
RELATING COALGEBRAIC NOTIONS OF BISIMULATION*

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ABSTRACT. The theory of coalgebras, for an endofunctor on a category, has been proposed as a general theory of transition systems. We investigate and relate four generalizations of bisimulation to this setting, providing conditions under which the four different generalizations coincide. We study transfinite sequences whose limits are the greatest bisimulations.

INTRODUCTION

Notions of bisimulation play a central role in the theory of transition systems. The theory of coalgebras provides a setting in which different notions of system can be understood at a general level. In this article I investigate notions of bisimulation at this general level.

To explain the generalization from transition systems to coalgebras, we begin with the traditional presentation of a labelled transition system,

$$(X, (\rightarrow_X) \subseteq X \times L \times X)$$

(for some set L of labels). A labelled transition system can be considered ‘coalgebraically’ as a set X of states equipped with a function $X \rightarrow \mathcal{P}(L \times X)$, into the powerset of $(L \times X)$, assigning to each state $x \in X$ the set $\{(l, x') \mid x \xrightarrow{l}_X x'\}$. Generalizing, we are led to consider an arbitrary category \mathcal{C} and an endofunctor B on it; then a coalgebra is an object $X \in \mathcal{C}$ of ‘states’, and a ‘next-state’ morphism $X \rightarrow B(X)$.

Coalgebras in different categories. Coalgebras appear as generalized transition systems in various settings. For instance: transition systems for name and value passing process calculi have been studied in terms of coalgebras in categories of presheaves (e.g. [19, 22, 48]); probabilistic transition systems have been modelled by coalgebras for a probability-distribution monad (e.g. [6, 53]); descriptive frames and concepts from modal logic have been studied in terms of coalgebras over Stone spaces (e.g. [1, 9, 32]); basic process calculi with recursion have been modelled using coalgebras over categories of domains [31, 44]; and

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stochastic transition systems have been studied in terms of coalgebras over metric and measurable spaces (see e.g. [12, 16, 52, 53]). Finally, there are questions about the conventional theory of labelled transition systems in a more constructive universe of sets (e.g. [7]).

Notions of bisimulation. Once coalgebras are understood as generalized transition systems, we can consider bisimulation relations for these systems. Recall that, for labelled transition systems (X, \rightarrow_X) and (Y, \rightarrow_Y) , a relation $R \subseteq X \times Y$ is a *bisimulation* if, whenever $x R y$, then for all $l \in L$:

- For $x' \in X$, if $x \xrightarrow{l}_X x'$ then there is $y' \in Y$ such that $y \xrightarrow{l}_Y y'$ and $x' R y'$;
- For $y' \in Y$, if $y \xrightarrow{l}_Y y'$ then there is $x' \in X$ such that $x \xrightarrow{l}_X x'$ and $x' R y'$.

How should the notion of bisimulation be generalized to the case of coalgebras for endofunctors on arbitrary categories? In this article, we identify four notions of bisimulation that have been proposed in the coalgebraic context.

- (1) A relation over which a suitably compatible transition structure can be defined, as proposed by Aczel and Mendler [2];
- (2) A relation that is compatible for a suitable ‘relation-lifting’ of the endofunctor, as proposed by Hermida and Jacobs [25];
- (3) A relation satisfying a ‘congruence’ condition, proposed by Aczel and Mendler [2] and used to obtain their general final coalgebra theorem;
- (4) A relation which is the kernel of a common compatible refinement of the two systems.

The four notions coincide for the particular case of labelled transition systems. Under certain conditions, the notions are related in the more general setting of coalgebras.

Relationship with the terminal sequence. Various authors have constructed terminal coalgebras as a limit of a transfinite sequence; the initial part of the sequence is:

$$1 \xleftarrow{!} B(1) \xleftarrow{B(!)} B(B(1)) \xleftarrow{B(B(!))} B(B(B(1))) \leftarrow \dots \leftarrow \dots$$

Of the notions of bisimulation mentioned above, notions (2) and (3) can often be characterized as post-fixed points of a monotone operator Φ on a lattice of relations. In this setting, by Tarski’s fixed point theorem, there is a maximum bisimulation (‘bisimilarity’). It is given explicitly as a limit of a transfinite sequence; the initial part of the sequence is:

$$X \times Y \supseteq \Phi(X \times Y) \supseteq \Phi(\Phi(X \times Y)) \supseteq \Phi(\Phi(\Phi(X \times Y))) \supseteq \dots \supseteq \dots$$

starting with the maximal relation, that relates everything. Under certain conditions, the steps of the terminal coalgebra sequence are precisely related with the steps of this relation refinement sequence.

Other approaches not considered. In this article we are concerned with internal relations between the state objects of two fixed coalgebras. A relation is itself an object of the base category.

Some authors (e.g. [16]) are concerned with defining an equivalence relation on the class of all coalgebras, by setting two coalgebras as bisimilar if there is a span of surjective homomorphisms between them. Others work with relations as bimodules (e.g. [12, 45, 54]). We will not discuss these approaches here.

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1. COALGEBRAS: DEFINITIONS AND EXAMPLES

Recall the definition of a coalgebra for an endofunctor:

Definition 1.1. Consider an endofunctor B on a category \mathcal{C} . A B -coalgebra is given by an object X of \mathcal{C} together with morphism $X \rightarrow B(X)$ in \mathcal{C} .

A *homomorphism* of B -coalgebras, from (X, h) to (Y, k) , is a morphism $f : X \rightarrow Y$ that respects the coalgebra structures, i.e. such that $Bf \circ h = k \circ f$.

1.1. Examples. We collect some examples of concepts that arise as coalgebras for endofunctors. For further motivation, see [4, 26, 46].

Coinductive datatypes. Coinductive datatypes can be understood in terms of coalgebras for polynomial endofunctors. A polynomial endofunctor on a category with sums and products is a functor of the following form. (See e.g. Rutten [46, Sec. 10].)

$$X \mapsto \sum_{i \in I} A_i \times X^{n_i}$$

(Here, each A_i is an object of the category, and each n_i is a natural number.)

Transition systems. In the introduction we discussed the correspondence between labelled transition systems and coalgebras for the endofunctor $\mathcal{P}(L \times (-))$. Here, \mathcal{P} is the powerset functor, that acts by direct image. For finite non-determinism, and image-finite transition systems, one can instead consider the endofunctor

$$\mathcal{P}_f(L \times (-))$$

where \mathcal{P}_f is the finite powerset functor, the free semilattice.

Transition systems in toposes and name-passing calculi. Recall that a topos is a category with finite limits and a powerobject construction. By definition, the powerobjects classify relations, and so the coalgebraic characterization of labelled transition systems is relevant in any topos.

In process calculi such as the π -calculus [41], transitions occur between terms with free variables, and those free variables play an important role. Conventional labelled transition systems in the category of sets are inadequate for such calculi. Instead, one can work in a category of covariant presheaves ($\mathbb{C} \rightarrow \mathbf{Set}$). Various categories have been proposed for \mathbb{C} . We will focus on two examples: the category \mathbb{I} of finite sets and injections between them, and the category \mathbb{F}^+ of non-empty finite sets and all functions between them. More sophisticated models of process calculi are found by taking presheaves over more elaborate categories (see e.g. [10, 11, 22, 49]).

In this setting, the object X of states is no longer a set, but a presheaf. For instance, if $X : \mathbb{I} \rightarrow \mathbf{Set}$, we think of $X(C)$ as the set of states involving the free variables in the set C , and the functorial action of X describes injective renaming of states.

The appropriate endofunctor on these presheaf categories typically has the following form

$$\mathcal{P}(B'(-)) \quad \text{where e.g. } B'(-) = (N \times (-)^N + N \times N \times (-) + (-)) \quad (1.1)$$

with the summands of B' representing input, output, and silent actions respectively. The presheaf N is a special object of names. The interesting question is: what is \mathcal{P} ?

- A natural choice is to let \mathcal{P} be the powerobject functor in the presheaf topos $[\mathbb{C}, \mathbf{Set}]$. For any presheaf $Y \in [\mathbb{C}, \mathbf{Set}]$, and any object $C \in \mathbb{C}$, $(\mathcal{P}(Y))(C)$ is the set of subpresheaves of $(\mathbb{C}(C, -) \times Y)$. A coalgebra $X \rightarrow \mathcal{P}(B'(X))$ is a natural transformation between presheaves, that assigns a behaviour to each state $x \in X(C)$, for $C \in \mathbb{C}$. This behaviour is not only a set of future states for x , but also the future states of $Xf(x)$ for any morphism $f : C \rightarrow C'$ in \mathbb{C} .

To understand this more formally, recall that the powerobject \mathcal{P} classifies relations in the following sense. For presheaves X and Y in $[\mathbb{C}, \mathbf{Set}]$, there is a bijective correspondence between natural transformations $r : X \rightarrow \mathcal{P}(Y)$ and subpresheaves $R \subseteq X \times Y$. A subpresheaf $R \subseteq X \times Y$ determines a natural transformation $r : X \rightarrow \mathcal{P}(Y)$; for $C \in \mathbb{C}$ and $x \in X(C)$, we have $r_C(x) \in (\mathcal{P}(Y))(C)$:

$$(r_C(x))(C') = \{(f, y) \mid (Xf(x), y) \in R(C')\} \quad .$$

- The powerobject $\mathcal{P}(X)$ accommodates infinite branching transition systems. To focus on finite branching, we can find a ‘finite’ subfunctor of \mathcal{P} .

The approach taken by Fiore and Turi [19] is to let \mathcal{P} be the free semi-lattice (henceforth \mathcal{P}_f). This is sometimes called ‘Kuratowski finiteness’. For any presheaf $Y \in [\mathbb{C}, \mathbf{Set}]$, and any object $C \in \mathbb{C}$, $(\mathcal{P}_f(Y))(C)$ is the set of finite subsets of $Y(C)$. We have an natural monomorphism $i_Y : \mathcal{P}_f(Y) \hookrightarrow \mathcal{P}(Y)$ into the full powerobject: for $C \in \mathbb{C}$, $S \subseteq (Y(C))$, we define a subpresheaf $i_{Y,C}(S) \in (\mathcal{P}(Y))(C)$:

$$(i_{Y,C}(S))(C') = \{(f, Yf(y)) \mid f : C \rightarrow C', y \in S\} \quad .$$

- The free semilattice is too naive on the presheaf category $[\mathbb{F}^+, \mathbf{Set}]$. For example, the π -calculus process $(\bar{a} \mid b)$ cannot perform a τ -step, but it can perform a τ -step after the substitution $\{a \mapsto b, b \mapsto b\}$. The construction $\mathcal{P}_f(X)$ is too small to allow this information to be recorded. Indeed, the π -calculus can be described as a coalgebra for the functor $\mathcal{P}(B'(-))$ on $[\mathbb{F}^+, \mathbf{Set}]$, but this coalgebra does not factor through the free semilattice, $\mathcal{P}_f(B'(-))$.

In this situation, a more appropriate finite powerset is the sub-join-semilattice of the powerobject $\mathcal{P}(Y)$ that is generated by the partial map classifier. We will write $\mathcal{P}_{\text{pf}}(Y)$ for this — Freyd [20] writes \tilde{K} . I gave an algebraic description of this construction in [50], and it has been used by Miculan in his model of the fusion calculus [37].

Frames in modal logic. Let the base category \mathcal{C} be the category of Stone spaces and continuous maps. (A Stone space is a compact Hausdorff space in which the clopen sets form a basis.) Let $K(X)$ be the space of compact subsets of X , with the finite (aka Vietoris) topology. The construction K is made into a functor, acting by direct image. Coalgebras for K can be understood as descriptive general frames. Just as the category of Stone spaces

is dual to the category of Boolean algebras, the category of K -coalgebras is dual to the category of modal algebras (i.e., Boolean algebras equipped with a meet-preserving operation) — see e.g. [1, 32].

Powersets in algebraic set theory. A general treatment of powersets is suggested by the algebraic set theory of Joyal and Moerdijk [29]. A model of algebraic set theory is a category \mathcal{C} together with a class of ‘small’ maps \mathcal{S} in \mathcal{C} , all subject to certain conditions. An intuition is that a map $f: X \rightarrow Y$ is small if its fibres $f^{-1}(y)$ are all small.

In such a situation, an \mathcal{S} -relation is a relation $R \subseteq X \times Y$ for which the projection $R \rightarrow X$ is in \mathcal{S} . An endofunctor $P_{\mathcal{S}}$ on \mathcal{C} is said to be the \mathcal{S} -powerset if there is an \mathcal{S} -relation $(\exists_Y) \subseteq P_{\mathcal{S}}(Y) \times Y$ inducing a bijective correspondence between \mathcal{S} -relations ($R \subseteq X \times Y$) and morphisms $X \rightarrow P_{\mathcal{S}}(Y)$. Further details are given in the appendix.

These ideas cater for the notions of power set discussed so far. For instance:

- Let \mathcal{C} be the category of sets, in the classical sense, and say that a function $f: X \rightarrow Y$ is small if for every $y \in Y$ the set $f^{-1}(y)$, i.e. $\{x \in X \mid f(x) = y\}$, is finite. This class \mathcal{S} of maps satisfies all the axioms for small maps given in the appendix. An \mathcal{S} -relation is precisely an image-finite one, and the \mathcal{S} -powerset is the finite powerset.
- For a presheaf category $[\mathbb{C}, \mathbf{Set}]$, the free semilattice construction $\mathcal{P}_{\mathbf{f}}$ is an \mathcal{S} -powerset where \mathcal{S} is the class of natural transformations between presheaves, $\phi: X \rightarrow Y$, such that (i) for each $C \in \mathbb{C}$, $y \in Y(C)$, the set $\{x \in X(C) \mid f_C(x) = y\}$ is finite, and (ii) each naturality square is a weak pullback, i.e., if $\phi_{C'}(x') = Yf(y)$ then there is $x \in X(C)$ such that $\phi_C(x) = y$ and $Xf(x) = x'$:

$$\begin{array}{ccc} X(C) & \xrightarrow{\phi_C} & Y(C) \\ x(f) \downarrow & & \downarrow Y(f) \\ X(C') & \xrightarrow{\phi_{C'}} & Y(C'). \end{array} \quad (1.2)$$

This class \mathcal{S} of morphisms always satisfies Axioms A1–A6 and A9 for small maps, but not (M): monos are not small unless \mathbb{C} is a groupoid.

In the presheaf category $[\mathbb{F}^+, \mathbf{Set}]$, the free semilattice generated by the partial map classifier, \mathcal{P}_{pf} , is more liberal: it classifies the natural transformations between presheaves that satisfy the finiteness condition (i) but the requirement on naturality (ii) is weakened to the situation when f is an injection. This class of morphisms satisfies all the axioms for small maps, including (M). (This argument is quite specific to \mathbb{F}^+ .)

- Let \mathcal{C} be the category of Stone spaces, and let $K(X)$ be the space of compact subsets of X . Recall that a continuous map is *open* if the direct image of an open set is open. The class \mathcal{S} of open maps in the category of Stone spaces satisfies Axioms A1–A6 and A9 for small maps. The evident relation $\exists_X \subseteq K(X) \times X$ is an \mathcal{S} -relation, and exhibits each $K(X)$ as an \mathcal{S} -powerset.

(Although the Vietoris construction can be considered over more general spaces, the characterization of K as an \mathcal{S} -powerset is specific to Stone spaces. This raises the question of how best to treat more expressive positive/topological set theories [36] in an algebraic setting.)

Probabilistic transition systems. For any set X , let $\mathcal{D}_f(X)$ be the set of sub-probability distribution functions on X , viz., functions from $d: X \rightarrow [0, 1]$ into the unit interval for which $\{x \in X \mid d(x) \neq 0\}$ is finite and $\sum_{x \in X} d(x) \leq 1$. This construction extends to an endofunctor on **Set**, with the covariant action given by summation. Coalgebras for \mathcal{D}_f are discrete probabilistic transition systems [6, 53].

Systems where the state space has more structure. For continuous stochastic systems, researchers have investigated coalgebras for probability distribution functions on categories of metric or measurable spaces (see e.g. [12, 16, 52, 53]).

For recursively defined systems, it is reasonable to investigate coalgebras for powerdomain constructions on a category of domains (in the bialgebraic context, see [31, 44]).

Levy and Worrell [35, 54] have considered endofunctors on categories of preorders, posets, and categories enriched in quantales, in their investigations of similarity.

2. BISIMULATION: FOUR DEFINITIONS

We now recall four notions of bisimulation on the state spaces of coalgebras. The four notions generalize the standard notion of bisimulation for labelled transition systems (i.e. coalgebras for $\mathcal{P}(L \times (-))$ on **Set**), due to Milner [40] and Park [43]. For all four notions, the maximal bisimulation is the usual notion of strong bisimilarity for labelled transition systems.

To some extent, the different notions of bisimulation have arisen from the examples in Section 1.1, as authors sought coalgebraic notions of bisimulation that were appropriate to the base category and endofunctor under consideration, as well as to the intended applications. For name-passing calculi, coalgebraic bisimulation can be used to capture the bisimulations of Milner, Parrow and Walker [41] and also the open bisimulation of Sangiorgi [47] (see e.g. [19, 50, Sec. 6]); for discrete-space probabilistic systems, coalgebraic bisimulation can describe the probabilistic bisimulation of Larsen and Skou [34] (see e.g. [53]).

Relations in categories. For objects X, Y of a category \mathcal{C} , we let $\mathbf{Rel}_{\mathcal{C}}(X, Y)$ be the preorder of relations, viz. jointly-monic spans $X \leftarrow R \rightarrow Y$, where $R \leq R'$ if R factors through R' . When \mathcal{C} has products, the preorder $\mathbf{Rel}_{\mathcal{C}}(X, Y)$ coincides with the preorder of monos into $(X \times Y)$. Relations are most well-behaved in regular categories (see Appendix).

Context. In this section we fix a category \mathcal{C} and consider an endofunctor B on \mathcal{C} . We fix two B -coalgebras, $h: X \rightarrow BX$ and $k: Y \rightarrow BY$.

2.1. The lifting-span bisimulation of Aczel and Mendler [2]. This notion is categorically the simplest. It directly dualizes the concept of congruence from universal algebra.

Definition 2.1. A relation $R \in \mathbf{Rel}_{\mathcal{C}}(X, Y)$ is an *AM-bisimulation* between (X, h) and (Y, k) if there exists a B -coalgebra structure on R that lifts it to a span of coalgebra homomorphisms, as in the following diagram.

$$\begin{array}{ccc} X & \longleftarrow R & \longrightarrow Y \\ h \downarrow & \text{\scriptsize } \exists \text{\scriptsize } \downarrow & \downarrow k \\ BX & \longleftarrow BR & \longrightarrow BY \end{array}$$

2.2. The relation-lifting bisimulation of Hermida and Jacobs [25]. The following is a simplification of the bisimulation of Hermida and Jacobs, who work in a more general fibrational setting.

Definition 2.2. Let \mathcal{C} have products and images. (See the Appendix for definition.) For any relation $R \in \mathbf{Rel}_{\mathcal{C}}(X, Y)$, we define the relation $\bar{B}R \in \mathbf{Rel}_{\mathcal{C}}(BX, BY)$ to be the image of the composite morphism $BR \rightarrow B(X \times Y) \rightarrow BX \times BY$. (The construction \bar{B} is called the “relation lifting” of B .)

A relation R in $\mathbf{Rel}_{\mathcal{C}}(X, Y)$ is an *HJ-bisimulation* if there is a morphism $R \rightarrow \bar{B}(R)$ making the following diagram commute.

$$\begin{array}{ccc} X & \longleftarrow R & \longrightarrow Y \\ h \downarrow & \text{\scriptsize } \downarrow & \downarrow k \\ BX & \longleftarrow \bar{B}R & \longrightarrow BY \end{array}$$

When \mathcal{C} has pullbacks, let $\Phi^{\text{HJ}}(R)$ be the following pullback:

$$\begin{array}{ccc} \Phi^{\text{HJ}}(R) & \longrightarrow & \bar{B}R \\ \downarrow \lrcorner & & \downarrow \\ X \times Y & \xrightarrow{h \times k} & BX \times BY \end{array}$$

By definition, a relation R is an *HJ-bisimulation* if and only if $R \leq \Phi^{\text{HJ}}(R)$.

Proposition 2.3. *The operator Φ^{HJ} on $\mathbf{Rel}_{\mathcal{C}}(X, Y)$ is monotone.* □

For illustration, we briefly return to the situation of transition systems, where $\mathcal{C} = \mathbf{Set}$ and $B = \mathcal{P}(L \times -)$. For any relation $R \in \mathbf{Rel}_{\mathcal{C}}(X, Y)$, the refined relation $\Phi^{\text{HJ}}(R)$ in $\mathbf{Rel}_{\mathcal{C}}(X, Y)$ is the set of all pairs $(x, y) \in X \times Y$ for which

- (i) $\forall (l, x') \in h(x). \exists y' \in Y. (l, y') \in k(y) \text{ and } (x', y') \in R$;
- (ii) $\forall (l, y') \in k(y). \exists x' \in X. (l, x') \in h(x) \text{ and } (x', y') \in R$.

Thus the operator Φ^{HJ} is the construction \mathcal{F} considered by Milner [39, Sec. 4].

3. PROPERTIES OF ENDOFUNCTORS

All the definitions of the previous section are relevant when \mathcal{C} is a regular category. From a categorical perspective, one might restrict attention to endofunctors that are regular, i.e. that preserve limits and covers. But none of the endofunctors in Section 1.1 are regular. We now recall five weaker conditions that might be assumed of our endofunctor B .

- (1) The image of a relation under a functor need not again be a relation, and one can restrict attention to endofunctors that *preserve relations*, i.e., for which a jointly-monic span is mapped to a jointly-monic span.

The remaining restrictions have to do with weak forms of pullback-preservation. To introduce them, we consider a cospan $(A_1 \rightarrow Z \leftarrow A_2)$ in \mathcal{C} , and in particular the mediating morphism $m: B(A_1 \times_Z A_2) \rightarrow (BA_1) \times_{BZ} (BA_2)$ from the image of the pullback to the pullback of the image:

$$\begin{array}{ccc}
 & & A_1 \\
 & \nearrow^{\pi_1} & \\
 A_1 \times_Z A_2 & \triangleright & \\
 & \searrow_{\pi_2} & \\
 & & A_2 \\
 & & \nearrow \\
 & & Z
 \end{array}
 \qquad
 \begin{array}{ccc}
 & & BA_1 \\
 & \nearrow^{B\pi_1} & \\
 B(A_1 \times_Z A_2) & \xrightarrow{m} & BA_1 \times_{BZ} BA_2 \\
 & \searrow_{B\pi_2} & \\
 & & BA_2 \\
 & & \nearrow \\
 & & BZ
 \end{array}$$

Here are some conditions on B , listed in order of decreasing strength.

- (2) B *preserves pullbacks* if m is always an isomorphism.
- (3) B *preserves weak pullbacks*, if m is always split epi. Gumm [23] describes several equivalent definitions of this term.
- (4) B *covers pullbacks* if m is always a cover. The terminology is due to [51]. Note that a split epi is always a cover.
- (5) B *preserves pullbacks along monos* if m is an isomorphism when $A_1 \rightarrow Z$ is monic.

Tying up with relation preservation (1): B preserves pullbacks if and only if it preserves relations and covers pullbacks (see Carboni et al. [14, Sec. 4.3]).

3.1. Relevance of the properties.

Proposition 3.1.

- (1) Let Ψ be one of the following properties: relation preservation, pullback preservation, weak pullback preservation, preservation of pullbacks along monos. The composition of two endofunctors satisfying Ψ also satisfies Ψ .
- (2) If B and B' both cover pullbacks and B' preserves covers, then the composite $(B'B)$ also covers pullbacks. \square

Proposition 3.2. *Every polynomial endofunctor on an extensive category preserves pullbacks and covers.* \square

Regarding powerset functors from algebraic set theory (see Appendix), we have the following general results.

Proposition 3.3. *Let \mathcal{C} be regular, and let \mathcal{S} be a class of open maps in \mathcal{C} . Suppose that $P_{\mathcal{S}}$ is an \mathcal{S} -powerset.*

- (1) *The functor $P_{\mathcal{S}}$ preserves pullbacks along monomorphisms.*
- (2) *If \mathcal{S} contains all monomorphisms (M) , then $P_{\mathcal{S}}$ preserves weak pullbacks.*

(3) Let \mathcal{C} also be extensive, and let \mathcal{S} satisfy the axioms for extensive categories (axioms (A1–6)) and also collection (A9), but not necessarily (M). The functor $P_{\mathcal{S}}$ covers pullbacks.

Proof of Propn. 3.3: I will sketch proofs in the set-theoretic notation. Translation into categorical language is straightforward.

For item (1), consider a pullback square, and its image under $P_{\mathcal{S}}$.

$$\begin{array}{ccc} f^{-1}(Z') \rightrightarrows & A & \\ \downarrow \lrcorner & \downarrow f & \\ Z' \rightrightarrows & Z & \end{array} \qquad \begin{array}{ccc} P_{\mathcal{S}}(f^{-1}(Z')) \longrightarrow & P_{\mathcal{S}}(A) & \\ \downarrow & \downarrow P_{\mathcal{S}}f & \\ P_{\mathcal{S}}(Z') \longrightarrow & P_{\mathcal{S}}(Z) & \end{array}$$

To see that the right-hand square is a pullback, consider S in $P_{\mathcal{S}}(A)$, for which the direct image $P_{\mathcal{S}}f(S)$ is in $P_{\mathcal{S}}(Z')$; then, by definition, S is in $P_{\mathcal{S}}(f^{-1}(Z'))$.

For items (2) and (3), consider a cospan $(A_1 \rightarrow Z \leftarrow A_2)$. For item (2), we must exhibit a section $s : P_{\mathcal{S}}(A_1) \times_{P_{\mathcal{S}}(Z)} P_{\mathcal{S}}(A_2) \rightarrow P_{\mathcal{S}}(A_1 \times_Z A_2)$ of the canonical morphism. For (S_1, S_2) in $P_{\mathcal{S}}(A_1) \times_{P_{\mathcal{S}}(Z)} P_{\mathcal{S}}(A_2)$, note that

$$\forall a_1 \in S_1. \exists a_2 \in S_2. f(a_1) = g(a_2) \quad \text{and} \quad \forall a_2 \in S_2. \exists a_1 \in S_1. f(a_1) = g(a_2)$$

and let

$$s(S_1, S_2) = \{(a_1, a_2) \in (S_1 \times S_2) \mid f(a_1) = g(a_2)\} \quad .$$

Here we have used the separation axiom, which is valid when all monomorphisms are small.

For item (3), we show that the canonical morphism

$$P_{\mathcal{S}}(A_1 \times_Z A_2) \rightarrow P_{\mathcal{S}}(A_1) \times_{P_{\mathcal{S}}(Z)} P_{\mathcal{S}}(A_2)$$

is a cover. Consider (S_1, S_2) in $P_{\mathcal{S}}(A_1) \times_{P_{\mathcal{S}}(Z)} P_{\mathcal{S}}(A_2)$; we must show that there is S in $P_{\mathcal{S}}(A_1 \times_Z A_2)$ whose direct image is (S_1, S_2) .

By the (strong) collection axiom, we have T_1 in $P_{\mathcal{S}}(A_1 \times_Z A_2)$ such that $\pi_1(T_1) = S_1$ and $\pi_2(T_1) \subseteq S_2$. Similarly, we also have T_2 in $P_{\mathcal{S}}(A_1 \times_Z A_2)$ such that $\pi_2(T_2) = S_2$ and $\pi_1(T_2) \subseteq S_1$. Thus $(T_1 \cup T_2)$ is in $P_{\mathcal{S}}(A_1 \times_Z A_2)$, and its direct image is (S_1, S_2) , as required. Thus Propn. 3.3 is proved. \square

The free semi-lattice functor \mathcal{P}_f on any topos covers pullbacks, but will only preserve weak pullbacks if the topos is Boolean. In general, the corresponding class of small maps does not contain all monomorphisms, as Johnstone et al. [27, Ex. 1.4] have observed. The counterexample of [27] is easily adapted to the settings of the presheaf categories for name passing, correcting oversights in [19, 22, 48].

On the other hand, the endofunctor \mathcal{P}_{pf} on the presheaf category $[\mathbb{F}^+, \mathbf{Set}]$ does preserve weak pullbacks.

The compact-subspace endofunctor K on Stone spaces covers pullbacks (a consequence of Propn. 3.3) although it does not preserve weak pullbacks, as observed by Bezhanishvili et al [9].

The probability distribution functor \mathcal{D}_f on \mathbf{Set} preserves weak pullbacks [42, 53].

More sophisticated continuous settings are problematic. Counterexamples to the weak-pullback-preservation of probability distributions on measurable spaces are discussed in [52]. Plotkin [44] discusses problems with coalgebraic bisimulation in categories of domains: the convex powerdomain does not even preserve monomorphisms. The endofunctors on posets that Levy considers [35] typically do not preserve monomorphisms either.

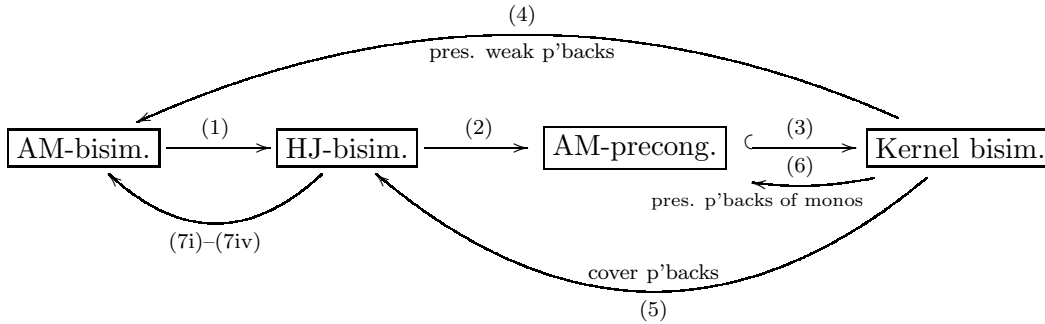
4. RELATING THE NOTIONS OF BISIMULATION

The purpose of this section is to relate the four notions of bisimulation introduced in Section 2.

Theorem 4.1. *Let B be an endofunctor on a category \mathcal{C} with finite limits and images.*

- (1) *Every AM-bisimulation is an HJ-bisimulation.*
- (2) *Every HJ-bisimulation is an AM-precongruence.*
- (3) *Every AM-precongruence is contained in a kernel bisimulation that is an AM-precongruence, provided \mathcal{C} has pushouts.*
- (4) *Every kernel bisimulation is an AM-bisimulation, provided B preserves weak pullbacks.*
- (5) *Every kernel bisimulation is an HJ-bisimulation, provided B covers pullbacks.*
- (6) *Every kernel bisimulation is an AM-precongruence, provided B preserves pullbacks along monos and \mathcal{C} is regular.*
- (7) *Every HJ-bisimulation is an AM-bisimulation, provided either*
 - (i) *every epi in \mathcal{C} is split, or*
 - (ii) *B preserves relations, or*
 - (iii) *\mathcal{C} is regular with a class \mathcal{S} of open maps containing all monomorphisms, and there is an \mathcal{S} -powerset $P_{\mathcal{S}}$, and $B(-) \cong P_{\mathcal{S}}(B'(-))$, for a relation preserving functor B' .*
 - (iv) *\mathcal{C} is a topos and $B \cong P(B'(-))$, where P is the powerobject of \mathcal{C} and B' is an arbitrary endofunctor.*

In summary:



Note that the different notions of bisimulation are not, in general, the same. For instance, Aczel and Mendler [2, p. 363] provide an example of an endofunctor on **Set** for which there is an AM-precongruence that is not an AM-bisimulation. Bezhanishvili et al. [9, Sec. 4] demonstrate that AM-bisimulation is different from HJ-bisimulation for the Vietoris construction on Stone spaces.

4.1. Proof of Theorem 4.1. Throughout the proof, we fix two B -coalgebras, $h : X \rightarrow BX$ and $k : Y \rightarrow BY$.

Item (1) is trivial.

For item (2), let R be an HJ-bisimulation. Let $(X \rightarrow Z \leftarrow Y)$ be a cone over the span $(X \leftarrow R \rightarrow Y)$. Consider the following commuting diagrams.

$$(a) \quad \begin{array}{ccccc} & & X & \xrightarrow{h} & BX \\ & \nearrow & & & \nearrow \\ R & \longrightarrow & \bar{B}R & & \\ & \searrow & & & \searrow \\ & & Y & \xrightarrow{k} & BY \\ & & & & \nearrow \\ & & & & BZ \end{array} \quad (b) \quad \begin{array}{ccc} & BX & \\ & \nearrow \quad \searrow & \\ BR & & BZ \\ & \searrow \quad \nearrow & \\ & BY & \end{array}$$

The left-hand squares of diagram (a) say that R is an HJ-bisimulation. The right-hand square of diagram (a) commutes since diagram (b) commutes, and $BR \rightarrow \bar{B}R$ is epi. Thus the whole of (a) commutes, and R is an AM-precongruence.

For item (3), let R be an AM-precongruence. Let $(X \rightarrow Z \leftarrow Y)$ be the pushout of the span $(X \leftarrow R \rightarrow Y)$. The following diagram commutes; the dotted morphism follows from universality of the pushout.

$$\begin{array}{ccccc} & & X & \xrightarrow{h} & BX \\ & \nearrow & & \searrow & \nearrow \\ R & & & & Z \cdots \longrightarrow BZ \\ & \searrow & & \nearrow & \\ & & Y & \xrightarrow{k} & BY \\ & & & & \nearrow \\ & & & & BZ \end{array}$$

Let $(X \leftarrow R' \rightarrow Y)$ be the pullback of $(X \rightarrow Z \leftarrow Y)$. By definition, it is a kernel bisimulation. Moreover, the pushout of $(X \leftarrow R' \rightarrow Y)$ is Z again, so R' is an AM-precongruence.

For items (4), (5) and (6), let R be a kernel bisimulation, the pullback of a cospan $(X \rightarrow Z \leftarrow Y)$, for some coalgebra (Z, z) . Note that the following diagram commutes.

$$\begin{array}{ccccc} & & X & \xrightarrow{h} & BX \\ & \nearrow & & & \nearrow \\ R & & & & BZ \\ & \searrow & & & \searrow \\ & & Y & \xrightarrow{k} & BY \end{array} \quad (4.1)$$

For item (4), we must show that R is an AM-bisimulation. We construct a coalgebra structure on R by considering the morphism $R \rightarrow BR$ induced since BR is a weak pullback, as in the following diagram.

$$\begin{array}{ccccc} & & X & \xrightarrow{h} & BX \\ & \nearrow & & & \nearrow \\ R & \cdots \longrightarrow & BR & & \\ & \searrow & & & \searrow \\ & & Y & \xrightarrow{k} & BY \\ & & & & \nearrow \\ & & & & BZ \end{array}$$

For item (5), we must show that R is an HJ-bisimulation. This follows from the following fact, which is immediate from the definition of $\bar{B}R$, and which is worth recording:

Fact 4.2. If B covers pullbacks, then

$$\text{if } \begin{array}{ccc} & X & \\ & \nearrow \quad \searrow & \\ R & & Z \\ & \searrow \quad \nearrow & \\ & Y & \end{array} \text{ is a pullback, so is } \begin{array}{ccc} & BX & \\ & \nearrow \quad \searrow & \\ \bar{B}R & & BZ \\ & \searrow \quad \nearrow & \\ & BY & \end{array} .$$

For item (6), we must show that R is an AM-precongruence. This is more involved. We make use of the following lemma.

Lemma 4.3. *Consider objects S, S', V, V' , and morphisms $p, q, p', q', f, g, f', g'$, making the following three diagrams commute.*

$$\begin{array}{ccc}
 \begin{array}{ccc} S & \begin{array}{c} \xrightarrow{p} \\ \searrow q \end{array} & \begin{array}{c} X \\ \xrightarrow{f} \\ \swarrow g \end{array} \\ & (a) & \\ & & V \end{array} & \begin{array}{ccc} S' & \begin{array}{c} \xrightarrow{p'} \\ \searrow q' \end{array} & \begin{array}{c} X \xrightarrow{h} BX \\ \xrightarrow{k} BY \end{array} \\ & (b) & \\ & & BV \end{array} & \begin{array}{ccc} S & \begin{array}{c} \xrightarrow{p} \\ \searrow q \end{array} & \begin{array}{c} X \\ \xrightarrow{f'} \\ \swarrow g' \end{array} \\ & (c) & \\ & & V' \end{array}
 \end{array}$$

If the left-hand diagram is a pullback, and B preserves pullbacks along monos, and \mathcal{C} is regular, then the following diagram also commutes.

$$\begin{array}{ccc} S' & \begin{array}{c} \xrightarrow{p'} \\ \searrow q' \end{array} & \begin{array}{c} X \xrightarrow{h} BX \\ \xrightarrow{k} BY \end{array} \\ & (d) & \\ & & BV' \end{array}$$

Proof of Lemma 4.3: Write $\text{im}(f)$ for the image of $f : X \rightarrow V$, etc.. Subdivide the pullback (a) as follows.

$$\begin{array}{ccccc} & & & X & \\ & & & \swarrow & \searrow \\ S & \begin{array}{c} \nearrow \\ \searrow \end{array} & \begin{array}{c} \text{im}(p) \\ \text{im}(q) \end{array} & \begin{array}{c} \nearrow \\ \searrow \\ \nearrow \\ \searrow \end{array} & \begin{array}{c} X \\ W \\ Y \end{array} \\ & & & \swarrow & \searrow \\ & & & \text{im}(f) & \text{im}(g) \\ & & & \nearrow & \searrow \\ & & & & V \end{array} \tag{4.2}$$

Since \mathcal{C} is regular, the composite $S \rightarrow W$ is regular epi. In this situation, the leftmost pullback is also a pushout, as indicated (see e.g. [13, Thm 5.2].)

Since diagram (b) commutes, and since B preserves pullbacks along monos, we have unique morphisms $S' \rightarrow B(\text{im}(p))$ and $S' \rightarrow B(\text{im}(q))$ making the following diagram commute.

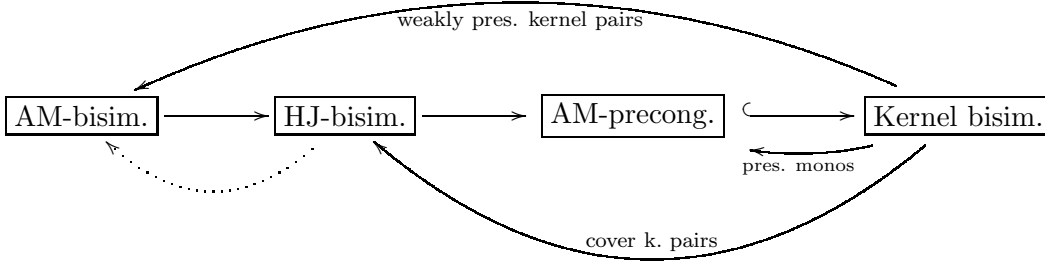
$$\begin{array}{ccccc} S' & \begin{array}{c} \xrightarrow{p'} \\ \searrow q' \end{array} & \begin{array}{c} X \xrightarrow{h} BX \\ \xrightarrow{k} BY \end{array} \\ & \nearrow & \begin{array}{c} B(\text{im}(p)) \\ B(\text{im}(q)) \end{array} & \begin{array}{c} \nearrow \\ \searrow \\ \nearrow \\ \searrow \end{array} & \begin{array}{c} BX \\ BW \\ BY \end{array} \\ & & & \swarrow & \searrow \\ & & & B(\text{im}(f)) & B(\text{im}(g)) \\ & & & \nearrow & \searrow \\ & & & & BV \end{array}$$

Now consider diagram (c). Since W is a pushout, we have a unique morphism $W \rightarrow V'$ making the following diagram commute.

$$\begin{array}{ccccc} S & \begin{array}{c} \nearrow \\ \searrow \end{array} & \begin{array}{c} \text{im}(p) \\ \text{im}(q) \end{array} & \begin{array}{c} \nearrow \\ \searrow \\ \nearrow \\ \searrow \end{array} & \begin{array}{c} X \\ W \\ Y \end{array} \\ & & & \swarrow & \searrow \\ & & & \text{im}(f') & \text{im}(g') \\ & & & \nearrow & \searrow \\ & & & & V' \end{array}$$

The conditions are connected: in an extensive regular category, a functor covers pullbacks if and only if it covers kernel pairs and preserves pullbacks along monos. (Gumm and Schröder [24] showed this for **Set**, and their argument is readily adapted to this more general setting.)

In summary, we have the following situation, when focusing on equivalence relations:



5. CONSTRUCTING BISIMILARITY

In this section we consider a procedure for constructing the maximal bisimulation. We relate it with the terminal sequence, which is used for finding final coalgebras.

Context. In this section we assume that the ambient category \mathcal{C} is complete and regular. We fix an endofunctor B on \mathcal{C} , and fix two B -coalgebras, $h : X \rightarrow BX$ and $k : Y \rightarrow BY$.

5.1. The relation refinement sequence. The greatest HJ-bisimulations can be understood as greatest fixed points of the operator Φ^{HJ} on $\mathbf{Rel}_{\mathcal{C}}(X, Y)$. We define an ordinal indexed cochain $(r_{\beta, \alpha} : R_{\beta}^{\text{HJ}} \rightrightarrows R_{\alpha}^{\text{HJ}})_{\alpha \leq \beta}$ in $\mathbf{Rel}_{\mathcal{C}}(X, Y)$, in the usual way:

- *Limiting case:* If λ is limiting, then let R_{λ}^{HJ} be $\bigcap_{\alpha < \lambda} R_{\alpha}^{\text{HJ}}$, i.e. the limit of the cochain $(r_{\beta, \alpha} : R_{\beta}^{\text{HJ}} \rightrightarrows R_{\alpha}^{\text{HJ}})_{\alpha \leq \beta < \lambda}$. In particular, let $R_0^{\text{HJ}} = X \times Y$.
- *Inductive case:* let $R_{\alpha+1}^{\text{HJ}} = \Phi^{\text{HJ}}(R_{\alpha}^{\text{HJ}})$.

We call this cochain the *relation refinement sequence*. If this sequence is eventually stationary then it achieves the maximal post-fixed point of Φ^{HJ} , the greatest HJ-bisimulation. (NB. The sequence always converges when $\mathbf{Rel}_{\mathcal{C}}(X, Y)$ is small, e.g. when \mathcal{C} is well-powered.)

For the case of the endofunctor $\mathcal{P}(L \times (-))$ on **Set**, the relation refinement sequence is a transfinite extension of Milner's sequence $\sim_0 \geq \sim_1 \geq \dots \geq \sim$ (e.g. [38, Sec. 5.7]), studied from an algorithmic perspective by Kanellakis and Smolka [30].

If \mathcal{C} has pushouts, we can also consider a cochain $(R_{\beta}^{\text{AM}} \rightrightarrows R_{\alpha}^{\text{AM}})_{\alpha \leq \beta}$ corresponding to the operator Φ^{AM} . As will be seen, the two sequences of relations often coincide.

(The other notions of bisimulation, AM-bisimulation and kernel bisimulation, cannot be characterized as post-fixed points, in general.)

5.2. The terminal sequence. There is another sequence that is often studied in the coalgebraic setting. The *terminal sequence* is an ordinal-indexed cochain

$$(z_{\beta,\alpha} : Z_\beta \rightarrow Z_\alpha)_{\alpha \leq \beta}$$

that can be used to construct a final coalgebra for an endofunctor (e.g. Worrell [55]). The idea is to begin with the terminal object, and then successively find *B-algebra* structures, so that if the sequence converges, i.e. the *B-algebra* structure is an isomorphism, then we have a *B-coalgebra* structure, and indeed the final such. The cochain commutes and satisfies the following conditions:

- *Limiting case:* $Z_\lambda = \lim \{z_{\beta,\alpha} : Z_\beta \rightarrow Z_\alpha \mid \alpha \leq \beta < \lambda\}$, if λ is limiting, in which case the cone $\{z_{\lambda,\alpha} : Z_\lambda \rightarrow Z_\alpha \mid \alpha < \lambda\}$ is the limiting one;
- *Inductive case:* $Z_{\alpha+1} = B(Z_\alpha)$; and $z_{\beta+1,\alpha+1} = B(z_{\beta,\alpha}) : Z_{\beta+1} \rightarrow Z_{\alpha+1}$.

5.3. Relating the relation and terminal sequences. The coalgebras (X, h) , (Y, k) determine two cones

$$(x_\alpha : X \rightarrow Z_\alpha)_\alpha \quad (y_\alpha : Y \rightarrow Z_\alpha)_\alpha$$

over the terminal sequence. The first cone, $(x_\alpha)_\alpha$ is given as follows.

- *Limiting case:* If λ is a limit ordinal then the morphisms $x_\alpha : X \rightarrow Z_\alpha$ for $\alpha < \lambda$ form a cocone over the cochain $(z_{\beta,\alpha} : Z_\beta \rightarrow Z_\alpha)_{\alpha \leq \beta < \lambda}$, with apex X . We let $x_\lambda : X \rightarrow Z_\lambda$ be the unique mediating morphism. For instance, when $\lambda = 0$, then $x_\lambda : X \rightarrow Z_\lambda$ is the terminal map $X \rightarrow 1$.
- *Inductive case:* Let $x_{\alpha+1}$ be the composite

$$X \xrightarrow{h} BX \xrightarrow{Bx_\alpha} BZ_\alpha = Z_{\alpha+1} \quad .$$

The other cone, $(y_\alpha : X \rightarrow Z_\alpha)_\alpha$, is defined similarly.

These cones determine another ordinal indexed cochain $(R_\beta \rightarrow R_\alpha)_{\alpha \leq \beta}$ in $\mathbf{Rel}_C(X, Y)$. For every ordinal α , let R_α be the following pullback.

$$\begin{array}{ccc} R_\alpha & \longrightarrow & Y \\ \downarrow \lrcorner & & \downarrow y_\alpha \\ X & \xrightarrow{x_\alpha} & Z_\alpha \end{array}$$

Theorem 5.1. *Consider an ordinal α .*

- (1) R_α contains all the kernel bisimulations and all the AM-precongruences.
- (2) If B covers pullbacks, then $R_\alpha = R_\alpha^{\text{HJ}}$.
- (3) If B preserves pullbacks along monos, and C has pushouts, then $R_\alpha = R_\alpha^{\text{AM}}$.

Proof. To see that R_α contains all the kernel bisimulations, notice that for a cospan of coalgebras, $(X \rightarrow Z \leftarrow Y)$, the coalgebra Z determines a cone over the terminal sequence, just as X and Y do. The relation R_α contains all the AM-precongruences by transfinite induction on α : the limit step is vacuous and the inductive step uses the definition of AM-precongruence. Statements (2) and (3) are also proved by transfinite induction on α : the limit steps use the fact that limits commute with limits; the inductive steps follow from Fact 4.2, and from Lemma 4.3, respectively. \square

Note. A consequence of Item (3) of Theorem 5.1 is that, if the sequence $(R_\beta \rightsquigarrow R_\alpha)_{\alpha \leq \beta}$ converges, then the result is the greatest AM-precongruence, provided the endofunctor preserves pullbacks along monos and \mathcal{C} has pushouts. In fact, this corollary still holds even if \mathcal{C} does not have all pushouts. This can be proved directly by transfinite induction; the inductive step uses Lemma 4.3.

5.4. Convergence for the relation refinement sequences. By Theorem 5.1, if the terminal sequence converges, then the relation refinement sequence does too. Of course, this is not a sufficient condition. Indeed, even when there is a final coalgebra, the relation refinement sequence may converge before the terminal sequence:

Proposition 5.2. *If, for some ordinal α , the morphism $z_{\alpha+1,\alpha} : Z_{\alpha+1} \rightarrow Z_\alpha$ is monic, then $R_\alpha = R_{\alpha+1}$.*

Proof. We have the following situation.

$$\begin{array}{ccccc}
 R_{\alpha+1} & \xrightarrow{\quad} & Z_{\alpha+1} & \xrightarrow{z_{\alpha+1,\alpha}} & Z_\alpha \\
 \downarrow \lrcorner & & \downarrow \Delta & & \downarrow \Delta \\
 X \times Y & \xrightarrow{x_{\alpha+1} \times y_{\alpha+1}} & Z_{\alpha+1} \times Z_{\alpha+1} & \xrightarrow{z_{\alpha+1,\alpha} \times z_{\alpha+1,\alpha}} & Z_\alpha \times Z_\alpha
 \end{array}$$

The left-hand square is a pullback — this is a rearrangement of the definition of $R_{\alpha+1}$. The right-hand square is a pullback if and only if $z_{\alpha+1,\alpha}$ is monic. Thus the outer square is a pullback. Now $x_\alpha = (x_{\alpha+1} \cdot z_{\alpha+1,\alpha})$ and $y_\alpha = (y_{\alpha+1} \cdot z_{\alpha+1,\alpha})$, which means that $R_{\alpha+1} = R_\alpha$. \square

If \mathcal{C} is **Set** and B preserves filtered colimits, then the terminal sequence does not converge until $(\omega + \omega)$, but it becomes monic at ω [55]. As is well-known, the relation refinement sequence for image-finite transition systems converges at ω .

For the case when $\mathcal{C} = \mathbf{Set}$ and sets X and Y are both finite, the relation refinement sequence will converge before ω because the skeleton of $\mathbf{Rel}_{\mathcal{C}}(X, Y)$ is finite. This is relevant in a slightly more general setting: In a Boolean Grothendieck topos, every descending ω -cochain of subobjects from a finitely presentable object is eventually constant.

The presheaf topos $[\mathbb{I}, \mathbf{Set}]$, used to model name-passing, is not Boolean. It does, however, have $\mathbf{Sh}_{\neg, \neg}(\mathbb{I})$ as a Boolean subcategory, and this is perhaps a more appropriate universe for name-passing calculi (see e.g. [18]). There, the finitely presentable objects are exactly those objects that can be described by finite ‘named-sets with symmetry’ [18, 21]. Thus the techniques of this article provide a general foundation for the coalgebra-inspired verification procedures of Ferrari, Montanari and Pistore [17].

APPENDIX A. SOME CONCEPTS FROM CATEGORICAL LOGIC

We recall some concepts from categorical logic: regular categories; and powersets from algebraic set theory.

A.1. Regular and extensive categories. An *image* of a morphism $f : A \rightarrow C$ is a monomorphism $m : B \rightarrow C$ through which f factors, which is minimal in the sense that, if f factors through any other mono, $B' \rightarrow C$, then B is a subobject of B' . In this setting, the factoring morphism $f : A \rightarrow B$ is called a *cover*; its image is $B \xrightarrow{\text{id}} B$. A category *has images* if every morphism has an image.

In a category with finite limits, covers are epimorphisms. They serve as a generalization of ‘surjective function’. In this setting, other authors refer to covers as *strong* epimorphisms.

A category with finite limits and images is said to be *regular* if covers are stable under pullback, i.e., if the following diagram is a pullback, and if f is a cover, then so is f' .

$$\begin{array}{ccc} A' & \xrightarrow{g'} & A \\ f' \downarrow & \lrcorner & \downarrow f \\ B' & \xrightarrow{g} & B \end{array} \quad (\text{A.1})$$

Recall that a category is said to be *extensive* if it has coproducts and they are disjoint and stable under pullback (see e.g. [15]).

A.2. Open maps, powersets, and algebraic set theory. We now recall an analysis of ‘smallness’ due to Joyal and Moerdijk [28, 29]. (For a more recent introduction to this area of research, see the articles by Awodey [5] and by van den Berg and Moerdijk [8].)

An intuition for this analysis is that a morphism $f : A \rightarrow B$ describes a B -indexed family of classes — informally, for each element b of B , we have a class $f^{-1}(b)$. When we say that a morphism $f : A \rightarrow B$ is small, an intuition is that each fibre $f^{-1}(b)$ is small. In particular, we say that an object A is small if the terminal map $A \rightarrow 1$ is small.

Open maps in regular categories. Let \mathcal{C} be a regular category. A class \mathcal{S} of morphisms in \mathcal{C} is a class of *open maps* if it satisfies the following four axioms. (This numbering follows [28].)

- (A1) \mathcal{S} is closed under composition, and all identity morphisms are in \mathcal{S} .
- (A2) \mathcal{S} is stable under pullback, i.e., in diagram (A.1), if f is in \mathcal{S} , then f' is also in \mathcal{S} .
- (A3) (‘Descent’) In diagram (A.1), if f' is in \mathcal{S} and g is a cover, then f is also in \mathcal{S} .
- (A6) In the following triangle, if f is in \mathcal{S} and e is a cover, then g is also in \mathcal{S} .

$$\begin{array}{ccc} A & \xrightarrow{e} & A' \\ & \searrow f & \downarrow g \\ & & B \end{array}$$

Most authors assume that \mathcal{C} has additional structure so that the universal quantifier can be interpreted in \mathcal{C} . We do not need that in this article.

Sums and open maps. In an extensive regular category, it is appropriate to assume the following additional axioms.

- (A4) The maps $0 \rightarrow 1$ and $1 + 1 \rightarrow 1$ are in \mathcal{S} .
- (A5) If $A \rightarrow A'$ and $B \rightarrow B'$ are in \mathcal{S} , then so is $(A + B) \rightarrow (A' + B')$.

Powersets. Given a regular category \mathcal{C} and a class of open maps \mathcal{S} , an \mathcal{S} -relation is a jointly monic span $(I \leftarrow R \rightarrow A)$ whose left projection is in \mathcal{S} .

An \mathcal{S} -powerset for an object A of \mathcal{C} is an object $P_{\mathcal{S}}(A)$ together with an \mathcal{S} -relation $(P_{\mathcal{S}}(A) \leftarrow \exists_A \rightarrow A)$ such that for every \mathcal{S} -relation $(I \leftarrow R \rightarrow A)$ there is a unique morphism $I \rightarrow P_{\mathcal{S}}(A)$ making $(R \rightarrow I \times A)$ a pullback of $(\exists_A \rightarrow P_{\mathcal{S}}(A) \times A)$:

$$\begin{array}{ccc} R & \longrightarrow & \exists_A \\ \downarrow & & \downarrow \\ I \times A & \longrightarrow & P_{\mathcal{S}}(A) \times A \end{array}$$

It follows from axiom (A2) that morphisms $I \rightarrow P_{\mathcal{S}}(A)$ are in bijective correspondence with \mathcal{S} -relations $(I \leftarrow R \rightarrow A)$.

If every object of \mathcal{C} has an \mathcal{S} -powerset, then the construction $P_{\mathcal{S}}$ extends straightforwardly to a covariant endofunctor on \mathcal{C} , as follows. For any morphism $f : A \rightarrow B$, the action $P_{\mathcal{S}}(f) : P_{\mathcal{S}}(A) \rightarrow P_{\mathcal{S}}(B)$ corresponds to the \mathcal{S} -relation in the image of the span $(P_{\mathcal{S}}(A) \leftarrow \exists_A \rightarrow A \rightarrow B)$, using axiom (A6).

Separation and collection. There are various axioms and axiom schema that can be assumed as principles for defining sets in a constructive setting. The notes by Aczel and Rathjen [3] provide an overview.

The separation axiom (also known as bounded comprehension) amounts to the following axiom on \mathcal{S} .

(M) All monomorphisms in \mathcal{C} are in \mathcal{S} .

The (strong) collection axiom has the following categorical counterpart, when there is an \mathcal{S} -powerset:

(A9) The endofunctor $P_{\mathcal{S}}$ preserves covers.

Further axioms. The axioms above are all that we need in this article. As a foundation of mathematics, these axioms are too weak: one would typically also require that there is a small natural numbers object; that each powerset $P_{\mathcal{S}}(X)$ is small; and that there is a class of all sets, a universal small map.

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