

SOUNDNESS OF UNRAVELINGS FOR CONDITIONAL TERM REWRITING SYSTEMS VIA ULTRA-PROPERTIES RELATED TO LINEARITY*

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ABSTRACT. Unravelings are transformations from conditional term rewriting systems (CTRSs) into unconditional term rewriting systems (TRSs) over extended signatures. They are complete, but in general, not sound w.r.t. reduction. Here, soundness w.r.t. reduction for a CTRS means that for every term over the original signature of the CTRS, if the corresponding unraveled TRS reduces the term to a term over the original signature, then so does the original CTRS. In this paper, we show that an optimized variant of Ohlebusch’s unraveling for deterministic CTRSs is sound w.r.t. reduction if the corresponding unraveled TRSs are left-linear, or both right-linear and non-erasing. Then, we show that soundness of the variant implies soundness of Ohlebusch’s unraveling, and show that soundness of Marchiori’s unravelings for join and normal CTRSs also implies soundness of Ohlebusch’s unraveling. Finally, we show that soundness of a transformation proposed by Şerbănuță and Roşu for deterministic CTRSs implies soundness of Ohlebusch’s unraveling.

1. INTRODUCTION

Unravelings are transformations from conditional term rewriting systems (CTRSs) into unconditional term rewriting systems (TRSs) over extended signatures of the original signatures for the CTRSs. They are complete w.r.t. reduction sequences of the original CTRSs [15], *i.e.*, for every derivation of the CTRSs, there exists a corresponding derivation of the unraveled TRSs. In this respect, the unraveled TRSs are over-approximations of the original CTRSs w.r.t. reduction, and the unraveled TRSs are useful for analyzing the properties of the original CTRSs, such as syntactic properties, *modularity*, and *operational termination* since TRSs are in general much easier to handle than CTRSs.

Marchiori proposed unravelings for *join* and *normal* CTRSs in order to analyze *ultra-properties* and modularity of the CTRSs [15], and he also proposed an unraveling for *deterministic* CTRSs (DCTRSs) [16]. The transformation technique used in his unravelings originates from [4, 8]. Afterwards, Ohlebusch presented an improved variant of Marchiori’s

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unraveling for DCTRSs in order to analyze termination of logic programs [25]—Marchiori’s and Ohlebusch’s unravelings are called *sequential unravelings* [10]. Termination of the unraveled TRSs is a sufficient condition for proving *operational termination* of the original CTRSs [14]. Later, a variant of Ohlebusch’s unraveling was proposed in both [18] and [7]. This variant is sometimes called *optimized*, in the sense that the variable-carrying arguments of *U symbols* introduced by means of the application of the unraveling are optimized, *i.e.*, U symbols propagate only values received by variables that are referred later.

Although the mechanism of unconditional rewriting is much simpler than that of conditional rewriting, the reduction of the unraveled TRS has never been used instead of the original CTRS in order to reduce terms over the original signature, until being used in program inversion methods [18, 22, 23] described later. This is because unravelings are not *sound w.r.t. reduction* in general [15, 25] while they are complete. Here, soundness w.r.t. reduction (simply, *soundness*) for a CTRS means that, for every term over the original signature of the CTRS, if the unraveled TRS reduces the term to a term over the original signature, then so does the original CTRS [15]. Several studies have been made on soundness conditions of unravelings—some syntactic properties and particular reduction strategies for the unraveled TRSs. Marchiori showed that his unraveling for normal CTRSs is sound for left-linear ones [15], and he also showed that his unraveling for DCTRSs is sound for DCTRSs that are semi-linear or confluent [16]. Nishida et al. showed that the combined reduction restriction of the *membership condition* [34] and *context-sensitive condition* [13] determined by means of the application of the optimized unraveling is sufficient for soundness [22]. Later, Schernhammer and Gramlich showed that the same context-sensitive condition without the membership condition is sufficient for soundness of Ohlebusch’s unraveling [28, 29] and Gmeiner et al. showed that Marchiori’s unraveling for normal CTRSs is sound for confluent, non-erasing, or weakly left-linear ones, and they presented some properties that are not sufficient for soundness [9].

As another kind of transformation from CTRSs to TRSs, Șerbănuță and Roșu proposed a complete transformation (*SR transformation*) from strongly or syntactically DCTRSs into TRSs [30, 31]. The SR transformation is sound if the DCTRSs are semi-linear or confluent, where function symbols in the original signatures are completely extended by increasing the arities of some function symbols. The SR transformation is based on Viry’s approach [35] that is another direction of developing transformations from CTRSs into TRSs, and that has been further studied in [1, 27]. The SR transformation provides *computationally equivalent* TRSs to the original DCTRSs if the original DCTRSs are operationally terminating and either semi-linear or (ground) confluent. On the other hand, the theoretical relationship between the SR transformation and the existing unravelings has never been discussed.

In this paper, we show two sufficient conditions of DCTRSs for soundness of the optimized unraveling: one condition is *ultra-left-linearity*, *i.e.*, that the unraveled TRSs are left-linear, and the second condition is the combination of *ultra-right-linearity* and *ultra-non-erasingness*, *i.e.*, that the unraveled TRSs are right-linear and non-erasing. We also provide necessary and sufficient conditions of DCTRSs under which the corresponding unraveled TRSs are left-linear, right-linear, and non-erasing, respectively. All the conditions are syntactic and it is decidable whether a DCTRS satisfies the conditions. Moreover, we show that soundness of the optimized unraveling implies soundness of Ohlebusch’s unraveling, *i.e.*, if the optimized unraveling is sound for a DCTRS, then so is Ohlebusch’s

unraveling. Finally, we show that soundness of the existing unravelings and the SR transformation respectively imply soundness of Ohlebusch’s unraveling. This paper is different from the preliminary version [24] in that we present

- abstract comparison methods for soundness of two transformations from CTRSSs into TRSs (Lemmas 5.3, 5.20 and Theorem 6.2),
- a comparison with other unravelings for join and normal DCTRSSs (Subsection 5.3) in terms of soundness, and
- a comparison with the SR transformation (Section 6) in terms of soundness.

The optimized unraveling has been employed in the (*full* or *partial*) *program inversion* methods for constructor TRSs [18, 22, 23]. The methods first transform a constructor TRS into a DCTRSS that defines (full or partial) inverses of functions defined in the constructor TRS, and then *unravel* the DCTRSS into a TRS (see Example 3.2). The resulting TRS may have extra variables since the intermediate DCTRSS may have extra variables that occur on the right-hand side, but not in the conditional part. For this reason, this paper allows TRSs to have extra variables. In applying a rewrite rule, extra variables of the rule are allowed to be instantiated with arbitrary terms. Since many instantiated terms of extra variables are meaningless and sometimes cause non-termination, we limit reduction sequences to meaningful ones by giving a restriction to reduction sequences of the resulting TRS. The restriction is *EV-safeness* [20, 18, 21] that is a relaxed variant of the *basicness* property [11, 17] of reduction sequences: when a TRS has extra variables, any redex introduced by means of extra variables is not reduced anywhere in the reduction sequences. In this paper, we discuss soundness of unravelings w.r.t. EV-safe derivations of the unraveled TRSs.

It has been shown that the optimized unraveling is sound for the intermediate DCTRSSs of the inversion methods in [18, 22, 23], where conditional rules of the intermediate DCTRSSs are of the restricted form: $l \rightarrow r \leftarrow s_1 \rightarrow t_1; \dots; s_k \rightarrow t_k$ where r, t_1, \dots, t_k are non-variable constructor terms and s_1, \dots, s_k are rooted by defined symbols. Although the optimized unraveling is known to be sound for the intermediate DCTRSSs, studies on soundness conditions of the (optimized) unraveling would be useful when the intermediate DCTRSSs are further transformed into more relaxed forms, *e.g.*, DCTRSSs obtained by removing a unary *tuple* symbol tp_1 (see Example 4.10). Roughly speaking, in applying the inversion method, the resulting TRS is often right-linear if the input constructor TRS is left-linear. Moreover, the resulting TRS is non-erasing if the input constructor TRS is fully inverted, and, in addition, the resulting TRS has no extra variable if the input is non-erasing. Note that injective functions are often defined by non-erasing TRSs and the class of injective functions is the most interesting as an object of program inversion. For the reasons mentioned above, the sufficient conditions shown in this paper can be used to guarantee that the resulting TRSs of the inversion method for left-linear constructor TRSs are definitely inverses of the constructor TRSs (see Example 4.13).

As mentioned previously, Ohlebusch’s unraveling is sound for any DCTRSS if we introduce the particular context-sensitive restriction to the reduction of the corresponding unraveled TRSs. Since recently context-sensitive reduction has been well investigated (*e.g.*, techniques to prove context-sensitive termination) and its interpreter can be easily implemented, the unraveled TRSs with the particular context-sensitivity can be used instead of the original CTRSSs to completely reduce terms over the original signature to terms over the original signature. However, sufficient (syntactic) properties for soundness without the

restriction to the reduction are useful for the use of the unraveled TRSs instead of the original CTRSs since context-sensitivity makes the reduction more complicated than ordinary reduction. Moreover, if the unraveling used in [18, 22, 23] is sound for the resulting TRS obtained by the inversion method without context-sensitivity, then we can apply the restricted version of *completion* [19] to the resulting TRS to make the resulting TRS convergent or to provide useful information for transforming the intermediate DCTRS into an equivalent functional program. For these reasons, soundness of unravelings without any restriction to the reduction is meaningful in order to employ the reduction of the unraveled TRSs instead of the original CTRSs.

In summary, the main contribution of this paper is to show the following:

- the optimized unraveling is sound for a DCTRS that is ultra-left-linear, or both ultra-right-linear and ultra-non-erasing (Theorems 4.3, 4.9),
- soundness of the existing unravelings and the SR transformation respectively implies soundness of Ohlebusch’s unraveling (Corollary 5.5 and Theorems 5.21, 6.12), and
- abstract comparison methods for soundness of two transformations from CTRSs into TRSs (Lemmas 5.3, 5.20 and Theorem 6.2).

All the soundness conditions are summarized at the end of this paper (Table 1 in Subsection 6.4).

This paper is organized as follows. In Section 2, we recall basic notions and notations of term rewriting. In Section 3, we review the existing unravelings for DCTRSs, and present syntactic properties of DCTRSs for some ultra-properties. In Section 4, we show that the optimized unraveling is sound for a DCTRS if the corresponding unraveled TRS is left-linear, or both right-linear and non-erasing. In Section 5, we show that soundness of the existing unravelings for join, normal, and deterministic CTRSs respectively implies soundness of Ohlebusch’s unraveling. In Section 6, we compare soundness of Ohlebusch’s unraveling with soundness of the SR transformation. In Section 7, we briefly describe related work and summarize soundness conditions of unravelings and the SR transformation. In Section 8, we conclude this paper and briefly describe future work on unravelings. Proofs of some technical results are included in the appendix.

2. PRELIMINARIES

In this section, we recall basic notions and notations of term rewriting [3, 26].

Let \rightarrow_L be a binary relation (over a set of A) with a label L . The reflexive closure of \rightarrow_L is denoted by $\rightarrow_L^=$, the transitive closure of \rightarrow_L by \rightarrow_L^+ , and the reflexive and transitive closure of \rightarrow_L by \rightarrow_L^* . The *joinability* relation w.r.t. \rightarrow_L is denoted by \downarrow_L : $\downarrow_L = \rightarrow_L^* \cdot \leftarrow_L^*$. An element $a \in A$ is called a *normal form w.r.t. \rightarrow_L* (or *w.r.t. L*) if there exists no element $b \in A$ such that $a \rightarrow_L b$.

Throughout the paper, we use \mathcal{V} as a countably infinite set of *variables*. Let \mathcal{F} be a *signature*, a finite set of *function symbols* each of which has its own fixed arity, and $\text{arity}(f)$ be the arity of function symbol f . The set of *terms* over \mathcal{F} and \mathcal{V} is denoted by $\mathcal{T}(\mathcal{F}, \mathcal{V})$, and the set of variables appearing in any of terms t_1, \dots, t_n is denoted by $\text{Var}(t_1, \dots, t_n)$. A term t is called *ground* if $\text{Var}(t) = \emptyset$. A term is called *linear* if any variable occurs in the term at most once, and called *linear w.r.t. a variable* if the variable appears at most once in t . The set of *positions* of term t is denoted by $\text{Pos}(t)$. The set of positions for function symbols in t is denoted by $\text{Pos}_{\mathcal{F}}(t)$, and the set of positions for variables in t is denoted by $\text{Pos}_{\mathcal{V}}(t)$. For term t and position p of t , the notation $t|_p$ represents the *subterm* of t at position p .

The function symbol at the *root* position ε of term t is denoted by $\mathbf{root}(t)$. Given an n -hole *context* $C[\]$ with parallel positions p_1, \dots, p_n , the notation $C[t_1, \dots, t_n]_{p_1, \dots, p_n}$ represents the term obtained by replacing hole \square at position p_i with term t_i for all $1 \leq i \leq n$. We may omit the subscription $_{p_1, \dots, p_n}$ from $C[\dots]_{p_1, \dots, p_n}$. For positions p and p' of a term, we write $p' \geq p$ if p is a prefix of p' (*i.e.*, there exists a sequence q such that $pq = p'$). Moreover, we write $p' > p$ if p is a proper prefix of p' .

The *domain* and *range* of a *substitution* σ are denoted by $\mathcal{D}om(\sigma)$ and $\mathcal{R}an(\sigma)$, respectively. We may denote σ by $\{x_1 \mapsto t_1, \dots, x_n \mapsto t_n\}$ if $\mathcal{D}om(\sigma) = \{x_1, \dots, x_n\}$ and $\sigma(x_i) = t_i$ for all $1 \leq i \leq n$. For a signature \mathcal{F} , the set of *substitutions* whose ranges are over \mathcal{F} and \mathcal{V} is denoted by $\mathcal{S}ub(\mathcal{F}, \mathcal{V})$: $\mathcal{S}ub(\mathcal{F}, \mathcal{V}) = \{\sigma \mid \mathcal{R}an(\sigma) \subseteq \mathcal{T}(\mathcal{F}, \mathcal{V})\}$. For a substitution σ and a term t , the application $\sigma(t)$ of σ to t is abbreviated to $t\sigma$, and $t\sigma$ is called an *instance* of t . Given a set X of variables, $\sigma|_X$ denotes the *restricted* substitution of σ w.r.t. X : $\sigma|_X = \{x \mapsto x\sigma \mid x \in \mathcal{D}om(\sigma) \cap X\}$. The *composition* $\sigma\theta$ of substitutions σ and θ is defined as $x(\sigma\theta) = (x\sigma)\theta$.

A *conditional rewrite rule* over a signature \mathcal{F} is a triple (l, r, c) , denoted by $l \rightarrow r \Leftarrow c$, such that the *left-hand side* l is a non-variable term in $\mathcal{T}(\mathcal{F}, \mathcal{V})$, the *right-hand side* r is a term in $\mathcal{T}(\mathcal{F}, \mathcal{V})$, and the *conditional part* c is a sequence $s_1 \approx t_1; \dots; s_k \approx t_k$ of term pairs ($k \geq 0$) where all of $s_1, t_1, \dots, s_k, t_k$ are terms in $\mathcal{T}(\mathcal{F}, \mathcal{V})$. In particular, a conditional rewrite rule is called *unconditional* if the conditional part is the empty sequence (*i.e.*, $k = 0$), and we may abbreviate it to $l \rightarrow r$. The conditional rewrite rule is called *extended* if the condition “ $l \notin \mathcal{V}$ ” is not imposed. We sometimes attach a unique label ρ to the conditional rewrite rule $l \rightarrow r \Leftarrow c$ by denoting $\rho : l \rightarrow r \Leftarrow c$, and we use the label to refer to the rewrite rule. The sets of variables in c and ρ are denoted by $\mathcal{V}ar(c)$ and $\mathcal{V}ar(\rho)$, respectively: $\mathcal{V}ar(s_1 \approx t_1; \dots; s_k \approx t_k) = \mathcal{V}ar(s_1, t_1, \dots, s_k, t_k)$ and $\mathcal{V}ar(\rho) = \mathcal{V}ar(l, r) \cup \mathcal{V}ar(c)$. A variable occurring in r or c is called an *extra variables of ρ* if it does not occur in l . The set of extra variables of ρ is denoted by $\mathcal{E}Var(\rho)$: $\mathcal{E}Var(\rho) = (\mathcal{V}ar(r) \cup \mathcal{V}ar(c)) \setminus \mathcal{V}ar(l)$.

A *conditional term rewriting system* (CTRS) over a signature \mathcal{F} is a set of conditional rules over \mathcal{F} . In particular, a CTRS is called an *EV-TRS* if all of its rules are unconditional, and called an *extended CTRS* (eCTRS) if the condition “ $l \notin \mathcal{V}$ ” of conditional rewrite rules $l \rightarrow r \Leftarrow c$ is not imposed. Moreover, a CTRS is called an (unconditional) *term rewriting system* (TRS) if every rule $l \rightarrow r \Leftarrow c$ in the CTRS is unconditional and satisfies $\mathcal{V}ar(l) \supseteq \mathcal{V}ar(r)$. Note that an eCTRS is called an *eTRS* if all of its rules are unconditional. The *underlying unconditional system* of a CTRS R is denoted by R_u : $R_u = \{l \rightarrow r \mid l \rightarrow r \Leftarrow c \in R\}$.

A CTRS R is called *oriented* if the symbol \approx in the conditions of its rewrite rules is interpreted as *reachability*: the *reduction relation* of R is defined as $\rightarrow_R = \bigcup_{n \geq 0} \rightarrow_{(n), R}$ where

- $\rightarrow_{(0), R} = \emptyset$, and
- $\rightarrow_{(i+1), R} = \rightarrow_{(i), R} \cup \{(C[l\sigma]_p, C[r\sigma]_p) \mid \rho : l \rightarrow r \Leftarrow s_1 \approx t_1; \dots; s_k \approx t_k \in R, s_1\sigma \rightarrow_{(i), R}^* t_1\sigma, \dots, s_k\sigma \rightarrow_{(i), R}^* t_k\sigma\}$ for $i \geq 0$.

Rewrite rules $l \rightarrow r \Leftarrow s_1 \approx t_1; \dots; s_k \approx t_k$ of oriented CTRSs are written as $l \rightarrow r \Leftarrow s_1 \rightarrow t_1; \dots; s_k \rightarrow t_k$. To specify the applied rule ρ and the position p where ρ is applied, we may write $\rightarrow_{p, \rho}$ or $\rightarrow_{p, R}$ instead of \rightarrow_R . Moreover, we may write $\rightarrow_{>\varepsilon, R}$ instead of $\rightarrow_{p, R}$ if $p > \varepsilon$. The *parallel reduction* \rightrightarrows_R is defined as $\rightrightarrows_R = \{(C[s_1, \dots, s_n]_{p_1, \dots, p_n}, C[t_1, \dots, t_n]_{p_1, \dots, p_n}) \mid s_1 \rightarrow_R t_1, \dots, s_n \rightarrow_R t_n\}$. To specify the positions p_1, \dots, p_n in the definition, we may write $\rightrightarrows_{\{p_1, \dots, p_n\}, R}$ instead of \rightrightarrows_R , and we may write $\rightrightarrows_{>\varepsilon, R}$ instead of \rightrightarrows_R if $p_i > \varepsilon$ for all

$1 \leq i \leq n$. Moreover, for a set P of parallel positions, we may write $\Rightarrow_{\geq P, R}$ instead of \Rightarrow_R if, for each position $p_i \in \{p_1, \dots, p_n\}$, there exists a position $p \in P$ such that $p \leq p_i$.

For an eCTRS R , a substitution σ is called *normalized w.r.t. R* if $x\sigma$ is a normal form w.r.t. R for every variable $x \in \text{Dom}(\sigma)$.

An (extended) conditional rewrite rule $l \rightarrow r \Leftarrow c$ is called

- *left-linear* (LL) if l is linear,
- *right-linear* (RL) if r is linear,
- *non-erasing* (NE) if $\text{Var}(l) \subseteq \text{Var}(r)$,
- *non-collapsing* or *non-right-variable* (non-RV) if the right-hand side r is not a variable, and
- *non-left-variable* (non-LV) if l is not a variable.

For a syntactic property P of conditional rewrite rules, we say that an eCTRS has the property P if all of its rules have the property P , *e.g.*, an eCTRS is called *left-linear* (LL) if all of its rules are LL. Note that a non-LV eCTRS is a CTRS.

An (extended) conditional rewrite rule $\rho : l \rightarrow r \Leftarrow s_1 \twoheadrightarrow t_1; \dots; s_k \twoheadrightarrow t_k$ is called *deterministic* if $\text{Var}(s_i) \subseteq \text{Var}(l, t_1, \dots, t_{i-1})$ for all $1 \leq i \leq k$. An (e)CTRS is called *deterministic*, an *(e)DCTRS* for short, if all of its rules are deterministic. Conditional rule ρ is classified according to the distribution of variables in the rule as follows:

- *Type 1* if $\text{Var}(r, s_1, t_1, \dots, s_k, t_k) \subseteq \text{Var}(l)$,
- *Type 2* if $\text{Var}(r) \subseteq \text{Var}(l)$,
- *Type 3* if $\text{Var}(r) \subseteq \text{Var}(l, s_1, t_1, \dots, s_k, t_k)$, and
- *Type 4* otherwise.

An (eD)CTRS is called an *i -(eD)CTRS* if all of its rules are of Type i . An eDCTRS R is called *normal* (or a *normal CTRS*) if, for every rule $l \rightarrow r \Leftarrow s_1 \twoheadrightarrow t_1; \dots; s_k \twoheadrightarrow t_k \in R$, all of t_1, \dots, t_k are ground normal forms w.r.t. R_u .

Let R be a CTRS over a signature \mathcal{F} . The sets of *defined symbols* and *constructors* of R are denoted by \mathcal{D}_R and \mathcal{C}_R , respectively: $\mathcal{D}_R = \{\text{root}(l) \mid l \rightarrow r \Leftarrow c \in R\}$ and $\mathcal{C}_R = \mathcal{F} \setminus \mathcal{D}_R$. Terms in $\mathcal{T}(\mathcal{C}_R, \mathcal{V})$ are *constructor terms of R* . R is called a *constructor system* if all proper subterms of the left-hand sides in R are constructor terms of R .

Let R be a CTRS. Two conditional rewrite rules $l_1 \rightarrow r_1 \Leftarrow c_1$ and $l_2 \rightarrow r_2 \Leftarrow c_2$ in R are called *overlapping* if there exists a context $C[\]$ and a non-variable term t such that $l_2 = C[t]$ and l_1 and t are unifiable, where we assume w.l.o.g. that these rules share no variable. Then, a conditional pair of terms $((C[r_1])\theta, r_2\theta) \Leftarrow c_1\theta; c_2\theta$ is called a *critical pair* of R where θ is a *most general unifier* of l_1 and t . A critical pair $(s, t) \Leftarrow c$ is called *trivial* if $s = t$, and called *infeasible* if for any substitution σ , c contains a condition $u \twoheadrightarrow v$ such that $u\sigma \not\rightarrow_R^* v\sigma$ [12] (*cf.*, [26]).

Let $\mathcal{F}_1, \mathcal{F}_2$ be signatures, $\mathcal{G} \subseteq \mathcal{F}_1 \cap \mathcal{F}_2$, and $\rightarrow_1, \rightarrow_2$ be binary relations on terms in $\mathcal{T}(\mathcal{F}_1, \mathcal{V})$ and $\mathcal{T}(\mathcal{F}_2, \mathcal{V})$, respectively. We say that $\rightarrow_1 \subseteq \rightarrow_2$ *on terms in $\mathcal{T}(\mathcal{G}, \mathcal{V})$* if, for all terms $s, t \in \mathcal{T}(\mathcal{G}, \mathcal{V})$, $s \rightarrow_2 t$ whenever $s \rightarrow_1 t$.

3. UNRAVELINGS FOR DETERMINISTIC CTRSs

In this section, we first recall unravelings for DCTRSs, and then show some syntactic properties of DCTRSs, that are related to the syntactic properties of the unraveled TRSs. The unravelings and some results in this section are straightforwardly extended to eDCTRSs.

We first recall the notion of unravelings. A computable transformation U from eCTRSs into eTRSs is called an *unraveling* if for every eCTRS R , we have $\rightarrow_R \subseteq \rightarrow_{U(R)}^*$ and $U(R' \cup R) = R' \cup U(R)$ whenever R' is an eTRS [15, 16].¹ Unraveling U is called *tidy* if it has *compositionality* ($U(R_1 \cup R_2) = U(R_1) \cup U(R_2)$), *finiteness* (if R is finite, then so is $U(R)$), and *emptiness* (if R is empty, then so is $U(R)$) [15]. Let R be an eCTRS over a signature \mathcal{F} , and $\Rightarrow_{U(R)}$ be a subrelation of $\rightarrow_{U(R)}$. U is called *sound w.r.t. reduction for R w.r.t. $\Rightarrow_{U(R)}$* (*simulation-sound* [21, 22], or simply *sound for R w.r.t. $\Rightarrow_{U(R)}$*) if $\Rightarrow_{U(R)}^* \subseteq \rightarrow_R^*$ on terms in $\mathcal{T}(\mathcal{F}, \mathcal{V})$ (i.e., for all terms s, t in $\mathcal{T}(\mathcal{F}, \mathcal{V})$, if $s \Rightarrow_{U(R)}^* t$, then $s \rightarrow_R^* t$). U is called *complete w.r.t. reduction for R w.r.t. $\Rightarrow_{U(R)}$* (or simply *complete for R w.r.t. $\Rightarrow_{U(R)}$*) if $\rightarrow_R^* \subseteq \Rightarrow_{U(R)}^*$ on terms in $\mathcal{T}(\mathcal{F}, \mathcal{V})$. We omit “w.r.t. $\rightarrow_{U(R)}$ ” if $\Rightarrow_{U(R)} = \rightarrow_{U(R)}$.

Next, we recall an unraveling for DCTRSs, proposed by Ohlebusch [25] that is a natural improvement of Marchiori’s unraveling [16]. For a finite set $T = \{t_1, \dots, t_n\}$, given some fixed ordering \prec such that $t_1 \prec \dots \prec t_n$, \vec{T} denotes the unique sequence t_1, \dots, t_n of elements in T .

Definition 3.1 (unraveling \mathbb{U} [25]). Let R be an eDCTRS over a signature \mathcal{F} . For every conditional rule $\rho : l \rightarrow r \Leftarrow s_1 \twoheadrightarrow t_1; \dots; s_k \twoheadrightarrow t_k$ in R , we prepare k fresh function symbols $U_1^\rho, \dots, U_k^\rho$, called *U symbols*, that do not appear in \mathcal{F} . We transform $\rho : l \rightarrow r \Leftarrow s_1 \twoheadrightarrow t_1; \dots; s_k \twoheadrightarrow t_k$ into a set $\mathbb{U}(\rho)$ of $k+1$ unconditional rewrite rules as follows:

$$\mathbb{U}(\rho) = \left\{ \begin{array}{l} l \rightarrow U_1^\rho(s_1, \vec{X}_1) \\ U_1^\rho(t_1, \vec{X}_1) \rightarrow U_2^\rho(s_2, \vec{X}_2) \\ \vdots \\ U_k^\rho(t_k, \vec{X}_k) \rightarrow r \end{array} \right\}$$

where $X_i = \text{Var}(l, t_1, \dots, t_{i-1})$ for all $1 \leq i \leq k$. Note that $\mathbb{U}(l' \rightarrow r') = \{l' \rightarrow r'\}$. \mathbb{U} is extended to eDCTRSs (i.e., $\mathbb{U}(R) = \bigcup_{\rho \in R} \mathbb{U}(\rho)$) and $\mathbb{U}(R)$ is an eTRS over the extended signature $\mathcal{F}_{\mathbb{U}(R)} = \mathcal{F} \cup \{U_i^\rho \mid \rho : l \rightarrow r \Leftarrow s_1 \twoheadrightarrow t_1; \dots; s_k \twoheadrightarrow t_k \in R, 1 \leq i \leq k\}$.

It is clear that $\rightarrow_R \subseteq \rightarrow_{\mathbb{U}(R)}^*$, and $\mathbb{U}(R' \uplus R) = R' \cup \mathbb{U}(R)$ if R' is unconditional. Moreover, by definition, \mathbb{U} has compositionality, finiteness, and emptiness. Thus, \mathbb{U} is a tidy unraveling for eDCTRSs.

The variant \mathbb{U}_{opt} of Ohlebusch’s unraveling \mathbb{U} is proposed in both [18] and [7]. For a conditional rewrite rule $\rho : l \rightarrow r \Leftarrow s_1 \twoheadrightarrow t_1; \dots; s_k \twoheadrightarrow t_k$, the set $\mathbb{U}_{\text{opt}}(\rho)$ of unconditional rewrite rules is defined by replacing \vec{X}_i in $\mathbb{U}(\rho)$ by $\overline{X_i \cap \text{Var}(r, t_i, s_{i+1}, t_{i+1}, \dots, s_k, t_k)}$ for all $1 \leq i \leq k$:

$$\mathbb{U}_{\text{opt}}(\rho) = \left\{ \begin{array}{l} l \rightarrow U_1^\rho(s_1, \overline{X_1 \cap Y_1}) \\ U_1^\rho(t_1, \overline{X_1 \cap Y_1}) \rightarrow U_2^\rho(s_2, \overline{X_2 \cap Y_2}) \\ \vdots \\ U_k^\rho(t_k, \overline{X_k \cap Y_k}) \rightarrow r \end{array} \right\}$$

where $Y_i = \text{Var}(r, t_i, s_{i+1}, t_{i+1}, \dots, s_k, t_k)$ for all $1 \leq i \leq k$. Note that $\mathbb{U}_{\text{opt}}(R)$ is an eTRS over the extended signature $\mathcal{F}_{\mathbb{U}_{\text{opt}}(R)}$ where $\mathcal{F}_{\mathbb{U}_{\text{opt}}(R)} = \mathcal{F} \cup \{U_i^\rho \mid \rho : l \rightarrow r \Leftarrow s_1 \twoheadrightarrow$

¹ In the original definition [15], not the property $\rightarrow_R \subseteq \rightarrow_{U(R)}^*$ but the property $\downarrow_R \subseteq \downarrow_{U(R)}$ is imposed. Under this property, unravelings are not complete in general. For example, if $\rightarrow_R \subseteq \leftarrow_{U(R)}^*$, then U is an unraveling. However, all the existing unravelings are designed so as to satisfy $\rightarrow_R \subseteq \rightarrow_{U(R)}^*$, that is implicitly required of unravelings. For this reason, this paper imposes the more restrictive property $\rightarrow_R \subseteq \rightarrow_{U(R)}^*$.

$t_1; \dots; s_k \rightarrow t_k \in R, 1 \leq i \leq k$.² Note also that \mathbb{U}_{opt} is a tidy unraveling for eDCTRSs. Y_i above, the set of variables appearing in any of $r, t_i, s_{i+1}, t_{i+1}, \dots, s_k, t_k$, is the set of variables that are referred after s_i is considered. Thus, $X_i \cap Y_i$ is the set of variables that appear in any of l, t_1, \dots, t_{i-1} and also appear after s_i is considered, and hence $\overrightarrow{X_i \cap Y_i}$ is used for propagating only the variables that are referred later. On the other hand, $\overrightarrow{X_i}$ in Definition 3.1 is used for propagating *all* the appeared variables. This is the only difference between \mathbb{U} and \mathbb{U}_{opt} , and the reason why \mathbb{U}_{opt} is sometimes called an *optimized* variant of \mathbb{U} . Note that all of the following are equivalent:

- R is of Type 3,
- $\mathbb{U}(R)$ has no extra variables, and
- $\mathbb{U}_{\text{opt}}(R)$ has no extra variables.

In the rest of the paper, unless noted otherwise, we use the label ρ for presenting a conditional rewrite rule $l \rightarrow r \Leftarrow s_1 \rightarrow t_1; \dots; s_k \rightarrow t_k$, and we denote the sets $\mathcal{V}ar(l, t_1, \dots, t_{i-1})$, $\mathcal{V}ar(r, t_i, s_{i+1}, t_{i+1}, \dots, s_k, t_k)$, and $X_i \cap Y_i$ by X_i , Y_i , and Z_i , respectively.

Example 3.2. Consider the following TRS defining addition and multiplication of natural numbers encoded as $0, s(0), s(s(0)), \dots$:

$$R_1 = \left\{ \begin{array}{l} \text{add}(0, y) \rightarrow y \\ \text{add}(s(x), y) \rightarrow s(\text{add}(x, y)) \\ \text{mult}(0, y) \rightarrow 0 \\ \text{mult}(x, 0) \rightarrow 0 \\ \text{mult}(s(x), s(y)) \rightarrow s(\text{add}(\text{mult}(x, s(y)), y)) \end{array} \right\}$$

The inversion method in [18] inverts this TRS to the following DCTRS R_2 where add^{-1} and mult^{-1} are function symbols that define the inverse relation of add and mult , respectively,³ and tp_2 is a binary constructor for representing tuples of two terms:

$$R_2 = \left\{ \begin{array}{l} \text{add}^{-1}(y) \rightarrow \text{tp}_2(0, y) \\ \text{add}^{-1}(s(z)) \rightarrow \text{tp}_2(s(x), y) \Leftarrow \text{add}^{-1}(z) \rightarrow \text{tp}_2(x, y) \\ \text{mult}^{-1}(0) \rightarrow \text{tp}_2(0, y) \\ \text{mult}^{-1}(0) \rightarrow \text{tp}_2(x, 0) \\ \text{mult}^{-1}(s(z)) \rightarrow \text{tp}_2(s(x), s(y)) \Leftarrow \text{add}^{-1}(z) \rightarrow \text{tp}_2(w, y); \text{mult}^{-1}(w) \rightarrow \text{tp}_2(x, s(y)) \end{array} \right\}$$

This DCTRS is unraveled by \mathbb{U} and \mathbb{U}_{opt} as follows:

$$\mathbb{U}(R_2) = \left\{ \begin{array}{l} \vdots \\ \text{add}^{-1}(s(z)) \rightarrow \text{U}_1(\text{add}^{-1}(z), z) \\ \text{U}_1(\text{tp}_2(x, y), z) \rightarrow \text{tp}_2(s(x), y) \\ \vdots \\ \text{mult}^{-1}(s(z)) \rightarrow \text{U}_2(\text{add}^{-1}(z), z) \\ \text{U}_2(\text{tp}_2(w, y), z) \rightarrow \text{U}_3(\text{mult}^{-1}(w), z, w, y) \\ \text{U}_3(\text{tp}_2(x, s(y)), z, w, y) \rightarrow \text{tp}_2(s(x), s(y)) \end{array} \right\}$$

² The extended signatures $\mathcal{F}_{\mathbb{U}(R)}$ and $\mathcal{F}_{\mathbb{U}_{\text{opt}}(R)}$ are not equivalent in terms of the arities of \mathbb{U} symbols (see, e.g., Example 3.2). We distinguish between these extended signatures since we deal with mappings from $\mathcal{T}(\mathcal{F}_{\mathbb{U}(R)}, \mathcal{V})$ to $\mathcal{T}(\mathcal{F}_{\mathbb{U}_{\text{opt}}(R)}, \mathcal{V})$ in Subsection 5.2.

³ As inverse computation of $\text{add}(s^m(0), s^n(0)) \rightarrow_{R_1}^* s^{m+n}(0)$ and $\text{mult}(s^m(0), s^n(0)) \rightarrow_{R_1}^* s^{m \times n}(0)$, we have the derivations $\text{add}^{-1}(s^{m+n}(0)) \rightarrow_{R_2}^* \text{tp}_2(s^m(0), s^n(0))$ and $\text{mult}^{-1}(s^{m \times n}(0)) \rightarrow_{R_2}^* \text{tp}_2(s^m(0), s^n(0))$.

$$\mathbb{U}_{\text{opt}}(R_2) = \left\{ \begin{array}{l} \vdots \\ \text{add}^{-1}(s(z)) \rightarrow \text{U}_1(\text{add}^{-1}(z)) \\ \text{U}_1(\text{tp}_2(x, y)) \rightarrow \text{tp}_2(s(x), y) \\ \vdots \\ \text{mult}^{-1}(s(z)) \rightarrow \text{U}_2(\text{add}^{-1}(z)) \\ \text{U}_2(\text{tp}_2(w, y)) \rightarrow \text{U}_3(\text{mult}^{-1}(w), y) \\ \text{U}_3(\text{tp}_2(x, s(y)), y) \rightarrow \text{tp}_2(s(x), s(y)) \end{array} \right\}$$

Unravelings are not sound in general. The CTRS shown in the following example is a counterexample against soundness of both \mathbb{U} and \mathbb{U}_{opt} .

Example 3.3 ([15, 26]). Consider the following 3-DCTRS R_3 and its unraveled TRSs:

$$R_3 = \left\{ \begin{array}{l} f(x) \rightarrow x \leftarrow x \rightarrow e \\ g(d, x, x) \rightarrow A \\ h(x, x) \rightarrow g(x, x, f(k)) \end{array} \right\} \cup R_0$$

$$\mathbb{U}(R_3) = \mathbb{U}_{\text{opt}}(R_3) = \left\{ \begin{array}{l} f(x) \rightarrow \text{U}_4(x, x) \\ \text{U}_4(e, x) \rightarrow x \\ \vdots \end{array} \right\} \cup R_0$$

where

$$R_0 = \left\{ \begin{array}{ccccc} a \rightarrow c & b \rightarrow c & c \rightarrow e & k \rightarrow l & d \rightarrow m \\ a \rightarrow d & b \rightarrow d & c \rightarrow l & k \rightarrow m & \end{array} \right\}$$

We have a reduction sequence of $\mathbb{U}_{\text{opt}}(R_3)$ from $h(f(a), f(b))$ to A :

$$\begin{aligned} h(f(a), f(b)) &\rightarrow_{\mathbb{U}_{\text{opt}}(R_3)}^* h(\text{U}_4(c, d), \text{U}_4(c, d)) \rightarrow_{\mathbb{U}_{\text{opt}}(R_3)} g(\text{U}_4(c, d), \text{U}_4(c, d), f(k)) \\ &\rightarrow_{\mathbb{U}_{\text{opt}}(R_3)}^* g(d, \text{U}_4(l, m), \text{U}_4(l, m)) \rightarrow_{\mathbb{U}_{\text{opt}}(R_3)} A \end{aligned}$$

However, we have no similar reduction sequence of R_3 , *i.e.*, $h(f(a), f(b)) \not\rightarrow_{R_3}^* A$. Thus, neither \mathbb{U} nor \mathbb{U}_{opt} is sound for R_3 . Note that being (ultra-)overlapping-systems is not sufficient for soundness of \mathbb{U}_{opt} and \mathbb{U} since $\mathbb{U}_{\text{opt}}(R_3)$ ($= \mathbb{U}(R_3)$) is an overlapping system.

Soundness of \mathbb{U} can be recovered by restricting the reduction of the unraveled TRSs to the *context-sensitive* reduction [13] with the *replacement mapping* determined by means of the application of \mathbb{U} [28, 29]: \mathbb{U} is sound for a 3-DCTRS R if the reduction of $\mathbb{U}(R)$ is restricted to context-sensitive rewriting with the replacement mapping μ such that $\mu(U_i^\rho) = \{1\}$ for any U symbol U_i^ρ —the replacement mapping forbids reducing any redex inside the second or later arguments of U symbols. This holds for \mathbb{U}_{opt} by restricting the context-sensitive reduction to the reduction with the *membership condition* [34], a very complicated restriction that soundness of \mathbb{U} does not require. In this respect, \mathbb{U}_{opt} does not look like an “optimized” variant of \mathbb{U} . The following examples show that neither the context-sensitive nor membership conditions above is sufficient on its own for soundness of \mathbb{U}_{opt} .

Example 3.4. Consider the following DCTRS and its unraveled TRSs:

$$R_4 = \{ f(x, y) \rightarrow x \leftarrow g(x) \rightarrow z; g(y) \rightarrow z \quad g(x) \rightarrow c \leftarrow d \rightarrow c \}$$

$$\mathbb{U}(R_4) = \left\{ \begin{array}{lll} f(x, y) \rightarrow \text{U}_5(g(x), x, y) & \text{U}_5(z, x, y) \rightarrow \text{U}_6(g(y), x, y, z) & \text{U}_6(z, x, y, z) \rightarrow x \\ g(x) \rightarrow \text{U}_7(d, x) & \text{U}_7(c, x) \rightarrow c & \end{array} \right\}$$

$$\mathbb{U}_{\text{opt}}(R_4) = \left\{ \begin{array}{lll} f(x, y) \rightarrow \text{U}_5(g(x), x, y) & \text{U}_5(z, x, y) \rightarrow \text{U}_6(g(y), x, z) & \text{U}_6(z, x, z) \rightarrow x \\ g(x) \rightarrow \text{U}_7(d) & \text{U}_7(c) \rightarrow c & \end{array} \right\}$$

For the context-sensitive condition mentioned above, we forbid reducing any redex inside the second or third arguments of U_5 and U_6 . We have the derivation $f(a, b) \rightarrow_{\mathbb{U}_{\text{opt}}(R_4)}^* U_6(U_7(d), a, U_7(d)) \rightarrow_{\mathbb{U}_{\text{opt}}(R_4)} a$ under the context-sensitive condition, but this derivation is not possible in R_4 . Therefore, the context-sensitive condition is not sufficient on its own for soundness of \mathbb{U}_{opt} . Note that the derivation $f(a, b) \rightarrow^* a$ does not hold in $\mathbb{U}(R_4)$ under the context-sensitive condition, either, since $f(a, b)$ can be reduced to both $U_6(U_7(d, b), a, g(a))$ and $U_6(U_7(d, a), a, U_7(d, b))$, but they are not reduced any more.

Example 3.5. Consider the following DCTRS R_5 and its unraveled TRSs:

$$R_5 = \{ f(x) \rightarrow x \Leftarrow x \rightarrow a; b \rightarrow x \quad a \rightarrow b \}$$

$$\mathbb{U}(R_5) = \mathbb{U}_{\text{opt}}(R_5) = \{ f(x) \rightarrow U_8(x, x) \quad U_8(a, x) \rightarrow U_9(b, x) \quad U_9(x, x) \rightarrow x \quad \dots \}$$

For the membership condition, we forbid reducing any redex that has a proper subterm containing U symbols. We have the derivation $f(a) \rightarrow_{\mathbb{U}_{\text{opt}}(R_5)}^* U_9(b, a) \rightarrow_{\mathbb{U}_{\text{opt}}(R_5)} U_9(b, b) \rightarrow_{\mathbb{U}_{\text{opt}}(R_5)} b$ under the membership condition, but this derivation is not possible in R_5 . Therefore, the membership condition is not sufficient on its own for soundness of either \mathbb{U}_{opt} or \mathbb{U} .

To analyze syntactic relationships between eDCTRS and the corresponding unraveled eTRSs, we recall *ultra-properties* of DCTRSs [15, 16], extending them to eDCTRSs.

Definition 3.6 (ultra-property [15, 16]). Let P be a property on (extended) conditional rewrite rules, and U be an unraveling. An (extended) conditional rewrite rule ρ is said to be *ultra- P w.r.t. U* (U - P) if all the rules in $U(\rho)$ satisfy the property P . An eDCTRS R is said to be *ultra- P w.r.t. U* (U - P) if all the rules in R are U - P .

Example 3.7. The DCTRS R_2 in Example 3.2 is non-LV and non-RV w.r.t. both \mathbb{U} and \mathbb{U}_{opt} , but R_2 is not \mathbb{U} -LL, \mathbb{U} -RL, or \mathbb{U} -NE either, while R_2 is \mathbb{U}_{opt} -RL and \mathbb{U}_{opt} -NE, but not \mathbb{U}_{opt} -LL.

Note that the \mathbb{U}_{opt} -LL property is the same as *semi-linearity* in [16]. Roughly speaking, the conditional parts of \mathbb{U}_{opt} -LL conditional rules correspond to the **let** structures of functional programs.

The \mathbb{U}_{opt} -LL, \mathbb{U}_{opt} -RL, and \mathbb{U}_{opt} -NE properties of conditional rewrite rules are identical with the following syntactic properties of DCTRSs, respectively.

Theorem 3.8. Let $\rho : l \rightarrow r \Leftarrow s_1 \rightarrow t_1; \dots; s_k \rightarrow t_k$ be an extended deterministic conditional rewrite rule. Then, all of the following hold:

- (1) ρ is \mathbb{U}_{opt} -LL iff all of l, t_1, \dots, t_k are linear and $\text{Var}(t_i) \cap X_i = \emptyset$ for all $1 \leq i \leq k$,
- (2) ρ is \mathbb{U}_{opt} -RL iff all of r, s_1, \dots, s_k are linear and $\text{Var}(s_i) \cap Y_i = \emptyset$ for all $1 \leq i \leq k$,
and
- (3) ρ is \mathbb{U}_{opt} -NE iff $\text{Var}(l) \subseteq \text{Var}(r, s_1, \dots, s_k)$ and $\text{Var}(t_i) \subseteq \text{Var}(r, s_{i+1}, \dots, s_k)$ for all $1 \leq i \leq k$.

Proof. The proof can be seen in Appendix A.1. □

The sufficient and necessary condition for the \mathbb{U}_{opt} -NE property in Theorem 3.8 is equivalent to the one shown in [18, 23] since the following are equivalent:

- $\mathcal{V}ar(l) \subseteq \mathcal{V}ar(r, s_1, t_1, \dots, s_k, t_k)$ and $\mathcal{V}ar(t_i) \subseteq \mathcal{V}ar(r, s_{i+1}, t_{i+1}, \dots, s_k, t_k)$ for all $1 \leq i \leq k$, and
- $\mathcal{V}ar(l) \subseteq \mathcal{V}ar(r, s_1, \dots, s_k)$ and $\mathcal{V}ar(t_i) \subseteq \mathcal{V}ar(r, s_{i+1}, \dots, s_k)$ for all $1 \leq i \leq k$.

Neither the second nor third claims in Theorem 3.8 holds for \mathbb{U} (*cf.*, Examples 3.2, 3.7), while the first one holds for \mathbb{U} . Quite restricted variants of the second and third claims hold for \mathbb{U} .

Theorem 3.9. *Let $\rho : l \rightarrow r \Leftarrow s_1 \rightarrow t_1; \dots; s_k \rightarrow t_k$ be an extended deterministic conditional rewrite rule. Then, all of the following hold:*

- (1) ρ is \mathbb{U} -LL iff all of l, t_1, \dots, t_k are linear and $\mathcal{V}ar(t_i) \cap X_i = \emptyset$ for all $1 \leq i \leq k$,
- (2) ρ is \mathbb{U} -RL iff r is linear and all of s_1, \dots, s_k are ground, and
- (3) ρ is \mathbb{U} -NE iff $\mathcal{V}ar(l, t_1, \dots, t_k) \subseteq \mathcal{V}ar(r)$.

Proof. The proof can be seen in Appendix A.2. □

Note that the \mathbb{U} -LL and \mathbb{U}_{opt} -LL properties are equivalent. Theorems 3.8, 3.9 lead to the following relationship between the ultra-RL and ultra-NE properties w.r.t. \mathbb{U} and \mathbb{U}_{opt} .

Corollary 3.10. *The \mathbb{U} -RL and \mathbb{U} -NE properties imply the \mathbb{U}_{opt} -RL and \mathbb{U}_{opt} -NE properties, respectively.*

As for the non-LV and non-RV properties, we have the following relationships between eDCTRSs and the corresponding unraveled eTRSs.

Proposition 3.11. *Let U be either \mathbb{U} or \mathbb{U}_{opt} , R be an eDCTRS, and ρ be an (extended) conditional rewrite rule. Then, all of the following hold:*

- ρ is non-LV iff so is $U(\rho)$,
- ρ is non-RV iff so is $U(\rho)$,
- R is non-LV iff so is $U(R)$, and
- R is non-RV iff so is $U(R)$.

Proof. Trivial by definition. □

We recognize from Proposition 3.11 that, for both \mathbb{U} and \mathbb{U}_{opt} , the non-LV and non-RV properties are equivalent to the ultra-non-LV and ultra-non-RV properties, respectively.

4. SOUNDNESS OF THE OPTIMIZED UNRAVELING

In this section, we first show that the optimized unraveling \mathbb{U}_{opt} is sound for \mathbb{U}_{opt} -LL 3-DCTRSs. Then, we show that \mathbb{U}_{opt} is sound for DCTRSs that are both \mathbb{U}_{opt} -RL and \mathbb{U}_{opt} -NE. Finally, we extend the result on soundness for \mathbb{U}_{opt} -LL 3-DCTRSs to \mathbb{U}_{opt} -LL DCTRSs, *i.e.*, \mathbb{U}_{opt} is sound for a \mathbb{U}_{opt} -LL DCTRS if the reduction of the corresponding unraveled EV-TRS is restricted to *EV-safe* ones (see Definition 4.14). In the rest of this paper, we write the terminology “RLNE” for “RL and NE”.

4.1. Soundness on Ultra-Left-Linearity. In this subsection, we first show that the LL property is not a soundness condition of either \mathbb{U}_{opt} or \mathbb{U} , and then we show that \mathbb{U}_{opt} is sound for \mathbb{U}_{opt} -LL 3-DCTRSs. This result also holds for arbitrary DCTRSs under some restriction to reduction. Although we first show the case of 3-DCTRSs to make the essential scheme of the proof clear, we will extend the result in this subsection to DCTRSs in Subsection 4.4.

As described in Section 1, the LL property is a soundness condition for unravelings for normal CTRSs. In contrast, the LL property is not a soundness condition for either \mathbb{U}_{opt} or \mathbb{U} .

Example 4.1. Consider the following DCTRS obtained from R_3 by left-linearizing:

$$R_6 = \left\{ \begin{array}{l} f(x) \rightarrow x \Leftarrow x \rightarrow e \\ g(d, x, y) \rightarrow A \Leftarrow y \rightarrow x \\ h(x, y) \rightarrow g(x, y, f(k)) \Leftarrow y \rightarrow x \end{array} \right\} \cup R_0$$

R_6 is unraveled by \mathbb{U}_{opt} and \mathbb{U} to the following TRSs:

$$\begin{aligned} \mathbb{U}_{\text{opt}}(R_6) &= \left\{ \begin{array}{ll} f(x) \rightarrow \mathbb{U}_{10}(x, x) & \mathbb{U}_{10}(e, x) \rightarrow x \\ g(d, x, y) \rightarrow \mathbb{U}_{11}(y, x) & \mathbb{U}_{11}(x, x) \rightarrow A \\ h(x, y) \rightarrow \mathbb{U}_{12}(y, x, y) & \mathbb{U}_{12}(x, x, y) \rightarrow g(x, y, f(k)) \end{array} \right\} \cup R_0 \\ \mathbb{U}(R_6) &= \left\{ \begin{array}{ll} f(x) \rightarrow \mathbb{U}_{10}(x, x) & \mathbb{U}_{10}(e, x) \rightarrow x \\ g(d, x, y) \rightarrow \mathbb{U}_{11}(y, x, y) & \mathbb{U}_{11}(x, x, y) \rightarrow A \\ h(x, y) \rightarrow \mathbb{U}_{12}(y, x, y) & \mathbb{U}_{12}(x, x, y) \rightarrow g(x, y, f(k)) \end{array} \right\} \cup R_0 \end{aligned}$$

As in Example 3.3, we have the derivations $h(f(a), f(b)) \rightarrow_{\mathbb{U}_{\text{opt}}(R_6)}^* A$ and $h(f(a), f(b)) \rightarrow_{\mathbb{U}(R_6)}^* A$, but $h(f(a), f(b))$ cannot be reduced by R_6 to A . Therefore, neither \mathbb{U}_{opt} nor \mathbb{U} is sound for R_6 .

The LL property of normal CTRSs is equivalent to the \mathbb{U}_{opt} -LL property since the right-hand sides n_i of conditions $s_i \rightarrow n_i$ are ground. In contrast, the LL property of DCTRSs is not equivalent to the ultra-LL property in general (see $\mathbb{U}_{\text{opt}}(R_6)$ and $\mathbb{U}(R_6)$ in Example 4.1). Moreover, the LL property of the unraveled TRSs plays an important role in the existing proof of soundness. Thus, the ultra-LL property seems a soundness condition for \mathbb{U}_{opt} (and also for \mathbb{U}).

The soundness result of this subsection is a consequence of the following key lemma: given a derivation $s \rightarrow_{\mathbb{U}_{\text{opt}}(R)}^* t\sigma$ with $s, t \in \mathcal{T}(\mathcal{F}, \mathcal{V})$, the lemma guarantees the existence of an intermediate term $t\theta \in \mathcal{T}(\mathcal{F}, \mathcal{V})$ such that $s \rightarrow_R^* t\theta \rightarrow_{\mathbb{U}_{\text{opt}}(R)}^* t\sigma$ and, moreover, $t\theta = t\sigma$ whenever $t\sigma \in \mathcal{T}(\mathcal{F}, \mathcal{V})$.

Lemma 4.2. *Let R be a \mathbb{U}_{opt} -LL 3-eDCTRS over a signature \mathcal{F} , s be a term in $\mathcal{T}(\mathcal{F}, \mathcal{V})$, t be a linear term in $\mathcal{T}(\mathcal{F}, \mathcal{V})$, and σ be a substitution in $\text{Sub}(\mathcal{F}_{\mathbb{U}_{\text{opt}}(R)}, \mathcal{V})$. Suppose that R is non-LV or non-RV. If $s \rightrightarrows_{\mathbb{U}_{\text{opt}}(R)}^n t\sigma$ for some $n \geq 0$, then there exists a substitution θ in $\text{Sub}(\mathcal{F}, \mathcal{V})$ such that*

- $s \rightarrow_R^* t\theta \rightrightarrows_{\geq \text{Pos}_{\mathcal{V}}(t), \mathbb{U}_{\text{opt}}(R)}^{n'} t\sigma$ for some $n' \leq n$, and
- if $t\sigma \in \mathcal{T}(\mathcal{F}, \mathcal{V})$, then $t\theta = t\sigma$.

Proof. The proof can be seen in Appendix A.3. □

As a consequence of Lemma 4.2, we show the main theorem of this subsection.

Theorem 4.3. \mathbb{U}_{opt} is sound for \mathbb{U}_{opt} -LL 3-eDCTRSs that are non-LV or non-RV.

Proof. Let R be a 3-eDCTRS over a signature \mathcal{F} , that is non-LV or non-RV. Suppose that $s \rightarrow_{\mathbb{U}_{\text{opt}}(R)}^* t$ and $s, t \in \mathcal{T}(\mathcal{F}, \mathcal{V})$. Since a single step of $\rightarrow_{\mathbb{U}_{\text{opt}}(R)}$ can be considered a single step of the parallel reduction, we have the derivation $s \rightrightarrows_{\mathbb{U}_{\text{opt}}(R)}^* t$. Let x be a variable and σ be a substitution such that $x\sigma = t$. Then, it follows from Lemma 4.2 that $s \rightarrow_R^* x\sigma = t$. \square

Example 4.4. Consider the following \mathbb{U}_{opt} -LL and non-LV DCTRS to define a splitting function `split` for lists of non-negative integers encoded as $0, \mathbf{s}(0), \dots, e.g., \mathbf{split}(3, [2, 5, 1, 4, 3]) = ([2, 1], [5, 4, 3])$:

$$R_7 = \left\{ \begin{array}{l} \mathbf{split}(x, \mathbf{nil}) \rightarrow \mathbf{tp}_2(\mathbf{nil}, \mathbf{nil}) \\ \mathbf{split}(x, \mathbf{cons}(y, ys)) \rightarrow \mathbf{tp}_2(zs_1, \mathbf{cons}(y, zs_2)) \Leftarrow \mathbf{split}(x, ys) \rightarrow \mathbf{tp}_2(zs_1, zs_2); \mathbf{le}(x, y) \rightarrow \mathbf{true} \\ \mathbf{split}(x, \mathbf{cons}(y, ys)) \rightarrow \mathbf{tp}_2(\mathbf{cons}(y, zs_1), zs_2) \Leftarrow \mathbf{split}(x, ys) \rightarrow \mathbf{tp}_2(zs_1, zs_2); \mathbf{le}(x, y) \rightarrow \mathbf{false} \\ \mathbf{le}(0, y) \rightarrow \mathbf{true} \\ \mathbf{le}(\mathbf{s}(x), 0) \rightarrow \mathbf{false} \\ \mathbf{le}(\mathbf{s}(x), \mathbf{s}(y)) \rightarrow \mathbf{le}(x, y) \end{array} \right.$$

R_7 is unraveled by \mathbb{U}_{opt} and \mathbb{U} into the following TRSs:

$$\mathbb{U}_{\text{opt}}(R_7) = \left\{ \begin{array}{l} \vdots \\ \mathbf{split}(x, \mathbf{cons}(y, ys)) \rightarrow \mathbb{U}_{13}(\mathbf{split}(x, ys), x, y) \\ \mathbb{U}_{13}(\mathbf{tp}_2(zs_1, zs_2), x, y) \rightarrow \mathbb{U}_{14}(\mathbf{le}(x, y), y, zs_1, zs_2) \\ \mathbb{U}_{14}(\mathbf{true}, y, zs_1, zs_2) \rightarrow \mathbf{tp}_2(zs_1, \mathbf{cons}(y, zs_2)) \\ \mathbf{split}(x, \mathbf{cons}(y, ys)) \rightarrow \mathbb{U}_{15}(\mathbf{split}(x, ys), x, y) \\ \mathbb{U}_{15}(\mathbf{tp}_2(zs_1, zs_2), x, y) \rightarrow \mathbb{U}_{16}(\mathbf{le}(x, y), y, zs_1, zs_2) \\ \mathbb{U}_{16}(\mathbf{false}, y, zs_1, zs_2) \rightarrow \mathbf{tp}_2(\mathbf{cons}(y, zs_1), zs_2) \\ \vdots \end{array} \right.$$

We recognize from Theorem 4.3 that \mathbb{U}_{opt} is sound for R_7 .

Due to the technical proof of Lemma 4.2, we assumed that eDCTRSs are non-LV or non-RV. It is not known yet whether this assumption can be relaxed (removed) or not. However, this assumption is not so restrictive since every DCTRS is non-LV. Theorem 4.3 is not a direct consequence of the result in [16] on soundness for \mathbb{U}_{opt} -LL 3-DCTRSs since \mathbb{U} symbols introduced by \mathbb{U}_{opt} have less arguments than those introduced by the unraveling in [16].

4.2. Observing Unsoundness of Marchiori's Counterexample to Soundness. In the previous subsection, we conjectured and proved that the ultra-LL property is a soundness condition of \mathbb{U}_{opt} since the property is already known to be a soundness condition of Marchiori's unraveling for normal CTRSSs. To find other soundness conditions, in this subsection, we take a close look at the derivation $\mathbf{h}(\mathbf{f}(\mathbf{a}), \mathbf{f}(\mathbf{b})) \rightarrow_{\mathbb{U}_{\text{opt}}(R_3)}^* \mathbf{A}$ in Example 3.3, observing the reason why \mathbb{U}_{opt} is not sound for R_3 in Example 3.3.

Recall the derivation $\mathbf{h}(\mathbf{f}(\mathbf{a}), \mathbf{f}(\mathbf{b})) \rightarrow_{\mathbb{U}_{\text{opt}}(R_3)}^* \mathbf{A}$ in Example 3.3:

$$\begin{aligned} \mathbf{h}(\mathbf{f}(\mathbf{a}), \mathbf{f}(\mathbf{b})) &\rightarrow_{\mathbb{U}_{\text{opt}}(R_3)}^* \mathbf{h}(\mathbb{U}_4(\mathbf{c}, \mathbf{d}), \mathbb{U}_4(\mathbf{c}, \mathbf{d})) \rightarrow_{\mathbb{U}_{\text{opt}}(R_3)} \mathbf{g}(\mathbb{U}_4(\mathbf{c}, \mathbf{d}), \mathbb{U}_4(\mathbf{c}, \mathbf{d}), \mathbf{f}(\mathbf{k})) \\ &\rightarrow_{\mathbb{U}_{\text{opt}}(R_3)}^* \mathbf{g}(\mathbf{d}, \mathbb{U}_4(\mathbf{l}, \mathbf{m}), \mathbb{U}_4(\mathbf{l}, \mathbf{m})) \rightarrow_{\mathbb{U}_{\text{opt}}(R_3)} \mathbf{A} \end{aligned}$$

To succeed in this derivation, the following subderivations are necessary:

- to apply the rule $\mathbf{g}(\mathbf{d}, x, x) \rightarrow \mathbf{A}$, the subterm $f(\mathbf{a})$ in the initial term is reduced to \mathbf{d} ,
- to apply the rule $\mathbf{h}(x, x) \rightarrow \mathbf{g}(x, x, f(\mathbf{k}))$, both the subterms $f(\mathbf{a})$ and $f(\mathbf{b})$ in the initial term are reduced to the same term, and
- to apply the rule $\mathbf{g}(\mathbf{d}, x, x) \rightarrow \mathbf{A}$, both the subterm $f(\mathbf{b})$ in the initial term and the term $f(\mathbf{k})$ derived from the application of $\mathbf{h}(x, x) \rightarrow \mathbf{g}(x, x, f(\mathbf{k}))$ are reduced to the same term.

As a consequence, all of the terms $f(\mathbf{a})$, $f(\mathbf{b})$, and $f(\mathbf{k})$ have to be reduced to the same term \mathbf{d} . However, this is impossible on the reduction of R_3 . Nevertheless, in the above derivation, $\mathbf{h}(x, x) \rightarrow \mathbf{g}(x, x, f(\mathbf{k}))$ is applied after reducing $f(\mathbf{a})$ and $f(\mathbf{b})$ to $\mathbf{U}_4(\mathbf{c}, \mathbf{d})$: the $\mathbf{U}_4(\mathbf{c}, \mathbf{d})$, that derives from $f(\mathbf{a})$, is reduced to \mathbf{d} , and the other $\mathbf{U}_4(\mathbf{c}, \mathbf{d})$, that derives from $f(\mathbf{b})$, is reduced to $\mathbf{U}_4(\mathbf{l}, \mathbf{m})$ in order to be the same as $f(\mathbf{k})$. Finally, $\mathbf{g}(\mathbf{d}, x, x) \rightarrow \mathbf{A}$ is applied. These undesired subderivations must be caused by the non-RL rule $\mathbf{h}(x, x) \rightarrow \mathbf{g}(x, x, f(\mathbf{k}))$ and the erasing rule $\mathbf{g}(\mathbf{d}, x, x) \rightarrow \mathbf{A}$ in $\mathbb{U}_{\text{opt}}(R_3)$. This stems from the following aspect:

- the application of $\mathbf{h}(x, x) \rightarrow \mathbf{g}(x, x, f(\mathbf{k}))$ to $\mathbf{h}(\mathbf{U}_4(\mathbf{c}, \mathbf{d}), \mathbf{U}_4(\mathbf{c}, \mathbf{d}))$ keeps two occurrences of $\mathbf{U}_4(\mathbf{c}, \mathbf{d})$ that are intermediate states of evaluating $f(\mathbf{a})$ and $f(\mathbf{b})$, respectively, and each of occurrence has a capability to be reduced to a different term later although they should be the same, and
- $\mathbf{g}(\mathbf{d}, x, x) \rightarrow \mathbf{A}$ erases the two occurrences of $\mathbf{U}_4(\mathbf{l}, \mathbf{m})$ as if they derive from the same term (in fact, they derive from the terms $f(\mathbf{b})$ and $f(\mathbf{k})$, respectively, although $f(\mathbf{b})$ and $f(\mathbf{k})$ should be reduced to different terms).

Viewed in this light, we conjecture that the RLNE property of the unraveled TRSs is a sufficient condition for soundness of \mathbb{U}_{opt} . Note that the above issue does not arise in the case of ultra-LL DCTRSs since the LL property does not require equivalence at all between subterms in redexes.

In the next subsection, we will prove the conjecture above, by reducing soundness for a \mathbb{U}_{opt} -RLNE DCTRS to that for a DCTRS obtained by simply inverting. The key feature is that if a DCTRS is \mathbb{U}_{opt} -NE, then,

- the unraveled TRS of the inverted one is equivalent to the inverted unraveled TRS of the DCTRS, and
- the inverted one is \mathbb{U}_{opt} -LL iff the DCTRS is \mathbb{U}_{opt} -RL.

The converse of this approach is impossible since the first property above needs the \mathbb{U}_{opt} -NE property and not all \mathbb{U}_{opt} -LL DCTRSs have the \mathbb{U}_{opt} -NE property.

4.3. Soundness on Ultra-RLNE Property. In this subsection, we show that the optimized unraveling \mathbb{U}_{opt} is sound for \mathbb{U}_{opt} -RLNE DCTRSs. To prove it, we reduce the soundness to that of \mathbb{U}_{opt} for ultra-LL DCTRSs. Moreover, we provide examples showing that neither \mathbb{U}_{opt} -RL nor \mathbb{U}_{opt} -NE properties is sufficient on its own for soundness of \mathbb{U}_{opt} .

We first define the operation to transform eDCTRSs into eDCTRSs that define the inverse relation of the former eDCTRSs. Note that the “inverse” here is slightly distinct from “inverse” in the sense of *program inversion*.

Definition 4.5. Let $\rho : l \rightarrow r \Leftarrow s_1 \twoheadrightarrow t_1; \dots; s_k \twoheadrightarrow t_k$ be an (extended) conditional rewrite rule. We define the operation $(\cdot)^{-1}$ as follows:

$$(l \rightarrow r \Leftarrow s_1 \twoheadrightarrow t_1; \dots; s_k \twoheadrightarrow t_k)^{-1} = r \rightarrow l \Leftarrow t_k \twoheadrightarrow s_k; \dots; t_1 \twoheadrightarrow s_1.$$

This operation is extended to eDCTRSs as $(R)^{-1} = \{(\rho)^{-1} \mid \rho \in R\}$.

For an eCTRS R , the inverse relation of \rightarrow_R is equivalent to the reduction of $(R)^{-1}$.

Proposition 4.6. *Let R be an eCTRS. Then, $\leftarrow_R = \rightarrow_{(R)^{-1}}$.*

Proof (Sketch). It suffices to show that $\leftarrow_{(n),R} = \rightarrow_{(n),(R)^{-1}}$ for all $n \geq 0$. This can be proved by induction on n . \square

Regarding the operation $(\cdot)^{-1}$ and the $\mathbb{U}_{\text{opt-NE}}$ property, the unraveled TRSs are equivalent and we have dual relationships between the $\mathbb{U}_{\text{opt-LL}}$ and $\mathbb{U}_{\text{opt-RL}}$ properties and between the non-LV and non-RV properties.

Theorem 4.7. *Let R be an eDCTRS. Then, all of the following hold:*

- (1) R is $\mathbb{U}_{\text{opt-NE}}$ iff $(R)^{-1}$ is a 3-eDCTRS,
- (2) if R is $\mathbb{U}_{\text{opt-NE}}$, then,
 - a. $\mathbb{U}_{\text{opt}}((R)^{-1}) = (\mathbb{U}_{\text{opt}}(R))^{-1}$ up to the renaming of U symbols (i.e., $U_i^\rho = U_{k-i+1}^{\rho^{-1}}$ for all $1 \leq i \leq k$),
 - b. R is $\mathbb{U}_{\text{opt-LL}}$ iff $(R)^{-1}$ is $\mathbb{U}_{\text{opt-RL}}$, and
 - c. R is $\mathbb{U}_{\text{opt-RL}}$ iff $(R)^{-1}$ is $\mathbb{U}_{\text{opt-LL}}$,
- (3) R is non-LV iff $(R)^{-1}$ is non-RV, and
- (4) R is non-RV iff $(R)^{-1}$ is non-LV.

Proof. The proof can be seen in Appendix A.4. \square

Note that the claim (2)-a in Theorem 4.7 does not hold for \mathbb{U} in general.

Example 4.8. Consider the following $\mathbb{U}_{\text{opt-NE}}$ 3-DCTRS R_8 and its unraveled TRSs:

$$R_8 = \{ \quad f(x) \rightarrow c(x, y) \Leftarrow g(x) \Rightarrow y; \quad y \Rightarrow h(x) \quad a \rightarrow b \quad g(a) \rightarrow h(b) \quad \}$$

$$\mathbb{U}_{\text{opt}}(R_8) = \mathbb{U}(R_8) = \left\{ \begin{array}{l} f(x) \rightarrow \mathbb{U}_{17}(g(x), x) \\ \mathbb{U}_{17}(y, x) \rightarrow \mathbb{U}_{18}(y, x, y) \\ \mathbb{U}_{18}(h(x), x, y) \rightarrow c(x, y) \\ \vdots \end{array} \right\}$$

The following TRS is obtained from R_8 by applying $(\cdot)^{-1}$:

$$(R_8)^{-1} = \{ \quad c(x, y) \rightarrow f(x) \Leftarrow h(x) \Rightarrow y; \quad y \Rightarrow g(x) \quad b \rightarrow a \quad h(b) \rightarrow g(a) \quad \}$$

The DCTRS $(R_8)^{-1}$ is unraveled by \mathbb{U}_{opt} and \mathbb{U} as follows:

$$\mathbb{U}_{\text{opt}}((R_8)^{-1}) = \left\{ \begin{array}{l} c(x, y) \rightarrow \mathbb{U}_{18}(h(x), x, y) \\ \mathbb{U}_{18}(y, x, y) \rightarrow \mathbb{U}_{17}(y, x) \\ \mathbb{U}_{17}(g(x), x) \rightarrow f(x) \\ \vdots \end{array} \right\}$$

$$\mathbb{U}((R_8)^{-1}) = \left\{ \begin{array}{l} c(x, y) \rightarrow \mathbb{U}_{18}(h(x), x, y) \\ \mathbb{U}_{18}(y, x, y) \rightarrow \mathbb{U}_{17}(y, x, y) \\ \mathbb{U}_{17}(g(x), x, y) \rightarrow f(x) \\ \vdots \end{array} \right\}$$

We have that $\mathbb{U}_{\text{opt}}((R_8)^{-1}) = (\mathbb{U}_{\text{opt}}(R_8))^{-1}$, but $\mathbb{U}((R_8)^{-1}) \neq (\mathbb{U}(R_8))^{-1}$.

Finally, we show soundness of \mathbb{U}_{opt} for a $\mathbb{U}_{\text{opt-RLNE}}$ eDCTRS R by reducing it to soundness for the $\mathbb{U}_{\text{opt-LL}}$ eDCTRS $(R)^{-1}$.

Theorem 4.9. \mathbb{U}_{opt} is sound for \mathbb{U}_{opt} -RLNE eDCTRSs that are non-LV or non-RV.

Proof. Let R be a \mathbb{U}_{opt} -RLNE eDCTRS over a signature \mathcal{F} . Then, it follows from Theorem 4.7 that $(R)^{-1}$ is a \mathbb{U}_{opt} -LL 3-eDCTRS which is non-RV or non-LV. Thus, it follows from Theorem 4.3 that \mathbb{U}_{opt} is sound for $(R)^{-1}$, i.e., $\rightarrow_{\mathbb{U}_{\text{opt}}((R)^{-1})}^* \subseteq \rightarrow_{(R)^{-1}}^*$ on terms in $\mathcal{T}(\mathcal{F}, \mathcal{V})$. It follows from Theorem 4.7 that $\mathbb{U}_{\text{opt}}((R)^{-1}) = (\mathbb{U}_{\text{opt}}(R))^{-1}$, and hence $\rightarrow_{\mathbb{U}_{\text{opt}}((R)^{-1})}^* = \rightarrow_{(\mathbb{U}_{\text{opt}}(R))^{-1}}^*$. It follows from Proposition 4.6 that $\rightarrow_{(\mathbb{U}_{\text{opt}}(R))^{-1}}^* = \leftarrow_{\mathbb{U}_{\text{opt}}(R)}^*$ and $\rightarrow_{(R)^{-1}}^* = \leftarrow_R^*$. Therefore, we have that $\rightarrow_{\mathbb{U}_{\text{opt}}(R)}^* \subseteq \rightarrow_R^*$ on terms in $\mathcal{T}(\mathcal{F}, \mathcal{V})$. \square

Example 4.10. Consider the following TRS defining a function `quad` that computes the quadruple of input natural numbers:

$$R_9 = \left\{ \begin{array}{l} \text{quad}(x) \rightarrow \text{twice}(\text{twice}(x)) \\ \text{twice}(x) \rightarrow \text{add}(x, x) \end{array} \right\} \cup R_1$$

The inversion method in [18] inverts this TRS to the following DCTRS R_{10} :

$$R_{10} = \left\{ \begin{array}{l} \text{quad}^{-1}(y) \rightarrow \text{tp}_1(x) \leftarrow \text{twice}^{-1}(y) \twoheadrightarrow \text{tp}_1(z); \text{twice}^{-1}(z) \twoheadrightarrow \text{tp}_1(x) \\ \text{twice}^{-1}(y) \rightarrow \text{tp}_1(x) \leftarrow \text{add}^{-1}(y) \twoheadrightarrow \text{tp}_2(x, x) \end{array} \right\} \cup R_2$$

This DCTRS is \mathbb{U}_{opt} -RLNE, and thus, we recognize from Theorem 4.9 that \mathbb{U}_{opt} is sound for R_{10} , while soundness of \mathbb{U}_{opt} for the resulting EV-TRSs (e.g., R_{10}) of the inversion method [18] has already been shown (cf., [22, 23]). On the other hand, soundness of \mathbb{U}_{opt} for DCTRSs obtained by removing the unary tuple symbol tp_1 that seems meaningless:

$$R'_{10} = \left\{ \begin{array}{l} \text{quad}^{-1}(y) \rightarrow x \leftarrow \text{twice}^{-1}(y) \twoheadrightarrow z; \text{twice}^{-1}(z) \twoheadrightarrow x \\ \text{twice}^{-1}(y) \rightarrow x \leftarrow \text{add}^{-1}(y) \twoheadrightarrow \text{tp}_2(x, x) \end{array} \right\} \cup R_2$$

The soundness results in [18, 22, 23] cannot guarantee that \mathbb{U}_{opt} is sound for R'_{10} . However, since this DCTRS R'_{10} is also \mathbb{U}_{opt} -RLNE, we recognize from Theorem 4.9 that \mathbb{U}_{opt} is sound for R'_{10} .

The open problem mentioned in [18] that \mathbb{U}_{opt} is sound for \mathbb{U}_{opt} -NE eDCTRSs does not hold in general. This indicates that the ultra-NE property on its own is not a soundness condition for either \mathbb{U}_{opt} or \mathbb{U} .

Example 4.11. Consider the 3-DCTRS R_8 and the unraveled TRS $\mathbb{U}_{\text{opt}}(R_8)$ in Example 4.8 again. R_8 is \mathbb{U}_{opt} -NE and \mathbb{U} -NE, but not \mathbb{U}_{opt} -RL or \mathbb{U} -RL, either. We have the derivation $f(a) \rightarrow_{\mathbb{U}_{\text{opt}}(R_8)}^* c(b, h(b))$, but $f(a)$ cannot be reduced by R_8 to $c(b, h(b))$. Therefore, \mathbb{U}_{opt} is not sound for every \mathbb{U}_{opt} -NE DCTRS. By the same token, \mathbb{U} is not sound for every \mathbb{U} -NE DCTRS since $\mathbb{U}_{\text{opt}}(R_8) = \mathbb{U}(R_8)$.

Moreover, the ultra-RL property on its own is not a soundness condition for \mathbb{U}_{opt} .

Example 4.12. Consider the following DCTRS R_{10} and its unraveled TRSs:

$$R_{10} = \left\{ \begin{array}{l} f(x) \rightarrow e \leftarrow d \twoheadrightarrow l \\ h(x, x) \rightarrow A \end{array} \right\}$$

$$\mathbb{U}_{\text{opt}}(R_{10}) = \left\{ \begin{array}{l} f(x) \rightarrow \mathbb{U}_{19}(d) \\ \mathbb{U}_{19}(l) \rightarrow e \\ \vdots \end{array} \right\} \quad \mathbb{U}(R_{10}) = \left\{ \begin{array}{l} f(x) \rightarrow \mathbb{U}_{19}(d, x) \\ \mathbb{U}_{19}(l, x) \rightarrow e \\ \vdots \end{array} \right\}$$

The DCTRS R_{10} is \mathbb{U}_{opt} -RL, but not \mathbb{U}_{opt} -NE. Although we have the derivation $\mathbf{h}(f(\mathbf{a}), f(\mathbf{b})) \rightarrow_{\mathbb{U}_{\text{opt}}(R_{10})}^2 \mathbf{h}(U_{19}(\mathbf{d}), U_{19}(\mathbf{d})) \rightarrow_{\mathbb{U}_{\text{opt}}(R_{10})} \mathbf{A}$, the term $\mathbf{h}(f(\mathbf{a}), f(\mathbf{b}))$ cannot be reduced by R_{10} to \mathbf{A} . Therefore, \mathbb{U}_{opt} is not sound for R_{10} while \mathbb{U} is sound for R_{10} .

It is possible to prove Theorem 4.9 directly [21], by using the feature that every reduction sequence of RL TRSs can be transformed to a *basic* reduction sequence [17]. As stated at the end of Subsection 4.2, however, Theorem 4.3 cannot be proved by using Theorem 4.9 since $\mathbb{U}_{\text{opt}}((R)^{-1}) = (\mathbb{U}_{\text{opt}}(R))^{-1}$ does not hold for every \mathbb{U}_{opt} -LL DCTRS R (see $\mathbb{U}_{\text{opt}}(R_{10})$ in Example 4.12).

In the proof of Theorem 4.3 (and also the direct proof of Theorem 4.9), linearity plays a very important role and finding other soundness conditions by means of a similar proof scheme is quite difficult.

Finally, we revisit the resulting system of the program inversion mentioned in Section 1.

Example 4.13. Consider the EV-TRS $\mathbb{U}_{\text{opt}}(R_2)$ in Example 3.2 again. The original TRS R_1 is left-linear, and thus, R_2 is \mathbb{U}_{opt} -RLNE [18, 23]. Theorem 4.9 guarantees that $\mathbb{U}_{\text{opt}}(R_2)$ is an *inverse* system of R_1 .

4.4. Soundness of Unraveled TRSs with Extra Variables. As we stated in Section 1, the optimized unraveling \mathbb{U}_{opt} is used in the program inversion method proposed in [18, 22, 23] and DCTRSs obtained by the inversion method are of Type 4 (not of Type 3) in general. For this reason, in this subsection, we extend Theorem 4.3 to 4-eDCTRSs. More precisely, we show that \mathbb{U}_{opt} is sound for \mathbb{U}_{opt} -LL DCTRSs if reduction sequences of the unraveled TRSs are restricted to *EV-safe* reduction sequences (see Definition 4.14). Roughly speaking, in an EV-safe reduction sequence, any redex introduced via extra variables at the application of rewrite rules is never reduced anywhere. In practical cases (*e.g.*, *inverse* TRSs [18, 22, 23, 20]), extra variables are instantiated with constructor terms. However, at the application of rewrite rules, extra variables in the unraveled eTRSs may introduce undesired terms, *e.g.*, terms rooted by \mathbb{U} symbols that are not reachable from terms over the original signature. As a consequence, \mathbb{U}_{opt} is not always sound w.r.t. non-EV-safe reduction sequences of the unraveled eTRSs (see Example 4.18).

We first define the notion of *EV-safe* reduction sequences of eTRSs [20, 18, 21]. This notion can be formalized by extending the notion of *basic* reduction sequences [11, 17].

Definition 4.14 (EV-safe reduction [20]). Let R be an eTRS and $\rho_i : l_i \rightarrow r_i \in R$ for all $i \geq 1$. Let $t_0 \rightarrow_{p_1, \rho_1} t_2 \rightarrow_{p_2, \rho_2} \dots$ be a reduction sequence of R , and $B_0 \subseteq \mathcal{Pos}_{\mathcal{F}}(t_0)$ such that B_0 is prefix closed (*i.e.*, if $p < q$ and $q \in B_0$, then $p \in B_0$). We define the sets B_1, B_2, \dots of positions from the sequence and B_0 inductively as

$$B_i = (B_{i-1} \setminus \{q \in B_{i-1} \mid q \geq p_i\}) \cup \{p_i q \mid q \in \mathcal{Pos}_{\mathcal{F}}(r_i)\} \\ \cup \{p_i p' q \mid p_i p q \in B_{i-1}, p \in \mathcal{Pos}_{\mathcal{V}}(l_i), l_i|_p = r_i|_{p'}\}$$

for all $i \geq 1$. Note that B_1, B_2, \dots are also prefix closed. For all $i \geq 0$, positions in B_i are referred as *basic positions of t_i w.r.t. extra variables*. The reduction sequence above is said to be *based on B_0 w.r.t. extra variables* if $p_i \in B_{i-1}$ for all $i \geq 1$. If the sequence is finite with length n , then we denote it by $B_0 : t_0 \xrightarrow{\text{evs}_R^*} B_n : t_n$ or $B_0 : t_0 \xrightarrow{\text{evs}_R^*} t_n$. In particular, the reduction sequence is called *safe w.r.t. extra variables* (EV-safe) if $B_0 = \mathcal{Pos}_{\mathcal{F}}(t_0)$. If the EV-safe sequence is finite with length n , then we denote it by $t_0 \xrightarrow{\text{evs}_R^*} t_n$.

Note that EV-safeness is different from *basicness* [11, 17] in the sense that all the basic positions are propagated at the application of rewrite rules, but none of the positions for extra variables are added to basic positions. A typical instance of EV-safe reduction sequences is a reduction sequence obtained by substituting a normal form for each extra variable when applying rewrite rules.

To specify a set of terms that extra variables are possibly instantiated at the rule application, we introduce the notion of *EV-instantiation on sets of terms*. Let R be an eTRS and T be a set of terms. A derivation $t_0 \rightarrow_{p_1, \rho_1} t_1 \rightarrow_{p_2, \rho_2} \cdots$ of R is called *EV-instantiated on T* if any extra variable of $\rho_i : l_i \rightarrow r_i$ is instantiated by a term in T , i.e., $t_i|_{p_i q} \in T$ for any $q \in \text{Pos}_{\mathcal{V}}(r_i)$ such that $r_i|_q \in \mathcal{E}\text{Var}(\rho_i)$. By the same token, the notion of the EV-instantiation property is defined for the parallel reduction of eTRSs. For any of the unraveled eTRSs, their EV-safe reduction sequences have the following property related to EV-instantiation on the set of terms over the original signature.

Lemma 4.15. *Let R be a $\mathbb{U}_{\text{opt}}\text{-LL}$ eDCTRS over a signature \mathcal{F} , and s, t be terms in $\mathcal{T}(\mathcal{F}, \mathcal{V})$. If $s \xrightarrow{\text{evs}^*_{\mathbb{U}_{\text{opt}}(R)}} t$, then there exists a derivation $s \xrightarrow{\text{evs}^*_{\mathbb{U}_{\text{opt}}(R)}} t$ that is EV-instantiated on $\mathcal{T}(\mathcal{F}, \mathcal{V})$.*

Proof. The proof can be seen in Appendix A.5. \square

Lemma 4.2, the key lemma for the case of $\mathbb{U}_{\text{opt}}\text{-LL}$ 3-DCTRSs, is adapted to 4-eDCTRSs as follows.

Lemma 4.16. *Let R be a $\mathbb{U}_{\text{opt}}\text{-LL}$ eDCTRS over a signature \mathcal{F} , s be a term in $\mathcal{T}(\mathcal{F}, \mathcal{V})$, t be a linear term in $\mathcal{T}(\mathcal{F}_{\mathbb{U}_{\text{opt}}(R)}, \mathcal{V})$, and σ be a substitution in $\text{Sub}(\mathcal{F}_{\mathbb{U}_{\text{opt}}(R)}, \mathcal{V})$. Suppose that R is non-LV or non-RV. If $s \xrightarrow{\text{evs}^n_{\mathbb{U}_{\text{opt}}(R)}} t\sigma$ for some $n \geq 0$ and the derivation is EV-instantiated on $\mathcal{T}(\mathcal{F}, \mathcal{V})$, then there exists a substitution θ in $\text{Sub}(\mathcal{F}, \mathcal{V})$ such that*

- $s \xrightarrow{*}_R t\theta \xrightarrow{\text{evs}^{n'}_{\geq \text{Pos}_{\mathcal{V}}(t), \mathbb{U}_{\text{opt}}(R)}} t\sigma$ for some $n' \leq n$,
- the derivation $t\theta \xrightarrow{\text{evs}^{n'}_{\geq \text{Pos}_{\mathcal{V}}(t), \mathbb{U}_{\text{opt}}(R)}} t\sigma$ is EV-instantiated on $\mathcal{T}(\mathcal{F}, \mathcal{V})$, and
- if $t\sigma \in \mathcal{T}(\mathcal{F}, \mathcal{V})$, then $t\theta = t\sigma$.

Proof. This lemma can be proved similarly to Lemma 4.2. \square

As a consequence of Lemma 4.16, we extend Theorem 4.3 to 4-eDCTRSs.

Theorem 4.17. *\mathbb{U}_{opt} is sound for a $\mathbb{U}_{\text{opt}}\text{-LL}$ eDCTRSs R over a signature \mathcal{F} w.r.t. $\xrightarrow{\text{evs}^*_{\mathbb{U}_{\text{opt}}(R)}}$ if R is non-LV or non-RV.*

Proof. Suppose that $s \xrightarrow{\text{evs}^*_{\mathbb{U}_{\text{opt}}(R)}} t$ and $s, t \in \mathcal{T}(\mathcal{F}, \mathcal{V})$. Then, it follows from Lemma 4.15 that there is a derivation $s \xrightarrow{\text{evs}^*_{\mathbb{U}_{\text{opt}}(R)}} t$ that is EV-instantiated on $\mathcal{T}(\mathcal{F}, \mathcal{V})$. Since a single step of $\xrightarrow{\text{evs}^*_{\mathbb{U}_{\text{opt}}(R)}}$ can be considered a single step of the parallel reduction, we have the derivation $s \xrightarrow{\text{evs}^*_{\mathbb{U}_{\text{opt}}(R)}} t$ that is EV-instantiated on $\mathcal{T}(\mathcal{F}, \mathcal{V})$. Let x be a variable and σ be a substitution such that $x\sigma = t$. Then, it follows from Lemma 4.16 that $s \xrightarrow{*}_R x\sigma = t$. \square

Note that Lemma 4.16 and Theorem 4.17 are strict extensions of Lemma 4.2 and Theorem 4.3, respectively.

Finally, we show a counterexample against Theorem 4.3 without the EV-safe property.

Example 4.18. Consider the DCTRS R_{10} and its unraveled TRSs in Example 4.12 again. Their inverted systems are as follows:

$$(R_{10})^{-1} = \left\{ \begin{array}{l} e \rightarrow f(x) \leftarrow l \rightarrow d \\ A \rightarrow h(x, x) \end{array} \right\}$$

$$\mathbb{U}_{\text{opt}}((R_{10})^{-1}) = \mathbb{U}((R_{10})^{-1}) = (\mathbb{U}_{\text{opt}}(R_{10}))^{-1} = \left\{ \begin{array}{l} e \rightarrow \mathbb{U}_{19}(l) \\ \mathbb{U}_{19}(d) \rightarrow f(x) \\ \vdots \end{array} \right\}$$

We have the derivation $A \rightarrow_{\mathbb{U}_{\text{opt}}((R_{10})^{-1})} h(\mathbb{U}_{19}(d), \mathbb{U}_{19}(d)) \rightarrow_{\mathbb{U}_{\text{opt}}^*((R_{10})^{-1})}^* h(f(a), f(b))$ that is not EV-safe: the term $\mathbb{U}_{19}(d)$ introduced by instantiating the extra variable x in the applied rule $A \rightarrow h(x, x)$ is reduced. However, A cannot be reduced by $(R_{10})^{-1}$ to $h(f(a), f(b))$. Therefore, \mathbb{U}_{opt} is not sound for $(R_{10})^{-1}$. Note that \mathbb{U} is not sound for $(R_{10})^{-1}$, either.

Note that if an unraveling U is sound for an eCTRS R , then U is sound for R w.r.t. $\xrightarrow{\text{evs}} U(R)$. For this reason, we need not discuss soundness of U for R w.r.t. $\xrightarrow{\text{evs}} U(R)$ when soundness of U for R w.r.t. $\rightarrow_{U(R)}$ has already been shown.

5. SOUNDNESS OF OTHER UNRAVELINGS

In this section, we show that soundness of \mathbb{U}_{opt} implies that of \mathbb{U} , and then we revisit soundness of the unravelings for join and normal CTRSSs.

We first recall the notion of *tree homomorphisms*. Let \mathcal{F} and \mathcal{G} be signatures and $\phi_{\mathcal{F}}$ be a mapping which, for an n -ary function symbol $f \in \mathcal{F}$, associates a term in $\mathcal{T}(\mathcal{G}, \{x_1, \dots, x_n\})$ where $x_1, \dots, x_n \in \mathcal{V}$. The *tree homomorphism* $\phi : \mathcal{T}(\mathcal{F}, \mathcal{V}) \rightarrow \mathcal{T}(\mathcal{G}, \mathcal{V})$ determined by $\phi_{\mathcal{F}}$ is defined as follows [33, 5]:

- $\phi(x) = x$ for $x \in \mathcal{V}$, and
- $\phi(f(t_1, \dots, t_n)) = \phi_{\mathcal{F}}(f)\{x_i \mapsto \phi(t_i) \mid 1 \leq i \leq n\}$ for an n -ary function symbol $f \in \mathcal{F}$.

When $\phi_{\mathcal{F}}(f)$ is not specified explicitly for an n -ary function symbol f , we let $\phi_{\mathcal{F}}(f) = f(x_1, \dots, x_n)$ with assuming that $f \in \mathcal{G}$. To declare $\phi_{\mathcal{F}}$ intelligibly, we may use the notation “ $\phi_{\mathcal{F}}(f(x_1, \dots, x_n)) = t$ ” instead of “ $\phi_{\mathcal{F}}(f) = t$ ”, e.g., $\phi_{\mathcal{F}}(f(x, y, z)) = g(y, g(x, z))$. The tree homomorphism ϕ is called *linear* if $\phi_{\mathcal{F}}(f)$ is linear for any function symbol $f \in \mathcal{F}$, and called *non-erasing* if $\text{Var}(\phi(f)) = \{x_1, \dots, x_n\}$ for any n -ary function symbol $f \in \mathcal{F}$. The tree homomorphism ϕ is extended to eCTRSs as follows: $\phi(R) = \{\phi(l) \rightarrow \phi(r) \leftarrow \phi(s_1) = \phi(t_1); \dots; \phi(s_k) = \phi(t_k) \mid l \rightarrow r \leftarrow s_1 \approx t_1; \dots; s_k \approx t_k \in R\}$. We extend it to a set of term pairs T (e.g., a binary relation) as follows: $\phi(T) = \{(\phi(s), \phi(t)) \mid (s, t) \in T\}$. For a substitution $\sigma \in \text{Sub}(\mathcal{F}, \mathcal{V})$, σ_{ϕ} denotes the substitution $\{x \mapsto \phi(x\sigma) \mid x \in \text{Dom}(\sigma)\}$. Tree homomorphisms have the following properties.

Lemma 5.1. *Let ϕ be a tree homomorphism.*

- (1) *Let t be a term and σ be a substitution in $\text{Sub}(\mathcal{F}, \mathcal{V})$. Then, $\phi(t\sigma) = (\phi(t))\sigma_{\phi}$.*
- (2) *Let t be a term and $C[\]$ be a one-hole context. Then, all of the following hold:*

- $\phi(C[t]) = \phi(C[\overbrace{\phi(t), \dots, \phi(t)}^n])$,⁴
- *if ϕ is non-erasing, then $\phi(C)$ has at least one hole,*

⁴ $\phi(C)$ has no hole (i.e., $n = 0$) if ϕ removes the hole from $C[\]$.

- if ϕ is linear, then $\phi(C)$ has at most one hole. That is, if ϕ is linear, then, for any term t and any one-hole context $C[\]$, either $\phi(C[t]) = \phi(C)$ or $\phi(C[t]) = \phi(C)[\phi(t)]$.
- (3) Let R be an eCTRS and s, t be terms in $\mathcal{T}(\mathcal{F}, \mathcal{V})$. If $s \rightarrow_R^* t$, then $\phi(s) \rightarrow_{\phi(R)}^* \phi(t)$. That is, $\phi(\rightarrow_R^*) \subseteq \rightarrow_{\phi(R)}^*$.

The proof of Lemma 5.1 is omitted since it can be easily proved by induction.

5.1. Abstract Comparison Method for Soundness of Unravelings. Before we discuss the relationship between soundness of two or more unravelings, we present a sufficient condition of two unravelings under which soundness of the first implies soundness of the other.

To show soundness of an unraveling U_2 by means of a sound unraveling U_1 , it suffices to show that all the derivations of U_2 on terms over the original signature are included in the derivations of U_1 , i.e., $\rightarrow_{U_2(R)}^* \subseteq \rightarrow_{U_1(R)}^*$ on terms in $\mathcal{T}(\mathcal{F}, \mathcal{V})$. Suppose that U_1 is sound for a CTRS R , i.e., $\rightarrow_{U_1(R)}^* \subseteq \rightarrow_R^*$ on terms in $\mathcal{T}(\mathcal{F}, \mathcal{V})$. Then, it follows from $\rightarrow_{U_2(R)}^* \subseteq \rightarrow_{U_1(R)}^*$ that $\rightarrow_{U_2(R)}^* \subseteq \rightarrow_R^*$ on terms in $\mathcal{T}(\mathcal{F}, \mathcal{V})$. Therefore, U_2 is sound for R .

To show that $\rightarrow_{U_2(R)}^* \subseteq \rightarrow_{U_1(R)}^*$ on terms in $\mathcal{T}(\mathcal{F}, \mathcal{V})$, it suffices to show the existence of a tree homomorphism ϕ for an extended signature of \mathcal{F} such that

- $U_1(R) = \phi(U_2(R))$, and
- $\phi(t) = t$ for any term $t \in \mathcal{T}(\mathcal{F}, \mathcal{V})$.

Moreover, since we consider soundness w.r.t. $\xrightarrow{\text{evs}}$, we are interested in a sufficient condition under which $\xrightarrow{\text{evs}}_{U_2(R)}^* \subseteq \xrightarrow{\text{evs}}_{U_1(R)}^*$. To show that $\xrightarrow{\text{evs}}_{U_2(R)}^* \subseteq \xrightarrow{\text{evs}}_{U_1(R)}^*$, it suffices to additionally show that $\mathcal{EVar}(\phi(l) \rightarrow \phi(r)) \subseteq \mathcal{EVar}(l \rightarrow r)$ for any rule $l \rightarrow r \in U_2(R)$.

For a set $\mathcal{G} \subseteq \mathcal{F}$ of function symbols, a tree homomorphism ϕ determined by a mapping $\phi_{\mathcal{F}}$ is called \mathcal{G} -identical if $\phi_{\mathcal{F}}(f) = f(x_1, \dots, x_n)$ for any n -ary function symbol $f \in \mathcal{G}$. Moreover, ϕ is called *EV-preserving for an eTRS R* if $\mathcal{EVar}(\phi(l) \rightarrow \phi(r)) = \mathcal{EVar}(l \rightarrow r)$ for any rule $l \rightarrow r \in R$.

Lemma 5.2. *Let R be an eTRS, s, t be terms, and ϕ be a tree homomorphism that is EV-preserving for R . If $s \xrightarrow{\text{evs}}_R^* t$, then $\phi(s) \xrightarrow{\text{evs}}_{\phi(R)}^* \phi(t)$.*

Proof. We first define the mapping ϕ_u from a position of a term u to a set of positions of $\phi(u)$, and extend it to sets of positions of u :

- $\phi_x(\varepsilon) = \{\varepsilon\}$ for $x \in \mathcal{V}$,
- $\phi_{f(u_1, \dots, u_n)}(\varepsilon) = \{\varepsilon\}$ for an n -ary function symbol f ,
- $\phi_{f(u_1, \dots, u_n)}(ip) = \{qp' \mid q \in \mathcal{Pos}_{\mathcal{V}}(\phi_{\mathcal{F}}(f)), \phi_{\mathcal{F}}(f)|_q = x_i, p' \in \phi_{u_i}(p)\}$ for an n -ary function symbol f , where $1 \leq i \leq n$ and $p \in \mathcal{Pos}(u_i)$, and
- $\phi_u(P) = \bigcup_{p \in P} \phi_u(p)$.

For a position p of u , we mean by $p' \in \phi_u(p)$ that the application of ϕ to u maps $u|_p$ to $\phi(u)|_{p'}$. Note that positions in $\phi_u(p)$ are parallel since variable positions of $\phi_{\mathcal{F}}(f)$ are parallel. We prove that if $B : s \xrightarrow{\text{evs}}_R^* B' : t$ and $\phi_s(B) \subseteq B_1 \subseteq \mathcal{Pos}(\phi(s))$, then $B_1 : \phi(s) \xrightarrow{\text{evs}}_{\phi(R)}^* B'_1 : \phi(t)$ and $\phi_t(B') \subseteq B'_1$. To prove this claim by induction on the length n of the derivation $B : s \xrightarrow{\text{evs}}_R^* B' : t$, it suffices to show that if $B : s \xrightarrow{\text{evs}}_{p,R} B' : t$, then $\phi_s(p)$ is defined and $\phi_s(B) : \phi(s) \xrightarrow{\text{evs}}_{q_1, \phi(R)} \cdots \xrightarrow{\text{evs}}_{q_m, \phi(R)} \phi_t(B') : \phi(t)$ where $\phi_s(p) = \{q_1, \dots, q_m\}$. This follows from the assumption and the definitions of $\xrightarrow{\text{evs}}$ and ϕ_t . \square

Accordingly, to show soundness of U_2 by means of soundness of U_1 , we obtain the following useful lemma.

Lemma 5.3. *Let U_1 and U_2 be unravelings, R be an eCTRS over a signature \mathcal{F} , and \mathcal{G} be an extended signature of \mathcal{F} such that $U_2(R)$ is defined over \mathcal{G} . Let ϕ be an \mathcal{F} -identical tree homomorphism determined by $\phi_{\mathcal{G}}$ such that $U_1(R) = \phi(U_2(R))$. Then, all of the following hold:*

- (1) $\rightarrow_{U_2(R)}^* \subseteq \rightarrow_{U_1(R)}^*$ on terms in $\mathcal{T}(\mathcal{F}, \mathcal{V})$,
- (2) if ϕ is non-erasing, then ϕ is EV-preserving for any eTRS, and
- (3) if ϕ is EV-preserving for $U_2(R)$, then $\xrightarrow{\text{evs}}_{U_2(R)}^* \subseteq \xrightarrow{\text{evs}}_{U_1(R)}^*$ on terms in $\mathcal{T}(\mathcal{F}, \mathcal{V})$.

That is, all of the following holds:

- (4) if U_1 is sound for R , then so is U_2 ,
- (5) if ϕ is non-erasing and U_1 is sound for R w.r.t. $\xrightarrow{\text{evs}}_{U_1(R)}$, then U_2 is sound for R w.r.t. $\xrightarrow{\text{evs}}_{U_2(R)}$, and
- (6) if ϕ is EV-preserving for $U_2(R)$ and U_1 is sound for R w.r.t. $\xrightarrow{\text{evs}}_{U_1(R)}$, then U_2 is sound for R w.r.t. $\xrightarrow{\text{evs}}_{U_2(R)}$.

Proof. We first prove the first claim $\rightarrow_{U_2(R)}^* \subseteq \rightarrow_{U_1(R)}^*$ on terms in $\mathcal{T}(\mathcal{F}, \mathcal{V})$. It follows from Lemma 5.1 and the assumption $U_1(R) = \phi(U_2(R))$ that $\phi(\rightarrow_{U_2(R)}^*) \subseteq \rightarrow_{U_1(R)}^*$. Since ϕ is \mathcal{F} -identical, we have that $\rightarrow_{U_2(R)}^* \subseteq \rightarrow_{U_1(R)}^*$ on terms in $\mathcal{T}(\mathcal{F}, \mathcal{V})$.

Let ϕ be determined by a mapping $\phi_{\mathcal{G}}$. To prove the second claim, it suffices to show that $\text{Var}(t) = \text{Var}(\phi(t))$ for any term t . We prove this claim by induction on the structure of t . Since the case that t is a variable is trivial, we only consider the remaining case that t is of the form $f(t_1, \dots, t_n)$. By the induction hypothesis, $\text{Var}(t_i) = \text{Var}(\phi(t_i))$ for all $1 \leq i \leq n$. It follows from the non-erasingness of ϕ that $\text{Var}(\phi(f(t_1, \dots, t_n))) = \text{Var}(\phi_{\mathcal{G}}(f)\{x_i \mapsto \phi(t_i) \mid 1 \leq i \leq n\}) = \bigcup_{i=1}^n \text{Var}(\phi(t_i))$, and hence $\text{Var}(\phi(t)) = \text{Var}(t)$.

The third claim follows from the first claim and Lemma 5.2. The remaining claims follow from the first, second, and third claims, and soundness of U_1 . \square

Due to Lemma 5.3, to show soundness of U_2 by soundness of U_1 , it suffices to show the existence of an \mathcal{F} -identical tree homomorphism ϕ satisfying that $U_1(R) = \phi(U_2(R))$. Moreover, for the case of soundness w.r.t. $\xrightarrow{\text{evs}}$, it suffices to additionally show that the tree homomorphism ϕ is non-erasing or EV-preserving for $U_2(R)$.

5.2. On Ohlebusch's Unraveling for DCTRSs. As stated in Section 3, the optimized unraveling \mathbb{U}_{opt} is a variant of the unraveling \mathbb{U} , in the sense that variables carried by \mathbb{U} symbols are optimized. For this reason, for a DCTRS R , it is easy to find a tree homomorphism ϕ such that $\mathbb{U}_{\text{opt}}(R) = \phi(\mathbb{U}(R))$. In the following, we assume that for every rule $\rho \in R$, the same \mathbb{U} symbols $U_1^\rho, \dots, U_k^\rho$ are introduced for $\mathbb{U}_{\text{opt}}(\rho)$ and $\mathbb{U}(\rho)$.

Lemma 5.4. *Let R be an eDCTRS over a signature \mathcal{F} . There exists an \mathcal{F} -identical tree homomorphism ϕ such that $\phi(\mathbb{U}(R)) = \mathbb{U}_{\text{opt}}(R)$ and ϕ is EV-preserving for $\mathbb{U}(R)$.*

Proof. Let ϕ be a tree homomorphism determined by $\phi_{\mathcal{F}_{\mathbb{U}(R)}}$ such that

$$\phi_{\mathcal{F}_{\mathbb{U}(R)}}(U_i^\rho(x_i, \vec{X}_i)) = U_i^\rho(x_i, \vec{Z}_i)$$

where x_i is a fresh variable such that $x_i \notin X_i$. Then, it is clear that $\phi(\mathbb{U}(\rho)) = \mathbb{U}_{\text{opt}}(\rho)$, and hence $\phi(\mathbb{U}(R)) = \mathbb{U}_{\text{opt}}(R)$.

Next, we show that $\phi_{\mathcal{F}_{\mathbb{U}(R)}}$ is EV-preserving for $\mathbb{U}(R)$. For unconditional rules $l_0 \rightarrow r_0$ in R , it is clear that $\mathcal{E}\mathcal{V}ar(l_0 \rightarrow r_0) = \mathcal{E}\mathcal{V}ar(\phi(l_0) \rightarrow \phi(r_0))$, since $l_0, r_0 \in \mathcal{T}(\mathcal{F}, \mathcal{V})$ and $\phi(t) = t$ for all $t \in \mathcal{T}(\mathcal{F}, \mathcal{V})$. Thus, we only consider the case of conditional rules $\rho : l \rightarrow r \Leftarrow s_1 \rightarrow t_1; \dots; s_k \rightarrow t_k \in R$. Rules in $\mathbb{U}_{\text{opt}}(\rho)$ and $\mathbb{U}(\rho)$ that may contain extra variables are rules of U_k^ρ , that is, $\rho_k : U_k^\rho(t_k, \vec{Z}_k) \rightarrow r \in \mathbb{U}_{\text{opt}}(R)$, $\rho'_k : U_k^\rho(t_k, \vec{X}_k) \rightarrow r \in \mathbb{U}(R)$. It follows from $X_k \cap Y_k \subseteq X_k$ and $\mathcal{V}ar(r) \cap Z_k = \mathcal{V}ar(r) \cap X_k$ that $\mathcal{E}\mathcal{V}ar(\rho_k) = \mathcal{V}ar(r) \setminus (\mathcal{V}ar(t_k) \cup (\mathcal{V}ar(l, t_1, \dots, t_{k-1}) \cap \mathcal{V}ar(r, t_k))) = \mathcal{V}ar(r) \setminus \mathcal{V}ar(l, t_1, \dots, t_k) = \mathcal{E}\mathcal{V}ar(\rho'_k)$. Therefore, $\phi_{\mathcal{F}_{\mathbb{U}(R)}}$ is EV-preserving for $\mathbb{U}(R)$. \square

As a consequence, we conclude that soundness of \mathbb{U}_{opt} implies soundness of \mathbb{U} .

Corollary 5.5. *Let R be an eDCTRS over a signature \mathcal{F} . \mathbb{U} is sound for R (w.r.t. $\xrightarrow{\text{evs}}_{\mathbb{U}(R)}$) if \mathbb{U}_{opt} is sound for R (w.r.t. $\xrightarrow{\text{evs}}_{\mathbb{U}_{\text{opt}}(R)}$).*

Example 5.6. Consider the DCTRS R'_{10} in Example 4.10 again. As stated in Example 5.6, \mathbb{U}_{opt} is sound for R'_{10} , and thus, we recognize from Corollary 5.5 that \mathbb{U} is also sound for R'_{10} .

The converse of Corollary 5.5 does not hold in general since, for a DCTRS R over a signature \mathcal{F} , $\rightarrow_{\mathbb{U}(R)}^* \neq \rightarrow_{\mathbb{U}_{\text{opt}}(R)}^*$ on terms in $\mathcal{T}(\mathcal{F}, \mathcal{V})$ in general (see Example 4.12). The reason why the converse of Corollary 5.5 does not hold must be that the \mathbb{U} symbols introduced via the application of \mathbb{U} have more variables (*i.e.*, information) than the corresponding \mathbb{U} symbols introduced by \mathbb{U}_{opt} . Thus, \mathbb{U} is sufficient to produce TRSs that can be used instead of the original DCTRSs. Nonetheless, \mathbb{U}_{opt} will be useful in investigating soundness of \mathbb{U} since the unraveled TRSs obtained by \mathbb{U}_{opt} are simpler than those obtained by \mathbb{U} .

5.3. On Unravelings for Join and Normal CTRSs. Join CTRSs can be converted into equivalent normal CTRSs that are special cases of DCTRSs, and normal CTRSs are join CTRSs since the conditions $s_i \rightarrow n_i$ and $s_i \downarrow n_i$ are identical:

$$\text{join CTRS} = \text{normal CTRS} \subset \text{DCTRS}$$

In this subsection, we show that the unraveling \mathbb{U}_J for join CTRSs [15] is sound for join CTRSs if the unraveling \mathbb{U}_N for normal CTRSs [26, 9] is sound for the corresponding normal CTRSs. Then, by using this result and the existing soundness condition of \mathbb{U}_N [9], we show that \mathbb{U}_J is sound for LL join CTRSs. We also show that \mathbb{U}_J is sound for join CTRSs that can be considered normal CTRSs. Moreover, we show that \mathbb{U}_N is sound for a normal CTRS if \mathbb{U}_J is sound for the normal CTRS that is considered as a join CTRS. Finally, we show that soundness of \mathbb{U}_N implies soundness of \mathbb{U} . As far as we know, soundness of \mathbb{U}_J has never been discussed, whereas soundness of \mathbb{U}_N has been investigated in some papers [15, 9]. For this reason, we show the soundness condition for \mathbb{U}_J and compare soundness of \mathbb{U}_J with \mathbb{U}_N .

A CTRS R is called *join* if the symbol \approx in the conditions of rewrite rules is interpreted as *joinability*: the *reduction relation* of R is defined as $\rightarrow_R = \bigcup_{n \geq 0} \rightarrow_{(n), R}$ where

- $\rightarrow_{(0), R} = \emptyset$, and
- $\rightarrow_{(i+1), R} = \rightarrow_{(i), R} \cup \{(C[l\sigma]_p, C[l\sigma]_p) \mid \rho : l \rightarrow r \Leftarrow s_1 \approx t_1; \dots; s_k \approx t_k \in R, s_1\sigma \downarrow_{(i), R} t_1\sigma, \dots, s_k\sigma \downarrow_{(i), R} t_k\sigma\}$ for $i \geq 0$.

From now on, rewrite rules $l \rightarrow r \Leftarrow s_1 \approx t_1; \dots; s_k \approx t_k$ of join CTRSs are written as $l \rightarrow r \Leftarrow s_1 \downarrow t_1; \dots; s_k \downarrow t_k$.

We first recall the definition of the unravelings \mathbb{U}_J and \mathbb{U}_N for join and normal CTRSs, that are variants of unravelings proposed by Marchiori [15].

Definition 5.7 ($\mathbb{U}_J, \mathbb{U}_N$ [26, 9]). Let R be a join or normal eCTRS over a signature \mathcal{F} . Introducing a \mathbb{U} symbol U^ρ , we transform $\rho : l \rightarrow r \Leftarrow s_1 \approx t_1; \dots; s_k \approx t_k$ into sets $\mathbb{U}_J(\rho)$ and $\mathbb{U}_N(\rho)$ of two unconditional rules as follows:

- $\mathbb{U}_J(\rho) = \{l \rightarrow U^\rho(s_1, t_1, \dots, s_k, t_k, \overrightarrow{\mathcal{V}ar(l)}), U^\rho(x_1, x_1, \dots, x_k, x_k, \overrightarrow{\mathcal{V}ar(l)}) \rightarrow r\}$ if R is join, and
- $\mathbb{U}_N(\rho) = \{l \rightarrow U^\rho(s_1, \dots, s_k, \overrightarrow{\mathcal{V}ar(l)}), U^\rho(t_1, \dots, t_k, \overrightarrow{\mathcal{V}ar(l)}) \rightarrow r\}$ if R is normal.

where x_1, \dots, x_k are different fresh variables. Note that $\mathbb{U}_J(l' \rightarrow r') = \mathbb{U}_N(l' \rightarrow r') = \{l' \rightarrow r'\}$. \mathbb{U}_J and \mathbb{U}_N are extended to join and normal CTRSs, respectively, *i.e.*, $\mathbb{U}_J(R) = \bigcup_{\rho \in R} \mathbb{U}_J(\rho)$ and $\mathbb{U}_N(R) = \bigcup_{\rho \in R} \mathbb{U}_N(\rho)$. We define the extended signatures $\mathcal{F}_{\mathbb{U}_J(R)}$ and $\mathcal{F}_{\mathbb{U}_N(R)}$ of \mathcal{F} as $\mathcal{F}_{\mathbb{U}_J(R)} = \mathcal{F}_{\mathbb{U}_N(R)} = \mathcal{F} \cup \{U^\rho \mid \rho \in R\}$.

Note that \mathbb{U}_J and \mathbb{U}_N are tidy unravelings for join and normal CTRSs, respectively.⁵ The difference from the original definition in [15] is the replacement of $\overrightarrow{\mathcal{V}ar(r)}$ by $\overrightarrow{\mathcal{V}ar(l)}$. We denote the original unravelings for join and normal CTRSs by \mathbb{U}_J^r and \mathbb{U}_N^r , respectively. \mathbb{U}_J^r and \mathbb{U}_N^r can be considered optimized variants of \mathbb{U}_J and \mathbb{U}_N , respectively, as well as the optimized variant \mathbb{U}_{opt} of \mathbb{U} . The relationship between \mathbb{U}_J^r and \mathbb{U}_J and between \mathbb{U}_N^r and \mathbb{U}_N is similar to that between \mathbb{U}_{opt} and \mathbb{U} , *i.e.*, if \mathbb{U}_J^r (\mathbb{U}_N^r) is sound for a join (normal) CTRS R , then so is \mathbb{U}_J (\mathbb{U}_N) (*cf.*, Corollary 5.5). Thus, in the following, we deal with \mathbb{U}_J and \mathbb{U}_N .

Example 5.8. Consider the following join CTRS defining odd and even that, given a natural number $s^n(0)$, return **true** and **false**, respectively, if n is odd, and return **false** and **true**, respectively, otherwise:

$$R_{12} = \left\{ \begin{array}{ll} \text{odd}(0) \rightarrow \text{false} & \text{even}(0) \rightarrow \text{true} \\ \text{odd}(s(x)) \rightarrow \text{true} \Leftarrow \text{even}(x) \downarrow \text{true} & \text{even}(s(x)) \rightarrow \text{true} \Leftarrow \text{odd}(x) \downarrow \text{true} \\ \text{odd}(s(x)) \rightarrow \text{false} \Leftarrow \text{even}(x) \downarrow \text{false} & \text{even}(s(x)) \rightarrow \text{false} \Leftarrow \text{odd}(x) \downarrow \text{false} \end{array} \right\}$$

This join CTRS is unraveled by \mathbb{U}_J into the following TRS:

$$\mathbb{U}_J(R_{12}) = \left\{ \begin{array}{ll} \vdots & \\ \text{odd}(s(x)) \rightarrow \mathbb{U}_{22}(\text{even}(x), \text{true}, x) & \mathbb{U}_{22}(y, y, x) \rightarrow \text{true} \\ \text{odd}(s(x)) \rightarrow \mathbb{U}_{23}(\text{even}(x), \text{false}, x) & \mathbb{U}_{23}(y, y, x) \rightarrow \text{false} \\ \vdots & \\ \text{even}(s(x)) \rightarrow \mathbb{U}_{24}(\text{odd}(x), \text{true}, x) & \mathbb{U}_{24}(y, y, x) \rightarrow \text{true} \\ \text{even}(s(x)) \rightarrow \mathbb{U}_{25}(\text{odd}(x), \text{false}, x) & \mathbb{U}_{25}(y, y, x) \rightarrow \text{false} \end{array} \right\}$$

⁵ Compared with sequential unravelings (*e.g.*, \mathbb{U} and \mathbb{U}_{opt}), \mathbb{U}_J and \mathbb{U}_N are called *simultaneous unravelings* [10].

When we consider R_{12} as a normal CTRS by replacing \downarrow by \rightarrow , the CTRS, denoted by R'_{12} below, is unraveled by \mathbb{U}_N as follows:

$$\mathbb{U}_N(R'_{12}) = \left\{ \begin{array}{l} \vdots \\ \text{odd}(s(x)) \rightarrow \text{U}_{22}(\text{even}(x), x) \quad \text{U}_{22}(\text{true}, x) \rightarrow \text{true} \\ \text{odd}(s(x)) \rightarrow \text{U}_{23}(\text{even}(x), x) \quad \text{U}_{23}(\text{false}, x) \rightarrow \text{false} \\ \vdots \\ \text{even}(s(x)) \rightarrow \text{U}_{24}(\text{odd}(x), x) \quad \text{U}_{24}(\text{true}, x) \rightarrow \text{true} \\ \text{even}(s(x)) \rightarrow \text{U}_{25}(\text{odd}(x), x) \quad \text{U}_{25}(\text{false}, x) \rightarrow \text{false} \end{array} \right\}$$

Next, we define a variant of a transformation from join CTRSs to normal ones, that is proposed in [6] (*cf.*, [26]).

Definition 5.9. Let R be a join CTRS over a signature \mathcal{F} . Introducing a fresh binary function symbol eq and a fresh constant \top , we define a transformation $\mathcal{N}orm$ as follows:

$$\begin{aligned} \mathcal{N}orm(l \rightarrow r \Leftarrow s_1 \downarrow t_1; \dots; s_k \downarrow t_k) \\ = l \rightarrow r \Leftarrow \text{eq}(s_1, t_1) \twoheadrightarrow \text{eq}(\top, \top); \dots; \text{eq}(s_k, t_k) \twoheadrightarrow \text{eq}(\top, \top) \end{aligned}$$

and

$$\mathcal{N}orm(R) = \{\text{eq}(x, x) \rightarrow \text{eq}(\top, \top)\} \cup \{\mathcal{N}orm(\rho) \mid \rho \in R\}$$

The added rule $\text{eq}(x, x) \rightarrow \text{eq}(\top, \top)$ results in non-termination, but non-termination does not affect the following discussion.⁶ The difference from the original transformation [6] is the use of $\text{eq}(\top, \top)$ instead of $\text{and } \top$. The reason of this difference is to make it simple to prove a theorem shown later (Theorem 5.13)—the original transformation, denoted by n in [26], can substitute for $\mathcal{N}orm$ since $\rightarrow_{n(R)} = \rightarrow_{\mathcal{N}orm(R)}$ on terms in $\mathcal{T}(\mathcal{F}, \mathcal{V})$. It is clear that $\mathcal{N}orm(R)$ is a normal CTRS over the signature $\mathcal{F} \cup \{\text{eq}, \top\}$, $\rightarrow_R^* \subseteq \rightarrow_{\mathcal{N}orm(R)}^*$, and especially $\rightarrow_R = \rightarrow_{\mathcal{N}orm(R)}$ on terms in $\mathcal{T}(\mathcal{F}, \mathcal{V})$. Note that the composed transformation $\mathbb{U}_N \circ \mathcal{N}orm$ is an unraveling for join CTRSs.

Example 5.10. The join CTRS R_{12} in Example 5.8 is transformed by $\mathcal{N}orm$ as follows:

$$\mathcal{N}orm(R_{12}) = \left\{ \begin{array}{l} \vdots \\ \text{odd}(s(x)) \rightarrow \text{true} \Leftarrow \text{eq}(\text{even}(x), \text{true}) \twoheadrightarrow \text{eq}(\top, \top) \\ \text{odd}(s(x)) \rightarrow \text{false} \Leftarrow \text{eq}(\text{even}(x), \text{false}) \twoheadrightarrow \text{eq}(\top, \top) \\ \vdots \\ \text{even}(s(x)) \rightarrow \text{true} \Leftarrow \text{eq}(\text{odd}(x), \text{true}) \twoheadrightarrow \text{eq}(\top, \top) \\ \text{even}(s(x)) \rightarrow \text{false} \Leftarrow \text{eq}(\text{odd}(x), \text{false}) \twoheadrightarrow \text{eq}(\top, \top) \\ \text{eq}(x, x) \rightarrow \text{eq}(\top, \top) \end{array} \right\}$$

⁶ To avoid non-termination caused by the added rule $\text{eq}(x, x) \rightarrow \text{eq}(\top, \top)$, we may introduce a unary constructor c_1 as follows:

$$\begin{aligned} \mathcal{N}orm'(l \rightarrow r \Leftarrow s_1 \downarrow t_1; \dots; s_k \downarrow t_k) = \\ l \rightarrow r \Leftarrow \text{eq}(\text{c}_1(s_1), \text{c}_1(t_1)) \twoheadrightarrow \text{eq}(\top, \top); \dots; \text{eq}(\text{c}_1(s_k), \text{c}_1(t_k)) \twoheadrightarrow \text{eq}(\top, \top) \end{aligned}$$

and

$$\mathcal{N}orm'(R) = \{\text{eq}(\text{c}_1(x), \text{c}_1(x)) \rightarrow \text{eq}(\top, \top)\} \cup \{\mathcal{N}orm'(\rho) \mid \rho \in R\}$$

This variant can substitute for $\mathcal{N}orm$ in the following discussion.

Here, we recall the notion of weak left-linearity. A normal 1-CTRS R is called *weakly left-linear* (WLL) [9] if any conditional rule with a non-empty condition in R is $\mathbb{U}_{\text{opt-LL}}$ and any unconditional rule in R is LL w.r.t. non-erasing variables (*i.e.*, for $l \rightarrow r$, the left-hand side is linear w.r.t. variables in $\mathcal{V}ar(l) \cap \mathcal{V}ar(r)$). For example, the normal CTRS R_{10} in Example 4.12 is WLL. R is called *ground conditional* if, for any rule $l \rightarrow r \Leftarrow s_1 \approx t_1; \dots; s_k \approx t_k$ in R , the terms $s_1, t_1, \dots, s_k, t_k$ in the conditional part are ground. As mentioned before, some soundness conditions for \mathbb{U}_N and \mathbb{U}_N^r are known, that are related to the (W)LL property, while no soundness condition for either \mathbb{U}_J or \mathbb{U}_J^r is known.

Theorem 5.11 ([9]). \mathbb{U}_N is sound for WLL normal 1-CTRSs.

The WLL property and Theorem 5.11 lead to the following soundness condition of the composed unraveling $\mathbb{U}_N \circ \text{Norm}$.

Lemma 5.12.

- If a join CTRS R is LL, then $\text{Norm}(R)$ is WLL.
- $\mathbb{U}_N \circ \text{Norm}$ is sound for LL join CTRSs.

Proof. The first claim is trivial by definition. The second claim follows from the first claim and Theorem 5.11. \square

TRSs obtained by $\mathbb{U}_N \circ \text{Norm}$ can completely derive reduction sequences of the corresponding TRSs obtained by \mathbb{U}_J , *i.e.*, if $\mathbb{U}_N \circ \text{Norm}$ is sound for a join CTRS R , then so is \mathbb{U}_J .

Theorem 5.13. Let R be a join CTRS over a signature \mathcal{F} . Then, there exists an \mathcal{F} -identical and non-erasing tree homomorphism ϕ such that $\mathbb{U}_N \circ \text{Norm}(R) = \phi(\mathbb{U}_J(R))$. That is, if $\mathbb{U}_N \circ \text{Norm}$ is sound for R (w.r.t. $\overrightarrow{\text{evs}}_{\mathbb{U}_N \circ \text{Norm}(R)}$), then \mathbb{U}_J is sound for R (w.r.t. $\overrightarrow{\text{evs}}_{\mathbb{U}_J(R)}$).

Proof. Let ϕ be an \mathcal{F} -identical and non-erasing tree homomorphism determined by $\phi_{\mathcal{F}_{\mathbb{U}_J(R)}}$ such that

$$\phi_{\mathcal{F}_{\mathbb{U}_J(R)}}(U^\rho(x_1, x'_1, \dots, x_k, x'_k, \overrightarrow{\mathcal{V}ar(l)})) = U^\rho(\text{eq}(x_1, x'_1), \dots, \text{eq}(x_k, x'_k), \overrightarrow{\mathcal{V}ar(l)})$$

where $\rho : l \rightarrow r \Leftarrow s_1 \downarrow t_1; \dots; s_k \downarrow t_k \in R$ and $x_1, x'_1, \dots, x_k, x'_k$ are fresh different variables. Then, it is clear that $\mathbb{U}_N \circ \text{Norm}(R) = \phi(\mathbb{U}_J(R))$. \square

Theorem 5.13 indicates that, for a join CTRS R , soundness conditions of \mathbb{U}_N for $\text{Norm}(R)$ are soundness conditions of \mathbb{U}_J for R . For example, as a consequence of Lemma 5.12 and Theorem 5.13, we conclude the following result on soundness of \mathbb{U}_J .

Corollary 5.14. \mathbb{U}_J is sound for LL join 3-CTRSs.

We recognize from Corollary 5.14 that \mathbb{U}_J is sound for R_{12} in Example 5.8.

As we mentioned before, normal CTRSs can be considered join CTRSs because the conditions $s_i \rightarrow n_i$ and $s_i \downarrow n_i$ with a ground normal form n_i are identical. Thus, soundness of \mathbb{U}_J implies soundness of \mathbb{U}_N .

Theorem 5.15. \mathbb{U}_N is sound for a normal CTRS R (w.r.t. $\overrightarrow{\text{evs}}_{\mathbb{U}_N(R)}$) if \mathbb{U}_J is sound for the the corresponding join CTRS $R' = \{l \rightarrow r \Leftarrow s_1 \downarrow n_1; \dots; s_k \downarrow n_k \mid l \rightarrow r \Leftarrow s_1 \rightarrow n_1; \dots; s_k \rightarrow n_k \in R\}$ (w.r.t. $\overrightarrow{\text{evs}}_{\mathbb{U}_J(R')}$).

Proof. Let R be over a signature \mathcal{F} and ϕ be an \mathcal{F} -identical and non-erasing tree homomorphism determined by $\phi_{\mathbb{U}_N(R)}$ such that

$$\phi_{\mathbb{U}_N(R)}(U^\rho(x_1, \dots, x_k, \overrightarrow{\text{Var}(l)})) = U^\rho(x_1, n_1, \dots, x_k, n_k, \overrightarrow{\text{Var}(l)})$$

where $\rho : l \rightarrow r \Leftarrow s_1 \downarrow n_1; \dots; s_k \downarrow n_k \in R$ and $x_1, x'_1, \dots, x_k, x'_k$ are different fresh variables. Then, it is clear that $\phi(\mathbb{U}_N(R)) = \mathbb{U}_J(R')$. \square

Example 5.16. Consider the join CTRS R_{12} and the corresponding normal CTRS R_{12} in Example 5.8 again. Let ϕ be a tree homomorphism determined by the mapping $\phi_{\mathcal{F}}$ such that $\phi_{\mathcal{F}}(\mathbb{U}_{22}(y, x)) = \mathbb{U}_{22}(y, \text{true}, x)$, $\phi_{\mathcal{F}}(\mathbb{U}_{23}(y, x)) = \mathbb{U}_{23}(y, \text{false}, x)$, $\phi_{\mathcal{F}}(\mathbb{U}_{24}(y, x)) = \mathbb{U}_{24}(y, \text{true}, x)$, and $\phi_{\mathcal{F}}(\mathbb{U}_{25}(y, x)) = \mathbb{U}_{25}(y, \text{false}, x)$. Then, $\phi(\mathbb{U}_N(R'_{12})) = \mathbb{U}_J(R_{12})$. Since \mathbb{U}_J is sound for R_{12} , we recognize from Theorem 5.15 that \mathbb{U}_N is sound for R'_{12} .

By the same token, a join CTRS R can be considered a normal CTRS if, for any rule $l \rightarrow r \Leftarrow s_1 \downarrow t_1; \dots; s_k \downarrow t_k \in R$ and for all $1 \leq i \leq k$, at least one s_i and t_i is a ground normal form w.r.t. R_u . We call such a join CTRS R *normal* and assume w.l.o.g. that t_i is a ground normal form w.r.t. R_u . Then, we obtain the following soundness condition of \mathbb{U}_J .

Theorem 5.17. \mathbb{U}_J is sound for a normal join CTRS R (w.r.t. $\xrightarrow{\text{evs}}_{\mathbb{U}_J(R)}$) if \mathbb{U}_N is sound for the corresponding normal CTRS $R' = \{l \rightarrow r \Leftarrow s_1 \rightarrow n_1; \dots; s_k \rightarrow n_k \mid l \rightarrow r \Leftarrow s_1 \downarrow n_1; \dots; s_k \downarrow n_k \in R\}$ (w.r.t. $\xrightarrow{\text{evs}}_{\mathbb{U}_N(R')}$).

Proof. Let R be over a signature \mathcal{F} and ϕ be an \mathcal{F} -identical and non-erasing tree homomorphism determined by $\phi_{\mathbb{U}_J(R)}$ such that

$$\phi_{\mathbb{U}_J(R)}(U^\rho(x_1, x'_1, \dots, x_k, x'_k, \overrightarrow{\text{Var}(l)})) = U^\rho(x_1, \dots, x_k, \overrightarrow{\text{Var}(l)})$$

where $\rho : l \rightarrow r \Leftarrow s_1 \downarrow n_1; \dots; s_k \downarrow n_k \in R$ and $x_1, x'_1, \dots, x_k, x'_k$ are different fresh variables. Then, it is clear that $\phi(\mathbb{U}_J(R)) = \mathbb{U}_N(R')$. \square

It is possible to transform join CTRSs into DCTRSs without adding the rule $\text{eq}(x, x) \rightarrow \text{eq}(\top, \top)$.

Definition 5.18. Let R be a join CTRS over a signature \mathcal{F} . Introducing a fresh $2k$ -ary constructor eq_k for each $\rho : l \rightarrow r \Leftarrow s_1 \downarrow t_1; \dots; s_k \downarrow t_k \in R$, we define a transformation Det as follows:

- $\text{Det}(l \rightarrow r \Leftarrow s_1 \downarrow t_1; \dots; s_k \downarrow t_k) = l \rightarrow r \Leftarrow \text{eq}_k(s_1, t_1, \dots, s_k, t_k) \rightarrow \text{eq}_k(x_1, x_1, \dots, x_k, x_k)$ where x_1, \dots, x_k are different fresh variables, and
- $\text{Det}(R) = \{\text{Det}(\rho) \mid \rho \in R\}$.

The reason why we introduced eq_k instead of $s_1 \rightarrow x_1; t_1 \rightarrow x_1; \dots; s_k \rightarrow x_k; t_k \rightarrow x_k$ is to make the number of the conditions in each rule of $\text{Det}(R)$ at most one. It is clear that $\mathbb{U} \circ \text{Det}$ is an unraveling for join CTRSs. It is also clear that $\rightarrow_{\text{Det}(R)} = \rightarrow_R$ on terms in $\mathcal{T}(\mathcal{F}, \mathcal{V})$. TRSs obtained by $\mathbb{U} \circ \text{Norm}$ can completely derive reduction sequences of the corresponding TRSs obtained by \mathbb{U}_J . This indicates that if $\mathbb{U} \circ \text{Det}$ is sound for a join CTRS R , then so is \mathbb{U}_J .

Theorem 5.19. Let R be a join CTRS over a signature \mathcal{F} . Then, there exists an \mathcal{F} -identical and non-erasing tree homomorphism ϕ such that $\mathbb{U} \circ \text{Det}(R) = \phi(\mathbb{U}_J(R))$. That is, if $\mathbb{U} \circ \text{Det}$ is sound for R (w.r.t. $\xrightarrow{\text{evs}}_{\mathbb{U} \circ \text{Det}(R)}$), then \mathbb{U}_J is sound for R (w.r.t. $\xrightarrow{\text{evs}}_{\mathbb{U}_J(R)}$).

Proof. Let ϕ be an \mathcal{F} -identical and non-erasing tree homomorphism determined by $\phi_{\mathcal{F}_{\mathbb{U}_J^r(R)}}$ such that

$$\phi_{\mathcal{F}_{\mathbb{U}_J(R)}}(U^\rho(x_1, x'_1, \dots, x_k, x'_k, \overrightarrow{\text{Var}(l)})) = U_1^\rho(\text{eq}_k(x_1, x'_1, \dots, x_k, x'_k), \overrightarrow{\text{Var}(l)})$$

where $\rho : l \rightarrow r \Leftarrow s_1 \downarrow t_1; \dots; s_k \downarrow t_k \in R$ and $x_1, x'_1, \dots, x_k, x'_k$ are fresh different variables. Then, it is clear that $\phi(\mathbb{U}_J(R)) = \mathbb{U} \circ \mathcal{D}et(R)$. \square

Note that it is easy to adapt Theorem 5.19 to \mathbb{U}_N and normal CTRSSs.

Normal CTRSSs are special cases of DCTRSSs, and thus, the unravelings \mathbb{U} and \mathbb{U}_{opt} for DCTRSSs are applicable to normal CTRSSs. Moreover, by definition, \mathbb{U}_N can be considered a special variant of \mathbb{U} while there is a slight difference: \mathbb{U}_N introduces at most one \mathbb{U} symbol for each rewrite rule, and \mathbb{U} introduces k \mathbb{U} symbols for each rewrite rule with k conditions. This difference prevents us from using Lemma 5.3 to prove that if \mathbb{U}_N is sound for R , then so is \mathbb{U} . For this reason, we extend Lemma 5.3 as follows.

Lemma 5.20. *Let U_1 and U_2 be unravelings, R be an eCTRS over a signature \mathcal{F} , and $\mathcal{G}_1, \mathcal{G}_2$ be extended signatures of \mathcal{F} such that $U_1(R)$ and $U_2(R)$ are defined over \mathcal{G}_1 and \mathcal{G}_2 , respectively. Let ϕ be an \mathcal{F} -identical tree homomorphism determined by $\phi_{\mathcal{G}}$ such that $U_1(R) = \phi(U_2(R)) \setminus \{t \rightarrow t \mid t \in \mathcal{T}(\mathcal{G}_1, \mathcal{V}) \setminus \mathcal{T}(\mathcal{F}, \mathcal{V})\}$. Then, all of the following hold:*

- (1) $\rightarrow_{U_2(R)}^* \subseteq \rightarrow_{U_1(R)}^*$ on terms in $\mathcal{T}(\mathcal{F}, \mathcal{V})$,
- (2) if ϕ is EV-preserving for $U_2(R)$, then $\overrightarrow{\text{evs}}_{U_2(R)}^* \subseteq \overrightarrow{\text{evs}}_{U_1(R)}^*$ on terms in $\mathcal{T}(\mathcal{F}, \mathcal{V})$.

That is, all of the following hold:

- (3) if U_1 is sound for R , then so is U_2 , and,
- (4) if ϕ is EV-preserving for $U_2(R)$ and U_1 is sound for R w.r.t. $\overrightarrow{\text{evs}}_{U_1(R)}$, then U_2 is sound for R w.r.t. $\overrightarrow{\text{evs}}_{U_2(R)}$.

Proof. We first prove the first claim $\rightarrow_{U_2(R)}^* \subseteq \rightarrow_{U_1(R)}^*$ on terms in $\mathcal{T}(\mathcal{F}, \mathcal{V})$. It follows from the assumption that $U_1(R) = (\phi(U_2(R)) \setminus \{t \rightarrow t \mid t \in \mathcal{T}(\mathcal{G}_1, \mathcal{V})\}) \cup R'$ for some eTRS $R' \subseteq \{t \rightarrow t \mid t \in \mathcal{T}(\mathcal{G}_1, \mathcal{V})\}$. Then, it follows from Lemma 5.1 that $\phi(\rightarrow_{U_2(R)}^*) \subseteq \rightarrow_{U_1(R) \cup R'}^*$. Since $\rightarrow_{R'}$ is the identity relation, we have that $\rightarrow_{U_1(R) \cup R'}^* = \rightarrow_{U_1(R)}^*$, and hence $\phi(\rightarrow_{U_2(R)}^*) \subseteq \rightarrow_{U_1(R)}^*$. Since ϕ is \mathcal{F} -identical, we have that $\rightarrow_{U_2(R)}^* \subseteq \rightarrow_{U_1(R)}^*$ on terms in $\mathcal{T}(\mathcal{F}, \mathcal{V})$.

The second claim follows from the first claim and Lemma 5.2. The third and fourth claims follow from the first and second claims, and soundness of U_1 . \square

Due to Lemma 5.20, we obtain the following theorem.

Theorem 5.21. *Let R be a normal CTRS over a signature \mathcal{F} . Then, there exists an \mathcal{F} -identical and non-erasing tree homomorphism ϕ such that $\mathbb{U}_N(R) = \phi(\mathbb{U}(R)) \setminus \{t \rightarrow t \mid t \in \mathcal{T}(\mathcal{F}_{\mathbb{U}(R)}, \mathcal{V}) \setminus \mathcal{T}(\mathcal{F}, \mathcal{V})\}$. That is, if \mathbb{U}_N is sound for a normal CTRS R (w.r.t. $\overrightarrow{\text{evs}}_{\mathbb{U}_N(R)}$), then \mathbb{U} is sound for R (w.r.t. $\overrightarrow{\text{evs}}_{\mathbb{U}(R)}$).*

Proof. Let ϕ be an \mathcal{F} -identical tree homomorphism determined by $\phi_{\mathcal{F}_{\mathbb{U}(R)}}$ such that

$$\phi_{\mathcal{F}_{\mathbb{U}(R)}}(U_i^\rho(x_i, \overrightarrow{X_i})) = U^\rho(n_1, \dots, n_{i-1}, x_i, s_{i+1}, \dots, s_k, \overrightarrow{\text{Var}(l)})$$

where $\rho : l \rightarrow r \Leftarrow s_1 \rightarrow n_1; \dots; s_k \rightarrow n_k \in R$ and x_i is a fresh variable. It is clear that $\mathbb{U}_N(R) = \phi(\mathbb{U}(R)) \setminus \{t \rightarrow t \mid t \in \mathcal{T}(\mathcal{F}_{\mathbb{U}(R)}, \mathcal{V}) \setminus \mathcal{T}(\mathcal{F}, \mathcal{V})\}$:

- $\phi(l \rightarrow U_1^\rho(s_1, \overrightarrow{X_1})) = l \rightarrow U^\rho(s_1, \dots, s_k, \overrightarrow{\text{Var}(l)}) \in \mathbb{U}_N(R)$,

- $\phi(U_i^\rho(n_i, \vec{X}_i) \rightarrow U_{i+1}^\rho(s_{i+1}, \vec{X}_{i+2})) = U^\rho(n_1, \dots, n_i, s_{i+1}, \dots, s_k, \overline{\mathcal{V}ar(l)}) \rightarrow U^\rho(n_1, \dots, n_i, s_{i+1}, \dots, s_k, \overline{\mathcal{V}ar(l)}) \in \{t \rightarrow t \mid t \in \mathcal{T}(\mathcal{F}_{\mathbb{U}_N(R)}, \mathcal{V}) \setminus \mathcal{T}(\mathcal{F}, \mathcal{V})\}$, and
- $\phi(U_k^\rho(n_k, \vec{X}_k) \rightarrow r) = U^\rho(n_1, \dots, n_k, \overline{\mathcal{V}ar(l)}) \rightarrow r \in \mathbb{U}_N(R)$.

Since R is normal, we have that $\mathcal{V}ar(n_1, \dots, n_k) = \emptyset$ and $\mathcal{V}ar(s_1, \dots, s_k) \subseteq \mathcal{V}ar(l)$, and hence $X_i = \mathcal{V}ar(l)$ for all $1 \leq i \leq k$. Thus, $\mathcal{V}ar(\phi_{\mathcal{F}_{\mathbb{U}_N(R)}}(U_i^\rho(x_i, \vec{X}_i))) = \{x_i\} \cup X_i = \{x_i\} \cup \mathcal{V}ar(l) = \mathcal{V}ar(U^\rho(n_1, \dots, n_{i-1}, x_i, s_{i+1}, \dots, s_k, \overline{\mathcal{V}ar(l)}))$, and hence ϕ is non-erasing. \square

It is not known whether the converse of Theorem 5.21 (*i.e.*, $\rightarrow_{\mathbb{U}_N(R)}^* \subseteq \rightarrow_{\mathbb{U}(R)}^*$) on terms in $\mathcal{T}(\mathcal{F}, \mathcal{V})$ holds or not. In other words, it is not known whether the following claim holds or not: if \mathbb{U} is sound for a normal CTRS, then so is \mathbb{U}_N . As we mentioned before, to show soundness of \mathbb{U}_N by means of \mathbb{U} , we would like to show that, for any normal CTRS R , all the derivations of $\mathbb{U}_N(R)$ can be derived by $\mathbb{U}(R)$. However, this is not true in general.

Example 5.22. Consider the following variant R'_3 of the DCTRS R_3 in Example 3.3, that is obtained by replacing the conditional part $x \rightarrow e$ of the first rule by $x \rightarrow e$; $x \rightarrow e'$ and by adding $c \rightarrow e'$ to the rules:

$$R'_3 = \left\{ \begin{array}{l} f(x) \rightarrow x \leftarrow x \rightarrow e; x \rightarrow e' \\ g(d, x, x) \rightarrow A \\ h(x, x) \rightarrow g(x, x, f(k)) \\ c \rightarrow e' \end{array} \right\} \cup R_0$$

The CTRS R'_3 is unraveled by \mathbb{U}_N and \mathbb{U} as follows:

$$\mathbb{U}_N(R'_3) = \left\{ \begin{array}{l} f(x) \rightarrow U_4(x, x, x) \\ U_4(e, e', x) \rightarrow x \\ \vdots \end{array} \right\} \cup R_0 \quad \mathbb{U}(R'_3) = \left\{ \begin{array}{l} f(x) \rightarrow U'_4(x, x) \\ U'_4(e, x) \rightarrow U''_4(x, x) \\ U''_4(e', x) \rightarrow x \\ \vdots \end{array} \right\} \cup R_0$$

We have that $h(f(a), f(b)) \rightarrow_{\mathbb{U}_N(R'_3)}^* A$, but $h(f(a), f(b)) \not\rightarrow_{\mathbb{U}(R'_3)}^* A$. This means that $\mathbb{U}(R'_3)$ cannot derive every reduction sequence of $\mathbb{U}_N(R'_3)$ that starts from terms over the original signature of R'_3 . \mathbb{U} seems sound for R'_3 . However, we have no sufficient condition to prove soundness of \mathbb{U} for R'_3 , and thus, it is not known whether \mathbb{U} is sound for R'_3 or not.

The symbols e and e' are used for the same role; therefore, this distinction is meaningless. Thus, the replacement of $x \rightarrow e$ with $x \rightarrow e$; $x \rightarrow e$ is sufficient for the purpose of this example. For the original CTRS R'_3 , this duplication of $x \rightarrow e$ is quite meaningless, but this greatly affects the reduction of $\mathbb{U}(R'_3)$. For this reason, this would be an interesting example for investigating soundness conditions of unravelings.

A trivial sufficient condition for the converse of Theorem 5.21 is that any rule of R has at most one condition: by considering $U^\rho = U_1^\rho$, we have that $\mathbb{U}_N(R) = \mathbb{U}(R)$.

As stated above, the relationship between \mathbb{U}_J^r , \mathbb{U}_N^r , and \mathbb{U}_{opt} is similar to that between \mathbb{U}_J , \mathbb{U}_N , and \mathbb{U} . For this reason, Theorems 5.15, 5.17, 5.21 also hold for \mathbb{U}_J^r , \mathbb{U}_N^r , and \mathbb{U}_{opt} .

6. COMPARISON WITH ŒERBĂNUŢĂ-ROŞU TRANSFORMATION

In this section, we compare the unraveling \mathbb{U} with the SR transformation, in terms of soundness, operational termination, confluence, computational equivalence, and so on.

6.1. Formalization of Transformations for CTRSSs. In this subsection, to make it easier to compare unravelings with other transformations, we first formalize transformations of CTRSSs into TRSSs, and also generalize the notion of soundness and completeness for unravelings. Then, we present relationship between soundness of two transformations by generalizing Lemma 5.3.

We first formalize transformations of CTRSSs and the notions of soundness and completeness.

Definition 6.1 (CTRS transformations). A *CTRS transformation* is a computable transformation T from eCTRSs into eTRSSs with injective mappings as follows: for an eCTRS R over a signature \mathcal{F} , the transformed eTRSS R_T over a signature \mathcal{G} is defined and the corresponding mapping $\phi_{T(R)}$ from $\mathcal{T}(\mathcal{F}, \mathcal{V})$ to $\mathcal{T}(\mathcal{G}, \mathcal{V})$ is also defined, *i.e.*, $T(R) = (R_T, \phi_{T(R)})$. The mapping $\phi_{T(R)}$ is called a *translation related to $T(R)$* .⁷ We extend $\phi_{T(R)}$ to pairs of terms in $\mathcal{T}(\mathcal{F}, \mathcal{V})$: for $S \subseteq \mathcal{T}(\mathcal{F}, \mathcal{V}) \times \mathcal{T}(\mathcal{F}, \mathcal{V})$, $\phi(S) = \{(\phi(s), \phi(t)) \mid (s, t) \in S\}$. Moreover, T is called *simple* if the related translation $\phi_{T(R)}$ is the identity mapping (*i.e.*, $\mathcal{F} \subseteq \mathcal{G}$ and $\phi_{T(R)}(t) = t$ for all $t \in \mathcal{T}(\mathcal{F}, \mathcal{V})$), and we abuse notation and write $T(R)$ as the transformed system R_T .

Let \Rightarrow_{R_T} be a subrelation of \rightarrow_{R_T} .

- T is called *sound for R w.r.t. \Rightarrow_{R_T}* if $\Rightarrow_{R_T}^* \subseteq \phi_{T(R)}(\rightarrow_R^*)$ on terms in $\mathcal{T}(\mathcal{F}, \mathcal{V})$ (*i.e.*, for all terms s, t in $\mathcal{T}(\mathcal{F}, \mathcal{V})$, if $\phi_{T(R)}(s) \Rightarrow_{R_T}^* \phi_{T(R)}(t)$, then $s \rightarrow_R^* t$).
- T is called *complete for R w.r.t. \Rightarrow_{R_T}* if $\phi_{T(R)}(\rightarrow_R^*) \subseteq \Rightarrow_{R_T}^*$ (*i.e.*, for all terms $s, t \in \mathcal{T}(\mathcal{F}, \mathcal{V})$, if $s \rightarrow_R^* t$, then $\phi_{T(R)}(s) \Rightarrow_{R_T}^* \phi_{T(R)}(t)$).

When T is sound and complete for R w.r.t. \rightarrow_{R_T} , we simply say that T is *sound and complete for R* , respectively. Moreover, T is called *sound (complete)* if T is sound (complete) for any eCTRS R such that $T(R)$ is defined.

Note that unravelings are complete simple CTRS transformations.

Next, we generalize Lemma 5.3 to two CTRS transformations, one of which is simple.

Theorem 6.2. *Let T be a CTRS transformation, U be a simple CTRS transformation, R be an eCTRS over a signature \mathcal{F} such that $T(R)$ and $U(R)$ are defined, R_T is over a signature \mathcal{G}_T , and $U(R)$ is over a signature \mathcal{G}_U . Then, all of the following hold:*

- *if T is sound for R and $\phi(\rightarrow_{U(R)}^*) \subseteq \rightarrow_{R_T}^*$,⁸ then U is sound for R ,*
- *if U is sound for R and $\rightarrow_{R_T}^* \subseteq \phi(\rightarrow_{U(R)}^*)$, then T is sound for R ,*
- *if T is complete for R and $\rightarrow_{R_T}^* \subseteq \phi(\rightarrow_{U(R)}^*)$, then U is complete for R , and*
- *if U is complete for R and $\phi(\rightarrow_{U(R)}^*) \subseteq \rightarrow_{R_T}^*$, then T is complete for R .*

Proof. We only prove the first claim since the other claims can be proved similarly to the first one. Let s, t be terms in $\mathcal{T}(\mathcal{F}, \mathcal{V})$. Suppose that $s \rightarrow_{U(R)}^* t$. Then, it follows from $\phi(\rightarrow_{U(R)}^*) \subseteq \rightarrow_{R_T}^*$ that $\phi(s) \rightarrow_{R_T}^* \phi(t)$. It follows from soundness of T for R that $s \rightarrow_R^* t$, and hence U is sound for R . \square

⁷ The mapping ϕ can be considered a translation from original terms for R into the corresponding ones for $T(R)$.

⁸ Note that $\phi(\rightarrow_{U(R)}^*) = \{(\phi(s), \phi(t)) \mid s, t \in \mathcal{T}(\mathcal{F}, \mathcal{V}), s \rightarrow_{U(R)}^* t\}$ since ϕ is not defined for any term containing a function symbol in $\mathcal{G}_U \setminus \mathcal{F}$.

6.2. Șerbănuță-Roșu Transformation. In this subsection, we recall the definition of the SR transformation proposed by Șerbănuță and Roșu [30, 31], which is basically applied to *strongly* or *syntactically* DCTRSs. We also recall some of its properties.

Let R be an eDCTRS. A term t is called *strongly irreducible w.r.t. R* if $t\sigma$ is a normal form w.r.t. R for every normalized substitution σ . R is called *strongly deterministic* (strongly DCTRS) if, for every rule $l \rightarrow r \Leftarrow s_1 \rightarrow t_1; \dots; s_k \rightarrow t_k \in R$, every term t_i is strongly irreducible w.r.t. R . R is called *syntactically deterministic* (syntactically DCTRS) if, for every rule $l \rightarrow r \Leftarrow s_1 \rightarrow t_1; \dots; s_k \rightarrow t_k \in R$, every term t_i is a constructor term or a ground normal form w.r.t. R_u . Note that normal CTRSs are syntactically DCTRSs, and syntactically DCTRSs are also strongly DCTRSs.

In the following, we assume that for each defined symbol f of R , there are n_f many f -rules in R that have non-empty conditions and are ordered. We denote the i -th conditional rewrite rule of f with a non-empty condition by $\rho_{f,i}$.

In the SR transformation \mathbb{SR} below, a fresh unary function symbol $\{\cdot\}$, a fresh constant \perp , and fresh k -ary constructors $[\cdot]_k$ are introduced and for a defined symbol f , a fresh function symbol \bar{f} is introduced by adding n_f arguments to f . The “ $n + i$ ”-th argument of \bar{f} is used for evaluating the conditions of the i -th conditional rule $\rho_{f,i} : l \rightarrow r \Leftarrow s_1 \rightarrow t_1; \dots; s_k \rightarrow t_k$, by initializing with \perp and by replacing \perp with an instance of $[\{\bar{s}_1\}, \perp, \dots, \perp]_k$ to start the evaluation, where \bar{s}_1 is the term obtained by replacing each defined symbol f by \bar{f} with filling extra arguments with \perp . The k -ary symbol $[\cdot]_k$ is used as a stack with k elements, *e.g.*, when $[\{\bar{s}_1\sigma}, \perp, \dots, \perp]_k$ is reduced to $[\{\bar{t}_1\theta\}, \perp, \dots, \perp]_k$, the evaluation of the second condition $s_2 \rightarrow t_2$ with θ starts from $[\{\bar{s}_2\theta\}, \bar{t}_1\theta, \perp, \dots, \perp]_k$.

Definition 6.3 (SR transformation \mathbb{SR} [31]). Let R be a strongly or syntactically DCTRS over a signature \mathcal{F} .⁹ For $f \in \mathcal{D}_R$, we prepare a function symbol \bar{f} with $\text{arity}(\bar{f}) = \text{arity}(f) + n_f$. Introducing a fresh unary function symbol $\{\cdot\}$, a fresh constant \perp and fresh j -ary constructors $[\cdot]_j$ with $j > 0$ ($[\cdot]_1, [\cdot]_2, \dots$ are sometimes abbreviated to $[\cdot]$) into the signature, the DCTRS R is transformed into the following TRS $\mathbb{SR}^\rightarrow(R)$:

$$\mathbb{SR}^{\text{rule}}(\rho_{f,i} : f(w_1, \dots, w_n) \rightarrow r \Leftarrow s_1 \rightarrow t_1; \dots; s_k \rightarrow t_k) = \left. \begin{array}{l} \bar{f}(\bar{w}_1, \dots, \bar{w}_n, z_1, \dots, z_{i-1}, \perp, z_{i+1}, \dots, z_{n_f}) \\ \quad \rightarrow \bar{f}(\bar{w}_1, \dots, \bar{w}_n, z_1, \dots, z_{i-1}, [\{\bar{s}_1\}, \perp, \dots, \perp]_k, z_{i+1}, \dots, z_{n_f}) \\ \bar{f}(\bar{w}_1, \dots, \bar{w}_n, z_1, \dots, z_{i-1}, [\{\bar{t}_1\}, \perp, \dots, \perp]_k, z_{i+1}, \dots, z_{n_f}) \\ \quad \rightarrow \bar{f}(\bar{w}_1, \dots, \bar{w}_n, z_1, \dots, z_{i-1}, [\{\bar{s}_2\}, \bar{t}_1, \dots, \perp]_k, z_{i+1}, \dots, z_{n_f}) \\ \quad \vdots \\ \bar{f}(\bar{w}_1, \dots, \bar{w}_n, z_1, \dots, z_{i-1}, [\{\bar{t}_{k-1}\}, \bar{t}_{k-2}, \dots, \bar{t}_1, \perp]_k, z_{i+1}, \dots, z_{n_f}) \\ \quad \rightarrow \bar{f}(\bar{w}_1, \dots, \bar{w}_n, z_1, \dots, z_{i-1}, [\{\bar{s}_k\}, \bar{t}_{k-1}, \dots, \bar{t}_1]_k, z_{i+1}, \dots, z_{n_f}) \\ \bar{f}(\bar{w}_1, \dots, \bar{w}_n, z_1, \dots, z_{i-1}, [\{\bar{t}_k\}, \bar{t}_{k-1}, \dots, \bar{t}_1]_k, z_{i+1}, \dots, z_{n_f}) \rightarrow \{\bar{r}\} \end{array} \right\}$$

$$\mathbb{SR}^{\text{rule}}(f(w_1, \dots, w_n) \rightarrow r) = \{ \bar{f}(\bar{w}_1, \dots, \bar{w}_n, z_1, \dots, z_{n_f}) \rightarrow \{\bar{r}\} \}$$

where z_1, \dots, z_{n_f} are fresh different variables and the operation $\bar{\cdot}$ is a linear non-erasing tree homomorphism determined by $\bar{\phi}$ such that

- $\bar{\phi}(c(x_1, \dots, x_n)) = c(x_1, \dots, x_n)$ for an n -ary constructor $c \in \mathcal{C}_R$, and

⁹ In [31], it is assumed that any deterministic conditional rule $l \rightarrow r \Leftarrow s_1 \rightarrow t_1; \dots; s_k \rightarrow t_k$ satisfies $\text{Var}(s_i) \not\subseteq \text{Var}(l, t_1, \dots, t_{i-2})$, *i.e.*, the i -th condition $s_i \rightarrow t_i$ cannot be evaluated before finishing the evaluation of the “ $i - 1$ ”-th condition $s_{i-1} \rightarrow t_{i-1}$. However, this is not essential for the definition of \mathbb{SR} .

- $\overline{\phi}(f(x_1, \dots, x_n)) = \overline{f}(x_1, \dots, x_n, \overbrace{\perp, \dots, \perp}^{n_f})$ for an n -ary defined symbol $f \in \mathcal{D}_R$.

Note that the operation $\overline{\cdot}$ is injective. The transformed TRS $\mathbb{S}\mathbb{R}^\rightarrow(R)$ is defined as follows:

$$\begin{aligned} \mathbb{S}\mathbb{R}^\rightarrow(R) = & \bigcup_{\rho \in R} \mathbb{S}\mathbb{R}^{\text{rule}}(\rho) \\ & \cup \{ \overline{f}(x_1, \dots, x_{i-1}, \{x_i\}, x_{i+1}, \dots, x_n, z_1, \dots, z_{n_f}) \rightarrow \{ \overline{f}(x_1, \dots, x_n, \perp, \dots, \perp) \} \mid f \in \mathcal{D}_R \} \\ & \cup \{ c(x_1, \dots, x_{i-1}, \{x_i\}, x_{i+1}, \dots, x_n) \rightarrow \{ c(x_1, \dots, x_n) \} \mid c \in \mathcal{C}_R \} \\ & \cup \{ \{ \{ x \} \} \rightarrow \{ x \} \} \end{aligned}$$

where $x_1, \dots, x_n, z_1, \dots, z_{n_f}$ are variables. Note that $\mathbb{S}\mathbb{R}^\rightarrow(R)$ is a TRS over $\mathcal{F}_{\mathbb{S}\mathbb{R}(R)} = \{ \perp, \{ \cdot \} \} \cup \{ \overline{f} \mid f \in \mathcal{D}_R \} \cup \mathcal{C}_R$. Moreover, a partial mapping $\widehat{\cdot}$ from $\mathcal{T}(\mathcal{F}_{\mathbb{S}\mathbb{R}(R)}, \mathcal{V})$ to $\mathcal{T}(\mathcal{F}, \mathcal{V})$ is defined as follows:

- $\widehat{x} = x$ for $x \in \mathcal{V}$,
- $\widehat{\{t\}} = \widehat{t}$,
- $c(\widehat{t_1}, \dots, \widehat{t_n}) = c(\widehat{t_1}, \dots, \widehat{t_n})$ for an n -ary constructor $c \in \mathcal{C}_R$, and
- $\overline{f}(\widehat{t_1}, \dots, \widehat{t_n}, \widehat{u_1}, \dots, \widehat{u_{n_f}}) = f(\widehat{t_1}, \dots, \widehat{t_n})$ for an n -ary defined symbol $f \in \mathcal{D}_R$.

Note that the operation $\widehat{\cdot}$ partially translates terms in $\mathcal{T}(\mathcal{F}_{\mathbb{S}\mathbb{R}(R)}, \mathcal{V})$ back into terms in $\mathcal{T}(\mathcal{F}, \mathcal{V})$. The *SR transformation* $\mathbb{S}\mathbb{R}$ is defined as $\mathbb{S}\mathbb{R}(R) = (\mathbb{S}\mathbb{R}^\rightarrow(R), \phi_{\mathbb{S}\mathbb{R}(R)})$, where the translation $\phi_{\mathbb{S}\mathbb{R}(R)}$ is defined as $\phi_{\mathbb{S}\mathbb{R}(R)}(t) = \{ \widehat{t} \}$. Moreover, a term t in $\mathcal{T}(\mathcal{F}_{\mathbb{S}\mathbb{R}(R)}, \mathcal{V})$ is called *reachable* if there exists a term s in $\mathcal{T}(\mathcal{F}, \mathcal{V})$ such that $\phi_{\mathbb{S}\mathbb{R}(R)}(s) \rightarrow_{\mathbb{S}\mathbb{R}^\rightarrow(R)}^* \{t\}$.

Note that $\mathbb{S}\mathbb{R}$ is a complete CTRS transformation [31]. By definition, it is clear that R is $\mathbb{U}_{\text{opt-LL}}$ iff $\mathbb{S}\mathbb{R}^\rightarrow(R)$ is LL. A reachable term s has the following property:

- every subterm of s , rooted by \overline{f} , is of the form $\overline{f}(s_1, \dots, s_n, u_1, \dots, u_{n_f})$ such that, for all $1 \leq j \leq n_f$, u_j is either \perp or of the form $[\{ \overline{t_i} \}, \overline{t_{i-1}}, \dots, \overline{t_1}, \perp, \dots, \perp]_k$ for some i , where $\rho_{f,j} : f(w_1, \dots, w_n) \rightarrow r \Leftarrow s_1 \rightarrow t_1; \dots; s_k \rightarrow t_k \in R$, and
- both the symbols \perp and $[\cdot]_k$ appear only as in the form mentioned in the previous case.

For a DCTRS R , the transformed TRS $\mathbb{S}\mathbb{R}^\rightarrow(R)$ is overlapping (not only at root position, but also at properly inner positions), thus not a constructor system, and all non-constant constructors of R are defined symbols of $\mathbb{S}\mathbb{R}^\rightarrow(R)$. However, critical pairs generated from rules to push out the special constructor $\{ \cdot \}$ are joinable and they are not so critical in terms of confluence.

Example 6.4 ([31]). Consider the $\mathcal{U}_{\text{opt-LL}}$ DCTRS R_7 in Example 4.4 again. R_7 is a syntactically DCTRS and it is transformed by $\mathbb{SR}^{\rightarrow}$ into the following TRS:

$$\mathbb{SR}^{\rightarrow}(R_7) = \left(\begin{array}{l} \overline{\text{split}}(x, \text{nil}, z_1, z_2) \rightarrow \{\text{tp}_2(\text{nil}, \text{nil})\} \\ \overline{\text{split}}(x, \text{cons}(y, ys), \perp, z_2) \rightarrow \overline{\text{split}}(x, \text{cons}(y, ys), [\{\overline{\text{split}}(x, ys, \perp, \perp)\}, \perp], z_2) \\ \overline{\text{split}}(x, \text{cons}(y, ys), [\{\text{tp}_2(zs_1, zs_2)\}, \perp], z_2) \rightarrow \overline{\text{split}}(x, \text{cons}(y, ys), [\{\overline{\text{le}}(x, y)\}, \text{tp}_2(zs_1, zs_2)], z_2) \\ \overline{\text{split}}(x, \text{cons}(y, ys), [\{\text{true}\}, \text{tp}_2(zs_1, zs_2)], z_2) \rightarrow \{\text{tp}_2(zs_1, \text{cons}(y, zs_2))\} \\ \overline{\text{split}}(x, \text{cons}(y, ys), z_1, \perp) \rightarrow \overline{\text{split}}(x, \text{cons}(y, ys), z_1, [\{\overline{\text{split}}(x, ys, \perp, \perp)\}, \perp]) \\ \overline{\text{split}}(x, \text{cons}(y, ys), z_1, [\{\text{tp}_2(zs_1, zs_2)\}, \perp]) \rightarrow \overline{\text{split}}(x, \text{cons}(y, ys), z_1, [\{\overline{\text{le}}(x, y)\}, \text{tp}_2(zs_1, zs_2)]) \\ \overline{\text{split}}(x, \text{cons}(y, ys), z_1, [\{\text{false}\}, \text{tp}_2(zs_1, zs_2)]) \rightarrow \{\text{tp}_2(\text{cons}(y, zs_1), zs_2)\} \\ \overline{\text{le}}(0, y) \rightarrow \{\text{true}\} \\ \overline{\text{le}}(s(x), 0) \rightarrow \{\text{false}\} \\ \overline{\text{split}}(\{x\}, ys, z_1, z_2) \rightarrow \{\overline{\text{split}}(x, ys, \perp, \perp)\} \\ \overline{\text{split}}(x, \{ys\}, z_1, z_2) \rightarrow \{\overline{\text{split}}(x, ys, \perp, \perp)\} \\ \text{cons}(\{x\}, xs) \rightarrow \{\text{cons}(x, xs)\} \\ \text{cons}(x, \{xs\}) \rightarrow \{\text{cons}(x, xs)\} \\ s(\{x\}) \rightarrow \{s(x)\} \end{array} \right)$$

Consider the term $\overline{\text{split}}(s(0), \text{cons}(0, \text{cons}(s(s(0)), \text{nil})))$. Starting from its translated term, we have the following derivation of $\mathbb{SR}^{\rightarrow}(R_7)$ under the leftmost innermost strategy that selects the topmost rules of applicable ones:

$$\begin{aligned} & \overline{\text{split}}(s(0), \text{cons}(0, \text{cons}(s(s(0)), \text{nil}))) \\ &= \overline{\text{split}}(s(0), \text{cons}(0, \text{cons}(s(s(0)), \text{nil})), \perp, \perp) \\ &\rightarrow_{\mathbb{SR}^{\rightarrow}(R_7)} \overline{\text{split}}(s(0), \text{cons}(0, \dots), [\{\overline{\text{split}}(s(0), \text{cons}(s(s(0)), \text{nil}), \perp, \perp)\}, \perp], \perp) \\ &\rightarrow_{\mathbb{SR}^{\rightarrow}(R_7)}^* \overline{\text{split}}(s(0), \text{cons}(0, \dots), [\{\text{tp}_2(\text{nil}, \text{cons}(s(s(0)), \text{nil}))\}, \perp], \perp) \\ &\rightarrow_{\mathbb{SR}^{\rightarrow}(R_7)} \overline{\text{split}}(s(0), \text{cons}(0, \dots), [\{\overline{\text{le}}(s(0), 0)\}, \text{tp}_2(\text{nil}, \text{cons}(s(s(0)), \text{nil}))], \perp) \\ &\rightarrow_{\mathbb{SR}^{\rightarrow}(R_7)} \overline{\text{split}}(s(0), \text{cons}(0, \dots), [\{\{\text{false}\}\}, \text{tp}_2(\text{nil}, \text{cons}(s(s(0)), \text{nil}))], \perp) \\ &\rightarrow_{\mathbb{SR}^{\rightarrow}(R_7)} \overline{\text{split}}(s(0), \text{cons}(0, \dots), [\{\text{false}\}, \text{tp}_2(\text{nil}, \text{cons}(s(s(0)), \text{nil}))], \perp) \\ &\rightarrow_{\mathbb{SR}^{\rightarrow}(R_7)} \overline{\text{split}}(s(0), \text{cons}(0, \dots), [\{\text{false}\}, \dots], [\{\overline{\text{split}}(s(0), \text{cons}(s(s(0)), \text{nil}), \perp, \perp)\}, \perp]) \\ &\rightarrow_{\mathbb{SR}^{\rightarrow}(R_7)}^* \overline{\text{split}}(s(0), \text{cons}(0, \dots), [\{\text{false}\}, \dots], [\{\text{tp}_2(\text{nil}, \text{cons}(s(s(0)), \text{nil}))\}, \perp]) \\ &\rightarrow_{\mathbb{SR}^{\rightarrow}(R_7)} \overline{\text{split}}(s(0), \text{cons}(0, \dots), [\{\text{false}\}, \dots], [\{\overline{\text{le}}(s(0), 0)\}, \text{tp}_2(\text{nil}, \text{cons}(s(s(0)), \text{nil}))]) \\ &\rightarrow_{\mathbb{SR}^{\rightarrow}(R_7)} \overline{\text{split}}(s(0), \text{cons}(0, \dots), [\{\text{false}\}, \dots], [\{\{\text{false}\}\}, \text{tp}_2(\text{nil}, \text{cons}(s(s(0)), \text{nil}))]) \\ &\rightarrow_{\mathbb{SR}^{\rightarrow}(R_7)} \overline{\text{split}}(s(0), \text{cons}(0, \dots), [\{\text{false}\}, \dots], [\{\text{false}\}, \text{tp}_2(\text{nil}, \text{cons}(s(s(0)), \text{nil}))]) \\ &\rightarrow_{\mathbb{SR}^{\rightarrow}(R_7)} \{\{\text{tp}_2(\text{cons}(0, \text{nil}), \text{cons}(s(s(0)), \text{nil}))\}\} \\ &\rightarrow_{\mathbb{SR}^{\rightarrow}(R_7)} \{\text{tp}_2(\text{cons}(0, \text{nil}), \text{cons}(s(s(0)), \text{nil}))\} \end{aligned}$$

By applying the translation-back mapping $\hat{\cdot}$ to $\{\text{tp}_2(\text{cons}(0, \text{nil}), \text{cons}(s(s(0)), \text{nil}))\}$, we obtain $\text{tp}_2(\text{cons}(0, \text{nil}), \text{cons}(s(s(0)), \text{nil}))$, a normal form of $\overline{\text{split}}(s(0), \text{cons}(0, \text{cons}(s(s(0)), \text{nil})))$ w.r.t. R_7 .

The SR transformation \mathbb{SR} has the following properties.

Theorem 6.5 ([31]). *Let R be a strongly or syntactically DCTRS. Then, all of the following hold:*

- \mathbb{SR} is sound for R if R is confluent¹⁰ or $\mathcal{U}_{\text{opt-LL}}$,
- if R is $\mathcal{U}_{\text{opt-LL}}$ and $\mathbb{SR}^{\rightarrow}(R)$ is confluent on reachable terms, then R is confluent, and

¹⁰ In [30, 31], soundness and completeness are discussed on ground reduction sequences only. In the proof of soundness and completeness, groundness of terms in derivations is only used with groundness in

- if R is $\mathbb{U}_{\text{opt}}\text{-LL}$ and confluent, then $\mathbb{S}\mathbb{R}^{\rightarrow}(R)$ is confluent on reachable terms.

We recognize from the second statement of Theorem 6.5 that confluence of $\mathbb{S}\mathbb{R}^{\rightarrow}(R)$ is a sufficient condition for confluence of R .

Example 6.6. Consider the DCTRS R_7 and the transformed TRS $\mathbb{S}\mathbb{R}(R_7)$ in Examples 4.4, 6.4 again. The DCTRS R_7 is operationally terminating since $\mathbb{S}\mathbb{R}^{\rightarrow}(R_7)$ is terminating [31]. We have only a critical pair of R_7 between the second and third rules. The critical pair is infeasible since there exists no terms s, t such that $\text{le}(s, t) \rightarrow_{R_7}^* \text{true}$ and $\text{le}(s, t) \rightarrow_{R_7}^* \text{false}$. Thus, we can see that R_7 is confluent [2] (*cf.*, [26]). Though, we have no formal method for proving confluence of R_7 . On the other hand, all the critical pairs of $\mathbb{S}\mathbb{R}(R_7)$ are joinable and $\mathbb{S}\mathbb{R}^{\rightarrow}(R_7)$ is terminating, and hence $\mathbb{S}\mathbb{R}^{\rightarrow}(R_7)$ is confluent. Due to Theorem 6.5, confluence of $\mathbb{S}\mathbb{R}^{\rightarrow}(R_7)$ guarantees confluence of R_7 .

Consider the unraveled TRS $\mathbb{U}(R_7)$:

$$\mathbb{U}(R_7) = \left\{ \begin{array}{c} \vdots \\ \text{split}(x, \text{cons}(y, ys)) \rightarrow \mathbb{U}_{13}(\text{split}(x, ys), x, y, ys) \\ \mathbb{U}_{13}(\text{tp}_2(zs_1, zs_2), x, y, ys) \rightarrow \mathbb{U}_{14}(\text{le}(x, y), x, y, ys, zs_1, zs_2) \\ \mathbb{U}_{14}(\text{true}, x, y, ys, zs_1, zs_2) \rightarrow \text{tp}_2(zs_1, \text{cons}(y, zs_2)) \\ \text{split}(x, \text{cons}(y, ys)) \rightarrow \mathbb{U}_{15}(\text{split}(x, ys), x, y, ys) \\ \mathbb{U}_{15}(\text{tp}_2(zs_1, zs_2), x, y, ys) \rightarrow \mathbb{U}_{16}(\text{le}(x, y), x, y, ys, zs_1, zs_2) \\ \mathbb{U}_{16}(\text{false}, x, y, ys, zs_1, zs_2) \rightarrow \text{tp}_2(\text{cons}(y, zs_1), zs_2) \\ \vdots \end{array} \right\}$$

Unlike $\mathbb{S}\mathbb{R}^{\rightarrow}(R_7)$, this unraveled TRS $\mathbb{U}(R_7)$ is not confluent since we have a *critical* peak, *e.g.*, $\text{tp}_2(\text{nil}, \text{cons}(0, \text{nil})) \leftarrow_{\mathbb{U}(R_7)}^* \text{split}(0, \text{cons}(0, \text{nil})) \rightarrow_{\mathbb{U}(R_7)}^* \mathbb{U}_{16}(\text{false}, 0, 0, \text{nil}, \text{nil}, \text{nil})$ that is not joinable. In this case, we can solve this non-confluence by replacing \mathbb{U}_{16} with \mathbb{U}_{14} since the only difference between the second and third rules of R_7 is whether $\text{le}(x, y)$ reduces to true or false. However, this simple solution is not possible in general.

Example 6.7. Consider the following TRS defining *snoc* that appends the element to the end of the list, *e.g.*, $\text{snoc}([1, 2, 3], 4) = [1, 2, 3, 4]$:

$$R_{11} = \left\{ \begin{array}{l} \text{snoc}(\text{nil}, y) \rightarrow \text{cons}(y, \text{nil}) \\ \text{snoc}(\text{cons}(x, xs), y) \rightarrow \text{cons}(x, \text{snoc}(xs, y)) \end{array} \right\}$$

The inversion method in [18] inverts this TRS to the following DCTRS R_{20} :

$$R_{20} = \left\{ \begin{array}{l} \text{snoc}^{-1}(\text{cons}(y, \text{nil})) \rightarrow \text{tp}_2(\text{nil}, y) \\ \text{snoc}^{-1}(\text{cons}(x, ys)) \rightarrow \text{tp}_2(\text{cons}(x, xs), y) \leftarrow \text{snoc}^{-1}(ys) \rightarrow \text{tp}_2(xs, y) \end{array} \right\}$$

This DCTRS R_{20} is unraveled by \mathbb{U} as follows:

$$\mathbb{U}(R_{20}) = \left\{ \begin{array}{c} \vdots \\ \text{snoc}^{-1}(\text{cons}(x, ys)) \rightarrow \mathbb{U}_{21}(\text{snoc}^{-1}(ys), x, ys) \\ \mathbb{U}_{21}(\text{tp}_2(xs, y), x, ys) \rightarrow \text{tp}_2(\text{cons}(x, xs), y) \end{array} \right\}$$

The DCTRS R_{20} is confluent, but the unraveled TRS $\mathbb{U}(R_{20})$ is not since we have a *critical* peak $\mathbb{U}_{21}(\text{snoc}^{-1}(\text{nil}), x, \text{nil}) \leftarrow_{\mathbb{U}(R_{20})} \cdot \rightarrow_{\mathbb{U}(R_{20})} \text{tp}_2(\text{nil}, x)$ that is not joinable. The simple solution described in Example 6.6 cannot solve non-confluence of $\mathbb{U}(R_{20})$.

“ground confluence”. For this reason, confluence is a soundness condition for the case of arbitrary reduction sequences.

Finally, we show some properties and a notion related to reachable terms that are helpful to compare the SR transformation with unravelings.

Definition 6.8 ([30, 31]). Let R be a DCTRS over a signature \mathcal{F} . For a reachable term s in $\mathcal{T}(\mathcal{F}_{\mathbb{SR}(R)}, \mathcal{V})$, we define the set $\mathcal{Pos}_{\text{str}}(s)$ of *structural positions* for s as follows:

- $\mathcal{Pos}_{\text{str}}(x) = \{\varepsilon\}$ for $x \in \mathcal{V}$,
- $\mathcal{Pos}_{\text{str}}(\{t\}) = \{1p \mid p \in \mathcal{Pos}_{\text{str}}(t)\}$,
- $\mathcal{Pos}_{\text{str}}(c(t_1, \dots, t_n)) = \{ip \mid 1 \leq i \leq n, p \in \mathcal{Pos}_{\text{str}}(t_i)\}$ for $c \in \mathcal{C}_R$, and
- $\mathcal{Pos}_{\text{str}}(\overline{f}(t_1, \dots, t_n, u_1, \dots, u_{n_f})) = \{\varepsilon\} \cup \{ip \mid 1 \leq i \leq n, p \in \mathcal{Pos}_{\text{str}}(t_i)\}$ for an n -ary defined symbol $f \in \mathcal{D}_R$.

Note that $\mathcal{Pos}_{\text{str}}$ is well-defined for reachable terms while it is not defined for the symbols \perp and $[\cdot]$.

Example 6.9. Consider the following term related to $\mathbb{SR}^\rightarrow(R_7)$ in Example 6.6:

$$\{\overline{\text{split}}(s(0), \text{cons}(0, \text{cons}(s(s(0))), \text{nil})), \{\overline{\text{split}}(s(0), \text{cons}(s(s(0))), \text{nil}), \perp, \perp\}, \perp, \perp\}$$

The structural positions of this terms are 1, 1.1, 1.1.1, 1.2, 1.2.1, 1.2.2, 1.2.2.1, 1.2.2.1.1, 1.2.2.1.1.1, and 1.2.2.2.

By definition, structural positions have the following property related to contexts.

Lemma 6.10. Let R be a DCTRS over a signature \mathcal{F} , t be a term in $\mathcal{T}(\mathcal{F}_{\mathbb{SR}(R)}, \mathcal{V})$, and $C[\]_p$ be a one-hole context over $\mathcal{F}_{\mathbb{SR}(R)}$ such that $p \in \mathcal{Pos}_{\text{str}}(C[\])$. Then, $C[\{t\}] \rightarrow_{\mathbb{SR}^\rightarrow(R)}^* \{C[t]\}$.

The proof of Lemma 6.10 is omitted since it can be easily proved by induction.

6.3. Relationship between Soundness. In this subsection, we show that if \mathbb{SR} is sound for a DCTRS, then so is \mathbb{U} . To this end, as in Section 5, we show that all the derivations of \mathbb{U} on terms over the original signature are included in the derivations of \mathbb{SR} .

In rewrite rules obtained from \mathbb{SR} , the conditional parts related to the same defined symbol are evaluated in parallel, and thus, the system $\mathbb{SR}(R)$ is more reasonable than the system $\mathbb{U}(R)$. Due to the parallel evaluation of conditional parts, $\mathbb{SR}(R)$ can derive all the reduction sequences of $\mathbb{U}(R)$, and thus, soundness of \mathbb{SR} implies that of \mathbb{U} .

Lemma 6.11. Let R be a DCTRS over a signature \mathcal{F} . Then, $\phi_{\mathbb{SR}(R)}(\rightarrow_{\mathbb{U}(R)}^*) \subseteq \rightarrow_{\mathbb{SR}^\rightarrow(R)}^*$ on terms in $\mathcal{T}(\mathcal{F}, \mathcal{V})$.

Proof. The proof can be seen in Appendix A.6. □

Due to Lemma 6.11, we obtain the following theorem.

Theorem 6.12. If \mathbb{SR} is sound for a syntactically or strongly DCTRS, then so is \mathbb{U} .

Proof. Suppose that \mathbb{SR} is sound for a syntactically or strongly DCTRS R . Then, it follows from Lemma 6.11 that $\phi_{\mathbb{SR}(R)}(\rightarrow_{\mathbb{U}(R)}^*) \subseteq \rightarrow_{\mathbb{SR}^\rightarrow(R)}^*$. Therefore, it follows from Theorem 6.2 that \mathbb{U} is sound for R . □

It is not known whether the converse of Theorem 6.12 holds or not.

Similarly to \mathbb{U} , the LL property of DCTRSs is not a soundness condition of \mathbb{SR} ; Suppose that \mathbb{SR} is sound for LL DCTRSs; Then, it follows from Theorem 6.12 that \mathbb{U} is sound for LL DCTRSs, but \mathbb{U} is not sound for every LL DCTRS (see Example 4.1).

Example 6.13. The DCTRS R_6 in Example 4.1 is transformed by \mathbb{SR}^\rightarrow into the following TRS:

$$\mathbb{SR}^\rightarrow(R_6) = \left\{ \begin{array}{ll} \bar{f}(x, \perp) \rightarrow \bar{f}(x, [\{x\}]) & \bar{f}(x, [\{e\}]) \rightarrow \{x\} \\ \bar{g}(d, x, y, \perp) \rightarrow \bar{g}(d, x, y, [\{y\}]) & \bar{g}(d, x, y, [\{x\}]) \rightarrow \{A\} \\ \bar{h}(x, y, \perp) \rightarrow \bar{h}(x, y, [\{y\}]) & \bar{h}(x, y, [\{x\}]) \rightarrow \{\bar{g}(x, y, \bar{f}(\bar{k}, \perp), \perp)\} \\ \bar{a} \rightarrow \{\bar{c}\} & \bar{a} \rightarrow \{d\} \\ \bar{b} \rightarrow \{\bar{c}\} & \bar{b} \rightarrow \{d\} \\ \bar{c} \rightarrow \{e\} & \bar{c} \rightarrow \{l\} \\ \bar{d} \rightarrow \{m\} & \bar{k} \rightarrow \{l\} \\ \bar{k} \rightarrow \{m\} & \{\{x\}\} \rightarrow \{x\} \\ \bar{f}(\{x\}, z_1) \rightarrow \{\bar{f}(x, \perp)\} & \bar{g}(\{x\}, y, z, z_1) \rightarrow \{\bar{g}(x, y, z, \perp)\} \\ \bar{g}(x, \{y\}, z, z_1) \rightarrow \{\bar{g}(x, y, z, \perp)\} & \bar{g}(x, y, \{z\}, z_1) \rightarrow \{\bar{g}(x, y, z, \perp)\} \\ \bar{h}(\{x\}, y, z_1) \rightarrow \{\bar{h}(x, y, \perp)\} & \bar{h}(x, \{y\}, z_1) \rightarrow \{\bar{h}(x, y, \perp)\} \end{array} \right.$$

We have the derivation $\bar{h}(\bar{f}(\bar{a}, \perp), \bar{f}(\bar{a}, \perp)) \rightarrow_{\mathbb{SR}^\rightarrow(R_6)}^* \{A\}$, but $h(f(a), f(a)) \not\rightarrow_{R_6} A$. Thus, the LL property is not a sufficient condition for soundness of \mathbb{SR} .

6.4. A Comparison from Several Viewpoints. Finally, we compare unravelings with the SR transformation, in terms of the following points.

- *Proving Operational Termination.* Both the unravelings and the SR transformation can be used for proving operational termination: if the transformed TRS is terminating, then the original CTRS is operationally terminating [14, 30, 31, 28, 29].
- *Soundness.* As shown in Theorem 6.12, for strongly or syntactically DCTRSs, soundness of \mathbb{SR} implies soundness of \mathbb{U} . The known soundness conditions are the $\mathbb{U}_{\text{opt-LL}}$ property and confluence only. These conditions are also the ones for unravelings and more soundness conditions for unravelings are known than those for \mathbb{SR} (see Table 1 in Section 7).
- *Strong Soundness.* A CTRS transformation T is called *strongly sound* for an eCTRS R over a signature \mathcal{F} if there exists a (partial)¹¹ mapping ψ as an inverse to ϕ (i.e., $\psi(\phi(t)) = t$ for $t \in \mathcal{T}(\mathcal{F}, \mathcal{V})$) such that, for all terms $s \in \mathcal{T}(\mathcal{F}, \mathcal{V})$ and $t \in \mathcal{T}(\mathcal{G}, \mathcal{V})$, $\phi(s) \rightarrow_{R_T}^* t$ implies $s \rightarrow_R^* \psi(t)$, where \mathcal{G} is a signature over which R_T is defined. The well-designed rules obtained by the SR transformation provide strong soundness from soundness, that plays an important role in the points below. On the other hand, strong soundness of unravelings has never been discussed, and soundness of unravelings does not imply strong soundness of the unravelings in general.
- *Proving Confluence.* As stated in Theorem 6.5, the SR transformation provides a method for proving confluence of strongly or syntactically \mathbb{U} -LL DCTRSs. For unravelings, this has never been discussed, and furthermore, for any overlapping confluent DCTRS, usual

¹¹ The mapping ψ only needs to translate resulting terms (terms reachable from $\phi(s)$ for some original term s) for $T(R)$ back into the corresponding terms for R .

unravelings (e.g., \mathbb{U} and \mathbb{U}_{opt}) do not preserve confluence, *i.e.*, the unraveled TRS is not confluent (see Examples 6.6, 6.7).

- *Computing Normal Forms.* For a strongly or syntactically DCTRS R , the normal forms of $\mathbb{SR}^{\rightarrow}(R)$ can be converted to the corresponding normal forms of R if \mathbb{SR} is strongly sound for R . Thus, $\mathbb{SR}(R)$ can be used for the normalizing reduction of R . Moreover, the obtained normal form is a unique one if R is confluent. In general, this is impossible for unravelings.
- *Computational Equivalence.* For a CTRS transformation T and an eCTRS R , the transformed eTRS R_T is called *computationally equivalent to R* if, whenever R terminates on s admitting a unique normal form t (*i.e.*, $t' = t$ for all normal forms t' of s), R_T also terminates on $\phi(s)$ and for any of its normal forms t' , we have that $\psi(t') = t$ [30, 31]. $\mathbb{SR}^{\rightarrow}(R)$ is computationally equivalent to R if R is finite, confluent, and operationally terminating [31]. Thus, for such a DCTRS R , $\mathbb{SR}(R)$ can be used as a rewriting engine for R in terms of reduction. This is the main advantage of \mathbb{SR} and has never been discussed for unravelings.

In summary, when the SR transformation is sound for a strongly or syntactically DCTRS with confluence and operational termination, the SR transformation seems better to use as a reasonable rewriting engine for the DCTRS than the unravelings mentioned in this paper. On the other hand, unravelings are good tools for investigating soundness conditions of CTRS transformations, which is required for computational equivalence. Moreover, as stated in Section 1, unravelings are useful in order to analyze or modify DCTRSs. Currently, for DCTRSs that are neither strongly nor syntactically DCTRSs, unravelings are more useful than the SR transformation since it is not known whether \mathbb{SR} provides computational equivalence (and even soundness) for such DCTRSs or not.

7. SUMMARY AND RELATED WORK OF SOUNDNESS CONDITIONS

In this section, we briefly describe related work on soundness of unravelings and we summarize positive and negative results on soundness conditions of unravelings and the SR transformation.

First, we briefly describe a comparison with related work, in terms of the approach to the proof of soundness related to the \mathbb{U}_{opt} -LL property (Subsection 4.1). For an LL normal CTRS R over a signature \mathcal{F} , the approach to the proof of soundness in [9] is the use of the transformation ∇ from $\mathcal{T}(\mathcal{F}_{\mathbb{U}_N(R)}, \mathcal{V})$ to $\mathcal{T}(\mathcal{F}, \mathcal{V})$, proving that for any term $s \in \mathcal{T}(\mathcal{F}, \mathcal{V})$ and term $t \in \mathcal{T}(\mathcal{F}_{\mathbb{U}_N(R)}, \mathcal{V})$, if $s \rightarrow_{\mathbb{U}_N(R)}^* t$, then $s \rightarrow_R^* \nabla(t)$ [26]. Note that the transformation ∇ has been extended to \mathbb{U} [25] (*cf.*, [26]). The transformation ∇ was introduced in [25] to discuss innermost termination. Unlike the case of normal CTRSs, however, ∇ has never been used to show soundness. The transformation ∇ cannot be defined well for \mathbb{U}_{opt} since not all the variables in l appear in $U_i^{\rho}(t_i, \vec{Z}_i)$. For this reason, the proof in this paper takes a direct approach to the proof of soundness for \mathbb{U}_{opt} -LL DCTRSs (*cf.*, Lemma 4.2).

Extending the results in [9], Gmeiner et al. have shown that \mathbb{U} is sound for confluent and *right-stable* 3-DCTRSs w.r.t. the reduction to normal forms,¹² and \mathbb{U} is sound for \mathbb{U} -RL

¹² A syntactically DCTRS R is called *right-stable* [32, 10] if for every rule $l \rightarrow r \Leftarrow s_1 \rightarrow t_1; \dots; s_k \rightarrow t_k$ and for all $1 \leq i \leq n$, t_i is linear and $X_i \cap \text{Var}(t_i) = \emptyset$.

or WLL 3-DCTRSs [10].¹³ For the case of \mathbb{U} -RL 3-DCTRSs, this result is incompatible with Theorem 4.9 since \mathbb{U} -RL is strictly more restrictive than \mathbb{U}_{opt} -RL. For example, the DCTRS R'_{10} is \mathbb{U}_{opt} -RLNE, but not \mathbb{U} -RL. This indicates that soundness of \mathbb{U} for R'_{10} cannot be proved by using the result in [10], while the soundness can be proved by the results in this paper (see Example 5.6). For the case of WLL 3-DCTRS, this result strictly contains the combination of Theorem 4.3 and Corollary 5.5 since \mathbb{U}_{opt} -LL 3-DCTRSs are WLL. On the other hand, the WLL property is not a soundness condition of \mathbb{U}_{opt} since \mathbb{U}_{opt} is not sound for the WLL DCTRS R_{10} shown in Example 4.12. Furthermore, Gmeiner et al. have also shown that \mathbb{U}_{opt} is sound for \mathbb{U}_{opt} -NE and *right-separated* 2-DCTRSs [10].¹⁴ The \mathbb{U}_{opt} -RL property is incompatible with the right-separated property even if the DCTRSs are \mathbb{U}_{opt} -NE and of Type 2.¹⁵ For this reason, the soundness result on \mathbb{U}_{opt} in [10] is incomparable with Theorem 4.9.

Finally, we summarize the positive and negative results on soundness in Table 1, *i.e.*, sufficient and insufficient conditions for soundness of the CTRS transformations mentioned in this paper. We can recognize from Example 4.12 that neither confluence nor \mathbb{U}_{opt} -confluence is sufficient on its own for soundness of \mathbb{U}_{opt} . As we have seen, soundness of the unraveling \mathbb{U} is provided by the other transformations \mathbb{U}_J , \mathbb{U}_N , \mathbb{U}_{opt} , and $\mathbb{S}\mathbb{R}$. In summary, many sufficient and insufficient conditions for soundness of unravelings are investigated and all the soundness conditions of $\mathbb{S}\mathbb{R}$ are soundness conditions of unravelings.

8. CONCLUSION

In this paper, we showed that the optimized unraveling for DCTRSs is sound for ultra-LL or ultra-RLNE DCTRSs, and showed that if the optimized unraveling is sound for a DCTRS, then so is Ohlebusch's unraveling. We also presented necessary and sufficient syntactic conditions for ultra-LL, ultra-RL, and ultra-NE, respectively, and soundness conditions of unravelings for join and normal CTRSs. Moreover, we showed that soundness of the existing unravelings and the SR transformation respectively implies soundness of Ohlebusch's unraveling.

Our future work is to solve the remaining open problems, *e.g.*, either to show soundness of \mathbb{U} and \mathbb{U}_{opt} for R'_3 in Example 5.22, or to prove the converse of Theorem 5.21. We are also interested in a study on strong soundness and computational equivalence of unravelings.

There seems to be room for discussing sufficient conditions of unravelings related to confluence, *e.g.*, under which confluence of the unraveled TRSs implies that of the original CTRSs, or under which confluence of the original CTRSs implies that of the unraveled TRSs. For a confluent DCTRS R , a trivial such condition is that R is \mathbb{U}_{opt} -LL and non-overlapping, *i.e.*, $\mathbb{U}(R)$ and $\mathbb{U}_{\text{opt}}(R)$ are LL and non-overlapping. In many cases, however, neither $\mathbb{U}(R)$ nor $\mathbb{U}_{\text{opt}}(R)$ is confluent even if R is confluent (see Example 6.6). Viewed in

¹³ A 3-DCTRS R is called *weakly left-linear* (WLL) [10] if for every rule $\rho : l \rightarrow r \leftarrow s_1 \twoheadrightarrow t_1; \dots; s_k \twoheadrightarrow t_k \in R$ and all variables $x \in \text{Var}(\rho)$, x does not appear in any of r, s_1, \dots, s_k whenever x appears at least twice in l, t_1, \dots, t_k . Note that this WLL property for 3-DCTRSs is an extension of the WLL property for normal 1-CTRSs.

¹⁴ A DCTRS R is called *right-separated* [10] if for every rule $l \rightarrow r \leftarrow s_1 \twoheadrightarrow t_1; \dots; s_k \twoheadrightarrow t_k$ and all $1 \leq i \leq k$, $\text{Var}(t_i) \cap X_i = \emptyset$.

¹⁵ The rule $f(x, y, z) \rightarrow x \leftarrow g(y) \twoheadrightarrow w; g(z) \twoheadrightarrow y; h(y) \twoheadrightarrow a$ is \mathbb{U}_{opt} -RLNE and of Type 2, but not right-separated. On the other hand, the rule $f(x, y) \rightarrow x \leftarrow g(y) \twoheadrightarrow z; g(z) \twoheadrightarrow a; h(z) \twoheadrightarrow b$ is \mathbb{U}_{opt} -NE, right-separated, and of Type 2, but not \mathbb{U}_{opt} -RLNE.

Table 1: Results on soundness conditions for CTRS transformations.

		soundness	insufficient for soundness
J	\mathbb{U}_J	LL (Corollary 5.14) soundness of $\mathbb{U}_N \circ \text{Norm}$ (Theorem 5.13) soundness of \mathbb{U}_N (Theorem 5.17) soundness of $\mathbb{U} \circ \text{Det}$ (Theorem 5.19)	
N	\mathbb{U}_N	LL (= $\mathbb{U}_{\text{opt-LL}}$) [9] confluence [9] NE [9] groundness of all conditions [9] WLL (\supset LL) [9] soundness of \mathbb{U}_J (Theorem 5.15)	constructor systems [9] overlay systems [9] non-RV [9] RL [9] overlappingness [9] UN [9] UN^\rightarrow [9]
S	\mathbb{SR}	$\mathbb{U}_{\text{opt-LL}}$ [31] confluence [31]	LL (Example 6.13)
D	\mathbb{U}	confluence and right-stability [10]	confluence [10]
		$\mathbb{U}_{\text{opt-LL}}$ [16] soundness of \mathbb{U}_{opt} (Corollary 5.5) soundness of \mathbb{U}_N (Theorem 5.21) soundness of \mathbb{SR} (Theorem 6.12) \mathbb{U} -RL [10] WLL (\supset $\mathbb{U}_{\text{opt-LL}}$) [10]	confluence [10] \mathbb{U} -NE [10]
	\mathbb{U}_{opt}	$\mathbb{U}_{\text{opt-LL}}$ (Theorem 4.3) $\mathbb{U}_{\text{opt-NE-RL}}$ (Theorem 4.9) $\mathbb{U}_{\text{opt-NE}}$, right-separation, and Type 2 [10]	LL (Example 4.1) $\mathbb{U}_{\text{opt-NE}}$ (Example 4.11) WLL (Example 4.12) $\mathbb{U}_{\text{opt-RL}}$ (Example 4.12) soundness of \mathbb{U} (Example 4.12) confluence (Example 4.12) $\mathbb{U}_{\text{opt-confluence}}$ (Example 4.12)

- “J”, “N”, “D”, and “S” in the first column represent “join CTRSs”, “normal CTRSs”, “DCTRSs”, and “strongly or syntactically DCTRSs”, respectively.
- “soundness of \mathbb{U}_N ” means that the target is (or can be considered) a normal CTRS and \mathbb{U}_N is sound for the target.
- “soundness of \mathbb{SR} ” means that the target is a strongly or syntactically DCTRS and \mathbb{SR} is sound for the target.

this light, an interesting further direction related to confluence will be to improve unraveling transformations themselves, *e.g.*, to optimize introduction of \mathbb{U} symbols as stated in Example 6.6. Such an optimization has been already discussed in [30, 31]. For unravelings, however, it is not clear what the optimization leads to. What has to be noticed in this direction is that the improvement is in agreement with the SR transformation.

As stated in the comparison with the SR transformation, soundness conditions of unravelings are better studied than soundness of the SR transformation and it must be easier to investigate soundness of unravelings than that of the SR transformation. Thus, it is still worth investigating unravelings while the SR transformation provides a reasonable rewriting engine in terms of computational equivalence to the original CTRSs. On the other

hand, if the converse of Theorem 6.12 holds, then unravelings would be useful tools to show soundness of the SR transformation. A further direction of this research will be to prove or disprove the converse of Theorem 6.12.

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APPENDIX A. PROOFS OF TECHNICAL RESULTS

In this appendix, we show missing proofs of some technical results.

A.1. Proof of Theorem 3.8.

Theorem 3.8. *Let $\rho : l \rightarrow r \Leftarrow s_1 \Rightarrow t_1; \dots; s_k \Rightarrow t_k$ be an extended deterministic conditional rewrite rule. Then, all of the following hold:*

- (1) ρ is $\mathbb{U}_{\text{opt-LL}}$ iff all of l, t_1, \dots, t_k are linear and $\mathcal{V}ar(t_i) \cap X_i = \emptyset$ for all $1 \leq i \leq k$,
- (2) ρ is $\mathbb{U}_{\text{opt-RL}}$ iff all of r, s_1, \dots, s_k are linear and $\mathcal{V}ar(s_i) \cap Y_i = \emptyset$ for all $1 \leq i \leq k$,
and
- (3) ρ is $\mathbb{U}_{\text{opt-NE}}$ iff $\mathcal{V}ar(l) \subseteq \mathcal{V}ar(r, s_1, \dots, s_k)$ and $\mathcal{V}ar(t_i) \subseteq \mathcal{V}ar(r, s_{i+1}, \dots, s_k)$ for all $1 \leq i \leq k$.

Proof. The case that ρ is unconditional is trivial, so let $k > 0$. Recall that $\mathbb{U}_{\text{opt}}(\rho) = \{ l \rightarrow U_1^\rho(s_1, \vec{Z}_1), U_1^\rho(t_1, \vec{Z}_1) \rightarrow U_2^\rho(s_2, \vec{Z}_2), \dots, U_k^\rho(t_k, \vec{Z}_k) \rightarrow r \}$.

- (1) Suppose that ρ is $\mathbb{U}_{\text{opt-LL}}$. Then, by definition, all of $l, U_1^\rho(t_1, \vec{Z}_1), \dots, U_k^\rho(t_k, \vec{Z}_k)$ are linear. Thus, all of l, t_1, \dots, t_k are linear and $\mathcal{V}ar(t_i) \cap Z_i = \emptyset$ for all $1 \leq i \leq k$, and hence $\mathcal{V}ar(t_i) \cap X_i = \emptyset$ for all $1 \leq i \leq k$. Therefore, the *only-if* part holds.

Suppose that ρ is not $\mathbb{U}_{\text{opt-LL}}$, all of l, t_1, \dots, t_k are linear, and $\mathcal{V}ar(t_i) \cap X_i = \emptyset$ for all $1 \leq i \leq k$. Then, $\mathcal{V}ar(t_j) \cap Z_j \neq \emptyset$ for some j since all of l, t_1, \dots, t_k are linear and the sequence \vec{Z}_i is linear w.r.t. variable occurrences for all $1 \leq i \leq k$. Since $Z_i \subseteq X_i$ for all $1 \leq i \leq k$ by definition, we have $\mathcal{V}ar(t_j) \cap X_j \neq \emptyset$. This contradicts the assumption that $\mathcal{V}ar(t_i) \cap X_i = \emptyset$ for all $1 \leq i \leq k$. Therefore, the *if* part holds.

- (2) Suppose that ρ is $\mathbb{U}_{\text{opt-RL}}$. Then, by definition, all of $r, U_1^\rho(s_1, \vec{Z}_1), \dots, U_k^\rho(s_k, \vec{Z}_k)$ are linear and $\mathcal{V}ar(s_i) \cap Z_i = \emptyset$ for all $1 \leq i \leq k$, and hence all of r, s_1, \dots, s_k are linear. Since ρ is deterministic, we have $\mathcal{V}ar(s_i) \subseteq X_i$, and hence $\mathcal{V}ar(s_i) \cap Y_i = \emptyset$ for all $1 \leq i \leq k$. Therefore, the *only-if* part holds.

Suppose that ρ is not $\mathbb{U}_{\text{opt-RL}}$, all of r, s_1, \dots, s_k are linear, and $\mathcal{V}ar(s_i) \cap Y_i = \emptyset$ for all $1 \leq i \leq k$. Then, by definition, $\mathcal{V}ar(s_j) \cap Z_j \neq \emptyset$ for some j since all of r, s_1, \dots, s_k are linear and the variable sequence \vec{Z}_i is linear w.r.t. variable occurrences for all $1 \leq i \leq k$. Since $Z_i \subseteq Y_i$ for all $1 \leq i \leq k$ by definition, it follows from $Z_j \subseteq Y_j$ that $\mathcal{V}ar(s_j) \cap Y_j \neq \emptyset$. This contradicts the assumption that $\mathcal{V}ar(s_i) \cap Y_i = \emptyset$ for all $1 \leq i \leq k$. Therefore, the *if* part holds.

- (3) Suppose that ρ is $\mathbb{U}_{\text{opt-NE}}$. Then, by definition, $\mathcal{V}ar(l) \subseteq \mathcal{V}ar(s_1) \cup Z_1$, $\mathcal{V}ar(t_i) \cup Z_i \subseteq \mathcal{V}ar(s_{i+1}) \cup Z_{i+1}$ for all $1 \leq i < k$, and $\mathcal{V}ar(t_k) \cup Z_k \subseteq \mathcal{V}ar(r)$, and hence $\mathcal{V}ar(t_k) \subseteq \mathcal{V}ar(r)$, $Z_k \subseteq \mathcal{V}ar(r)$, $\mathcal{V}ar(t_i) \subseteq \mathcal{V}ar(s_{i+1}) \cup Z_{i+1}$ and $Z_i \subseteq \mathcal{V}ar(s_{i+1}) \cup Z_{i+1}$ for all $1 \leq i < k$. Thus, $Z_i \subseteq \mathcal{V}ar(s_{i+1}) \cup Z_{i+1} \subseteq \mathcal{V}ar(s_{i+1}, s_{i+2}) \cup Z_{i+2} \subseteq \dots \subseteq \mathcal{V}ar(s_{i+1}, \dots, s_k) \cup \mathcal{V}ar(r)$ for all $1 \leq i < k$, and hence $\mathcal{V}ar(t_i) \subseteq \mathcal{V}ar(r, s_{i+1}, \dots, s_k)$ for all $1 \leq i \leq k$. Moreover, $\mathcal{V}ar(l) \subseteq \mathcal{V}ar(s_1) \cup Z_1 \subseteq \mathcal{V}ar(s_1) \cup \mathcal{V}ar(r, s_2, \dots, s_k) = \mathcal{V}ar(r, s_1, \dots, s_k)$. Therefore, the *only-if* part holds.

Suppose that $\mathcal{V}ar(l) \subseteq \mathcal{V}ar(r, s_1, \dots, s_k)$ and $\mathcal{V}ar(t_i) \subseteq \mathcal{V}ar(r, s_{i+1}, \dots, s_k)$ for all $1 \leq i \leq k$. Then, by the definition of Y_i , we have $Y_i = \mathcal{V}ar(r, s_{i+1}, \dots, s_k)$ for all $1 \leq i \leq k$.

- Consider the rule $U_k^\rho(t_k, \vec{Z}_k) \rightarrow r \in \mathbb{U}_{\text{opt}}(\rho)$. It follows from $\mathcal{V}ar(t_k) \subseteq \mathcal{V}ar(r)$ and $Y_k = \mathcal{V}ar(r)$ that $\mathcal{V}ar(t_k) \cup Z_k = \mathcal{V}ar(t_k) \cup (X_k \cap Y_k) \subseteq \mathcal{V}ar(r)$. Thus, $U_k^\rho(t_k, \vec{Z}_k) \rightarrow r \in \mathbb{U}_{\text{opt}}(\rho)$ is NE.
- Consider the rule $U_i^\rho(t_i, \vec{Z}_i) \rightarrow U_{i+1}^\rho(s_{i+1}, \vec{Z}_{i+1}) \in \mathbb{U}_{\text{opt}}(\rho)$ with $1 \leq i < k$. Suppose that $\mathcal{V}ar(t_i) \cup Z_i \not\subseteq \mathcal{V}ar(s_{i+1}) \cup Z_{i+1}$. Then, there exists a variable $x \in \mathcal{V}ar(t_i) \cup Z_i$ such that $x \notin \mathcal{V}ar(s_{i+1}) \cup Z_{i+1}$, and hence $x \notin \mathcal{V}ar(s_{i+1}) \cup Y_{i+1}$. It follows from $Y_{i+1} = \mathcal{V}ar(r, s_{i+2}, \dots, s_k)$ that $x \notin \mathcal{V}ar(r, s_{i+1}, \dots, s_k)$.
 - Suppose that $x \in \mathcal{V}ar(t_i)$. Then, it follows from $\mathcal{V}ar(t_i) \subseteq \mathcal{V}ar(r, s_{i+1}, \dots, s_k)$ that $x \in \mathcal{V}ar(r, s_{i+1}, \dots, s_k)$. This contradicts the fact that $x \notin \mathcal{V}ar(r, s_{i+1}, \dots, s_k)$.
 - Suppose that $x \notin \mathcal{V}ar(t_i)$. Then, $x \in Z_i = X_i \cap Y_i$, and hence $x \in X_i$ and $x \in Y_i = \mathcal{V}ar(r, s_{i+1}, \dots, s_k)$. This contradicts the fact that $x \notin \mathcal{V}ar(r, s_{i+1}, \dots, s_k)$.
 Thus, $U_i^\rho(t_i, \vec{Z}_i) \rightarrow U_{i+1}^\rho(s_{i+1}, \vec{Z}_{i+1})$ is NE.
- Consider the remaining rule $l \rightarrow U_1^\rho(s_1, \vec{Z}_1) \in \mathbb{U}_{\text{opt}}(\rho)$. Suppose that $\mathcal{V}ar(l) \not\subseteq \mathcal{V}ar(s_1) \cup Z_1$. Then, there exists a variable $x \in \mathcal{V}ar(l)$ such that $x \notin \mathcal{V}ar(s_1) \cup Z_1$, and hence $x \notin \mathcal{V}ar(s_1) \cup Y_1$. It follows from $Y_1 = \mathcal{V}ar(r, s_2, \dots, s_k)$ that $x \notin \mathcal{V}ar(r, s_1, \dots, s_k)$. This contradicts the fact that $\mathcal{V}ar(l) \subseteq \mathcal{V}ar(r, s_1, \dots, s_k)$. Thus, $l \rightarrow U_1^\rho(s_1, \vec{Z}_1)$ is NE.

Therefore, $\mathbb{U}_{\text{opt}}(\rho)$ is NE, and hence the *if* part holds. \square

A.2. Proof of Theorem 3.9.

Theorem 3.9. *Let $\rho : l \rightarrow r \Leftarrow s_1 \Rightarrow t_1; \dots; s_k \Rightarrow t_k$ be an extended deterministic conditional rewrite rule. Then, all of the following hold:*

- (1) ρ is \mathbb{U} -LL iff all of l, t_1, \dots, t_k are linear and $\mathcal{V}ar(t_i) \cap X_i = \emptyset$ for all $1 \leq i \leq k$,
- (2) ρ is \mathbb{U} -RL iff r is linear and all of s_1, \dots, s_k are ground, and
- (3) ρ is \mathbb{U} -NE iff $\mathcal{V}ar(l, t_1, \dots, t_k) \subseteq \mathcal{V}ar(r)$.

Proof. The case that ρ is unconditional is trivial, so let $k > 0$. Recall that $\mathbb{U}(\rho) = \{ l \rightarrow U_1^\rho(s_1, \vec{X}_1), U_1^\rho(t_1, \vec{X}_1) \rightarrow U_2^\rho(s_2, \vec{X}_2), \dots, U_k^\rho(t_k, \vec{X}_k) \rightarrow r \}$.

- (1) This claim can be proved similarly to Theorem 3.8 (1).
- (2) Suppose that ρ is \mathbb{U} -RL. Then, by definition, all of $r, U_1^\rho(s_1, \vec{X}_1), \dots, U_k^\rho(s_k, \vec{X}_k)$ are linear. Suppose that s_j is not ground for some j . Then, there exists a variable $x \in \mathcal{V}ar(s_j)$. Since ρ is deterministic, x appears in any of l, t_1, \dots, t_{j-1} , and hence $x \in X_j$. Thus, $U_j^\rho(s_j, \vec{X}_j)$ is not linear, and hence $\mathbb{U}(\rho)$ is not RL, *i.e.*, ρ is not \mathbb{U} -RL. This contradicts the assumption that ρ is \mathbb{U} -RL. Therefore, all of s_1, \dots, s_k are ground, and hence the *only-if* part holds.

Suppose that ρ is not \mathbb{U} -RL, r is linear, and all of s_1, \dots, s_k are ground. Then, by definition, $U_j^\rho(s_j, \vec{X}_j)$ is not linear for some j since r is linear. It follows from groundness of s_j that $\mathcal{V}ar(s_j) \cap X_j = \emptyset$. Moreover, since the variable sequence \vec{X}_j is linear w.r.t. variable occurrences, the term $U_j^\rho(s_j, \vec{X}_j)$ is linear. This contradicts the non-linearity of $U_j^\rho(s_j, \vec{X}_j)$. Therefore, the *if* part holds.

- (3) Suppose that ρ is \mathbb{U} -NE. Then, by definition, the rule $U_k^\rho(t_k, \vec{X}_k) \rightarrow r$ is NE, and hence $\mathcal{V}ar(l, t_1, \dots, t_k) = \mathcal{V}ar(t_k) \cup X_k \subseteq \mathcal{V}ar(r)$. Therefore, the *only-if* part holds.

Suppose that $\mathcal{V}ar(l, t_1, \dots, t_n) \subseteq \mathcal{V}ar(r)$. Then, since $X_k = \mathcal{V}ar(l, t_1, \dots, t_{k-1})$ by definition, $U_k^\rho(t_k, \overrightarrow{X_k}) \rightarrow r$ is NE. It follows from $\mathcal{V}ar(l) = X_1$ that $l \rightarrow U_1^\rho(s_1, \overrightarrow{X_1})$ is NE. Since $\mathcal{V}ar(t_i) \cup X_i = X_{i+1}$ for all $1 \leq i < k$ by definition, $U_i^\rho(t_i, \overrightarrow{X_i}) \rightarrow U_{i+1}^\rho(s_{i+1}, \overrightarrow{X_{i+1}})$ is NE for all $1 \leq i < k$. Thus, $\mathbb{U}(\rho)$ is NE, and hence ρ is \mathbb{U} -NE. Therefore, the *if* part holds. \square

A.3. Proof of Lemma 4.2. We first prepare a technical lemma to help us to prove Lemma 4.2. Let X be a finite set of variables, σ and θ be substitutions, and \rightarrow be a binary relation on terms. Then, we write $X\sigma \rightarrow X\theta$ if $x\sigma \rightarrow x\theta$ for any $x \in X$.

Lemma A.1. *Let R be an eDCTRS, $\rho : l \rightarrow r \leftarrow s_1 \rightarrow t_1; \dots; s_k \rightarrow t_k$ be a $\mathbb{U}_{\text{opt-LL}}$ conditional rewrite rule in R , and $\sigma_1, \dots, \sigma_{k+1}$ be substitutions. If $s_i\sigma_i \rightarrow_R^* t_i\sigma_{i+1}$ and $Z_i\sigma_i \rightarrow_R^* Z_i\sigma_{i+1}$ for all $1 \leq i \leq k$, then $l\sigma_1 \rightarrow_R^+ r\sigma_{k+1}$.*

Proof. Let σ be the substitution $\sigma_1|_{\mathcal{V}ar(l)} \cup \sigma_2|_{Z_1 \setminus \mathcal{V}ar(l)} \cup \dots \cup \sigma_k|_{Z_k \setminus Z_{k-1}} \cup \sigma_{k+1}|_{\mathcal{V}ar(t_k, r) \setminus Z_k}$. Then, $l\sigma = l\sigma_1$. It follows from $Z_i\sigma_i \rightarrow_R^* Z_i\sigma_{i+1}$ that $Z_i\sigma \rightarrow_R^* Z_i\sigma_{i+1}$ for all $1 \leq i \leq k$. Moreover, it follows from the $\mathbb{U}_{\text{opt-LL}}$ property and Theorem 3.8 that $\mathcal{V}ar(t_i) \cap (\text{Dom}(\sigma_1|_{\mathcal{V}ar(l)}) \cup \dots \cup \text{Dom}(\sigma_{i-1}|_{Z_{i-1} \setminus Z_{i-2}})) = \emptyset$ for all $1 \leq i \leq k$, and hence $t_i\sigma_i = t_i\sigma$ for all $1 \leq i \leq k$.

Now we show that $s_i\sigma \rightarrow_R^* s_i\sigma_i$ for all $1 \leq i \leq k$, *i.e.*, $x\sigma \rightarrow_R^* x\sigma_i$ for all variables $x \in \mathcal{V}ar(s_i)$. The case that $i = 1$ is trivial, so let $i > 1$. We make a case distinction depending on where x appears.

- Consider the case that $x \in \mathcal{V}ar(l)$. By definition, $x \in Z_j$ for all $1 \leq j < i$, and hence we have the derivation $x\sigma = x\sigma_1 \rightarrow_R^* x\sigma_2 \rightarrow_R^* \dots \rightarrow_R^* x\sigma_i$.
- Consider the remaining case that $x \in \mathcal{V}ar(t_j)$ for some j with $1 \leq j < i$. It follows from the $\mathbb{U}_{\text{opt-LL}}$ property of ρ that $x \in Z_j \setminus Z_{j-1}$. By definition, $x \in Z_{j'}$ for all $j \leq j' < i$, and hence we have the derivation $x\sigma = x\sigma_j \rightarrow_R^* x\sigma_{j+1} \rightarrow_R^* \dots \rightarrow_R^* x\sigma_i$.

Thus, $x\sigma \rightarrow_R^* x\sigma_i$ for all variables $x \in \mathcal{V}ar(s_i)$, and hence $s_i\sigma \rightarrow_R^* s_i\sigma_i$. It follows from the assumption that $s_i\sigma \rightarrow_R^* s_i\sigma_i \rightarrow_R^* t_i\sigma_{i+1} = t_i\sigma$ for all $1 \leq i \leq k$. Similarly, we have the derivation $r\sigma \rightarrow_R^* r\sigma_{k+1}$. Therefore, we have the derivation $l\sigma_1 = l\sigma \rightarrow_R r\sigma \rightarrow_R^* r\sigma_{k+1}$. \square

Next, we show the proof of Lemma 4.2.

Lemma 4.2. *Let R be a $\mathbb{U}_{\text{opt-LL}}$ 3-eDCTRS over a signature \mathcal{F} , s be a term in $\mathcal{T}(\mathcal{F}, \mathcal{V})$, t be a linear term in $\mathcal{T}(\mathcal{F}, \mathcal{V})$, and σ be a substitution in $\text{Sub}(\mathcal{F}_{\mathbb{U}_{\text{opt}}(R)}, \mathcal{V})$. Suppose that R is non-LV or non-RV. If $s \rightrightarrows_{\mathbb{U}_{\text{opt}}(R)}^n t\sigma$ for some $n \geq 0$, then there exists a substitution θ in $\text{Sub}(\mathcal{F}, \mathcal{V})$ such that*

- $s \rightarrow_R^* t\theta \rightrightarrows_{\geq \mathcal{P}os_{\mathcal{V}}(t), \mathbb{U}_{\text{opt}}(R)}^{n'} t\sigma$ for some $n' \leq n$, and
- if $t\sigma \in \mathcal{T}(\mathcal{F}, \mathcal{V})$, then $t\theta = t\sigma$.

Proof. We prove this lemma by induction on the lexicographic product (n, s) of the length n and the structure of s . The case that $n = 0$ is trivial, so let $n > 0$.

We first consider the case that $s \rightrightarrows_{\mathbb{U}_{\text{opt}}(R)}^n t\sigma$ does not contain any reduction step at the root position. In this case, s is not a variable. Let s be of the form $f(s_1, \dots, s_m)$ with $f \in \mathcal{F}$. We make a case distinction depending on whether t is a variable or not.

- Consider the case that t is not a variable. In this case, $s = f(s_1, \dots, s_m) \rightrightarrows_{\mathbb{U}_{\text{opt}}(R)}^n f(t_1, \dots, t_m)\sigma = t\sigma$, and thus, $s_i \rightrightarrows_{\mathbb{U}_{\text{opt}}(R)}^{n_i} t_i\sigma$, where $n_i \leq n$, for all $1 \leq i \leq m$. By the

induction hypothesis, for all $1 \leq i \leq m$, there exists a substitution $\theta_i \in \mathcal{S}ub(\mathcal{F}, \mathcal{V})$ such that

- $s_i \rightarrow_R^* t_i \theta_i \rightrightarrows_{\geq \mathcal{P}os_{\mathcal{V}}(t_i), \mathbb{U}_{opt}(R)}^{n'_i} t_i \sigma$ for some $n'_i \leq n_i$, and
- if $t_i \sigma \in \mathcal{T}(\mathcal{F}, \mathcal{V})$, then $t_i \theta_i = t_i \sigma$.

Let $\theta = \theta_1|_{\mathcal{V}ar(t_1)} \cup \dots \cup \theta_m|_{\mathcal{V}ar(t_m)}$. Then, it follows from the linearity of t that θ is a substitution in $\mathcal{S}ub(\mathcal{F}, \mathcal{V})$. Thus, we have the derivation $s = f(s_1, \dots, s_m) \rightarrow_R^* f(t_1, \dots, t_m)\theta = t\theta \rightrightarrows_{\geq \mathcal{P}os_{\mathcal{V}}(t), \mathbb{U}_{opt}(R)}^{n'} t\sigma$ where

- n' is the maximum of n'_1, \dots, n'_m , and
- if $t\sigma \in \mathcal{T}(\mathcal{F}, \mathcal{V})$, then $t\theta = t\sigma$.

Moreover, it follows from $n'_i \leq n$ that $n' \leq n$.

- Consider the remaining case that t is a variable x . In this case, we can let $\sigma = \{x \mapsto f(t_1, \dots, t_m)\}$. Now, let t' be a linear term $f(x_1, \dots, x_m)$ with $x_1, \dots, x_m \in \mathcal{V}$, and $\sigma' = \{x_i \mapsto t_i \mid 1 \leq i \leq m\}$. Then, $s = f(s_1, \dots, s_m) \rightrightarrows_{> \varepsilon, \mathbb{U}_{opt}(R)}^n f(x_1, \dots, x_m)\sigma'$. Similarly to the previous case, we have a substitution θ' such that
 - $f(s_1, \dots, s_m) \rightarrow_R^* t'\theta' \rightrightarrows_{\geq \mathcal{P}os_{\mathcal{V}}(f(x_1, \dots, x_m)), \mathbb{U}_{opt}(R)}^{n'} t'\sigma'$, and
 - if $t'\sigma' \in \mathcal{T}(\mathcal{F}, \mathcal{V})$, then $t'\theta' = t'\sigma'$,
 for some $n' \leq n$. Let $\theta = \{x \mapsto t'\theta'\}$. Then, we have the derivation $s = f(s_1, \dots, s_m) \rightarrow_R^* t'\theta' = t\theta \rightrightarrows_{\geq \{\varepsilon\}, \mathbb{U}_{opt}(R)}^{n'} t\sigma$ with $n' \leq n$, and $t\theta = t'\theta' = t\sigma$ whenever $t\sigma \in \mathcal{T}(\mathcal{F}, \mathcal{V})$.

Next we consider the remaining case that at least one rule is applied at the root position. In the following, we make a case distinction depending on whether R is non-LV or non-RV. In the case that R is non-LV, we focus on the first rule applied at the root position, and otherwise (*i.e.*, R is non-RV), we focus on the last rule applied at the root position. The case that the focused rule does not contain a \mathbb{U} symbol is simpler than the other case that the focused rule contains a \mathbb{U} symbol since the rule is contained not only in $\mathbb{U}_{opt}(R)$ but also in R . For this reason, we only consider the case that the focused rule contains a \mathbb{U} symbol. Now we assume that the focused rule is of the following form:

- $l \rightarrow U_1^\rho(s_1, \vec{Z}_1)$ if R is non-LV, and
- $U_i^\rho(t_i, \vec{Z}_i) \rightarrow U_{i+1}^\rho(s_{i+1}, \vec{Z}_{i+1})$ or $U_k^\rho(t_k, \vec{Z}_k) \rightarrow r$ if R is non-RV.

For the sake of readability, we assume w.l.o.g. that $k = 2$.

Let us start the case distinction mentioned above.

- (1) Consider the case that R is non-LV. In this case, we have the following subcases depending on where $t\sigma$ appears.

- a. Consider the case that

$$s \rightrightarrows_{> \varepsilon, \mathbb{U}_{opt}(R)}^{n_0} l\sigma_1 \rightarrow_{\varepsilon, \mathbb{U}_{opt}(R)} U_1^\rho(s_1, \vec{Z}_1)\sigma_1 \rightrightarrows_{> \varepsilon, \mathbb{U}_{opt}(R)}^{n_1} t\sigma$$

where $n_0 + n'' + 1 = n$. By the induction hypothesis, there exists a substitution $\theta_1 \in \mathcal{S}ub(\mathcal{F}, \mathcal{V})$ such that $s \rightarrow_R^* l\theta_1 \rightrightarrows_{\geq \mathcal{P}os_{\mathcal{V}}(l), \mathbb{U}_{opt}(R)}^{n'_1} l\sigma_1$ for some $n'_1 \leq n_0$. Thus, $l\theta_1 \rightrightarrows_{\mathbb{U}_{opt}(R)}^{n'_1} l\sigma_1 \rightarrow_{\varepsilon, \mathbb{U}_{opt}(R)} U_1^\rho(s_1, \vec{Z}_1)\sigma_1 \rightrightarrows_{> \varepsilon, \mathbb{U}_{opt}(R)}^{n''} t\sigma$ with $n'_1 + 1 + n'' \leq n$. Since $U_1^\rho(s_1, \vec{Z}_1)\sigma_1 \rightrightarrows_{> \varepsilon, \mathbb{U}_{opt}(R)}^{n''} t\sigma$ does not contain a rewrite step at the root position, $\mathbf{root}(t\sigma) = U_1^\rho$. It follows from the assumption $t \in \mathcal{T}(\mathcal{F}, \mathcal{V})$ that t is a variable x . Now let $\theta = \{x \mapsto l\theta_1\}$. Then, θ is a substitution in $\mathcal{S}ub(\mathcal{F}, \mathcal{V})$ such that $s \rightarrow_R^* l\theta_1 = t\theta \rightrightarrows_{> \{\varepsilon\}, \mathbb{U}_{opt}(R)}^{n'_1 + 1 + n''} t\sigma$ with $n'_1 + 1 + n'' \leq n$ and $t\theta \notin \mathcal{T}(\mathcal{F}, \mathcal{V})$.

b. Consider the case that

$$\begin{aligned} s \rightrightarrows_{>\varepsilon, \mathbb{U}_{\text{opt}}(R)}^{n_0} l\sigma_1 &\rightarrow_{\varepsilon, \mathbb{U}_{\text{opt}}(R)} U_1^\rho(s_1, \vec{Z}_1)\sigma_1 \rightrightarrows_{>\varepsilon, \mathbb{U}_{\text{opt}}(R)}^{n_1} U_1^\rho(t_1, \vec{Z}_1)\sigma_2 \\ &\rightarrow_{\varepsilon, \mathbb{U}_{\text{opt}}(R)} U_2^\rho(s_2, \vec{Z}_2)\sigma_2 \rightrightarrows_{>\varepsilon, \mathbb{U}_{\text{opt}}(R)}^{n''} t\sigma \end{aligned}$$

where $n_0 + n_1 + n'' + 2 = n$. This case is proved similarly to the previous case.

c. Consider the remaining case that

$$\begin{aligned} s \rightrightarrows_{>\varepsilon, \mathbb{U}_{\text{opt}}(R)}^{n_0} l\sigma_1 &\rightarrow_{\varepsilon, \mathbb{U}_{\text{opt}}(R)} U_1^\rho(s_1, \vec{Z}_1)\sigma_1 \rightrightarrows_{>\varepsilon, \mathbb{U}_{\text{opt}}(R)}^{n_1} U_1^\rho(t_1, \vec{Z}_1)\sigma_2 \\ &\rightarrow_{\varepsilon, \mathbb{U}_{\text{opt}}(R)} U_2^\rho(s_2, \vec{Z}_2)\sigma_2 \rightrightarrows_{>\varepsilon, \mathbb{U}_{\text{opt}}(R)}^{n_2} U_2^\rho(t_2, \vec{Z}_2)\sigma_3 \\ &\rightarrow_{\varepsilon, \mathbb{U}_{\text{opt}}(R)} r\sigma_3 \rightrightarrows_{\mathbb{U}_{\text{opt}}(R)}^{n''} t\sigma \end{aligned}$$

where $n_0 + n_1 + n_2 + n'' + 3 = n$. By the induction hypothesis, there exists a substitution $\theta_1 \in \text{Sub}(\mathcal{F}, \mathcal{V})$ such that $s \rightarrow_R^* l\theta_1 \rightrightarrows_{\geq \mathcal{P}_{\text{osv}}(l), \mathbb{U}_{\text{opt}}(R)}^{n'_1} l\sigma_1$ for some $n'_1 \leq n_0$. Since $l\theta_1 \rightrightarrows_{\geq \mathcal{P}_{\text{osv}}(l), \mathbb{U}_{\text{opt}}(R)}^{n'_1} l\sigma_1$, it follows from the well-known standard property of the parallel reduction [3, Lemma 6.4.2] that $U_1^\rho(s_1, \vec{Z}_1)\theta_1 \rightrightarrows_{\geq \mathcal{P}_{\text{osv}}(U_1^\rho(s_1, \vec{Z}_1)), \mathbb{U}_{\text{opt}}(R)}^{n'_1} U_1^\rho(s_1, \vec{Z}_1)\sigma_1$. Thus, $U_1^\rho(s_1, \vec{Z}_1)\theta_1 \rightrightarrows_{>\varepsilon, \mathbb{U}_{\text{opt}}(R)}^{n'_1} U_1^\rho(s_1, \vec{Z}_1)\sigma_1 \rightrightarrows_{>\varepsilon, \mathbb{U}_{\text{opt}}(R)}^{n_1} U_1^\rho(t_1, \vec{Z}_1)\sigma_2$ with $n'_1 + n_1 < n$, and hence $s_1\theta_1 \rightrightarrows_{\mathbb{U}_{\text{opt}}(R)}^{n'_1+n_1} t_1\sigma_2$ and $Z_1\theta_1 \rightrightarrows_{\mathbb{U}_{\text{opt}}(R)}^{n'_1+n_1} Z_1\sigma_2$. Since the \mathbb{U}_{opt} -LL property provides the linearity of t_1 , by the induction hypothesis, there exists a substitution $\theta'_2 \in \text{Sub}(\mathcal{F}, \mathcal{V})$ such that $s_1\theta_1 \rightarrow_R^* t_1\theta'_2 \rightrightarrows_{\geq \mathcal{P}_{\text{osv}}(t_1), \mathbb{U}_{\text{opt}}(R)}^{n''_2} t_1\sigma_2$ for some $n''_2 \leq n'_1 + n_1$. Also, by the induction hypothesis, for any variable $y \in Z_1$, there exists a substitution $\delta_y \in \text{Sub}(\mathcal{F}, \mathcal{V})$ such that $y\theta_1 \rightarrow_R^* y\delta_y \rightrightarrows_{\geq \{\varepsilon\}, \mathbb{U}_{\text{opt}}(R)}^{j_y} y\sigma_2$ for some $j_y \leq n'_1 + n_1$. Let $\theta_2 = \theta'_2|_{\text{Var}(t_1)} \cup \{y \mapsto y\delta_y \mid y \in Z_1\}$. Then, since the \mathbb{U}_{opt} -LL property provides $\text{Var}(t_1) \cap Z_1 = \emptyset$, we have the derivations $s_1\theta_1 \rightarrow_R^* t_1\theta_2 \rightrightarrows_{\mathbb{U}_{\text{opt}}(R)}^{n'_2} t_1\sigma_2$ and $Z_1\theta_1 \rightarrow_R^* Z_1\theta_2 \rightrightarrows_{\mathbb{U}_{\text{opt}}(R)}^{n'_2} Z_1\sigma_2$ for some $n'_2 \leq n'_1 + n_1$ that is the maximum of n''_2 and j_y for $y \in Z_1$. Thus, we have the derivation $U_2^\rho(s_2, \vec{Z}_2)\theta_2 \rightrightarrows_{>\varepsilon, \mathbb{U}_{\text{opt}}(R)}^{n'_2+n_2} U_2^\rho(t_2, \vec{Z}_2)\sigma_3$ with $n'_2 + n_2 < n$.

In the same way, we obtain a substitution θ_3 in $\text{Sub}(\mathcal{F}, \mathcal{V})$ such that $s_2\theta_2 \rightarrow_R^* t_2\theta_3$, $Z_2\theta_2 \rightarrow_R^* Z_2\theta_3$, $U_2^\rho(t_2, \vec{Z}_2)\theta_3 \rightrightarrows_{>\varepsilon, \mathbb{U}_{\text{opt}}(R)}^{n'_3} U_2^\rho(t_2, \vec{Z}_2)\sigma_3$ for some $n'_3 \leq n'_2 + n_2$. Moreover, in the same way, we obtain a substitution θ in $\text{Sub}(\mathcal{F}, \mathcal{V})$ such that $r\theta_3 \rightarrow_R^* t\theta \rightrightarrows_{\geq \mathcal{P}_{\text{osv}}(t), \mathbb{U}_{\text{opt}}(R)}^{n'}$ $t\sigma$ and $t\sigma \in \mathcal{T}(\mathcal{F}, \mathcal{V})$ implies $t\theta = t\sigma$, where $n' \leq n'_3 + n'' < n$. It follows from Lemma A.1 that $l\theta_1 \rightarrow_R^+ r\theta_3$. Therefore, we have the derivation $s \rightarrow_R^* l\theta_1 \rightarrow_R^+ r\theta_3 \rightarrow_R^* t\theta \rightrightarrows_{\geq \mathcal{P}_{\text{osv}}(t), \mathbb{U}_{\text{opt}}(R)}^{n'}$ $t\sigma$ with $n' \leq n$, and $t\theta = t\sigma$ whenever $t\sigma \in \mathcal{T}(\mathcal{F}, \mathcal{V})$.

(2) Consider the remaining case (*i.e.*, R is non-LV). Similarly to Case (1), we have the following subcases.

a. Consider the case that

$$s \rightrightarrows_{\mathbb{U}_{\text{opt}}(R)}^{n_0} l\sigma_1 \rightarrow_{\mathbb{U}_{\text{opt}}(R)} U_1^\rho(s_1, \vec{Z}_1)\sigma_1 \rightrightarrows_{>\varepsilon, \mathbb{U}_{\text{opt}}(R)}^{n_1} t\sigma$$

where $n_0 + n'' + 1 = n$. The only difference from Case (1)-a is that $s \rightrightarrows_{\mathbb{U}_{\text{opt}}(R)}^{n_0} l\sigma_1$ may contain a rewrite step at the root position. This case can be proved similarly to Case (1)-a.

b. Consider the case that

$$\begin{aligned} s \rightrightarrows_{\mathbb{U}_{\text{opt}}(R)}^{n_0} l\sigma_1 &\xrightarrow{\varepsilon, \mathbb{U}_{\text{opt}}(R)} U_1^\rho(s_1, \vec{Z}_1)\sigma_1 \rightrightarrows_{>\varepsilon, \mathbb{U}_{\text{opt}}(R)}^{n_1} U_1^\rho(t_1, \vec{Z}_1)\sigma_2 \\ &\xrightarrow{\varepsilon, \mathbb{U}_{\text{opt}}(R)} U_k^\rho(s_2, \vec{Z}_2)\sigma_2 \rightrightarrows_{>\varepsilon, \mathbb{U}_{\text{opt}}(R)}^{n''} t\sigma \end{aligned}$$

where $n_0 + n_1 + n'' + 2 = n$. Again, the only difference from Case (1)-b is that $s \rightrightarrows_{\mathbb{U}_{\text{opt}}(R)}^{n_0} l\sigma_1$ may contain a rewrite step at the root position. This case can be proved similarly to Case (1)-b.

c. Consider the remaining case that

$$\begin{aligned} s \rightrightarrows_{\mathbb{U}_{\text{opt}}(R)}^{n_0} l\sigma_1 &\xrightarrow{\varepsilon, \mathbb{U}_{\text{opt}}(R)} U_1^\rho(s_1, \vec{Z}_1)\sigma_1 \rightrightarrows_{>\varepsilon, \mathbb{U}_{\text{opt}}(R)}^{n_1} U_1^\rho(t_1, \vec{Z}_1)\sigma_2 \\ &\xrightarrow{\varepsilon, \mathbb{U}_{\text{opt}}(R)} U_k^\rho(s_2, \vec{Z}_2)\sigma_2 \rightrightarrows_{>\varepsilon, \mathbb{U}_{\text{opt}}(R)}^{n_2} U_k^\rho(t_2, \vec{Z}_2)\sigma_3 \\ &\xrightarrow{\varepsilon, \mathbb{U}_{\text{opt}}(R)} r\sigma_3 \rightrightarrows_{>\varepsilon, \mathbb{U}_{\text{opt}}(R)}^{n''} t\sigma \end{aligned}$$

where $n_0 + n_1 + n_2 + n'' + 3 = n$. The only difference from Case (1)-c is that $s \rightrightarrows_{\mathbb{U}_{\text{opt}}(R)}^{n_0} l\sigma_1$ may contain a rewrite step at the root position, but $r\sigma_3 \rightrightarrows_{\mathbb{U}_{\text{opt}}(R)}^{n''} t\sigma$ does not contain any rewrite step at the root position. This case can be proved similarly to Case (1)-c. \square

A.4. Proof of Theorem 4.7. Theorem 4.7 is a direct consequence of the following lemma, a rule-based variant of Theorem 4.7.

Lemma A.2. *Let $\rho : l \rightarrow r \leftarrow s_1 \rightarrow t_1; \dots; s_k \rightarrow t_k$ be an (extended) deterministic rewrite rule. Then all of the following hold:*

- (1) $\text{Var}(t_i) \subseteq \text{Var}(r, s_{i+1}, \dots, s_k)$ for all $1 \leq i \leq k$ iff $(\rho)^{-1}$ is deterministic,
- (2) $\text{Var}(l) \subseteq \text{Var}(r, s_1, \dots, s_k)$ iff $(\rho)^{-1}$ is of Type 3,
- (3) if $\text{Var}(t_i) \subseteq \text{Var}(r, s_{i+1}, \dots, s_k)$ for all $1 \leq i \leq k$, then
 - a. $\mathbb{U}_{\text{opt}}((\rho)^{-1}) = (\mathbb{U}_{\text{opt}}(\rho))^{-1}$ up to the renaming of U symbols,
 - b. ρ is $\mathbb{U}_{\text{opt}}\text{-LL}$ iff $(\rho)^{-1}$ is $\mathbb{U}_{\text{opt}}\text{-RL}$, and
 - c. ρ is $\mathbb{U}_{\text{opt}}\text{-RL}$ iff $(\rho)^{-1}$ is $\mathbb{U}_{\text{opt}}\text{-LL}$,
- (4) ρ is non-LV iff $(\rho)^{-1}$ is non-RV, and
- (5) ρ is non-RV iff $(\rho)^{-1}$ is non-LV.

Proof. Since the first, second, fourth, and fifth claims are trivial, we only prove the third claim. Recall that $\mathbb{U}_{\text{opt}}(\rho) = \{ l \rightarrow U_1^\rho(s_1, \vec{Z}_1), U_1^\rho(t_1, \vec{Z}_1) \rightarrow U_2^\rho(s_2, \vec{Z}_2), \dots, U_k^\rho(t_k, \vec{Z}_k) \rightarrow r \}$. We assume w.l.o.g. that $\mathbb{U}_{\text{opt}}((\rho)^{-1}) = \{ r \rightarrow U_k^\rho(t_k, \vec{V}_k), \dots, U_2^\rho(s_2, \vec{V}_2) \rightarrow U_1^\rho(t_1, \vec{V}_1), U_1^\rho(s_1, \vec{V}_1) \rightarrow l \}$ where $V_i = \text{Var}(r, s_k, \dots, s_{i+1}) \cap \text{Var}(l, s_i, t_{i-1}, s_{i-1}, \dots, t_1, s_1)$.

Since ρ is deterministic, we have $V_i = \text{Var}(r, s_k, \dots, s_{i+1}) \cap X_i$. Moreover, it follows from the assumption that $Y_i = \text{Var}(r, t_i, s_{i+1}, t_{i+1}, \dots, s_k, t_k) = \text{Var}(r, s_{i+1}, \dots, s_k)$, and hence $V_i = X_i \cap Y_i = Z_i$ for all $1 \leq i \leq k$. Therefore, the claim (3)-a holds.

By the definition of $(\cdot)^{-1}$, $\mathbb{U}_{\text{opt}}(R)$ is LL (RL) iff $(\mathbb{U}_{\text{opt}}(R))^{-1}$ is RL (LL). Therefore, the claims (3)-b,c follow from the claim (3)-a. \square

A.5. Proof of Lemma 4.15. We first prepare some technical lemmas to prove Lemma 4.15.

Lemma A.3. *Let l be a linear term with U -symbol-free proper subterms, and θ, σ, η be substitutions such that $\theta \in \text{Sub}(\mathcal{F}, \mathcal{V})$ and $\text{root}(x\eta)$ is a U symbol for any $x \in \text{Dom}(\eta)$. If $s\theta\eta = l\sigma$, then there exists a substitution σ' such that $s\theta = l\sigma'$ and $l\sigma = l\sigma'\eta$.*

Proof (Sketch). This claim can be proved by induction on the term structure of s . \square

Lemma 4.15 is a direct consequence of the following lemma.

Lemma A.4. *Let s be a term in $\mathcal{T}(\mathcal{F}_{\mathbb{U}_{\text{opt}}(R)}, \mathcal{V})$, t be a term in $\mathcal{T}(\mathcal{F}, \mathcal{V})$, and θ, η be substitutions such that $\theta \in \text{Sub}(\mathcal{F}_{\mathbb{U}_{\text{opt}}(R)}, \mathcal{V})$ and $\text{root}(x\eta)$ is a U symbol for any $x \in \text{Dom}(\eta)$. If $\text{Pos}_{\mathcal{F}}(s\theta) : s\theta\eta \xrightarrow{\text{evs}^*_{\mathbb{U}_{\text{opt}}(R)}} t$, then there exists a substitution $\sigma \in \text{Sub}(\mathcal{F}, \mathcal{V})$ such that*

- $s\theta\sigma \xrightarrow{\text{evs}^n_{\mathbb{U}_{\text{opt}}(R)}} t$, and
- the derivation is EV-instantiated on $\mathcal{T}(\mathcal{F}, \mathcal{V})$.

Proof. We prove this lemma by induction on the length n of the derivation $\text{Pos}_{\mathcal{F}}(s\theta) : s\theta\eta \xrightarrow{\text{evs}^*_{\mathbb{U}_{\text{opt}}(R)}} t$. The case that $n = 0$ is trivial, so let $n > 0$.

From the EV-safe property of the derivation and Lemma A.3, we can assume w.l.o.g. that

- $s\theta$ is of the form $C[s']_p$ with $s' = l\delta$ and $p \in \text{Pos}_{\mathcal{F}}(s\theta)$,
- $s\theta\eta = C\theta\eta[s'\eta] = C\theta\eta[l\delta\eta]_p \xrightarrow{\text{evs}_{p, l \rightarrow r}} C\theta\eta[r\delta\eta] \xrightarrow{\text{evs}^{n-1}_{\mathbb{U}_{\text{opt}}(R)}} t$, and
- $\delta \in \text{Sub}(\mathcal{F}, \mathcal{V})$,

where

- $l \rightarrow r \in \mathbb{U}_{\text{opt}}(R)$ with $\text{Var}(l, r) \cap \text{Var}(s\theta) = \emptyset$,
- the set B of EV-safe positions in $C\theta\eta[r\delta\eta]$ is $(\text{Pos}_{\mathcal{F}}(s\theta) \setminus \{q \in \text{Pos}_{\mathcal{F}}(s\theta) \mid p \leq q\}) \cup \{pq \mid q \in \text{Pos}_{\mathcal{F}}(r)\} \cup \{pp'q \mid pp'q \in \text{Pos}_{\mathcal{F}}(s\theta), p'' \in \text{Pos}_{\mathcal{V}}(l), l|_{p''} = r|_{p''}\}$, and
- $B : C\theta\eta[r\delta\eta] \xrightarrow{\text{evs}^{n-1}_{\mathbb{U}_{\text{opt}}(R)}} t$.

Let δ' and δ'' be substitutions such that $\delta' \in \text{Sub}(\mathcal{F}, \mathcal{V})$, $\delta|_{\mathcal{E}\text{Var}(l \rightarrow r)} = \delta'\eta$, $\text{Dom}(\delta'') \cap (\text{Var}(l, r) \cup \text{Dom}(\eta)) = \emptyset$, and $\text{root}(x\delta'')$ is a U symbol for any $x \in \text{Dom}(\delta'')$.

Let $\theta' = \theta|_{\text{Var}(C[\])} \cup \delta|_{\text{Var}(l)} \cup \delta'|_{\mathcal{E}\text{Var}(l \rightarrow r)}$ and $\eta' = \eta \cup \delta''$. Then, θ' and η' are substitutions such that $C\theta\eta[r\delta] = (C[r])\theta'\eta'$. It follows from the definition of the EV-safe property that $B = \text{Pos}_{\mathcal{F}}((C[r])\theta')$. Thus, by the induction hypothesis, there exists a substitution σ in $\text{Sub}(\mathcal{F}, \mathcal{V})$ such that $(C[r])\theta'\sigma \xrightarrow{\text{evs}^{n-1}_{\mathbb{U}_{\text{opt}}(R)}} t$ and the derivation is EV-instantiated on $\mathcal{T}(\mathcal{F}, \mathcal{V})$. Now, we have the derivation $s\theta\sigma = (C\theta[s']\sigma) = (C\theta[l\delta])\sigma = (C\theta'[l\theta'])\sigma \xrightarrow{\text{evs}_{\mathbb{U}_{\text{opt}}(R)}} (C\theta'[r\theta'])\sigma = (C[r])\theta'\sigma' \xrightarrow{\text{evs}^{n-1}_{\mathbb{U}_{\text{opt}}(R)}} t$. Since θ and σ are in $\text{Sub}(\mathcal{F}, \mathcal{V})$, any extra variable in r is instantiated by a term in $\mathcal{T}(\mathcal{F}, \mathcal{V})$. Therefore, this derivation is EV-instantiated on $\mathcal{T}(\mathcal{F}, \mathcal{V})$. \square

A.6. Proof of Lemma 6.11.

Lemma 6.11. *Let R be a DCTRS over a signature \mathcal{F} . Then, $\phi_{\text{SR}(R)}(\rightarrow^*_{\mathbb{U}(R)}) \subseteq \rightarrow^*_{\text{SR}(R)}$ on terms in $\mathcal{T}(\mathcal{F}, \mathcal{V})$.*

Proof. We extend the operation $\bar{\cdot}$ by adding the following clause to the definition of $\bar{\phi}$ in Definition 6.3:

$$\begin{aligned} \bar{\phi}(U_i^{\rho_{f,j}}(x_i, \overrightarrow{\text{Var}(f(w_1, \dots, w_n), t_1, \dots, t_{i-1})})) \\ = \bar{f}(\overline{w_1}, \dots, \overline{w_n}, \underbrace{\perp, \dots, \perp}_{j-1}, [\{x_i\}, \overline{t_{i-1}}, \dots, \overline{t_1}, \perp, \dots, \perp]_k, \underbrace{\perp, \dots, \perp}_{n_f-j}) \end{aligned}$$

where $\rho_{f,j} : f(w_1, \dots, w_n) \rightarrow r \Leftarrow s_1 \Rightarrow t_1; \dots; s_k \Rightarrow t_k \in R$, x_i is a fresh variable, and $\overline{w_1}, \dots, \overline{w_n}, \overline{t_1}, \dots, \overline{t_{i-1}}$ are terms obtained by applying the original operation $\bar{\cdot}$ to $w_1, \dots, w_n, t_1, \dots, t_{i-1}$, respectively. We also extend $\phi_{\text{SR}(R)}$ by introducing the extension of $\bar{\cdot}$.

To prove this lemma, it suffices to show that for terms $s, t \in \mathcal{T}(\mathcal{F}_{\mathbb{U}(R)}, \mathcal{V})$, if $s \rightarrow_{\mathbb{U}(R)}^* t$, then $\phi_{\text{SR}(R)}(s) \rightarrow_{\text{SR} \rightarrow (R)}^* \phi_{\text{SR}(R)}(t)$. We prove this claim by induction on the length m of $s \rightarrow_{\mathbb{U}(R)}^* t$. The case $m = 0$ is trivial, so let $m > 0$.

Suppose that $s = C[u\sigma] \rightarrow_{\mathbb{U}(R)} C[v\sigma] \rightarrow_{\mathbb{U}(R)}^{m-1} t$ and $\bar{\sigma} = \{x \mapsto \overline{x\sigma} \mid x \in \text{Dom}(\sigma)\}$. In applying $\phi_{\text{SR}(R)}$ to $C[\]$, by definition, the hole in $C[\]$ is neither erased nor duplicated. Thus, $\phi_{\text{SR}(R)}(C[\])$ is a context of the form $\{\overline{C}\}[\]$. Moreover, by definition, the position p of the hole in $\{\overline{C}\}[\]$ is structural. Now, we make a case distinction depending on what $u \rightarrow v \in \mathbb{U}(R)$ is.

- Consider the case that $u \rightarrow v$ is $f(w_1, \dots, w_n) \rightarrow r \in R$. Since $\bar{f}(\overline{w_1}, \dots, \overline{w_n}, z_1, \dots, z_{n_f}) \rightarrow \{\bar{r}\} \in \text{SR} \rightarrow (R)$ by definition, we have the derivation $\overline{u\sigma} = \bar{f}(\overline{w_1}, \dots, \overline{w_n}, z_1, \dots, z_{n_f})\bar{\sigma} \rightarrow_{\text{SR} \rightarrow (R)} \{\bar{r}\bar{\sigma}\} = \{\overline{v\sigma}\}$.
- Consider the case that $u \rightarrow v$ is $f(w_1, \dots, w_n) \rightarrow U_1^{\rho_{f,j}}(s_1, \overrightarrow{\text{Var}(l)}) \in \mathbb{U}(R)$. By definition, $\overline{u\sigma} = \bar{f}(\overline{w_1}, \dots, \overline{w_n}, \perp, \dots, \perp)\bar{\sigma}$ and

$$\begin{aligned} \bar{f}(\overline{w_1}, \dots, \overline{w_n}, z_1, \dots, z_{j-1}, \perp, z_{j+1}, \dots, z_{n_f}) \rightarrow \\ \bar{f}(\overline{w_1}, \dots, \overline{w_n}, z_1, \dots, z_{j-1}, [\{\overline{s_1}\}, \perp, \dots, \perp], z_{j+1}, \dots, z_{n_f}) \in \text{SR} \rightarrow (R). \end{aligned}$$

Therefore, we have the derivation

$$\begin{aligned} \overline{u\sigma} &= \bar{f}(\overline{w_1}, \dots, \overline{w_n}, \perp, \dots, \perp)\bar{\sigma} \\ &\rightarrow_{\text{SR} \rightarrow (R)} \bar{f}(\overline{w_1}, \dots, \overline{w_n}, \perp, \dots, \perp, [\{\overline{s_1}\}, \perp, \dots, \perp], \perp, \dots, \perp)\bar{\sigma} \\ &= U_1^{\rho_{f,j}}(s_1, \overrightarrow{\text{Var}(l)})\bar{\sigma} = U_1^{\rho_{f,j}}(s_1, \overrightarrow{\text{Var}(l)})\sigma = \overline{v\sigma}. \end{aligned}$$

- Consider the case that $u \rightarrow v$ is the rule $U_i^{\rho_{f,j}}(t_i, \overrightarrow{\text{Var}(l, t_1, \dots, t_{i-1})}) \rightarrow U_{i+1}^{\rho_{f,j}}(s_{i+1}, \overrightarrow{\text{Var}(l, t_1, \dots, t_i)}) \in \mathbb{U}(R)$. By definition,

$$\overline{u\sigma} = \bar{f}(\overline{w_1}, \dots, \overline{w_n}, \perp, \dots, \perp, [\{\overline{t_i}\}, \overline{t_{i-1}}, \dots, \overline{t_1}, \perp, \dots, \perp], \perp, \dots, \perp)\bar{\sigma}$$

and

$$\begin{aligned} \bar{f}(\overline{w_1}, \dots, \overline{w_n}, z_1, \dots, z_{j-1}, [\{\overline{t_i}\}, \overline{t_{i-1}}, \dots, \overline{t_1}, \perp, \dots, \perp], z_{j+1}, \dots, z_{n_f}) \rightarrow \\ \bar{f}(\overline{w_1}, \dots, \overline{w_n}, z_1, \dots, z_{j-1}, [\{\overline{s_{i+1}}\}, \overline{t_i}, \dots, \overline{t_1}, \perp, \dots, \perp], z_{j+1}, \dots, z_{n_f}) \in \text{SR} \rightarrow (R). \end{aligned}$$

Therefore, we have the derivation

$$\begin{aligned} \overline{u\sigma} &= \bar{f}(\overline{w_1}, \dots, \overline{w_n}, \perp, \dots, \perp, [\{\overline{t_i}\}, \overline{t_{i-1}}, \dots, \overline{t_1}, \perp, \dots, \perp], \perp, \dots, \perp)\bar{\sigma} \\ &\rightarrow_{\text{SR} \rightarrow (R)} \bar{f}(\overline{w_1}, \dots, \overline{w_n}, \perp, \dots, \perp, [\{\overline{s_{i+1}}\}, \overline{t_i}, \dots, \overline{t_1}, \perp, \dots, \perp], \perp, \dots, \perp)\bar{\sigma} \\ &= U_{i+1}^{\rho_{f,j}}(s_{i+1}, \overrightarrow{\text{Var}(l, t_1, \dots, t_i)})\bar{\sigma} = U_{i+1}^{\rho_{f,j}}(s_{i+1}, \overrightarrow{\text{Var}(l, t_1, \dots, t_i)})\sigma = \overline{v\sigma}. \end{aligned}$$

- Consider the remaining case that $u \rightarrow v$ is $U_k^{\rho f, j}(t_k, \overline{\mathcal{V}ar}(l, t_1, \dots, t_{k-1})) \rightarrow r \in \mathbb{U}(R)$. By definition,

$$\overline{u\sigma} = \overline{f}(\overline{w_1}, \dots, \overline{w_n}, \perp, \dots, \perp, \{\overline{t_1}, \overline{t_{k-1}} \dots, \overline{t_1}\}, \perp, \dots, \perp) \overline{\sigma}$$

and $\overline{f}(\overline{w_1}, \dots, \overline{w_n}, z_1, \dots, z_{j-1}, \{\overline{t_k}, \overline{t_{k-1}}, \dots, \overline{t_1}\}, z_{j+1}, \dots, z_{n_f}) \rightarrow \{\overline{r}\} \in \mathbb{S}\mathbb{R}^{\rightarrow}(R)$. Therefore, we have the derivation $\overline{u\sigma} = \overline{f}(\overline{w_1}, \dots, \overline{w_n}, \perp, \dots, \perp, \{\overline{t_1}, \overline{t_{k-1}} \dots, \overline{t_1}\}, \perp, \dots, \perp) \overline{\sigma} \rightarrow_{\mathbb{S}\mathbb{R}^{\rightarrow}(R)} \{\overline{r}\} \overline{\sigma} = \{\overline{r\sigma}\} = \{\overline{v\sigma}\}$.

Now, we have either $\overline{u\sigma} \rightarrow_{\mathbb{S}\mathbb{R}^{\rightarrow}(R)} \{\overline{v\sigma}\}$ or $\overline{u\sigma} \rightarrow_{\mathbb{S}\mathbb{R}^{\rightarrow}(R)} \overline{v\sigma}$.

- Consider the case that $\overline{u\sigma} \rightarrow_{\mathbb{S}\mathbb{R}^{\rightarrow}(R)} \{\overline{v\sigma}\}$. Since p is a structural position, it follows from Lemma 6.10 that $\phi_{\mathbb{S}\mathbb{R}(R)}(s) = \{\overline{C[u\sigma]}\} = \{\overline{C}\}[\overline{u\sigma}] \rightarrow_{\mathbb{S}\mathbb{R}^{\rightarrow}(R)} \{\overline{C}\}[\{\overline{v\sigma}\}] \rightarrow_{\mathbb{S}\mathbb{R}^{\rightarrow}(R)}^* \{\{\overline{C}\}[\overline{v\sigma}]\} = \{\{\overline{C[v\sigma]}\}\} \rightarrow_{\mathbb{S}\mathbb{R}^{\rightarrow}(R)} \{\overline{C[v\sigma]}\} = \phi_{\mathbb{S}\mathbb{R}(R)}(C[v\sigma])$.
- Consider the remaining case that $\overline{u\sigma} \rightarrow_{\mathbb{S}\mathbb{R}^{\rightarrow}(R)} \overline{v\sigma}$. Then, $\phi_{\mathbb{S}\mathbb{R}(R)}(s) = \{\overline{C[u\sigma]}\} = \{\overline{C}\}[\overline{u\sigma}] \rightarrow_{\mathbb{S}\mathbb{R}^{\rightarrow}(R)} \{\overline{C}\}[\overline{v\sigma}] = \{\overline{C[v\sigma]}\} = \phi_{\mathbb{S}\mathbb{R}(R)}(C[v\sigma])$.

By the induction hypothesis, $\phi_{\mathbb{S}\mathbb{R}(R)}(C[v\sigma]) \rightarrow_{\mathbb{S}\mathbb{R}^{\rightarrow}(R)}^* \phi_{\mathbb{S}\mathbb{R}(R)}(t)$. Therefore, we have the derivation $\phi_{\mathbb{S}\mathbb{R}(R)}(s) \rightarrow_{\mathbb{S}\mathbb{R}^{\rightarrow}(R)}^* \phi_{\mathbb{S}\mathbb{R}^{\rightarrow}(R)}(C[v\sigma]) \rightarrow_{\mathbb{S}\mathbb{R}^{\rightarrow}(R)}^* \phi_{\mathbb{S}\mathbb{R}(R)}(t)$. \square