : S \to \mathcal{T}_s$ defined by guarded corecursion from $h$ and $g_1, \ldots, g_m$.*

*Proof.* Let $f_0 : S \to \mathcal{T}_s$ be an arbitrary function. Define $f_{n+1}$ for $n \in \mathbb{N}$ by $f_{n+1}(x) = h(x, f_n(g_1(x)), \ldots, f_n(g_m(x)))$. Using the fact that $h$ is constructor-guarded, one shows by induction on $n$ that:

$$f_{n+1}(x)(p) = f_n(x)(p) \text{ for } x \in S \text{ and } p \in \mathbb{N}^* \text{ with } |p| < n \tag{$\star$}$$

where $|p|$ denotes the length of $p$. Indeed, the base is obvious. We show the inductive step. Let $x \in S$. Because $h$ is constructor-guarded, we have for instance

$$f_{n+2}(x) = h(x, f_{n+1}(g_1(x)), \ldots, f_{n+1}(g_m(x))) = c_1(c_2, c_3(w, f_{n+1}(g_1(x))))$$

Let $p \in \mathbb{N}^*$ with $|p| \le n$. The only interesting case is when $p = 11p'$, i.e., when $p$ points inside $f_{n+1}(g_1(x))$. But then $|p'| < |p| \le n$, so by the inductive hypothesis $f_{n+1}(g_1(x))(p') = f_n(g_1(x))(p')$. Thus $f_{n+2}(x)(p) = f_{n+1}(g_1(x))(p') = f_n(g_1(x))(p') = f_{n+1}(x)(p)$.

Now we define $f : S \to \mathcal{T}_s$ by

$$f(x)(p) = f_{|p|+1}(x)(p)$$

for $x \in S$, $p \in \mathbb{N}^*$. Using ($\star$) it is routine to check that $f(x)$ is a well-defined infinitary term for each $x \in S$. To show that $f : S \to \mathcal{T}_s$ is defined by guarded corecursion from $h$ and $g_1, \ldots, g_m$, using ($\star$) one shows by induction on the length of $p \in \mathbb{N}^*$ that for any $x \in S$:

$$f(x)(p) = h(x, f(g_1(x)), \ldots, f(g_m(x)))(p).$$

To prove that $f$ is unique it suffices to show that it does not depend on $f_0$. For this purpose, using ($\star$) one shows by induction on the length of $p \in \mathbb{N}^*$ that $f(x)(p)$ does not depend on $f_0$ for any $x \in S$. $\square$

We shall often use the above theorem implicitly, just mentioning that some equations define a function by guarded corecursion.

**Example 2.5.** Consider the equation

$$\mathtt{even}(x : y : t) = x : \mathtt{even}(t)$$

It may be rewritten as

$$\mathtt{even}(t) = \mathtt{hd}(t) : \mathtt{even}(\mathtt{tl}(\mathtt{tl}(t)))$$

So $\mathtt{even} : A^\omega \to A^\omega$ is defined by guarded corecursion from $h : A^\omega \times A^\omega \to A^\omega$ given by

$$h(t, t') = \mathtt{hd}(t) : t'$$

and $g : A^\omega \to A^\omega$ given by

$$g(t) = \mathtt{tl}(\mathtt{tl}(t))$$

By Theorem 2.4 there is a unique function $\mathtt{even} : A^\omega \to A^\omega$ satisfying the original equation.

Another example of a function defined by guarded corecursion is $\mathtt{zip} : A^\omega \times A^\omega \to A^\omega$:

$$\mathtt{zip}(x : t, s) = x : \mathtt{zip}(s, t)$$

The following function $\mathtt{merge} : \mathbb{N}^\omega \times \mathbb{N}^\omega \to \mathbb{N}^\omega$ is also defined by guarded corecursion:

$$\mathtt{merge}(x : t_1, y : t_2) = \begin{cases} x : \mathtt{merge}(t_1, y : t_2) & \text{if } x \leq y \\ y : \mathtt{merge}(x : t_1, t_2) & \text{otherwise} \end{cases}$$

## 3. Coinduction

In this section[1] we give a brief explanation of coinduction as it is used in the present paper. Our presentation of coinductive proofs is similar to e.g. [19, 8, 37, 34, 31].

There are many ways in which our coinductive proofs could be justified. Since we formalised our main results (see Section 7), the proofs may be understood as a paper presentation of formal Coq proofs. They can also be justified by appealing to one of a number of established coinduction principles. With enough patience one could, in principle, reformulate all proofs to directly employ the usual coinduction principle in set theory based on the Knaster-Tarski fixpoint theorem [39]. One could probably also use the coinduction principle from [31]. Finally, one may justify our proofs by indicating how to interpret them in ordinary set theory, which is what we do in this section.

The purpose of this section is to explain how to justify our proofs by reducing coinduction to transfinite induction. The present section does not provide a formal coinduction proof principle as such, but only indicates how one could elaborate the proofs so as to eliminate the use of coinduction. Naturally, such an elaboration would introduce some tedious details. The point is that all these details are essentially the same for each coinductive proof. The advantage of using coinduction is that the details need not be provided each time. A similar elaboration could be done to directly employ any of a number of established coinduction principles, but as far as we know elaborating the proofs in the way explained here requires the least amount of effort in comparison to reformulating them to directly employ an established coinduction principle. In fact, we do not wish to explicitly commit to any single formal proof principle, because we do not think that choosing a specific principle has an essential impact on the content of our proofs, except by making it more or less straightforward to translate the proofs into a form which uses the principle directly.

A reader not satisfied with the level of rigour of the explanations of coinduction below is referred to our formalisation (see Section 7). The present section provides one way in which

[1] This section is largely based on [12, Section 2].

our proofs can be understood and verified without resorting to a formalisation. To make the observations of this section completely precise and general one would need to introduce formal notions of "proof" and "statement". In other words, one would need to formulate a system of logic with a capacity for coinductive proofs. We do not want to do this here, because this paper is about a coinductive confluence proof for infinitary lambda calculus, not about foundations of coinduction. It would require some work, but should not be too difficult, to create a formal system based on the present section in which our coinductive proofs could be interpreted reasonably directly. We defer this to future work. The status of the present section is that of a "meta-explanation", analogous to an explanation of how, e.g., the informal presentations of inductive constructions found in the literature may be encoded in ZFC set theory.

**Example 3.1.** Let $T$ be the set of all finite and infinite terms defined coinductively by

$$T ::= V \parallel A(T) \parallel B(T,T)$$

where $V$ is a countable set of variables, and $A$, $B$ are constructors. By $x, y, \ldots$ we denote variables, and by $t, s, \ldots$ we denote elements of $T$. Define a binary relation $\to$ on $T$ coinductively by the following rules.

$$\frac{}{x \to x}\;(1) \qquad \frac{t \to t'}{A(t) \to A(t')}\;(2) \qquad \frac{s \to s' \quad t \to t'}{B(s,t) \to B(s',t')}\;(3) \qquad \frac{t \to t'}{A(t) \to B(t',t')}\;(4)$$

Formally, the relation $\to$ is the greatest fixpoint of a monotone function

$$F : \mathcal{P}(T \times T) \to \mathcal{P}(T \times T)$$

defined by

$$F(R) = \left\{ \langle t_1, t_2 \rangle \mid \exists_{x \in V}(t_1 \equiv t_2 \equiv x) \vee \exists_{t,t' \in T}(t_1 \equiv A(t) \wedge t_2 \equiv A(t') \wedge R(t,t')) \vee \ldots \right\}.$$

Alternatively, using the Knaster-Tarski fixpoint theorem, the relation $\to$ may be characterised as the greatest binary relation on $T$ (i.e. the greatest subset of $T \times T$ w.r.t. set inclusion) such that $\to \subseteq F(\to)$, i.e., such that for every $t_1, t_2 \in T$ with $t_1 \to t_2$ one of the following holds:

(1) $t_1 \equiv t_2 \equiv x$ for some variable $x \in V$,
(2) $t_1 \equiv A(t)$, $t_2 \equiv A(t')$ with $t \to t'$,
(3) $t_1 \equiv B(s,t)$, $t_2 \equiv B(s',t')$ with $s \to s'$ and $t \to t'$,
(4) $t_1 \equiv A(t)$, $t_2 \equiv B(t',t')$ with $t \to t'$.

Yet another way to think about $\to$ is that $t_1 \to t_2$ holds if and only if there exists a *potentially infinite* derivation tree of $t_1 \to t_2$ built using the rules $(1) - (4)$.

The rules $(1) - (4)$ could also be interpreted inductively to yield the least fixpoint of $F$. This is the conventional interpretation, and it is indicated with a single line in each rule separating premises from the conclusion. A coinductive interpretation is indicated with double lines.

The greatest fixpoint $\to$ of $F$ may be obtained by transfinitely iterating $F$ starting with $T \times T$. More precisely, define an ordinal-indexed sequence $(\to^\gamma)_\gamma$ by:

- $\to^0 = T \times T$,
- $\to^{\gamma+1} = F(\to^\gamma)$,
- $\to^\delta = \bigcap_{\gamma<\delta} \to^\gamma$ for a limit ordinal $\delta$.

Then there exists an ordinal $\zeta$ such that $\to = \to^\zeta$. The least such ordinal is called the *closure ordinal*. Note also that $\to^\gamma \subseteq \to^\delta$ for $\gamma \geq \delta$ (we often use this fact implicitly). See e.g. [14, Chapter 8]. The relation $\to^\gamma$ is called the $\gamma$-*approximant*. Note that the $\gamma$-approximants depend on a particular definition of $\to$ (i.e. on the function $F$), not solely on the relation $\to$ itself. We use $R^\gamma$ for the $\gamma$-approximant of a relation $R$.

It is instructive to note that the coinductive rules for $\to$ may also be interpreted as giving rules for the $\gamma+1$-approximants, for any ordinal $\gamma$.

$$\frac{}{x \to^{\gamma+1} x}\ (1) \quad \frac{t \to^\gamma t'}{A(t) \to^{\gamma+1} A(t')}\ (2) \quad \frac{s \to^\gamma s' \quad t \to^\gamma t'}{B(s,t) \to^{\gamma+1} B(s',t')}\ (3) \quad \frac{t \to^\gamma t'}{A(t) \to^{\gamma+1} B(t',t')}\ (4)$$

Usually, the closure ordinal for the definition of a coinductive relation is $\omega$, as is the case with all coinductive definitions appearing in this paper. In general, however, it is not difficult to come up with a coinductive definition whose closure ordinal is greater than $\omega$. For instance, consider the relation $R \subseteq \mathbb{N} \cup \{\infty\}$ defined coinductively by the following two rules.

$$\frac{R(n) \quad n \in \mathbb{N}}{R(n+1)} \qquad \frac{\exists n \in \mathbb{N}.R(n)}{R(\infty)}$$

We have $R = \emptyset$, $R^n = \{m \in \mathbb{N} \mid m \geq n\} \cup \{\infty\}$ for $n \in \mathbb{N}$, $R^\omega = \{\infty\}$, and only $R^{\omega+1} = \emptyset$. Thus the closure ordinal of this definition is $\omega + 1$.

In this paper we are interested in proving by coinduction statements of the form $\psi(R_1, \ldots, R_m)$ where

$$\psi(X_1, \ldots, X_m) \equiv \forall x_1 \ldots x_n . \varphi(\vec{x}) \to X_1(g_1(\vec{x}), \ldots, g_k(\vec{x})) \wedge \ldots \wedge X_m(g_1(\vec{x}), \ldots, g_k(\vec{x})).$$

and $R_1, \ldots, R_m$ are coinductive relations on $T$, i.e, relations defined as the greatest fixpoints of some monotone functions on the powerset of an appropriate cartesian product of $T$, and $\psi(R_1, \ldots, R_m)$ is $\psi(X_1, \ldots, X_m)$ with $R_i$ substituted for $X_i$. Statements with an existential quantifier may be reduced to statements of this form by skolemising, as explained in Example 3.3 below.

Here $X_1, \ldots, X_m$ are meta-variables for which relations on $T$ may be substituted. In the statement $\varphi(\vec{x})$ only $x_1, \ldots, x_n$ occur free. The meta-variables $X_1, \ldots, X_m$ *are not allowed to occur* in $\varphi(\vec{x})$. In general, we abbreviate $x_1, \ldots, x_n$ with $\vec{x}$. The symbols $g_1, \ldots, g_k$ denote some functions of $\vec{x}$.

To prove $\psi(R_1, \ldots, R_m)$ it suffices to show by transfinite induction that $\psi(R_1^\gamma, \ldots, R_m^\gamma)$ holds for each ordinal $\gamma \leq \zeta$, where $R_i^\gamma$ is the $\gamma$-approximant of $R_i$. It is an easy exercise to check that because of the special form of $\psi$ (in particular because $\varphi$ does not contain $X_1, \ldots, X_m$) and the fact that each $R_i^0$ is the full relation, the base case $\gamma = 0$ and the case of $\gamma$ a limit ordinal hold. They hold for *any* $\psi$ of the above form, *irrespective* of $\varphi, R_1, \ldots, R_m$. Note that $\varphi(\vec{x})$ is the same in all $\psi(R_1^\gamma, \ldots, R_m^\gamma)$ for any $\gamma$, i.e., it does not refer to the $\gamma$-approximants or the ordinal $\gamma$. Hence it remains to show the inductive step for $\gamma$ a successor ordinal. It turns out that a coinductive proof of $\psi$ may be interpreted as a proof of this inductive step for a successor ordinal, with the ordinals left implicit and the phrase "coinductive hypothesis" used instead of "inductive hypothesis".

**Example 3.2.** On terms from $T$ (see Example 3.1) we define the operation of substitution by guarded corecursion.

$$\begin{array}{rclll} y[t/x] & = & y & \text{if } x \neq y & (A(s))[t/x] = A(s[t/x]) \\ x[t/x] & = & t & & (B(s_1, s_2))[t/x] = B(s_1[t/x], s_2[t/x]) \end{array}$$

We show by coinduction: if $s \to s'$ and $t \to t'$ then $s[t/x] \to s'[t'/x]$, where $\to$ is the relation from Example 3.1. Formally, the statement we show by transfinite induction on $\gamma \leq \zeta$ is: for $s, s', t, t' \in T$, if $s \to s'$ and $t \to t'$ then $s[t/x] \to^\gamma s'[t'/x]$. For illustrative purposes, we indicate the $\gamma$-approximants with appropriate ordinal superscripts, but it is customary to omit these superscripts.

Let us proceed with the proof. The proof is by coinduction with case analysis on $s \to s'$. If $s \equiv s' \equiv y$ with $y \neq x$, then $s[t/x] \equiv y \equiv s'[t'/x]$. If $s \equiv s' \equiv x$ then $s[t/x] \equiv t \to^{\gamma+1} t' \equiv s'[t'/x]$ (note that $\to \equiv \to^\zeta \subseteq \to^{\gamma+1}$). If $s \equiv A(s_1)$, $s' \equiv A(s_1')$ and $s_1 \to s_1'$, then $s_1[t/x] \to^\gamma s_1'[t'/x]$ by the coinductive hypothesis. Thus $s[t/x] \equiv A(s_1[t/x]) \to^{\gamma+1} A(s_1'[t'/x]) \equiv s'[t'/x]$ by rule (2). If $s \equiv B(s_1, s_2)$, $s' \equiv B(s_1', s_2')$ then the proof is analogous. If $s \equiv A(s_1)$, $s' \equiv B(s_1', s_1')$ and $s_1 \to s_1'$, then the proof is also similar. Indeed, by the coinductive hypothesis we have $s_1[t/x] \to^\gamma s_1'[t'/x]$, so $s[t/x] \equiv A(s_1[t/x]) \to^{\gamma+1} B(s_1'[t'/x], s_1'[t'/x]) \equiv s'[t'/x]$ by rule (4).

With the following example we explain how our proofs of existential statements should be interpreted.

**Example 3.3.** Let $T$ and $\to$ be as in Example 3.1. We want to show: for all $s, t, t' \in T$, if $s \to t$ and $s \to t'$ then there exists $s' \in T$ with $t \to s'$ and $t' \to s'$. The idea is to skolemise this statement. So we need to find a Skolem function $f : T^3 \to T$ which will allow us to prove the Skolem normal form:

$$\text{if } s \to t \text{ and } s \to t' \text{ then } t \to f(s, t, t') \text{ and } t' \to f(s, t, t'). \tag{$\star$}$$

The rules for $\to$ suggest a definition of $f$:

$$\begin{array}{rcl}
f(x, x, x) & = & x \\
f(A(s), A(t), A(t')) & = & A(f(s, t, t')) \\
f(A(s), A(t), B(t', t')) & = & B(f(s, t, t'), f(s, t, t')) \\
f(A(s), B(t, t), A(t')) & = & B(f(s, t, t'), f(s, t, t')) \\
f(A(s), B(t, t), B(t', t')) & = & B(f(s, t, t'), f(s, t, t')) \\
f(B(s_1, s_2), B(t_1, t_2), B(t_1', t_2')) & = & B(f(s_1, t_1, t_1'), f(s_2, t_2, t_2')) \\
f(s, t, t') & = & \text{some fixed term if none of the above matches}
\end{array}$$

This is a definition by guarded corecursion, so there exists a unique function $f : T^3 \to T$ satisfying the above equations. The last case in the above definition of $f$ corresponds to the case in Definition 2.3 where all $u_i$ are constant functions. Note that any fixed term has a fixed constructor (in the sense of Definition 2.3) at the root. In the sense of Definition 2.3 also the elements of $V$ are nullary constructors.

We now proceed with a coinductive proof of ($\star$). Assume $s \to t$ and $s \to t'$. If $s \equiv t \equiv t' \equiv x$ then $f(s, t, t') \equiv x$, and $x \to x$ by rule (1). If $s \equiv A(s_1)$, $t \equiv A(t_1)$ and $t' \equiv A(t_1')$ with $s_1 \to t_1$ and $s_1 \to t_1'$, then by the coinductive hypothesis $t_1 \to f(s_1, t_1, t_1')$ and $t_1' \to f(s_1, t_1, t_1')$. We have $f(s, t, t') \equiv A(f(s_1, t_1, t_1'))$. Hence $t \equiv A(t_1) \to f(s, t, t')$ and $t \equiv A(t_1') \to f(s, t, t')$, by rule (2). If $s \equiv B(s_1, s_2)$, $t \equiv B(t_1, t_2)$ and $t' \equiv B(t_1', t_2')$, with $s_1 \to t_1$, $s_1 \to t_1'$, $s_2 \to t_2$ and $s_2 \to t_2'$, then by the coinductive hypothesis we have $t_1 \to f(s_1, t_1, t_1')$, $t_1' \to f(s_1, t_1, t_1')$, $t_2 \to f(s_2, t_2, t_2')$ and $t_2' \to f(s_2, t_2, t_2')$. Hence $t \equiv B(t_1, t_2) \to B(f(s_1, t_1, t_1'), f(s_2, t_2, t_2')) \equiv f(s, t, t')$ by rule (3). Analogously, $t' \to f(s, t, t')$ by rule (3). Other cases are similar.

Usually, it is inconvenient to invent the Skolem function beforehand, because the definition of the Skolem function and the coinductive proof of the Skolem normal form are typically interdependent. Therefore, we adopt an informal style of doing a proof by coinduction of a statement

$$\begin{aligned}\psi(R_1,\ldots,R_m) \quad = \quad & \forall_{x_1,\ldots,x_n\in T}\,.\,\varphi(\vec{x}) \to \\ & \exists_{y\in T}.R_1(g_1(\vec{x}),\ldots,g_k(\vec{x}),y) \wedge \ldots \wedge R_m(g_1(\vec{x}),\ldots,g_k(\vec{x}),y)\end{aligned}$$

with an existential quantifier. We intertwine the corecursive definition of the Skolem function $f$ with a coinductive proof of the Skolem normal form

$$\begin{aligned}&\forall_{x_1,\ldots,x_n\in T}\,.\,\varphi(\vec{x}) \to \\ &\quad R_1(g_1(\vec{x}),\ldots,g_k(\vec{x}),f(\vec{x})) \wedge \ldots \wedge R_m(g_1(\vec{x}),\ldots,g_k(\vec{x}),f(\vec{x}))\end{aligned}$$

We proceed as if the coinductive hypothesis was $\psi(R_1^\gamma,\ldots,R_m^\gamma)$ (it really is the Skolem normal form). Each element obtained from the existential quantifier in the coinductive hypothesis is interpreted as a corecursive invocation of the Skolem function. When later we exhibit an element to show the existential subformula of $\psi(R_1^{\gamma+1},\ldots,R_m^{\gamma+1})$, we interpret this as the definition of the Skolem function in the case specified by the assumptions currently active in the proof. Note that this exhibited element may (or may not) depend on some elements obtained from the existential quantifier in the coinductive hypothesis, i.e., the definition of the Skolem function may involve corecursive invocations.

To illustrate our style of doing coinductive proofs of statements with an existential quantifier, we redo the proof done above. For illustrative purposes, we indicate the arguments of the Skolem function, i.e., we write $s'_{s,t,t'}$ in place of $f(s,t,t')$. These subscripts $s,t,t'$ are normally omitted.

We show by coinduction that if $s \to t$ and $s \to t'$ then there exists $s' \in T$ with $t \to s'$ and $t' \to s'$. Assume $s \to t$ and $s \to t'$. If $s \equiv t \equiv t' \equiv x$ then take $s'_{x,x,x} \equiv x$. If $s \equiv A(s_1)$, $t \equiv A(t_1)$ and $t' \equiv A(t_1')$ with $s_1 \to t_1$ and $s_1 \to t_1'$, then by the coinductive hypothesis we obtain $s'_{s_1,t_1,t_1'}$ with $t_1 \to s'_{s_1,t_1,t_1'}$ and $t_1' \to s'_{s_1,t_1,t_1'}$. More precisely: by corecursively applying the Skolem function to $s_1,t_1,t_1'$ we obtain $s'_{s_1,t_1,t_1'}$, and by the coinductive hypothesis we have $t_1 \to s'_{s_1,t_1,t_1'}$ and $t_1' \to s'_{s_1,t_1,t_1'}$. Hence $t \equiv A(t_1) \to A(s'_{s_1,t_1,t_1'})$ and $t \equiv A(t_1') \to A(s'_{s_1,t_1,t_1'})$, by rule (2). Thus we may take $s'_{s,t,t'} \equiv A(s'_{s_1,t_1,t_1'})$. If $s \equiv B(s_1,s_2)$, $t \equiv B(t_1,t_2)$ and $t' \equiv B(t_1',t_2')$, with $s_1 \to t_1$, $s_1 \to t_1'$, $s_2 \to t_2$ and $s_2 \to t_2'$, then by the coinductive hypothesis we obtain $s'_{s_1,t_1,t_1'}$ and $s'_{s_2,t_2,t_2'}$ with $t_1 \to s'_{s_1,t_1,t_1'}$, $t_1' \to s'_{s_1,t_1,t_1'}$, $t_2 \to s'_{s_2,t_2,t_2'}$ and $t_2' \to s'_{s_2,t_2,t_2'}$. Hence $t \equiv B(t_1,t_2) \to B(s'_{s_1,t_1,t_1'},s'_{s_2,t_2,t_2'})$ by rule (3). Analogously, $t' \to B(s'_{s_1,t_1,t_1'},s'_{s_2,t_2,t_2'})$ by rule (3). Thus we may take $s'_{s,t,t'} \equiv B(s'_{s_1,t_1,t_1'},s'_{s_2,t_2,t_2'})$. Other cases are similar.

It is clear that the above informal proof, when interpreted in the way outlined before, implicitly defines the Skolem function $f$. It should be kept in mind that in every case the definition of the Skolem function needs to be guarded. We do not explicitly mention this each time, but verifying this is part of verifying the proof.

When doing proofs by coinduction the following criteria need to be kept in mind in order to be able to justify the proofs according to the above explanations.

- When we conclude from the coinductive hypothesis that some relation $R(t_1,\ldots,t_n)$ holds, this really means that only its approximant $R^\gamma(t_1,\ldots,t_n)$ holds. Usually, we need to infer that the next approximant $R^{\gamma+1}(s_1,\ldots,s_n)$ holds (for some other elements $s_1,\ldots,s_n$) by using $R^\gamma(t_1,\ldots,t_n)$ as a premise of an appropriate rule. But we cannot, e.g., inspect (do case reasoning on) $R^\gamma(t_1,\ldots,t_n)$, use it in any lemmas, or otherwise treat it as $R(t_1,\ldots,t_n)$.

- An element $e$ obtained from an existential quantifier in the coinductive hypothesis is not really the element itself, but a corecursive invocation of the implicit Skolem function. Usually, we need to put it inside some constructor $c$, e.g. producing $c(e)$, and then exhibit $c(e)$ in the proof of an existential statement. Applying at least one constructor to $e$ is necessary to ensure guardedness of the implicit Skolem function. But we cannot, e.g., inspect $e$, apply some previously defined functions to it, or otherwise treat it as if it was really given to us.
- In the proofs of existential statements, the implicit Skolem function cannot depend on the ordinal $\gamma$. However, this is the case as long as we do not violate the first point, because if the ordinals are never mentioned and we do not inspect the approximants obtained from the coinductive hypothesis, then there is no way in which we could possibly introduce a dependency on $\gamma$.

Equality on infinitary terms may be characterised coinductively.

**Definition 3.4.** Let $\Sigma$ be a many-sorted algebraic signature, as in Definition 2.1. Let $\mathcal{T} = \mathcal{T}(\Sigma)$. Define on $\mathcal{T}$ a binary relation $=$ of *bisimilarity* by the coinductive rules

$$\frac{t_1 = s_1 \quad \dots \quad t_n = s_n}{f(t_1, \dots, t_n) = f(s_1, \dots, s_n)}$$

for each constructor $f \in \Sigma_c$.

It is intuitively obvious that on infinitary terms bisimilary is the same as identity. The following easy (and well-known) proposition makes this precise.

**Proposition 3.5.** *For $t, s \in \mathcal{T}$ we have: $t = s$ iff $t \equiv s$.*

*Proof.* Recall that each term is formally a partial function from $\mathbb{N}^*$ to $\Sigma_c$. We write $t(p) \approx s(p)$ if either both $t(p), s(p)$ are defined and equal, or both are undefined.

Assume $t = s$. It suffices to show by induction of the length of $p \in \mathbb{N}^*$ that $t_{|p} = s_{|p}$ or $t(p)\uparrow, s(p)\uparrow$, where by $t_{|p}$ we denote the subterm of $t$ at position $p$. For $p = \epsilon$ this is obvious. Assume $p = p'j$. By the inductive hypothesis, $t_{|p'} = s_{|p'}$ or $t(p')\uparrow, s(p')\uparrow$. If $t_{|p'} = s_{|p'}$ then $t_{|p'} \equiv f(t_0, \dots, t_n)$ and $s_{|p'} \equiv f(s_0, \dots, s_n)$ for some $f \in \Sigma_c$ with $t_i = s_i$ for $i = 0, \dots, n$. If $0 \le j \le n$ then $t_{|p} \equiv t_j = s_j = s_{|p}$. Otherwise, if $j > n$ or if $t(p')\uparrow, s(p')\uparrow$, then $t(p)\uparrow, s(p)\uparrow$ by the definition of infinitary terms.

For the other direction, we show by coinduction that for any $t \in \mathcal{T}$ we have $t = t$. If $t \in \mathcal{T}$ then $t \equiv f(t_1, \dots, t_n)$ for some $f \in \Sigma_c$. By the coinductive hypothesis we obtain $t_i = t_i$ for $i = 1, \dots, n$. Hence $t = t$ by the rule for $f$. $\square$

For infinitary terms $t, s \in \mathcal{T}$, we shall therefore use the notations $t = s$ and $t \equiv s$ interchangeably, employing Proposition 3.5 implicitly. In particular, the above coinductive characterisation of term equality is used implicitly in the proof of Lemma 5.29.

**Example 3.6.** Recall the coinductive definitions of `zip` and `even` from Example 2.5.

$$\begin{array}{rcl} \mathtt{even}(x : y : t) & = & x : \mathtt{even}(t) \\ \mathtt{zip}(x : t, s) & = & x : \mathtt{zip}(s, t) \end{array}$$

By coinduction we show

$$\mathtt{zip}(\mathtt{even}(t), \mathtt{even}(\mathtt{tl}(t))) = t$$

for any stream $t \in A^\omega$. Let $t \in A^\omega$. Then $t = x : y : s$ for some $x, y \in A$ and $s \in A^\omega$. We have

$$\begin{array}{rcl} \mathtt{zip}(\mathtt{even}(t), \mathtt{even}(\mathtt{tl}(t))) & = & \mathtt{zip}(\mathtt{even}(x : y : s), \mathtt{even}(y : s)) \\ & = & \mathtt{zip}(x : \mathtt{even}(s), \mathtt{even}(y : s)) \\ & = & x : \mathtt{zip}(\mathtt{even}(y : s), \mathtt{even}(s)) \\ & = & x : y : s \quad \text{(by CH)} \\ & = & t \end{array}$$

In the equality marked with (by CH) we use the coinductive hypothesis, and implicitly a bisimilarity rule from Definition 3.4.

The above explanation of coinduction is generalised and elaborated in much more detail in [11]. Also [31] may be helpful as it gives many examples of coinductive proofs written in a style similar to the one used here. The book [39] is an elementary introduction to coinduction and bisimulation (but the proofs there are presented in a different way than here, not referring to the coinductive hypothesis but explicitly constructing a backward-closed set). The chapters [7, 9] explain coinduction in Coq from a practical viewpoint. A reader interested in foundational matters should also consult [20, 38] which deal with the coalgebraic approach to coinduction.

In the rest of this paper we shall freely use coinduction, giving routine coinductive proofs in as much (or as little) detail as it is customary with inductive proofs of analogous difficulty.

## 4. Definitions and basic properties

In this section we define infinitary lambda terms and the various notions of infinitary reductions.

**Definition 4.1.** The set of *infinitary lambda terms* is defined coinductively:

$$\Lambda^\infty \quad ::= \quad C \parallel V \parallel \Lambda^\infty \Lambda^\infty \parallel \lambda V.\Lambda^\infty$$

where $V$ is an infinite set of *variables* and $C$ is a set of *constants* such that $V \cap C = \emptyset$. An *atom* is a variable or a constant. We use the symbols $x, y, z, \ldots$ for variables, and $c, c', c_1, \ldots$ for constants, and $a, a', a_1, \ldots$ for atoms, and $t, s, \ldots$ for terms. By $\mathrm{FV}(t)$ we denote the set of variables occurring free in $t$. Formally, $\mathrm{FV}(t)$ could be defined using coinduction.

We define substitution by guarded corecursion.

$$\begin{array}{rcl} x[t/x] & = & t \\ a[t/x] & = & a \quad \text{if } a \neq x \\ (t_1 t_2)[t/x] & = & (t_1[t/x])(t_2[t/x]) \\ (\lambda y.s)[t/x] & = & \lambda y.s[t/x] \quad \text{if } y \notin \mathrm{FV}(t, x) \end{array}$$

In our formalisation we use a de Bruijn representation of infinitary lambda terms, defined analogously to the de Bruijn representation of finite lambda terms [15]. Hence, infinitary lambda terms here may be understood as a human-readable presentation of infinitary lambda terms based on de Bruijn indices. Strictly speaking, also the definition of substitution above is not completely precise, because it implicitly treats lambda terms up to renaming of bound variables and we have not given a precise definition of free variables. The definition of substitution can be understood as a human-readable presentation of substitution defined on infinitary lambda terms based on de Bruijn indices.

Infinitary lambda terms could be precisely defined as the $\alpha$-equivalence classes of the terms given in Definition 4.1, with a coinductively defined $\alpha$-equivalence relation $=_\alpha$. Such a definition involves some technical issues. If the set of variables $V$ is countable, then it may be impossible to choose a "fresh" variable $x \notin \mathrm{FV}(t)$ for a term $t \in \Lambda^\infty$, because $t$ may contain all variables free. This presents a difficulty when trying to precisely define substitution. See also [32, 33]. There are two ways of resolving this situation:

(1) assume that $V$ is uncountable,
(2) consider only terms with finitely many free variables.

Assuming that a fresh variable may always be chosen, one may precisely define substitution and use coinductive techniques to prove: if $t =_\alpha t'$ and $s =_\alpha s'$ then $s[t/x] =_\alpha s'[t'/x]$. This implies that substitution lifts to a function on the $\alpha$-equivalence classes, which is also trivially true for application and abstraction. Therefore, all functions defined by guarded corecursion using only the operations of substitution, application and abstraction lift to functions on $\alpha$-equivalence classes (provided the same defining equation is used for all terms within the same $\alpha$-equivalence class). This justifies the use of Barendregt's variable convention [5, 2.1.13] (under the assumption that we may always choose a fresh variable).

Since our formalisation is based on de Bruijn indices, we omit explicit treatment of $\alpha$-equivalence in this paper.

We also mention that another principled and precise way of dealing with the renaming of bound variables is to define the set of infinitary lambda terms as the final coalgebra of an appropriate functor in the category of nominal sets [32, 33].

**Definition 4.2.** Let $R \subseteq \Lambda^\infty \times \Lambda^\infty$ be a binary relation on infinitary lambda terms. The *compatible closure* of $R$, denoted $\to_R$, is defined inductively by the following rules.

$$\frac{\langle s,t\rangle \in R}{s \to_R t} \qquad \frac{s \to_R s'}{st \to_R s't} \qquad \frac{t \to_R t'}{st \to_R st'} \qquad \frac{s \to_R s'}{\lambda x.s \to_R \lambda x.s'}$$

If $\langle t,s\rangle \in R$ then $t$ is an *$R$-redex*. A term $t \in \Lambda^\infty$ is in *$R$-normal form* if there is no $s \in \Lambda^\infty$ with $t \to_R s$, or equivalently if it contains no $R$-redexes. The *parallel closure* of $R$, usually denoted $\Rightarrow_R$, is defined coinductively by the following rules.

$$\frac{\langle s,t\rangle \in R}{s \Rightarrow_R t} \qquad \frac{}{a \Rightarrow_R a} \qquad \frac{s_1 \Rightarrow_R t_1 \quad s_2 \Rightarrow_R t_2}{s_1 s_2 \Rightarrow_R t_1 t_2} \qquad \frac{s \Rightarrow_R s'}{\lambda x.s \Rightarrow_R \lambda x.s'}$$

Let $\to \subseteq \Lambda^\infty \times \Lambda^\infty$. By $\to^*$ we denote the transitive-reflexive closure of $\to$, and by $\to^\equiv$ the reflexive closure of $\to$. The *infinitary closure* of $\to$, denoted $\to^\infty$, is defined coinductively by the following rules.

$$\frac{s \to^* a}{s \to^\infty a} \qquad \frac{s \to^* t_1 t_2 \quad t_1 \to^\infty t_1' \quad t_2 \to^\infty t_2'}{s \to^\infty t_1' t_2'} \qquad \frac{s \to^* \lambda x.r \quad r \to^\infty r'}{s \to^\infty \lambda x.r'}$$

Let $R_\beta = \{\langle(\lambda x.s)t, s[t/x]\rangle \mid t,s \in \Lambda^\infty\}$. The relation $\to_\beta$ of *one-step $\beta$-reduction* is defined as the compatible closure of $R_\beta$. The relation $\to_\beta^*$ of *$\beta$-reduction* is the transitive-reflexive closure of $\to_\beta$. The relation $\to_\beta^\infty$ of *infinitary $\beta$-reduction* is defined as the infinitary closure of $\to_\beta$. This gives the same coinductive definition of infinitary $\beta$-reduction as in [19].

The relation $\to_w$ of *one-step weak head reduction* is defined inductively by the following rules.

$$\frac{}{(\lambda x.s)t \to_w s[t/x]} \qquad \frac{s \to_w s'}{st \to_w s't}$$

The relations $\to_w^*$, $\to_w^\equiv$ and $\to_w^\infty$ are defined accordingly. In a term $(\lambda x.s)tt_1 \ldots t_m$ the subterm $(\lambda x.s)t$ is the *weak head redex*. So $\to_w$ may contract only the weak head redex.

**Definition 4.3.** Let $\bot$ be a distinguished constant. A $\Lambda^\infty$-term $t$ is in *root normal form* (rnf) if:

- $t \equiv a$ with $a \not\equiv \bot$, or
- $t \equiv \lambda x.t'$, or
- $t \equiv t_1 t_2$ and there is no $s$ with $t_1 \to_\beta^\infty \lambda x.s$ (equivalently, there is no $s$ with $t_1 \to_\beta^* \lambda x.s$).

In other words, a term $t$ is in rnf if $t \not\equiv \bot$ and $t$ does not infinitarily $\beta$-reduce to a $\beta$-redex. We say that $t$ *has rnf* if $t \to_\beta^\infty t'$ for some $t'$ in rnf. In particular, $\bot$ has no rnf. A term with no rnf is also called *root-active*. By $\mathcal{R}$ we denote the set of all root-active terms.

**Definition 4.4.** A set $\mathcal{U} \subseteq \Lambda^\infty$ is a set of *meaningless terms* (see [22]) if it satisfies the following axioms.

- **Closure:** if $t \in \mathcal{U}$ and $t \to_\beta^\infty s$ then $s \in \mathcal{U}$.
- **Substitution:** if $t \in \mathcal{U}$ then $t[s/x] \in \mathcal{U}$ for any term $s$.
- **Overlap:** if $\lambda x.s \in \mathcal{U}$ then $(\lambda x.s)t \in \mathcal{U}$.
- **Root-activeness:** $\mathcal{R} \subseteq \mathcal{U}$.
- **Indiscernibility:** if $t \in \mathcal{U}$ and $t \sim_\mathcal{U} s$ then $s \in \mathcal{U}$, where $\sim_\mathcal{U}$ is the parallel closure of $\mathcal{U} \times \mathcal{U}$.

A set $\mathcal{U}$ of meaningless terms is a set of *strongly meaningless terms* if it additionally satisfies the following expansion axiom.

- **Expansion:** if $t \in \mathcal{U}$ and $s \to_\beta^\infty t$ then $s \in \mathcal{U}$.

Let $\mathcal{U} \subseteq \Lambda^\infty$. Let $R_{\bot_\mathcal{U}} = \{\langle t, \bot \rangle \mid t \in \mathcal{U} \text{ and } t \not\equiv \bot\}$. We define the relation $\to_{\beta\bot_\mathcal{U}}$ of *one-step $\beta\bot_\mathcal{U}$-reduction* as the compatible closure of $R_{\beta\bot_\mathcal{U}} = R_\beta \cup R_{\bot_\mathcal{U}}$. A term $t$ is in *$\beta\bot_\mathcal{U}$-normal form* if it is in $R_{\beta\bot_\mathcal{U}}$-normal form. The relation $\to_{\beta\bot_\mathcal{U}}^*$ of *$\beta\bot_\mathcal{U}$-reduction* is the transitive-reflexive closure of $\to_{\beta\bot_\mathcal{U}}$. The relation $\to_{\beta\bot_\mathcal{U}}^\infty$ of *infinitary $\beta\bot_\mathcal{U}$-reduction*, or *Böhm reduction* (over $\mathcal{U}$), is the infinitary closure of $\to_{\beta\bot_\mathcal{U}}$. The relation $\Rightarrow_{\bot_\mathcal{U}}$ of *parallel $\bot_\mathcal{U}$-reduction* is the parallel closure of $R_{\bot_\mathcal{U}}$.

In general, relations on infinitary terms need to be defined coinductively. However, if the relation depends only on finite initial parts of the terms then it may often be defined inductively. Because induction is generally better understood than coinduction, we prefer to give inductive definitions whenever it is possible to give such a definition in a natural way, like with the definition of compatible closure or one-step weak head reduction. This is in contrast to e.g. the definition of infinitary reduction $\to^\infty$, which intuitively may contain infinitely many reduction steps, and thus must be defined by coinduction.

The idea with the definition of the infinitary closure $\to^\infty$ of a one-step reduction relation $\to$ is that the depth at which a redex is contracted should tend to infinity. This is achieved by defining $\to^\infty$ in such a way that always after finitely many reduction steps the subsequent contractions may be performed only under a constructor. So the depth of the contracted redex always ultimately increases. The idea for the definition of $\to^\infty$ comes from [19, 16, 17]. For infinitary $\beta$-reduction $\to_\beta^\infty$ the definition is the same as in [19]. To each derivation of $t \to^\infty s$ corresponds a strongly convergent reduction sequence of length at most $\omega$ obtained by concatenating the finite $\to^*$-reductions in the prefixes. See the proof of Theorem 6.4.

Our definition of meaningless terms differs from [22] in that it treats terms with the $\bot$ constant, but it is equivalent to the original definition, in the following sense. Let $\Lambda_0^\infty$ be the set of infinitary-lambda terms without $\bot$. If $\mathcal{U}$ is a set of meaningless terms defined as in [22] on $\Lambda_0^\infty$, then $\mathcal{U}_\bot$ (the set of terms from $\mathcal{U}$ with some subterms in $\mathcal{U}$ replaced by $\bot$) is a set of meaningless terms according to our definition. Conversely, if $\mathcal{U}$ is a set of meaningless terms according to our definition, then $\mathcal{U} = \mathcal{U}'_\perp$ where $\mathcal{U}' = \mathcal{U} \cap \Lambda^\infty_0$ ($\mathcal{U}'$ then satisfies the axioms of [22]).

To show confluence of Böhm reduction over $\mathcal{U}$ we also need the expansion axiom. The reason is purely technical. In the present coinductive framework there is no way of talking about infinitary reductions of arbitrary ordinal length, only about reductions of length $\omega$. We need the expansion axiom to show that $t \to^\infty_{\beta\perp_\mathcal{U}} t' \Rightarrow_{\perp_\mathcal{U}} s$ implies $t \to^\infty_{\beta\perp_\mathcal{U}} s$.

The expansion axiom is necessary for this implication. Let $\mathsf{O}$ be the *ogre* [41] satisfying $\mathsf{O} \equiv \lambda x.\mathsf{O}$, i.e., $\mathsf{O} \equiv \lambda x_1.\lambda x_2.\lambda x_3 \ldots$. A term $t$ is *head-active* [41] if $t \equiv \lambda x_1 \ldots x_n.rt_1 \ldots t_m$ with $r \in \mathcal{R}$ and $n, m \geq 0$. Define $\mathcal{H} = \{t \in \Lambda^\infty \mid t \to^*_\beta t' \text{ with } t' \text{ head-active}\}$, $\mathcal{O} = \{t \in \Lambda^\infty \mid t \to^*_\beta \mathsf{O}\}$ and $\mathcal{U} = \mathcal{H} \cup \mathcal{O}$. One can show that $\mathcal{U}$ is a set of meaningless terms (see the appendix). Consider $\Omega_\mathsf{O} = (\lambda xy.xx)(\lambda xy.xx)$. We have $\Omega_\mathsf{O} \to^\infty_\beta \mathsf{O} \in \mathcal{O}$. But $\Omega_\mathsf{O} \notin \mathcal{U}$, so $\mathcal{U}$ does not satisfy the expansion axiom. Now, $\Omega_\mathsf{O} \to^\infty_{\beta\perp_\mathcal{U}} \mathsf{O} \Rightarrow_{\perp_\mathcal{U}} \perp$, but $\Omega_\mathsf{O} \not\to^\infty_{\beta\perp_\mathcal{U}} \perp$, because no finite $\beta$-reduct of $\Omega_\mathsf{O}$ is in $\mathcal{U}$.

The expansion axiom could probably be weakened slightly, but the present formulation is simple and it already appeared in the literature [42, 41, 25]. Sets of meaningless terms which do not satisfy the expansion axiom tend to be artificial. A notion of a set of strongly meaningless terms equivalent to ours appears in [41]. In the presence of the expansion axiom, the indiscernibility axiom may be weakened [42, 41].

In the setup of [16, 17] it is possible to talk about reductions of arbitrary ordinal length, but we have not investigated the possibility of adapting the framework of [16, 17] to the needs of the present paper.

The axioms of a set $\mathcal{U}$ of meaningless terms are sufficient for confluence and normalisation of Böhm reduction over $\mathcal{U}$. However, they are not necessary. The paper [43] gives axioms necessary and sufficient for confluence and normalisation.

The following two simple lemmas will often be used implicitly.

**Lemma 4.5.** *Let $\to^\infty$ be the infinitary and $\to^*$ the transitive-reflexive closure of $\to$. Then the following conditions hold for all $t, s, s' \in \Lambda^\infty$:*
1. *$t \to^\infty t$,*
2. *if $t \to^* s \to^\infty s'$ then $t \to^\infty s'$,*
3. *if $t \to^* s$ then $t \to^\infty s$.*

*Proof.* The first point follows by coinduction. The second point follows by case analysis on $s \to^\infty s'$. The last point follows from the previous two.

The proof of the first point is straightforward, but to illustrate the coinductive technique we give this proof in detail. A reader not familiar with coinduction is invited to study this proof and insert the implicit ordinals as in Section 3.

Let $t \in \Lambda^\infty$. There are three cases. If $t \equiv a$ then $a \to^* a$, so $t \to^\infty t$ by the definition of $\to^\infty$. If $t \equiv t_1 t_2$ then $t_1 \to^\infty t_1$ and $t_2 \to^\infty t_2$ by the coinductive hypothesis. Since also $t \to^* t_1 t_2$, we conclude $t \to^\infty t$. If $t \equiv \lambda x.t'$ then $t' \to^\infty t'$ by the coinductive hypothesis. Since also $t \to^* \lambda x.t'$, we conclude $t \to^\infty t$. ∎

**Lemma 4.6.** *If $R \subseteq S \subseteq \Lambda^\infty \times \Lambda^\infty$ then $\to^\infty_R \subseteq \to^\infty_S$.*

*Proof.* By coinduction. ∎

The next three lemmas have essentially been shown in [19, Lemma 4.3–4.5].

**Lemma 4.7.** *If $s \to^\infty_\beta s'$ and $t \to^\infty_\beta t'$ then $s[t/x] \to^\infty_\beta s'[t'/x]$.*

*Proof.* By coinduction, with case analysis on $s \to_\beta^\infty s'$, using that $t_1 \to_\beta^* t_2$ implies $t_1[t/x] \to_\beta^* t_2[t/x]$. ☐

**Lemma 4.8.** *If $t_1 \to_\beta^\infty t_2 \to_\beta t_3$ then $t_1 \to_\beta^\infty t_3$.*

*Proof.* Induction on $t_2 \to_\beta t_3$, using Lemma 4.7. ☐

**Lemma 4.9.** *If $t_1 \to_\beta^\infty t_2 \to_\beta^\infty t_3$ then $t_1 \to_\beta^\infty t_3$.*

*Proof.* By coinduction, with case analysis on $t_2 \to_\beta^\infty t_3$, using Lemma 4.8. ☐

**Lemma 4.10.** *If $t$ is in rnf and $t \to_\beta^\infty s$ then $s$ is in rnf.*

*Proof.* Suppose $s$ is not in rnf, i.e., $s \equiv \bot$ or $s \equiv s_1 s_2$ with $s_1 \to_\beta^* \lambda x.u$. If $s \equiv \bot$ then $t \to_\beta^* \bot$, and thus either $t \equiv \bot$ or it $\beta$-reduces to a redex. So $t$ is not in rnf. If $s \equiv s_1 s_2$ with $s_1 \to_\beta^\infty \lambda x.u'$, then $t \to_\beta^* t_1 t_2$ with $t_i \to_\beta^\infty s_i$. By Lemma 4.8 we have $t_1 \to_\beta^\infty \lambda x.u$. Thus $t$ reduces to a redex $(\lambda x.u)t_2$. Hence $t$ is not in rnf. ☐

## 5. Confluence and normalisation of Böhm reductions

In this section we use coinductive techniques to prove confluence and normalisation of Böhm reduction over an arbitrary set of strongly meaningless terms $\mathcal{U}$. The infinitary lambda calculus we are concerned with, including the $\bot_\mathcal{U}$-reductions to $\bot$, shall be called the $\lambda^\infty_{\beta\bot_\mathcal{U}}$-calculus.

More precisely, our aim is to prove the following theorems.

**Theorem 5.33** (Confluence of the $\lambda^\infty_{\beta\bot_\mathcal{U}}$-calculus)**.**
*If $t \to^\infty_{\beta\bot_\mathcal{U}} t_1$ and $t \to^\infty_{\beta\bot_\mathcal{U}} t_2$ then there exists $t_3$ such that $t_1 \to^\infty_{\beta\bot_\mathcal{U}} t_3$ and $t_2 \to^\infty_{\beta\bot_\mathcal{U}} t_3$.*
**Theorem 5.34** (Normalisation of the $\lambda^\infty_{\beta\bot_\mathcal{U}}$-calculus)**.**
*For every $t \in \Lambda^\infty$ there exists a unique $s \in \Lambda^\infty$ in $\beta\bot_\mathcal{U}$-normal form such that $t \to^\infty_{\beta\bot_\mathcal{U}} s$.*

In what follows we assume that $\mathcal{U}$ is an arbitrary fixed set of strongly meaningless terms, unless specified otherwise. Actually, almost all lemmas are valid for $\mathcal{U}$ being a set of meaningless terms, without the expansion axiom. Unless explicitly mentioned before the statement of a lemma, the proofs do not use the expansion axiom. To show confluence modulo $\sim_\mathcal{U}$ (Theorem 5.49), it suffices that $\mathcal{U}$ is a set of meaningless terms. Confluence and normalisation of the $\lambda_{\beta\bot_\mathcal{U}}$-calculus (Theorem 5.33 and Theorem 5.34), however, require the expansion axiom. But this is only because in the present coinductive framework we are not able to talk about infinite reductions of arbitrary ordinal length. Essentially, we need the expansion axiom to compress the Böhm reductions to length $\omega$.

The idea of the proof is to show that for every term there exists a certain standard infinitary $\beta\bot_\mathcal{U}$-reduction to normal form. This reduction is called an infinitary $N_\mathcal{U}$-reduction (Definition 5.26 and Lemma 5.28). We show that the normal forms obtained through infinitary $N_\mathcal{U}$-reductions are unique (Lemma 5.29). Then we prove that prepending infinitary $\beta\bot_\mathcal{U}$-reduction to an $N_\mathcal{U}$-reduction results in an $N_\mathcal{U}$-reduction (Theorem 5.32). Since an $N_\mathcal{U}$-reduction is an infinitary $\beta\bot_\mathcal{U}$-reduction of a special form (Lemma 5.27), these results immediately imply confluence (Theorem 5.33) and normalisation (Theorem 5.34) of infinitary $\beta\bot_\mathcal{U}$-reduction. Hence, in essence we derive confluence from a strengthened normalisation result. See Figure 1.

$$\begin{tikzcd}
t_1 \arrow[d, squiggly, "N_{\mathcal{U}}"'] & & t \arrow[ll, "\infty"', "\beta\bot_{\mathcal{U}}"] \arrow[rr, "\infty", "\beta\bot_{\mathcal{U}}"'] \arrow[dll, squiggly, "N_{\mathcal{U}}"] \arrow[drr, squiggly, "N_{\mathcal{U}}"'] & & t_2 \arrow[d, squiggly, "N_{\mathcal{U}}"] \\
t_1' & & \equiv & & t_2'
\end{tikzcd}$$

Figure 1: Confluence of infinitary Böhm reduction.

In our proof we use a standardisation result for infinitary $\beta$-reductions from [19] (Theorem 5.20). In particular, this theorem is needed to show uniqueness of canonical root normal forms (Definition 5.24). Theorem 5.32 depends on this. Even when counting in the results of [19] only referenced here, our confluence proof may be considered simpler than previous proofs of related results. In particular, it is much easier for formalise.

We also show that the set of root-active terms is strongly meaningless. Together with the previous theorems this implies confluence and normalisation of the $\lambda^{\infty}_{\beta\bot_{\mathcal{R}}}$-calculus. Confluence of the $\lambda^{\infty}_{\beta\bot_{\mathcal{R}}}$-calculus in turn implies confluence of $\to^{\infty}_{\beta}$ modulo equivalence of meaningless terms. The following theorem does not require the expansion axiom.

**Theorem 5.49** (Confluence modulo equivalence of meaningless terms)**.**
*If $t \sim_{\mathcal{U}} t'$, $t \to^{\infty}_{\beta} s$ and $t' \to^{\infty}_{\beta} s'$ then there exist $r, r'$ such that $r \sim_{\mathcal{U}} r'$, $s \to^{\infty}_{\beta} r$ and $s' \to^{\infty}_{\beta} r'$.*

Note that our overall proof strategy is different from [10, 22, 24]. We first derive a strengthened normalisation result for Böhm reduction, from this we derive confluence of Böhm reduction, then we show that root-active terms are strongly meaningless thus specialising the confluence result, and only using that we show confluence modulo equivalence of meaningless terms. In [10, 22, 24] first confluence modulo equivalence of meaningless terms is shown, and from that confluence of Böhm reduction is derived. Of course, some intermediate lemmas we prove have analogons in [10, 22, 24], but we believe the general proof strategy to be fundamentally different.

5.1. **Properties of $\sim_{\mathcal{U}}$.** In this subsection $\mathcal{U}$ is an arbitrary fixed set of meaningless terms, and $\sim_{\mathcal{U}}$ is the parallel closure of $\mathcal{U} \times \mathcal{U}$. The expansion axiom is not used in this subsection.

**Lemma 5.1.** *If $t \sim_{\mathcal{U}} t'$ and $s \sim_{\mathcal{U}} s'$ then $t[s/x] \sim_{\mathcal{U}} t'[s'/x]$.*

*Proof.* By coinduction, using the substitution axiom. □

**Lemma 5.2.** *If $t \to_{\beta} s$ and $t \sim_{\mathcal{U}} t'$ then there is $s'$ with $t' \to^{\equiv}_{\beta} s'$ and $s \sim_{\mathcal{U}} s'$.*

*Proof.* Induction on $t \to_{\beta} s$. If the case $t, t' \in \mathcal{U}$ in the definition of $t \sim_{\mathcal{U}} t'$ holds then $s \in \mathcal{U}$ by the closure axiom, so $t' \sim_{\mathcal{U}} s$ and we may take $s' \equiv t'$. Thus assume otherwise. Then all cases follow directly from the inductive hypothesis, except when $t$ is the contracted $\beta$-redex. Then $t \equiv (\lambda x.t_1)t_2$ and $s \equiv t_1[t_2/x]$. First assume $t \in \mathcal{U}$. Then also $t' \in \mathcal{U}$ by the indiscernibility axiom (note this does not imply that the first case in the definition of $t \sim_{\mathcal{U}} t'$ holds). Also $s \in \mathcal{U}$ by the closure axiom, so $t' \sim_{\mathcal{U}} s$ and we may take $s' \equiv t'$. So assume $t \notin \mathcal{U}$. Then $\lambda x.t_1 \notin \mathcal{U}$ by the overlap axiom. Hence $t' \equiv (\lambda x.t_1')t_2'$ with $t_i \sim_{\mathcal{U}} t_i'$. Thus $t_1[t_2/x] \sim_{\mathcal{U}} t_1'[t_2'/x]$ by Lemma 5.1. So we may take $s' \equiv t_1'[t_2'/x]$. □

**Lemma 5.3.** *If $t \to^{\infty}_{\beta} s$ and $t \sim_{\mathcal{U}} t'$ then there is $s'$ with $t' \to^{\infty}_{\beta} s'$ and $s \sim_{\mathcal{U}} s'$.*

*Proof.* By coinduction. If $s \equiv a$ then $t \to_\beta^* s$ and the claim follows from Lemma 5.2. If $s \equiv s_1 s_2$ then $t \to_\beta^* t_1 t_2$ with $t_i \to_\beta^\infty s_i$. By Lemma 5.2 there is $u$ with $t_1 t_2 \sim_{\mathcal{U}} u$ and $t' \to_\beta^* u$. If $t_1 t_2, u \in \mathcal{U}$ then $s \in \mathcal{U}$ by the closure axiom, and thus we may take $s' \equiv u$. Otherwise $u \equiv u_1 u_2$ with $t_i \sim_{\mathcal{U}} u_i$. By the coinductive hypothesis we obtain $s_1', s_2'$ with $u_i \to_\beta^\infty s_i'$ and $s_i \sim_{\mathcal{U}} s_i'$. Take $s' \equiv s_1' s_2'$. Then $t' \to_\beta^\infty s'$ and $s \sim_{\mathcal{U}} s'$. If $s \equiv \lambda x.s'$ then the argument is analogous to the previous case. □

**Lemma 5.4.** *If $t \sim_{\mathcal{U}} s$ and $s \sim_{\mathcal{U}} u$ then $t \sim_{\mathcal{U}} s$.*

*Proof.* By coinduction, using the indiscernibility axiom. □

**Lemma 5.5.** *If $t \sim_{\mathcal{U}} s$ then there is $r$ with $t \Rightarrow_{\perp_{\mathcal{U}}} r$ and $s \Rightarrow_{\perp_{\mathcal{U}}} r$.*

*Proof.* By coinduction. □

5.2. **Properties of parallel $\perp_{\mathcal{U}}$-reduction.** Recall that $\mathcal{U}$ is an arbitrary fixed set of strongly meaningless terms. The expansion axiom is not used in this subsection except for Corollary 5.16, Lemma 5.17, Corollary 5.18 and Lemma 5.19.

**Lemma 5.6.** *If $s \Rightarrow_{\perp_{\mathcal{U}}} s'$ and $t \Rightarrow_{\perp_{\mathcal{U}}} t'$ then $s[t/x] \Rightarrow_{\perp_{\mathcal{U}}} s'[t'/x]$.*

*Proof.* Coinduction with case analysis on $s \Rightarrow_{\perp_{\mathcal{U}}} s'$, using the substitution axiom. □

**Lemma 5.7.** *If $t \Rightarrow_{\perp_{\mathcal{U}}} s$ then $t \to_{\beta\perp_{\mathcal{U}}}^\infty s$.*

*Proof.* By coinduction. □

**Lemma 5.8.** *If $t \in \mathcal{U}$ and $t \Rightarrow_{\perp_{\mathcal{U}}} s$ or $s \Rightarrow_{\perp_{\mathcal{U}}} t$ then $s \in \mathcal{U}$.*

*Proof.* Using the root-activeness axiom and that $\perp$ is root-active, show by coinduction that $t \sim_{\mathcal{U}} s$. Then use the indiscernibility axiom. □

**Lemma 5.9.** *If $t_1 \Rightarrow_{\perp_{\mathcal{U}}} t_2 \Rightarrow_{\perp_{\mathcal{U}}} t_3$ then $t_1 \Rightarrow_{\perp_{\mathcal{U}}} t_3$.*

*Proof.* Coinduction with case analysis on $t_2 \Rightarrow_{\perp_{\mathcal{U}}} t_3$, using Lemma 5.8. □

**Lemma 5.10.** *If $t_1 \Rightarrow_{\perp_{\mathcal{U}}} t_2 \to_\beta t_3$ then there exists $t_1'$ such that $t_1 \to_\beta t_1' \Rightarrow_{\perp_{\mathcal{U}}} t_3$.*

*Proof.* Induction on $t_2 \to_\beta t_3$. The only interesting case is when $t_2 \equiv (\lambda x.s_1)s_2$ and $t_3 \equiv s_1[s_2/x]$. Then $t_1 \equiv (\lambda x.u_1)u_2$ with $u_i \Rightarrow_{\perp_{\mathcal{U}}} s_i$. By Lemma 5.6, $u_1[u_2/x] \Rightarrow_{\perp_{\mathcal{U}}} s_1[s_2/x]$. Thus take $t_1' \equiv u_1[u_2/x]$. □

**Lemma 5.11.** *If $s \to_{\beta\perp_{\mathcal{U}}}^* t$ then there exists $r$ such that $s \to_\beta^* r \Rightarrow_{\perp_{\mathcal{U}}} t$.*

*Proof.* Induction on the length of $s \to_{\beta\perp_{\mathcal{U}}}^* t$, using Lemma 5.10 and Lemma 5.9. □

**Corollary 5.12.** *If $t_1 \Rightarrow_{\perp_{\mathcal{U}}} t_2 \to_{\beta\perp_{\mathcal{U}}}^* t_3$ then there is $s$ with $t_1 \to_\beta^* s \Rightarrow_{\perp_{\mathcal{U}}} t_3$.*

*Proof.* Follows from Lemmas 5.11, 5.10, 5.9. □

**Lemma 5.13.** *If $t_1 \Rightarrow_{\perp_{\mathcal{U}}} t_2 \to_{\beta\perp_{\mathcal{U}}}^\infty t_3$ then $t_1 \to_{\beta\perp_{\mathcal{U}}}^\infty t_3$.*

*Proof.* By coinduction. There are three cases.

- $t_3 \equiv a$. Then $t_1 \Rightarrow_{\perp_{\mathcal{U}}} t_2 \to_{\beta\perp_{\mathcal{U}}}^* a$. By Corollary 5.12 there is $s$ with $t_1 \to_\beta^* s \Rightarrow_{\perp_{\mathcal{U}}} a$. By Lemma 5.7 we have $s \to_{\beta\perp_{\mathcal{U}}}^\infty a$. Thus $t_1 \to_{\beta\perp_{\mathcal{U}}}^\infty a$.

- $t_3 \equiv s_1 s_2$. Then $t_1 \Rightarrow_{\bot_\mathcal{U}} t_2 \to^*_{\beta\bot_\mathcal{U}} s_1' s_2'$ with $s_i' \to^\infty_{\beta\bot_\mathcal{U}} s_i$. By Corollary 5.12 there is $u$ with $t_1 \to^*_\beta u \Rightarrow_{\bot_\mathcal{U}} s_1' s_2'$. Then $u \equiv u_1 u_2$ with $u_i \Rightarrow_{\bot_\mathcal{U}} s_i' \to^\infty_{\beta\bot_\mathcal{U}} s_i$. By the coinductive hypothesis $u_i \to^\infty_{\beta\bot_\mathcal{U}} s_i$. Thus $t_1 \to^\infty_{\beta\bot_\mathcal{U}} s_1 s_2 \equiv t_3$.
- $t_3 \equiv \lambda x.r$. The argument is analogous to the previous case. □

The following lemma is an analogon of [22, Lemma 12.9.22].

**Lemma 5.14** (Postponement of parallel $\bot_\mathcal{U}$-reduction)**.** *If $t \to^\infty_{\beta\bot_\mathcal{U}} s$ then there exists $r$ such that $t \to^\infty_\beta r \Rightarrow_{\bot_\mathcal{U}} s$.*

*Proof.* By coinduction with case analysis on $t \to^\infty_{\beta\bot_\mathcal{U}} s$, using Lemmas 5.11, 5.13.

Since this is the first of our coinductive proofs involving an implicit Skolem function (see Example 3.3), we give it in detail. The reader is invited to extract from this proof an explicit corecursive definition of the Skolem function.

Assume $t \to^\infty_{\beta\bot_\mathcal{U}} s$. There are three cases.

- $s \equiv a$ and $t \to^*_{\beta\bot_\mathcal{U}} a$. Then the claim follows from Lemma 5.11.
- $s \equiv s_1 s_2$ and $t \to^*_{\beta\bot_\mathcal{U}} t_1 t_2$ and $t_i \to^\infty_{\beta\bot_\mathcal{U}} s_i$. By Lemma 5.11 there is $t'$ with $t \to^*_\beta t' \Rightarrow_{\bot_\mathcal{U}} t_1 t_2$. Because $t_1 t_2 \not\equiv \bot$, we must have $t' \equiv t_1' t_2'$ with $t_i' \Rightarrow_{\bot_\mathcal{U}} t_i$. By Lemma 5.13 we have $t_i' \to^\infty_{\beta\bot_\mathcal{U}} s_i$. By the coinductive hypothesis we obtain $s_1', s_2'$ such that $t_i' \to^\infty_\beta s_i' \Rightarrow_{\bot_\mathcal{U}} s_i$. Hence $t \to^\infty_\beta s_1' s_2' \Rightarrow_{\bot_\mathcal{U}} s_1 s_2 \equiv s$.
- $s \equiv \lambda x.s'$ and $t \to^*_{\beta\bot_\mathcal{U}} \lambda x.t'$ and $t' \to^\infty_{\beta\bot_\mathcal{U}} s'$. By Lemma 5.11 there is $u$ with $t \to^*_\beta u \Rightarrow_{\bot_\mathcal{U}} \lambda x.t'$. Then $u \equiv \lambda x.u'$ with $u' \Rightarrow_{\bot_\mathcal{U}} t'$. By Lemma 5.13 we have $u' \to^\infty_{\beta\bot_\mathcal{U}} s'$. By the coinductive hypothesis we obtain $w$ such that $u' \to^\infty_\beta w \Rightarrow_{\bot_\mathcal{U}} s'$. Hence $t \to^\infty_\beta \lambda x.w \Rightarrow_{\bot_\mathcal{U}} \lambda x.s' \equiv s$. □

**Corollary 5.15.** *If $t \in \mathcal{U}$ and $t \to^\infty_{\beta\bot_\mathcal{U}} s$ then $s \in \mathcal{U}$.*

*Proof.* Follows from Lemma 5.14, the closure axiom and Lemma 5.8. □

The following depend on the expansion axiom.

**Corollary 5.16.** *If $s \in \mathcal{U}$ and $t \to^\infty_{\beta\bot_\mathcal{U}} s$ then $t \in \mathcal{U}$.*

*Proof.* Follows from Lemma 5.14, Lemma 5.8 and the expansion axiom. □

**Lemma 5.17.** *If $t \to^\infty_{\beta\bot_\mathcal{U}} t' \Rightarrow_{\bot_\mathcal{U}} s$ then $t \to^\infty_{\beta\bot_\mathcal{U}} s$.*

*Proof.* By coinduction, analysing $t' \Rightarrow_{\bot_\mathcal{U}} s$. All cases follow directly from the coinductive hypothesis, except when $s \equiv \bot$ and $t' \in \mathcal{U}$. But then $t \in \mathcal{U}$ by Corollary 5.16, so $t \Rightarrow_{\bot_\mathcal{U}} s$, and thus $t \to^\infty_{\beta\bot_\mathcal{U}} s$ by Lemma 5.7. □

**Corollary 5.18.** *If $t \to^\infty_{\beta\bot_\mathcal{U}} s \to^*_\beta r$ then $t \to^\infty_{\beta\bot_\mathcal{U}} r$.*

*Proof.* By Lemma 5.14 we have $t \to^\infty_\beta t' \Rightarrow_{\bot_\mathcal{U}} s \to^*_\beta r$. By Lemma 5.10 there is $s'$ with $t' \to^*_\beta s' \Rightarrow_{\bot_\mathcal{U}} r$. By Lemma 4.8 we have $t \to^\infty_\beta s'$, and thus $t \to^\infty_{\beta\bot_\mathcal{U}} s'$. By Lemma 5.17 we finally obtain $t \to^\infty_{\beta\bot_\mathcal{U}} r$. □

**Lemma 5.19.** *If $t \notin U$ and $t \Rightarrow_{\bot_\mathcal{U}} s$ and $s$ is in rnf, then $t$ is in rnf.*

*Proof.* We consider possible forms of $s$.

- $s \equiv a$ with $a \not\equiv \bot$. Then $t \equiv a$ and $t$ is in rnf.
- $s \equiv \lambda x.s'$. Then $t \equiv \lambda x.t'$ with $t' \Rightarrow_{\bot_\mathcal{U}} s'$, so $t$ is in rnf.

- $s \equiv s_1 s_2$ and there is no $r$ with $s_1 \to_\beta^\infty \lambda x.r$. Then $t \equiv t_1 t_2$ with $t_i \Rightarrow_{\bot_\mathcal{U}} s_i$. Then also $t_1 \sim_\mathcal{U} s_1$. Suppose $t_1 \to_\beta^\infty \lambda x.r$. By Lemma 5.3 there is $r'$ with $s_1 \to_\beta^\infty r' \sim_\mathcal{U} \lambda x.r$. There are two cases.
  - $r', \lambda x.r \in \mathcal{U}$. Then $(\lambda x.r)t_2 \in \mathcal{U}$ by the overlap axiom, and thus $t \in \mathcal{U}$ by the expansion axiom. Contradiction.
  - $r' \equiv \lambda x.r''$ with $r \sim_\mathcal{U} r''$. But then $s_1 \to_\beta^\infty \lambda x.r''$. Contradiction. □

### 5.3. **Weak head reduction.**

**Theorem 5.20** (Endrullis, Polonsky [19])**.** *$t \to_\beta^\infty s$ iff $t \to_w^\infty s$.*

Strictly speaking, in [19] the above theorem is shown for a different set of infinitary lambda terms which do not contain constants. However, it is clear that for the purposes of [19] constants may be treated as variables not occuring bound, and thus the proof of the above theorem may be used in our setting. We omit the proof of this theorem here, but we included the proof in our formalisation.

**Lemma 5.21.** *If $t \to_w^\infty t_1$ and $t \to_w t_2$ then there is $t_3$ with $t_2 \to_w^\infty t_3$ and $t_1 \to_w^\equiv t_3$.*

*Proof.* If the weak head redex in $t$ is contracted in $t \to_w^\infty t_1$ then $t \to_w t_2 \to_w^\infty t_1$ and we may take $t_3 \equiv t_1$. Otherwise $t \equiv (\lambda x.s)uu_1 \dots u_m$, $t_2 \equiv s[u/x]u_1 \dots u_m$ and $t_1 \equiv (\lambda x.s')u'u'_1 \dots u'_m$ with $s \to_w^\infty s'$, $u \to_w^\infty u'$ and $u_i \to_w^\infty u'_i$ for $i = 1, \dots, m$. By Theorem 5.20 and Lemma 4.7 we obtain $s[u/x] \to_w^\infty s'[u'/x]$. Take $t_3 \equiv s'[u'/x]u'_1 \dots u'_m$. Then $t_2 \to_w^\infty t_3$ and $t_1 \to_w t_3$. □

**Lemma 5.22.** *If $t \to_\beta^\infty s$ with $s$ in rnf, then there is $s'$ in rnf with $t \to_w^* s' \to_w^\infty s$.*

*Proof.* By Theorem 5.20 we have $t \to_w^\infty s$. Because $s$ is in rnf, there are three cases.

- $s \equiv a$ with $a \not\equiv \bot$. Then $t \to_w^* s$ and we may take $s' \equiv s$.
- $s \equiv \lambda x.s_0$. Then $t \to_w^* \lambda x.t_0$ with $t_0 \to_w^\infty s_0$. So take $s' \equiv \lambda x.t_0$.
- $s \equiv s_1 s_2$ and there is no $r$ with $s_1 \to_\beta^\infty \lambda x.r$. Then $t \to_w^* t_1 t_2$ with $t_i \to_w^\infty s_i$. Suppose $t_1 \to_\beta^\infty \lambda x.u$. Then $t_1 \to_w^* \lambda x.u'$ for some $u'$, by Theorem 5.20. By Lemma 5.21 there is $r$ with $\lambda x.u' \to_w^\infty r$ and $s_1 \to_w^* r$. But then $r \equiv \lambda x.r'$, so $s_1$ reduces to an abstraction. Contradiction. Hence $t_1 t_2$ is in rnf, so we may take $s' \equiv t_1 t_2$. □

**Lemma 5.23.** *If $t \to_w^* s_1$, $t \to_w^* s_2$ and these reductions have the same length, then $s_1 \equiv s_2$.*

*Proof.* By induction on the length of the reduction, using the fact that weak head redexes are unique if they exist. □

**Definition 5.24.** The *canonical root normal form* (crnf) of a term $t$ is an rnf $s$ such that $t \to_w^* s$ and this reduction is shortest among all finitary weak head reductions of $t$ to root normal form.

It follows from Lemma 5.22 and Lemma 5.23 that if $t$ has a rnf then it has a unique crnf. We shall denote this crnf by $\mathrm{crnf}(t)$.

**Lemma 5.25.** *If $t \to_\beta^\infty s$ with $s$ in rnf, then $t \to_w^* \mathrm{crnf}(t) \to_w^\infty s$.*

*Proof.* Follows from Lemma 5.22 and Lemma 5.23. □

5.4. **Infinitary $N_{\mathcal{U}}$-reduction.** In the $\lambda^{\infty}_{\beta\bot_{\mathcal{U}}}$-calculus every term has a unique normal form. This normal form may be obtained through an infinitary $N_{\mathcal{U}}$-reduction, defined below.

**Definition 5.26.** The relation $\rightsquigarrow_{N_{\mathcal{U}}}$ is defined coinductively.

$$\frac{t \notin \mathcal{U} \quad \mathrm{crnf}(t) \equiv a}{t \rightsquigarrow_{N_{\mathcal{U}}} a} \qquad \frac{t \notin \mathcal{U} \quad \mathrm{crnf}(t) \equiv t_1 t_2 \quad t_1 \rightsquigarrow_{N_{\mathcal{U}}} s_1 \quad t_2 \rightsquigarrow_{N_{\mathcal{U}}} s_2}{t \rightsquigarrow_{N_{\mathcal{U}}} s_1 s_2}$$

$$\frac{t \notin \mathcal{U} \quad \mathrm{crnf}(t) \equiv \lambda x.t' \quad t' \rightsquigarrow_{N_{\mathcal{U}}} s}{t \rightsquigarrow_{N_{\mathcal{U}}} \lambda x.s} \qquad \frac{t \in \mathcal{U}}{t \rightsquigarrow_{N_{\mathcal{U}}} \bot}$$

Note that because $\mathcal{R} \subseteq \mathcal{U}$, every term $t \notin \mathcal{U}$ has a rnf, so $\mathrm{crnf}(t)$ is defined for $t \notin \mathcal{U}$. Also note that $\rightsquigarrow_{N_{\mathcal{U}}}$ is not closed under contexts, e.g., $t \rightsquigarrow_{N_{\mathcal{U}}} t'$ does *not* imply $ts \rightsquigarrow_{N_{\mathcal{U}}} t's$.

The infinitary $N_{\mathcal{U}}$-reduction $\rightsquigarrow_{N_{\mathcal{U}}}$ reduces a term to its normal form — its Böhm-like tree. It is a "standard" reduction with a specifically regular structure, which allows us to prove Theorem 5.32: if $t \to^{\infty}_{\beta\bot_{\mathcal{U}}} t' \rightsquigarrow_{N_{\mathcal{U}}} s$ then $t \rightsquigarrow_{N_{\mathcal{U}}} s$. This property allows us to derive confluence from the fact that every term has a unique normal form reachable by an infinitary $N_{\mathcal{U}}$-reduction. See Figure 1. It is crucial here that canonical root normal forms are unique, and that Lemma 5.25 holds. This depends on Theorem 5.20 — the standardisation result shown by Endrullis and Polonsky.

**Lemma 5.27.** *If $t \rightsquigarrow_{N_{\mathcal{U}}} s$ then $t \to^{\infty}_{\beta\bot_{\mathcal{U}}} s$.*

*Proof.* By coinduction. □

**Lemma 5.28.** *For every term $t \in \Lambda^{\infty}$ there is $s$ with $t \rightsquigarrow_{N_{\mathcal{U}}} s$.*

*Proof.* By coinduction. If $t \in \mathcal{U}$ then $t \rightsquigarrow_{N_{\mathcal{U}}} \bot$ and we may take $s \equiv \bot$. Otherwise there are three cases depending on the form of $\mathrm{crnf}(t)$.

- $\mathrm{crnf}(t) \equiv a$. Then $t \rightsquigarrow_{N_{\mathcal{U}}} a$ by the first rule, so we may take $s \equiv a$.
- $\mathrm{crnf}(t) \equiv t_1 t_2$. Then by the coinductive hypothesis we obtain $s_1, s_2$ with $t_i \rightsquigarrow_{N_{\mathcal{U}}} s_i$. Take $s \equiv s_1 s_2$. Then $t \rightsquigarrow_{N_{\mathcal{U}}} s$.
- $\mathrm{crnf}(t) \equiv \lambda x.t'$. Analogous to the previous case. □

**Lemma 5.29.** *If $t \rightsquigarrow_{N_{\mathcal{U}}} s_1$ and $t \rightsquigarrow_{N_{\mathcal{U}}} s_2$ then $s_1 \equiv s_2$.*

*Proof.* By coinduction. If $s_1 \equiv \bot$ then $t \in \mathcal{U}$, so we must also have $s_2 \equiv \bot$. Otherwise there are three cases, depending on the form of $\mathrm{crnf}(t)$. Suppose $\mathrm{crnf}(t) \equiv t_1 t_2$, other cases being similar. Then $s_1 \equiv u_1 u_2$ with $t_i \rightsquigarrow_{N_{\mathcal{U}}} u_i$ and $s_2 \equiv w_1 w_2$ with $t_i \rightsquigarrow_{N_{\mathcal{U}}} w_i$. By the coinductive hypothesis $u_i \equiv w_i$. Thus $s_1 \equiv u_1 u_2 \equiv w_1 w_2 \equiv s_2$. □

The next two lemmas and the theorem depend on the expansion axiom.

**Lemma 5.30.** *If $t \rightsquigarrow_{N_{\mathcal{U}}} s$ then $s$ is in $\beta\bot_{\mathcal{U}}$-normal form.*

*Proof.* Suppose $s$ contains a $\beta\bot_{\mathcal{U}}$-redex. Without loss of generality, assume the redex is at the root. First assume that $s$ is a $\bot_{\mathcal{U}}$-redex, i.e., $s \in \mathcal{U}$ and $s \not\equiv \bot$. Using Lemma 5.27 we conclude $t \to^{\infty}_{\beta\bot_{\mathcal{U}}} s$. Then $t \in \mathcal{U}$ by Corollary 5.16. Thus $s \equiv \bot$, giving a contradiction. So assume $s$ is a $\beta$-redex, i.e., $s \equiv (\lambda x.s_1)s_2$. But by inspecting the definition of $t \rightsquigarrow_{N_{\mathcal{U}}} s$ one sees that this is only possible when $\mathrm{crnf}(t)$ is a $\beta$-redex. Contradiction. □

**Lemma 5.31.** *Suppose $t \to^{\infty}_{\beta\bot_{\mathcal{U}}} s$ and $t, s$ are in rnf.*

- *If $s \equiv a$ then $t \equiv s$.*

- *If* $s \equiv s_1 s_2$ *then* $t \equiv t_1 t_2$ *with* $t_i \to^\infty_{\beta\bot_\mathcal{U}} s_i$.
- *If* $s \equiv \lambda x.s'$ *then* $t \equiv \lambda x.t'$ *with* $t' \to^\infty_{\beta\bot_\mathcal{U}} s'$.

*Proof.* First note that by Lemma 5.14 there is $r$ with $t \to^\infty_\beta r \Rightarrow_{\bot_\mathcal{U}} s$.

- If $s \equiv a$ then $a \not\equiv \bot$ and $r \equiv a$, and thus $t \to^*_\beta a$. But because $t$ is in rnf it does not reduce to a $\beta$-redex, so in fact $t \equiv a$.
- If $s \equiv s_1 s_2$ then $r \equiv r_1 r_2$ with $r_i \Rightarrow_{\bot_\mathcal{U}} s_i$. Thus $t \to^*_\beta t'_1 t'_2$ where $t'_i \to^\infty_\beta r_i$. Because $t$ is in rnf, we must in fact have $t \equiv t_1 t_2$ with $t_i \to^*_\beta t'_i$. Then $t_i \to^\infty_\beta r_i \Rightarrow_{\bot_\mathcal{U}} s_i$, so $t_i \to^\infty_{\beta\bot_\mathcal{U}} s_i$ by Lemma 5.17.
- The case $s \equiv \lambda x.s'$ is analogous to the previous one. ☐

**Theorem 5.32.** *If* $t \to^\infty_{\beta\bot_\mathcal{U}} t' \rightsquigarrow_{N_\mathcal{U}} s$ *then* $t \rightsquigarrow_{N_\mathcal{U}} s$.

*Proof.* By coinduction. If $s \equiv \bot$ then $t' \in \mathcal{U}$. By Corollary 5.16 also $t \in \mathcal{U}$. Hence $t \rightsquigarrow_{N_\mathcal{U}} \bot \equiv s$. If $s \not\equiv \bot$ then $t' \notin \mathcal{U}$ and $t' \to^*_w \mathrm{crnf}(t')$. By Corollary 5.18 we have $t \to^\infty_{\beta\bot_\mathcal{U}} \mathrm{crnf}(t')$. By Lemma 5.14 there is $r$ with $t \to^\infty_\beta r \Rightarrow_{\bot_\mathcal{U}} \mathrm{crnf}(t')$. We have $t \notin \mathcal{U}$ by Corollary 5.15. Then $r$ is in rnf by Lemma 5.19. Hence $t \to^*_w \mathrm{crnf}(t) \to^\infty_{\beta\bot_\mathcal{U}} \mathrm{crnf}(t')$ by Lemma 5.25 and Lemma 5.17. Now there are three cases depending on the form of $\mathrm{crnf}(t')$.

- $\mathrm{crnf}(t') \equiv a$. Then $s \equiv a$, and $\mathrm{crnf}(t) \equiv a$ by Lemma 5.31. Thus $t \rightsquigarrow_{N_\mathcal{U}} a \equiv s$.
- $\mathrm{crnf}(t') \equiv t'_1 t'_2$. Then $s \equiv s_1 s_2$ with $t'_i \rightsquigarrow_{N_\mathcal{U}} s_i$. By Lemma 5.31 we have $\mathrm{crnf}(t) \equiv t_1 t_2$ with $t_i \to^\infty_{\beta\bot_\mathcal{U}} t'_i$. By the coinductive hypothesis $t_i \rightsquigarrow_{N_\mathcal{U}} s_i$. Hence $t \rightsquigarrow_{N_\mathcal{U}} s_1 s_2 \equiv s$.
- The case $\mathrm{crnf}(t') \equiv \lambda x.u$ is analogous to the previous one. ☐

### 5.5. **Confluence and normalisation.** 

Recall that $\mathcal{U}$ is an arbitrary fixed set of strongly meaningless terms.

**Theorem 5.33** (Confluence of the $\lambda^\infty_{\beta\bot_\mathcal{U}}$-calculus)**.**
*If* $t \to^\infty_{\beta\bot_\mathcal{U}} t_1$ *and* $t \to^\infty_{\beta\bot_\mathcal{U}} t_2$ *then there exists* $t_3$ *such that* $t_1 \to^\infty_{\beta\bot_\mathcal{U}} t_3$ *and* $t_2 \to^\infty_{\beta\bot_\mathcal{U}} t_3$.

*Proof.* By Lemma 5.28 there are $t'_1, t'_2$ with $t_i \rightsquigarrow_{N_\mathcal{U}} t'_i$ for $i = 1, 2$. By Theorem 5.32 we have $t \rightsquigarrow_{N_\mathcal{U}} t'_i$ for $i = 1, 2$. By Lemma 5.29 we have $t'_1 \equiv t'_2$. Take $t_3 \equiv t'_1 \equiv t'_2$. We have $t_i \rightsquigarrow_{N_\mathcal{U}} t_3$ for $i = 1, 2$, so $t_1 \to^\infty_{\beta\bot_\mathcal{U}} t_3$ and $t_2 \to^\infty_{\beta\bot_\mathcal{U}} t_3$ by Lemma 5.27. ☐

**Theorem 5.34** (Normalisation of the $\lambda^\infty_{\beta\bot_\mathcal{U}}$-calculus)**.**
*For every* $t \in \Lambda^\infty$ *there exists a unique* $s \in \Lambda^\infty$ *in* $\beta\bot_\mathcal{U}$*-normal form such that* $t \to^\infty_{\beta\bot_\mathcal{U}} s$.

*Proof.* By Lemma 5.28 there is $s$ with $t \rightsquigarrow_{N_\mathcal{U}} s$. By Lemma 5.30, $s$ is in $\beta\bot_\mathcal{U}$-normal form. By Lemma 5.27 we have $t \to^\infty_{\beta\bot_\mathcal{U}} s$. The uniqueness of $s$ follows from Theorem 5.33. ☐

### 5.6. **Root-active terms are strongly meaningless.**

**Definition 5.35.** We define the relation $\succ_x$ coinductively

$$\frac{u_1, \ldots, u_n \in \Lambda^\infty}{t \succ_x x u_1 \ldots u_n} \qquad \frac{}{a \succ_x a} \qquad \frac{t \succ_x t' \quad s \succ_x s'}{ts \succ_x t's'} \qquad \frac{t \succ_x t' \quad x \neq y}{\lambda y.t \succ_x \lambda y.t'}$$

In other words, $s \succ_x s'$ iff $s'$ may be obtained from $s$ by changing some arbitrary subterms in $s$ into some terms having the form $x u_1 \ldots u_n$.

**Lemma 5.36.** *If* $t \succ_x t'$, $s \succ_x s'$ *and* $x \neq y$ *then* $t[s/y] \succ_x t'[s'/y]$.

*Proof.* By coinduction, analysing $t \succ_x t'$. □

**Lemma 5.37.** *If $t \succ_x s$ and $t \to_\beta t'$ then there is $s'$ with $t' \succ_x s'$ and $s \to_\beta^\equiv s'$.*

*Proof.* Induction on $t \to_\beta t'$. The interesting case is when $t \equiv (\lambda y.t_1)t_2$, $t' \equiv t_1[t_2/y]$, $s \equiv s_1 s_2$, $\lambda y.t_1 \succ_x s_1$ and $t_2 \succ_x s_2$. If $s_1 \equiv x u_1 \dots u_m$ then $t' \succ_x x u_1 \dots u_m s_2$ and we may take $s' \equiv s$. Otherwise $s_1 \equiv \lambda y.s_1'$ with $t_1 \succ_x s_1'$ (by the variable convention $x \neq y$). Then $t' \equiv t_1[t_2/y] \succ_x s_1'[s_2/y]$ by Lemma 5.36. We may thus take $s' \equiv s_1'[s_2/y]$. □

**Lemma 5.38.** *If $t \succ_x s$ and $s \to_\beta s'$ then there is $t'$ with $t' \succ_x s'$ and $t \to_\beta^\equiv t'$.*

*Proof.* Induction on $s \to_\beta s'$, using Lemma 5.36 for the redex case. □

**Lemma 5.39.** *If $t \succ_x t'$ and $t$ is in rnf, then so is $t'$.*

*Proof.* Assume $t'$ is not in rnf. Then $t' \equiv \bot$ or $t' \equiv t_1' t_2'$ with $t_1'$ reducing to an abstraction. If $t' \equiv \bot$ then $t \equiv \bot$, so assume $t' \equiv t_1' t_2'$ and $t_1' \to_\beta^* \lambda y.u'$ with $x \neq y$. Then $t \equiv t_1 t_2$ with $t_i \succ_x t_i'$. By Lemma 5.38 there is $u$ with $t_1 \to_\beta^* \lambda y.u$ and $u \succ_x u'$. But this implies that $t \equiv t_1 t_2$ is not in rnf. Contradiction. □

**Lemma 5.40.** *If $t_1, t_2 \in \Lambda^\infty$ and $t_1$ has no rnf, then neither does $t_1[t_2/x]$.*

*Proof.* Assume $t_1[t_2/x]$ has a rnf. Then $t_1[t_2/x] \to_\beta^* s$ for some $s$ in rnf, by Lemma 5.22. By the variable convention $t_1[t_2/x] \succ_x t_1$. Hence by Lemma 5.37 there is $s'$ such that $t_1 \to_\beta^* s'$ and $s \succ_x s'$. Since $s$ is in rnf, so is $s'$, by Lemma 5.39. Thus $t_1$ has a rnf. Contradiction. □

**Lemma 5.41.** *If $t \sim_\mathcal{R} t'$ and $s \sim_\mathcal{R} s'$ then $t[s/x] \sim_\mathcal{R} t'[s'/x]$.*

*Proof.* By coinduction, using Lemma 5.40. □

**Lemma 5.42.** *If $t \to_\beta t'$ and $t \sim_\mathcal{R} s$ then there is $s'$ with $s \to_\beta^\equiv s'$ and $t' \sim_\mathcal{R} s'$.*

*Proof.* Induction on $t \to_\beta t'$. There are two interesting cases.
- $t, s \in \mathcal{R}$, i.e., they have no rnf. Then also $t' \in \mathcal{R}$, so we may take $s' \equiv s$.
- $t \equiv (\lambda x.t_1)t_2$, $t' \equiv t_1[t_2/x]$, $s \equiv (\lambda x.s_1)s_2$ and $t_i \sim_\mathcal{R} s_i$. Then $t' \sim_\mathcal{R} s_1[s_2/x]$ by Lemma 5.41. Hence we may take $s' \equiv s_1[s_2/x]$. □

**Lemma 5.43.** *If $t$ is in rnf and $t \sim_\mathcal{R} s$, then so is $s$.*

*Proof.* Because $t$ is in rnf, there are three cases.
- $t \equiv a$ with $a \not\equiv \bot$. Then $s \equiv t$, so it is in rnf.
- $t \equiv \lambda x.t'$. Then $s \equiv \lambda x.s'$, so $s$ is in rnf.
- $t \equiv t_1 t_2$ and $t_1$ does not $\beta$-reduce to an abstraction. Then $s \equiv s_1 s_2$ with $t_i \sim_\mathcal{R} s_i$. Assume $s_1 \to_\beta^* \lambda x.s'$. Then by Lemma 5.42 there is $t'$ with $t_1 \to_\beta^* t' \sim_\mathcal{R} \lambda x.s'$. But then $t'$ must be an abstraction. Contradiction. □

**Corollary 5.44.** *If $t$ has a rnf and $t \sim_\mathcal{R} s$, then so does $s$.*

*Proof.* Follows from Lemma 5.42 and Lemma 5.43. □

**Lemma 5.45.** *If $t \to_\beta^\infty s$ and $t$ has a rnf, then so does $s$.*

*Proof.* Suppose $t$ has a rnf. Then by Lemma 5.22 there is $t'$ in rnf with $t \to_w^* t'$. By Theorem 5.20 and Lemma 5.21 there is $r$ with $s \to_w^* r$ and $t' \to_\beta^\infty r$. Since $t'$ is in rnf, by Lemma 4.10 so is $r$. Hence $s$ has a rnf $r$. □

**Theorem 5.46.** *The set $\mathcal{R}$ of root-active terms is a set of strongly meaningless terms.*

*Proof.* We check the axioms. The root-activeness axiom is obvious. The closure axiom follows from Lemma 4.9. The substitution axiom follows from Lemma 5.40. The overlap axiom follows from the fact that lambda abstractions are in rnf. The indiscernibility axiom follows from Corollary 5.44. The expansion axiom follows from Lemma 5.45. □

**Corollary 5.47** (Confluence of the $\lambda^\infty_{\beta\bot_\mathcal{R}}$-calculus)**.**
*If $t \to^\infty_{\beta\bot_\mathcal{R}} t_1$ and $t \to^\infty_{\beta\bot_\mathcal{R}} t_2$ then there exists $t_3$ such that $t_1 \to^\infty_{\beta\bot_\mathcal{R}} t_3$ and $t_2 \to^\infty_{\beta\bot_\mathcal{R}} t_3$.*

**Corollary 5.48** (Normalisation of the $\lambda^\infty_{\beta\bot_\mathcal{R}}$-calculus)**.**
*For every $t \in \Lambda^\infty$ there exists a unique $s \in \Lambda^\infty$ in $\beta\bot_\mathcal{R}$-normal form such that $t \to^\infty_{\beta\bot_\mathcal{R}} s$.*

5.7. **Confluence modulo equivalence of meaningless terms.** From confluence of the $\lambda^\infty_{\beta\bot_\mathcal{R}}$-calculus we may derive confluence of infinitary $\beta$-reduction $\to^\infty_\beta$ modulo equivalence of meaningless terms. The expansion axiom in not needed for the proof of the following theorem.

**Theorem 5.49** (Confluence modulo equivalence of meaningless terms)**.**
*If $t \sim_\mathcal{U} t'$, $t \to^\infty_\beta s$ and $t' \to^\infty_\beta s'$ then there exist $r, r'$ such that $r \sim_\mathcal{U} r'$, $s \to^\infty_\beta r$ and $s' \to^\infty_\beta r'$.*

*Proof.* By Lemma 5.3 and Lemma 5.4 it suffices to consider the case $t \equiv t'$. By Corollary 5.47 there is $u$ with $s \to^\infty_{\beta\bot_\mathcal{R}} u$ and $s' \to^\infty_{\beta\bot_\mathcal{R}} u$. By Theorem 5.46 and Lemma 5.14 there are $r, r'$ with $s \to^\infty_\beta r \sim_\mathcal{R} u$ and $s' \to^\infty_\beta r' \sim_\mathcal{R} u$. Because $\mathcal{R} \subseteq \mathcal{U}$, by Lemma 5.4 we obtain $r \sim_\mathcal{U} r'$. □

## 6. Strongly convergent reductions

In this section we prove that the existence of coinductive infinitary reductions is equivalent to the existence of strongly convergent reductions, under certain assumptions. As a corollary, this also yields $\omega$-compression of strongly convergent reductions, under certain assumptions. The equivalence proof is virtually the same as in [19]. The notion of strongly convergent reductions is the standard notion of infinitary reductions used in non-coinductive treatments of infinitary lambda calculus.

**Definition 6.1.** On the set of infinitary lambda terms we define a metric $d$ by

$$d(t, s) = \inf\{2^{-n} \mid t^{\restriction n} \equiv s^{\restriction n}\}$$

where $r^{\restriction n}$ for $r \in \Lambda^\infty$ is defined as the infinitary lambda term obtained by replacing all subterms of $r$ at depth $n$ by $\bot$. This defines a metric topology on the set of infinitary lambda terms. Let $R \subseteq \Lambda^\infty \times \Lambda^\infty$ and let $\zeta$ be an ordinal. A map $f : \{\gamma \le \zeta\} \to \Lambda^\infty$ together with reduction steps $\sigma_\gamma : f(\gamma) \to_R f(\gamma + 1)$ for $\gamma < \zeta$ is a *strongly convergent $R$-reduction sequence of length $\zeta$ from $f(0)$ to $f(\zeta)$* if the following conditions hold:

(1) if $\delta \le \zeta$ is a limit ordinal then $f(\delta)$ is the limit in the metric topology on infinite terms of the ordinal-indexed sequence $(f(\gamma))_{\gamma<\delta}$,
(2) if $\delta \le \zeta$ is a limit ordinal then for every $d \in \mathbb{N}$ there exists $\gamma < \delta$ such that for all $\gamma'$ with $\gamma \le \gamma' < \delta$ the redex contracted in the step $\sigma_{\gamma'}$ occurs at depth greater than $d$.

We write $s \xrightarrow{S,\zeta}_R t$ if $S$ is a strongly convergent $R$-reduction sequence of length $\zeta$ from $s$ to $t$.

A relation $\to \subseteq \Lambda^\infty \times \Lambda^\infty$ is *appendable* if $t_1 \to^\infty t_2 \to t_3$ implies $t_1 \to^\infty t_3$. We define $\to^{2\infty}$ as the infinitary closure of $\to^\infty$. We write $\to^{\infty*}$ for the transitive-reflexive closure of $\to^\infty$.

**Lemma 6.2.** *If $\to$ is appendable then $t_1 \to^\infty t_2 \to^\infty t_3$ implies $t_1 \to^\infty t_3$.*

*Proof.* By coinduction. This has essentially been shown in [19, Lemma 4.5]. □

**Lemma 6.3.** *If $\to$ is appendable then $s \to^{2\infty} t$ implies $s \to^\infty t$.*

*Proof.* By coinduction. There are three cases.

- $t \equiv a$. Then $s \to^{\infty*} a$, so $s \to^\infty a$ by Lemma 6.2.
- $t \equiv t_1t_2$. Then there are $t_1', t_2'$ with $s \to^{\infty*} t_1't_2'$ and $t_i' \to^{2\infty} t_i$. By Lemma 6.2 we have $s \to^\infty t_1't_2'$, so there are $u_1, u_2$ with $s \to^* u_1u_2$ and $u_i \to^\infty t_i'$. Then $u_i \to^{2\infty} t_i$. By the coinductive hypothesis $u_i \to^\infty t_i$. Hence $s \to^\infty t_1t_2 \equiv t$.
- $t \equiv \lambda x.r$. Then by Lemma 6.2 there is $s'$ with $s \to^\infty \lambda x.s'$ and $s' \to^{2\infty} r$. So there is $s_0$ with $s \to^* \lambda x.s_0$ and $s_0 \to^\infty s'$. Then also $s_0 \to^{2\infty} r$. By the coinductive hypothesis $s_0 \to^\infty r$. Thus $s \to^\infty \lambda x.r \equiv t$. □

**Theorem 6.4.** *For every $R \subseteq \Lambda^\infty \times \Lambda^\infty$ such that $\to_R$ is appendable, and for all $s, t \in \Lambda^\infty$, we have the equivalence: $s \to_R^\infty t$ iff there exists a strongly convergent $R$-reduction sequence from $s$ to $t$. Moreover, if $s \to_R^\infty t$ then the sequence may be chosen to have length at most $\omega$.*

*Proof.* The proof is a straightforward generalisation of the proof of Theorem 3 in [19].

Suppose that $s \to_R^\infty t$. By traversing the infinite derivation tree of $s \to_R^\infty t$ and accumulating the finite prefixes by concatenation, we obtain a reduction sequence of length at most $\omega$ which satisfies the depth requirement by construction.

For the other direction, by induction on $\zeta$ we show that if $s \xrightarrow{S,\zeta}_R t$ then $s \to_R^{2\infty} t$, which suffices for $s \to_R^\infty t$ by Lemma 6.3. There are three cases.

- $\zeta = 0$. If $s \xrightarrow{S,0}_R t$ then $s \equiv t$, so $s \to_R^{2\infty} t$.
- $\zeta = \gamma + 1$. If $s \xrightarrow{S,\gamma+1}_R t$ then $s \xrightarrow{S',\gamma}_R s' \to_R t$. Hence $s \to_R^{2\infty} s'$ by the inductive hypothesis. Then $s \to_R^\infty s' \to_R t$ by Lemma 6.3. So $s \to_R^\infty t$ because $\to_R$ is appendable.
- $\zeta$ is a limit ordinal. By coinduction we show that if $s \xrightarrow{S,\zeta}_R t$ then $s \to_R^{2\infty} t$. By the depth condition there is $\gamma < \zeta$ such that for every $\delta \geq \gamma$ the redex contracted in $S$ at $\delta$ occurs at depth greater than zero. Let $t_\gamma$ be the term at index $\gamma$ in $S$. Then by the inductive hypothesis we have $s \to_R^{2\infty} t_\gamma$, and thus $s \to_R^\infty t_\gamma$ by Lemma 6.3. There are three cases.
  - $t_\gamma \equiv a$. This is impossible because then there can be no reduction of $t_\gamma$ at depth greater than zero.
  - $t_\gamma \equiv \lambda x.r$. Then $t \equiv \lambda x.u$ and $r \xrightarrow{S',\delta}_R u$ with $\delta \leq \zeta$. Hence $r \to_R^{2\infty} u$ by the coinductive hypothesis if $\delta = \zeta$, or by the inductive hypothesis if $\delta < \zeta$. Since $s \to_R^\infty \lambda x.r$ we obtain $s \to_R^{2\infty} \lambda x.u \equiv t$.
  - $t_\gamma \equiv t_1t_2$. Then $t \equiv u_1u_2$ and the tail of the reduction $S$ past $\gamma$ may be split into two parts: $t_i \xrightarrow{S_i,\delta_i}_R u_i$ with $\delta_i \leq \zeta$ for $i = 0, 1$. Then $t_i \to_R^{2\infty} u_i$ by the inductive and/or the coinductive hypothesis. Since $s \to_R^\infty t_1t_2$ we obtain $s \to_R^{2\infty} u_1u_2 \equiv t$. □

**Corollary 6.5** ($\omega$-compression)**.** *If $\to_R$ is appendable and there exists a strongly convergent $R$-reduction sequence from $s$ to $t$ then there exists such a sequence of length at most $\omega$.*

**Corollary 6.6.** *Let $\mathcal{U}$ be a set of strongly meaningless terms.*

- $s \to^\infty_{\beta\bot_\mathcal{U}} t$ *iff there exists a strongly convergent $\beta\bot_\mathcal{U}$-reduction sequence from $s$ to $t$.*
- $s \to^\infty_\beta t$ *iff there exists a strongly convergent $\beta$-reduction sequence from $s$ to $t$.*

*Proof.* By Theorem 6.4 it suffices to show that $\to_{\beta\bot_\mathcal{U}}$ and $\to_\beta$ are appendable. For $\to_{\beta\bot_\mathcal{U}}$ this follows from Lemma 5.17 and Corollary 5.18. For $\to_\beta$ this follows from Lemma 4.8. □

## 7. The formalisation

The results of this paper have been formalised in the Coq proof assistant. The formalisation is available at:

`https://github.com/lukaszcz/infinitary-confluence`

The formalisation contains all results of Section 5. We did not formalise the proof from Section 6 of the equivalence between the coinductive definition of the infinitary reduction relation and the standard notion of strongly convergent reductions.

In our formalisation we use a representation of infinitary lambda terms with de Bruijn indices, and we do not allow constants except $\bot$. Hence, the results about $\alpha$-conversion alluded to in Section 4 are not formalised either. Because the formalisation is based on de Bruijn indices, many tedious lifting lemmas need to be proved. These lemmas are present only in the formalisation, but not in the paper.

In general, the formalisation follows closely the text of the paper. Each lemma from Section 5 has a corresponding statement in the formalisation (annotated with the lemma number from the paper). There are, however, some subtleties, described below.

One difficulty with a Coq formalisation of our results is that in Coq the coinductively defined equality (bisimilarity) $=$ on infinite terms (see Definition 3.4) is not identical with Coq's definitional equality $\equiv$. In the paper we use $\equiv$ and $=$ interchangeably, following Proposition 3.5. In the formalisation we needed to formulate our definitions "modulo" bisimilarity. For instance, the inductive definition of the transitive-reflexive closure $R^*$ of a relation $R$ on infinite terms is as follows.

(1) If $t_1 = t_2$ then $R^* t_1 t_2$ (where $=$ denotes bisimilarity coinductively defined like in Definition 3.4).
(2) If $R t_1 t_2$ and $R^* t_2 t_3$ then $R^* t_1 t_3$.

Changing the first point to

(1) $R^* tt$ for any term $t$

would not work with our formalisation. Similarly, the formal definition of the compatible closure of a relation $R$ follows the inductive rules

$$\frac{\langle s,t\rangle \in R}{s \to_R t} \qquad \frac{s \to_R s' \quad t = t'}{st \to_R s't'} \qquad \frac{t \to_R t' \quad s = s'}{st \to_R s't'} \qquad \frac{s \to_R s'}{\lambda x.s \to_R \lambda x.s'}$$

where $=$ denotes the coinductively defined bisimilarity relation.

Another limitation of Coq is that it is not possible to directly prove by coinduction statements of the form $\forall \vec{x}.\varphi(\vec{x}) \to R_1(\vec{x}) \wedge R_2(\vec{x})$, i.e., statements with a conjuction of two coinductive predicates. Instead, we show $\forall \vec{x}.\varphi(\vec{x}) \to R_1(\vec{x})$ and $\forall \vec{x}.\varphi(\vec{x}) \to R_2(\vec{x})$ separately. In all our coinductive proofs we use the coinductive hypothesis in a way that makes this separation possible.

The formalisation assumes the following axioms.

(1) The constructive indefinite description axiom:

$$\forall A : \text{Type}.\forall P : A \to \text{Prop}.(\exists x : A.Px) \to \{x : A \mid Px\}.$$

This axiom states that if there exists an object $x$ of type $A$ satisfying the predicate $P$, then it is possible to choose a fixed such object. This is not provable in the standard logic of Coq. We need this assumption to be able to define the implicit functions in some coinductive proofs which show the existence of an infinite object, when the form of this object depends on which case in the definition of some (co)inductive predicate holds. More precisely, the indefinite description axiom is needed in the proof of Lemma 5.14, in the definition of canonical root normal forms (Definition 5.24), and in the proofs of Lemma 5.3, Lemma 5.5 and Lemma 5.28.

(2) Excluded middle for the property of being in root normal form: for every term $t$, either $t$ is in root normal form or not.

(3) Excluded middle for the property of having a root normal form: for every term $t$, either $t$ has a root normal form or not.

(4) Excluded middle for the property of belonging to a set of strongly meaningless terms: for any set of strongly meaningless terms $\mathcal{U}$ and any term $t$, either $t \in \mathcal{U}$ or $t \notin \mathcal{U}$.

Note that the last axiom does not constructively imply the third. We define being root-active as not having a root normal form. In fact, we need the third axiom to show that if a term does not belong to a set of meaningless terms then it has a root normal form.

The first axiom could probably be avoided by making the reduction relations Set-valued instead of Prop-valued. We do not use the impredicativity of Prop. The reason why we chose to define the relations as Prop-valued is that certain automated proof search tactics work better with Prop-valued relations, which makes the formalisation easier to carry out.

Because the $\perp_{\mathcal{U}}$-reduction rule, for any set of meaningless terms $\mathcal{U}$, requires an oracle to check whether it is applicable, the present setup is inherently classical. It is an interesting research question to devise a constructive theory of meaningless terms.

Aside of the axioms (1)–(4), everything else from Section 5 is formalised in the constructive logic of Coq, including the proof of Theorem 5.20 only cited in this paper. Our formalisation of Theorem 5.20 closely follows [19].

In our formalisation we extensively used the CoqHammer tool [13].

## 8. Conclusions

We presented new and formal coinductive proofs of the following results.

(1) Confluence and normalisation of Böhm reduction over any set of strongly meaningless terms.

(2) Confluence and normalisation of Böhm reduction over root-active terms, by showing that root-active terms are strongly meaningless.

(3) Confluence of infinitary $\beta$-reduction modulo any set of meaningless terms (expansion axiom not needed).

We formalised these results in Coq. Our formalisation uses a definition of infinitary lambda terms based on de Bruijn indices. Strictly speaking, the precise relation of this definition to other definitions of infinitary lambda terms in the literature has not been established. We leave this for future work. The issue of the equivalence of various definitions of infinitary lambda terms is not necessarily trivial [32, 33].

By a straightforward generalisation of a result in [19] we also proved equivalence, in the sense of existence, of the coinductive definitions of infinitary rewriting relations with the standard definitions based on strong convergence. However, we did not formalise this result.

In Section 3 we explained how to elaborate our coinductive proofs by reducing them to proofs by transfinite induction and thus eliminating coinduction. This provides one way to understand and verify our proofs without resorting to a formalisation. After properly understanding the observations of Section 3 it should be "clear" that coinduction may in principle be eliminated in the described manner. We use the word "clear" in the same sense that it is "clear" that the more sophisticated inductive constructions commonly used in the literature can be formalised in ZFC set theory. Of course, this notion of "clear" may always be debated. The only way to make this completely precise is to create a formal system based on Section 3 in which our proofs could be interpreted reasonably directly. We do not consider the observations of Section 3 to be novel or particularly insightful. However, distilling them into a formal system could perhaps arise some interest. This is left for future work.

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## Appendix A. A set of meaningless terms not satisfying the expansion axiom

We recall the definitions from page 14. Let $\mathsf{O}$ be the *ogre* satisfying $\mathsf{O} \equiv \lambda x.\mathsf{O}$. A term $t$ is *head-active* if $t \equiv \lambda x_1 \dots x_n.rt_1 \dots t_m$ with $r \in \mathcal{R}$ and $n, m \geq 0$. Define:

- $\mathcal{H} = \{t \in \Lambda^\infty \mid t \to_\beta^* t' \text{ with } t' \text{ head-active}\}$,
- $\mathcal{O} = \{t \in \Lambda^\infty \mid t \to_\beta^* \mathsf{O}\}$,
- $\mathcal{U} = \mathcal{H} \cup \mathcal{O}$.

We will show that $\mathcal{U}$ is a set of meaningless terms. The proofs in this appendix rely on the results established in the paper. In particular, we use that $\mathcal{R}$ is a set of strongly meaningless terms and that $\to_\beta^\infty$ is confluent modulo $\mathcal{R}$.

**Lemma A.1.** (1) *If $t \in \mathcal{U}$ then $ts \in \mathcal{U}$.*
(2) *If $t \in \mathcal{U}$ then $\lambda x.t \in \mathcal{U}$.*
(3) *If $t \in \mathcal{U}$ then $t[s/x] \in \mathcal{U}$.*

*Proof.* Follows from definitions and the fact that $\mathcal{R}$ satisfies the substitution axiom. □

**Lemma A.2.** *If $t \to_\beta^\infty t'$ and $t \in \mathcal{H}$ then $t' \in \mathcal{H}$.*

*Proof.* We have $t \to_\beta^* u$ with $u$ head-active. By confluence modulo $\mathcal{R}$ there are $s, s'$ with $u \to_\beta^\infty s$, $t' \to_\beta^\infty s'$ and $s \sim_\mathcal{R} s'$. It follows from the closure and indiscernibility axioms for $\mathcal{R}$ that $s'$ is head-active. Now, using the expansion axiom for $\mathcal{R}$ one may show that there is a head-active $w$ such that $t' \to_\beta^* w \to_\beta^\infty s'$. □

**Lemma A.3.** *If $t \to_\beta^\infty t'$ and $t \in \mathcal{O}$ then $t' \in \mathcal{O}$.*

*Proof.* We have $t \to_\beta^* \mathsf{O}$. Because the reduction is finite, no $\beta$-contractions occur below a fixed depth $d$. We can write $\mathsf{O} \equiv \lambda \vec{x}.\mathsf{O}$ where on the right side $\mathsf{O}$ occurs below depth $d$. Then there is $u$ with exactly one occurrence of $z$ (where $z \notin \vec{x}$) such that $u \to_\beta^* \lambda \vec{x}.z$ and $t \equiv u[\mathsf{O}/z]$.

Write $s \succ_z^\mathsf{O} s'$ if $s'$ may be obtained from $s$ by changing some $\mathsf{O}$ subterms in $s$ into some terms having the form $zu_1 \dots u_n$, defined coinductively analogously to Definition 5.35. One shows:

(1) if $s \succ_z^\mathsf{O} s'$ and $s \to_\beta^* w$ then there is $w'$ with $s' \to_\beta^* w'$ and $w \succ_z^\mathsf{O} w'$,
(2) if $s \succ_z^\mathsf{O} s'$ and $s' \to_\beta^* w'$ then there is $w$ with $s \to_\beta^* w$ and $w \succ_z^\mathsf{O} w'$,
(3) if $s \succ_z^\mathsf{O} s'$ and $s \to_\beta^\infty w$ then there is $w'$ with $s' \to_\beta^\infty w'$ and $w \succ_z^\mathsf{O} w'$,

The first two points are proved by induction. The third one follows from the first one using coinduction.

Hence, there exists $u'$ such that $u \to_\beta^\infty u'$, and $t' \succ_z^\mathsf{O} u'$. By confluence modulo $\mathcal{R}$ and the fact that $\lambda \vec{x}.z$ is a finite normal form, we have $u' \to_\beta^* \lambda \vec{x}.z$. Thus there is $w$ with $t' \to_\beta^* w \succ_z^\mathsf{O} \lambda \vec{x}.z$. This is possible only when $w \equiv \mathsf{O}$. □

**Corollary A.4.** *If $t \to_\beta^\infty t'$ and $t \in \mathcal{U}$ then $t' \in \mathcal{U}$.*

**Lemma A.5.** *If $t_1 \sim_\mathcal{U} s_1$ and $t_2 \sim_\mathcal{U} s_2$ then $t_1[t_2/x] \sim_\mathcal{U} s_1[s_2/x]$.*

*Proof.* By coinduction, using Lemma A.1(3). □

**Lemma A.6.** *If $t \to_\beta t'$ and $t \sim_\mathcal{U} s$ then there is $s'$ with $s \to_\beta^\equiv s'$ and $t' \sim_\mathcal{U} s'$.*

*Proof.* Induction on $t \to_\beta t'$. If the case $t, s \in \mathcal{U}$ in the definition of $t \sim_\mathcal{U} s$ holds, then $t' \in \mathcal{U}$ by Corollary A.4, so $t' \sim_\mathcal{U} s$ and we may take $s' \equiv s$. So assume otherwise. Then all cases follow directly from the inductive hypothesis except when $t$ is the contracted $\beta$-redex. Then $t \equiv (\lambda x.t_1)t_2$ and $t' \equiv t_1[t_2/x]$ and $s = s_0 s_2$ with $\lambda x.t_1 \sim_\mathcal{U} s_0$ and $t_2 \sim_\mathcal{U} s_2$. If $\lambda x.t_1, s_0 \in \mathcal{U}$ then $t, s \in \mathcal{U}$ by Lemma A.1, and thus $t' \in \mathcal{U}$ by Corollary A.4, and thus $t' \sim_\mathcal{U} s$ and we may take $s' \equiv s$. Otherwise, $s_0 \equiv \lambda x.s_1$ with $t_1 \sim_\mathcal{U} s_1$. By Lemma A.5 we have $s \to_\beta s_1[s_2/x] \sim_\mathcal{U} t_1[t_2/x] \equiv t'$, so we take $s' \equiv s_1[s_2/x]$. □

**Lemma A.7.** *If $t \in \mathcal{R}$ and $t \sim_\mathcal{U} t'$ then $t' \in \mathcal{U}$.*

*Proof.* Assume $t' \to_\beta^\infty s_0$ with $s_0$ in rnf. Then by Lemma 5.22 there is $s'$ in rnf with $t' \to_\beta^* s'$. By Lemma A.6 there is $s$ with $t \to_\beta^* s$ and $s \sim_\mathcal{U} s'$. If $s, s' \in \mathcal{U}$ then $t' \in \mathcal{U}$. Otherwise, because $t \in \mathcal{R}$, we must have $s \equiv s_1 s_2$, $s' \equiv s_1' s_2'$ with $s_i \sim_\mathcal{U} s_i'$. Because $t \in \mathcal{R}$ and $t \to_\beta^* s_1 s_2$, there exists $u$ such that $s_1 \to_\beta^* \lambda x.u$. Then by Lemma A.6 there is $u_0$ with $s_1' \to_\beta^* u_0 \sim_\mathcal{U} \lambda x.u$. If $u_0, \lambda.u \in \mathcal{U}$ then $s_1' \in \mathcal{U}$, and thus $s' \in \mathcal{U}$, and thus $t' \in \mathcal{U}$. Otherwise $u_0 \equiv \lambda x.u'$ with $u \sim_\mathcal{U} u'$. But this contradicts that $s'$ is in rnf. □

**Lemma A.8.** *If $t \in \mathcal{U}$ and $t \sim_\mathcal{U} t'$ then $t' \in \mathcal{U}$.*

*Proof.* First assume $t \in \mathcal{H}$, i.e., $t \to_\beta^* u \equiv \lambda x_1 \ldots x_n.r t_1 \ldots t_m$ with $r \in \mathcal{R}$. By Lemma A.6 there is $u$ with $t' \to_\beta^* u' \sim_\mathcal{U} u$. We may assume $u' \equiv \lambda x_1 \ldots x_n.r' t_1' \ldots t_m'$ with $r' \sim_\mathcal{U} r$ (otherwise $u' \in \mathcal{U}$, using Lemma A.1, so $t' \in \mathcal{U}$). But then $r' \in \mathcal{R}$ by Lemma A.7. Hence $t' \in \mathcal{H} \subseteq \mathcal{U}$.

Now assume $t \in \mathcal{O}$, i.e., $t \to_\beta^* \mathsf{O}$. By Lemma A.6 there is $u'$ with $t' \to_\beta^* u' \sim_\mathcal{U} \mathsf{O}$. Using Lemma A.1 one checks that $u' \sim_\mathcal{U} \mathsf{O}$ implies $u' \in \mathcal{U}$. Then also $t' \in \mathcal{U}$. □

**Theorem A.9.** *$\mathcal{U}$ is a set of meaningless terms.*

*Proof.* The closure axiom follows from Corollary A.4. The substitution axiom follows from Lemma A.1(3). The overlap axiom follows from Lemma A.1(1). The root-activeness axiom follows from $\mathcal{R} \subseteq \mathcal{H} \subseteq \mathcal{U}$. The indiscernibility axiom follows from Lemma A.8. □