

CONSTRUCTIVE CANONICITY OF INDUCTIVE INEQUALITIES

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ABSTRACT. We prove the canonicity of inductive inequalities in a constructive meta-theory, for classes of logics algebraically captured by varieties of normal and regular lattice expansions. This result encompasses Ghilardi-Meloni’s and Suzuki’s constructive canonicity results for Sahlqvist formulas and inequalities, and is based on an application of the tools of unified correspondence theory. Specifically, we provide an alternative interpretation of the language of the algorithm ALBA for lattice expansions: nominal and conominal variables are respectively interpreted as closed and open elements of canonical extensions of normal/regular lattice expansions, rather than as completely join-irreducible and meet-irreducible elements of perfect normal/regular lattice expansions. We show the correctness of ALBA with respect to this interpretation. From this fact, the constructive canonicity of the inequalities on which ALBA succeeds follows by an adaptation of the standard argument. The claimed result then follows as a consequence of the success of ALBA on inductive inequalities.

1. INTRODUCTION

Perhaps the most important uniform methodology for proving Kripke completeness for modal logics is the notion of canonicity, which, thanks to duality, can be studied both frame-theoretically and algebraically. Frame-theoretically, canonicity can be formulated as *d-persistence*, i.e. preservation of validity from any given descriptive general frame to its underlying Kripke frame (or in other words, equivalence between validity w.r.t. admissible assignments and w.r.t. arbitrary assignments); algebraically, as preservation of validity from any given modal algebra to its canonical extension. The study of canonicity has been extended from classical normal modal logic to its many neighbouring logics, and has given

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rise to a rich literature. Particularly relevant to the present paper are two general methods for canonicity, pioneered by Sambin and Vaccaro [49] and by Ghilardi and Meloni [31]. Sambin and Vaccaro obtain canonicity for Sahlqvist formulas of classical modal logic in a frame-theoretic setting as a byproduct of correspondence. The core of their proof strategy is the observation that, whenever it exists, the first-order correspondent of a modal formula provides an equivalent rewriting of the modal formula with no occurring propositional variables, so that its validity w.r.t. admissible assignments is tantamount to its validity w.r.t. arbitrary assignments, which proves d-persistence. Sambin-Vaccaro’s proof strategy, sometimes called *canonicity-via-correspondence*, has been imported to the algorithmic proof of canonicity given in [12], which achieves a uniform proof for the widest class known so far, the so-called *inductive* formulas, which significantly extends the class of Sahlqvist inequalities. Here it is the algorithm SQEMA which produces the equivalent rewriting (i.e. the first-order correspondent) of the modal formula, and this is something it does successfully for (at least) all inductive formulas.

Ghilardi and Meloni’s work [31] shows that canonicity can be meaningfully investigated purely algebraically, in a constructive meta-theory where correspondence is not even defined, in general. Indeed, in [31], the canonical extension construction for certain bi-intuitionistic modal algebras, later applied also to general lattice and poset expansions in [29, 25], is formulated in terms of general filters and ideals, and does not depend on any form of the axiom of choice (such as the existence of ‘enough’ optimal filter-ideal pairs). Thus, while the constructive canonical extension need not be perfect in the sense of [40], the canonical embedding map, sending the original algebra into its canonical extension, retains the properties of denseness and compactness (cf. Definition 2.10). These properties make it possible for the authors of [31] to identify a class of formulas which are *constructively canonical* (i.e. the validity of which is preserved under constructive canonical extension). These inequalities are identified from the order-theoretic properties of the induced term-functions, and the preservation of their validity from an algebra to its constructive canonical extension is shown in two steps: first, from the elements of a given algebra to the closed/open elements of its canonical extension, and then from the closed/open elements to arbitrary elements. This proof strategy is very similar to that of Jónsson [39], but was developed independently. It should be noted however that, in terms of the classes of formulas to which they apply, the results of both [31] and [39] fall within the scope of applicability of the canonicity-via-correspondence method.

The approaches of Sambin-Vaccaro, on the one hand, and Ghilardi-Meloni, on the other, have been very influential, and have contributed to a certain binary divide detectable in the literature between correspondence and canonicity, namely: correspondence being typically developed on frames, and canonicity on algebras. Moreover, subsequent algebraic proofs of canonicity have mostly remained restricted to Sahlqvist formulas, rather than considering e.g. the wider class of inductive formulas [32]. For the latter class, the first algebraic proof of canonicity in the style of Jónsson-Ghilardi-Meloni appeared only very recently [46]. *Unified correspondence theory* [11], to which the contributions of the present paper belong, bridges this divide in the sense that will be explained below, and by doing this, succeeds in importing Sambin-Vaccaro’s proof strategy to the constructive setting of Ghilardi-Meloni, thus providing the conceptual unification of these very different perspectives which had remained a prominent open problem in the subsequent literature. For instance, the intermediate step of [31] can be recognized as the equivalent rewriting, independent of the evaluation of proposition variables, pursued in [49].

At the core of unified correspondence is an *algebraic* reformulation of *correspondence* theory [18], with its ensuing algorithmic canonicity-via-correspondence argument [13]. This reformulation makes it possible to construe the computation of first-order correspondents in two phases: reduction, and translation (cf. [14] for an expanded discussion on this point). Formulas/inequalities are interpreted in the canonical extension \mathbb{A}^δ of a given algebra \mathbb{A} , and a calculus of rules (captured by the ALBA algorithm) is applied to rewrite them into equivalent expressions with no occurring propositional variables, called pure. The pure expressions may, however, contain (non-propositional) variables known as nominals and co-nominals. If successful, achieving pure expressions completes the reduction phase. Pure expressions are already enough to implement Sambin-Vaccaro’s canonicity strategy: indeed, the validity of pure expressions under assignments sending propositional variables into \mathbb{A} (identified with the *admissible* assignments on \mathbb{A}^δ) is tantamount to their validity w.r.t. arbitrary assignments on \mathbb{A}^δ , and this establishes the canonicity of the original formula or inequality.

In a non-constructive setting, the nominals and co-nominals are interpreted as ranging over the completely join-irreducible or meet-irreducible elements of \mathbb{A}^δ . The soundness of the rewrite rules is based, in part, on the fact that the completely join-irreducible and meet-irreducible elements respectively join-generate and meet-generate the ambient algebra \mathbb{A}^δ . Moreover, in this setting, completely meet- and join-irreducible elements correspond, via discrete duality, to first-order definable subsets of the dual relational semantics. Thus the first-order frame correspondent of the original formula or inequality can be obtained by simply applying the appropriate standard translation to the pure expressions. This is known as the translation phase.

In a constructive setting, the situation just described is changed by the fact that we can no longer rely on the completely join-irreducible and meet-irreducible elements to respectively join-generate and meet-generate \mathbb{A}^δ . However, we may fall back on the closed and open elements of \mathbb{A}^δ as complete join- and meet-generators, and adjust the interpretations of nominals and co-nominals accordingly. By doing this, the reduction phase remains sound in the non-constructive setting, and still yields canonicity. As expected, however, discrete duality is not available in general, and the possibility of translating to the relational semantics only applies modulo restriction to the setting in which discrete duality is available.

In the present paper, we expand on the ideas outlined above and give formal proofs of their correctness. We will do this in the setting of arbitrary normal or regular lattice expansions, and prove the constructive canonicity of inductive inequalities in the appropriate signature. The inductive inequalities significantly enlarge the set of Sahlqvist inequalities, accounting for important axiomatic instances¹, and the canonicity proof for inductive inequalities presented here subsumes and unifies those of [31, 49].

The results of the present paper contribute to the realization of one of the core objectives of unified correspondence theory, which is to create an environment in which different proof techniques for canonicity and correspondence can be systematically compared and connected to each other. For instance, in [20], the algorithmic route to correspondence and canonicity for distributive lattice based logics has been compared to the route via reduction to the Boolean setting, by means of Gödel-type translations. In the specific case of canonicity, the results of the present paper build on techniques developed investigating the phenomenon of canonicity via pseudocorrespondence [19], and concur to complete the picture begun in

¹Well known formulas such as Frege’s axiom are inductive but not Sahlqvist (cf. Example 3.6).

[46], where Sambin-Vaccaro’s methodology for canonicity has been systematically connected to Jónsson’s algebraic but non-constructive canonicity technique. These results also serve as base for proving constructive canonicity in the setting of mu-calculi [6], a result which bridges constructive canonicity with algebraic and algorithmic correspondence and canonicity for mu-calculi [3, 7]. Moreover, the present results are of direct relevance to the problem of canonicity for possibility semantics in modal logic (cf. [37, 53], more about this in the conclusions).

Recently, systematic connections have been established between unified correspondence and the theory of proper display calculi for certain axiomatic extensions of basic normal DLE-logics [33]. These connections have been also put to use in defining analytic Gentzen calculi for subintuitionistic logics [44]. In particular, a uniform argument for proving that the resulting proper display calculi are sound and conservative crucially uses the canonicity of (a certain subclass of) inductive inequalities. From these and other results, a perspective in modern structural proof theory has emerged which is based on the systematic integration of results in algebraic logic into the design of analytic calculi; the results and insights of the present paper naturally fit also in this line of investigation (more about this in the conclusions).

It is interesting to observe that, while the core tools of unified correspondence, and the algorithm ALBA in particular, have proven their versatility in settings as diverse as hybrid logics [21], many valued logics [42], and monotone modal logic [27], through the development of applications such as [46, 33, 19], these tools have acquired novel conceptual significance, which cannot be reduced exclusively to their original purpose as computational tools for correspondence theory. In this respect, the results of the present paper are yet another instance of the potential of ALBA to be used as a general-purpose computational tool, capable of meaningfully contributing to more general and different issues than pure correspondence.

Structure of the paper. In Section 2 we provide some necessary preliminaries. Particularly, we introduce the logical languages we will consider, we provide them with algebraic semantics in the form of lattices expanded with normal and regular operations, and discuss the constructive canonical extensions of the latter. In Section 3 we define the Inductive and Sahlqvist inequalities in the setting of lattices with normal and regular operations. Section 4 contains the specification of the constructive version of ALBA, and the partial correctness of this algorithm is proved in Section 5. The latter relies on some technical lemmas which have been relegated to Appendix A. We next show, in Section 6, that constructive ALBA successfully reduces all inductive and Sahlqvist inequalities. In Section 7 we prove that all inequalities on which constructive ALBA succeeds are constructively canonical. From this our main theorem follows, i.e. that all inductive inequalities are constructively canonical. We conclude in Section 8.

2. PRELIMINARIES

2.1. Language. Our base language is an unspecified but fixed language \mathcal{L}_{LE} , to be interpreted over lattice expansions of compatible similarity type. We will make heavy use of the following auxiliary definition: an *order-type* over $n \in \mathbb{N}$ is an n -tuple $\varepsilon \in \{1, \partial\}^n$. We will write $\varepsilon(i)$ or ε_i for the i -th component of order type ε with $1 \leq i \leq n$. We denote its *opposite* order type by ε^∂ , that is, $\varepsilon_i^\partial = 1$ iff $\varepsilon_i = \partial$ for every $1 \leq i \leq n$. Throughout the paper,

order-types will be typically associated with arrays of variables $\vec{p} := (p_1, \dots, p_n)$. When the order of the variables in \vec{p} is not specified, we will sometimes abuse notation and write $\varepsilon(p) = 1$ or $\varepsilon(p) = \partial$. For any lattice \mathbb{A} , we let $\mathbb{A}^1 := \mathbb{A}$ and \mathbb{A}^∂ be the dual lattice, that is, the lattice associated with the converse partial order of \mathbb{A} . Accordingly, we write \leq^∂ for \geq , \geq^∂ for \leq , \wedge^∂ for \vee , \vee^∂ for \wedge , \top^∂ for \perp and \perp^∂ for \top , while \star^1 denotes \star for any symbol $\star \in \{\leq, \geq, \wedge, \vee, \top, \perp\}$. For any order type ε , we let $\mathbb{A}^\varepsilon := \prod_{i=1}^n \mathbb{A}^{\varepsilon_i}$.

The language $\mathcal{L}_{\text{LE}}(\mathcal{F}, \mathcal{G})$ (from now on abbreviated as \mathcal{L}_{LE}) takes as parameters: 1) a denumerable set PROP of proposition letters, elements of which are denoted p, q, r , possibly with indexes; 2) disjoint sets of connectives \mathcal{F} and \mathcal{G} such that $\mathcal{F} := \mathcal{F}_r \uplus \mathcal{F}_n$ and $\mathcal{G} := \mathcal{G}_r \uplus \mathcal{G}_n$. Each $f \in \mathcal{F}$ and $g \in \mathcal{G}$ has arity $n_f \in \mathbb{N}$ (resp. $n_g \in \mathbb{N}$) and is associated with some order-type ε_f over n_f (resp. ε_g over n_g).² Connectives belonging to \mathcal{F}_r or \mathcal{G}_r are always unary.

The motivation behind splitting the signature into these disjoint sets will become clearer in the next subsection but, in brief, the idea is that the algebraic interpretation of members $f \in \mathcal{F}$ (resp. $g \in \mathcal{G}$) preserve finite joins (resp. meets) in arguments i with $\varepsilon_f(i) = 1$ (resp. $\varepsilon_g(i) = 1$) while turning finite meets (resp. joins) in argument i into joins if $\varepsilon_f(i) = 1\partial$ (resp. $\varepsilon_g(i) = 1$). For $f \in \mathcal{F}_r$ and $g \in \mathcal{G}_r$ this only applies to non-empty finite meets and joins.

The terms (formulas) of \mathcal{L}_{LE} are defined recursively as follows:

$$\varphi ::= p \mid \perp \mid \top \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid f(\vec{\varphi}) \mid g(\vec{\varphi})$$

where $p \in \text{PROP}$, $f \in \mathcal{F}$, $g \in \mathcal{G}$. Terms in \mathcal{L}_{LE} will be denoted either by s, t , or by lowercase Greek letters such as φ, ψ, γ etc.

Remark 2.1. The above presentation of the language \mathcal{L}_{LE} follows the conventions established in a series of papers, including [14, 20, 33]. However, as should be clear by the comments above, the intended interpretations of the connectives belonging to families \mathcal{F} and \mathcal{G} are order-dual to one another. As such it is possible to present them more compactly, viz. by specifying that a signature consists of a family \mathcal{H} of connectives, such that each $h \in \mathcal{H}$ has arity $n_h \in \mathbb{N}$, and *adjunction-type* $\alpha_h \in \{1, \partial\}$. (Relating this to the standard terminology, above, $\mathcal{F} = \{h \in \mathcal{H} \mid \alpha_h = 1\}$ and $\mathcal{G} = \{h \in \mathcal{H} \mid \alpha_h = \partial\}$.) A subset, denoted \mathcal{H}_n , of the connectives in \mathcal{H} are designated as *normal*, while the remainder, $\mathcal{H}_r = \mathcal{H} \setminus \mathcal{H}_n$, are *regular*. Every regular connective $h \in \mathcal{H}_r$ has arity $n_h = 1$. A number of notions can be formulated mores succinctly using this convention, see Remarks 2.3, 2.7 and 2.8.

2.2. Lattice expansions, and their canonical extensions. The following definition captures the algebraic setting of the present paper, which generalizes the normal lattice expansions of [14] and the regular distributive lattice expansions of [47]. In what follows, we will refer to these algebras simply as *lattice expansions*.

Definition 2.2. For any tuple $(\mathcal{F}, \mathcal{G})$ of function symbols as above, a *lattice expansion* (LE) is a tuple $\mathbb{A} = (L, \mathcal{F}^\mathbb{A}, \mathcal{G}^\mathbb{A})$ such that L is a bounded lattice, $\mathcal{F}^\mathbb{A} = \{f^\mathbb{A} \mid f \in \mathcal{F}\}$ and $\mathcal{G}^\mathbb{A} = \{g^\mathbb{A} \mid g \in \mathcal{G}\}$, such that every $f^\mathbb{A} \in \mathcal{F}^\mathbb{A}$ (resp. $g^\mathbb{A} \in \mathcal{G}^\mathbb{A}$) is an n_f -ary (resp. n_g -ary) operation on \mathbb{A} , and moreover,

- (1) every $f^\mathbb{A} \in \mathcal{F}_n^\mathbb{A}$ (resp. $g^\mathbb{A} \in \mathcal{G}_n^\mathbb{A}$) preserves finite (hence also empty) joins (resp. meets) in each coordinate with $\varepsilon_f(i) = 1$ (resp. $\varepsilon_g(i) = 1$) and reverses finite (hence also empty) meets (resp. joins) in each coordinate with $\varepsilon_f(i) = \partial$ (resp. $\varepsilon_g(i) = \partial$).

²Unary f (resp. g) will be sometimes denoted as \diamond (resp. \square) if the order-type is 1, and \triangleleft (resp. \triangleright) if the order-type is ∂ .

(2) every $f^{\mathbb{A}} \in \mathcal{F}_r^{\mathbb{A}}$ (resp. $g^{\mathbb{A}} \in \mathcal{G}_r^{\mathbb{A}}$) preserves finite nonempty joins (resp. meets) if $\varepsilon_f = 1$ (resp. $\varepsilon_g = 1$) and reverses finite nonempty meets (resp. joins) if $\varepsilon_f = \partial$ (resp. $\varepsilon_g = \partial$).

Let \mathbb{LE} be the class of LEs. Sometimes we will refer to certain LEs as $\mathcal{L}_{\mathbb{LE}}$ -algebras when we wish to emphasize that these algebras have a compatible signature with the logical language we have fixed.

Remark 2.3. Following Remark 2.1, for any signature \mathcal{H} , an LE algebra can be defined as a structure $\mathbb{A} = (L, \mathcal{H}^{\mathbb{A}})$ such that L is a bounded lattice and $\mathcal{H}^{\mathbb{A}} = \{h^{\mathbb{A}} \mid h \in \mathcal{H}\}$ is a family of operations on L such that $h^{\mathbb{A}}$ takes n_h arguments, and $h^{\mathbb{A}}(a_1, \dots, a(\vee^{\varepsilon_h(i)})^{\alpha(h)} b, \dots, a_{n_h}) = h^{\mathbb{A}}(a_1, \dots, a, \dots, a_{n_h}) \vee^{\alpha_h} h^{\mathbb{A}}(a_1, \dots, b, \dots, a_{n_h})$. Moreover, if $h \in \mathcal{H}_n$, then we have that $h^{\mathbb{A}}(a_1, \dots, (\perp^{\varepsilon_h(i)})^{\alpha(h)}, \dots, a_{n_h}) = \perp^{\alpha_h}$. This clearly makes possible analogous compact presentations of the identities below as well as the axioms and rules in Definition 2.4.

In the remainder of the paper, we will abuse notation and write e.g. f for $f^{\mathbb{A}}$ when this causes no confusion. The class of all LEs is equational, and can be axiomatized by the usual lattice identities and the following equations for any $f \in \mathcal{F}$ (resp. $g \in \mathcal{G}$) and $1 \leq i \leq n_f$ (resp. for each $1 \leq j \leq n_g$):

- if $\varepsilon_f(i) = 1$, then $f(p_1, \dots, p \vee q, \dots, p_{n_f}) = f(p_1, \dots, p, \dots, p_{n_f}) \vee f(p_1, \dots, q, \dots, p_{n_f})$;
moreover if $f \in \mathcal{F}_n$, then $f(p_1, \dots, \perp, \dots, p_{n_f}) = \perp$,
- if $\varepsilon_f(i) = \partial$, then $f(p_1, \dots, p \wedge q, \dots, p_{n_f}) = f(p_1, \dots, p, \dots, p_{n_f}) \vee f(p_1, \dots, q, \dots, p_{n_f})$;
moreover if $f \in \mathcal{F}_n$, then $f(p_1, \dots, \top, \dots, p_{n_f}) = \perp$,
- if $\varepsilon_g(j) = 1$, then $g(p_1, \dots, p \wedge q, \dots, p_{n_g}) = g(p_1, \dots, p, \dots, p_{n_g}) \wedge g(p_1, \dots, q, \dots, p_{n_g})$;
moreover if $g \in \mathcal{G}_n$, then $g(p_1, \dots, \top, \dots, p_{n_g}) = \top$,
- if $\varepsilon_g(j) = \partial$, then $g(p_1, \dots, p \vee q, \dots, p_{n_g}) = g(p_1, \dots, p, \dots, p_{n_g}) \wedge g(p_1, \dots, q, \dots, p_{n_g})$;
moreover if $g \in \mathcal{G}_n$, then $g(p_1, \dots, \perp, \dots, p_{n_g}) = \top$.

Each language $\mathcal{L}_{\mathbb{LE}}$ is interpreted in the appropriate class of LEs. In particular, for every LE \mathbb{A} , each operation $f^{\mathbb{A}} \in \mathcal{F}_n^{\mathbb{A}}$ (resp. $g^{\mathbb{A}} \in \mathcal{G}_n^{\mathbb{A}}$) is finitely join-preserving (resp. meet-preserving) in each coordinate when regarded as a map $f^{\mathbb{A}} : \mathbb{A}^{\varepsilon_f} \rightarrow \mathbb{A}$ (resp. $g^{\mathbb{A}} : \mathbb{A}^{\varepsilon_g} \rightarrow \mathbb{A}$), and each operation $f^{\mathbb{A}} \in \mathcal{F}_r^{\mathbb{A}}$ (resp. $g^{\mathbb{A}} \in \mathcal{G}_r^{\mathbb{A}}$) preserves nonempty joins (resp. nonempty meets) in each coordinate when regarded as a map $f^{\mathbb{A}} : \mathbb{A}^{\varepsilon_f} \rightarrow \mathbb{A}$ (resp. $g^{\mathbb{A}} : \mathbb{A}^{\varepsilon_g} \rightarrow \mathbb{A}$).

Definition 2.4. For any language $\mathcal{L}_{\mathbb{LE}} = \mathcal{L}_{\mathbb{LE}}(\mathcal{F}, \mathcal{G})$, the *basic*, or *minimal* $\mathcal{L}_{\mathbb{LE}}$ -logic is a set of sequents $\varphi \vdash \psi$, with $\varphi, \psi \in \mathcal{L}_{\mathbb{LE}}$, which contains the following axioms:

- Sequents for lattice operations:

$$\begin{array}{lll} p \vdash p, & \perp \vdash p, & p \vdash \top, \\ p \vdash p \vee q, & q \vdash p \vee q, & p \wedge q \vdash p, \quad p \wedge q \vdash q, \end{array}$$

- Sequents for $f \in \mathcal{F}$ and $g \in \mathcal{G}$:

$$\begin{array}{l} f(p_1, \dots, p \vee q, \dots, p_{n_f}) \vdash f(p_1, \dots, p, \dots, p_{n_f}) \vee f(p_1, \dots, q, \dots, p_{n_f}), \text{ for } \varepsilon_f(i) = 1, \\ f(p_1, \dots, p \wedge q, \dots, p_{n_f}) \vdash f(p_1, \dots, p, \dots, p_{n_f}) \vee f(p_1, \dots, q, \dots, p_{n_f}), \text{ for } \varepsilon_f(i) = \partial, \\ g(p_1, \dots, p, \dots, p_{n_g}) \wedge g(p_1, \dots, q, \dots, p_{n_g}) \vdash g(p_1, \dots, p \wedge q, \dots, p_{n_g}), \text{ for } \varepsilon_g(i) = 1, \\ g(p_1, \dots, p, \dots, p_{n_g}) \wedge g(p_1, \dots, q, \dots, p_{n_g}) \vdash g(p_1, \dots, p \vee q, \dots, p_{n_g}), \text{ for } \varepsilon_g(i) = \partial, \end{array}$$

- Additional sequents for $f \in \mathcal{F}_n$ and $g \in \mathcal{G}_n$:

$$\begin{aligned} f(p_1, \dots, \perp, \dots, p_{n_f}) \vdash \perp, & \text{ for } \varepsilon_f(i) = 1, \\ f(p_1, \dots, \top, \dots, p_{n_f}) \vdash \perp, & \text{ for } \varepsilon_f(i) = \partial, \\ \top \vdash g(p_1, \dots, \top, \dots, p_{n_g}), & \text{ for } \varepsilon_g(i) = 1, \\ \top \vdash g(p_1, \dots, \perp, \dots, p_{n_g}), & \text{ for } \varepsilon_g(i) = \partial, \end{aligned}$$

and is closed under the following inference rules:

$$\begin{array}{c} \frac{\varphi \vdash \chi \quad \chi \vdash \psi}{\varphi \vdash \psi} \quad \frac{\varphi \vdash \psi}{\varphi(\chi/p) \vdash \psi(\chi/p)} \quad \frac{\chi \vdash \varphi \quad \chi \vdash \psi}{\chi \vdash \varphi \wedge \psi} \quad \frac{\varphi \vdash \chi \quad \psi \vdash \chi}{\varphi \vee \psi \vdash \chi} \\ \hline \frac{\varphi \vdash \psi}{f(p_1, \dots, \varphi, \dots, p_n) \vdash f(p_1, \dots, \psi, \dots, p_n)} (\varepsilon_f(i) = 1) \\ \frac{\varphi \vdash \psi}{f(p_1, \dots, \psi, \dots, p_n) \vdash f(p_1, \dots, \varphi, \dots, p_n)} (\varepsilon_f(i) = \partial) \\ \frac{\varphi \vdash \psi}{g(p_1, \dots, \varphi, \dots, p_n) \vdash g(p_1, \dots, \psi, \dots, p_n)} (\varepsilon_g(i) = 1) \\ \frac{\varphi \vdash \psi}{g(p_1, \dots, \psi, \dots, p_n) \vdash g(p_1, \dots, \varphi, \dots, p_n)} (\varepsilon_g(i) = \partial). \end{array}$$

The minimal LE-logic is denoted by \mathbf{L}_{LE} . For any LE-language \mathcal{L}_{LE} , by an LE-logic we understand any axiomatic extension of the basic \mathcal{L}_{LE} -logic in \mathcal{L}_{LE} .

For every LE \mathbb{A} , the symbol \vdash is interpreted as the lattice order \leq . A sequent $\varphi \vdash \psi$ is valid in \mathbb{A} if $h(\varphi) \leq h(\psi)$ for every homomorphism h from the \mathcal{L}_{LE} -algebra of formulas over \mathbf{PROP} to \mathbb{A} . The notation $\mathbb{LE} \models \varphi \vdash \psi$ indicates that $\varphi \vdash \psi$ is valid in every LE. Then, by means of a routine Lindenbaum-Tarski construction, it can be shown that the minimal LE-logic \mathbf{L}_{LE} is sound and complete with respect to its correspondent class of algebras \mathbb{LE} , i.e. that any sequent $\varphi \vdash \psi$ is provable in \mathbf{L}_{LE} iff $\mathbb{LE} \models \varphi \vdash \psi$.

Definition 2.5. For every \mathcal{L}_{LE} -algebra \mathbb{A} and all $f \in \mathcal{F}_r$ and $g \in \mathcal{G}_r$, the *normalizations* of $f^{\mathbb{A}}$ and $g^{\mathbb{A}}$ are the operations defined as follows: if $\varepsilon_f = \varepsilon_g = 1$,

$$\diamond_f(u) := \begin{cases} f(u) & \text{if } u \neq \perp \\ \perp & \text{if } u = \perp \end{cases} \quad \square_g(u) := \begin{cases} g(u) & \text{if } u \neq \top \\ \top & \text{if } u = \top \end{cases}$$

if $\varepsilon_f = \varepsilon_g = \partial$,

$$\triangleleft_f(u) := \begin{cases} f(u) & \text{if } u \neq \top \\ \perp & \text{if } u = \top \end{cases} \quad \triangleright_g(u) := \begin{cases} g(u) & \text{if } u \neq \perp \\ \top & \text{if } u = \perp \end{cases}$$

Lemma 2.6. For every \mathcal{L}_{LE} -algebra \mathbb{A} and all $f \in \mathcal{F}_r$ and $g \in \mathcal{G}_r$,

- (1) if $\varepsilon_f = 1$ then \diamond_f preserves finite (hence also empty) joins and $f(u) = f(\perp) \vee \diamond_f u$ for every $u \in \mathbb{A}$;
- (2) if $\varepsilon_g = 1$ then \square_g preserves finite (hence also empty) meets and $g(u) = g(\top) \wedge \square_g u$ for every $u \in \mathbb{A}$;
- (3) if $\varepsilon_f = \partial$ then \triangleleft_f reverses finite (hence also empty) meets and $f(u) = f(\top) \vee \triangleleft_f u$ for every $u \in \mathbb{A}$;
- (4) if $\varepsilon_g = \partial$ then \triangleright_g reverses finite (hence also empty) joins and $g(u) = g(\perp) \wedge \triangleright_g u$ for every $u \in \mathbb{A}$.

Proof. 1. If $u \vee v = \perp$, then $u = \perp = v$, and the claim immediately follows by definition of \diamond_f . If $u \vee v \neq \perp$, then $\diamond_f(u \vee v) = f(u \vee v) = f(u) \vee f(v)$. If $u \neq \perp \neq v$ then the claim immediately follows by definition of \diamond_f . If $u = \perp$ and $v \neq \perp$, then $u \leq v$ and the claim follows by definition of \diamond_f and the monotonicity of f . Analogously if $u \neq \perp$ and $v = \perp$. The second part of the claim immediately follows from the definition of \diamond_f . The remaining items are order-variants and their proof is omitted. \square

Remark 2.7. Using the formulation adumbrated in Remarks 2.1 and 2.3, Definition 2.5 can be equivalently given by specifying that, for all regular connectives $h \in \mathcal{H}_r$, the normalization of $h^{\mathbb{A}}$ is the operation \heartsuit_h given by

$$\heartsuit_h(u) = \begin{cases} h(u) & u \neq (\perp^{\alpha_h})^{\varepsilon_h(1)} \\ \perp^{\alpha_h} & u = (\perp^{\alpha_h})^{\varepsilon_h(1)} \end{cases}$$

Using this formulation, Lemma 2.6 can be given by stating that, for all finite, possibly empty subsets S of elements of an LE-algebra \mathbb{A} , $\heartsuit_h((\bigvee_{s \in S}^{\alpha_h})^{\varepsilon_h(1)} S) = \bigvee_{s \in S}^{\alpha_h} \heartsuit_h(s)$, and moreover $h(u) = h((\perp^{\alpha_h})^{\varepsilon_h(1)} \vee^{\alpha_h} \heartsuit_h(u))$.

2.3. The expanded language $\mathcal{L}_{\text{LE}}^*$. We now introduce an expansion of LE-languages, which adds connectives intended to be interpreted as the normalized counterparts of all regular connectives, as well as connectives to be interpreted as the left and right residuals of interpretations of all normal connectives, including the introduced normalized counterparts. Formally, any given language $\mathcal{L}_{\text{LE}} = \mathcal{L}_{\text{LE}}(\mathcal{F}, \mathcal{G})$ can be associated with the language $\mathcal{L}_{\text{LE}}^* = \mathcal{L}_{\text{LE}}(\mathcal{F}^*, \mathcal{G}^*)$, where $\mathcal{F}^* \supseteq \mathcal{F}$ and $\mathcal{G}^* \supseteq \mathcal{G}$ are obtained by expanding \mathcal{L}_{LE} in two steps as follows: as to the first step, let $\mathcal{F}' \supseteq \mathcal{F}_n$ and $\mathcal{G}' \supseteq \mathcal{G}_n$ be obtained by adding:

- (1) for each $f \in \mathcal{F}_r$ s.t. $\varepsilon_f = 1$, the unary connective \diamond_f with $\varepsilon_{\diamond_f} = 1$, the intended interpretation of which is the normalization of f (cf. Definition 2.5);
- (2) for each $f \in \mathcal{F}_r$ s.t. $\varepsilon_f = \partial$, the unary connective \triangleleft_f with $\varepsilon_{\triangleleft_f} = \partial$, the intended interpretation of which is the normalization of f (cf. Definition 2.5);
- (3) for each $g \in \mathcal{G}_r$ s.t. $\varepsilon_g = 1$, the unary connective \square_g with $\varepsilon_{\square_g} = 1$, the intended interpretation of which is the normalization of g (cf. Definition 2.5);
- (4) for each $g \in \mathcal{G}_r$ s.t. $\varepsilon_g = \partial$, the unary connective \triangleright_g with $\varepsilon_{\triangleright_g} = \partial$, the intended interpretation of which is the normalization of g (cf. Definition 2.5).

As to the second step, let $\mathcal{F}_n^* \supseteq \mathcal{F}'$ and $\mathcal{G}_n^* \supseteq \mathcal{G}'$ be obtained by adding:

- (1) for $f \in \mathcal{F}'$ and $0 \leq i \leq n_f$, the n_f -ary connective f_i^\sharp , the intended interpretation of which is the right residual of $f \in \mathcal{F}_n$ in its i th coordinate if $\varepsilon_f(i) = 1$ (resp. its Galois-adjoint if $\varepsilon_f(i) = \partial$);
- (2) for $g \in \mathcal{G}'$ and $0 \leq i \leq n_g$, the n_g -ary connective g_i^b , the intended interpretation of which is the left residual of $g \in \mathcal{G}_n$ in its i th coordinate if $\varepsilon_g(i) = 1$ (resp. its Galois-adjoint if $\varepsilon_g(i) = \partial$).³

³We reserve the symbols \diamond and \triangleleft to denote unary connectives in \mathcal{F}_n such that $\varepsilon_\diamond = 1$ and $\varepsilon_{\triangleleft} = \partial$ and \square and \triangleright to denote unary connectives in \mathcal{G}_n such that $\varepsilon_\square = 1$ and $\varepsilon_{\triangleright} = \partial$. The adjoints of \square , \diamond , \triangleleft and \triangleright are denoted \blacklozenge , \blacksquare , \blacktriangleleft_f and \blacktriangleright_g , respectively. For every $f \in \mathcal{F}_r$ and $g \in \mathcal{G}_r$, we let \blacksquare_f and \blacklozenge_g denote the right and left adjoint of \diamond_f and \square_g respectively if $\varepsilon_f = \varepsilon_g = 1$, and \blacktriangleleft_f and \blacktriangleright_g denote the Galois-adjoints of \triangleleft_f and \triangleright_g if $\varepsilon_f = \varepsilon_g = \partial$.

We stipulate that $f_i^\sharp \in \mathcal{G}_n^*$ if $\varepsilon_f(i) = 1$, and $f_i^\sharp \in \mathcal{F}_n^*$ if $\varepsilon_f(i) = \partial$. Dually, $g_i^b \in \mathcal{F}_n^*$ if $\varepsilon_g(i) = 1$, and $g_i^b \in \mathcal{G}_n^*$ if $\varepsilon_g(i) = \partial$. The order-type assigned to the additional connectives is predicated on the order-type of their intended interpretations. That is, for any $f \in \mathcal{F}'$ and $g \in \mathcal{G}'$,

- (1) if $\varepsilon_f(i) = 1$, then $\varepsilon_{f_i^\sharp}(i) = 1$ and $\varepsilon_{f_i^\sharp}(j) = (\varepsilon_f(j))^\partial$ for any $j \neq i$.
- (2) if $\varepsilon_f(i) = \partial$, then $\varepsilon_{f_i^\sharp}(i) = \partial$ and $\varepsilon_{f_i^\sharp}(j) = \varepsilon_f(j)$ for any $j \neq i$.
- (3) if $\varepsilon_g(i) = 1$, then $\varepsilon_{g_i^b}(i) = 1$ and $\varepsilon_{g_i^b}(j) = (\varepsilon_g(j))^\partial$ for any $j \neq i$.
- (4) if $\varepsilon_g(i) = \partial$, then $\varepsilon_{g_i^b}(i) = \partial$ and $\varepsilon_{g_i^b}(j) = \varepsilon_g(j)$ for any $j \neq i$.

Finally,⁴ let $\mathcal{F}^* := \mathcal{F}_n^* \uplus \mathcal{F}_r$ and $\mathcal{G}^* := \mathcal{G}_n^* \uplus \mathcal{G}_r$.

Remark 2.8. Following Remarks 2.1, 2.3 and 2.7, the language \mathcal{L}_{LE} can alternatively be defined as follows: let \mathcal{H}' be obtained from \mathcal{H} by adding \heartsuit_h for each $h \in \mathcal{H}_r$, with intended interpretation as given in Remark 2.7. Next, obtain \mathcal{H}^* by adding: (1) for each $h \in \mathcal{H}'$ with $\alpha_h = 1$, the n_h -ary connectives h_i^\sharp for $1 \leq i \leq n_h$ where $\alpha_{h_i^\sharp} = (\alpha_h^\partial)^{\varepsilon_h(i)}$ and $\varepsilon_{h_i^\sharp}(i) = \varepsilon_h(i)$ while $\varepsilon_{h_i^\sharp}(j) = ((\varepsilon_h(j))^\partial)^{\varepsilon_h(i)}$ for $j \neq i$; (2) for each $h \in \mathcal{H}'$ with $\alpha_h = \partial$, the n_h -ary connectives h_i^b for $1 \leq i \leq n_h$ where $\alpha_{h_i^b} = (\alpha_h^\partial)^{\varepsilon_h(i)}$ and $\varepsilon_{h_i^b}(i) = \varepsilon_h(i)$ while $\varepsilon_{h_i^b}(j) = ((\varepsilon_h(j))^\partial)^{\varepsilon_h(i)}$ for $j \neq i$.

Definition 2.9. For any language $\mathcal{L}_{LE}(\mathcal{F}, \mathcal{G})$, the *basic expanded \mathcal{L}_{LE} -logic* is defined by specializing Definition 2.4 to the language $\mathcal{L}_{LE}^* = \mathcal{L}_{LE}(\mathcal{F}^*, \mathcal{G}^*)$ and closing under the following residuation rules for each $f \in \mathcal{F}'$ and $g \in \mathcal{G}'$:

$$\begin{aligned} (\varepsilon_f(i) = 1) & \frac{f(\varphi_1, \dots, \varphi, \dots, \varphi_{n_f}) \vdash \psi}{\varphi \vdash f_i^\sharp(\varphi_1, \dots, \psi, \dots, \varphi_{n_f})} & \frac{\varphi \vdash g(\varphi_1, \dots, \psi, \dots, \varphi_{n_g})}{g_i^b(\varphi_1, \dots, \varphi, \dots, \varphi_{n_g}) \vdash \psi} & (\varepsilon_g(i) = 1) \\ (\varepsilon_f(i) = \partial) & \frac{f(\varphi_1, \dots, \varphi, \dots, \varphi_{n_f}) \vdash \psi}{f_i^\sharp(\varphi_1, \dots, \psi, \dots, \varphi_{n_f}) \vdash \varphi} & \frac{\varphi \vdash g(\varphi_1, \dots, \psi, \dots, \varphi_{n_g})}{\psi \vdash g_i^b(\varphi_1, \dots, \varphi, \dots, \varphi_{n_g})} & (\varepsilon_g(i) = \partial) \end{aligned}$$

The double line in each rule above indicates that the rule should be read both top-to-bottom and bottom-to-top. Let \mathbf{L}_{LE}^* be the minimal expanded \mathcal{L}_{LE} -logic. For any language \mathcal{L}_{LE} , by a *expanded \mathcal{L}_{LE} -logic* we understand any axiomatic extension of the basic expanded \mathcal{L}_{LE} -logic in \mathcal{L}_{LE}^* .

The algebraic semantics of \mathbf{L}_{LE}^* is given by the class of \mathcal{L}_{LE} -algebras defined as tuples $\mathbb{A} = (L, \mathcal{F}^*, \mathcal{G}^*)$ such that L is a lattice, and moreover,

- (1) for every $f \in \mathcal{F}'$ s.t. $n_f \geq 1$, all $a_1, \dots, a_{n_f} \in L$ and $b \in L$, and each $1 \leq i \leq n_f$,
 - if $\varepsilon_f(i) = 1$, then $f(a_1, \dots, a_i, \dots, a_{n_f}) \leq b$ iff $a_i \leq f_i^\sharp(a_1, \dots, b, \dots, a_{n_f})$;
 - if $\varepsilon_f(i) = \partial$, then $f(a_1, \dots, a_i, \dots, a_{n_f}) \leq b$ iff $a_i \leq^\partial f_i^\sharp(a_1, \dots, b, \dots, a_{n_f})$.
- (2) for every $g \in \mathcal{G}'$ s.t. $n_g \geq 1$, any $a_1, \dots, a_{n_g} \in L$ and $b \in L$, and each $1 \leq i \leq n_g$,
 - if $\varepsilon_g(i) = 1$, then $b \leq g(a_1, \dots, a_i, \dots, a_{n_g})$ iff $g_i^b(a_1, \dots, b, \dots, a_{n_g}) \leq a_i$.

⁴ For instance, if f and g are binary connectives such that $\varepsilon_f = (1, \partial)$ and $\varepsilon_g = (\partial, 1)$, then $\varepsilon_{f_1^\sharp} = (1, 1)$, $\varepsilon_{f_2^\sharp} = (1, \partial)$, $\varepsilon_{g_1^b} = (\partial, 1)$ and $\varepsilon_{g_2^b} = (1, 1)$. Warning: notice that this notation heavily depends from the connective which is taken as primitive, and needs to be carefully adapted to well known cases. For instance, consider the ‘fusion’ connective \circ (which, when denoted as f , is such that $\varepsilon_f = (1, 1)$). Its residuals f_1^\sharp and f_2^\sharp are commonly denoted $/$ and \backslash respectively. However, if \backslash is taken as the primitive connective g , then g_2^b is $\circ = f$, and $g_1^b(x_1, x_2) := x_2/x_1 = f_1^\sharp(x_2, x_1)$. This example shows that, when identifying g_1^b and f_1^\sharp , the conventional order of the coordinates is not preserved, and depends of which connective is taken as primitive.

- if $\varepsilon_g(i) = \partial$, then $b \leq g(a_1, \dots, a_i, \dots, a_{n_g})$ iff $g_i^b(a_1, \dots, b, \dots, a_{n_g}) \leq^\partial a_i$.

It is also routine to prove using the Lindenbaum-Tarski construction that \mathbf{L}_{LE}^* (as well as any of its sound axiomatic extensions) is sound and complete w.r.t. the class of expanded \mathcal{L}_{LE} -algebras (w.r.t. the suitably defined equational subclass, respectively).

2.4. Canonical extensions, constructively. Canonical extensions provide a purely algebraic encoding of Stone-type dualities, and indeed, the existence of the canonical extensions of the best-known varieties of LEs can be proven via preexisting dualities. However, alternative, purely algebraic constructions are available, such as those of [29, 25]. These constructions are in fact more general, in that their definition does not rely on principles such as Zorn's lemma. In what follows we will adapt them to the setting of LEs introduced above.

Definition 2.10. Let \mathbb{A} be a (bounded) sublattice of a complete lattice \mathbb{A}' .

- (1) \mathbb{A} is *dense* in \mathbb{A}' if every element of \mathbb{A}' can be expressed both as a join of meets and as a meet of joins of elements from \mathbb{A} .
- (2) \mathbb{A} is *compact* in \mathbb{A}' if, for all $S, T \subseteq \mathbb{A}$, if $\bigwedge S \leq \bigvee T$ then $\bigwedge S' \leq \bigvee T'$ for some finite $S' \subseteq S$ and $T' \subseteq T$.
- (3) The *canonical extension* of a lattice \mathbb{A} is a complete lattice \mathbb{A}^δ containing \mathbb{A} as a dense and compact sublattice.

Let $K(\mathbb{A}^\delta)$ and $O(\mathbb{A}^\delta)$ denote the meet-closure and the join-closure of \mathbb{A} in \mathbb{A}^δ respectively. The elements of $K(\mathbb{A}^\delta)$ are referred to as *closed* elements, and elements of $O(\mathbb{A}^\delta)$ as *open* elements.

Theorem 2.11 (Propositions 2.6 and 2.7 in [29]). *The canonical extension of a bounded lattice \mathbb{A} exists and is unique up to any isomorphism fixing \mathbb{A} .*

Proof. We expand on the existence, since it is relevant to the present paper. Let \mathbf{I} and \mathbf{F} be the collections of the ideals and filters of \mathbb{A} respectively. Consider the polarity $(\mathbf{F}, \mathbf{I}, \leq)$, where $F \leq I$ iff $F \cap I \neq \emptyset$ for every $F \in \mathbf{F}$ and $I \in \mathbf{I}$. As is well known (cf. [22]), this polarity induces a Galois connection $(u : \mathcal{P}(\mathbf{F}) \rightarrow \mathcal{P}(\mathbf{I}), \ell : \mathcal{P}(\mathbf{I}) \rightarrow \mathcal{P}(\mathbf{F}))$, with u and ℓ defined by the assignments $X \mapsto \{I \mid F \leq I \text{ for all } F \in X\}$, and $\mathcal{Y} \mapsto \{F \mid F \leq I \text{ for all } I \in \mathcal{Y}\}$, respectively. Hence the maps $\ell \circ u$ and $u \circ \ell$ are closure operators on $\mathcal{P}(\mathbf{F})$ and $\mathcal{P}(\mathbf{I})$ respectively. The collections of Galois-stable sets of $\ell \circ u$ and $u \circ \ell$ form complete \bigcap -subsemilattices \mathcal{G}_F and \mathcal{G}_I of $\mathcal{P}(\mathbf{F})$ and $\mathcal{P}(\mathbf{I})$ respectively. These semilattices are then complete lattices, and are dually order-isomorphic to each other via the appropriate restrictions of u and ℓ . The maps $\mathbb{A} \rightarrow \mathcal{G}_F$ and $\mathbb{A} \rightarrow \mathcal{G}_I$ defined by the assignments $a \mapsto \ell \circ u(a\uparrow)$ and $a \mapsto u \circ \ell(a\downarrow)$ are dense and compact order-embeddings of \mathbb{A} . \square

In meta-theoretic settings in which Zorn's lemma is available, the fact that \mathbf{F} and \mathbf{I} are closed under taking unions of \subseteq -chains guarantees that the canonical extension of a lattice \mathbb{A} is a *perfect* lattice. That is, in addition to being complete, is both completely join-generated by the set $J^\infty(\mathbb{A}^\delta)$ of the completely join-irreducible elements of \mathbb{A}^δ , and completely meet-generated by the set $M^\infty(\mathbb{A}^\delta)$ of the completely meet-irreducible elements of \mathbb{A}^δ . In our present, constructive setting, canonical extensions are not perfect in general, since in general they do not have 'enough' join-irreducibles and meet-irreducibles, as specified above.

The canonical extension of an LE \mathbb{A} will be defined as a suitable expansion of the canonical extension of the underlying lattice of \mathbb{A} . Before turning to this definition, recall

that taking the canonical extension of a lattice commutes with taking order duals and products, namely: $(\mathbb{A}^\partial)^\delta = (\mathbb{A}^\delta)^\partial$ and $(\mathbb{A}_1 \times \mathbb{A}_2)^\delta = \mathbb{A}_1^\delta \times \mathbb{A}_2^\delta$ (cf. [25, Theorem 2.8]). Hence, $(\mathbb{A}^\partial)^\delta$ can be identified with $(\mathbb{A}^\delta)^\partial$, $(\mathbb{A}^n)^\delta$ with $(\mathbb{A}^\delta)^n$, and $(\mathbb{A}^\varepsilon)^\delta$ with $(\mathbb{A}^\delta)^\varepsilon$ for any order type ε . Thanks to these identifications, in order to extend operations of any arity and which are monotone or antitone in each coordinate from a lattice \mathbb{A} to its canonical extension, treating the case of *monotone* and *unary* operations suffices:

Definition 2.12. For every unary, order-preserving operation $f : \mathbb{A} \rightarrow \mathbb{A}$, the σ -extension of f is defined firstly by declaring, for every $k \in K(\mathbb{A}^\delta)$,

$$f^\sigma(k) := \bigwedge \{f(a) \mid a \in \mathbb{A} \text{ and } k \leq a\},$$

and then, for every $u \in \mathbb{A}^\delta$,

$$f^\sigma(u) := \bigvee \{f^\sigma(k) \mid k \in K(\mathbb{A}^\delta) \text{ and } k \leq u\}.$$

The π -extension of f is defined firstly by declaring, for every $o \in O(\mathbb{A}^\delta)$,

$$f^\pi(o) := \bigvee \{f(a) \mid a \in \mathbb{A} \text{ and } a \leq o\},$$

and then, for every $u \in \mathbb{A}^\delta$,

$$f^\pi(u) := \bigwedge \{f^\pi(o) \mid o \in O(\mathbb{A}^\delta) \text{ and } u \leq o\}.$$

Note that the operations σ and π are order dual. It is easy to see that the σ - and π -extensions of ε -monotone maps are ε -monotone. More remarkably, the σ -extension of a map which sends finite (resp. finite nonempty) joins or meets in the domain to finite (resp. finite nonempty) joins in the codomain sends *arbitrary* (resp. *arbitrary nonempty*) joins or meets in the domain to *arbitrary* (resp. *arbitrary nonempty*) joins in the codomain. Dually, the π -extension of a map which sends finite (resp. finite nonempty) joins or meets in the domain to finite (resp. finite nonempty) meets in the codomain sends *arbitrary* (resp. *arbitrary nonempty*) joins or meets in the domain to *arbitrary* (resp. *arbitrary nonempty*) meets in the codomain (cf. [29, Lemma 4.6]; notice that the proof given there holds in a constructive meta-theory). Therefore, depending on the properties of the original operation, it is more convenient to use one or the other extension. This justifies the following

Definition 2.13. The canonical extension of an \mathcal{L}_{LE} -algebra $\mathbb{A} = (L, \mathcal{F}^{\mathbb{A}}, \mathcal{G}^{\mathbb{A}})$ is the \mathcal{L}_{LE} -algebra $\mathbb{A}^\delta := (L^\delta, \mathcal{F}^{\mathbb{A}^\delta}, \mathcal{G}^{\mathbb{A}^\delta})$ such that $f^{\mathbb{A}^\delta}$ and $g^{\mathbb{A}^\delta}$ are defined as the σ -extension of $f^{\mathbb{A}}$ and as the π -extension of $g^{\mathbb{A}}$ respectively, for all $f \in \mathcal{F}$ and $g \in \mathcal{G}$.

The canonical extension of an LE \mathbb{A} is a *quasi-perfect* LE:

Definition 2.14. An LE $\mathbb{A} = (L, \mathcal{F}^{\mathbb{A}}, \mathcal{G}^{\mathbb{A}})$ is quasi-perfect if L is a complete lattice, the following infinitary distribution laws are satisfied for each $f \in \mathcal{F}_n$, $g \in \mathcal{G}_n$, $1 \leq i \leq n_f$ and $1 \leq j \leq n_g$: for every $S \subseteq L$,

$$\begin{aligned} f(x_1, \dots, \bigvee S, \dots, x_{n_f}) &= \bigvee \{f(x_1, \dots, x, \dots, x_{n_f}) \mid x \in S\} & \text{if } \varepsilon_f(i) = 1 \\ f(x_1, \dots, \bigwedge S, \dots, x_{n_f}) &= \bigvee \{f(x_1, \dots, x, \dots, x_{n_f}) \mid x \in S\} & \text{if } \varepsilon_f(i) = \partial \\ g(x_1, \dots, \bigwedge S, \dots, x_{n_g}) &= \bigwedge \{g(x_1, \dots, x, \dots, x_{n_g}) \mid x \in S\} & \text{if } \varepsilon_g(i) = 1 \\ f(x_1, \dots, \bigvee S, \dots, x_{n_f}) &= \bigwedge \{g(x_1, \dots, x, \dots, x_{n_f}) \mid x \in S\} & \text{if } \varepsilon_g(i) = \partial, \end{aligned}$$

and analogous identities hold for every $f \in \mathcal{F}_r$ and $g \in \mathcal{F}_r$, restricted to $S \neq \emptyset$.

Before finishing the present subsection, let us spell out and further simplify the definitions of the extended operations. First of all, we recall that taking order-duals interchanges closed and open elements: $K((\mathbb{A}^\delta)^\partial) = O(\mathbb{A}^\delta)$ and $O((\mathbb{A}^\delta)^\partial) = K(\mathbb{A}^\delta)$; similarly, $K((\mathbb{A}^n)^\delta) = K(\mathbb{A}^\delta)^n$, and $O((\mathbb{A}^n)^\delta) = O(\mathbb{A}^\delta)^n$. Hence, $K((\mathbb{A}^\delta)^\varepsilon) = \prod_i K(\mathbb{A}^\delta)^{\varepsilon(i)}$ and $O((\mathbb{A}^\delta)^\varepsilon) = \prod_i O(\mathbb{A}^\delta)^{\varepsilon(i)}$ for every LE \mathbb{A} and every order-type ε on any $n \in \mathbb{N}$, where

$$K(\mathbb{A}^\delta)^{\varepsilon(i)} := \begin{cases} K(\mathbb{A}^\delta) & \text{if } \varepsilon(i) = 1 \\ O(\mathbb{A}^\delta) & \text{if } \varepsilon(i) = \partial \end{cases} \quad O(\mathbb{A}^\delta)^{\varepsilon(i)} := \begin{cases} O(\mathbb{A}^\delta) & \text{if } \varepsilon(i) = 1 \\ K(\mathbb{A}^\delta) & \text{if } \varepsilon(i) = \partial. \end{cases}$$

Denoting by \leq^ε the product order on $(\mathbb{A}^\delta)^\varepsilon$, we have for every $f \in \mathcal{F}$, $g \in \mathcal{G}$, $\bar{u} \in (\mathbb{A}^\delta)^{n_f}$ and $\bar{v} \in (\mathbb{A}^\delta)^{n_g}$,

$$\begin{aligned} f^\sigma(\bar{k}) &:= \bigwedge \{f(\bar{a}) \mid \bar{a} \in (\mathbb{A}^\delta)^{\varepsilon_f} \text{ and } \bar{k} \leq^{\varepsilon_f} \bar{a}\} \\ f^\sigma(\bar{u}) &:= \bigvee \{f^\sigma(\bar{k}) \mid \bar{k} \in K((\mathbb{A}^\delta)^{\varepsilon_f}) \text{ and } \bar{k} \leq^{\varepsilon_f} \bar{u}\} \\ g^\pi(\bar{o}) &:= \bigvee \{g(\bar{a}) \mid \bar{a} \in (\mathbb{A}^\delta)^{\varepsilon_g} \text{ and } \bar{a} \leq^{\varepsilon_g} \bar{o}\} \\ g^\pi(\bar{v}) &:= \bigwedge \{g^\pi(\bar{o}) \mid \bar{o} \in O((\mathbb{A}^\delta)^{\varepsilon_g}) \text{ and } \bar{v} \leq^{\varepsilon_g} \bar{o}\} \end{aligned}$$

Notice that the algebraic completeness of the logics \mathbf{L}_{LE} and \mathbf{L}_{LE}^* and the canonical embedding of LEs into their canonical extensions immediately give completeness of \mathbf{L}_{LE} and \mathbf{L}_{LE}^* w.r.t. the appropriate class of perfect LEs.

2.5. Constructive canonical extensions are natural \mathcal{L}_{LE}^* -algebras. The aim of the present subsection is showing that the constructive canonical extension of any \mathcal{L}_{LE} -algebra \mathbb{A} supports the interpretation of the language \mathcal{L}_{LE}^* (cf. Section 2.3). This will be done in two steps: Firstly, we need to verify that taking the normalization of any $f \in \mathcal{F}_r^{\mathbb{A}^\delta}$ and $g \in \mathcal{G}_r^{\mathbb{A}^\delta}$ commutes with taking canonical extensions:

Lemma 2.15. *For all $f \in \mathcal{F}_r$ and $g \in \mathcal{G}_r$,*

- (1) *if $\varepsilon_f = 1$, then $\diamond_f^\sigma u = \diamond_{f^\sigma} u$ for every $u \in \mathbb{A}^\delta$.*
- (2) *if $\varepsilon_g = 1$, then $\square_g^\pi u = \square_{g^\pi} u$ for every $u \in \mathbb{A}^\delta$.*
- (3) *if $\varepsilon_f = \partial$, then $\triangleleft_f^\sigma u = \triangleleft_{f^\sigma} u$ for every $u \in \mathbb{A}^\delta$.*
- (4) *if $\varepsilon_g = \partial$, then $\triangleright_g^\pi u = \triangleright_{g^\pi} u$ for every $u \in \mathbb{A}^\delta$.*

Proof. 1. By nonempty join-preservation, it is enough to show that if $k \in K(\mathbb{A}^\delta)$ and $k \neq \perp$, then $\diamond_f^\sigma k = f^\sigma(k) =: \diamond_{f^\sigma} u$. By denseness, $\{k \mid k \in K(\mathbb{A}^\delta) \text{ and } k \leq u\} \neq \emptyset$. Recalling that in \mathbb{A}^δ the interpretation of any $f \in \mathcal{F}_r$ with $\varepsilon_f = 1$ preserves arbitrary nonempty joins, the following chain of identities holds:

$$\begin{aligned} \diamond_f^\sigma k &= \bigwedge \{\diamond_f a \mid a \in \mathbb{A} \text{ and } k \leq a\} \quad (\text{def. of } \sigma\text{-extension}) \\ &= \bigwedge \{f(a) \mid a \in \mathbb{A} \text{ and } k \leq a\} \quad (k \neq \perp) \\ &= f^\sigma(k). \quad (\text{def. of } \sigma\text{-extension}) \end{aligned}$$

The remaining items are order-variants and hence their proof is omitted. \square

Since \mathbb{A}^δ is a complete lattice, by general and well known order-theoretic facts, all the connectives in $\mathcal{F}' \supseteq \mathcal{F}_n$ and in $\mathcal{G}' \supseteq \mathcal{G}_n$ (cf. Subsection 2.3), have (coordinatewise) *adjoints*. This implies that the constructive canonical extension of any \mathcal{L}_{LE} -algebra supports the

interpretation of the connectives in \mathcal{F}^* and in \mathcal{G}^* (cf. Subsection 2.3), and can hence be endowed with the structure of an \mathcal{L}_{LE}^* -algebra, as required.

2.6. The language of constructive ALBA for LEs. The expanded language manipulated by ALBA includes the \mathcal{L}_{LE}^* -connectives, as well as a denumerably infinite set of sorted variables NOM called *nominals*, and a denumerably infinite set of sorted variables CO-NOM, called *co-nominals*. The elements of NOM will be denoted with \mathbf{i}, \mathbf{j} , possibly indexed, and those of CO-NOM with \mathbf{m}, \mathbf{n} , possibly indexed. While in the non-constructive setting nominals and co-nominals range over the completely join-irreducible and the completely meet-irreducible elements of perfect LEs, respectively, in the present, constructive setting, nominals and co-nominals will be interpreted as elements of $K(\mathbb{A}^\delta)$ and $O(\mathbb{A}^\delta)$, respectively.

Let us introduce the expanded language formally: the *formulas* φ of \mathcal{L}_{LE}^+ are given by the following recursive definition:

$$\varphi ::= \mathbf{j} \mid \mathbf{m} \mid \psi \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid f(\bar{\varphi}) \mid g(\bar{\varphi})$$

with $\psi \in \mathcal{L}_{LE}$, $\mathbf{j} \in \text{NOM}$ and $\mathbf{m} \in \text{CO-NOM}$, $f \in \mathcal{F}^*$ and $g \in \mathcal{G}^*$.

As in the case of \mathcal{L}_{LE} , we can form inequalities and quasi-inequalities based on \mathcal{L}_{LE}^+ . Formulas, inequalities and quasi-inequalities in \mathcal{L}_{LE}^+ not containing any propositional variables (but possibly containing nominals and co-nominals) will be called *pure*.

In the previous section, we showed that constructive canonical extensions of \mathcal{L}_{LE} -algebras can be naturally endowed with the structure of \mathcal{L}_{LE}^* -algebras. Building on this fact, we can use constructive canonical extensions of \mathcal{L}_{LE} -algebras as a semantic environment for the language \mathcal{L}_{LE}^+ as follows. If \mathbb{A} is an \mathcal{L}_{LE} -algebra, then an *assignment* for \mathcal{L}_{LE}^+ on \mathbb{A}^δ is a map $v : \text{PROP} \cup \text{NOM} \cup \text{CO-NOM} \rightarrow \mathbb{A}^\delta$ sending propositional variables to elements of \mathbb{A}^δ , sending nominals to $K(\mathbb{A}^\delta)$ and co-nominals to $O(\mathbb{A}^\delta)$. An *admissible assignment* for \mathcal{L}_{LE}^+ on \mathbb{A}^δ is an assignment v for \mathcal{L}_{LE}^+ on \mathbb{A}^δ , such that $v(p) \in \mathbb{A}$ for each $p \in \text{PROP}$. In other words, the assignment v sends propositional variables to elements of the subalgebra \mathbb{A} , while nominals and co-nominals get sent to closed and open elements of \mathbb{A}^δ , respectively. This means that the value of \mathcal{L}_{LE} -terms under an admissible assignment will belong to \mathbb{A} , whereas \mathcal{L}_{LE}^+ -terms in general will not.

3. INDUCTIVE AND SAHLQVIST INEQUALITIES

In this section we introduce the notion of inductive $\mathcal{L}_{LE}(\mathcal{F}, \mathcal{G})$ -inequalities, which, as discussed in the introduction, is one of the main tools of unified correspondence. Its characteristic feature is being parametric in every signature, and being formulated purely in terms of the order-theoretic properties of the interpretation of the logical connectives. However, in the present setting, unlike most of the others, the parametric signature witnesses the coexistence of normal and regular connectives, and hence we warn the reader that Definition 3.3 below, although verbatim the same as the analogous one given in e.g. [14], is actually a proper generalization of e.g. [14, Definition 3.4], since the definition of good branch on which it depends is not the same as the one given in [14]. We deliberately keep the same terminology (good branch, inductive) and reduce the notational changes to a minimum, since the conceptual principles motivating these notions are the same in every setting, and the present definitions extend the ones relative to all more restrictive settings and project onto each of them. The constructive canonicity of inductive inequalities will follow from

their ALBA-reducibility and the fact that all ALBA-reducible inequalities are constructively canonical.

Definition 3.1 (Signed Generation Tree). The *positive* (resp. *negative*) *generation tree* of any \mathcal{L}_{LE} -term s is defined by labelling the root node of the generation tree of s with the sign $+$ (resp. $-$), and then propagating the labelling on each remaining node as follows:

- For any node labelled with \vee or \wedge , assign the same sign to its children nodes.
- For any node labelled with $h \in \mathcal{F} \cup \mathcal{G}$ of arity $n_h \geq 1$, and for any $1 \leq i \leq n_h$, assign the same (resp. the opposite) sign to its i th child node if $\varepsilon_h(i) = 1$ (resp. if $\varepsilon_h(i) = \partial$).

Nodes in signed generation trees are *positive* (resp. *negative*) if are signed $+$ (resp. $-$).

Signed generation trees will be mostly used in the context of term inequalities $s \leq t$. In this context we will typically consider the positive generation tree $+s$ for the left-hand side and the negative one $-t$ for the right-hand side. We will also say that a term-inequality $s \leq t$ is *uniform* in a given variable p if all occurrences of p in both $+s$ and $-t$ have the same sign, and that $s \leq t$ is ε -*uniform* in a (sub)array \vec{p} of its variables if $s \leq t$ is uniform in p , occurring with the sign indicated by ε , for every⁵ p in \vec{p} .

For any term $s(p_1, \dots, p_n)$, any order type ε over n , and any $1 \leq i \leq n$, an ε -*critical node* in a signed generation tree of s is a leaf node $+p_i$ with $\varepsilon_i = 1$ or $-p_i$ with $\varepsilon_i = \partial$. An ε -*critical branch* in the tree is a branch from an ε -critical node. The intuition, which will be built upon later, is that variable occurrences corresponding to ε -critical nodes are *to be solved for*, according to ε .

For every term $s(p_1, \dots, p_n)$ and every order type ε , we say that $+s$ (resp. $-s$) *agrees with* ε , and write $\varepsilon(+s)$ (resp. $\varepsilon(-s)$), if every leaf in the signed generation tree of $+s$ (resp. $-s$) is ε -critical. In other words, $\varepsilon(+s)$ (resp. $\varepsilon(-s)$) means that all variable occurrences corresponding to leaves of $+s$ (resp. $-s$) are to be solved for according to ε . We will also write $+s' < *s$ (resp. $-s' < *s$) to indicate that the subterm s' inherits the positive (resp. negative) sign from the signed generation tree $*s$. Finally, we will write $\varepsilon(\gamma) < *s$ (resp. $\varepsilon^\partial(\gamma_h) < *s$) to indicate that the signed subtree γ , with the sign inherited from $*s$, agrees with ε (resp. with ε^∂).

Definition 3.2. Nodes in signed generation trees will be called Δ -*adjoints*, *syntactically additive coordinatewise (SAC)*, *syntactically right residual (SRR)*, and *syntactically multiplicative in the product (SMP)*⁶, according to the specification given in Table 1. A branch in a signed generation tree $*s$, with $* \in \{+, -\}$, is called a *good branch* if it is the concatenation of two paths P_1 and P_2 , one of which may possibly be of length 0, such that P_1 is a path from the leaf consisting (apart from variable nodes) only of PIA-nodes, and P_2 consists (apart from variable nodes) only of Skeleton-nodes. A branch is *excellent* if it is good and in P_1 there are only SMP-nodes. A good branch is *Skeleton* if the length of P_1 is 0 (hence Skeleton branches are excellent), and is *SAC*, or *definite*, if P_2 only contains SAC nodes.

⁵The following observation will be used at various points in the remainder of the present paper: if a term inequality $s(\vec{p}, \vec{q}) \leq t(\vec{p}, \vec{q})$ is ε -uniform in \vec{p} (cf. discussion after Definition 3.1), then the validity of $s \leq t$ is equivalent to the validity of $s(\overline{\tau^{\varepsilon(i)}}\vec{p}, \vec{q}) \leq t(\overline{\tau^{\varepsilon(i)}}\vec{p}, \vec{q})$, where $\tau^{\varepsilon(i)} = \top$ if $\varepsilon(i) = 1$ and $\tau^{\varepsilon(i)} = \perp$ if $\varepsilon(i) = \partial$.

⁶This division reflects the order-theoretic properties of the interpretation of these connectives (join/meet preservation/reversal, adjunction, residuation) which will be exploited in the reduction phase of the constructive ALBA-algorithm, introduced below in Section 4

Skeleton		PIA	
Δ -adjoints		(SMP)	
+	\vee	+	$\wedge \quad g \in \mathcal{G} \quad \text{with } n_g = 1$
-	\wedge	-	$\vee \quad f \in \mathcal{F} \quad \text{with } n_f = 1$
(SAC)		(SRR)	
+	$f \in \mathcal{F}$	+	$g \in \mathcal{G}_n \quad \text{with } n_g \geq 2$
-	$g \in \mathcal{G}$	-	$f \in \mathcal{F}_n \quad \text{with } n_f \geq 2$

TABLE 1. Skeleton and PIA nodes for LE.

Definition 3.3 (Inductive inequalities). For any order type ε and any irreflexive and transitive relation Ω on p_1, \dots, p_n , the signed generation tree $*s$ ($* \in \{-, +\}$) of a term $s(p_1, \dots, p_n)$ is (Ω, ε) -inductive if

- (1) for all $1 \leq i \leq n$, every ε -critical branch with leaf p_i is good (cf. Definition 3.2);
- (2) every m -ary SRR-node occurring in the critical branch is of the form $h(\gamma_1, \dots, \gamma_{j-1}, \beta, \gamma_{j+1}, \dots, \gamma_m)$, where for any $\ell \in \{1, \dots, m\} \setminus j$:
 - (a) $\varepsilon^\partial(\gamma_\ell) < *s$ (cf. discussion before Definition 3.2), and
 - (b) $p_k \Omega p_i$ for every p_k occurring in γ_ℓ .

We will refer to Ω as the *dependency order* on the variables. An inequality $s \leq t$ is (Ω, ε) -inductive if the signed generation trees $+s$ and $-t$ are (Ω, ε) -inductive. An inequality $s \leq t$ is *inductive* if it is (Ω, ε) -inductive for some Ω and ε .

In what follows, we will find it useful to refer to formulas φ such that only PIA nodes occur in $+\varphi$ (resp. $-\varphi$) as *positive* (resp. *negative*) *PIA-formulas*, and to formulas ξ such that only Skeleton nodes occur in $+\xi$ (resp. $-\xi$) as *positive* (resp. *negative*) *Skeleton-formulas*.

Definition 3.4. Given an order type ε , the signed generation tree $*s$, $* \in \{-, +\}$, of a term $s(p_1, \dots, p_n)$ is ε -Sahlqvist if every ε -critical branch is excellent. An inequality $s \leq t$ is ε -Sahlqvist if the trees $+s$ and $-t$ are both ε -Sahlqvist. An inequality $s \leq t$ is *Sahlqvist* if it is ε -Sahlqvist for some ε .

Remark 3.5. In [51], Suzuki proves a constructive canonicity result in a setting of lattice expansions given by the signature (h, k, \neg, l, r, c) such that c is a constant, $-h$ and $+k$ are SMP of arbitrary arity and order-type, $+l$ and $-r$ are SAC of arbitrary arity $n_l = n_r$ and order-type, and \neg is a unary connective s.t. $\varepsilon_\neg = \partial$, which is an involution ($\neg\neg x = x$) and moreover $+\neg$ and $-\neg$ are both SAC. In some cases, l and r are intended to be residual to each other in some coordinate, thus being promoted from SAC to SRR in that coordinate.

$$\begin{aligned}
\cup\text{-terms} \quad s &::= b \mid s \vee s \mid h(\bar{s}^{\varepsilon h}) \mid l(\bar{s}^{\varepsilon l}) \\
\cap\text{-terms} \quad t &::= d \mid t \wedge t \mid k(\bar{t}^{\varepsilon k}) \mid r(\bar{t}^{\varepsilon r}) \\
b\text{-terms} \quad b &::= p \mid \top \mid \perp \mid b \wedge b \mid k(\bar{b}^{\varepsilon k}) \mid r(\bar{x}^{\varepsilon r(i)}) \\
d\text{-terms} \quad d &::= p \mid \top \mid \perp \mid d \vee d \mid h(\bar{d}^{\varepsilon h}) \mid l(\bar{y}^{\varepsilon l(i)}),
\end{aligned}$$

where

- the i th element of the n_h -tuple $\bar{s}^{\varepsilon h}$ (resp. $\bar{d}^{\varepsilon h}$) is a \cup -term (resp. a d -term) if $\varepsilon_h(i) = 1$ and is a \cap -term (resp. a b -term) if $\varepsilon_h(i) = \partial$;

- the i th element of the n_l -tuple \bar{s}^{ε_l} is a \cup -term if $\varepsilon_l(i) = 1$ and is a \cap -term if $\varepsilon_l(i) = \partial$;
- the i th element of the n_k -tuple \bar{t}^{ε_k} (resp. \bar{b}^{ε_k}) is a \cap -term (resp. b -term) if $\varepsilon_k(i) = 1$ and is a \cup -term (resp. a d -term) if $\varepsilon_k(i) = \partial$;
- the i th element of the n_r -tuple \bar{t}^{ε_r} is a \cap -term if $\varepsilon_r(i) = 1$ and is a \cup -term if $\varepsilon_r(i) = \partial$;
- $\bar{x}^{\varepsilon_r(i)}$ is an n_r -tuple such that every coordinate but the i th is a constant term, and the i th element is a b -term if $\varepsilon_r(i) = 1$, and is a d -term if $\varepsilon_r(i) = \partial$;
- $\bar{y}^{\varepsilon_l(i)}$ is an n_l -tuple such that every coordinate but the i th is a constant term, and the i th element is a d -term if $\varepsilon_l(i) = 1$, and is a b -term if $\varepsilon_l(i) = \partial$.

The inequalities proven to be canonical in [52, Theorem 5.10] are of the form $\varphi \leq \psi$, such that

$$\varphi = s(\bar{x}, \overline{a_1(\bar{x})/\bar{z}}) \quad \psi = t(\bar{x}, \overline{a_2(\bar{x})/\bar{z}})$$

where $s(\bar{x}, \bar{z})$ is a \cup -term, $t(\bar{x}, \bar{z})$ is a \cap -term, and there exists some order-type ε on \bar{x} such that $\varepsilon^\partial(a_1(\bar{x})) < +\varphi$ and $\varepsilon^\partial(a_2(\bar{x})) < -\psi$ for each a_1 in $\overline{a_1}$ and each a_2 in $\overline{a_2}$.

We would now like to compare these inequalities with the Sahlqvist and inductive inequalities as defined above. The first point to note is that we do not include any non-unary SMP nodes other than $+\wedge$ and $-\vee$ in Table 1. The reason for this is that, apart from conjunction and disjunction, we are not aware of any non-unary connective occurring in a well-known logic, the interpretation of which has this order-theoretic property. This being said, it is possible to lift the restriction that $n_g = 1$ and $n_f = 1$ in the SMP cell of Table 1 without affecting any of the results in the present paper. Doing so does mean we need to add n -ary versions of the residuation rules for regular connectives to ALBA, but this is straightforward and will be discussed in Remark 4.2.

Given this modification to Table 1, it is not difficult to see that the inequality $\varphi \leq \psi$ described above falls under the definition of ε -Sahlqvist inequalities (cf. Definition 3.4), when one takes into account the following considerations. Firstly, any operation that is multiplicative in the product (as always, modulo some order-type), is also multiplicative coordinatewise, therefore all SMP nodes are SAC (of course, preservation of empty meets in the product does not imply preservation of empty meets coordinatewise, but that does not concern us now). Secondly, it follows that taking any operation that is coordinatewise multiplicative and holding all but one of its coordinates constant turns it, in effect, into one that is multiplicative in the (now one-fold) product. Therefore, instantiating all arguments of a SAC node save one with constants turns it into an SMP node.

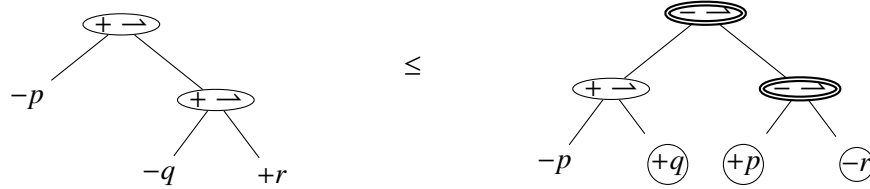
Taking this into account, the trees $+b$ and $-d$ can be recognized as consisting entirely of PIA-nodes, and in particular entirely of SMP nodes. Further, the generation trees $+s$ and $-t$ of \cup - and \cap -terms are constructed by taking trees consisting entirely of skeleton nodes and inserting subtrees $+b < +s$, $-t$ and $-d < +s$, $-t$ at leaves. Thus *all* branches in $+s$ and $-t$ are excellent. Substituting $\overline{a_1(\bar{x})}$ and $\overline{a_2(\bar{x})}$ into s and t as indicated possibly introduces non-good branches into $+s(\bar{x}, \overline{a_1(\bar{x})/\bar{z}})$ and $-t(\bar{x}, \overline{a_2(\bar{x})/\bar{z}})$, but these will be non-critical according to ε and therefore both trees will be ε -Sahlqvist.

The same analysis applies to the scope of the Sahlqvist canonicity result of Ghilardi and Meloni [31] (cf. [19, Remark 12]). Indeed, Suzuki's treatment extracts the syntactic definition from Ghilardi-Meloni's order-theoretic insights and transfers it to the setting of general lattice expansions.

Example 3.6. Let $\mathcal{F} = \emptyset = \mathcal{G}_r$, and $\mathcal{G}_n = \{\rightarrow\}$, with \rightarrow binary and of order-type $(\partial, 1)$. As observed in [13], the Frege inequality

$$p \multimap (q \multimap r) \leq (p \multimap q) \multimap (p \multimap r)$$

is not Sahlqvist for any order type, but is (Ω, ε) -inductive, e.g. for $r <_{\Omega} p <_{\Omega} q$ and $\varepsilon(p, q, r) = (1, 1, \partial)$, as can be seen from the signed generation trees below.



In the picture above, the circled variable occurrences are the ε -critical ones, the doubly circled nodes are the Skeleton ones and the single-circle ones are PIA. In the intuitionistic setting of [31], the Frege inequality with \multimap interpreted as Heyting implication is a validity, and hence is trivially canonical. In the lattice/poset-based setting of [51, 52], the Frege inequality is not a validity, but because proposition variables occur on both coordinates of SRR nodes, the Frege inequality falls outside the fragment accounted for in [51, 52]. In Example 4.1, we will give a successful execution of ALBA on the Frege inequality.

4. CONSTRUCTIVE ALBA

In this section, we present the second main tool of unified correspondence, namely the algorithm ALBA, in a version suitable to the constructive setting. Like all versions of ALBA, this one proceeds in three stages: preprocessing, reduction/elimination, and output. Since the current setting is general (i.e. non necessarily distributive) lattice expansions, the algorithm discussed here is more similar to the one introduced in [14] than to versions based on distributive lattices. However, unlike the version of [14], the constructive ALBA includes special ‘residuation rules’ for regular connectives along with residuation rules proper. This is due to the fact that we need to accommodate a signature with different order-theoretic properties than the one in [14], and, in particular, the residuation rules for normal connectives are not sound when applied to regular connectives. The other rules, although syntactically the same as in [14], take on a different semantic meaning in the present setting, as we will see in the next section.

ALBA takes \mathcal{L}_{LE} -inequalities $\varphi \leq \psi$ as input and then proceeds in three stages. The first stage preprocesses $\varphi \leq \psi$ by eliminating all uniformly occurring propositional variables, and applying distribution and splitting rules exhaustively. This produces a finite set of inequalities, $\varphi'_i \leq \psi'_i$, $1 \leq i \leq n$.

Now ALBA forms the *initial quasi-inequalities* $\& S_i \Rightarrow \text{Ineq}_i$, compactly represented as tuples (S_i, Ineq_i) referred as *systems*, with each S_i initialized to the empty set and Ineq_i initialized to $\varphi'_i \leq \psi'_i$.

The second stage (called the reduction stage) transforms S_i and Ineq_i through the application of transformation rules, which are listed below. The aim is to eliminate all propositional variables from S_i and Ineq_i in favour of nominals and co-nominals. A system for which this has been done will be called *pure* or *purified*. The actual eliminations are effected through the Ackermann-rules, while the other rules are used to bring S_i and Ineq_i into the appropriate shape which make these applications possible. Once all propositional

variables have been eliminated, this phase terminates and returns the pure quasi-inequalities $\&\mathcal{S}_i \Rightarrow \text{lneq}_i$.

The third stage either reports failure if some system could not be purified, or else returns the conjunction of the pure quasi-inequalities $\&\mathcal{S}_i \Rightarrow \text{lneq}_i$, which we denote by $\text{ALBA}(\varphi \leq \psi)$.

We now outline each of the three stages in more detail:

4.1. Stage 1: Preprocessing and initialization. ALBA receives an \mathcal{L}_{LE} -inequality $\varphi \leq \psi$ as input. It applies the following **rules for elimination of monotone variables** to $\varphi \leq \psi$ exhaustively, in order to eliminate any propositional variables which occur uniformly:

$$\frac{\alpha(p) \leq \beta(p)}{\alpha(\perp) \leq \beta(\perp)} \quad \frac{\gamma(p) \leq \delta(p)}{\gamma(\top) \leq \delta(\top)}$$

for $\alpha(p) \leq \beta(p)$ positive and $\gamma(p) \leq \delta(p)$ negative in p , respectively (see footnote 5).

Next, ALBA exhaustively distributes $f \in \mathcal{F}$ over $+\vee$, and $g \in \mathcal{F}$ over $-\wedge$, so as to bring occurrences of $+\vee$ and $-\wedge$ to the surface wherever this is possible, and then eliminate them via exhaustive applications of *splitting* rules.

Splitting-rules.

$$\frac{\alpha \leq \beta \wedge \gamma}{\alpha \leq \beta \quad \alpha \leq \gamma} \quad \frac{\alpha \vee \beta \leq \gamma}{\alpha \leq \gamma \quad \beta \leq \gamma}$$

This gives rise to a set of inequalities $\{\varphi'_i \leq \psi'_i \mid 1 \leq i \leq n\}$. Now ALBA forms the *initial quasi-inequalities* $\&\mathcal{S}_i \Rightarrow \text{lneq}_i$, compactly represented as tuples $(\mathcal{S}_i, \text{lneq}_i)$ referred as *systems*, with each \mathcal{S}_i initialized to the empty set and lneq_i initialized to $\varphi'_i \leq \psi'_i$. Each initial system is passed separately to stage 2, described below, where we will suppress indices i .

4.2. Stage 2: Reduction and elimination. The aim of this stage is to eliminate all occurring propositional variables from a given system $(\mathcal{S}, \text{lneq})$. This is done by means of the following *approximation rules*, *residuation rules*, *splitting rules*, and *Ackermann-rules*, collectively called *reduction rules*. The terms and inequalities in this subsection are from $\mathcal{L}_{\text{LE}}^+$.

Approximation rules. There are four approximation rules. Each of these rules functions by simplifying lneq and adding an inequality to \mathcal{S} .

Left-positive approximation rule.:

$$\frac{(S, \varphi'(\gamma/!x) \leq \psi)}{(S \cup \{\mathbf{j} \leq \gamma\}, \varphi'(\mathbf{j}/!x) \leq \psi)} (L^+A)$$

with $+x < +\varphi'(!x)$, the branch of $+\varphi'(!x)$ starting at $+x$ being SAC (cf. definition 3.2), γ belonging to the original language \mathcal{L}_{LE} and \mathbf{j} being the first nominal variable not occurring in S or $\varphi'(\gamma/!x) \leq \psi$.

Left-negative approximation rule.:

$$\frac{(S, \varphi'(\gamma/!x) \leq \psi)}{(S \cup \{\gamma \leq \mathbf{m}\}, \varphi'(\mathbf{m}/!x) \leq \psi)} (L^-A)$$

with $-x < +\varphi'(!x)$, the branch of $+\varphi'(!x)$ starting at $-x$ being SAC, γ belonging to the original language \mathcal{L}_{LE} and \mathbf{m} being the first co-nominal not occurring in S or $\varphi'(\gamma/!x) \leq \psi$.

Right-positive approximation rule.:

$$\frac{(S, \varphi \leq \psi'(\gamma/!x))}{(S \cup \{\mathbf{j} \leq \gamma\}, \varphi \leq \psi'(\mathbf{j}/!x))} (R^+ A)$$

with $+x < -\psi'(!x)$, the branch of $-\psi'(!x)$ starting at $+x$ being SAC, γ belonging to the original language \mathcal{L}_{LE} and \mathbf{j} being the first nominal not occurring in S or $\varphi \leq \psi'(\gamma/!x)$.

Right-negative approximation rule.:

$$\frac{(S, \varphi \leq \psi'(\gamma/!x))}{(S \cup \{\gamma \leq \mathbf{m}\}, \varphi \leq \psi'(\mathbf{m}/!x))} (R^- A)$$

with $-x < -\psi'(!x)$, the branch of $-\psi'(!x)$ starting at $-x$ being SAC, γ belonging to the original language \mathcal{L}_{LE} and \mathbf{m} being the first co-nominal not occurring in S or $\varphi \leq \psi'(\gamma/!x)$.

The approximation rules above, as stated, will be shown to be sound both under admissible and under arbitrary assignments (cf. Proposition 5.5). However, their liberal application gives rise to topological complications in the proof of canonicity. Therefore, we will restrict the applications of approximation rules to nodes $!x$ giving rise to *maximal* SAC branches. Such applications will be called *pivotal*. Also, executions of ALBA in which approximation rules are applied only pivotally will be referred to as *pivotal*.

Residuation rules. These rules operate on the inequalities in S , by rewriting a chosen inequality in S into another inequality. For every $f \in \mathcal{F}_n$ and $g \in \mathcal{G}_n$, and any $1 \leq i \leq n_f$ and $1 \leq j \leq n_g$,

$$\begin{array}{cc} \frac{f(\varphi_1, \dots, \varphi_i, \dots, \varphi_{n_f}) \leq \psi}{\varphi_i \leq f_i^\sharp(\varphi_1, \dots, \psi, \dots, \varphi_{n_f})} \varepsilon_f(i) = 1 & \frac{f(\varphi_1, \dots, \varphi_i, \dots, \varphi_{n_f}) \leq \psi}{f_i^\sharp(\varphi_1, \dots, \psi, \dots, \varphi_{n_f}) \leq \varphi_i} \varepsilon_f(i) = \partial \\ \frac{\psi \leq g(\varphi_1, \dots, \varphi_i, \dots, \varphi_{n_g})}{g_i^b(\varphi_1, \dots, \psi, \dots, \varphi_{n_g}) \leq \varphi_i} \varepsilon_g(i) = 1 & \frac{\psi \leq g(\varphi_1, \dots, \varphi_i, \dots, \varphi_{n_g})}{\varphi_i \leq g_i^b(\varphi_1, \dots, \psi, \dots, \varphi_{n_g})} \varepsilon_g(i) = \partial \end{array}$$

For every $f \in \mathcal{F}_r$ and $g \in \mathcal{G}_r$,

$$\begin{array}{cc} \frac{f(\varphi) \leq \psi}{f(\perp) \leq \psi \quad \varphi \leq \blacksquare_f \psi} \text{ (if } \varepsilon_f = 1) & \frac{\varphi \leq g(\psi)}{\varphi \leq g(\top) \quad \blacklozenge_g \varphi \leq \psi} \text{ (if } \varepsilon_g = 1) \\ \frac{f(\varphi) \leq \psi}{f(\top) \leq \psi \quad \blacktriangleleft_f \psi \leq \varphi} \text{ (if } \varepsilon_f = \partial) & \frac{\varphi \leq g(\psi)}{\varphi \leq g(\perp) \quad \psi \leq \blacktriangleright_g \varphi} \text{ (if } \varepsilon_g = \partial) \end{array}$$

In a given system, each of these rules replaces an instance of the upper inequality with the corresponding instances of the two lower inequalities.

The leftmost inequalities in each rule above will be referred to as the *side condition*.

Right Ackermann-rule.

$$\frac{(\{\alpha_i \leq p \mid 1 \leq i \leq n\} \cup \{\beta_j(p) \leq \gamma_j(p) \mid 1 \leq j \leq m\}, \text{Ineq})}{(\{\beta_j(\bigvee_{i=1}^n \alpha_i) \leq \gamma_j(\bigvee_{i=1}^n \alpha_i) \mid 1 \leq j \leq m\}, \text{Ineq})} (RAR)$$

where:

- p does not occur in $\alpha_1, \dots, \alpha_n$ or in Ineq,
- $\beta_1(p), \dots, \beta_m(p)$ are positive in p , and
- $\gamma_1(p), \dots, \gamma_m(p)$ are negative in p .

Left Ackermann-rule.

$$\frac{(\{p \leq \alpha_i \mid 1 \leq i \leq n\} \cup \{\beta_j(p) \leq \gamma_j(p) \mid 1 \leq j \leq m\}, \text{Ineq})}{(\{\beta_j(\bigwedge_{i=1}^n \alpha_i) \leq \gamma_j(\bigwedge_{i=1}^n \alpha_i) \mid 1 \leq j \leq m\}, \text{Ineq})} \text{ (LAR)}$$

where:

- p does not occur in $\alpha_1, \dots, \alpha_n$ or in Ineq ,
- $\beta_1(p), \dots, \beta_m(p)$ are negative in p , and
- $\gamma_1(p), \dots, \gamma_m(p)$ are positive in p .

4.3. Stage 3: Success, failure and output. If stage 2 succeeded in eliminating all propositional variables from each system, the algorithm returns the conjunction of these purified quasi-inequalities, denoted by $\text{ALBA}(\varphi \leq \psi)$. Otherwise, the algorithm reports failure and terminates.

Example 4.1. As mentioned in Example 3.6,

$$p \rightarrow (q \rightarrow r) \leq (p \rightarrow q) \rightarrow (p \rightarrow r)$$

is (Ω, ε) -inductive for $r <_{\Omega} p <_{\Omega} q$ and $\varepsilon(p, q, r) = (1, 1, \partial)$. A pivotal execution of ALBA according to this choice of Ω and ε , is given below. The variables $\mathbf{i}, \mathbf{j}, \mathbf{h}$ are nominals, \mathbf{m} is a co-nominal, \bullet is the left adjoint of \rightarrow in its second coordinate and \leftarrow is the Galois-adjoint of \rightarrow in its first coordinate.⁷

$$\begin{aligned} & \forall p \forall q \forall r [p \rightarrow (q \rightarrow r) \leq (p \rightarrow q) \rightarrow (p \rightarrow r)] \\ \text{iff } & \forall p \forall q \forall r \forall \mathbf{i} \forall \mathbf{j} \forall \mathbf{m} \forall \mathbf{h} [(\mathbf{h} \leq p \rightarrow (q \rightarrow r) \ \& \ \mathbf{i} \leq p \rightarrow q \ \& \ \mathbf{j} \leq p \ \& \ r \leq \mathbf{m}) \Rightarrow \mathbf{h} \leq \mathbf{i} \rightarrow (\mathbf{j} \rightarrow \mathbf{m})] \\ \text{iff } & \forall p \forall q \forall \mathbf{i} \forall \mathbf{j} \forall \mathbf{h} \forall \mathbf{m} [(\mathbf{h} \leq p \rightarrow (q \rightarrow \mathbf{m}) \ \& \ \mathbf{i} \leq p \rightarrow q \ \& \ \mathbf{j} \leq p) \Rightarrow \mathbf{h} \leq \mathbf{i} \rightarrow (\mathbf{j} \rightarrow \mathbf{m})] \\ \text{iff } & \forall q \forall \mathbf{i} \forall \mathbf{j} \forall \mathbf{h} \forall \mathbf{m} [(\mathbf{h} \leq \mathbf{j} \rightarrow (q \rightarrow \mathbf{m}) \ \& \ \mathbf{i} \leq \mathbf{j} \rightarrow q) \Rightarrow \mathbf{h} \leq \mathbf{i} \rightarrow (\mathbf{j} \rightarrow \mathbf{m})] \\ \text{iff } & \forall q \forall \mathbf{i} \forall \mathbf{j} \forall \mathbf{h} \forall \mathbf{m} [(\mathbf{j} \bullet \mathbf{h} \leq q \rightarrow \mathbf{m} \ \& \ \mathbf{i} \leq \mathbf{j} \rightarrow q) \Rightarrow \mathbf{h} \leq \mathbf{i} \rightarrow (\mathbf{j} \rightarrow \mathbf{m})] \\ \text{iff } & \forall q \forall \mathbf{i} \forall \mathbf{j} \forall \mathbf{h} \forall \mathbf{m} [(q \bullet (\mathbf{j} \bullet \mathbf{h}) \leq \mathbf{m} \ \& \ \mathbf{i} \leq \mathbf{j} \rightarrow q) \Rightarrow \mathbf{h} \leq \mathbf{i} \rightarrow (\mathbf{j} \rightarrow \mathbf{m})] \\ \text{iff } & \forall q \forall \mathbf{i} \forall \mathbf{j} \forall \mathbf{h} \forall \mathbf{m} [(q \leq \mathbf{m} \leftarrow (\mathbf{j} \bullet \mathbf{h}) \ \& \ \mathbf{i} \leq \mathbf{j} \rightarrow q) \Rightarrow \mathbf{h} \leq \mathbf{i} \rightarrow (\mathbf{j} \rightarrow \mathbf{m})] \\ \text{iff } & \forall \mathbf{i} \forall \mathbf{j} \forall \mathbf{h} \forall \mathbf{m} [\mathbf{i} \leq \mathbf{j} \rightarrow (\mathbf{m} \leftarrow (\mathbf{j} \bullet \mathbf{h})) \Rightarrow \mathbf{h} \leq \mathbf{i} \rightarrow (\mathbf{j} \rightarrow \mathbf{m})] \\ \text{iff } & \forall \mathbf{i} \forall \mathbf{j} \forall \mathbf{h} \forall \mathbf{m} [\mathbf{i} \leq \mathbf{j} \rightarrow (\mathbf{m} \leftarrow (\mathbf{j} \bullet \mathbf{h})) \Rightarrow \mathbf{i} \bullet \mathbf{h} \leq \mathbf{j} \rightarrow \mathbf{m}] \\ \text{iff } & \forall \mathbf{i} \forall \mathbf{j} \forall \mathbf{h} \forall \mathbf{m} [\mathbf{i} \leq \mathbf{j} \rightarrow (\mathbf{m} \leftarrow (\mathbf{j} \bullet \mathbf{h})) \Rightarrow \mathbf{i} \leq (\mathbf{j} \rightarrow \mathbf{m}) \leftarrow \mathbf{h}] \\ \text{iff } & \forall \mathbf{j} \forall \mathbf{h} \forall \mathbf{m} [\mathbf{j} \rightarrow (\mathbf{m} \leftarrow (\mathbf{j} \bullet \mathbf{h})) \leq (\mathbf{j} \rightarrow \mathbf{m}) \leftarrow \mathbf{h}]. \end{aligned}$$

Remark 4.2. As discussed in Remark 3.5, it would be possible to accommodate, both in the definition of the inductive formulas and in ALBA, n -ary connectives which preserve meets and joins in the product. We could add sets $\mathcal{F}_p \subseteq \mathcal{F}$ and $\mathcal{G}_p \subseteq \mathcal{G}$ of such connectives. In every $(\mathcal{F}, \mathcal{G})$ -expanded lattice \mathbb{A} we would have $f^{\mathbb{A}}(\bigvee_{i \in I}^{\varepsilon_f} \bar{a}_i) = \bigvee_{i \in I} f(\bar{a}_i)$ for all nonempty subsets $\{\bar{a}_i \mid i \in I\} \subseteq \mathbb{A}^{\varepsilon_f}$ and, similarly, $g^{\mathbb{A}}(\bigwedge_{i \in I}^{\varepsilon_g} \bar{a}_i) = \bigwedge_{i \in I} g(\bar{a}_i)$ for all nonempty subsets $\{\bar{a}_i \mid i \in I\} \subseteq \mathbb{A}^{\varepsilon_g}$. The accompanying expanded languages would contain connectives Δ_f and ∇_g for the normalizations of each $f \in \mathcal{F}_p$ and $g \in \mathcal{G}_p$. The intended interpretations of these are such that $f(\bar{a}) = f(\perp^{\varepsilon_f}) \vee \Delta_f(\bar{a})$ and $g(\bar{a}) = g(\top^{\varepsilon_g}) \wedge \nabla_g(\bar{a})$. For each $f \in \mathcal{F}_p$ there would be adjoints \blacktriangledown_i , $1 \leq i \leq n_f$, such that $\Delta_f(\bar{a}) \leq b$ iff $\bar{a} \leq^{\varepsilon_f} \blacktriangledown_i b$ iff $a_i \leq^{\varepsilon_f(i)} \blacktriangledown_i b$ for each $1 \leq i \leq n_f$. Dually, for each $g \in \mathcal{G}_p$ there would be adjoints \blacktriangle_i , $1 \leq i \leq n_g$, such that $a \leq \Delta_g(\bar{b})$ iff $\blacktriangle_i a \leq^{\varepsilon_g} \bar{b}$ iff $\blacktriangledown_i a \leq^{\varepsilon_g(i)} b_i$ for each $1 \leq i \leq n_g$.

⁷Notice, however, that the Galois-adjoint of \rightarrow in its first coordinate is, modulo inversion of coordinates, also the right adjoint of \bullet in its first coordinate (cf. Footnote 4). The symbol \leftarrow is more appropriate for the latter role.

The appropriate residuation (or rather, *adjunction*) rules for these connective to be added to ALBA would be as follows:

$$\frac{f(\varphi_1, \dots, \varphi_n) \leq \psi}{\frac{f(\perp^{\varepsilon_f}) \leq \psi \quad \varphi_1 \leq \varepsilon_f(1) \quad \nabla_1^f \psi \quad \dots \quad \varphi_n \leq \varepsilon_f(n_f) \quad \nabla_{n_f}^f \psi}{\varphi \leq g(\psi_1, \dots, \psi_n)}}{\frac{\varphi \leq g(\overline{\top}^{\varepsilon_g}) \quad \blacktriangle_1^g \varphi \leq \varepsilon_g(1) \quad \varphi_1 \quad \dots \quad \blacktriangle_{n_g}^g \varphi \leq \varepsilon_g(n_g) \quad \varphi_{n_g}}$$

5. PARTIAL CORRECTNESS OF ALBA

In this section, we prove that ALBA is partially correct, in the sense that whenever it succeeds in eliminating all propositional variables from an inequality $\varphi \leq \psi$, the conjunction of the quasi-inequalities returned is equivalent to $\varphi \leq \psi$ on the constructive canonical extension \mathbb{A}^δ of any \mathcal{L}_{LE} -algebra \mathbb{A} , for an arbitrarily fixed language \mathcal{L}_{LE} . Our treatment follows the same structure as the proof of correctness in [14]. Accordingly, we rely on [14] whenever possible, and mostly expand on the parts addressing the differences.

Fix a \mathcal{L}_{LE} -algebra $\mathbb{A} = (L, \mathcal{F}^{\mathbb{A}}, \mathcal{G}^{\mathbb{A}})$ for the remainder of this section. We first give the statement of the correctness theorem and its proof, and subsequently prove the lemmas needed in the proof.

Theorem 5.1 (Partial Correctness). *If ALBA succeeds in reducing an \mathcal{L}_{LE} -inequality $\varphi \leq \psi$ and yields $\text{ALBA}(\varphi \leq \psi)$, then $\mathbb{A}^\delta \models \varphi \leq \psi$ iff $\mathbb{A}^\delta \models \text{ALBA}(\varphi \leq \psi)$.*

Proof. Let $\varphi_i \leq \psi_i$, $1 \leq i \leq n$ be the quasi-inequalities produced by preprocessing $\varphi \leq \psi$. Consider the chain of statements (5.1) to (5.5) below. The proof will be complete if we can show that they are all equivalent.

$$(5.1) \quad \mathbb{A}^\delta \models \varphi \leq \psi$$

$$(5.2) \quad \mathbb{A}^\delta \models \varphi_i \leq \psi_i, \quad 1 \leq i \leq n$$

$$(5.3) \quad \mathbb{A}^\delta \models \& \emptyset \Rightarrow \varphi_i \leq \psi_i, \quad 1 \leq i \leq n$$

$$(5.4) \quad \mathbb{A}^\delta \models \& S_i \Rightarrow \text{Ineq}_i, \quad 1 \leq i \leq n$$

$$(5.5) \quad \mathbb{A}^\delta \models \text{ALBA}(\varphi \leq \psi)$$

For the equivalence of (5.1) and (5.2) we need to verify that the rules for the elimination of uniform variables, distribution and splitting preserve validity on \mathbb{A}^δ . Distribution and splitting are immediate. As to elimination, if $\alpha(p) \leq \beta(p)$ is positive in p , then for all $a \in \mathbb{A}$ it is the case that $\alpha(a) \leq \alpha(\perp)$ and $\beta(\perp) \leq \beta(a)$. Hence if $\alpha(\perp) \leq \beta(\perp)$ then $\alpha(a) \leq \beta(a)$, and hence $\mathbb{A}^\delta \models \alpha(p) \leq \beta(p)$ iff $\mathbb{A}^\delta \models \alpha(\perp) \leq \beta(\perp)$. The case for $\gamma(p) \leq \delta(p)$ negative in p is similar.

That (5.2) and (5.3) are equivalent is immediate. The bi-implication between (5.3) and (5.4) follows from proposition 5.5, while (5.4) and (5.5) are the same by definition. \square

Lemma 5.2 (Distribution lemma). *If $\varphi(!x), \psi(!x), \xi(!x), \chi(!x) \in \mathcal{L}_{\text{LE}}^+$ and $\emptyset \neq \{a_j\}_{j \in I} \subseteq \mathbb{A}^\delta$, then*

- (1) $\varphi(\bigvee_{j \in I} a_j) = \bigvee \{\varphi(a_j) \mid j \in I\}$, when $+x < +\varphi(!x)$ and in $+\varphi(!x)$ the branch ending in $+x$ is SAC;
- (2) $\psi(\bigwedge_{j \in I} a_j) = \bigvee \{\psi(a_j) \mid j \in I\}$, when $-x < +\psi(!x)$ and in $+\psi(!x)$ the branch ending in $-x$ is SAC;

- (3) $\xi(\bigwedge_{j \in I} a_j) = \bigwedge \{\xi(a_j) \mid j \in I\}$, when $-x < -\xi(!x)$ and in $-\xi(!x)$ the branch ending in $-x$ is SAC;
- (4) $\chi(\bigvee_{j \in I} a_j) = \bigwedge \{\chi(a_j) \mid j \in I\}$, when $+x < -\chi(!x)$ and in $-\chi(!x)$ the branch ending in $+x$ is SAC.

Proof. The proof is very similar to [14, Lemma 6.2] and proceeds by simultaneous induction on φ , ψ , ξ and χ . The base cases for \perp , \top , and x , when applicable, are trivial. We check the inductive cases for φ , and list the other inductive cases, which all follow in a similar way.

φ of the form:

$f(\varphi_1, \dots, \varphi_i(!x), \dots, \varphi_{n_f})$ with $f \in \mathcal{F}$ and $\varepsilon_f(i) = 1$:: By the assumption of a unique occurrence of x in φ , the variable x occurs in φ_i for exactly one index $1 \leq i \leq n_f$. The assumption that $\varepsilon_f(i) = 1$ implies that $+x < +\varphi_i$. Then

$$\begin{aligned} \varphi(\bigvee_{j \in I} a_j) &= f(\varphi_1, \dots, \varphi_i(\bigvee_{j \in I} a_j), \dots, \varphi_{n_f}) \\ &= f(\varphi_1, \dots, \bigvee_{j \in I} \varphi_i(a_j), \dots, \varphi_{n_f}) \\ &= \bigvee_{j \in I} f(\varphi_1, \dots, \varphi_i(a_j), \dots, \varphi_{n_f}) \quad (\emptyset \neq \{a_j\}_{j \in I}) \\ &= \bigvee_{j \in I} \varphi(a_j), \end{aligned}$$

where the second equality holds by the inductive hypothesis, since the branch of $+\varphi$ ending in $+x$ is SAC, and it traverses $+\varphi_i$.

$f(\varphi_1, \dots, \psi_i(!x), \dots, \varphi_{n_f})$ with $f \in \mathcal{F}$ and $\varepsilon_f(i) = \partial$:: By the assumption of a unique occurrence of x in φ , the variable x occurs in ψ_i for exactly one index $1 \leq i \leq n_f$. The assumption that $\varepsilon_f(i) = \partial$ implies that $-x < +\psi_i$. Then

$$\begin{aligned} \varphi(\bigvee_{j \in I} a_j) &= f(\varphi_1, \dots, \psi_i(\bigvee_{j \in I} a_j), \dots, \varphi_{n_f}) \\ &= f(\varphi_1, \dots, \bigwedge_{j \in I} \psi_i(a_j), \dots, \varphi_{n_f}) \\ &= \bigvee_{j \in I} f(\varphi_1, \dots, \psi_i(a_j), \dots, \varphi_{n_f}) \quad (\emptyset \neq \{a_j\}_{j \in I}) \\ &= \bigvee_{j \in I} \varphi(a_j), \end{aligned}$$

where the second equality holds by the inductive hypothesis, since the branch of $+\varphi$ ending in $+x$ is SAC, and it traverses $-\psi_i$.

ψ of the form: $f(\psi_1, \dots, \psi_i(!x), \dots, \psi_{n_f})$ with $f \in \mathcal{F}$ and $\varepsilon_f(i) = 1$ or $f(\psi_1, \dots, \xi_i(!x), \dots, \varphi_{n_f})$ with $f \in \mathcal{F}$ and $\varepsilon_f(i) = \partial$.

ξ of the form: $g(\xi_1, \dots, \xi_i(!x), \dots, \xi_{n_g})$ with $g \in \mathcal{G}$ and $\varepsilon_g(i) = 1$ or $g(\xi_1, \dots, \chi_i(!x), \dots, \xi_{n_g})$ with $g \in \mathcal{G}$ and $\varepsilon_g(i) = \partial$.

χ of the form: $g(\chi_1, \dots, \chi_i(!x), \dots, \chi_{n_g})$ with $g \in \mathcal{G}$ and $\varepsilon_g(i) = 1$ or $g(\chi_1, \dots, \varphi_i(!x), \dots, \xi_{n_g})$ with $g \in \mathcal{G}$ and $\varepsilon_g(i) = \partial$. \square

Lemma 5.3 (Right Ackermann Lemma). *Let $\alpha_1, \dots, \alpha_n \in \mathcal{L}_{LE}^+$ with $p \notin \text{PROP}(\alpha_i)$, $1 \leq i \leq n$, let $\beta_1(p), \dots, \beta_m(p) \in \mathcal{L}_{LE}^+$ be positive in p , and let $\gamma_1(p), \dots, \gamma_m(p) \in \mathcal{L}_{LE}^+$ be negative in p . Let v be any assignment on the constructive canonical extension of an \mathcal{L}_{LE} -algebra \mathbb{A} . Then*

$$\mathbb{A}^\delta, v \models \beta_i(\alpha_1 \vee \dots \vee \alpha_n/p) \leq \gamma_i(\alpha_1 \vee \dots \vee \alpha_n/p), \text{ for all } 1 \leq i \leq m$$

iff there exists a variant $v' \sim_p v$ such that

$$\mathbb{A}^\delta, v' \models \alpha_i \leq p \text{ for all } 1 \leq i \leq n, \text{ and } \mathbb{A}, v' \models \beta_i(p) \leq \gamma_i(p), \text{ for all } 1 \leq i \leq m.$$

Proof. For the implication from top to bottom, let $v'(p) = v(\alpha_1 \vee \dots \vee \alpha_n)$. Since the α_i does not contain p , we have $v(\alpha_i) = v'(\alpha_i)$, and hence that $v'(\alpha_i) \leq v(\alpha_1) \vee \dots \vee v(\alpha_n) = v(\alpha_1 \vee \dots \vee \alpha_n) = v'(p)$. Moreover, $v'(\beta_i(p)) = v(\beta_i(\alpha_1 \vee \dots \vee \alpha_n/p)) \leq v(\gamma_i(\alpha_1 \vee \dots \vee \alpha_n/p)) = v'(\gamma_i(p))$.

For the implication from bottom to top, we make use of the fact that the β_i are monotone (since positive) in p , while the γ_i are antitone (since negative) in p . Since

$v(\alpha_i) = v'(\alpha_i) \leq v'(p)$ for all $1 \leq i \leq n$, we have $v(\alpha_1 \vee \cdots \vee \alpha_n) \leq v'(p)$, and hence $v(\beta_i(\alpha_1 \vee \cdots \vee \alpha_n/p)) \leq v'(\beta_i(p)) \leq v'(\gamma_i(p)) \leq (\gamma_i(\alpha_1 \vee \cdots \vee \alpha_n/p))$. \square

The proof of the following version of the lemma is similar.

Lemma 5.4 (Left Ackermann Lemma). *Let $\alpha_1, \dots, \alpha_n \in \mathcal{L}_{\text{LE}}^+$ with $p \notin \text{PROP}(\alpha_i)$, $1 \leq i \leq n$, let $\beta_1(p), \dots, \beta_m(p) \in \mathcal{L}_{\text{LE}}^+$ be negative in p , and let $\gamma_1(p), \dots, \gamma_m(p) \in \mathcal{L}_{\text{LE}}^+$ be positive in p . Let v be any assignment on the constructive canonical extension of an \mathcal{L}_{LE} -algebra \mathbb{A} . Then*

$$\mathbb{A}^\delta, v \models \beta_i(\alpha_1 \wedge \cdots \wedge \alpha_n/p) \leq \gamma_i(\alpha_1 \wedge \cdots \wedge \alpha_n/p), \text{ for all } 1 \leq i \leq m$$

iff there exists a variant $v' \sim_p v$ such that

$$\mathbb{A}^\delta, v' \models p \leq \alpha_i \text{ for all } 1 \leq i \leq n, \text{ and } \mathbb{A}, v' \models \beta_i(p) \leq \gamma_i(p), \text{ for all } 1 \leq i \leq m.$$

Proposition 5.5. *If a system (S, Ineq) is obtained from a system (S_0, Ineq_0) by the application of reduction rules, then*

$$\mathbb{A}^\delta \models \forall \overline{\text{var}}_0 [\mathcal{E}' S_0 \Rightarrow \text{Ineq}_0] \text{ iff } \mathbb{A}^\delta \models \forall \overline{\text{var}} [\mathcal{E}' S \Rightarrow \text{Ineq}],$$

where $\overline{\text{var}}_0$ and $\overline{\text{var}}$ are the vectors of all variables occurring in $\mathcal{E}' S_0 \Rightarrow \text{Ineq}_0$ and $\mathcal{E}' S \Rightarrow \text{Ineq}$, respectively.

Proof. It is sufficient to verify that each rule preserves this equivalence, i.e., that if S and Ineq are obtained from S' and Ineq' by the application of a single transformation rule then

$$\mathbb{A}^\delta \models \forall \overline{\text{var}}' [\& S' \Rightarrow \text{Ineq}'] \text{ iff } \mathbb{A}^\delta \models \forall \overline{\text{var}} [\& S \Rightarrow \text{Ineq}].$$

Left-positive approximation rule: Let Ineq be $\varphi'(\gamma!/x) \leq \psi$, with $+x < +\varphi'(!x)$, and the branch of $+\varphi'(!x)$ starting at $+x$ being SAC. Then, under any assignment to the occurring variables, $\varphi'(\gamma!/x) = \varphi'(\bigvee \{k \in K(\mathbb{A}^\delta) \mid k \leq \gamma\}) = \bigvee \{\varphi'(k) \mid \gamma \geq k \in K(\mathbb{A}^\delta)\}$, where the latter equality holds by lemma 5.2.1, given that $\perp \in \{k \in K(\mathbb{A}^\delta) \mid k \leq \gamma\} \neq \emptyset$. But then

$$\mathbb{A} \models \forall \overline{\text{var}}' [\& S' \Rightarrow \varphi'(\gamma!/x) \leq \psi] \text{ iff } \mathbb{A} \models \forall \overline{\text{var}}' \forall \mathbf{j} [\& S' \& \mathbf{j} \leq \gamma \Rightarrow \varphi'(\mathbf{j}) \leq \psi].$$

The other approximation rules are justified in a similar manner, appealing to the other clauses of lemma 5.2.

The residuation rules for $f \in \mathcal{F}_n$ (resp. $g \in \mathcal{G}_n$) are justified by the fact that (the algebraic interpretation of) every such f (resp. g) is a left (resp. right) residual in each positive coordinate and left (resp. right) Galois-adjoint in each negative coordinate. The residuation rules for $f \in \mathcal{F}_r$ (resp. $g \in \mathcal{G}_r$) are justified as follows: recall that by Lemma 2.15,

$$\begin{aligned} \mathbb{A}^\delta \models f(p) = f(\perp) \vee \diamond_f(p) \text{ if } \varepsilon_f = 1 & \quad \mathbb{A}^\delta \models g(p) = g(\top) \wedge \square_g(p) \text{ if } \varepsilon_g = 1 \\ \mathbb{A}^\delta \models f(p) = f(\top) \vee \triangleleft_f(p) \text{ if } \varepsilon_f = \partial & \quad \mathbb{A}^\delta \models g(p) = g(\perp) \wedge \triangleright_g(p) \text{ if } \varepsilon_g = \partial. \end{aligned}$$

This, together with the the adjunction properties enjoyed by the normalized operations, guarantees that the rules above are sound, in that they can be ‘derived’ by means of equivalent substitution and adjunction for the normalized connectives. Below we provide examples of how this is done for $f \in \mathcal{F}_r$ with $\varepsilon_f = \partial$ (left-hand side) and $\varepsilon_f = 1$ (right-hand side).

$$\begin{array}{c}
\frac{f(\varphi) \leq \psi}{f(\top) \vee \triangleleft_f \varphi \leq \psi} \\
\frac{f(\top) \leq \psi \quad \triangleleft_f \varphi \leq \psi}{f(\top) \leq \psi \quad \blacktriangleleft_f \psi \leq \varphi}
\end{array}
\qquad
\begin{array}{c}
\frac{f(\varphi) \leq \psi}{f(\perp) \vee \diamond_f \varphi \leq \psi} \\
\frac{f(\perp) \leq \psi \quad \diamond_f \varphi \leq \psi}{f(\perp) \leq \psi \quad \varphi \leq \blacksquare_f \psi}
\end{array}$$

The Ackermann-rules are justified by the Ackermann lemmas 5.3 and 5.4. \square

Remark 5.6. The proof of the soundness of the approximation rules given above pivots on three ingredients: (1) the denseness of the canonical extension, guaranteeing e.g. that every element in \mathbb{A}^δ is the join of the closed elements below it; (2) the fact that the set of the closed elements below a given one is always nonempty, since \perp always belongs to it; (3) the defining property of the SAC connectives, guaranteeing the preservation of joins of nonempty collections. If nominals and co-nominals are interpreted as completely join-irreducible and meet-irreducible elements respectively, as in the usual context in which ALBA aims at correspondence, the set of the completely join-irreducible elements below a given element might be empty, and hence, these approximation rules are sound only at the price of strengthening the requirements on the Skeleton connectives different from Δ -adjoints, and insisting that they be SLR, and not just SAC. In [47], more complicated approximation rules had to be given, precisely to account for correspondence in a context in which Skeleton connectives other than Δ -adjoints are SAC.

Remark 5.7. In Section 7 we will prove that, for each language \mathcal{L}_{LE} , all \mathcal{L}_{LE} -inequalities on which ALBA succeeds are canonical. For this it is necessary to show that ALBA transformations preserve validity with respect to *admissible* assignments on canonical extensions of LEs. Towards this, it is an easy observation that all ALBA transformations other than the Ackermann rules preserve validity under admissible assignments. The potential difficulty with the Ackermann rules stems from the fact that they, unlike all other rules, involve changing the assignments to propositional variables.

6. ALBA SUCCESSFULLY REDUCES ALL INDUCTIVE INEQUALITIES

The aim of the present section is showing that ALBA is successful on all inductive \mathcal{L}_{LE} -inequalities, and that in fact *safe* and *pivotal* executions suffice. The arguments presented here are largely a synthesis of those made in [19] and [14]. We need to provide a certain amount of detail to show that different parts of the ALBA runs shown to exist satisfy the properties of safety and pivotality, either separately or in combination.

Definition 6.1. An execution of ALBA is *safe* if no side conditions (cf. Page 19) introduced by applications of adjunction rules for connectives in $\mathcal{F}_r \cup \mathcal{G}_r$ are further modified, except for receiving Ackermann substitutions.

Definition 6.2. An (Ω, ε) -inductive inequality is *definite* if in its critical branches, all Skeleton nodes are SAC nodes.

Lemma 6.3. *Let $\{\varphi_i \leq \psi_i\}$ be the set of inequalities obtained by preprocessing an (Ω, ε) -inductive \mathcal{L}_{LE} -inequality $\varphi \leq \psi$. Then each $\varphi_i \leq \psi_i$ is a definite (Ω, ε) -inductive inequality.*

Proof. Notice that the distribution during preprocessing only swap the order of Skeleton nodes on (critical) paths, and hence does not affect the goodness of critical branches. Moreover, PIA parts are entirely unaffected, and in particular the side conditions on SRR nodes of

critical branches are maintained. Finally, notice that SAC nodes commute exhaustively with Δ -adjoints, and hence all Δ -adjoints are effectively surfaced and eliminated via splitting, thus producing definite inductive inequalities. \square

The following definition is intended to capture the state of a system after approximation rules have been applied pivotally until no propositional variable remains in `lneq`:

Definition 6.4. Call a system (S, lneq) (Ω, ε) -stripped if `lneq` is pure, and for each $\xi \leq \chi \in S$ the following conditions hold:

- (1) one of $-\xi$ and $+\chi$ is pure, and the other is (Ω, ε) -inductive;
- (2) every ε -critical branch in $-\xi$ and $+\chi$ is PIA.

Lemma 6.5. *For any definite (Ω, ε) -inductive inequality $\varphi \leq \psi$ the system $(\mathcal{O}, \varphi \leq \psi)$ can be transformed into an (Ω, ε) -stripped system by the pivotal and safe application of approximation rules.*

Proof. By assumption, $+\varphi$ and $-\psi$ are both definite (Ω, ε) -inductive. Hence, for any propositional variable occurrences p , we can apply an approximation rule on the first non SAC-node in the path which goes from the root to p .

Let us show that the resulting system (S, lneq) is (Ω, ε) -stripped. Clearly, the procedure reduces `lneq` to a pure inequality. Each inequality in S is generated by the application of some approximation rule, and is therefore either of the form $\mathbf{j} \leq \alpha$ or $\beta \leq \mathbf{m}$, where $+\alpha$ and $-\beta$ are subtrees of (Ω, ε) -inductive trees, and hence are (Ω, ε) -inductive, as required by item 1 of the definition.

Next, if in an inequality in S contained a critical variable occurrence such that its associated path has an SAC node, then, because such a path is by assumption good, this would contradict the fact that the inequality has been generated by a pivotal application of an approximation rule. This shows item 2.

Lastly, we note that no approximation rule can alter a previously introduced side condition and that therefore the rule applications described in this lemma are safe. \square

Definition 6.6. An (Ω, ε) -stripped system (S, lneq) is *Ackermann-ready* with respect to a propositional variable p_i with $\varepsilon_i = 1$ (respectively, $\varepsilon_i = \partial$) if every inequality $\xi \leq \chi \in S$ is of one of the following forms:

- (1) $\xi \leq p$ where ξ is pure (respectively, $p \leq \chi$ where χ is pure), or
- (2) $\xi \leq \chi$ where neither $-\xi$ nor $+\chi$ contain any $+p_i$ (respectively, $-p_i$) leaves.

Note that the right or left Ackermann-rule (depending on whether $\varepsilon_i = 1$ or $\varepsilon_i = \partial$) is applicable to a system which is Ackermann-ready with respect to p_i . In fact, this would still have been the case had we weakened the requirement in clause 1 that ξ and χ must be pure to simply require that they do not contain p_i .

Lemma 6.7. *If (S, lneq) is (Ω, ε) -stripped and p_i is Ω -minimal among propositional variables occurring in (S, lneq) , then (S, lneq) can be transformed, through the application of residuation- and splitting-rules, into a system which is Ackermann-ready with respect to p_i . Moreover, this can be done in such a way that*

- (1) all side conditions introduced are pure inequalities, and
- (2) if all side conditions occurring in S are pure, then the all rule applications in this process are safe.

Proof. If $\xi \leq \chi \in S$ and $-\xi$ and $+\chi$ contain no ε -critical p_i -nodes then this inequality already satisfies condition 2 of Definition 6.6. So suppose that $-\xi$ and $+\chi$ contain some ε -critical p_i -node among them. This means $\xi \leq \chi$ is of the form $\alpha \leq \text{Pure}$ with the ε -critical p_i -node in α and Pure pure, or of the form $\text{Pure} \leq \delta$ with Pure pure and the ε -critical p_i -node in δ . We can now prove by simultaneous induction on α and δ that these inequalities can be transformed into the form specified by clause 1 of definition 6.6.

The base cases are when $-\alpha = -p_i$ and $+\delta = +p_i$. Here the inequalities are in desired shape and no rules need be applied to them. We will only check a few of the inductive cases. If $-\alpha = -(\alpha_1 \vee \alpha_2)$, then applying the \vee -splitting rule we transform $\alpha_1 \vee \alpha_2 \leq \text{Pure}$ into $\alpha_1 \leq \text{Pure}$ and $\alpha_2 \leq \text{Pure}$. The resulting system is clearly still (Ω, ε) -stripped, and we may apply the inductive hypothesis to $\alpha_1 \leq \text{Pure}$ and $\alpha_2 \leq \text{Pure}$. If $-\alpha = -f(\bar{\alpha})$ for some $f \in \mathcal{F}_n$, then, as per definition of inductive inequalities and given that p_i is by assumption Ω -minimal, exactly one of the formulas in $\bar{\alpha}$ contains an ε -critical node, and all the others (if any) are pure. Assume that the critical node is in α_j for $1 \leq j \leq n_f$. Then, applying the appropriate f -residuation rule transforms $f(\bar{\alpha}) \leq \text{Pure}$ into either $\alpha_j \leq f_j^\sharp(\alpha_1, \dots, \text{Pure}, \dots, \alpha_{n_f})$ if $\varepsilon_f = 1$ or $f_j^\sharp(\alpha_1, \dots, \text{Pure}, \dots, \alpha_{n_f}) \leq \alpha_j$ if $\varepsilon_f = \partial$, yielding an (Ω, ε) -stripped system, and the inductive hypothesis is applicable.

The case when $+\delta = +g(\bar{\delta})$ with $g \in \mathcal{G}_n$ is similar. We still need to consider the cases when $-\alpha$ is $-f(\alpha')$ for some $f \in \mathcal{F}_r$, and when $+\delta$ is $+g(\delta')$ for some $g \in \mathcal{G}_r$. As these cases are order-dual, we will only treat the latter. If $\varepsilon_g = 1$, then the residuation rule replaces $\text{Pure} \leq g(\delta')$ with the two inequalities $\text{Pure} \leq g(\top)$ and $\blacklozenge_g \text{Pure} \leq \delta'$, to which the inductive hypothesis is applicable. Notice that the introduced side condition is pure. If $\varepsilon_g = \partial$, then the residuation rule replaces $\text{Pure} \leq g(\delta')$ with $\text{Pure} \leq g(\perp)$ and $\delta \leq \blacktriangleright_g \text{Pure} \leq \delta'$, to which the inductive hypothesis is applicable. The introduced side condition is again pure.

In all the cases considered above, rules are applied only to non-pure inequalities, so the safety of these rule applications follow immediately from the assumption that all occurring side conditions are pure. \square

Lemma 6.8. *Applying the appropriate Ackermann-rule with respect to p_i to an (Ω, ε) -stripped system which is Ackermann-ready with respect to p_i , again yields an (Ω, ε) -stripped system.*

Proof. Let $(S, \varphi \leq \psi)$ be an (Ω, ε) -stripped system which is Ackermann-ready with respect to p_i . We only consider the case in which the right Ackermann-rule is applied, the case for the left Ackermann-rule being dual. This means that $S = \{\alpha_k \leq p \mid 1 \leq k \leq n\} \cup \{\beta_j(p_i) \leq \gamma_j(p_i) \mid 1 \leq j \leq m\}$ where the α s are pure and the $-\beta$ s and $+\gamma$ s contain no $+p_i$ nodes. Let us denote the pure formula $\bigvee_{k=1}^n \alpha_k$ by α . It is sufficient to show that for each $1 \leq j \leq m$, the trees $-\beta(\alpha/p_i)$ and $+\gamma(\alpha/p_i)$ satisfy the conditions of Definition 6.4. Condition 2 follows immediately once we notice that, since α is pure and is being substituted everywhere for variable occurrences corresponding to non-critical nodes, $-\beta(\alpha/p_i)$ and $+\gamma(\alpha/p_i)$ have exactly the same ε -critical paths as $-\beta(p_i)$ and $+\gamma(p_i)$, respectively. Condition 1, namely that $-\beta(\alpha/p_i)$ and $+\gamma(\alpha/p_i)$ are (Ω, ε) -inductive, also follows using additionally the observation that all new paths that arose from the substitution are variable free. \square

Theorem 6.9. *ALBA succeeds on all inductive inequalities, and safe, pivotal executions suffice.*

Proof. Let $\varphi \leq \psi$ be an (Ω, ε) -inductive inequality. By Lemma 6.3, applying preprocessing yields a finite set of definite (Ω, ε) -inductive inequalities, each of which gives rise to an initial

system $(\emptyset, \varphi' \leq \psi')$. Since preprocessing does not introduce side conditions, each initial system is free of them. By Lemma 6.5, pivotal applications of the approximation rules convert this system into an (Ω, ε) -stripped system, say $(S_1, \varphi'' \leq \psi'')$. These systems are still free of side conditions, since approximation rules do not introduce side conditions. Now pick any Ω -minimal variable occurring in $(S_1, \varphi'' \leq \psi'')$, say p . By lemma 6.7, the system can be made Ackermann-ready with respect to p by applying residuation and splitting rules while introducing only pure side conditions. Now apply the appropriate Ackermann-rule to eliminate p from the system. Since all occurring side conditions are pure, this is a safe rule application. By lemma 6.8, the result is again an (Ω, ε) -stripped system, now containing one propositional variable less, and to which lemma 6.7 can again be applied. This process is iterated until all occurring propositional variables are eliminated and a pure system is obtained. \square

7. CONSTRUCTIVE CANONICITY

This section is devoted to proving that all inequalities on which constructive ALBA succeeds are *constructively canonical* (i.e. their validity is preserved under the constructive canonical extension). The arguments presented here amalgamate the constructive generalizations of the canonicity arguments used in [19, 47] and [14]. Indeed, in the current setting, we need to appeal to (the constructive counterparts of) both the pivotality and safety of ALBA execution.

Fix an \mathcal{L}_{LE} -algebra \mathbb{A} and let \mathbb{A}^δ be its canonical extension. We write $\mathbb{A}^\delta \models_{\mathbb{A}} \varphi \leq \psi$ to indicate that $\mathbb{A}^\delta, \nu \models \varphi \leq \psi$ for all *admissible* assignments ν , as defined in Subsection 2.6, page 13. Recall that pivotal executions of ALBA are defined on page 19 in Section 4.

Theorem 7.1. *All \mathcal{L}_{LE} -inequalities on which ALBA succeeds safely and pivotally are canonical.*

Proof. Let $\varphi \leq \psi$ be an \mathcal{L}_{LE} -inequality for which some execution of ALBA which is both safe and pivotal exists. The required canonicity proof is summarized in the following U-shaped diagram:

$$\begin{array}{ccc}
 \mathbb{A} \models \varphi \leq \psi & & \mathbb{A}^\delta \models \varphi \leq \psi \\
 \Downarrow & & \Downarrow \\
 \mathbb{A}^\delta \models_{\mathbb{A}} \varphi \leq \psi & & \\
 \Downarrow & & \\
 \mathbb{A}^\delta \models_{\mathbb{A}} \text{ALBA}(\varphi \leq \psi) & \Leftrightarrow & \mathbb{A}^\delta \models \text{ALBA}(\varphi \leq \psi)
 \end{array}$$

The uppermost bi-implication on the left is given by the definition of validity on algebras and \mathbb{A} being a subalgebra of \mathbb{A}^δ . The lower bi-implication on the left is given by Proposition 7.7 below. The horizontal bi-implication follows from the facts that, by assumption, $\text{ALBA}(\varphi \leq \psi)$ is pure, and that, when restricted to pure formulas, the ranges of admissible and arbitrary assignments coincide. The bi-implication on the right is given by Theorem 5.1. \square

Towards the proof of Proposition 7.7, the following definitions and lemmas will be useful:

Definition 7.2 (Syntactically closed and open \mathcal{L}_{LE}^+ -formulas). (1) A \mathcal{L}_{LE}^+ formula is *syntactically closed* if all occurrences of nominals, f_i^\sharp for any $f \in \mathcal{F}_n$ with $\varepsilon_f(i) = \partial$, g_i^\flat for any $g \in \mathcal{G}_n$ with $\varepsilon_g(i) = 1$, \blacktriangleleft_f for any $f \in \mathcal{F}_r$ with $\varepsilon_f = \partial$, \blacklozenge_g for any $g \in \mathcal{G}_r$ with $\varepsilon_g = 1$

are positive, and all occurrences of co-nominals, $f_i^\#$ for any $f \in \mathcal{F}_n$ with $\varepsilon_f(i) = 1$, g_i^\flat for any $g \in \mathcal{G}_n$ with $\varepsilon_g(i) = \partial$, \blacktriangleright_g for any $g \in \mathcal{G}_r$ with $\varepsilon_g = \partial$, and $\blacksquare_f f \in \mathcal{F}_r$ with $\varepsilon_f = 1$ are negative;

- (2) A $\mathcal{L}_{\text{LE}}^+$ formula is *syntactically closed* if all occurrences of nominals, $f_i^\#$ for any $f \in \mathcal{F}_n$ with $\varepsilon_f(i) = \partial$, g_i^\flat for any $g \in \mathcal{G}_n$ with $\varepsilon_g(i) = 1$, \blacktriangleleft_f for any $f \in \mathcal{F}_r$ with $\varepsilon_f = \partial$, \blacklozenge_g for any $g \in \mathcal{G}_r$ with $\varepsilon_g = 1$ are negative, and all occurrences of co-nominals, $f_i^\#$ for any $f \in \mathcal{F}_n$ with $\varepsilon_f(i) = 1$, g_i^\flat for any $g \in \mathcal{G}_n$ with $\varepsilon_g(i) = \partial$, \blacktriangleright_g for any $g \in \mathcal{G}_r$ with $\varepsilon_g = \partial$, and $\blacksquare_f f \in \mathcal{F}_r$ with $\varepsilon_f = 1$ are positive.

A $\mathcal{L}_{\text{LE}}^+$ formula which is both syntactically open and syntactically closed is called *syntactically clopen*.

A system S of $\mathcal{L}_{\text{LE}}^+$ -inequalities is *compact-appropriate* if the left-hand side of each non-pure inequality in S is syntactically closed and the right-hand side of each non-pure inequality in S is syntactically open.

Definition 7.3. A system S of $\mathcal{L}_{\text{LE}}^+$ -inequalities is *topologically adequate* when the following conditions hold: for every $f \in \mathcal{F}_r$ and $g \in \mathcal{G}_r$,

- (1) if $\varepsilon_f = 1$ and $\varphi \leq \blacksquare_f \psi$ is in S , then $f(\perp) \leq \psi$ is in S ;
- (2) if $\varepsilon_g = 1$ and $\blacklozenge_g \varphi \leq \psi$ is in S , then $g(\top) \geq \varphi$ is in S ;
- (3) if $\varepsilon_f = \partial$ and $\blacktriangleleft_f \varphi \leq \psi$ is in S , then $f(\top) \leq \varphi$ is in S ;
- (4) if $\varepsilon_g = \partial$ and $\varphi \leq \blacktriangleright_g \psi$ is in S , then $g(\perp) \geq \varphi$ is in S .

Lemma 7.4. *Topological adequacy is an invariant of safe executions of ALBA.*

Proof. Preprocessing vacuously preserves the topological adequacy of any input inequality. The topological adequacy is vacuously satisfied up to the first application of an adjunction rule introducing any connective \blacksquare_f for $f \in \mathcal{F}_r$ with $\varepsilon_f = 1$, or \blacklozenge_g for $g \in \mathcal{G}_r$ with $\varepsilon_g = 1$, or \blacktriangleleft_f for $f \in \mathcal{F}_r$ with $\varepsilon_f = \partial$, or \blacktriangleright_g for $g \in \mathcal{G}_r$ with $\varepsilon_g = \partial$. Each such application introduces two inequalities, one of which contains the new black connective, and the other one is exactly the side condition required by the definition of topological adequacy for the first inequality to be non-offending. Moreover, at any later stage, safe executions of ALBA do not modify the side conditions, except for substituting minimal valuations. This, together with the fact that ALBA does not contain any rules which allow to manipulate any of \blacksquare_g , \blacklozenge_f , \blacktriangleleft_g , \blacktriangleright_f , guarantees the preservation of topological adequacy. Indeed, if e.g. $f(\perp) \leq \psi$ and $\varphi \leq \blacksquare_f \psi$ are both in a topologically adequate quasi-inequality, then the variables occurring in ψ in both inequalities have the same polarity, and in a safe execution, the only way in which they could be modified is if they both receive the same minimal valuations under applications of Ackermann rules. Hence, after such an application, they would respectively be transformed into $f(\perp) \leq \psi'$ and $\varphi' \leq \blacksquare_f \psi'$ for the *same* ψ' . Thus, the topological adequacy of the quasi-inequality is preserved. \square

Lemma 7.5. *Compact-appropriateness is an invariant of ALBA^e executions.*

Proof. The proof proceeds by checking, case-by-case, that each rule of ALBA preserves compact-appropriateness, and is entirely analogous to the proof of [13, Lemma 9.5]. \square

Lemma 7.6. *If the system (S, Ineq) is obtained by running ALBA pivotally on some \mathcal{L}_{LE} -inequality $\varphi \leq \psi$, then (S, Ineq) is compact-appropriate and topologically adequate.*

Proof. Since $\varphi \leq \psi$ comes from the base language \mathcal{L}_{LE} , it is immediate that φ and ψ are both syntactically clopen. Since preprocessing does not introduce any symbols not in \mathcal{L}_{LE} ,

in any inequality $\varphi' \leq \psi'$ resulting from the preprocessing, φ' and ψ' are both syntactically clopen. Thus, the claim holds for each initial system $(\emptyset, \varphi' \leq \psi')$. In order to complete the proof, it now remains to check that each reduction rule preserves the desired syntactic shape. This is straightforward for the residuation rules. By way of illustration we will consider the right-negative approximation rule and the righthand Ackermann-rule.

The right-negative approximation rule transforms a system $(S, s \leq t'(\gamma!/x))$ with $-x < -t'(!x)$ into $(S \cup \{\gamma \leq \mathbf{m}\}, s \leq t'(\mathbf{m}/!x))$. If γ belongs to the original language, it is both syntactically clopen. Hence, $\gamma \leq \mathbf{m}$ is of the right shape. Moreover, $-x < -t'(!x)$ implies that x occurs positively in t' , which by assumption is syntactically open. Hence, $t'(\mathbf{m}/!x)$ is syntactically open. If γ does not belong to \mathcal{L}_{LE} , then the assumption that all approximation rules are applied pivotally guarantees that γ must be a conominal \mathbf{n} (which has been introduced by some previous application of the same approximation rule). Hence $\gamma \leq \mathbf{m}$ is pure.

As for the righthand Ackermann rule, it transforms a system

$$(\{\alpha_1 \leq p, \dots, \alpha_n \leq p, \beta_1 \leq \gamma_1, \dots, \beta_m \leq \gamma_m\}, \text{Ineq})$$

into

$$(\{\beta_1(\alpha/p) \leq \gamma_1(\alpha/p), \dots, \beta_m(\alpha/p) \leq \gamma_m(\alpha/p)\}, \text{Ineq}),$$

where $\alpha = \alpha_1 \vee \dots \vee \alpha_n$. Firstly, note that all the pure inequalities among the $\beta_i \leq \gamma_i$ remain unaffected by the rule, and hence remain pure. For non-pure $\beta_i \leq \gamma_i$, we have by assumption that β_i is syntactically closed and positive in p while γ_i is syntactically open and negative in p . Thus, in $\beta_i(\alpha/p)$ each occurrence of a symbol within any occurrence of the subformula α has the same polarity as it had in α before substitution. Hence, since α is syntactically closed, $\beta_i(\alpha/p)$ is syntactically closed. Similarly, in $\gamma_i(\alpha/p)$ any occurrence of a symbol within each occurrence of the subformula α has the opposite polarity from that which it had in α before substitution. Hence, $\gamma_i(\alpha/p)$ is syntactically open. \square

Proposition 7.7 (Correctness of safe and pivotal executions of ALBA under admissible assignments). *If ALBA succeeds in reducing an \mathcal{L}_{LE} -inequality $\varphi \leq \psi$ through some safe and pivotal execution and yields $\text{ALBA}(\varphi \leq \psi)$, then $\mathbb{A}^\delta \models_{\mathbb{A}} \varphi \leq \psi$ iff $\mathbb{A}^\delta \models_{\mathbb{A}} \text{ALBA}(\varphi \leq \psi)$.*

Proof. It has already been indicated in Remark 5.7 that the proof is essentially the same as that of Theorem 5.1. The only difficulty that arises is that the Ackermann-rules are generally not invertible under admissible assignments (cf. [13, Example 9.1]). However, by Propositions A.8 and A.9, in the special case that the system is topologically adequate and the left and right hand sides of all non-pure inequalities involved in the application of an Ackermann-rule are, respectively, syntactically closed and open, the rule is sound and invertible under admissible assignments. By Lemmas 7.4 and 7.6, these requirements are always satisfied when the rule is applied in safe and pivotal executions of ALBA. \square

Our main theorem follows as a corollary of Theorem 6.9 and Proposition 7.7.

Theorem 7.8 (Main). *All inductive \mathcal{L}_{LE} -inequalities are constructively canonical.*

8. CONCLUSIONS AND FURTHER DIRECTIONS

About the setting of the present paper. We have chosen to work in the setting of arbitrary lattices expanded with normal and/or regular operations. However, all results in

this paper generalize readily to posets expanded with operations of similar order-theoretic properties. Indeed, just like the canonical extension of a lattice, the canonical extension of a poset is a complete lattice into which the poset embeds in a dense and compact way [25]. Thus, all the properties we need to prove the soundness of (constructive) ALBA are still available in the poset case. This remains true even when one takes into account that, for posets, the notion of canonical extension is parametric in the choice of a definition of filters and ideals [45, 30].

On the other hand, the choice of the general (non-distributive) lattice setting, rather than the distributive lattice setting, is particularly advantageous for the formulation of a constructive version of ALBA. Indeed, the approximation rules employed in ALBA in the distributive setting [13] make essential use of the fact that the canonical extension there is completely join-generated (resp. meet-generated) by its completely join-*prime* (resp. meet-*prime*) elements. The approximation rules of distributive ALBA rely on the primeness of nominals and co-nominals and therefore fail utterly in a constructive environment, where this generation by sets of join- and meet-prime elements is unavailable. In the (non-constructive) non-distributive setting we still have complete join- and meet-generation by completely join- and meet-irreducible elements, but these elements need no longer be completely join- and meet-prime. Moreover, the soundness of the approximation rules of non-distributive ALBA [14] relies only on the fact that completely join- and meet-irreducibles are complete join- and meet generators, and remain sound when we replace them with any other sets of complete join- and meet generators. Such a replacement is, in fact, all we needed to do in order to salvage the soundness of these rules for the constructive case.

Our choice to extend our treatment to *regular* modal operators reflects the importance of these operators from the viewpoint of applications in formal philosophy, AI, and the social sciences. Indeed, many influential and very diversely motivated formalisms have appeared, also very recently (STIT logics [1], logics for conditionals [50], and for agent organizations [24] are a sample from the three areas), in which necessitation might not be plausible. Also in classical modal logic, necessitation has been observed to be incompatible w.r.t. certain core readings of the modal operators, such as the epistemic and the deontic (cf. Lemmon's work [43]). Regular modal logics cannot be accommodated semantically by standard relational models of normal modal logics, and a variety of competing semantic frameworks for regular logics exists in the literature, ranging from impossible worlds [41] to neighbourhood models [36]. This poses the problem that a separate Sahlqvist-type theory would need to be developed for each of these semantic frameworks. Via duality, the results in the present paper immediately imply the canonicity of inductive formulas of logical frameworks including regular operators in each of these semantic frameworks, which sorts out the canonicity issue simultaneously in each such setting. The logical systems mentioned above are mostly on a classical propositional base. Specifically concerning the lattice-based setting, an epistemic interpretation has been proposed very recently for the logic of lattice expansions with unary modal operators [9]. Specifically, this logic has been proposed as a formalism for reasoning about *categories* (understood as 'identity classes' of objects, and formalized as concepts in formal concept analysis [28]) and the way they are perceived, known and understood by different agents. In view of this interpretation, the treatment of regular modal operators on general lattices developed in the present paper contributes to a more accurate and flexible modelling of the epistemic reasoning of agents concerning categories.

The relational semantics developed in [9] is based on enriched polarities (referred to as polarity-based frames or enriched formal contexts) upon which certain conditions are imposed in order to ensure a duality with perfect lattice expansions. These conditions are dropped in the subsequent paper [10], leading to a constructive adjunction with complete (but not necessarily perfect) lattices and making the results of the present paper directly applicable. The epistemic interpretation of [9] and [10] is not inherent to polarity-based semantics, which can also support other interpretations, such as those proposed in e.g. in [8], where the semantic framework of polarity-based frames for lattice-based modal logic are used to generalize Rough Set Theory [48] to “rough formal concept theory”.

Alternative, graph-based semantics for non-distributive logics, also based on constructive adjunctions with complete lattice expansions, was introduced and studied in [5] and [17] where, in contrast to the polarity based semantics, formulas again denote propositions admitting truth values rather than denoting concepts, and where a natural hyper-constructivist reading of non-distributive logics emerges. Both the polarity and graph-based semantic paradigms can be naturally generalized to many-valued version, as has been done in [4], [15] and [16].

Constructive canonicity and possibility semantics. Possibility semantics [38] is an alternative and more general interpretation of classical (modal) logic on relational structures the states of which are not constrained to encode complete information. Thus, in a possibility model M , the consistent set $\{\varphi \in \mathcal{L} \mid M, w \Vdash \varphi\}$ does not need to be maximal for every state w in M . In this setting, the completeness of classical (modal) logic can be proven without appeal to any of the equivalent formulations of the axiom of choice. In [37], duality, correspondence and canonicity are developed for classical modal logic on possibility frames, where *filter-canonicity* is understood as the preservation of the validity of formulas from filter-descriptive (cf. [37, Definition 5.39]) to full possibility frames. As observed by Holliday, the BAO dually arising from the filter frame⁸ (cf. [37, Definition 5.30]) associated with any BAO \mathbb{A} is (isomorphic to) the constructive canonical extension of \mathbb{A} (cf. [37, Theorem 5.46]). Hence, it follows from the duality results developed in the same paper that filter-canonicity corresponds algebraically to the notion of constructive canonicity. Thus, again via duality, the constructive canonicity of inductive inequalities provides a straightforward route to proving that inductive formulas of classical (modal) logic are filter-canonical (cf. [37, Theorem 7.20]). The algorithmic treatment of canonicity and correspondence for possibility semantics is developed in [53].

Further directions. In [33], the main tools of unified correspondence theory have been used to characterize the axiomatic extensions of basic DLE-logics which admit proper display calculi in terms of a proper subclass of inductive inequalities (referred to as *analytic inductive inequalities*), and to effectively compute analytic rules corresponding to these axioms. Very recently, the class of analytic inductive inequalities has been shown to have interesting formal-topological properties, which can be formulated in terms of a generalized form of (both non-constructive and constructive) canonicity, referred to as *slanted canonicity* [23]. These results and insights have formed the base for the systematic development of analytic sequent calculi (cf. e.g. [34, 35, 2, 26]) endowed with a package of basic properties (soundness, completeness, conservativity, cut elimination and subformula property) which can be established uniformly also thanks to unified correspondence techniques. In particular, the canonicity of inductive inequalities is crucial for the uniform proof of soundness and

⁸By definition, any filter frame is a full possibility frame.

conservativity of these calculi. Generalizing canonicity to a constructive setting opens the possibility of endowing these calculi with a category-theoretic semantics, an amenable environment for addressing the long-standing issue of establishing a theory of isomorphisms of derivations. In particular, in [35], a proper display calculus has been introduced for the basic environment of the logic of general (i.e. not necessarily distributive) lattices, which can serve as a base for investigating these issues for LE logics.

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APPENDIX A. APPENDIX

Section A.1 below adapts parts of [19, Section 4.1] to the present LE setting. Sections A.2 and A.3 are based on [19, Section 10] and adapt it to the present setting. We include them for the sake of self-containedness.

Throughout this section, we let \mathbb{A} and \mathbb{B} be LEs, and \mathbb{A}^δ and \mathbb{B}^δ be their constructive canonical extensions.

A.1. Constructive conditional Esakia lemma.

Definition A.1. For all maps $f, g : \mathbb{A}^\delta \rightarrow \mathbb{B}^\delta$,

- (1) f is *positive closed Esakia* if it preserves down-directed meets of closed elements of \mathbb{A}^δ , that is:

$$f\left(\bigwedge \{c_i : i \in I\}\right) = \bigwedge \{f(c_i) : i \in I\}$$

for any downward-directed collection $\{c_i : i \in I\} \subseteq K(\mathbb{A}^\delta)$;

- (2) g is *positive open Esakia* if it preserves upward-directed joins of open elements of \mathbb{A}^δ , that is:

$$g\left(\bigvee \{o_i : i \in I\}\right) = \bigvee \{g(o_i) : i \in I\}$$

for any upward-directed collection $\{o_i : i \in I\} \subseteq O(\mathbb{A}^\delta)$.

- (3) f is *negative closed Esakia* if it reverses upward-directed joins of open elements of \mathbb{A}^δ , that is:

$$f\left(\bigvee \{o_i : i \in I\}\right) = \bigwedge \{f(o_i) : i \in I\}$$

for any upward-directed collection $\{o_i : i \in I\} \subseteq O(\mathbb{A}^\delta)$;

- (4) g is *negative open Esakia* if it reverses down-directed meets of closed elements of \mathbb{A}^δ , that is:

$$g\left(\bigwedge \{c_i : i \in I\}\right) = \bigvee \{g(c_i) : i \in I\}$$

for any downward-directed collection $\{c_i : i \in I\} \subseteq K(\mathbb{A}^\delta)$.

Lemma A.2. Let $f, g : \mathbb{A}^\delta \rightarrow \mathbb{B}^\delta$. For any $o \in O(\mathbb{B}^\delta)$ and $k \in K(\mathbb{B}^\delta)$,

- (1) if f is positive closed Esakia, $f(a) \in K(\mathbb{B}^\delta)$ for every $a \in \mathbb{A}$, and $f(\perp) \leq o$, then $\blacksquare_f(o) = \bigvee \{a \in \mathbb{A} \mid a \leq \blacksquare_f(o)\} \in O(\mathbb{A}^\delta)$.
- (2) if g is positive open Esakia, $g(a) \in O(\mathbb{B}^\delta)$ for every $a \in \mathbb{A}$, and $k \leq g(\top)$, then $\blacklozenge_g(k) = \bigwedge \{a \in \mathbb{A} \mid \blacklozenge_g(k) \leq a\} \in K(\mathbb{A}^\delta)$.
- (3) if f is negative closed Esakia, $f(a) \in K(\mathbb{B}^\delta)$ for every $a \in \mathbb{A}$, and $f(\top) \leq o$, then $\blacktriangleleft_f(o) = \bigwedge \{a \in \mathbb{A} \mid \blacktriangleleft_f(o) \leq a\} \in K(\mathbb{A}^\delta)$.
- (4) if g is negative open Esakia, $g(a) \in O(\mathbb{B}^\delta)$ for every $a \in \mathbb{A}$, and $k \leq g(\perp)$, then $\blacktriangleright_g(k) = \bigvee \{a \in \mathbb{A} \mid a \leq \blacktriangleright_g(k)\} \in O(\mathbb{A}^\delta)$.

Proof. 1. To prove the statement, it is enough to show that if $c \in K(\mathbb{A}^\delta)$ and $c \leq \blacksquare_f(o)$, then $c \leq a$ for some $a \in \mathbb{A}$ such that $a \leq \blacksquare_f(o)$. By adjunction, $c \leq \blacksquare_f(o)$ is equivalent to $\diamond_f(c) \leq o$. Then, by assumption, $f(\perp) \vee \diamond_f(c) \leq o$. The definition of \diamond_f implies that $f(c) \leq f(\perp) \vee \diamond_f(c) \leq o$. Since f is closed Esakia, $f(c) = \bigwedge \{f(a) \mid a \in \mathbb{A} \text{ and } c \leq a\}$. Moreover, by assumption, $f(a) \in K(\mathbb{B}^\delta)$ for every $a \in \mathbb{A}$. Hence by compactness, $\bigwedge_{i=1}^n f(a_i) \leq o$ for some $a_1, \dots, a_n \in \mathbb{A}$ s.t. $c \leq a_i$, $1 \leq i \leq n$. Let $a = \bigwedge_{i=1}^n a_i$. Clearly, $c \leq a$ and $a \in \mathbb{A}$; moreover, by the monotonicity of f and Lemma 2.6.1, we have $\diamond_f(a) \leq f(a) \leq \bigwedge_{i=1}^n f(a_i) \leq o$, and hence, by adjunction, $a \leq \blacksquare_f(o)$.

Clauses 2, 3 and 4 are order-variants of 1. \square

The following Esakia-type result is similar to [19, Proposition 30]. There it was called *conditional Esakia lemma*, since, unlike the usual versions, it crucially relies on additional assumptions (on $f(\perp)$ and $g(\top)$).

Proposition A.3 (Conditional Esakia Lemma). *Let $f, g : \mathbb{A}^\delta \rightarrow \mathbb{B}^\delta$ such that f is closed Esakia and $f(a) \in K(\mathbb{B}^\delta)$ for every $a \in \mathbb{A}$, and g is open Esakia and $g(a) \in O(\mathbb{B}^\delta)$ for every $a \in \mathbb{A}$. For any upward-directed collection $\mathcal{O} \subseteq O(\mathbb{B}^\delta)$ and any downward-directed collection $\mathcal{C} \subseteq K(\mathbb{B}^\delta)$,*

- (1) *if $f(\perp) \leq \bigvee \mathcal{O}$, then $\blacksquare_f(\bigvee \mathcal{O}) = \bigvee \{\blacksquare_f o \mid o \in \mathcal{O}\}$. Moreover, there exists some upward-directed subcollection $\mathcal{O}' \subseteq \mathcal{O}$ such that $\bigvee \mathcal{O}' = \bigvee \mathcal{O}$, and $\blacksquare_f o \in O(\mathbb{A}^\delta)$ for each $o \in \mathcal{O}'$, and $\bigvee \{\blacksquare_f o \mid o \in \mathcal{O}'\} = \bigvee \{\blacksquare_f o \mid o \in \mathcal{O}\}$.*
- (2) *if $g(\top) \geq \bigwedge \mathcal{C}$, then $\blacklozenge_g(\bigwedge \mathcal{C}) = \bigwedge \{\blacklozenge_g c \mid c \in \mathcal{C}\}$. Moreover, there exists some downward-directed subcollection $\mathcal{C}' \subseteq \mathcal{C}$ such that $\bigwedge \mathcal{C}' = \bigwedge \mathcal{C}$, and $\blacklozenge_g c \in K(\mathbb{A}^\delta)$ for each $c \in \mathcal{C}'$, and $\bigwedge \{\blacklozenge_g c \mid c \in \mathcal{C}'\} = \bigwedge \{\blacklozenge_g c \mid c \in \mathcal{C}\}$.*

Proof. 1. For the first part, it is enough to show that, if $k \in K(\mathbb{A}^\delta)$ and $k \leq \blacksquare_f(\bigvee \mathcal{O})$, then $k \leq \blacksquare_f(o_k)$ for some $o_k \in \mathcal{O}$. The assumption $k \leq \blacksquare_f(\bigvee \mathcal{O})$ can be rewritten as $\diamond_f k \leq \bigvee \mathcal{O}$, which together with $f(\perp) \leq \bigvee \mathcal{O}$ yields $f(\perp) \vee \diamond_f k \leq \bigvee \mathcal{O}$. By Lemma 2.6.1, this inequality can be rewritten as $f(k) \leq \bigvee \mathcal{O}$. From f being closed Esakia and $f(a) \in K(\mathbb{B}^\delta)$ for every $a \in \mathbb{A}$, it follows that $f(k) \in K(\mathbb{B}^\delta)$. By compactness, $f(k) \leq \bigvee_{i=1}^n o_i$ for some $o_1, \dots, o_n \in \mathcal{O}$. Since \mathcal{O} is upward-directed, $o_k \geq \bigvee_{i=1}^n o_i$ for some $o_k \in \mathcal{O}$. Then $f(\perp) \vee \diamond_f k = f(k) \leq o_k$, which yields $\diamond_f k \leq o_k$, which by adjunction can be rewritten as $k \leq \blacksquare_f(o_k)$ as required.

As to the second part of the statement, notice that the assumption $f(\perp) \leq \bigvee \mathcal{O}$ is too weak to imply that $f(\perp) \leq o$ for each $o \in \mathcal{O}$, and hence we cannot conclude, by way of Lemma A.2, that $\blacksquare_f o \in O(\mathbb{A}^\delta)$ for every $o \in \mathcal{O}$. However, let

$$\mathcal{O}' := \{o \in \mathcal{O} \mid o \geq o_k \text{ for some } k \in K(\mathbb{A}^\delta) \text{ s.t. } k \leq \blacksquare_f(\bigvee \mathcal{O})\}.$$

Clearly, by construction we have that \mathcal{O}' is upward-directed and $\bigvee \mathcal{O}' = \bigvee \mathcal{O}$, and moreover for each $o \in \mathcal{O}'$, we have that $f(\perp) \leq o_k \leq o$ for some $k \in K(\mathbb{A}^\delta)$ s.t. $k \leq \blacksquare_f(\bigvee \mathcal{O})$. Hence, by Lemma A.2, $\blacksquare_f o \in O(\mathbb{A}^\delta)$ for each $o \in \mathcal{O}'$. Moreover, the monotonicity of \blacksquare_f and the previous part of the statement imply that $\bigvee \{\blacksquare_f o \mid o \in \mathcal{O}'\} = \blacksquare_f(\bigvee \mathcal{O}) = \bigvee \{\blacksquare_f o \mid o \in \mathcal{O}\}$.

2. is order-dual. \square

A.2. Intersection lemmas for \mathcal{L}_{LE}^+ -formulas. Recall that $\blacksquare_f, \blacklozenge_g, \blacktriangleleft_f, \blacktriangleright_g : \mathbb{A}^\delta \rightarrow \mathbb{A}^\delta$ respectively denote the adjoints of the normalization maps \diamond_f for any $f \in \mathcal{F}_r$ with $\varepsilon_f = 1$, \square_g for any $g \in \mathcal{G}_r$ with $\varepsilon_g = 1$, \blacktriangleleft_f for any $f \in \mathcal{F}_r$ with $\varepsilon_f = \partial$, \blacktriangleright_g for any $g \in \mathcal{G}_r$ with $\varepsilon_g = \partial$. Then Lemma A.2 immediately implies the following facts, which will be needed for the soundness of the topological Ackermann rule:

- Lemma A.4.** (1) If $f \in \mathcal{F}_r$ with $\varepsilon_f = 1$, and $o \in O(\mathbb{A}^\delta)$ s.t. $f(\perp) \leq o$, then $\blacksquare_f(o) = \bigvee\{a \mid a \leq \blacksquare_f(o)\} \in O(\mathbb{A}^\delta)$.
- (2) If $g \in \mathcal{G}_r$ with $\varepsilon_g = 1$, and $k \in K(\mathbb{A}^\delta)$ s.t. $g(\top) \geq k$, then $\blacklozenge_g(k) = \bigwedge\{a \mid a \geq \blacklozenge_g(k)\} \in K(\mathbb{A}^\delta)$.
- (3) If $f \in \mathcal{F}_r$ with $\varepsilon_f = \partial$, and $o \in O(\mathbb{A}^\delta)$ s.t. $f(\top) \leq o$, then $\blacktriangleleft_f(o) = \bigwedge\{a \mid a \geq \blacktriangleleft_f(o)\} \in K(\mathbb{A}^\delta)$.
- (4) If $g \in \mathcal{G}_r$ with $\varepsilon_g = \partial$, and $k \in K(\mathbb{A}^\delta)$ s.t. $g(\perp) \geq k$, then $\blacktriangleright_g(k) = \bigvee\{a \mid a \leq \blacktriangleright_g(k)\} \in O(\mathbb{A}^\delta)$.

In what follows, we work under the assumption that the values of all parameters (propositional variables, nominals and conominals) occurring in the term functions mentioned in the statements of propositions and lemmas are given by admissible assignments.

Lemma A.5. Let $\varphi(p)$ be syntactically closed, $\psi(p)$ syntactically open (cf. Definition 7.2), $c \in K(\mathbb{A}^\delta)$ and $o \in O(\mathbb{A}^\delta)$.

- (1) If $\varphi(p)$ is positive in p , $\psi(p)$ is negative in p , and for all $f \in \mathcal{F}_r$ and $g \in \mathcal{G}_r$,
- (a) if $\varepsilon_f = 1$, then $f(\perp) \leq \psi'^{\mathbb{A}^\delta}(c)$ for any subformula $\blacksquare_f\psi'(p)$ in $\varphi(p)$ and $\psi(p)$;
 - (b) if $\varepsilon_g = 1$, then $g(\top) \geq \psi'^{\mathbb{A}^\delta}(c)$ for any subformula $\blacklozenge_g\psi'(p)$ in $\varphi(p)$ and $\psi(p)$;
 - (c) if $\varepsilon_f = \partial$, then $f(\top) \leq \psi'^{\mathbb{A}^\delta}(c)$ for any subformula $\blacktriangleleft_f\psi'(p)$ in $\varphi(p)$ and $\psi(p)$;
 - (d) if $\varepsilon_g = \partial$, then $g(\perp) \geq \psi'^{\mathbb{A}^\delta}(c)$ for any subformula $\blacktriangleright_g\psi'(p)$ in $\varphi(p)$ and $\psi(p)$;
- then $\varphi(c) \in K(\mathbb{A}^\delta)$ and $\psi(c) \in O(\mathbb{A}^\delta)$ for each $c \in K(\mathbb{A}^\delta)$.
- (2) If $\varphi(p)$ is negative in p , $\psi(p)$ is positive in p , and for all $f \in \mathcal{F}_r$ and $g \in \mathcal{G}_r$,
- (a) if $\varepsilon_f = 1$, then $f(\perp) \leq \psi'^{\mathbb{A}^\delta}(o)$ for any subformula $\blacksquare_f\psi'(p)$ in $\varphi(p)$ and $\psi(p)$;
 - (b) if $\varepsilon_g = 1$, then $g(\top) \geq \psi'^{\mathbb{A}^\delta}(o)$ for any subformula $\blacklozenge_g\psi'(p)$ in $\varphi(p)$ and $\psi(p)$;
 - (c) if $\varepsilon_f = \partial$, then $f(\top) \leq \psi'^{\mathbb{A}^\delta}(o)$ for any subformula $\blacktriangleleft_f\psi'(p)$ in $\varphi(p)$ and $\psi(p)$;
 - (d) if $\varepsilon_g = \partial$, then $g(\perp) \geq \psi'^{\mathbb{A}^\delta}(o)$ for any subformula $\blacktriangleright_g\psi'(p)$ in $\varphi(p)$ and $\psi(p)$;
- then $\varphi(o) \in K(\mathbb{A}^\delta)$ and $\psi(o) \in O(\mathbb{A}^\delta)$ for each $o \in O(\mathbb{A}^\delta)$.

Proof. 1. The proof proceeds by simultaneous induction on φ and ψ . It is easy to see that φ cannot be \mathbf{m} , and the outermost connective of φ cannot be f_i^\sharp for any $f \in \mathcal{F}_n$ with $\varepsilon_f(i) = 1$, or g_j^b for any $g \in \mathcal{G}_n$ with $\varepsilon_g(j) = \partial$, or \blacksquare_f for any $f \in \mathcal{F}_r$ with $\varepsilon_f = 1$, or \blacktriangleright_g for any $g \in \mathcal{G}_r$ with $\varepsilon_g = \partial$. Similarly, ψ cannot be \mathbf{i} , and the outermost connective of ψ cannot be g_j^b for any $g \in \mathcal{G}_n$ with $\varepsilon_g(j) = 1$, or f_i^\sharp for any $f \in \mathcal{F}_n$ with $\varepsilon_f(i) = \partial$, or \blacklozenge_g for any $g \in \mathcal{G}_r$ with $\varepsilon_g = 1$, or \blacktriangleleft_f for any $f \in \mathcal{F}_r$ with $\varepsilon_f = \partial$.

The basic cases, that is, $\varphi = \perp, \top, p, q, \mathbf{i}$ and $\psi = \perp, \top, p, q, \mathbf{m}$ are straightforward.

Assume that $\varphi(p) = \blacklozenge_g\varphi'(p)$. Since $\varphi(p)$ is positive in p , the subformula $\varphi'(p)$ is syntactically closed and positive in p , and assumptions 1(a)-1(d) hold also for $\varphi'(p)$. Hence, by inductive hypothesis, $\varphi'(c) \in K(\mathbb{A}^\delta)$ for any $c \in K(\mathbb{A}^\delta)$. In particular, assumption 1(b) implies that $g(\top) \geq \varphi'(c)$. Hence, by Lemma A.4, $\blacklozenge_g\varphi'(c) \in K(\mathbb{A}^\delta)$, as required. The case in which $\varphi(p)$ is negative in p is argued order-dually.

The cases in which $\varphi(p) = \blacksquare_f\varphi'(p), \blacktriangleleft_f\varphi'(p), \blacktriangleright_g\varphi'(p)$ are similar to the one above.

The cases of the remaining connectives are treated as in [13, Lemma 11.9] and the corresponding proofs are omitted. Of course, in [13] nominals and co-nominals are evaluated to completely join- and meet-irreducibles, respectively. However, the only property of completely join- and meet-irreducibles used there is the fact that they are, respectively, closed and open, and therefore the proof is directly transferable to the present setting. \square

Lemma A.6 (Intersection lemma). *Let $\varphi(p)$ be syntactically closed, $\psi(p)$ syntactically open, $C \subseteq K(\mathbb{A}^\delta)$ downward-directed, $O \subseteq O(\mathbb{A}^\delta)$ upward-directed. Then*

- (1) *if $\varphi(p)$ is positive in p , $\psi(p)$ is negative in p , and for all $f \in \mathcal{F}_r$ and $g \in \mathcal{G}_r$,*
 - (a) *if $\varepsilon_f = 1$, then $f(\perp) \leq \psi'^{\mathbb{A}^\delta}(\wedge C)$ for any subformula $\blacksquare_f \psi'(p)$ in $\varphi(p)$ and $\psi(p)$,*
 - (b) *if $\varepsilon_g = 1$, then $g(\top) \geq \psi'^{\mathbb{A}^\delta}(\wedge C)$ for any subformula $\blacklozenge_g \psi'(p)$ in $\varphi(p)$ and $\psi(p)$,*
 - (c) *if $\varepsilon_f = \partial$, then $f(\top) \leq \psi'^{\mathbb{A}^\delta}(\wedge C)$ for any subformula $\blacktriangleleft_f \psi'(p)$ in $\varphi(p)$ and $\psi(p)$,*
 - (d) *if $\varepsilon_g = \partial$, then $g(\perp) \geq \psi'^{\mathbb{A}^\delta}(\wedge C)$ for any subformula $\blacktriangleright_g \psi'(p)$ in $\varphi(p)$ and $\psi(p)$,*
then
 - (a) $\varphi^{\mathbb{A}^\delta}(\wedge C) = \wedge \{\varphi^{\mathbb{A}^\delta}(c) : c \in C'\}$ *for some down-directed subcollection $C' \subseteq C$ such that $\varphi^{\mathbb{A}^\delta}(c) \in K(\mathbb{A}^\delta)$ for each $c \in C'$.*
 - (b) $\psi^{\mathbb{A}^\delta}(\wedge C) = \vee \{\psi^{\mathbb{A}^\delta}(c) : c \in C'\}$ *for some down-directed subcollection $C' \subseteq C$ such that $\psi^{\mathbb{A}^\delta}(c) \in O(\mathbb{A}^\delta)$ for each $c \in C'$.*
- (2) *If $\varphi(p)$ is negative in p , $\psi(p)$ is positive in p , and for all $f \in \mathcal{F}_r$ and $g \in \mathcal{G}_r$,*
 - (a) *if $\varepsilon_f = 1$, then $f(\perp) \leq \psi'^{\mathbb{A}^\delta}(\vee O)$ for any subformula $\blacksquare_f \psi'(p)$ in $\varphi(p)$ and $\psi(p)$,*
 - (b) *if $\varepsilon_g = 1$, then $g(\top) \geq \psi'^{\mathbb{A}^\delta}(\vee O)$ for any subformula $\blacklozenge_g \psi'(p)$ in $\varphi(p)$ and $\psi(p)$,*
 - (c) *if $\varepsilon_f = \partial$, then $f(\top) \leq \psi'^{\mathbb{A}^\delta}(\vee O)$ for any subformula $\blacktriangleleft_f \psi'(p)$ in $\varphi(p)$ and $\psi(p)$,*
 - (d) *if $\varepsilon_g = \partial$, then $g(\perp) \geq \psi'^{\mathbb{A}^\delta}(\vee O)$ for any subformula $\blacktriangleright_g \psi'(p)$ in $\varphi(p)$ and $\psi(p)$,*
then
 - (a) $\varphi^{\mathbb{A}^\delta}(\vee O) = \wedge \{\varphi^{\mathbb{A}^\delta}(o) : o \in O'\}$ *for some up-directed subcollection $O' \subseteq O$ such that $\varphi^{\mathbb{A}^\delta}(o) \in K(\mathbb{A}^\delta)$ for each $o \in O'$.*
 - (b) $\psi^{\mathbb{A}^\delta}(\vee O) = \vee \{\psi^{\mathbb{A}^\delta}(o) : o \in O'\}$ *for some up-directed subcollection $O' \subseteq O$ such that $\psi^{\mathbb{A}^\delta}(o) \in O(\mathbb{A}^\delta)$ for each $o \in O'$.*

Proof. The proof proceeds by simultaneous induction on φ and ψ . It is easy to see that φ cannot be \mathbf{m} , and the outermost connective of φ cannot be f_i^\sharp for any $f \in \mathcal{F}_n$ with $\varepsilon_f(i) = 1$, or g_j^b for any $g \in \mathcal{G}_n$ with $\varepsilon_g(j) = \partial$, or \blacksquare_f for any $f \in \mathcal{F}_r$ with $\varepsilon_f = 1$, or \blacktriangleright_g for any $g \in \mathcal{G}_r$ with $\varepsilon_g = \partial$. Similarly, ψ cannot be \mathbf{i} , and the outermost connective of ψ cannot be g_j^b for any $g \in \mathcal{G}_n$ with $\varepsilon_g(j) = 1$, or f_i^\sharp for any $f \in \mathcal{F}_n$ with $\varepsilon_f(i) = \partial$, or \blacklozenge_g for any $g \in \mathcal{G}_r$ with $\varepsilon_g = 1$, or \blacktriangleleft_f for any $f \in \mathcal{F}_r$ with $\varepsilon_f = \partial$.

The basic cases in which $\varphi = \perp, \top, p, q, \mathbf{i}$ and $\psi = \perp, \top, p, q, \mathbf{m}$ are straightforward.

Assume that $\varphi(p) = \blacklozenge_g \varphi'(p)$ for some $g \in \mathcal{G}_r$ with $\varepsilon_g = 1$. Since $\varphi(p)$ is positive in p , the subformula $\varphi'(p)$ is syntactically closed and positive in p , and assumptions 1(a)-1(d) hold also for $\varphi'(p)$. Hence, by inductive hypothesis, $\varphi'(\wedge C) = \wedge \{\varphi'(c) \mid c \in C''\}$ for some down-directed subcollection $C'' \subseteq C$ such that $\varphi'(c) \in K(\mathbb{A}^\delta)$ for each $c \in C''$. In particular, assumption 1(b) implies that $g(\top) \geq \varphi'(\wedge C) = \wedge \{\varphi'(c) \mid c \in C''\}$. Notice that $\varphi'(p)$ being positive in p and C'' being down-directed imply that $\{\varphi'(c) \mid c \in C''\}$ is down-directed. Hence, by Proposition A.3 applied to $\{\varphi'(c) \mid c \in C''\}$, we get that $\blacklozenge_g \varphi'(\wedge C) = \blacklozenge_g(\wedge \{\varphi'(c) \mid c \in C''\}) = \wedge \{\blacklozenge_g \varphi'(c) \mid c \in C''\}$. Moreover, there exists some down-directed subcollection $C' \subseteq C''$ such that $\blacklozenge_g \varphi'(c) \in K(\mathbb{A}^\delta)$ for each $c \in C'$ and $\wedge \{\blacklozenge_g \varphi'(c) \mid c \in C'\} = \wedge \{\blacklozenge_g \varphi'(c) \mid c \in C''\}$. This gives us $\blacklozenge_g \varphi'(\wedge C) = \wedge \{\blacklozenge_g \varphi'(c) \mid c \in C'\}$, as required. The case in which $\varphi(p)$ is negative in p is argued order-dually.

The cases in which $\varphi(p) = \blacksquare_g \varphi'(p), \blacktriangleleft_f \varphi'(p), \blacktriangleright_g \varphi'(p)$ are similar to the one above.

The cases of the remaining connectives are treated as in [13, Lemma 11.10] and the corresponding proofs are omitted. \square

A.3. Topological Ackermann Lemmas for $\mathcal{L}_{\text{LE}}^+$.

Definition A.7. A system S of $\mathcal{L}_{\text{LE}}^+$ -inequalities is *topologically adequate* when the following conditions hold for all $f \in \mathcal{F}_r$ and $g \in \mathcal{G}_r$:

- (1) if $\varepsilon_f = 1$ and $\varphi \leq \blacksquare_f \psi$ is in S , then $f(\perp) \leq \psi$ is in S ;
- (2) if $\varepsilon_g = 1$ and $\blacklozenge_g \varphi \leq \psi$ is in S , then $g(\top) \geq \varphi$ is in S ;
- (3) if $\varepsilon_f = \partial$ and $\blacktriangleleft_f \varphi \leq \psi$ is in S , then $f(\top) \leq \varphi$ is in S ;
- (4) if $\varepsilon_g = \partial$ and $\varphi \leq \blacktriangleright_g \psi$ is in S , then $g(\perp) \geq \psi$ is in S .

Proposition A.8 (Right-handed Topological Ackermann Lemma). *Let S be a topologically adequate system of $\mathcal{L}_{\text{LE}}^+$ -inequalities which is the union of the following disjoint subsets:*

- S_1 consists only of inequalities in which p does not occur;
- S_2 consists of inequalities of the type $\alpha \leq p$, where α is syntactically closed and p does not occur in α ;
- S_3 consists of inequalities of the type $\beta(p) \leq \gamma(p)$ where $\beta(p)$ is syntactically closed and positive in p , and $\gamma(p)$ be syntactically open and negative in p ,

Then the following are equivalent:

- (1) $\beta^{\mathbb{A}^\delta}(\bigvee \alpha^{\mathbb{A}^\delta}) \leq \gamma^{\mathbb{A}^\delta}(\bigvee \alpha^{\mathbb{A}^\delta})$ for all inequalities in S_3 , where $\bigvee \alpha$ abbreviates $\bigvee \{\alpha \mid \alpha \leq p \in S_2\}$;
- (2) There exists $a_0 \in \mathbb{A}$ such that $\bigvee \alpha^{\mathbb{A}^\delta} \leq a_0$ and $\beta^{\mathbb{A}^\delta}(a_0) \leq \gamma^{\mathbb{A}^\delta}(a_0)$ for all inequalities in S_3 .

Proof. (\Leftarrow) By the monotonicity of $\beta_i(p)$ and antitonicity of $\gamma_i(p)$ in p for $1 \leq i \leq n$, together with $\alpha^{\mathbb{A}^\delta} \leq a_0$, we have that $\beta_i^{\mathbb{A}^\delta}(\alpha^{\mathbb{A}^\delta}) \leq \beta_i^{\mathbb{A}^\delta}(a_0) \leq \gamma_i^{\mathbb{A}^\delta}(a_0) \leq \gamma_i^{\mathbb{A}^\delta}(\alpha^{\mathbb{A}^\delta})$.

(\Rightarrow) Since the quasi-inequality is topologically adequate, by Lemma A.6.1, $\alpha^{\mathbb{A}^\delta} \in K(\mathbb{A}^\delta)$.

Hence, $\alpha^{\mathbb{A}^\delta} = \bigwedge \{a \in \mathbb{A} : \alpha^{\mathbb{A}^\delta} \leq a\}$, making it the meet of a downward-directed set of clopen elements. Therefore, we can rewrite each inequality in S_3 as

$$\beta^{\mathbb{A}^\delta}(\bigwedge \{a \in \mathbb{A} : \alpha^{\mathbb{A}^\delta} \leq a\}) \leq \gamma^{\mathbb{A}^\delta}(\bigwedge \{a \in \mathbb{A} : \alpha^{\mathbb{A}^\delta} \leq a\}).$$

Since β is syntactically closed and positive in p , γ is syntactically open and negative in p , again by topological adequacy, we can apply Lemma A.6 and get that

$$\bigwedge \{\beta^{\mathbb{A}^\delta}(a) : a \in \mathcal{A}_1\} \leq \bigvee \{\gamma_i^{\mathbb{A}^\delta}(b) : b \in \mathcal{A}_2\}$$

for some $\mathcal{A}_1, \mathcal{A}_2 \subseteq \{a \in \mathbb{A} : \alpha^{\mathbb{A}^\delta} \leq a\}$ such that $\beta^{\mathbb{A}^\delta}(a) \in K(\mathbb{A}^\delta)$ for each $a \in \mathcal{A}_1$, and $\gamma^{\mathbb{A}^\delta}(b) \in O(\mathbb{A}^\delta)$ for each $b \in \mathcal{A}_2$. By compactness,

$$\bigwedge \{\beta_i^{\mathbb{A}^\delta}(a) : a \in \mathcal{A}'_1\} \leq \bigvee \{\gamma_i^{\mathbb{A}^\delta}(b) : b \in \mathcal{A}'_2\}$$

for some finite subsets $\mathcal{A}'_1 \subseteq \mathcal{A}_1$ and $\mathcal{A}'_2 \subseteq \mathcal{A}_2$. Then let $a^* = \bigwedge \{\bigwedge \mathcal{A}'_1 \wedge \bigwedge \mathcal{A}'_2 \mid \beta \leq \gamma \in S_3\}$. Clearly, $a^* \in \mathbb{A}$, and $\alpha^{\mathbb{A}^\delta} \leq a^*$. By the monotonicity of $\beta(p)$ and the antitonicity of $\gamma(p)$ in p for each $\beta \leq \gamma$ in S_3 , we have $\beta^{\mathbb{A}^\delta}(a^*) \leq \beta^{\mathbb{A}^\delta}(a)$ and $\gamma_i^{\mathbb{A}^\delta}(b) \leq \gamma_i^{\mathbb{A}^\delta}(a^*)$ for all $a \in \mathcal{A}'_1$ and all $b \in \mathcal{A}'_2$. Therefore,

$$\beta_i^{\mathbb{A}^\delta}(a^*) \leq \bigwedge \{\beta_i^{\mathbb{A}^\delta}(a) : a \in \mathcal{A}'_1\} \leq \bigvee \{\gamma_i^{\mathbb{A}^\delta}(b) : b \in \mathcal{A}'_2\} \leq \gamma_i^{\mathbb{A}^\delta}(a^*)$$

for each $\beta \leq \gamma$ in S_3 . □

Proposition A.9 (Left-handed Topological Ackermann Lemma). *Let S be a topologically adequate system of \mathcal{L}_{LE}^+ -inequalities which is the union of the following disjoint subsets:*

- S_1 consists only of inequalities in which p does not occur;
- S_2 consists of inequalities of the type $p \leq \alpha$, where α is syntactically open and p does not occur in α ;
- S_3 consists of inequalities of the type $\beta(p) \leq \gamma(p)$ where $\beta(p)$ is syntactically closed and negative in p , and $\gamma(p)$ be syntactically open and positive in p ,

Then the following are equivalent:

- (1) $\beta^{A^\delta}(\bigwedge \alpha^{A^\delta}) \leq \gamma^{A^\delta}(\bigwedge \alpha^{A^\delta})$ for all inequalities in S_3 , where $\bigwedge \alpha$ abbreviates $\bigwedge \{\alpha \mid p \leq \alpha \in S_2\}$;
- (2) *There exists $a_0 \in \mathbb{A}$ such that $a_0 \leq \bigwedge \alpha^{A^\delta}$ and $\beta^{A^\delta}(a_0) \leq \gamma^{A^\delta}(a_0)$ for all inequalities in S_3 .*

Proof. The proof is similar to the proof of the right-handed Ackermann lemma and is omitted. □