

## AUTOMATA MINIMIZATION: A FUNCTORIAL APPROACH\*

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**ABSTRACT.** In this paper we regard languages and their acceptors – such as deterministic or weighted automata, transducers, or monoids – as functors from input categories that specify the type of the languages and of the machines to categories that specify the type of outputs.

Our results are as follows: a) We provide sufficient conditions on the output category so that minimization of the corresponding automata is guaranteed. b) We show how to lift adjunctions between the categories for output values to adjunctions between categories of automata. c) We show how this framework can be instantiated to unify several phenomena in automata theory, starting with determinization, minimization and syntactic algebras. We provide explanations of Choffrut’s minimization algorithm for subsequential transducers and of Brzozowski’s minimization algorithm in this setting.

### 1. INTRODUCTION

There is a long tradition of interpreting results of automata theory through the lens of category theory. Typical instances of this scheme interpret automata as algebras (together with a final map) as put forward in [2, 4, 15], or as coalgebras (together with an initial map), see for example [17, 22]. This dual narrative proved very useful [7] in explaining at an abstract level Brzozowski’s minimization algorithm and the duality between reachability and observability (which goes back all the way to the work of Arbib and Manes [4] and Kalman [18]).

In this paper, we adopt a slightly different approach, and we define directly the notion of an automaton (over finite words) as a functor from a category representing input words, to a category representing the computation and output spaces. For example, deterministic automata are represented as functors valued in the category of sets and functions, non-deterministic automata as functors valued in the category of sets and relations, while

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*Key words and phrases*: automata minimization, functor automata, minimization of subsequential transducers, Brzozowski’s minimization algorithm.

\* This is the journal version of [13].

This work was supported by the European Research Council (ERC) under the European Union’s Horizon 2020 research and innovation programme (grant agreement No.670624), and by the DeLTA ANR project (ANR-16-CE40-0007). The authors also thank the Simons Institute for the Theory of Computing where this work has been partly developed.

weighted automata over a semiring  $S$  as functors valued in the category of  $S$ -modules. The notions of a language and of language accepted by an automaton are adapted along the same pattern.

We provide several developments around this idea. First, we recall (see [12]) that the existence of a minimal automaton for a language is guaranteed by the existence of an initial and a final automaton in combination with a factorization system. The idea of using factorization systems in the context of minimization has of course a long history, going back at least to Goguen [15]. However, the functorial presentation that we adopted allows us to give a unifying perspective of the minimization of various forms of automata and algebraic structures used for language recognition. Additionally, we explain how, in the functor presentation that we have adopted, the existence of initial and final automata for a language can be phrased in terms of Kan extensions. As an immediate corollary, we identify sufficient conditions on the output category for the existence of the corresponding minimal automaton: existence of certain limits and colimits, as well as of a suitable factorization system.

We also show how adjunctions between categories can be lifted to the level of automata for languages in these categories (Lemma 3.4). This lifting accounts for several constructions in automata theory, determinization to start with. Indeed, determinization of automata can be understood via a lifting of the Kleisli adjunction between the categories  $\text{Rel}$  (of sets and relations) and  $\text{Set}$  (of sets and functions); and reversing non-deterministic automata can be understood via a lifting of the self-duality of  $\text{Rel}$ .

We then use this framework in order to explain several well-known constructions in automata theory.

The most involved contribution (Theorem 4.5) is to rephrase in this framework the minimization result for subsequential transducers due to Choffrut [10]. We do this by instantiating the category of outputs with the Kleisli category for the monad  $\mathcal{TX} = B^* \times X + 1$ , where  $B$  is the output alphabet of the transducers. In this case, despite the lack of completeness of the ambient category, one can still prove the existence of an initial and of a final automaton, as well as, surprisingly, of a factorization system.

The second concrete application presented in Section 5 is a proof of correctness of Brzozowski's minimization algorithm, for both deterministic and weighted automata. Brzozowski's minimization algorithm for deterministic automata can be understood by lifting the adjunctions between  $\text{Set}$  and its opposite category  $\text{Set}^{op}$ , as an immediate application of Lemma 3.4. Similarly, upon viewing weighted automata as functors valued in the category  $S\text{-Mod}$  of  $S$ -modules, a weighted version of Brzozowski's minimization algorithm described in [7] can be explained by lifting the adjunction between  $S\text{-Mod}$  and its opposite  $S\text{-Mod}^{op}$ .

Lastly, in Section 6 we show how the syntactic monoid for a language can be obtained in the same spirit as the minimal automaton. To this end, we replace the category representing finite words with one suitable for representing biaction and monoid recognizers of languages.

**Related work.** Many of the constructions outlined here have already been explained from a category-theoretic perspective, using various techniques. For example, the relationship between minimization and duality was subject to numerous papers, see for example [6–8] and the references therein. The coalgebraic perspective on minimization was also emphasized in papers such as [1, 3, 23]. We briefly mention the relationship with well-pointed coalgebras in Remark 3.3. However, we argue that in some instances the functorial approach may be better

suitable for explaining minimization, an example being that of subsequential transducers, see Section 4.

In [16] subsequential structures (i.e. subsequential transducers without initial state and initial prefix) are modeled as coalgebras for an endofunctor on  $\mathbf{Set}$ . However, the corresponding notion of coalgebra morphism does not accurately capture the suitable notion of subsequential morphisms. In Section 4 we model subsequential transducers as functors valued in a Kleisli category  $\mathbf{Kl}(\mathcal{T})$ . This category does not have powers, hence working with coalgebras for an endofunctor on  $\mathbf{Kl}(\mathcal{T})$  is not possible, see Remark 3.3. On the other hand, if one uses instead coalgebras for a  $\mathbf{Set}$ -endofunctor, as in [16], then only certain “normalized” subsequential structures can be fully dealt with coalgebraically.

Understanding determinization and codeterminization by lifting adjunctions to coalgebras was considered in [19], and is related to our results from Section 3.3.

The paper which is closest in spirit to our work is a seemingly forgotten paper [5]. However, in this work, Bainbridge models the *state space* of the machines as a functor. Left and right Kan extensions are featured in connection with the initial and final automata, but in a slightly different setting. Lemma 3.4, which albeit technically simple, has surprisingly many applications, builds directly on his work.

The functorial approach to non-deterministic automata presented in this paper is reminiscent of the work on automata in quantaloid-enriched categories developed in [21]. However, in this paper we do not consider neither relational presheaves (which are *lax* functors) nor enriched categories. We also aimed to keep the requirements on the output category as simple as possible.

## 2. LANGUAGES AND AUTOMATA AS FUNCTORS

In this section, we introduce the notion of automata via functors, and this is the common denominator of the different contributions of this paper. We then discuss automata minimization in this generic setting.

**2.1. Automata as functors.** We introduce automata as functors starting from the special case of classical deterministic automata. In the standard definition, a *deterministic automaton* is a tuple:

$$\langle Q, A, q_0, F, \delta \rangle$$

where  $Q$  is a set of *states*,  $A$  is an alphabet (not necessarily finite),  $q_0 \in Q$  is the *initial state*,  $F \subseteq Q$  is the set of *final states*, and  $\delta_a: Q \rightarrow Q$  is the *transition map* for all letters  $a \in A$ . The semantic of an automaton is to define what is a *run* over an input word  $u \in A^*$ , and whether it is accepting or not. Given a word  $e = a_1 \dots a_n$ , the automaton *accepts* the word if  $\delta_{a_n} \circ \dots \circ \delta_{a_1}(q_0) \in F$ , and otherwise *rejects it*.

If we see  $q_0$  as a map *init* from the one element set  $1 = \{0\}$  to  $Q$ , that maps 0 to  $q_0$ , and  $F$  as a map *final* from  $Q$  to the set  $2 = \{0, 1\}$ , where 1 means ‘accept’ and 0 means ‘reject’, then the semantic of the automaton is to associate to each word  $u = a_1 \dots a_n$  the map from 1 to 2 defined as  $final \circ \delta_{a_n} \circ \dots \circ \delta_{a_1} \circ init$ . If this map is (constant equal to) 1, this means that the word is accepted, and otherwise it is rejected.

Pushing this idea further, we can see the semantics of the automaton as a functor from the category  $\mathcal{I}_{\text{word}}$  spanned by the graph of vertices  $\{\text{in}, \text{states}, \text{out}\}$  in Figure 1 to  $\mathbf{Set}$ , and more precisely one that sends the object in to 1 and out to 2. The arrows of the

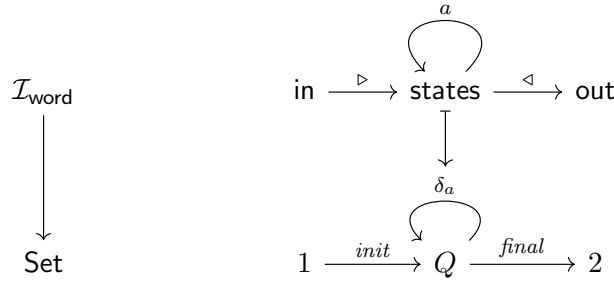


Figure 1: Deterministic automata as functors

three-object category  $\mathcal{I}_{\text{word}}$  are spanned by  $\triangleright, \triangleleft$  and  $a$  for all  $a \in A$ , and the composite of  $\text{states} \xrightarrow{w} \text{states} \xrightarrow{w'} \text{states}$  is given by the concatenation  $ww'$ . Notice the left to right order of composition.

In the above category, the arrows from  $\text{in}$  to  $\text{out}$  are of the form  $\triangleright w \triangleleft$  for  $w$  an arbitrary word in  $A^*$ . Furthermore, since a *language* can be seen as a map from  $A^*$  to the set  $1 \rightarrow 2$  of functions from 1 to 2, we can model it as a functor from the full subcategory  $\mathcal{O}_{\text{word}}$  on objects  $\text{in}$  and  $\text{out}$  to the category  $\text{Set}$ , which maps  $\text{in}$  to 1 and  $\text{out}$  to 2.

In this section we fix an arbitrary small category  $\mathcal{I}$  and a full subcategory  $\mathcal{O}$ . We denote by  $\iota$  the inclusion functor

$$\mathcal{O} \xleftarrow{\iota} \mathcal{I}.$$

We think of  $\mathcal{I}$  as a specification of the inner computations that an automaton can perform, including black box behavior, not observable from the outside. On the other hand, the full subcategory  $\mathcal{O}$  specifies the observable behavior of the automaton, that is, the language it accepts. In this interpretation, a machine/automaton  $\mathcal{A}$  is a functor from  $\mathcal{I}$  to a category of outputs  $\mathcal{C}$ , and the “behavior” or “language” of  $\mathcal{A}$  is the functor  $\mathcal{L}(\mathcal{A})$  obtained by precomposition with the inclusion  $\mathcal{O} \xleftarrow{\iota} \mathcal{I}$ . We obtain the following definition:

**Definition 2.1** (*C-languages and C-automata*). A *C-language* is a functor  $\mathcal{L}: \mathcal{O} \rightarrow \mathcal{C}$  and a *C-automaton* is a functor  $\mathcal{A}: \mathcal{I} \rightarrow \mathcal{C}$ . A *C-automaton*  $\mathcal{A}$  *accepts* a *C-language*  $\mathcal{L}$  when  $\mathcal{A} \circ \iota = \mathcal{L}$ ; i.e. the following diagram commutes:

$$\begin{array}{ccc} \mathcal{O} & \xrightarrow{\mathcal{L}} & \mathcal{C} \\ \iota \downarrow & \nearrow \mathcal{A} & \\ \mathcal{I} & & \end{array}$$

We write  $\text{Auto}(\mathcal{L})$  for the subcategory of the functor category  $[\mathcal{I}, \mathcal{C}]$  where

- (1) objects are *C-automata* that accept  $\mathcal{L}$ , and
- (2) arrows are natural transformations  $\alpha: \mathcal{A} \rightarrow \mathcal{B}$  so that the natural transformation obtained by composition with the inclusion functor  $\iota$  is the identity natural transformation on  $\mathcal{L}$ , that is,  $\alpha \circ \iota = id_{\mathcal{L}}$ .

**Example 2.2** (word automata and their languages). We can model various forms of word automata and their languages using the input categories  $\mathcal{O}_{\text{word}} \xleftarrow{\iota} \mathcal{I}_{\text{word}}$  and varying the category of outputs:

- (1) As described in Figure 1, deterministic automata can be seen as  $\text{Set}$ -automata, i.e. as functors  $\mathcal{A}: \mathcal{I}_{\text{word}} \rightarrow \text{Set}$  that map  $\text{in}$  to 1 and  $\text{out}$  to 2.

The language accepted by  $\mathcal{A}$  is the composite  $\mathcal{L}: \mathcal{O}_{\text{word}} \hookrightarrow \mathcal{I}_{\text{word}} \xrightarrow{\mathcal{A}} \text{Set}$ , which essentially specifies for each word  $w \in A^*$  a function  $\mathcal{L}(\triangleright w \triangleleft): 1 \rightarrow 2$ , establishing whether the word  $w$  is accepted or not. Indeed, if  $w = a_1 \dots a_n$ , then  $\mathcal{L}(\triangleright w \triangleleft)$  is exactly the function  $final \circ \delta_{a_n} \circ \dots \circ \delta_{a_1} \circ init$  described in the introduction of this section.

- (2) Non-deterministic automata can be modeled as Rel-automata, where Rel is the category whose objects are sets and maps are relations between them.

Indeed, a *non-deterministic automaton* is completely determined by the relations described in the next diagram, where the set of initial states is modeled as a relation from 1 to the set of states  $Q$ , the set of final states as a relation from  $Q$  to 1 and the transition relation by any input letter  $a$ , as a relation on  $Q$ :

$$\begin{array}{ccc} & \delta_a & \\ & \curvearrowright & \\ 1 & \xrightarrow{init} & Q & \xrightarrow{final} & 1 \end{array}$$

Explicitly, we consider Rel-automata  $\mathcal{A}: \mathcal{I}_{\text{word}} \rightarrow \text{Rel}$  so that  $\mathcal{A}(\text{in}) = 1$  and  $\mathcal{A}(\text{out}) = 1$ .

The language accepted by  $\mathcal{A}$  is the composite  $\mathcal{L}: \mathcal{O}_{\text{word}} \hookrightarrow \mathcal{I}_{\text{word}} \xrightarrow{\mathcal{A}} \text{Rel}$ . This functor specifies for each word  $w \in A^*$  a relation  $\mathcal{L}(\triangleright w \triangleleft): 1 \dashrightarrow 1$ . Notice that the set  $\text{Rel}(1, 1)$  of relations on the set 1 is isomorphic to 2, and the relation  $\mathcal{L}(\triangleright w \triangleleft)$  simply models whether the word  $w$  is accepted by the automaton or not.

- (3) Weighted automata over a semiring  $S$  can be modeled as functors  $\mathcal{A}: \mathcal{I}_{\text{word}} \rightarrow S\text{-Mod}$  valued in the category  $S\text{-Mod}$  of  $S$ -modules and  $S$ -linear morphisms and mapping both in and out to  $S$  (seen as a module over itself). Indeed, such an automaton is determined by the linear maps described in the next diagram, where the state space  $Q$  has an  $S$ -module structure.

$$\begin{array}{ccc} & \delta_a & \\ & \curvearrowright & \\ S & \xrightarrow{init} & Q & \xrightarrow{final} & S \end{array}$$

Indeed, to give a linear map  $init: S \rightarrow Q$  amounts to giving one element of the module  $Q$ , i.e. an initial state for the automaton.

The language accepted by  $\mathcal{A}$ , i.e. the composite  $\mathcal{L}: \mathcal{O}_{\text{word}} \hookrightarrow \mathcal{I}_{\text{word}} \xrightarrow{\mathcal{A}} S\text{-Mod}$  specifies for each word  $w \in A^*$  a linear transformation  $\mathcal{L}(\triangleright w \triangleleft): S \rightarrow S$ . Up to isomorphism, this is the same as specifying one scalar in  $S$  for each word in  $A^*$ , hence we obtain the weighted language  $A^* \rightarrow S$  classically accepted by the automaton.

**2.2. Minimization of  $\mathcal{C}$ -automata.** In this section we show that the notion of a minimal automaton is an instance of a more generic notion of minimal object that can be defined in an arbitrary category  $\mathcal{K}$  whenever there exist an initial object, a final object, and a factorization system  $(\mathcal{E}, \mathcal{M})$ .

Let  $X, Y$  be two objects of  $\mathcal{K}$ . We say that:

$$X \text{ } (\mathcal{E}, \mathcal{M})\text{-divides } Y \quad \text{if} \quad X \text{ is an } \mathcal{E}\text{-quotient of an } \mathcal{M}\text{-subobject of } Y.$$

Let us note immediately that in general this notion of  $(\mathcal{E}, \mathcal{M})$ -divisibility may not be transitive<sup>1</sup>. It is now natural to define an object  $M$  to be  $(\mathcal{E}, \mathcal{M})$ -*minimal* in the category, if it  $(\mathcal{E}, \mathcal{M})$ -divides all objects of the category. Note that there is no reason a priori that an  $(\mathcal{E}, \mathcal{M})$ -minimal object in a category, if it exists, be unique up to isomorphism. Nevertheless, in our case, when the category has both initial and a final object, we can state the following minimization lemma:

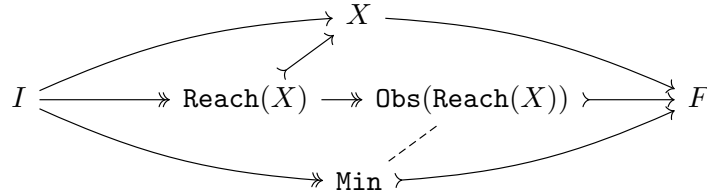
**Lemma 2.3.** *Let  $\mathcal{K}$  be a category with initial object  $I$  and final object  $F$  and let  $(\mathcal{E}, \mathcal{M})$  be a factorization system for  $\mathcal{K}$ . Define for every object  $X$ :*

- **Min** to be the factorization of the unique arrow from  $I$  to  $F$ ,
- **Reach**( $X$ ) to be the factorization of the unique arrow from  $I$  to  $X$ , and **Obs**( $X$ ) to be the factorization of the unique arrow from  $X$  to  $F$ .

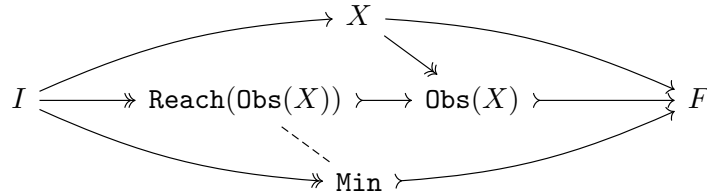
Then

- **Min** is  $(\mathcal{E}, \mathcal{M})$ -minimal, and
- **Min** is isomorphic to both **Obs**(**Reach**( $X$ )) and **Reach**(**Obs**( $X$ )) for all objects  $X$ .

*Proof.* The proof essentially consists of a diagram:



Using the definition of **Reach** and **Obs**, and the fact that  $\mathcal{E}$  is closed under composition, we obtain that **Obs**(**Reach**( $X$ )) is an  $(\mathcal{E}, \mathcal{M})$ -factorization of the unique arrow from  $I$  to  $F$ . Thus, thanks to the diagonal property of a factorization system, **Min** and **Obs**(**Reach**( $X$ )) are isomorphic. Hence, furthermore, since **Obs**(**Reach**( $X$ ))  $(\mathcal{E}, \mathcal{M})$ -divides  $X$  by construction, the same holds for **Min**. In a symmetric way, we have the next diagram:



This shows that **Reach**(**Obs**( $X$ )) is also isomorphic to **Min**. □

An object  $X$  of  $\mathcal{K}$  is called *reachable* when  $X$  is isomorphic to **Reach**( $X$ ). We denote by **Reach**( $\mathcal{K}$ ) the full subcategory of  $\mathcal{K}$  consisting of reachable objects. Similarly, an object  $X$  of  $\mathcal{K}$  is called *observable* when  $X$  is isomorphic to **Obs**( $X$ ). We denote by **Obs**( $\mathcal{K}$ ) the full subcategory of  $\mathcal{K}$  consisting of observable objects.

We can express reachability **Reach** and observability **Obs** as the right, respectively the left adjoint to the inclusion of **Reach**( $\mathcal{K}$ ), respectively of **Obs**( $\mathcal{K}$ ) into  $\mathcal{K}$ . It is indeed a standard fact that factorization systems give rise to reflective subcategories, see [9]. In

<sup>1</sup>There are nevertheless many situations for which it is the case; in particular when the category is regular, and  $\mathcal{E}$  happens to be the class of regular epis. This covers in particular the case of all algebraic categories with  $\mathcal{E}$ -quotients being the standard quotients of algebras, and  $\mathcal{M}$ -subobjects being the standard subalgebras.

our case, this is the reflective subcategory  $\mathbf{Obs}(\mathcal{K})$  of  $\mathcal{K}$ . By a dual argument, the category  $\mathbf{Reach}(\mathcal{K})$  is coreflective in  $\mathcal{K}$ . We can summarize these facts in the next lemma.

**Lemma 2.4.** *Let  $\mathcal{K}$  be a category with initial object  $I$  and final object  $F$  and let  $(\mathcal{E}, \mathcal{M})$  be a factorization system for  $\mathcal{K}$ . We have the adjunctions*

$$\begin{array}{ccc} \mathbf{Reach}(\mathcal{K}) & \begin{array}{c} \xrightarrow{\quad} \\ \perp \\ \xleftarrow{\quad} \end{array} & \mathcal{K} \\ & \text{Reach} & \\ & & \begin{array}{c} \xrightarrow{\text{Obs}} \\ \perp \\ \xleftarrow{\quad} \end{array} \\ & & \mathbf{Obs}(\mathcal{K}). \end{array} \quad (2.1)$$

In what follows we will instantiate  $\mathcal{K}$  with the category  $\mathbf{Auto}(\mathcal{L})$  of  $\mathcal{C}$ -automata accepting a language  $\mathcal{L}$ . Assuming the existence of an *initial* and a *final automaton* for  $\mathcal{L}$  – denoted by  $\mathcal{A}^{init}(\mathcal{L})$ , respectively  $\mathcal{A}^{final}(\mathcal{L})$  – and, of a factorization system, we obtain the functorial version of the usual notions of reachable sub-automaton  $\mathbf{Reach}(\mathcal{A})$  and observable quotient automaton  $\mathbf{Obs}(\mathcal{A})$  of an automaton  $\mathcal{A}$ . The minimal automaton  $\mathbf{Min}(\mathcal{L})$  for the language  $\mathcal{L}$  is obtained via the factorization

$$\mathcal{A}^{init}(\mathcal{L}) \twoheadrightarrow \mathbf{Min}(\mathcal{L}) \twoheadrightarrow \mathcal{A}^{final}(\mathcal{L}). \quad (2.2)$$

Lemma 2.3 implies that the minimal automaton divides any other automaton recognizing the language, while a particular instance of Lemma 2.4 pertaining to deterministic automata is given in [7, Section 9.4].

**Remark 2.5.** The duality between reachability and observability can be stated as the duality between  $\mathbf{Reach}(\mathcal{K})$  and  $\mathbf{Obs}(\mathcal{K}^{op})$ . Indeed, if we consider the factorization system  $(\mathcal{M}, \mathcal{E})$  on  $\mathcal{K}^{op}$ , then it immediately follows that  $\mathbf{Obs}(\mathcal{K}^{op})$  is isomorphic to  $\mathbf{Reach}(\mathcal{K})^{op}$ . Hence the two adjunctions from (2.1) are dual to each other.

**Remark 2.6** (Minimization via adjunctions). As a consequence of Lemma 2.3, minimization can be seen as an endofunctor  $\mathbf{Min}: \mathcal{K} \rightarrow \mathcal{K}$ , isomorphic to the functors obtained by considering any circuit in diagram (2.1).

We will come back to this observation of regarding minimization via adjunctions, in Section 5, where we will show how Brzozowski’s algorithms fits in the same conceptual approach, using however a longer chain of adjunctions.

**2.3. Minimization of  $\mathcal{C}$ -automata: sufficient conditions on  $\mathcal{C}$ .** In this section we provide sufficient conditions on  $\mathcal{C}$  so that the category  $\mathbf{Auto}(\mathcal{L})$  of  $\mathcal{C}$ -automata accepting a  $\mathcal{C}$ -language  $\mathcal{L}$  satisfies the three conditions of Lemma 2.3. The sufficient conditions on  $\mathcal{C}$  are as follows

- (1) completeness
- (2) cocompleteness
- (3) existence of a factorization system

In Corollary 2.10 below we show that when this conditions are satisfied then the initial and final automata for a language exist and the minimal automaton can be obtained via the factorization described in diagram (2.2).

**Remark 2.7.** Before proceeding to the technical details, a few remarks are in order.

- (1) First, this notion of minimization is parametric in the factorization system one chooses on  $\mathcal{C}$ .

- (2) Second, we emphasize that these conditions are only sufficient. In Section 4 we consider the example of sequential transducers and we instantiate  $\mathcal{C}$  with a Kleisli category. Although this category is not complete, the final automaton exists.
- (3) Finally, depending on the category  $\mathcal{I}$ , we may relax the conditions in Corollary 2.10, see Lemma 3.2. The reader may skip the rest of this section and consider Example 3.1.

We consider now the sufficient conditions on  $\mathcal{C}$  and we start with the factorization system. It is well known that given a factorization system  $(\mathcal{E}, \mathcal{M})$  on  $\mathcal{C}$ , we can extend it to a factorization system  $(\mathcal{E}_{[\mathcal{I}, \mathcal{C}]}, \mathcal{M}_{[\mathcal{I}, \mathcal{C}]})$  on the functor category  $[\mathcal{I}, \mathcal{C}]$  in a point-wise fashion. That is, a natural transformation is in  $\mathcal{E}_{[\mathcal{I}, \mathcal{C}]}$  if all its components are in  $\mathcal{E}$ , and analogously, a natural transformation is in  $\mathcal{M}_{[\mathcal{I}, \mathcal{C}]}$  if all its components are in  $\mathcal{M}$ . In turn, the factorization system on the functor category  $[\mathcal{I}, \mathcal{C}]$  induces a factorization system on its subcategory  $\text{Auto}(\mathcal{L})$  for an arbitrary language  $\mathcal{L}$ .

**Lemma 2.8.** *If  $\mathcal{C}$  has a factorization system  $(\mathcal{E}, \mathcal{M})$ , then  $\text{Auto}(\mathcal{L})$  has a factorization system  $(\mathcal{E}_{\text{Auto}(\mathcal{L})}, \mathcal{M}_{\text{Auto}(\mathcal{L})})$ , where  $\mathcal{E}_{\text{Auto}(\mathcal{L})}$  consists of all the natural transformations with components in  $\mathcal{E}$  and  $\mathcal{M}_{\text{Auto}(\mathcal{L})}$  consists of all natural transformations with components in  $\mathcal{M}$ .*

The proof of Lemma 2.8 is the same as the classical one that shows that factorization systems can be lifted to functor categories.

As for the existence of the initial and final automaton accepting a given language, we first notice that these can be stated in terms of Kan extensions, see [20].

**Lemma 2.9.** *If the left Kan extension  $\text{Lan}_\iota \mathcal{L}$  of  $\mathcal{L}$  along  $\iota$  exists, then it is an initial object in  $\text{Auto}(\mathcal{L})$ , that is,  $\mathcal{A}^{\text{init}}(\mathcal{L})$  exists and is isomorphic to  $\text{Lan}_\iota \mathcal{L}$ .*

*Dually, if the right Kan extension  $\text{Ran}_\iota \mathcal{L}$  of  $\mathcal{L}$  along  $\iota$  exists, then so does the final object  $\mathcal{A}^{\text{final}}(\mathcal{L})$  of  $\text{Auto}(\mathcal{L})$  and  $\mathcal{A}^{\text{final}}(\mathcal{L})$  is isomorphic to  $\text{Ran}_\iota \mathcal{L}$ .*

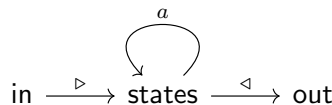
*Proof Sketch.* Assume the left Kan extension exists. Then the canonical natural transformation  $\mathcal{L} \rightarrow \text{Lan}_\iota \mathcal{L} \circ \iota$  is an isomorphism since  $\iota$  is full and faithful. Whenever  $\mathcal{A}$  accepts  $\mathcal{L}$ , that is,  $\mathcal{A} \circ \iota = \mathcal{L}$ , we obtain the required unique morphism  $\text{Lan}_\iota \mathcal{L} \rightarrow \mathcal{A}$  using the universal property of the Kan extension. The argument for the right Kan extension follows by duality.  $\square$

**Corollary 2.10.** *Assume  $\mathcal{C}$  is complete, cocomplete and has a factorization system and let  $\mathcal{L}$  be a  $\mathcal{C}$ -language. Then the initial  $\mathcal{L}$ -automaton and the final  $\mathcal{L}$ -automaton exist and are given by the left, respectively right Kan extensions of  $\mathcal{L}$  along  $\iota$ . Furthermore, the minimal  $\mathcal{C}$ -automaton  $\text{Min}(\mathcal{L})$  accepting  $\mathcal{L}$  is obtained via the factorization*

$$\text{Lan}_\iota \mathcal{L} \longrightarrow \text{Min}(\mathcal{L}) \triangleright \longrightarrow \text{Ran}_\iota \mathcal{L}.$$

### 3. WORD AUTOMATA

In Sections 3 to 5 we restrict our attention to the case of word automata, for which we recall the input category  $\mathcal{I}_{\text{word}}$  from Figure 1 (i.e., the three-object category with arrows spanned by  $\triangleright$ ,  $\triangleleft$  and  $a$  for all  $a \in A$ ) and its full subcategory  $\mathcal{O}_{\text{word}}$  on objects in and out.





We consider  $\mathcal{C}$ -languages, which are now functors  $\mathcal{L}: \mathcal{O}_{\text{word}} \rightarrow \mathcal{C}$ . If  $\mathcal{L}(\text{in}) = X$  and  $\mathcal{L}(\text{out}) = Y$  we call  $\mathcal{L}$  a  $(\mathcal{C}, X, Y)$ -*language*. Similarly, we consider  $\mathcal{C}$ -automata that are functors  $\mathcal{A}: \mathcal{I}_{\text{word}} \rightarrow \mathcal{C}$ . If  $\mathcal{A}(\text{in}) = X$  and  $\mathcal{A}(\text{out}) = Y$  we call  $\mathcal{A}$  a  $(\mathcal{C}, X, Y)$ -*automaton*.

**3.1. Minimization of word automata.** We first provide a couple of instances of the generic minimization results given in the previous section and then we show how they can be refined considering the particular structure of the category  $\mathcal{I}_{\text{word}}$ .

**Example 3.1.** (1) *Deterministic automata, i.e., (Set, 1, 2)-automata.* Since **Set** is complete and cocomplete, the initial and final automaton accepting a language  $\mathcal{L}$  can be computed as in Corollary 2.10.

- The initial automaton  $\mathcal{A}^{\text{init}}(\mathcal{L})$  is described in the next diagram.

$$1 \xrightarrow{\varepsilon} A^* \xrightarrow{\mathcal{L}^?} 2$$

Its state space is the set  $A^*$  of all words, the initial state is the empty word  $\varepsilon$ , the set of final states consist of the words belonging to the accepted language, and, for each input letter  $a$ , the transition map  $\delta_a$  is defined by  $w \mapsto wa$ .

- The final automaton  $\mathcal{A}^{\text{final}}(\mathcal{L})$  is described in the next diagram.

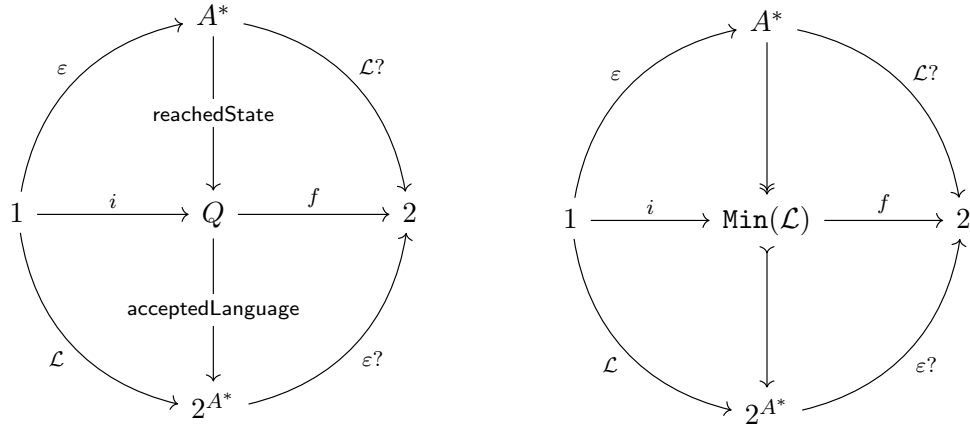
$$1 \xrightarrow{\mathcal{L}} 2^{A^*} \xrightarrow{\varepsilon^?} 2$$

Its state space is the set  $2^{A^*}$  of all languages, the initial state is the language accepted by the automaton, the final state consists of all the languages that contain the empty word, and, for each input letter  $a \in A$ , the transition map  $\delta_a$  is taking left quotients of a language by the letter  $a$ , that is,  $A^* \ni K \mapsto a^{-1}K = \{u \in A^* \mid au \in K\}$ .

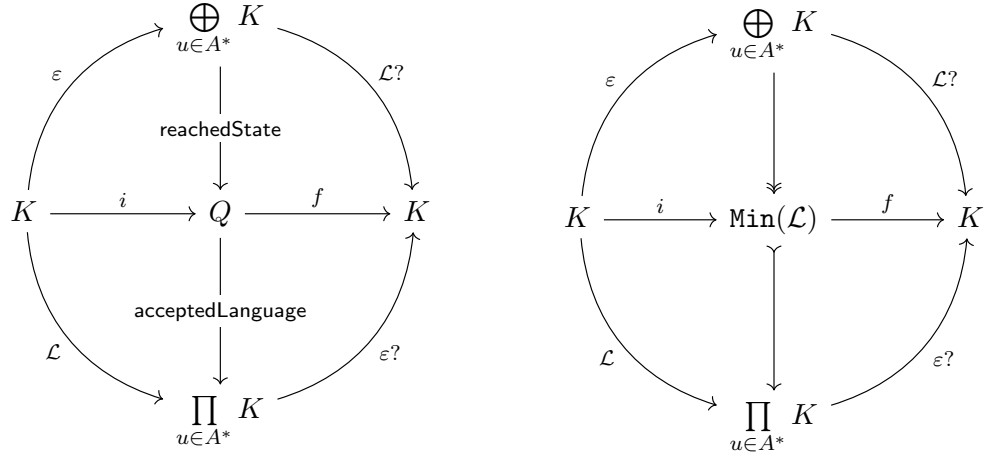
- The factorization system of  $\text{Auto}(\mathcal{L})$  is inherited from **Set**, consisting of surjective, respectively injective functions.

Indeed,  $\mathcal{A}^{\text{init}}(\mathcal{L})$  and  $\mathcal{A}^{\text{final}}(\mathcal{L})$  are the initial, respectively the final objects in the category  $\text{Auto}(\mathcal{L})$ . To see this, notice that for any other automaton  $\mathcal{A}$  in  $\text{Auto}(\mathcal{L})$  with state space  $\mathcal{A}(\text{states}) = Q$ , we have a unique morphism from  $\mathcal{A}^{\text{init}}(\mathcal{L})$  to  $\mathcal{A}$ , as described in the next diagram on the left. This is a natural transformation, determined by its component on the object **states**, i.e., by the function **reachedState**:  $A^* \rightarrow Q$ , which maps a word  $w \in A^*$  to the state of  $Q$  reached by reading  $w$ . Similarly, there is a unique automata morphism from  $\mathcal{A}$  to  $\mathcal{A}^{\text{final}}(\mathcal{L})$ , determined by the function **acceptedLanguage**:  $Q \rightarrow 2^{A^*}$  which maps a state  $q \in Q$  to the language accepted by  $q$ .

In particular, as shown in the next diagram (on the right), the unique morphism from  $\mathcal{A}^{\text{init}}(\mathcal{L})$  to  $\mathcal{A}^{\text{final}}(\mathcal{L})$  is determined by the function  $A^* \rightarrow 2^{A^*}$  which maps a word  $w \in A^*$  to the quotient language  $w^{-1}\mathcal{L}$ . If we factorize this map, we obtain the quotient of  $A^*$  by the syntactic equivalence  $\sim$  defined by  $w \sim w'$  if and only if  $w^{-1}\mathcal{L} = w'^{-1}\mathcal{L}$ , that is, we obtain the state space of the minimal automaton accepting  $\mathcal{L}$ .

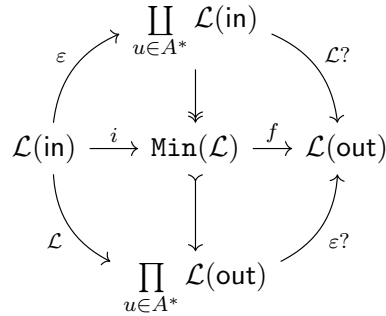


- (2) *Non-deterministic automata, i.e., (Rel, 1, 1)-automata.* The category Rel has countable products and coproducts, so the initial and final automata do exist and both have as state space the set  $A^*$ . However, the missing ingredient for obtaining a meaningful notion of minimal automaton in this case is a suitable factorization system.
- (3) *Weighted automata over a field  $K$ , i.e.,  $(K\text{-Mod}, K, K)$ -automata.* We have a similar situation depicted in the two diagrams below. The initial automaton has as state space the vector space of finitely supported functions from  $A^* \rightarrow K$ , while the final automaton has as state space the vector space of all functions  $A^* \rightarrow K$ . These are precisely the coproduct, respectively the product of  $A^*$  many copies of  $K$  in the category  $K\text{-Mod}$ . In the diagram below, the linear transformations  $\varepsilon$  and  $\varepsilon?$  are precisely the injection into the coproduct, respectively the projection from the product which correspond to the  $\varepsilon$ -component. The linear transformation, denoted (by an abuse) by  $\mathcal{L}$ , maps the unit of the field to the weighted language  $\mathcal{L}: A^* \rightarrow K$  accepted by the automaton. While the linear map  $\mathcal{L}?$  can be defined on a basis of  $\bigoplus_{u \in A^*} K$  as follows: for a given  $u \in A^*$ , it maps the unit of  $K$  of the  $u$ -component of the coproduct to  $\mathcal{L}(u)$ . If we factorize the unique linear transformation from  $\bigoplus_{u \in A^*} K$  to  $\prod_{u \in A^*} K$  we obtain precisely the vector space of the minimal automaton accepting  $\mathcal{L}$ .



The examples above are instances of the next generic lemma, which refines the statement of Corollary 2.10 taking into account the particular structure of the input category  $\mathcal{I}_{\text{word}}$ .

**Lemma 3.2** (from [12]). *If  $\mathcal{C}$  has countable products and countable coproducts, and a factorization system, then the minimal  $\mathcal{C}$ -automaton accepting  $\mathcal{L}$  is obtained via the factorization in the next diagram.*



The initial automaton  $\mathcal{A}^{\text{init}}(\mathcal{L})$  has as state space the copower  $\prod_{u \in A^*} \mathcal{L}(\text{in})$ . The map

$$\varepsilon = \mathcal{A}^{\text{init}}(\mathcal{L})(\triangleright): \mathcal{L}(\text{in}) \rightarrow \prod_{u \in A^*} \mathcal{L}(\text{in})$$

is the coproduct injection corresponding to  $\varepsilon \in A^*$ . The map

$$\mathcal{L}^? = \mathcal{A}^{\text{init}}(\mathcal{L})(\triangleleft): \prod_{u \in A^*} \mathcal{L}(\text{in}) \rightarrow \mathcal{L}(\text{out})$$

is given on the component of the coproduct corresponding to  $u \in A^*$  by  $\mathcal{L}(\triangleright u \triangleleft)$ . Lastly, for each  $a \in A$  the map  $\mathcal{A}^{\text{init}}(\mathcal{L})(a)$  is given on the component of the coproduct that corresponds to  $u \in A^*$  as the coproduct injection corresponding to the word  $ua$ .

The final automaton  $\mathcal{A}^{\text{final}}(\mathcal{L})$  can be obtained by a duality argument. It has as state space the power  $\prod_{u \in A^*} \mathcal{L}(\text{out})$ . The map

$$\mathcal{L} = \mathcal{A}^{\text{final}}(\mathcal{L})(\triangleright): \mathcal{L}(\text{in}) \rightarrow \prod_{u \in A^*} \mathcal{L}(\text{out})$$

is obtained using the universal property of the product by considering for each  $u \in A^*$  the map  $\mathcal{L}(\triangleright u \triangleleft): \mathcal{L}(\text{in}) \rightarrow \mathcal{L}(\text{out})$ . The map

$$\varepsilon^? = \mathcal{A}^{final}(\mathcal{L})(\triangleleft): \prod_{u \in A^*} \mathcal{L}(\text{out}) \rightarrow \mathcal{L}(\text{out})$$

is the projection corresponding to the  $\varepsilon$  component of the product. Lastly, for each  $a \in A$  the map  $\mathcal{A}^{final}(\mathcal{L})(a)$  is obtained by taking the product by  $u \in A^*$  of the projections  $\prod_{u \in A^*} \mathcal{L}(\text{out}) \rightarrow \mathcal{L}(\text{out})$  on the  $au$  component.

In [12] we gave a direct proof for the initiality of  $\mathcal{A}^{init}(\mathcal{L})$ . Here we can also notice that this is exactly the result of colimit computation of the left Kan extension of  $\mathcal{L}$  along  $\iota$  mentioned in Corollary 2.10. Indeed, we can use the fact that there are no morphisms from **out** to **states** in  $\mathcal{I}$  and the only morphism on which you take the colimit are of the form  $\triangleright w: \text{in} \rightarrow \text{states}$  for all  $w \in A^*$ .

For the final automaton, the proof follows by duality.

**Remark 3.3.** When the category  $\mathcal{C}$  has copowers, then a  $(\mathcal{C}, X, Y)$ -automaton gives rise to a pair  $(\alpha, f)$  consisting of an algebra  $\alpha: FQ \rightarrow Q$  for the functor  $F: \mathcal{C} \rightarrow \mathcal{C}$ ,  $FZ = X + A \cdot Z$  and of a morphism  $f: Q \rightarrow Y$ .

Dually, when the category  $\mathcal{C}$  has powers, then a  $(\mathcal{C}, X, Y)$ -automaton gives rise to a pair  $(\xi, i)$  consisting of a coalgebra  $\xi: Q \rightarrow HQ$  for the functor  $H: \mathcal{C} \rightarrow \mathcal{C}$ ,  $HZ = Y \times Z^A$  and of a morphism  $i: X \rightarrow Q$ . This is a mild generalization of the notion of pointed colagebras, see e.g. [1], where  $\mathcal{C}$  is assumed concrete over **Set** and the map  $i$  corresponds to selecting an element in  $Q$ . In [1] minimal automata are seen as *well-pointed coalgebras*. However, in Section 4 we will see an example of automata in a Kleisli category that does not have powers, and where the equivalence between the functorial approach and the coalgebraic one breaks.

**3.2. Lifting Adjunctions to Categories of Automata.** In Example 2.2 we have seen how languages  $L \subseteq A^*$  can be modeled as functors from  $\mathcal{O}_{\text{word}}$  to either **Set** or **Rel**, using the fact that  $\mathbf{Set}(1, 2) \simeq \mathbf{Rel}(1, 1) \simeq 2$ . We will see how the relationship between deterministic and non-deterministic automata, i.e.,  $(\mathbf{Set}, 1, 2)$ -automata and  $(\mathbf{Rel}, 1, 1)$ -automata, can be derived from the relationship between the categories **Set** and **Rel**, and, is a special instance of a more general phenomenon. This is the subject of the present section, in which we will juggle with languages and automata interpreted over different output categories connected via adjunctions.

Assume we have an adjunction between two categories  $\mathcal{C}$  and  $\mathcal{D}$

$$\mathcal{C} \begin{array}{c} \xrightarrow{F} \\ \perp \\ \xleftarrow{G} \end{array} \mathcal{D},$$

with  $F \dashv G: \mathcal{D} \rightarrow \mathcal{C}$ . Let  $(-)^*$  and  $(-)_*$  denote the induced natural isomorphisms between the homsets. In particular, given objects  $I$  in  $\mathcal{C}$  and  $O$  in  $\mathcal{D}$ , we have bijections

$$\mathcal{C}(I, GO) \begin{array}{c} \xrightarrow{(-)^*} \\ \xleftarrow{(-)_*} \end{array} \mathcal{D}(FI, O) \quad (3.1)$$

These bijections induce a one-to-one correspondence between  $(\mathcal{C}, I, GO)$ -languages and  $(\mathcal{D}, FI, O)$ -languages, which by an abuse of notation we denote by the same symbols:

$$(\mathcal{C}, I, GO)\text{-languages} \begin{array}{c} \xrightarrow{(-)^*} \\ \xleftarrow{(-)_*} \end{array} (\mathcal{D}, FI, O)\text{-languages}$$

Indeed, given a  $(\mathcal{C}, I, GO)$ -language  $\mathcal{L}: \mathcal{O} \rightarrow \mathcal{C}$  we obtain a  $(\mathcal{D}, FI, O)$ -language  $\mathcal{L}^*: \mathcal{O} \rightarrow \mathcal{D}$  by setting  $\mathcal{L}^*(\triangleright w \triangleleft) = (\mathcal{L}(\triangleright w \triangleleft))^* \in \mathcal{D}(FI, O)$ . Conversely, given a  $(\mathcal{D}, FI, O)$ -language  $\mathcal{L}'$  we obtain a  $(\mathcal{C}, I, GO)$ -language  $(\mathcal{L}')_*$  by setting  $(\mathcal{L}')_*(\triangleright w \triangleleft) = (\mathcal{L}'(\triangleright w \triangleleft))_*$ .

The next lemma shows how we can lift an adjunction between the output categories  $\mathcal{C}$  and  $\mathcal{D}$  to an adjunction between categories of automata that accept essentially the same language, admitting two equivalent representations in  $\mathcal{C}$  and  $\mathcal{D}$ .

**Lemma 3.4.** *Assume  $\mathcal{L}_{\mathcal{C}}$  and  $\mathcal{L}_{\mathcal{D}}$  are  $(\mathcal{C}, I, GO)$ -, respectively  $(\mathcal{D}, FI, O)$ -languages so that  $\mathcal{L}_{\mathcal{D}} = (\mathcal{L}_{\mathcal{C}})^*$ . Then the adjunction  $F \dashv G$  lifts to an adjunction  $\overline{F} \dashv \overline{G}: \text{Auto}(\mathcal{L}_{\mathcal{D}}) \rightarrow \text{Auto}(\mathcal{L}_{\mathcal{C}})$ . The lifted functors  $\overline{F}$  and  $\overline{G}$  are defined as  $F$ , resp.  $G$  on the state object, that is, the following diagram commutes*

$$\begin{array}{ccc} & \overline{F} & \\ \text{Auto}(\mathcal{L}_{\mathcal{C}}) & \xrightarrow{\quad} & \text{Auto}(\mathcal{L}_{\mathcal{D}}) \\ & \perp & \\ & \overline{G} & \\ & \xleftarrow{\quad} & \\ \downarrow \text{State} & & \downarrow \text{State} \\ \mathcal{C} & \begin{array}{c} \xrightarrow{\quad} \\ \perp \\ \xleftarrow{\quad} \\ G \end{array} & \mathcal{D} \end{array}$$

where the functor  $\text{State}: \text{Auto}(\mathcal{L}_{\mathcal{C}}) \rightarrow \mathcal{C}$  is the evaluation at states, that is, it sends an automaton  $\mathcal{A}: \mathcal{I}_{\text{word}} \rightarrow \mathcal{C}$  to  $\mathcal{A}(\text{states})$ .

*Proof sketch.* The functor  $\overline{F}$  maps an  $\mathcal{C}$ -automaton  $\mathcal{A}: \mathcal{I}_{\text{word}} \rightarrow \mathcal{C}$  from  $\text{Auto}(\mathcal{L}_{\mathcal{C}})$  to the  $\mathcal{D}$ -automaton  $\overline{F}\mathcal{A}: \mathcal{I}_{\text{word}} \rightarrow \mathcal{D}$  mapping  $\triangleright: \text{in} \rightarrow \text{states}$  to  $F(\mathcal{A}(\triangleright))$ ,  $a: \text{states} \rightarrow \text{states}$  to  $F(\mathcal{A}(a))$  and  $\triangleleft: \text{states} \rightarrow \text{out}$  to the adjoint transpose  $(\mathcal{A}(\triangleleft))^*: F\mathcal{A}(\text{states}) \rightarrow O$  of  $\mathcal{A}(\triangleleft): \mathcal{A}(\text{states}) \rightarrow GO$ . In a diagram

$$\begin{array}{ccc} \begin{array}{c} \mathcal{A}(a) \\ \curvearrowright \\ \mathcal{A}(\text{states}) \end{array} & \begin{array}{c} \xrightarrow{\mathcal{A}(\triangleright)} \\ \xrightarrow{\mathcal{A}(\triangleleft)} \end{array} & GO \\ \downarrow \text{State} & \xrightarrow{\overline{F}} & \downarrow \text{State} \\ \mathcal{C} & & \mathcal{D} \end{array}$$

The functor  $\overline{G}$  is defined similarly on an  $\mathcal{D}$ -automaton  $\mathcal{B}$ .

$$\begin{array}{ccc} \begin{array}{c} \mathcal{B}(a) \\ \curvearrowright \\ \mathcal{B}(\text{states}) \end{array} & \begin{array}{c} \xrightarrow{\mathcal{B}(\triangleright)} \\ \xrightarrow{\mathcal{B}(\triangleleft)} \end{array} & O \\ \downarrow \text{State} & \xrightarrow{\overline{G}} & \downarrow \text{State} \\ \mathcal{D} & & \mathcal{C} \end{array}$$

We show next that we have an isomorphism

$$\text{Auto}(\mathcal{L}_{\mathcal{D}})(\overline{F}\mathcal{A}, \mathcal{B}) \cong \text{Auto}(\mathcal{L}_{\mathcal{C}})(\mathcal{A}, \overline{G}\mathcal{B})$$

Indeed, consider a morphism  $\alpha: \overline{F}\mathcal{A} \rightarrow \mathcal{B}$  in  $\text{Auto}(\mathcal{L}_{\mathcal{D}})$ . We define a natural transformation  $\alpha_*: \mathcal{A} \rightarrow \overline{G}\mathcal{B}$  by setting its component at states as the adjoint transpose  $(\alpha_{\text{states}})_*$  of

$$\alpha_{\text{states}}: F\mathcal{A}(\text{states}) \rightarrow \mathcal{B}(\text{states}).$$

It is now easy to verify that  $\alpha_*$  is indeed an automata morphism in  $\text{Auto}(\mathcal{L}_{\mathcal{C}})$  and that the mapping  $\alpha \mapsto \alpha_*$  gives rise to the desired isomorphism.  $\square$

**3.3. Application: non-deterministic automata and (co)determinization.** As a first application of Lemma 3.4, we see how determinization of non-deterministic automata can be seen as a right adjoint to the inclusion of deterministic automata into non-deterministic ones. Similarly, codeterminization is a left adjoint to the inclusion of codeterministic automata into non-deterministic ones.

Given a language  $L \subseteq A^*$  we can model it in several equivalent ways: as a  $(\text{Set}, 1, 2)$ -language  $\mathcal{L}_{\text{Set}}$ , or as a  $(\text{Set}^{op}, 2, 1)$ -language  $\mathcal{L}_{\text{Set}^{op}}$ , or, lastly as a  $(\text{Rel}, 1, 1)$ -language  $\mathcal{L}_{\text{Rel}}$ . This is because we can model the fact  $w \in L$  using a morphisms in either of the three isomorphic homsets

$$\text{Set}(1, 2) \cong \text{Set}^{op}(2, 1) \cong \text{Rel}(1, 1) \cong 2. \quad (3.2)$$

These isomorphisms, can be seen in turn as arising from the next two adjunctions:

$$\begin{array}{ccccc} \text{Set} & \begin{array}{c} \xrightarrow{F_{\mathcal{P}}} \\ \perp \\ \xleftarrow{U_{\mathcal{P}}} \end{array} & \text{Rel} & \begin{array}{c} \xrightarrow{U_{\mathcal{P}}^{op}} \\ \perp \\ \xleftarrow{F_{\mathcal{P}}^{op}} \end{array} & \text{Set}^{op} \end{array}$$

The adjunction between  $\text{Set}$  and  $\text{Rel}$  is the Kleisli adjunction for the powerset monad:  $F_{\mathcal{P}}$  is identity on objects as maps a function  $f: X \rightarrow Y$  to itself  $f: X \dashrightarrow Y$ , but seen as a relation. The functor  $U_{\mathcal{P}}$  maps  $X$  to its powerset  $\mathcal{P}(X)$ , and a relation  $R: X \rightarrow Y$  to the function  $U_{\mathcal{P}}(R): \mathcal{P}(X) \rightarrow \mathcal{P}(Y)$  mapping  $A \subseteq X$  to  $\{y \in Y \mid \exists x \in X. (x, y) \in R\}$ . The adjunction between  $\text{Set}^{op}$  and  $\text{Rel}$  is the dual of the previous one, composed with the self-duality of  $\text{Rel}$ .

The isomorphisms of homsets from (3.2) can be rephrased as

$$\text{Set}(1, U_{\mathcal{P}}1) \cong \text{Rel}(F_{\mathcal{P}}1, 1) \quad \text{and} \quad \text{Set}^{op}(U_{\mathcal{P}}^{op}1, 1) \cong \text{Rel}(1, F_{\mathcal{P}}^{op}1).$$

In particular, we can regard  $\mathcal{L}_{\text{Set}}$  above as a  $(\text{Set}, 1, U_{\mathcal{P}}1)$ -language and  $\mathcal{L}_{\text{Rel}}$  as a  $(\text{Rel}, F_{\mathcal{P}}1, 1)$ -language. Using the notations from Lemma 3.4, we have that  $\mathcal{L}_{\text{Rel}} = (\mathcal{L}_{\text{Set}})^*$ . Similarly, we obtain that  $\mathcal{L}_{\text{Rel}} = (\mathcal{L}_{\text{Set}^{op}})_*$ .

Determinization and codeterminization (without restriction to reachable states as in the operations `determinize` and `codeterminize` introduced in Section 5) of a  $\text{Rel}$ -automaton can be seen as applications of Lemma 3.4 and are obtained by lifting the adjunctions between  $\text{Set}$ ,  $\text{Rel}$  and  $\text{Set}^{op}$  as in the next diagram. The left adjoint  $\overline{F_{\mathcal{P}}}$  transforms a deterministic automaton into a non-deterministic one, while the right adjoint  $\overline{U_{\mathcal{P}}}$  is the *determinization functor*. On the other hand, the left adjoint functor  $\overline{U_{\mathcal{P}}^{op}}$  is the *codeterminization functor*.

$$\begin{array}{ccccc}
& \xrightarrow{\overline{F_{\mathcal{P}}}} & & \xrightarrow{\overline{U_{\mathcal{P}}^{op}}} & \\
\text{Auto}(\mathcal{L}_{\text{Set}}) & \xrightarrow{\perp} & \text{Auto}(\mathcal{L}_{\text{Rel}}) & \xrightarrow{\perp} & \text{Auto}(\mathcal{L}_{\text{Set}^{op}}) \\
& \xleftarrow{\overline{U_{\mathcal{P}}}} & & \xleftarrow{\overline{F_{\mathcal{P}}^{op}}} & \\
\downarrow & & \downarrow & & \downarrow \\
\text{Set} & \xrightarrow{F_{\mathcal{P}}} & \text{Rel} & \xrightarrow{U_{\mathcal{P}}^{op}} & \text{Set}^{op} \\
& \xleftarrow{U_{\mathcal{P}}} & & \xleftarrow{F_{\mathcal{P}}^{op}} & \\
& \perp & & \perp & 
\end{array} \tag{3.3}$$

#### 4. CHOFFRUT'S MINIMIZATION OF SUBSEQUENTIAL TRANSDUCERS

In [10, 11] Choffrut establishes a minimality result for subsequential transducers, which are deterministic automata that output a word while processing their input. In this section, we show that this result can be established in the functorial framework of this paper.

We first present the model of subsequential transducers in Section 4.1, show how these can be identified with automata in the Kleisli category of a suitably chosen monad, and state the minimization result, Theorem 4.5. The subsequent sections provide the necessary material for proving the theorem.

**4.1. Subsequential transducers and automata in a Kleisli category.** Subsequential transducers are (finite state) machines that compute partial functions from input words in some alphabet  $A$  to output words in some other alphabet  $B$ . In this section, we recall the classical definition of these objects, and show how it can be phrased categorically.

**Definition 4.1.** A *subsequential transducer* is a tuple

$$T = (Q, A, B, q_0, t, u_0, (- \cdot a)_{a \in A}, (- * a)_{a \in A}),$$

where

- $A$  is the *input alphabet* and  $B$  the *output one*,
- $Q$  is a (finite) set of *states*.
- $q_0$  is either undefined or belongs to  $Q$  and is called the *initial state of the transducer*.
- $t: Q \rightarrow B^*$  is a *partial termination function*.
- $u_0 \in B^*$  is either undefined and is defined if and only if  $q_0$  is, and is the *initialization value*.
- $- \cdot a: Q \rightarrow Q$  is the *partial transition function for the letter  $a$* , for all  $a \in A$ .
- $- * a: Q \rightarrow B^*$  is the *partial production function for the letter  $a$*  for all  $a \in A$ ; it is required that  $q * a$  be defined if and only if  $(q \cdot a)$  is.

The subsequential transducer computes a partial function  $\llbracket T \rrbracket: A^* \rightarrow B^*$  defined as:

$$\llbracket T \rrbracket(a_1 \dots a_n) = u_0(q_0 * a_1)(q_1 * a_2) \dots (q_{n-1} * a_n)t(q_n) \quad \text{for all } a_1 \dots a_n \in A^*,$$

where each  $q_i$  is either undefined or belongs to  $Q$ , with  $q_0$  inherited from the definition of  $T$ , and  $q_i = q_{i-1} \cdot a_i$  for all  $i = 1 \dots n$ .

These subsequential transducers are modeled in our framework as automata in the category of free algebras for the monad  $\mathcal{T}$ , that we describe now.

**Definition 4.2.** The monad  $\mathcal{T}: \text{Set} \rightarrow \text{Set}$  is defined by

$$\mathcal{T}X = B^* \times X + 1$$

with unit  $\eta_X$  and multiplication  $\mu_X$  defined for all  $x \in X$  and  $w, u \in B^*$  as:

$$\begin{array}{ll} \eta_X: & X \rightarrow B^* \times X + 1 \\ & x \mapsto (\varepsilon, x) \\ \mu_X: & \mathcal{T}^2X \rightarrow \mathcal{T}X \\ & (w, (u, x)) \mapsto (wu, x) \\ & (w, \perp) \mapsto \perp \\ & \perp \mapsto \perp \end{array}$$

where we denote by  $\perp$  the unique element of  $1$  (used to model the partiality of functions).

Recall that the category of free  $\mathcal{T}$ -algebras, i.e., the *Kleisli category*  $\text{Kl}(\mathcal{T})$  for  $\mathcal{T}$ , has as objects sets  $X, Y, \dots$  and as *morphisms*  $f: X \dashrightarrow Y$  functions  $f: X \rightarrow B^* \times X + 1$  in  $\text{Set}$  (that is partial functions from  $X$  to  $B^* \times X + 1$ ).

Let  $T$  be a subsequential transducer. The initial state of the transducer  $q_0$  and the initialization value  $u_0$  together form a morphism  $i: 1 \dashrightarrow Q$  in the category  $\text{Kl}(\mathcal{T})$ . Similarly, the partial transition function and the partial production function for a letter  $a$  of the input alphabet  $A$  are naturally identified to Kleisli morphisms  $\delta_a: Q \dashrightarrow Q$  in  $\text{Kl}(\mathcal{T})$ . Finally, the partial termination function together with the partial production function are nothing but a Kleisli morphism of the form  $t: Q \dashrightarrow 1$ . To summarize, we obtained that a subsequential transducer  $T$  in the sense of [11] is specified by the following morphisms in  $\text{Kl}(\mathcal{T})$ :

$$\begin{array}{c} \delta_a \\ \curvearrowright \\ 1 \xrightarrow{i} Q \xrightarrow{t} 1 \end{array}$$

that is, by a functor  $\mathcal{A}_T: \mathcal{I}_{\text{word}} \rightarrow \text{Kl}(\mathcal{T})$  or equivalently, a  $(\text{Kl}(\mathcal{T}), 1, 1)$ -automaton. The subsequential function realized by the transducer  $T$  is a partial function  $A^* \dashrightarrow B^*$  and is fully captured by the  $(\text{Kl}(\mathcal{T}), 1, 1)$ -language  $\mathcal{L}_T: \mathcal{O}_{\text{word}} \rightarrow \text{Kl}(\mathcal{T})$  accepted by  $\mathcal{A}_T$ , which is obtained as  $\mathcal{A}_T \circ \iota$ . Indeed, this  $\text{Kl}(\mathcal{T})$ -language gives for each word  $w \in A^*$  a Kleisli morphism  $\mathcal{L}_T(\triangleright w \triangleleft): 1 \dashrightarrow 1$ , or equivalently, outputs for each word in  $A^*$  either a word in  $B^*$  or the undefined element  $\perp$ .

Putting all this together, we can state the following lemma, which validates the categorical encoding of subsequential transducers:

**Lemma 4.3.** *Subsequential transducers are in one to one correspondence with  $(\text{Kl}(\mathcal{T}), 1, 1)$ -automata, and partial maps from  $A^*$  to  $B^*$  are in one to one correspondence with  $(\text{Kl}(\mathcal{T}), 1, 1)$ -languages. Furthermore, the acceptance of languages is preserved under these bijections.*

**Remark 4.4.** The morphisms of  $(\text{Kl}(\mathcal{T}), 1, 1)$ -automata as in Definition 2.1 are very similar to the morphisms of subsequential structures provided in [16, Definition 3.4], with the sole exception that now we also have to take into account the initial states of the transducers. On the other hand, Choffrut [11] considers morphisms of transducers that may also output formal inverses of words in  $B^*$ . Nevertheless, as discussed in [16, Remark 3.12] and as follows from the development of the present paper, this is not necessary for minimization.

In the rest of this section we will see how to obtain Choffrut's minimization result as an application of Lemma 2.3. I.e., we have to provide in the category of  $(\text{Kl}(\mathcal{T}), 1, 1)$ -automata, (1) an initial object,



- (2) a final object, and,
- (3) a factorization system.

The existence of the initial transducer is addressed in Section 4.3, the one of the final transducer is the subject of Section 4.4. In Section 4.5 we show how to construct a factorization system. Putting together all these results, we obtain:

**Theorem 4.5** (Categorical version of [10,11]). *For every  $(\mathbf{Kl}(\mathcal{T}), 1, 1)$ -language, there exists a minimal  $(\mathbf{Kl}(\mathcal{T}), 1, 1)$ -automaton for it.*

Let us note that only the existence of the automaton is mentioned in this statement, and the way to compute it effectively is not addressed as opposed to Choffrut's work. Nevertheless, Lemma 2.3 describes what are the basic functions that have to be implemented, namely **Reach** and **Obs**.

The rest of this section is devoted to establish the three above mentioned points. Unfortunately, as it is usually the case with Kleisli categories,  $\mathbf{Kl}(\mathcal{T})$  is neither complete, nor cocomplete. It does not even have binary products, let alone countable powers. Also, the existence of a non-trivial factorization system does not generally hold in Kleisli categories. Hence, providing the above three pieces of information requires a bit of work.

In the next section we present an adjunction between  $(\mathbf{Kl}(\mathcal{T}), 1, 1)$ -automata and  $(\mathbf{Set}, 1, B^* + 1)$ -automata which is then used in the subsequent ones for proving the existence of initial and final automata. We finish the proof with a presentation of the factorization system.

**4.2. Back and forth to automata in set.** In order to understand what are the properties of the category of  $(\mathbf{Kl}(\mathcal{T}), 1, 1)$ -automata, an important tool will be the ability to see alternatively a subsequential transducer as an automaton in  $\mathbf{Kl}(\mathcal{T})$  as we have seen above, or as an automaton in **Set**, since **Set** is much better behaved than  $\mathbf{Kl}(\mathcal{T})$ . These two points of view are related through an adjunction, making use of the results of Section 3.2 and Lemma 2.8.

Indeed, we start from the well known adjunction between **Set** and  $\mathbf{Kl}(\mathcal{T})$ :

$$\begin{array}{ccc}
 & F_{\mathcal{T}} & \\
 \text{Set} & \begin{array}{c} \curvearrowright \\ \perp \\ \curvearrowleft \end{array} & \mathbf{Kl}(\mathcal{T}) \\
 & U_{\mathcal{T}} &
 \end{array} \quad (4.1)$$

We recall that the free functor  $F_{\mathcal{T}}$  is defined as the identity on objects, while for any function  $f: X \rightarrow Y$  the morphism  $F_{\mathcal{T}}f: X \rightarrow Y$  is defined as  $\eta_Y \circ f: X \rightarrow \mathcal{T}Y$ . For the other direction, the functor  $U_{\mathcal{T}}$  maps an object  $X$  in  $\mathbf{Kl}(\mathcal{T})$  to  $\mathcal{T}X$  and a morphism  $f: X \rightarrow Y$  (which is seen here as a function  $f: X \rightarrow \mathcal{T}Y$ ) to  $\mu_Y \circ \mathcal{T}f: \mathcal{T}X \rightarrow \mathcal{T}Y$ .

A simple, yet important observation is that the language of interest, which is a partial function  $L: A^* \rightarrow B^*$  can be modeled either as a  $(\mathbf{Kl}(\mathcal{T}), 1, 1)$ -language  $\mathcal{L}_{\mathbf{Kl}(\mathcal{T})}$ , or, as a  $(\mathbf{Set}, 1, B^* + 1)$ -language  $\mathcal{L}_{\mathbf{Set}}$ . This is because for each  $w \in A^*$  we can identify  $L(w)$  either

with an element of  $\text{Kl}(\mathcal{T})(1, 1)$  or, equivalently, as an element of  $\text{Set}(1, B^* + 1)$ .

$$\begin{array}{ll}
\mathcal{L}_{\text{Kl}(\mathcal{T})}: & \mathcal{O}_{\text{word}} \rightarrow \text{Kl}(\mathcal{T}) \\
& \text{in} \mapsto 1 \\
& \text{out} \mapsto 1 \\
& \triangleright w \triangleleft \mapsto L(w): 1 \dashv 1 \\
\mathcal{L}_{\text{Set}}: & \mathcal{O}_{\text{word}} \rightarrow \text{Set} \\
& \text{in} \mapsto 1 \\
& \text{out} \mapsto B^* + 1 \\
& \triangleright w \triangleleft \mapsto L(w): 1 \rightarrow B^* + 1
\end{array}$$

To see how this fits in the scope of Section 3.2, notice that  $\mathcal{L}_{\text{Kl}(\mathcal{T})}$  is an  $(\text{Kl}(\mathcal{T}), F_{\mathcal{T}}1, 1)$ -language, while  $\mathcal{L}_{\text{Set}}$  is an  $(\text{Set}, 1, U_{\mathcal{T}}1)$ -language and they correspond to each other via the bijections described in (3.1).

Applying Lemma 3.4 for the Kleisli adjunction (4.1) we obtain an adjunction  $\overline{F_{\mathcal{T}}} \dashv \overline{U_{\mathcal{T}}}$  between the categories of  $\text{Kl}(\mathcal{T})$ -automata for  $\mathcal{L}_{\text{Kl}(\mathcal{T})}$  and of  $\text{Set}$ -automata accepting  $\mathcal{L}_{\text{Set}}$ , as depicted in the diagram below. We will make heavy use of this correspondence in what follows.

$$\begin{array}{ccc}
\text{Auto}(\mathcal{L}_{\text{Set}}) & \begin{array}{c} \xrightarrow{\overline{F_{\mathcal{T}}}} \\ \perp \\ \xleftarrow{\overline{U_{\mathcal{T}}}} \end{array} & \text{Auto}(\mathcal{L}_{\text{Kl}(\mathcal{T})}) \\
\downarrow \text{State} & & \downarrow \text{State} \\
\text{Set} & \begin{array}{c} \xrightarrow{F_{\mathcal{T}}} \\ \perp \\ \xleftarrow{U_{\mathcal{T}}} \end{array} & \text{Kl}(\mathcal{T}).
\end{array}$$

**4.3. The initial  $\text{Kl}(\mathcal{T})$ -automaton for the language  $\mathcal{L}_{\text{Kl}(\mathcal{T})}$ .** The functor  $\overline{F_{\mathcal{T}}}$  is a left adjoint and consequently preserves colimits and in particular the initial object. We thus obtain that the initial  $\mathcal{L}_{\text{Kl}(\mathcal{T})}$ -automaton is  $\overline{F_{\mathcal{T}}}(\mathcal{A}^{\text{init}}(\mathcal{L}_{\text{Set}}))$ , where  $\mathcal{A}^{\text{init}}(\mathcal{L}_{\text{Set}})$  is the initial object of  $\text{Auto}(\mathcal{L}_{\text{Set}})$ . This automaton can be obtained by Lemma 3.2 as the functor  $\mathcal{A}^{\text{init}}(\mathcal{L}_{\text{Set}}): \mathcal{I}_{\text{word}} \rightarrow \text{Set}$  specified by  $\mathcal{A}^{\text{init}}(\mathcal{L}_{\text{Set}})(\text{states}) = A^*$  and for all  $a \in A$

$$\begin{array}{lll}
\mathcal{A}^{\text{init}}(\mathcal{L}_{\text{Set}})(\triangleright): 1 \rightarrow A^* & \mathcal{A}^{\text{init}}(\mathcal{L}_{\text{Set}})(\triangleleft): A^* \rightarrow B^* + 1 & \mathcal{A}^{\text{init}}(\mathcal{L}_{\text{Set}})(a): A^* \rightarrow A^* \\
0 \mapsto \varepsilon & w \mapsto L(w) & w \mapsto wa
\end{array}$$

Hence, by computing the image of  $\mathcal{A}^{\text{init}}(\mathcal{L}_{\text{Set}})$  under  $\overline{F_{\mathcal{T}}}$ , we obtain the following description of the initial  $\text{Kl}(\mathcal{T})$ -automaton  $\mathcal{A}^{\text{init}}(\mathcal{L}_{\text{Kl}(\mathcal{T})})$  accepting  $\mathcal{L}_{\text{Kl}(\mathcal{T})}$ :  $\mathcal{A}^{\text{init}}(\mathcal{L}_{\text{Kl}(\mathcal{T})})(\text{states}) = A^*$  and for all  $a \in A$

$$\begin{array}{lll}
\mathcal{A}^{\text{init}}(\mathcal{L}_{\text{Kl}(\mathcal{T})})(\triangleright): 1 \dashv A^* & \mathcal{A}^{\text{init}}(\mathcal{L}_{\text{Kl}(\mathcal{T})})(\triangleleft): A^* \dashv 1 & \mathcal{A}^{\text{init}}(\mathcal{L}_{\text{Kl}(\mathcal{T})})(a): A^* \dashv A^* \\
0 \mapsto (\varepsilon, \varepsilon) & w \mapsto L(w) & w \mapsto (\varepsilon, wa)
\end{array}$$

**4.4. The final  $\text{Kl}(\mathcal{T})$ -automaton for the language  $\mathcal{L}_{\text{Kl}(\mathcal{T})}$ .** The case of the final  $\text{Kl}(\mathcal{T})$ -automaton is more complicated, since it is not constructed as easily. However, assuming the final automaton exists, it has to be sent by  $\overline{U_{\mathcal{T}}}$  to a final  $\text{Set}$ -automaton, since  $\overline{U_{\mathcal{T}}}$  preserves limits. We shall see in Lemma 4.6 that  $\overline{U_{\mathcal{T}}}: \text{Auto}(\mathcal{L}_{\text{Kl}(\mathcal{T})}) \rightarrow \text{Auto}(\mathcal{L}_{\text{Set}})$  reflects final objects, and hence in order to prove that a given  $\text{Kl}(\mathcal{T})$ -automaton  $\mathcal{A}$  is a final object of  $\text{Auto}(\mathcal{L}_{\text{Kl}(\mathcal{T})})$  it suffices to show that  $\overline{U_{\mathcal{T}}}(\mathcal{A})$  is the final object in  $\text{Auto}(\mathcal{L}_{\text{Set}})$ . The proof of the following lemma generalizes the fact that  $U_{\mathcal{T}}$  reflects final objects and can be proved in the same spirit.

**Lemma 4.6.** *The functor  $\overline{U_{\mathcal{T}}}: \text{Auto}(\mathcal{L}_{\text{Kl}(\mathcal{T})}) \rightarrow \text{Auto}(\mathcal{L}_{\text{Set}})$  reflects final objects.*

*Proof.* Recall that we have the following two adjunctions for the categories  $\text{Kl}(\mathcal{T})$  of Kleisli algebras, respectively  $\text{EM}(\mathcal{T})$  of Eilenberg-Moore algebras, and the comparison functor  $K: \text{Kl}(\mathcal{T}) \rightarrow \text{EM}(\mathcal{T})$  between them.

$$\begin{array}{ccc}
 \text{Kl}(\mathcal{T}) & \xrightarrow{K} & \text{EM}(\mathcal{T}) \\
 & \searrow U_{\mathcal{T}} & \nearrow F^{\mathcal{T}} \\
 & & \text{Set} \\
 & \nearrow F_{\mathcal{T}} & \searrow U^{\mathcal{T}}
 \end{array} \tag{4.2}$$

The partial function  $L: A^* \rightarrow B^*$  from Section 4.2 can also be modeled as an  $(\text{EM}(\mathcal{T}), T1, T1)$ -language  $\mathcal{L}_{\text{EM}(\mathcal{T})}: \mathcal{O} \rightarrow \text{EM}(\mathcal{T})$ . Applying Lemma 3.4 for the adjunction  $F^{\mathcal{T}} \dashv U^{\mathcal{T}}$  we obtain an adjunction  $\overline{F^{\mathcal{T}}} \dashv \overline{U^{\mathcal{T}}}$  between the categories of  $\text{EM}(\mathcal{T})$ -automata for  $\mathcal{L}_{\text{EM}(\mathcal{T})}$  and of  $\text{Set}$ -automata for  $\mathcal{L}_{\text{Set}}$ . We also have a lifting  $\overline{K}: \text{Auto}(\mathcal{L}_{\text{Kl}(\mathcal{T})}) \rightarrow \text{Auto}(\mathcal{L}_{\text{EM}(\mathcal{T})})$  of the comparison functor  $K$ , which maps a  $\text{Kl}(\mathcal{T})$ -automaton  $\mathcal{A}$  to the  $\text{EM}(\mathcal{T})$ -automaton  $K \circ \mathcal{A}$ . We obtain the following situation, which is just a lifting of diagram (4.2) to the categories of automata.

$$\begin{array}{ccc}
 \text{Auto}(\mathcal{L}_{\text{Kl}(\mathcal{T})}) & \xrightarrow{\overline{K}} & \text{Auto}(\mathcal{L}_{\text{EM}(\mathcal{T})}) \\
 & \searrow \overline{U_{\mathcal{T}}} & \nearrow \overline{F^{\mathcal{T}}} \\
 & & \text{Auto}(\mathcal{L}_{\text{Set}}) \\
 & \nearrow \overline{F_{\mathcal{T}}} & \searrow \overline{U^{\mathcal{T}}}
 \end{array}$$

One can readily check that the functor  $\overline{U_{\mathcal{T}}}$  is the composite  $\overline{U^{\mathcal{T}}} \circ \overline{K}$ . The functor  $\overline{K}$  is full and faithful (a property inherited from  $K$ ) and thus reflects final objects. On the other hand, the final object in  $\text{Auto}(\mathcal{L}_{\text{EM}(\mathcal{T})})$  can be computed using Lemma 3.2, since the underlying category  $\text{EM}(\mathcal{T})$  has all limits. Moreover, this final automaton is the reflection of the final  $\text{Set}$ -automaton  $\mathcal{A}^{\text{final}}(\mathcal{L}_{\text{Set}})$ .  $\square$

We are now ready to describe the final  $\text{Kl}(\mathcal{T})$ -automaton. The final object in  $\text{Auto}(\mathcal{L}_{\text{Set}})$  is the automaton  $\mathcal{A}^{\text{final}}(\mathcal{L}_{\text{Set}})$  as described using Lemma 3.2. The functor  $\mathcal{A}^{\text{final}}(\mathcal{L}_{\text{Set}}): \mathcal{I} \rightarrow \text{Set}$

specified by

$$\begin{aligned} \mathcal{A}^{final}(\mathcal{L}_{\text{Set}})(\text{states}) &= (B^* + 1)^{A^*} & \mathcal{A}^{final}(\mathcal{L}_{\text{Set}})(\triangleleft) &: (B^* + 1)^{A^*} \rightarrow B^* + 1 \\ & & & K \mapsto K(\varepsilon) \\ \mathcal{A}^{final}(\mathcal{L}_{\text{Set}})(\triangleright) &: 1 \rightarrow (B^* + 1)^{A^*} & \mathcal{A}^{final}(\mathcal{L}_{\text{Set}})(a) &: (B^* + 1)^{A^*} \rightarrow (B^* + 1)^{A^*} \\ & 0 \mapsto L & & K \mapsto \lambda w. K(aw) \end{aligned}$$

To describe the set of states of the final automaton in  $\text{Auto}(\mathcal{L}_{\text{Kl}(\mathcal{T})})$  we need to introduce a few notations. Essentially we are looking for a set of states  $Q$  so that  $B^* \times Q + 1$  is isomorphic to  $(B^* + 1)^{A^*}$ . The intuitive idea is to decompose each function in  $K \in (B^* + 1)^{A^*}$  (except for the one which is nowhere defined, that is the function  $\kappa_{\perp} = \lambda w. \perp$ ) into a word in  $B^*$ , the common prefix of all the  $B^*$ -words in the image of  $K$ , and an irreducible function, i.e., a function such that the common prefix of all the words in the codomain is empty.

For  $v \in B^*$  and a function  $K \neq \kappa_{\perp}$  in  $(B^* + 1)^{A^*}$ , denote by  $v \star K$  the function defined for all  $u \in A^*$  by  $(v \star K)(u) = v K(u)$  if  $K(u) \in B^*$  and  $(v \star K)(u) = \perp$  otherwise.

Define also the *longest common prefix* of  $K$ ,  $\text{lcp}(K) \in B^*$ , as the longest word that is prefix of all  $K(u) \neq \perp$  for  $u$  in  $A^*$  (this is well defined since  $K \neq \kappa_{\perp}$ ). The *reduction* of  $K$ ,  $\text{red}(K)$ , is defined as:

$$\text{red}(K)(u) = \begin{cases} v & \text{if } K(u) = \text{lcp}(K) v, \\ \perp & \text{otherwise.} \end{cases}$$

Finally,  $K$  is called *irreducible* if  $\text{lcp}(K) = \varepsilon$  (or equivalently if  $K = \text{red}(K)$ ). We denote by  $\text{lrr}(A^*, B^*)$  the irreducible functions in  $(B^* + 1)^{A^*}$ .

What we have constructed is a bijection  $\varphi$  between

$$\mathcal{T}(\text{lrr}(A^*, B^*)) = B^* \times \text{lrr}(A^*, B^*) + 1 \quad \text{and} \quad (B^* + 1)^{A^*},$$

that is defined as

$$\begin{aligned} \varphi: B^* \times \text{lrr}(A^*, B^*) + 1 &\rightarrow (B^* + 1)^{A^*} \\ (u, K) &\mapsto u \star K \\ \perp &\mapsto \kappa_{\perp}, \end{aligned} \tag{4.3}$$

and the converse of which maps every  $K \neq \kappa_{\perp}$  to  $(\text{lcp}(K), \text{red}(K))$ , and  $\kappa_{\perp}$  to  $\perp$ .

Given  $a \in A$  and  $K \in (B^* + 1)^{A^*}$  we denote by  $a^{-1}K$  the function in  $(B^* + 1)^{A^*}$  that maps  $w \in A^*$  to  $K(aw)$ .

We can now define the automaton  $\mathcal{A}^{final}(\mathcal{L}_{\text{Kl}(\mathcal{T})}) : \mathcal{I} \rightarrow \text{Kl}(\mathcal{T})$  by setting

- $\mathcal{A}^{final}(\mathcal{L}_{\text{Kl}(\mathcal{T})})(\text{in}) = 1$ ,
- $\mathcal{A}^{final}(\mathcal{L}_{\text{Kl}(\mathcal{T})})(\text{states}) = \text{lrr}(A^*, B^*)$ , and,
- $\mathcal{A}^{final}(\mathcal{L}_{\text{Kl}(\mathcal{T})})(\text{out}) = 1$ ,

and defining  $\mathcal{A}^{final}(\mathcal{L}_{\text{Kl}(\mathcal{T})})$  on arrows as follows

$$\begin{aligned} \mathcal{A}^{final}(\mathcal{L}_{\text{Kl}(\mathcal{T})})(\triangleright) &: 1 \rightarrow \text{lrr}(A^*, B^*) & 0 &\mapsto (\text{lcp}(L), \text{red}(L)) \\ \mathcal{A}^{final}(\mathcal{L}_{\text{Kl}(\mathcal{T})})(\triangleleft) &: \text{lrr}(A^*, B^*) \rightarrow 1 & K &\mapsto K(\varepsilon) \\ \mathcal{A}^{final}(\mathcal{L}_{\text{Kl}(\mathcal{T})})(a) &: \text{lrr}(A^*, B^*) \rightarrow \text{lrr}(A^*, B^*) & K &\mapsto \begin{cases} \perp & \text{if } a^{-1}K = \kappa_{\perp}, \\ (\text{lcp}(a^{-1}K), \text{red}(a^{-1}K)) & \text{otherwise} \end{cases} \end{aligned}$$

**Lemma 4.7.** *The  $\text{Kl}(\mathcal{T})$ -automaton  $\mathcal{A}^{final}(\mathcal{L}_{\text{Kl}(\mathcal{T})})$  is a final object in  $\text{Auto}(\mathcal{L}_{\text{Kl}(\mathcal{T})})$ .*

*Proof.* We show that  $\overline{U_{\mathcal{T}}}(\mathcal{A}^{final}(\mathcal{L}_{\text{Kl}(\mathcal{T})}))$  is isomorphic to the final automaton  $\mathcal{A}^{final}(\mathcal{L}_{\text{Set}})$ . Indeed, at the level of objects the bijection between the sets  $\overline{U_{\mathcal{T}}}(\mathcal{A}^{final}(\mathcal{L}_{\text{Kl}(\mathcal{T})}))(\text{states})$  and  $\mathcal{A}^{final}(\mathcal{L}_{\text{Set}})(\text{states})$  is given by the function  $\varphi$  defined in (4.3). It is easy to check that also on arrows  $\overline{U_{\mathcal{T}}}(\mathcal{A}^{final}(\mathcal{L}_{\text{Kl}(\mathcal{T})}))$  is the same as  $\mathcal{A}^{final}(\mathcal{L}_{\text{Set}})$  up to the correspondence given by the function  $\varphi$ .  $\square$

**4.5. A factorization system on  $\text{Auto}(\mathcal{L}_{\text{Kl}(\mathcal{T})})$ .** The factorization system on  $\text{Auto}(\mathcal{L}_{\text{Kl}(\mathcal{T})})$  is obtained using Lemma 2.8 from a factorization system on  $\text{Kl}(\mathcal{T})$ . There are several non-trivial factorization systems on  $\text{Kl}(\mathcal{T})$ , one of which is obtained from the regular epi-mono factorization system on  $\text{Set}$ , or equivalently, from the regular epi-mono factorization system on the category of Eilenberg-Moore algebras for  $\mathcal{T}$ . Notice that this is a specific result for the monad  $\mathcal{T}$  since in general, there is no reason that the Eilenberg-Moore algebra obtained by factorizing a morphism between free algebras be free itself. Nevertheless, in order to capture precisely the syntactic transducer defined by Choffrut [10, 11], we will provide yet another factorization system  $(\mathcal{E}_{\text{Kl}(\mathcal{T})}, \mathcal{M}_{\text{Kl}(\mathcal{T})})$ , which we define concretely as follows. Given a morphism  $f: X \twoheadrightarrow Y$  in  $\text{Kl}(\mathcal{T})$  we write  $\pi_1(f): X \rightarrow B^* + \{\perp\}$  and  $\pi_2(f): X \rightarrow Y + \{\perp\}$  for the ‘projections’ of  $f$ , defined by

$$\pi_1(f)(x) = \begin{cases} u & \text{if } f(x) = (u, y), \\ \perp & \text{otherwise,} \end{cases} \quad \text{and} \quad \pi_2(f)(x) = \begin{cases} y & \text{if } f(x) = (u, y), \\ \perp & \text{otherwise.} \end{cases}$$

We say that a partial function  $g: X \rightarrow Y + \{\perp\}$  is surjective when for every  $y \in Y$  there exists  $x \in X$  so that  $g(x) = y$ .

The class  $\mathcal{E}_{\text{Kl}(\mathcal{T})}$  consists of all the morphisms of the form  $e: X \twoheadrightarrow Y$  such that  $\pi_2(e)$  is surjective and the class  $\mathcal{M}_{\text{Kl}(\mathcal{T})}$  consists of all the morphisms of the form  $m: X \twoheadrightarrow Y$  such that  $\pi_2(m)$  is injective and  $\pi_1(m)$  is the constant function mapping every  $x \in X$  to  $\varepsilon$ .

**Lemma 4.8.**  $(\mathcal{E}_{\text{Kl}(\mathcal{T})}, \mathcal{M}_{\text{Kl}(\mathcal{T})})$  is a factorization system on  $\text{Kl}(\mathcal{T})$ .

*Proof.* Notice that  $f$  is an isomorphism in  $\text{Kl}(\mathcal{T})$  if and only if  $f \in \mathcal{E}_{\text{Kl}(\mathcal{T})} \cap \mathcal{M}_{\text{Kl}(\mathcal{T})}$ .

If  $f: X \twoheadrightarrow Y$  is a morphism in  $\text{Kl}(\mathcal{T})$  then we can define

$$Z = \{y \in Y \mid \exists x \in X. \exists u \in B^*. f(x) = (u, y)\}.$$

We define  $e: X \twoheadrightarrow Z$  by  $e(x) = f(x)$  and  $m: Z \twoheadrightarrow Y$  by  $m(y) = (\varepsilon, y)$ . One can easily check that  $f = m \circ e$  in  $\text{Kl}(\mathcal{T})$ .

Lastly, we can show that the diagonal property holds. Assume we have a commuting square in  $\text{Kl}(\mathcal{T})$ .

$$\begin{array}{ccc} X & \xrightarrow{e} & Y \\ f \downarrow & \dashrightarrow d & \downarrow g \\ Z & \xrightarrow{m} & W \end{array}$$

We will prove the existence of  $d: Y \twoheadrightarrow Z$  so that  $d \circ e = f$  and  $m \circ d = g$ . Assume  $y \in Y$ . If  $g(y) = \perp$  we set  $d(y) = \perp$ . Otherwise assume  $g(y) = (v, t)$ , for some  $v \in B^*$  and  $t \in W$ . Since  $e \in \mathcal{E}_{\text{Kl}(\mathcal{T})}$ , there exists  $u \in B^*$  and  $x \in X$  so that  $e(x) = (u, y)$ . Assume  $f(x) = (w, z)$  for some  $w \in B^*$  and  $z \in Z$ . We set  $d(y) = (v, z)$ . First, we have to prove that this definition does not depend on the choice of  $x$ .

Assume that we have another  $x' \in X$  so that  $e(x') = (u', y)$  and assume  $f(x') = (w', z')$ . Using the fact that  $m \in \mathcal{M}_{\text{KI}(\mathcal{T})}$ , we will show that  $z = z'$ , and thus  $d(y)$  is well defined. Indeed, notice that

$$\begin{cases} g \circ e(x) = (uv, t) \\ g \circ e(x') = (u'v, t), \end{cases} \quad \text{or equivalently,} \quad \begin{cases} m \circ f(x) = (uv, t) \\ m \circ f(x') = (u'v, t), \end{cases}$$

Assume that  $m(z) = (\varepsilon, t_1)$  and  $m(z') = (\varepsilon, t_2)$ . This entails

$$\begin{cases} m \circ f(x) = (uv, t) = (w, t_1) \\ m \circ f(x') = (u'v, t) = (w', t_2). \end{cases}$$

We obtain that  $t_1 = t_2 = t$ . Since  $m \in \mathcal{M}_{\text{KI}(\mathcal{T})}$  (and thus  $\pi_2(m)$  is injective) we get that  $z = z'$ , which is what we wanted to prove. It is easy to verify that  $d \circ e = f$  and  $m \circ d = g$ .  $\square$

This completes the proof of Theorem 4.5.

## 5. BRZOWSKI'S MINIMIZATION ALGORITHM

**5.1. Presentation.** *Brzowski's algorithm* is a minimization algorithm for automata. It takes as input a non-deterministic automaton  $\mathcal{A}$ , and computes the deterministic automaton:

$$\text{determinize}(\text{transpose}(\text{determinize}(\text{transpose}(\mathcal{A}))))),$$

in which

- **determinize** is the operation from classical automata theory that takes as input a deterministic automaton, applies a powerset construction and at the same time restricts to the reachable states, yielding a deterministic automaton, and
- **transpose** is the operation that takes as input a non-deterministic automaton reverses all its edges, and swaps the role of initial and final states (it accepts the mirrored language).

In this section, we will establish the correctness of Brzowski's algorithm: this sequence of operations yields the minimal automaton for the language. For easing the presentation we shall present the algorithm in the form:

$$\text{determinize}(\text{codeterminize}(\mathcal{A})),$$

in which **codeterminize** is the operation that takes a non-deterministic automaton, and constructs a backward deterministic one (it is equivalent to the sequence **transpose**  $\circ$  **determinize**  $\circ$  **transpose**).

In the next section, we show how **determinize** and **codeterminize** can be seen as adjunctions, and we use it immediately after in a correctness proof of Brzowski's algorithm.

We will use the representation of non-deterministic automata as  $(\text{Rel}, 1, 1)$ -automata (see Example 2.2) and the fact that determinization, respectively codeterminization, can be seen as right, respectively left, adjoints, as discussed in Section 3.3.

**5.2. Brzozowski's minimization algorithm.** The correctness of Brzozowski's algorithm can be seen in the following chain of adjunctions from Lemma 2.4 and diagram (3.3) (that all correspond to equivalences at the level of languages):

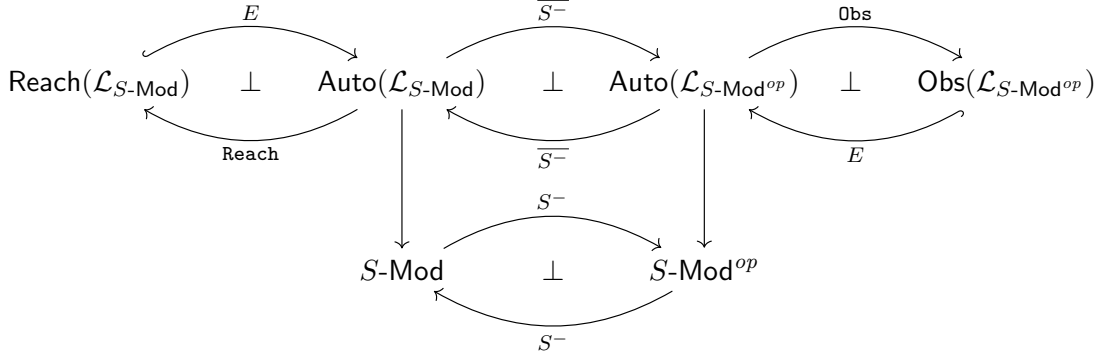
$$\begin{array}{ccccccc}
 & \xrightarrow{E} & & \xrightarrow{\overline{F_{\mathcal{P}}}} & & \xrightarrow{\overline{U_{\mathcal{P}}^{op}}} & & \xrightarrow{\text{Obs}} \\
 \text{Reach}(\mathcal{L}_{\text{Set}}) & \perp & \text{Auto}(\mathcal{L}_{\text{Set}}) & \perp & \text{Auto}(\mathcal{L}_{\text{Rel}}) & \perp & \text{Auto}(\mathcal{L}_{\text{Set}^{op}}) & \perp & \text{Obs}(\mathcal{L}_{\text{Set}^{op}}) \\
 & \xleftarrow{\text{Reach}} & & \xleftarrow{\overline{U_{\mathcal{P}}}} & & \xleftarrow{\overline{F_{\mathcal{P}}^{op}}} & & \xleftarrow{E}
 \end{array}$$

A path in this diagram corresponds to a sequence of transformations of automata. It happens that when **Obs** is taken, the resulting automaton is observable, i.e., there is an injection from it to the final object. This property is preserved under the sequence of right adjoints  $\text{Reach} \circ \overline{U_{\mathcal{P}}} \circ \overline{F_{\mathcal{P}}^{op}} \circ E$ . Furthermore, after application of **Reach**, the automaton is also reachable. This means that applying the sequence  $\text{Reach} \circ \overline{U_{\mathcal{P}}} \circ \overline{F_{\mathcal{P}}^{op}} \circ E \circ \text{Obs} \circ \overline{U_{\mathcal{P}}^{op}}$  to a non-deterministic automaton produces a deterministic and minimal one for the same language. We check for concluding that the sequence  $\text{Obs} \circ \overline{U_{\mathcal{P}}^{op}}$  is what is implemented by **codeterminize**, that the composite  $\overline{F_{\mathcal{P}}^{op}} \circ E$  essentially transforms a backward deterministic observable automaton into a non-deterministic one, and that finally  $\text{Reach} \circ \overline{U_{\mathcal{P}}}$  is what is implemented by **determinize**. Hence, this indeed is Brzozowski's algorithm.

**Remark 5.1.** The composite of the two adjunctions in (3.3) is almost the adjunction of [7, Corollary 9.2] upon noticing that the category  $\text{Auto}(\mathcal{L}_{\text{Set}^{op}})$  of  $\text{Set}^{op}$ -automata accepting a language  $\mathcal{L}_{\text{Set}^{op}}$  is isomorphic to the opposite of the category  $\text{Auto}(\mathcal{L}_{\text{Set}}^{\text{rev}})$  of  $\text{Set}$ -automata that accept the reversed language seen as functor  $\mathcal{L}_{\text{Set}^{op}}$ . This observation in turn can be proved using the symmetry of the input category  $\mathcal{I}$ .

**5.3. Weighted Brzozowski's minimization algorithm.** The weighted version of Brzozowski's minimization algorithm presented in [7] can also be explained in our framework using the chain of adjunctions described in the next diagram. Given a semiring  $S$ , a weighted language  $L: A^* \rightarrow S$ , can be modeled either as a functor  $\mathcal{L}_{S\text{-Mod}}: \mathcal{O}_{\text{word}} \rightarrow S\text{-Mod}$  or, equivalently, as a functor  $\mathcal{L}_{S\text{-Mod}^{op}}: \mathcal{O}_{\text{word}} \rightarrow S\text{-Mod}^{op}$ , using the observation that  $S\text{-Mod}(S, S) \cong S\text{-Mod}^{op}(S, S) \cong S$ . Notice that the category  $\text{Auto}(\mathcal{L}_{S\text{-Mod}^{op}})$  is the opposite of the category of automata accepting the reversed language  $L^{\text{rev}}$ , defined by  $L^{\text{rev}}(w) = L(w^{\text{rev}})$ . The adjunction between  $S\text{-Mod}$  and its opposite obtained by taking the dual spaces, lifts by virtue of Lemma 3.4 to an adjunction between the corresponding categories of automata, where the lifting  $\overline{S^-}$  is essentially reversing the automaton. Thus the operations of the weighted Brzozowski's algorithm correspond to a path  $E \circ \text{Reach} \circ \overline{S^-} \circ E \circ \text{Obs} \circ \overline{S^-}$

in the next diagram.



## 6. MONOIDS FOR LANGUAGE RECOGNITION

In this section we show that the notion of a syntactic monoid for a given language  $L \subseteq A^*$  fits in the functorial framework introduced in this paper. We argue that the syntactic monoid can be obtained using the generic principles outlined in Section 2 by changing accordingly the input category. However, it is not the monoids recognizing a language that will be modeled as Set-valued functors, but rather biaction recognizers. We will prove that we have initial and final biactions recognizing a language and that the minimal biaction recognizing a language (obtained via an epi-mono factorization) can be in fact equipped with a monoid structure and yields precisely the syntactic monoid for that language.

We start with the definitions of monoid and biaction recognizers.

**Definition 6.1.** We call *monoid recognizer* a tuple  $(\phi: A^* \rightarrow M, P \subseteq M)$  consisting of a monoid homomorphism  $\phi$  and a subset  $P$  of  $M$ . It recognizes the language  $\{w \in A^* \mid \phi(w) \in P\}$ . We say that a monoid recognizer  $(\phi: A^* \rightarrow M, P \subseteq M)$  is a *surjective monoid recognizer* when the morphism  $\phi$  is surjective. A morphism between monoid recognizers  $(\phi: A^* \rightarrow M, P \subseteq M)$  and  $(\phi': A^* \rightarrow M', P' \subseteq M')$  is a monoid morphism  $h: M \rightarrow M'$  such that  $h \circ \phi = \phi'$  and  $h^{-1}(P') = P$ .

**Definition 6.2.** An  *$A^*$ -biaction* is a set  $X$  equipped with left and right  $A^*$ -actions<sup>2</sup>  $\cdot: A^* \times X \rightarrow X$  and  $\cdot: X \times A^* \rightarrow X$  which commute, that is,  $(u \cdot x) \cdot v = u \cdot (x \cdot v)$  for all  $u, v \in A^*$  and  $x \in X$ . Morphisms of  $A^*$ -biactions are functions that are morphisms of both the left and right  $A^*$ -actions.

**Definition 6.3.** An  *$A^*$ -biaction recognizer* is a tuple  $(\phi: A^* \rightarrow X, P \subseteq X)$  where  $\phi$  is a morphism of  $A^*$ -biactions and  $P$  is a subset of  $X$ . We call the elements of  $P$ , the *accepting elements* of  $X$ . The language recognized by  $(\phi: A^* \rightarrow X, P \subseteq X)$  is the set  $\{w \in A^* \mid \phi(w) \in P\}$ . The  $A^*$ -biaction recognizer is called *surjective* when  $\phi$  is so. A morphism between  $A^*$ -biaction recognizers  $(\phi: A^* \rightarrow X, P \subseteq M)$  and  $(\phi': A^* \rightarrow X', P' \subseteq M')$  is an  $A^*$ -biaction morphism  $h: X \rightarrow X'$  such that  $h \circ \phi = \phi'$  and  $h^{-1}(P') = P$ .

The proof of the next lemma is straightforward. The second part was used in [14].

**Lemma 6.4.** *Any monoid recognizer is an  $A^*$ -biaction recognizer. Conversely, any surjective  $A^*$ -biaction recognizer is a surjective monoid recognizer.*

<sup>2</sup>both denoted by  $\cdot$  by abuse of notation



In order to describe  $A^*$ -biactions as functors, we will consider the category  $\mathcal{I}_{\text{Mon}}$  described in the diagram below. Just as the input category for word automata,  $\mathcal{I}_{\text{Mon}}$  has three objects: in, states and out. The homsets in this category can be intuitively described as follows:

- $\mathcal{I}_{\text{Mon}}(\text{in}, \text{states})$  and  $\mathcal{I}_{\text{Mon}}(\text{in}, \text{out})$  are both isomorphic to the set of finite words over  $A$ ;
- the sets  $\mathcal{I}_{\text{Mon}}(\text{states}, \text{states})$  and  $\mathcal{I}_{\text{Mon}}(\text{states}, \text{out})$  consist of the finite words over  $A$  with one “hole”.

Concretely, for each  $w \in A^*$  we have morphisms  $\mathbf{w}: \text{in} \rightarrow \text{states}$  and  $\bar{\mathbf{w}}: \text{in} \rightarrow \text{out}$ . For every two words  $u, v \in A^*$  we have morphisms  $\mathbf{u}\square\mathbf{v}: \text{states} \rightarrow \text{states}$  and  $\bar{\mathbf{u}}\square\bar{\mathbf{v}}: \text{states} \rightarrow \text{out}$ .

$$\begin{array}{ccc}
 & \text{u}\square\mathbf{v} & \\
 & \downarrow & \\
 \text{in} & \xrightarrow{\mathbf{w}} & \text{states} \xrightarrow{\mathbf{u}'\square\mathbf{v}'} & \text{out} \\
 & \searrow & \uparrow & \\
 & & \bar{\mathbf{w}}' & 
 \end{array} \tag{6.1}$$

The composition defined as a substitution of the  $\square$  symbol. We define it formally as follows:

$$\begin{aligned}
 \mathbf{u}\square\mathbf{v} \circ \mathbf{w} &= \mathbf{uwv}: \text{in} \rightarrow \text{states}, & \mathbf{u}'\square\mathbf{v}' \circ \mathbf{u}\square\mathbf{v} &= \mathbf{u}'\mathbf{u}\square\mathbf{v}\mathbf{v}': \text{states} \rightarrow \text{states}, \\
 \bar{\mathbf{u}}\square\bar{\mathbf{v}} \circ \mathbf{w} &= \bar{\mathbf{u}\bar{\mathbf{w}}\bar{\mathbf{v}}}: \text{in} \rightarrow \text{out}, & \bar{\mathbf{u}}\square\bar{\mathbf{v}} \circ \mathbf{u}\square\mathbf{v} &= \bar{\mathbf{u}'\mathbf{u}\square\mathbf{v}\mathbf{v}'}: \text{states} \rightarrow \text{out}
 \end{aligned}$$

**Definition 6.5.** An  $(\mathcal{I}_{\text{Mon}}, \text{Set}, 1, 2)$ -automaton is a functor  $\mathcal{A}: \mathcal{I}_{\text{Mon}} \rightarrow \text{Set}$  such that  $\mathcal{A}(\text{in}) = 1$  and  $\mathcal{A}(\text{out}) = 2$ . A morphism of  $(\mathcal{I}_{\text{Mon}}, \text{Set}, 1, 2)$ -automata is a natural transformation  $\alpha: \mathcal{A} \rightarrow \mathcal{B}$  so that both  $\alpha_{\text{in}}$  and  $\alpha_{\text{out}}$  are the identity morphisms on 1, respectively 2.

Let  $\mathcal{O}_{\text{Mon}}$  be the full subcategory of  $\mathcal{I}_{\text{Mon}}$  on objects in and out. We denote by  $\iota$  the inclusion

$$\mathcal{O}_{\text{Mon}} \xhookrightarrow{\iota} \mathcal{I}_{\text{Mon}}.$$

The language accepted by an  $(\mathcal{I}_{\text{Mon}}, \text{Set}, 1, 2)$ -automaton  $\mathcal{A}: \mathcal{I}_{\text{Mon}} \rightarrow \text{Set}$  is the composite functor  $\mathcal{L} = \mathcal{A} \circ \iota: \mathcal{O}_{\text{Mon}} \rightarrow \text{Set}$ . Just as for word automata, the functor  $\mathcal{L}$  encodes a language  $L \subseteq A^*$  consisting of the words  $w \in A^*$  such that the map  $\mathcal{L}(\bar{\mathbf{w}}): 1 \rightarrow 2$  is constant to 1, that is,  $\mathcal{L}(\bar{\mathbf{w}})(0) = 1$ .

**Lemma 6.6.** *The category of  $(\mathcal{I}_{\text{Mon}}, \text{Set}, 1, 2)$ -automata is equivalent to that of  $A^*$ -bifunction recognizers.*

*Proof.* Consider an  $(\mathcal{I}_{\text{Mon}}, \text{Set}, 1, 2)$ -automaton  $\mathcal{A}: \mathcal{I}_{\text{Mon}} \rightarrow \text{Set}$ . Then, the set  $Q = \mathcal{A}(\text{states})$  can be equipped with commuting left and right  $A^*$ -actions. Indeed, we define the left action  $\cdot: A^* \times Q \rightarrow Q$  by  $u \cdot q = \mathcal{A}(\mathbf{u}\square\epsilon)(q)$  and the right action  $\cdot: Q \times A^* \rightarrow Q$  by  $q \cdot v = \mathcal{A}(\epsilon\square\mathbf{v})(q)$ . We define  $\phi: A^* \rightarrow Q$  by  $\phi(w) = \mathcal{A}(\bar{\mathbf{w}})(0)$ .

One can check that  $\mathcal{A}$  being a functor entails that these are well defined and commuting left and right actions and that  $\phi$  is a morphism of biactions. Furthermore, we define the subset of accepting elements of  $Q$  as the subset whose characteristic function is the morphism  $\mathcal{A}(\bar{\epsilon}\square\bar{\epsilon}): Q \rightarrow 2$ .

Conversely, given an  $A^*$ -bifunction recognizer  $(\phi: A^* \rightarrow X, P \subseteq X)$  we define  $\mathcal{A}: \mathcal{I}_{\text{Mon}} \rightarrow \text{Set}$  as follows. We define  $\mathcal{A}(\text{states}) = X$  and we put

- $\mathcal{A}(\bar{\mathbf{w}})(0) = \phi(w)$ ;
- $\mathcal{A}(\mathbf{u}\square\mathbf{v})(x) = \phi((u \cdot x) \cdot v)$ ;
- $\mathcal{A}(\bar{\mathbf{u}}\square\bar{\mathbf{v}})(x) = \chi_P((u \cdot x) \cdot v)$ ;

where  $\chi_P$  is the characteristic function of  $P$ . □

**Remark 6.7.** The functors constructed in the proof of Lemma 6.6 preserve the accepted language, that is, for each language  $L \subseteq A^*$  we obtain an equivalence between the categories of  $(\mathcal{I}_{\text{Mon}}, \text{Set}, 1, 2)$ -automata accepting  $L$ , respectively of  $A^*$ -baction recognizers for  $L$ .

The following lemma immediately follows from Corollary 2.10.

**Lemma 6.8.** *Given a language  $\mathcal{L}: \mathcal{O}_{\text{Mon}} \rightarrow \text{Set}$ , the initial and final  $(\mathcal{I}_{\text{Mon}}, \text{Set}, 1, 2)$ -automata accepting  $\mathcal{L}$  exist and can be computed as left, respectively right Kan extensions of  $\mathcal{L}$  along  $\iota: \mathcal{O}_{\text{Mon}} \rightarrow \mathcal{I}_{\text{Mon}}$ .*

*Furthermore, the minimal  $(\mathcal{I}_{\text{Mon}}, \text{Set}, 1, 2)$ -automaton accepting  $\mathcal{L}$  is obtained via the factorization*

$$\text{Lan}_\iota \mathcal{L} \longrightarrow \text{Min}(\mathcal{L}) \twoheadrightarrow \text{Ran}_\iota \mathcal{L}.$$

Using the colimit computation of the left Kan extension  $\text{Lan}_\iota \mathcal{L}$ , we obtain the concrete description of the initial automaton  $\mathcal{A}^{\text{init}}(\mathcal{L})$  accepting  $\mathcal{L}$ . We have that

$$\mathcal{A}^{\text{init}}(\mathcal{L})(\text{states}) = \coprod_{u \in A^*} 1 \simeq A^*$$

and for all  $w, u, v \in A^*$  we have

- $\mathcal{A}^{\text{init}}(\mathcal{L})(\bar{w}): 1 \rightarrow A^*$ ,  $0 \in 1 \mapsto w \in A^*$ ;
- $\mathcal{A}^{\text{init}}(\mathcal{L})(\bar{u \square v}): A^* \rightarrow A^*$ ,  $w \mapsto u w v$ ;
- $\mathcal{A}^{\text{init}}(\mathcal{L})(\bar{u \square \bar{v}}): A^* \rightarrow 2$ ,  $w \mapsto \mathcal{L}(\bar{u w \bar{v}})(0)$ .

Dually, using the limit computation of the right Kan extension  $\text{Ran}_\iota \mathcal{L}$ , we obtain the concrete description of the final automaton  $\mathcal{A}^{\text{final}}(\mathcal{L})$  accepting  $\mathcal{L}$ . We have that

$$\mathcal{A}^{\text{final}}(\mathcal{L})(\text{states}) = \prod_{u, v \in A^*} 2 \simeq 2^{A^* \times A^*}$$

and for all  $w, u, v \in A^*$  we have

- $\mathcal{A}^{\text{final}}(\mathcal{L})(\bar{w}): 1 \rightarrow 2^{A^* \times A^*}$ ,  $0 \in 1 \mapsto \{(u, v) \in A^* \times A^* \mid \mathcal{L}(\bar{u w \bar{v}})(0) = 1\}$ ;
- $\mathcal{A}^{\text{final}}(\mathcal{L})(\bar{u \square v}): 2^{A^* \times A^*} \rightarrow 2^{A^* \times A^*}$ ,  $B \mapsto \{(u', v') \in A^* \times A^* \mid (u u', v' v) \in B\}$ ;
- $\mathcal{A}^{\text{final}}(\mathcal{L})(\bar{u \square \bar{v}}): 2^{A^* \times A^*} \rightarrow 2$ ,  $B \mapsto \begin{cases} 1, & (u, v) \in B \\ 0, & \text{otherwise} \end{cases}$

We thus obtain a diagram similar to that for word automata below Lemma 3.2.

$$\begin{array}{ccc}
 & \prod_{u \in A^*} 1 & \\
 & \downarrow & \\
 1 & \xrightarrow{i} \text{Min}(\mathcal{L}) \xrightarrow{f} 2 & \\
 & \downarrow & \\
 & \prod_{(u, v) \in A^* \times A^*} 2 & \\
 & \uparrow & \\
 & & 
 \end{array}
 \begin{array}{l}
 \text{curved arrow } w: 1 \rightarrow \prod_{u \in A^*} 1 \\
 \text{curved arrow } \mathcal{L}(u-v)?: \prod_{u \in A^*} 1 \rightarrow 2 \\
 \text{curved arrow } \mathcal{L}(-w-): 1 \rightarrow \prod_{(u, v) \in A^* \times A^*} 2 \\
 \text{curved arrow } (u, v)?: \prod_{(u, v) \in A^* \times A^*} 2 \rightarrow 2
 \end{array}
 \tag{6.2}$$

The unique natural transformation  $\alpha$  from the initial automaton  $\mathcal{A}^{\text{init}}(\mathcal{L})$  to the final automaton  $\mathcal{A}^{\text{final}}(\mathcal{L})$  is determined by the function  $\alpha_{\text{states}}: A^* \rightarrow 2^{A^* \times A^*}$  defined by

$$w \in A^* \mapsto \{(u, v) \in A^* \times A^* \mid u w v \in L\}$$

The epi-mono factorization of this maps yields precisely the quotient of  $A^*$  by the syntactic congruence

$$w \sim_L w' \text{ if and only if } \forall u, v \in A^* \quad uwv \in L \Leftrightarrow uw'v \in L,$$

that is, the carrier set of the syntactic monoid for the language  $L$ .

**Theorem 6.9.** *The minimal automaton  $\text{Min}(\mathcal{L})$  corresponds to the syntactic monoid  $\text{Syn}(L)$  of the language  $L$ .*

*Proof.* The minimal  $(\mathcal{I}_{\text{Mon}}, \text{Set}, 1, 2)$ -automaton accepting  $\mathcal{L}$  corresponds via the equivalence of Lemma 6.6 to a surjective  $A^*$ -baction recognizer. Thus, using Lemma 6.4, we obtain that  $\text{Min}(\mathcal{L})$  corresponds to a surjective monoid recognizer for  $L$ ,  $\text{Syn}(L)$ , with carrier set  $\text{Min}(\mathcal{L})(\text{states})$ . Consider any other monoid recognizer  $(\phi: A^* \rightarrow M, P \subseteq M)$  for  $L$ . By the first part of Lemma 6.4, any monoid recognizer can be seen as an  $A^*$ -baction recognizer, and thus as an  $(\mathcal{I}_{\text{Mon}}, \text{Set}, 1, 2)$ -automaton  $\mathcal{A}$ . By Lemma 2.3, we know that  $\text{Min}(\mathcal{L})$  is isomorphic to  $\text{Obs}(\text{Reach}(\mathcal{A}))$ , that is, to a quotient of a sub-automaton of  $\mathcal{A}$ . One can easily check that  $\text{Reach}(\mathcal{A})$  corresponds to a surjective  $A^*$ -baction recognizer, and thus to a surjective monoid recognizer for  $L$ . It follows that  $M$  is the quotient of a submonoid of  $\text{Syn}(L)$ .  $\square$

## 7. CONCLUSION

In this paper we propose a view of automata as functors and we showed how to recast well understood classical constructions in this setting, and in particular minimization of subsequential transducers. We argue that this perspective gives a unified view of language recognition and syntactic objects.

In a similar vein to the developments of Section 6, we can obtain the syntactic algebras recognizing languages for any algebraic theory over the category  $\text{Set}$ . The category  $\mathcal{I}_{\text{Mon}}$  is specific to the algebraic theory of monoids: one can notice that the hom-sets  $\mathcal{I}_{\text{Mon}}(\text{states}, \text{out})$ , respectively  $\mathcal{I}_{\text{Mon}}(\text{states}, \text{in})$  are isomorphic to “contexts (or terms) with one hole” also known as linear unary polynomials in universal algebra. In order to obtain a similar treatment for syntactic algebras for an arbitrary algebraic theory, one should change the input category and replace  $\mathcal{I}_{\text{Mon}}$  by a similar category, but where the morphisms are either terms or linear unary polynomials. A simpler input category could be obtained by considering the notion of “unary presentation” developed in [24].

We can go beyond regular languages and obtain in this fashion the “syntactic space with an internal monoid” of a possibly non-regular language [14]. To this end one would just have to compute the product and the coproduct in (6.2) in the category of Stone spaces.

We hope we can extend the framework to work with tree automata in monoidal categories. We discussed mostly NFA determinization, but we can obtain a variation of the generalized powerset construction [23] in this framework.

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