Logical Methods in Computer Science
Volume 16, Issue 2, 2020, pp. 5:1–5:10
https://lmcs.episciences.org/

Submitted May 01, 2019
Published May 14, 2020

ON THE INCOMPUTABILITY OF COMPUTABLE DIMENSION

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Abstract. Using an iterative tree construction we show that for simple computable subsets of the Cantor space Hausdorff, constructive and computable dimensions might be incomputable.

Computable dimension along with constructive dimension was introduced by Lutz [Lut03a, Lut03b] as a means for measuring the complexity of sets of infinite strings (ω-words). Since then and prior to this constructive and computable dimension were investigated in connection with Hausdorff dimension (for a detailed account see [DH10, Section 13]). The results of [Hit05, Sta93, Sta07] show that the Hausdorff, constructive and computable dimensions of automaton definable sets of infinite strings (regular ω-languages) are computable. In contrast to this Ko [Ko98] derived examples of computable ω-languages with an incomputable Hausdorff dimension.

In this paper we derive examples of computable ω-languages of a simple structure which have not only incomputable Hausdorff dimension but also incomputable computable dimension. To this end we use an iteration of finite trees which resembles the tree construction of Furstenberg [Fur70] (see also [MSS18])

Lutz [Lut03a, Lut03b] defines computable and constructive dimension via σ-(super)gales. Terwijn [Ter04, CST06] observed that this can also be done using Schnorr's concept of combining martingales with (exponential) order functions [Sch71, Section 17]. For the computable ω-languages constructed in this paper we can show that Schnorr's concept is in some details more precise than Lutz's approach.

1. Notation

In this section we introduce the notation used throughout the paper. By $\mathbb{N} = \{0, 1, 2, \ldots\}$ we denote the set of natural numbers, by $\mathbb{Q}$ the set of rational numbers, and $\mathbb{R}$ are the real numbers.

Let X be an alphabet of cardinality $|X| \geq 2$. By X^* we denote the set of finite words on X, including the *empty word* e, and X^ω is the set of infinite strings (ω-words) over X. Subsets of X^* will be referred to as *languages* and subsets of X^ω as *ω-languages*.

For $w \in X^*$ and $\eta \in X^* \cup X^\omega$ let $w \cdot \eta$ be their *concatenation*. This concatenation product extends in an obvious way to subsets $W \subseteq X^*$ and $B \subseteq X^* \cup X^\omega$. We denote by $|w|$ the *length* of the word $w \in X^*$ and $\mathbf{pref}(B)$ is the set of all finite prefixes of strings in $B \subseteq X^* \cup X^\omega$.

DOI:10.23638/LMCS-16(2:5)2020

It is sometimes convenient to regard X^ω as Cantor space, that is, as the product space of the (discrete space) X. Here *open* sets in X^ω are those of the form $W \cdot X^\omega$ with $W \subseteq X^*$. *Closed* are sets $F \subseteq X^\omega$ which satisfy the condition $F = \{\xi : \mathbf{pref}(\xi) \subseteq \mathbf{pref}(F)\}$.

For a computable domain $\mathcal{D}$, such as $\mathbb{N}$, $\mathbb{Q}$ or X^*, we refer to a function $f : \mathcal{D} \to \mathbb{R}$ as *left-computable* (or *approximable from below*) provided the set $\{(d,q) : d \in \mathcal{D} \wedge q \in \mathbb{Q} \wedge q < f(d)\}$ is computably enumerable. Accordingly, a function $f : \mathcal{D} \to \mathbb{R}$ is called *right-computable* (or *approximable from above*) if the set $\{(d,q) : d \in \mathcal{D} \wedge q \in \mathbb{Q} \wedge q > f(d)\}$ is computably enumerable, and f is *computable* if f is right- and left-computable. If we refer to a function $f : \mathcal{D} \to \mathbb{Q}$ as computable we usually mean that it maps the domain $\mathcal{D}$ to the domain $\mathbb{Q}$, that is, it returns the exact value $f(d) \in \mathbb{Q}$. If $\mathcal{D} = \mathbb{N}$ we write f as a sequence $(q_i)_{i\in\mathbb{N}}$.

A real number $\alpha \in \mathbb{R}$ is left-computable, right computable or computable provided the constant function $c_\alpha(t) = \alpha$ is left-computable, right-computable or computable, respectively. $\alpha \in \mathbb{R}$ is referred to as *computably approximable* if $\alpha = \lim_{i\to\infty} q_i$ for a computable sequence $(q_i)_{i\in\mathbb{N}}$ of rationals. It is well-known (see e.g. [ZW01]) that there are left-computable which are not right-computable and vice versa, and that there are computably approximable reals which are neither left-computable nor right-computable.

The following approximation property is easily verified.

Property 1.1. Let $(q_i)_{i\in\mathbb{N}}$ be a computable family of rationals converging to α and let $(q'_i)_{i\in\mathbb{N}}, q'_i > 0$, be a computable family of rationals converging to 0. If α is not right-computable then there are infinitely many $i \in \mathbb{N}$ such that $\alpha - q_i > q'_i$. $\square$

For, otherwise, α as the limit of $(q_i + q'_i)_{i\in\mathbb{N}}$ would be right-computable.

2. Gales and Martingales

Hausdorff [Hau18] introduced a notion of dimension of a subset Y of a metric space which is now known as its *Hausdorff dimension*, $\dim Y$; Falconer [Fal03] provides an overview and introduction to this subject. In the case of the Cantor space X^ω, Lutz [Lut03b] (see also [DH10, Section 13.2]) has found an equivalent definition of Hausdorff dimension via generalisations of martingales.

Following Lutz a mapping $d : X^* \to [0,\infty)$ will be called an *σ-supergale* provided

$$\forall w(w \in X^* \to |X|^\sigma \cdot d(w) \geq \sum_{x\in X} d(wx)) \,. \tag{2.1}$$

A σ-supergale d is called an σ-gale if, for all $w \in X^*$, Eq. (2.1) is satisfied with equality. (*Super*)*Martingales* are 1-(super)gales.

From Eq. (2.1) one easily infers that if $d, \mathcal{V} : X^* \to [0,\infty)$ satisfy

$$\forall w(w \in X^* \to \frac{\mathcal{V}(w)}{|X|^{(1-\sigma)\cdot|w|}} = d(w)) \tag{2.2}$$

then d is a σ-(super)gale if and only if $\mathcal{V}$ is a (super)martingale. Thus (super)gales can be viewed as a combination of (super)martingales with exponential order functions in the sense of Schnorr [Sch71, Section 17] (see also [Ter04, CST06] or [DH10, Section 13.3]).

Following Lutz [Lut03b] we define as follows.

Definition 2.1. Let $F \subseteq X^\omega$. Then α is the *Hausdorff dimension* $\dim F$ of F provided
(1) for all $\sigma > \alpha$ there is a σ-supergale d such that $\forall \xi(\xi \in F \to \limsup_{w \to \xi} d(w) = \infty)$, and[1]
(2) for all $\sigma < \alpha$ and all σ-supergales d it holds $\exists \xi(\xi \in F \wedge \limsup_{w \to \xi} d(w) < \infty)$.

If the ω-language $F \subseteq X^\omega$ is closed in Cantor space and satisfies a certain balance condition Theorem 4 of [Sta89] shows that the calculation of its Hausdorff dimension can be simplified. For the purposes of our investigations the following special case will suffice.

Proposition 2.2. *Let $F \subseteq X^\omega$ be non-empty and satisfy the conditions*
(1) $F = \{\xi : \mathbf{pref}(\xi) \subseteq \mathbf{pref}(F)\}$ *and*
(2) $|\mathbf{pref}(F) \cap w \cdot X^k| = |\mathbf{pref}(F) \cap v \cdot X^k|$ *for all $k \in \mathbb{N}$ and $w, v \in \mathbf{pref}(F)$ with $|w| = |v|$.*
Then $\dim F = \liminf_{n \to \infty} \dfrac{\log_{|X|} |\mathbf{pref}(F) \cap X^n|}{n}$. □

3. Iterative Tree Construction

The aim of this section is, given a sequence of rationals $(q_i)_{i \in \mathbb{N}}, 0 < q_i < 1$, to construct an ω-language $F \subseteq X^\omega$ with Hausdorff dimension $\dim F = \liminf_{i \to \infty} q_i$ satisfying the conditions (1) and (2) of Proposition 2.2.

3.1. Preliminaries.

As a preparation we show how to find sequences of natural numbers $(k_i)_{i \in \mathbb{N}}$ and $(\ell_i)_{i \in \mathbb{N}}$ with appropriate properties such that $q_i = k_i / \ell_i$.

Lemma 3.1. *Let $(q_i)_{i \in \mathbb{N}}$, $0 < q_i < 1$, $q_i \neq q_{i+1}$, be a family of positive rationals. Then there are families of natural numbers $(k_i)_{i \in \mathbb{N}}$, $(\ell_i)_{i \in \mathbb{N}}$, $(\kappa_i)_{i \in \mathbb{N}}$, $(p_i)_{i \in \mathbb{N}}$ and $(r_i)_{i \in \mathbb{N}}$, such that* $q_i = k_i/\ell_i$, $q_{i+1} = \dfrac{r_i \cdot k_i + \kappa_i \cdot \ell_i}{r_i \cdot \ell_i + p_i \cdot \ell_i}$ *where* $\kappa_i = \begin{cases} 0, & \text{if } q_i > q_{i+1} \text{ and} \\ p_i, & \text{if } q_i < q_{i+1}. \end{cases}$

Moreover, for $0 \leq t \leq p_i \cdot \ell_i$ we have
$$q_i \geq \frac{r_i \cdot k_i}{r_i \cdot \ell_i + t} \geq q_{i+1}, \ \text{if } q_i > q_{i+1} \text{ and} \tag{3.1}$$
$$q_i \leq \frac{r_i \cdot k_i + t}{r_i \cdot \ell_i + t} \leq q_{i+1}, \ \text{if } q_i < q_{i+1}. \tag{3.2}$$

Proof. Let $q_i = k_i/\ell_i$ and $q_{i+1} = a/b \cdot q_i = \dfrac{a \cdot k_i}{b \cdot \ell_i}$, with $a, b \in \mathbb{N} \setminus \{0\}, a \neq b$. Since $1 > q_{i+1}$ we have $b \cdot \ell_i - a \cdot k_i = a \cdot \frac{q_i}{q_{i+1}} \cdot (1 - q_{i+1}) \cdot \ell_i > 0$.

Assume $q_i > q_{i+1}$. Then $b > a$ and the equation
$$\frac{r_i \cdot k_i + \kappa_i \cdot \ell_i}{r_i \cdot \ell_i + p_i \cdot \ell_i} = \frac{a \cdot k_i}{b \cdot \ell_i} \tag{3.3}$$
has the solutions $r_i = a$, and $p_i = (b - a) = a \cdot (\frac{q_i}{q_{i+1}} - 1)$ and $\kappa_i = 0$.

If $q_i < q_{i+1}$ then $a > b$ and $r_i := b \cdot \ell_i - a \cdot k_i = a \cdot (\frac{q_i}{q_{i+1}} \cdot \ell_i - k_i) = a \cdot q_i \cdot (\frac{1}{q_{i+1}} - 1) \cdot \ell_i$ and $p_i = \kappa_i := (a - b) \cdot k_i = a \cdot q_i \cdot (1 - \frac{q_i}{q_{i+1}}) \cdot \ell_i$ are solutions of Eq. (3.3).

[1] Here $\limsup_{w \to \xi} d(w)$ is an abbreviation for $\lim_{n \to \infty} \sup\{d(w) : w \in \mathbf{pref}(\xi) \wedge |w| \geq n\}$.

In view of $\kappa_i = 0$ Eq. (3.1) is obvious. Eq. (3.2) follows inductively from $\frac{k+1}{\ell+1} \geq \frac{k}{\ell}$ whenever $0 \leq k < \ell$. □

If the family $(q_i)_{i\in\mathbb{N}}$ is a computable one then the families in Lemma 3.1 can be chosen to be computable. In addition, the values ℓ_i and ℓ_{i+1}/ℓ_i can be made arbitrarily large.

3.2. **Tree construction.** The ω-language F will be the limit of the following sequence of finite trees T_i. These trees have a property similar to the one in Proposition 2.2 (2) which is referred to as *spherical symmetry* in [Fur70].

We define the following auxiliary languages $T_i \subseteq X^{\ell_i}$ and $U_i \subseteq X^{p_i\cdot\ell_i}$.

Let $T_0 := X^{k_0} \cdot 0^{\ell_0-k_0}$ or $T_0 := 0^{\ell_0-k_0} \cdot X^{k_0}$ and set

$$T_{i+1} := T_i^{r_i} \cdot U_i \text{ with } U_i := \begin{cases} X^{p_i\cdot\ell_i}, & \text{if } q_{i+1} \geq q_i \text{ and} \\ \{u_i\}, & \text{otherwise} \end{cases} \tag{3.4}$$

where $u_i \in X^{p_i}$ is a fixed word. Then $\ell_{i+1} = (r_i+p_i)\cdot\ell_i$. Thus T_{i+1} consists of a concatenation of r_i copies of T_i plus an appendix U_i of length $p_i \cdot \ell_i$. The values r_i and p_i are referred to as repetition or prolongation factors, respectively.

By induction one proves

$$|T_i| = |X|^{q_i\cdot\ell_i}\,. \tag{3.5}$$

Property 3.2. The trees T_i have the following properties. Let $\ell \leq \ell_i$.

(1) Prefix property: $\mathbf{pref}(T_{i+1}) = \bigcup_{j=0}^{r_i-1} T_i^j \cdot \mathbf{pref}(T_i) \cup T_i^{r_i} \cdot \mathbf{pref}(U_i)$,

(2) Extension property: $\mathbf{pref}(T_i) \cap X^\ell = \mathbf{pref}(T_{i+1}) \cap X^\ell$, and

(3) Spherical symmetry: $\mathbf{pref}(T_i) \cap X^\ell = (\mathbf{pref}(T_i) \cap X^{\ell-1}) \cdot X$ or
$$|\mathbf{pref}(T_i) \cap X^\ell| = |\mathbf{pref}(T_i) \cap X^{\ell-1}|.$$ □

3.3. **The infinite tree.** We define our ω-language F having the properties mentioned in Proposition 2.2 as $F := \bigcap_{i\in\mathbb{N}} T_i \cdot X^\omega$ where the family $(T_i)_{i\in\mathbb{N}}$ satisfies Eq. (3.4).

Before we proceed to further properties of $(T_i)_{i\in\mathbb{N}}$ and F we mention a general property.

Lemma 3.3. *Let $T_i \subseteq X^*$, $T_{i+1} \subseteq T_i \cdot X \cdot X^*$, $T_i \subseteq \mathbf{pref}(T_{i+1})$ and $F := \bigcap_{i\in\mathbb{N}} T_i \cdot X^\omega$. Then $\mathbf{pref}(F) = \bigcup_{i\in\mathbb{N}} \mathbf{pref}(T_i)$.*

If, moreover, all T_i are finite then $F := \{\xi : \xi \in X^\omega \wedge \mathbf{pref}(\xi) \subseteq \bigcup_{i\in\mathbb{N}} \mathbf{pref}(T_i)\}$.

Proof. In view of $T_{i+1} \subseteq T_i \cdot X \cdot X^*$ we have $T_{i+1} \cdot X^\omega \subseteq T_i \cdot X^\omega$ and also $|w| \geq i$ for $w \in T_i$.

If $w \in \mathbf{pref}(F)$ then $w \in \mathbf{pref}(\xi)$ where $\xi \in F \subseteq T_i \cdot X^\omega$ for $i > |w|$. Consequently, $w \in \mathbf{pref}(T_i)$.

Using the condition $T_i \subseteq \mathbf{pref}(T_{i+1})$, by induction we obtain that for every $w \in \mathbf{pref}(T_i)$ there is an infinite chain $(w_j)_{j\geq i}$ such that $w_j \in T_j$ and $w \sqsubseteq w_i \sqsubset w_{i+1} \sqsubset \cdots$. Thus there is a $\xi \in F$ with $w \sqsubset \xi$.

If the languages T_i are finite $F = \bigcap_{i\in\mathbb{N}} T_i \cdot X^\omega$ is closed in the product topology of the space X^ω which implies $F := \{\xi : \xi \in X^\omega \wedge \mathbf{pref}(\xi) \subseteq \mathbf{pref}(F)\}$. □

Lemma 3.3 shows that $F := \{\xi : \xi \in X^\omega \wedge \mathbf{pref}(\xi) \subseteq \bigcup_{i\in\mathbb{N}} \mathbf{pref}(T_i)\}$ for the family $(T_i)_{i\in\mathbb{N}}$ defined in Section 3.2.

From the spherical symmetry of T_i (see Property 3.2 (3)) the ω-language $F = \bigcap_{i\in\mathbb{N}} T_i \cdot X^\omega$ inherits the following balance property of Proposition 2.2 (2).

Lemma 3.4. *Let $F = \bigcap_{i\in\mathbb{N}} T_i \cdot X^\omega$ where the T_i are defined by Eq. (3.4). Then for all $k \in \mathbb{N}$ and $w, v \in \mathbf{pref}(F)$ with $|w| = |v|$ we have*

$$|w \cdot X^k \cap \mathbf{pref}(F)| = |v \cdot X^k \cap \mathbf{pref}(F)|\,.$$

Proof. We proceed by induction on k. Let $k = 1$. Then for all $w, v \in \mathbf{pref}(F)$ with $|w| = |v|$ either $\mathbf{pref}(F) \cap X^{|u|+1} = (\mathbf{pref}(F) \cap X^{|u|}) \cdot X$ or $|\mathbf{pref}(F) \cap X^{|u|+1}| = |\mathbf{pref}(F) \cap X^{|u|}|$ ($u \in \{w, v\}$).

In the first case we have $|w \cdot X \cap \mathbf{pref}(F)| = |X| = |v \cdot X \cap \mathbf{pref}(F)|$ and in the second $|w \cdot X \cap \mathbf{pref}(F)| = 1 = |v \cdot X \cap \mathbf{pref}(F)|$.

Let the assertion be proved for k and all pairs $u, u' \in \mathbf{pref}(F)$ of the same length. Let $w, v \in \mathbf{pref}(F)$ with $|w| = |v|$ and consider words $w', v' \in X^k$ such that $w\cdot w', v\cdot v' \in \mathbf{pref}(F)$. Then from the spherical symmetry we obtain either $\mathbf{pref}(F) \cap X^{|u|+1} = (\mathbf{pref}(F) \cap X^{|u|}) \cdot X$ or $|\mathbf{pref}(F) \cap X^{|u|+1}| = |\mathbf{pref}(F) \cap X^{|u|}|$ for $u \in \{w \cdot w', v \cdot v'\}$ and we proceed as above.

Since, by our assumption $|\{w' : |w'| = k \wedge w \cdot w' \in \mathbf{pref}(F)\}| = |\{v' : |v'| = k \wedge v \cdot v' \in \mathbf{pref}(F)\}|$, the assertion follows. □

As a consequence of Lemmas 3.3, 3.4 and Proposition 2.2 we obtain the following.

Corollary 3.5. *Let $F = \bigcap_{i\in\mathbb{N}} T_i \cdot X^\omega$ where the T_i are defined by Eq. (3.4). Then* $\dim F = \liminf\limits_{n\to\infty} \frac{\log_{|X|} |\mathbf{pref}(F)\cap X^n|}{n}$. □

Next we investigate in more detail the structure function $s_F : \mathbb{N} \to \mathbb{N}$ where $s_F(\ell) := |\mathbf{pref}(F) \cap X^\ell|$. First, Lemma 3.3 implies

$$\mathbf{pref}(F) \cap X^\ell = \mathbf{pref}(T_i) \cap X^\ell \text{ whenever } \ell \le \ell_i\,. \tag{3.6}$$

From Eqs. (3.4) and (3.5) and the properties of the tree family $(T_i)_{i\in\mathbb{N}}$ we obtain for the intervals $\ell_i \le \ell \le \ell_{i+1}$:

Lemma 3.6. *Let $F = \bigcap_{i\in\mathbb{N}} T_i \cdot X^\omega$ where the T_i are defined by Eq. (3.4). Then the structure function $s_F : \mathbb{N} \to \mathbb{N}$ satisfies the following relations.*

(1) *In the interval $[j \cdot \ell_i, (j + 1) \cdot \ell_i]$ where $j < r_i$:*

$$s_F(j \cdot \ell_i + t) = s_F(\ell_i)^j \cdot s_F(t) \text{ for } 0 \le t \le \ell_i$$

(2) *In the subinterval $[j \cdot \ell_i + j' \cdot \ell_{i-1}, j \cdot \ell_i + (j' + 1) \cdot \ell_{i-1}]$ where $j' < r_{i-1}$:*

$$s_F(j \cdot \ell_i + j' \cdot \ell_{i-1} + t) = s_F(\ell_i)^j \cdot s_F(\ell_{i-1})^{j'} \cdot s_F(t) \text{ for } 0 \le t < \ell_{i-1}\,.$$

(3) *In the interval $[r_i \cdot \ell_i, \ell_{i+1}]$:*

$$s_F(r_i \cdot \ell_i + t) = \begin{cases} s_F(\ell_i)^{r_i}, & \text{if } |U_i| = 1 \text{ and} \\ s_F(\ell_i)^{r_i} \cdot |X|^t, & \text{if } U_i = X^{p_i\cdot\ell_i} \end{cases} \quad \text{for } 0 \le t \le p_i \cdot \ell_i\,.$$ □

This yields the following connection to the values q_i. In order to connect our considerations to the application of Proposition 2.2 we consider the values of $\frac{\log_{|X|} s_F(n)}{n}$ instead of $s_F(n)$.

From Eqs. (3.6) and (3.5) we obtain

$$\frac{\log_{|X|} s_F(j\cdot\ell_i)}{j\cdot\ell_i} = q_i\,. \tag{3.7}$$

Now we use the identities of Lemma 3.6 and Eqs. (3.1) and (3.2) to bound $\frac{\log_{|X|} s_F(\ell)}{\ell}$ in the range $\ell_i \le \ell \le \ell_{i+1} = r_i\cdot\ell_i + n_i\cdot\ell_i$.

For $\ell_i \le \ell < r_i\cdot\ell_i$ we have $\ell = j\cdot\ell_i + j'\cdot\ell_{i-1} + t$ where $0 \le t < \ell_{i-1}$, and Lemma 3.6 (1) and (2) yield

$$\begin{aligned}\frac{\log_{|X|} s_F(\ell)}{\ell} &\ge \frac{j\cdot\ell_i}{\ell}\cdot q_i + \frac{j'\cdot\ell_{i-1}}{\ell}\cdot q_{i-1}\\ &\ge \frac{j\cdot\ell_i + j'\cdot\ell_{i-1}}{\ell}\cdot\min\{q_{i-1},q_i\}\\ &\ge (1-\frac{\ell_{i-1}}{\ell_i})\cdot\min\{q_{i-1},q_i\}\end{aligned} \tag{3.8}$$

If $r_i\cdot\ell_i \le \ell \le \ell_{i+1}$, that is, for $\ell = r_i\cdot\ell_i + t$ where $t \le \ell_{i+1} - r_i\cdot\ell_i$, following Eqs. (3.1) and (3.2), respectively, we have according to Lemma 3.6 (3)

$$q_i \ge \frac{\log_{|X|} s_F(\ell)}{\ell} = \frac{\log_{|X|} s_F(r_i\cdot\ell_i)}{r_i\cdot\ell_i + t} \ge q_{i+1} \text{ if } q_i > q_{i+1} \tag{3.9}$$

$$q_i \le \frac{\log_{|X|} s_F(\ell)}{\ell} = \frac{\log_{|X|} s_F(r_i\cdot\ell_i) + t}{r_i\cdot\ell_i + t} \le q_{i+1} \text{ if } q_i < q_{i+1} \tag{3.10}$$

The considerations in Eqs. (3.7), (3.8), (3.9) and (3.10) show the following.

Lemma 3.7. *If the sequence $(\ell_i)_{i\in\mathbb{N}}$ is chosen in such a way that $\liminf_{i\to\infty} \frac{\ell_{i-1}}{\ell_i} = 0$ then*

$$\liminf_{\ell\to\infty} \frac{\log_{|X|} s_F(\ell)}{\ell} = \liminf_{i\to\infty} q_i\,.$$

Proof. In view of Eq. (3.7) the limit cannot exceed $\liminf_{i\to\infty} q_i$.

On the other hand, by Eqs. (3.8), (3.9) and (3.10), for $\ell_i \le \ell \le \ell_{i+1}$, the intermediate values satisfy $\frac{\log_{|X|} s_F(\ell)}{\ell} \ge (1-\frac{\ell_{i-1}}{\ell_i})\cdot\min\{q_{i-1},q_i,q_{i+1}\}$. □

3.4. **Monotone families $(q_i)_{i\in\mathbb{N}}$.** If the sequence $(q_i)_{i\in\mathbb{N}}$ is monotone we can simplify the above considerations of Eq. (3.8).

Proposition 3.8. *Let the sequence $(q_i)_{i\in\mathbb{N}}$ be monotone and $\lim_{i\to\infty} q_i = \alpha$.*

(1) *If $(q_i)_{i\in\mathbb{N}}$ is decreasing and $T_0 = X^{k_0}\cdot 0^{\ell_0-k_0}$ then $s_F(\ell) \ge |X|^{\alpha\cdot\ell}$, for all $\ell\in\mathbb{N}$.*
(2) *If $(q_i)_{i\in\mathbb{N}}$ is increasing and $T_0 = 0^{\ell_0-k_0}\cdot X^{k_0}$ then $s_F(\ell) \le |X|^{\alpha\cdot\ell}$, for all $\ell\in\mathbb{N}$.*

Proof. If $(q_i)_{i\in\mathbb{N}}$ is decreasing we start with $T_0 = X^{k_0}\cdot 0^{\ell_0-k_0}$ and have $s_F(\ell) \ge |X|^{q_0\cdot\ell} \ge |X|^{\alpha\cdot\ell}$ for $\ell \le \ell_0$. Then we use Eqs. (3.6) and (3.4) and proceed by induction.

$s_F(j\cdot\ell_i + t) = s_F(j\cdot\ell_i)\cdot s_F(t) \ge |X|^{q_i\cdot\ell_i}\cdot|X|^{\alpha\cdot t} \ge |X|^{\alpha\cdot\ell}$ for $j < r_i$. In the range $r_i\cdot\ell_i \le \ell \le \ell_{i+1}$ we have according to Eq. (3.9) $s_F(\ell) \ge |X|^{q_{i+1}\cdot\ell} \ge |X|^{\alpha\cdot\ell}$.

If $(q_i)_{i\in\mathbb{N}}$ is increasing we start with $T_0 = 0^{\ell_0-k_0}\cdot X^{k_0}$ and have $s_F(\ell) \ge |X|^{q_0\cdot\ell} \le |X|^{\alpha\cdot\ell}$ for $\ell \le \ell_0$. Again we use Eqs. (3.6) and (3.4) and proceed by induction.

$s_F(j\cdot\ell_i + t) = s_F(j\cdot\ell_i)\cdot s_F(t) \le |X|^{q_i\cdot\ell_i}\cdot|X|^{\alpha\cdot t} \le |X|^{\alpha\cdot\ell}$ for $j < r_i$. In the range $r_i\cdot\ell_i \le \ell \le \ell_{i+1}$ we have according to Eq. (3.10) $s_F(\ell) \le |X|^{q_{i+1}\cdot\ell} \le |X|^{\alpha\cdot\ell}$. □

4. Incomputable dimensions

4.1. **Hausdorff dimension.** In this section we provide the announced examples. First we have the following.

Lemma 4.1. *If the sequence $(q_i)_{i\in\mathbb{N}}$ of rationals $0 < q_i < 1, q_i \neq q_{i+1}$, is computable then one can construct an ω-language $F \subseteq X^\omega$ according to the tree construction such that $\mathbf{pref}(F)$ is a computable language.*

Proof. Construct from $(q_i)_{i\in\mathbb{N}}$ the numerator and denominator sequences $(k_i)_{i\in\mathbb{N}}$ and $(\ell_i)_{i\in\mathbb{N}}$ and the corresponding sequences for the repetition and prolongation factors $(r_i)_{i\in\mathbb{N}}$ and $(p_i)_{i\in\mathbb{N}}$. Then in view of Eq. (3.4) the assertion is obvious. □

Our lemma shows that the ω-language $F \subseteq X^\omega$ has a very simple computable structure (compare with [Sta07, Section 4.2]).

Next we show that the Hausdorff dimension of a computable ω-language $F \subseteq X^\omega$ as in Lemma 4.1 may be incomputable.

Theorem 4.2. *If the sequence $(q_i)_{i\in\mathbb{N}}$ of rationals $0 < q_i < 1, q_i \neq q_{i+1}$, is computable and $\alpha = \liminf_{i\to\infty} q_i$ then there is an ω-language $F \subseteq X^\omega$ such that $\mathbf{pref}(F)$ is a computable language and $\dim F = \alpha$.*

Proof. Construct from $(q_i)_{i\in\mathbb{N}}$ the numerator and denominator sequences $(k_i)_{i\in\mathbb{N}}$ and $(\ell_i)_{i\in\mathbb{N}}$ such that $\liminf_{i\to\infty} \frac{\ell_i}{\ell_{i+1}} = 0$. Then the assertion follows from Lemmas 3.7, 4.1 and Corollary 3.5. □

Theorem 3.4 of [Ko98] proves a similar result where the achieved Hausdorff dimension α is a computably approximable number. In [ZW01] it is shown that there are reals which are not computably approximable of the form $\liminf_{i\to\infty} q_i$ where $(q_i)_{i\in\mathbb{N}}$ is a computable sequence.

4.2. **Computable dimension.** If we require the supergales in Definition 2.1 to be computable mappings we obtain the definition of computable dimension $\dim_{\mathrm{comp}} F$ of [Hit05, Lut03b]. In view of Eq. (2.2) we may, as in Section 13.15 of [DH10], define the computable dimension of an ω-language $E \subseteq X^\omega$ via martingales.

Definition 4.3. Let $F \subseteq X^\omega$. Then α is the *computable dimension* of F provided

(1) for all $\sigma > \alpha$ there is a computable martingale $\mathcal{V}$ such that
$\forall\xi(\xi \in F \to \limsup\limits_{w\to\xi} \frac{\mathcal{V}(w)}{|X|^{(1-\sigma)\cdot|w|}} = \infty)$, and

(2) for all $\sigma < \alpha$ and all computable martingales $\mathcal{V}$ it holds
$\exists\xi(\xi \in F \wedge \limsup\limits_{w\to\xi} \frac{\mathcal{V}(w)}{|X|^{(1-\sigma)\cdot|w|}} < \infty)$.

The inequality $\dim F \leq \dim_{\mathrm{comp}} F$ is immediate.

We associate with every non-empty ω-language $E \subseteq X^\omega$ a martingale $\mathcal{V}_E$ in the following way.

Definition 4.4.

$$\begin{aligned}\mathcal{V}_E(e) &:= 1\\ \mathcal{V}_E(wx) &:= \begin{cases}\frac{|X|}{|\mathbf{pref}(E)\cap w\cdot X|}\cdot\mathcal{V}_E(w), & \text{if } x\in X \text{ and } wx\in\mathbf{pref}(E), \text{ and}\\ 0, & \text{otherwise.}\end{cases}\end{aligned}$$

In view of the spherical symmetry, for F defined as in Section 3.3, we obtain

$$\mathcal{V}_F(w) = |X|^{|w|}/s_F(|w|),\ \text{if } w\in\mathbf{pref}(F)\,. \tag{4.1}$$

Moreover, if $\mathbf{pref}(F)$ is computable then s_F and $\mathcal{V}_F$ are computable mappings.

Theorem 4.5. *If the sequence $(q_i)_{i\in\mathbb{N}}$ of rationals $0<q_i<1, q_i\neq q_{i+1}$, is computable and $\alpha=\liminf_{i\to\infty} q_i$ then there is an ω-language $F\subseteq X^\omega$ such that $\mathbf{pref}(F)$ is a computable language and $\dim F=\dim_{\mathrm{comp}} F=\alpha$.*

Proof. We use the ω-language F defined in the proof of Theorem 4.2 and the associated computable martingale $\mathcal{V}_F$.

Let $\sigma>\alpha=\liminf_{i\to\infty} q_i$. Then $(\sigma-q_i)>(\sigma-\alpha)/2>0$ for infinitely many $i\in\mathbb{N}$. Since $s_F(\ell_i)=|X|^{q_i\cdot\ell_i}$ (see Eq. (3.7)), we have $\mathcal{V}_F(w)/|X|^{(1-\sigma)\cdot|w|}=|X|^{(\sigma-q_i)}\geq|X|^{(\sigma-\alpha)/2}$ for $w\in\mathbf{pref}(F)\cap X^{\ell_i}$. This shows $\limsup_{w\to\xi}\mathcal{V}_F(w)/|X|^{(1-\sigma)\cdot|w|}=\infty$ for all $\xi\in F$, that is, $\dim_{\mathrm{comp}} F\leq\alpha$.

The other inequality follows from $\dim F\leq\dim_{\mathrm{comp}} F$ and Theorem 4.2. □

In certain cases we can achieve even the borderline value

$$\limsup_{w\to\xi}\frac{\mathcal{V}_F(w)}{|X|^{(1-\dim F)\cdot|w|}}=\limsup_{n\to\infty}\frac{|X|^{\dim F\cdot n}}{s_F(n)}=\infty \text{ for all } \xi\in F\,. \tag{4.2}$$

Theorem 4.6. *Let $(q_i)_{i\in\mathbb{N}}, 0<q_i<1, q_i\neq q_{i+1}$, be a computable sequence of rationals with $\liminf_{i\to\infty} q_i=\alpha$. If α is not right-computable then there is an ω-language $F\subseteq X^\omega$ such that $\mathbf{pref}(F)$ is a computable language, $\dim F=\dim_{\mathrm{comp}} F=\alpha$ and Eq. (4.2) is satisfied.*

Proof. We construct F as in the proof of Theorem 4.2 requiring additionally that $\ell_i\geq i^2$. Then $\mathbf{pref}(F)$ is computable and $\dim F=\dim_{\mathrm{comp}} F=\alpha$. In view of Property 1.1 there are infinitely many $i\in\mathbb{N}$ with $\alpha-\frac{1}{i}>q_i$ and, consequently, $s_F(\ell_i)=|X|^{q_i\cdot\ell_i}\leq|X|^{\alpha\cdot\ell_i-\ell_i/i}$. This shows $\limsup\limits_{n\to\infty}\frac{|X|^{\alpha\cdot n}}{s_F(n)}\geq\limsup\limits_{i\to\infty}|X|^{\ell_i/i}=\infty$. □

4.3. **Comparison of gales and martingales.** In this final part we compare the precision with which (super)gales and martingales achieve the value of computable dimension of a subset $E\subseteq X^\omega$. In Theorem 4.6 we have seen that there are ω-languages $F\subseteq X^\omega$ such that $\dim_{\mathrm{comp}} F=\alpha$ and $\limsup_{w\to\xi}\mathcal{V}_F(w)/|X|^{(1-\alpha)\cdot|w|}=\infty$ for all $\xi\in F$, that is, the computable martingale $\mathcal{V}_F$ "matches" exactly the value of the computable dimension of F. The following theorem shows that this is, in some cases, not possible for supergales.

First, observe that, for $\sigma'\geq\sigma$ any σ-supergale $d: X^*\to[0,\infty)$ is also a σ'-supergale. Thus computable σ-supergales exist for all $\sigma\in[0,1]$.

We define the *cut point* χ_d of a supergale d as the smallest value σ for which d can be an σ-supergale.

$$\chi_d := \inf\left\{\sigma : \forall w\left(|X|^\sigma \cdot d(w) \geq \sum\nolimits_{x\in X} d(wx)\right)\right\}. \tag{4.3}$$

If d is a computable mapping then χ_d as $\sup\{q : q \in \mathbb{Q} \wedge \exists w(|X|^q \cdot d(w) < \sum_{x\in X} d(wx))\}$ is a left-computable real number. For computable σ-gales d the cut point χ_d coincides with σ and is necessarily a computable real.

Theorem 4.7. *Let $(q_i)_{i\in\mathbb{N}}, 0 < q_i < 1, q_i \neq q_{i+1}$, be a computable sequence of rationals with $\liminf_{i\to\infty} q_i = \alpha$. If α is neither left- nor right-computable then there is an ω-language $F \subseteq X^\omega$ such that $\mathbf{pref}(F)$ is a computable language, $\alpha = \dim F = \dim_{\mathrm{comp}} F$, Eq. (4.2) is satisfied but there is no computable α-supergale with $\limsup_{w\to\xi} d(w) = \infty$ for all $\xi \in F$.*

Proof. In view of the preceding Theorems 4.2 and 4.6 it suffices to show that under the additional assumption that α is not left-computable no computable α-supergale satisfies $\limsup_{w\to\xi} d(w) = \infty$ for all $\xi \in F$.

Assume the contrary. Since α is not left-computable, the cut point χ_d of the computable α-supergale d cannot coincide with α. Hence $\alpha > \chi_d$, and we have some rational number $q, \alpha > q > \chi_d$. Consequently, d is a q-supergale with $\limsup_{w\to\xi} d(w) = \infty$ for all $\xi \in F$. This contradicts $q < \alpha = \dim_{\mathrm{comp}} F$ $\square$

Since there are computably approximable reals which are neither right- not left-computable Theorem 4.7 shows that in some cases Schnorr's [Sch71] combination of martingales with (exponential) order functions can be more precise than Lutz's approach via supergales.

5. Concluding remark

As the constructive dimension of subsets of X^ω is sandwiched between the computable and the Hausdorff dimension ([Lut03a, Lut03b, Hit05]) the result of Theorem 4.5 holds likewise for constructive dimension, too.

Acknowledgement

I wish to thank the anonymous reviewers for their suggestions for improving the presentation of this article.

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