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ROOTED DIVERGENCE-PRESERVING BRANCHING BISIMILARITY IS A CONGRUENCE

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Abstract. We prove that rooted divergence-preserving branching bisimilarity is a congruence for the process specification language consisting of nil, action prefix, choice, and the recursion construct.

1. Introduction

Branching bisimilarity [GW96] is a behavioural equivalence on processes that is compatible with abstraction from internal activity, while at the same time preserving the branching structure of processes in a strong sense [Gla01]. Branching bisimilarity abstracts to a large extent from *divergence* (i.e., infinite internal activity). For instance, it identifies a process, say P, that may perform some internal activity after which it returns to its initial state (i.e., P has a τ-loop) with a process, say P', that admits the same behaviour as P except that it cannot perform the internal activity leading to the initial state (i.e., P' is P without the τ-loop).

In situations where fairness principles apply, abstraction from divergence is often desirable. But there are circumstances in which abstraction from divergence is undesirable: A behavioural equivalence that abstracts from divergence is not compatible with any temporal logic featuring an *eventually* modality: for any desired state that P' will eventually reach, the mentioned internal activity of P may be performed forever, and thus prevent P from reaching this desired state. It is also generally not compatible with a process-algebraic priority operator (cf. [Vaa90, pp. 130–132]) or sequencing operator [BLY17]. Since a divergence may be exploited to simulate recursively enumerable branching in a computable transition system [Phi93], a divergence-insensitive behavioural equivalence may be considered too coarse for a theory that integrates computability and concurrency [BLvT13]. Preservation of divergence is widely considered an important correctness criterion when studying the relative expressiveness of process calculi [Gor10, Xu12, Fu16].

Key words and phrases: Process algebra; Recursion; Branching bisimulation; Divergence; Congruence.

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The notion of *branching bisimilarity with explicit divergence*, also stemming from [GW96], is a suitable refinement of branching bisimilarity that is compatible with the well-known branching-time temporal logic CTL* without the nexttime operator X (which is known to be incompatible with abstraction from internal activity). In fact, in [GLT09b] we have proved that it is the coarsest semantic equivalence on labelled transition systems with silent moves that is a congruence for parallel composition (as found in process algebras like CCS, CSP or ACP) and only equates processes satisfying the same CTL$^*_{-X}$ formulas. In [BLvT13], for stylistic reasons, *branching bisimilarity with explicit divergence* was named *divergence-preserving branching bisimilarity*; we shall henceforth use this term.

Divergence-preserving branching bisimilarity is the finest behavioural equivalence in the linear time – branching time spectrum [Gla93b]. It is the principal behavioural equivalence underlying the theory of executability [BLvT12, BLvT13, LY15, LY16]. Reduction modulo divergence-preserving branching bisimilarity is a part of methods for formal verification and analysis of the behaviour of systems [MW14, WE13, dPW17, ZSW$^+$16]. In [dFEKW16] a game-based characterisation of divergence-preserving branching bisimilarity is presented.

Processes are usually specified in some process specification language. For compositional reasoning it is then important that the behavioural equivalence used is a congruence with respect to the constructs of that language. Following Milner [Mil89], we consider the language *basic CCS with recursion*, i.e., the language consisting of $\mathbf{0}$, action prefix, and choice, extended with the recursion construct $\mu X.\_$; this language precisely allows the specification of finite-state behaviours. As for other weak behavioural equivalences, divergence-preserving branching bisimilarity is not a congruence for that language; in fact, it is not a congruence for any language that includes choice. The goal of this paper is to prove that adding the usual root condition suffices to obtain a congruence—and, in fact, the coarsest congruence—for the language under consideration that is included in divergence-preserving branching bisimilarity. Interestingly, the root condition is not only necessary to get a congruence for choice, but also for recursion: $\tau.X$ is divergence-preserving branching bisimilar to X, yet $\mu X.\tau.X$ diverges whereas $\mu X.X$ does not.

Recently, a congruence format was proposed for (rooted) divergence-preserving branching bisimilarity [FvGL19]. The operational rules for action prefix and choice are in this format. Unfortunately, however, this format does not support the recursion construct $\mu X.\_$. Interestingly, as far as we know, the recursion construct has not been covered at all in the rich literature on congruence formats, with the recent exception of [Gla17]. (The article [Gla17] differentiates between *lean* and *full* congruences for recursion; in this article we consider the full congruence.)

The congruence result obtained in this paper should serve as a stepping stone towards a complete axiomatisation of divergence-preserving branching bisimilarity for basic CCS with recursion. Such work, inspired by Milner's complete axiomatisation of weak bisimilarity [Mil89], would combine the adaptations of [Gla93a] to branching bisimilarity, and of [LDH05] to several divergence-sensitive variants of weak bisimilarity.

We originally thought that congruence for recursion could be obtained in the same spirit as Milner's ingenious proof in [Mil90] for strong bisimilarity, which cleverly makes use of an up-to technique. The proofs for weak and branching bisimilarity essentially reuse this idea [Mil90, Gla93a], but require the use of a weak step in the antecedent of the transfer condition. We were not able to generalise the idea to divergence-preserving branching bisimilarity until we included the root condition in the up-to technique.

We believe that the proofs of Corollaries 3.3 and 3.4, Propositions 3.7 and 3.11, and Lemma 3.16 contain novel twists. Although the other proofs are either routine or adaptations of the ones in [Mil90], we have included them for the convenience of the reader.

In this paper we do not study the CCS constructs for parallel composition, restriction and relabelling. However, combining our results with those of [FvGL19] yields a congruence result for full CCS with the proviso that parallel composition, restriction and relabelling are not allowed in the scope of a recursion. This spans most practical applications. The method employed in this paper does not generalise to obtain a congruence result for full CCS, featuring parallel composition in the scope of recursion. Although we conjecture that such a congruence result holds, proving it remains an open problem.

2. Rooted divergence-preserving branching bisimilarity

Let $\mathcal{A}$ be a non-empty set of *actions*, and let τ be a special action not in $\mathcal{A}$. Let $\mathcal{A}_\tau = \mathcal{A} \cup \{\tau\}$. Furthermore, let $\mathcal{V}$ be a set of *recursion variables.* The set of *process expressions* $\mathcal{E}$ is generated by the following grammar:

$$E ::= \mathbf{0} \;\mid\; X \;\mid\; \alpha.E \;\mid\; \mu X.E \;\mid\; E + E \qquad (\alpha \in \mathcal{A}_\tau,\ X \in \mathcal{V}) \ .$$

An occurrence of a recursion variable X in a process expression E is *bound* if it is in the scope of a $\mu X.\_$, and otherwise it is *free.* We denote by $FV(E)$ the set of variables with a free occurrence in E. If $\vec{X} = X_0, \ldots, X_n$ is a sequence of variables, and $\vec{F} = F_0, \ldots, F_n$ is a sequence of process expressions of the same length, then we write $E[\vec{F}/\vec{X}]$ for the process expression obtained from E by replacing all free occurrences of X_i in E by F_i $(i = 0, \ldots, n)$, applying α-conversion to E if necessary to avoid capture.

On $\mathcal{E}$ we define an $\mathcal{A}_\tau$-labelled transition relation $\longrightarrow \subseteq \mathcal{E} \times \mathcal{A}_\tau \times \mathcal{E}$ as the least ternary relation satisfying the following rules for all $\alpha \in \mathcal{A}_\tau$, $X \in \mathcal{V}$, and process expressions E, E', F and F':

$$1\,\frac{}{\alpha.E \xrightarrow{\alpha} E} \qquad 2\,\frac{E[\mu X.E/X] \xrightarrow{\alpha} E'}{\mu X.E \xrightarrow{\alpha} E'} \qquad 3\,\frac{E \xrightarrow{\alpha} E'}{E + F \xrightarrow{\alpha} E'} \qquad 4\,\frac{F \xrightarrow{\alpha} F'}{E + F \xrightarrow{\alpha} F'}$$

We write $E \xrightarrow{\alpha} E'$ for $(E, \alpha, E') \in \longrightarrow$ (as we already did in the rules above) and we abbreviate the statement '$E \xrightarrow{\alpha} E'$ or ($\alpha = \tau$ and $E = E'$)' by $E \xrightarrow{(\alpha)} E'$. Furthermore, we write $\twoheadrightarrow$ for the reflexive-transitive closure of $\xrightarrow{\tau}$, i.e., $E \twoheadrightarrow E'$ if there exist $E_0, E_1, \ldots, E_n \in \mathcal{E}$ such that $E = E_0 \xrightarrow{\tau} E_1 \xrightarrow{\tau} \cdots \xrightarrow{\tau} E_n = E'$.

A process expression is *closed* if it contains no free occurrences of recursion variables; we denote by $\mathcal{P}$ the subset of $\mathcal{E}$ consisting of all closed process expressions. It is easy to check that if P is a closed process expression and $P \xrightarrow{\alpha} E$, then E is a closed process expression too. Hence, the transition relation restricts in a natural way to closed process expressions, and thus associates with every closed process expression a behaviour. We proceed to define when two process expressions may be considered to represent the same behaviour.

Definition 2.1. A symmetric binary relation $\mathcal{R}$ on $\mathcal{P}$ is a *branching bisimulation* if it satisfies the following condition for all $P, Q \in \mathcal{P}$ and $\alpha \in \mathcal{A}_\tau$:

(T) if $P \mathrel{\mathcal{R}} Q$ and $P \xrightarrow{\alpha} P'$ for some closed process expression P', then there exist closed process expressions Q' and Q'' such that $Q \twoheadrightarrow Q'' \xrightarrow{(\alpha)} Q'$, $P \mathrel{\mathcal{R}} Q''$ and $P' \mathrel{\mathcal{R}} Q'$.

We say that a branching bisimulation $\mathcal{R}$ *preserves (internal) divergence* if

(D) if $P \mathrel{\mathcal{R}} Q$ and there is an infinite sequence of closed process expressions $(P_k)_{k\in\omega}$ such that $P = P_0$, $P_k \xrightarrow{\tau} P_{k+1}$ and $P_k \mathrel{\mathcal{R}} Q$ for all $k \in \omega$, then there is an infinite sequence of closed process expressions $(Q_\ell)_{\ell\in\omega}$ such that $Q = Q_0$, $Q_\ell \xrightarrow{\tau} Q_{\ell+1}$ and $P_k \mathrel{\mathcal{R}} Q_\ell$ for all $k, \ell \in \omega$.

We write $P \mathrel{\underline{\leftrightarrow}}_b^\Delta Q$ if there exists a divergence-preserving branching bisimulation $\mathcal{R}$ such that $P \mathrel{\mathcal{R}} Q$. The relation $\underline{\leftrightarrow}_b^\Delta$ was introduced in [GW96] under the name branching bisimilarity with explicit divergence and is here referred to as divergence-preserving branching bisimilarity.

The relation $\underline{\leftrightarrow}_b^\Delta$ was studied in detail in [GLT09a]; we recap some of the facts established *ibidem.*

First, the relation $\underline{\leftrightarrow}_b^\Delta$ is an equivalence relation. Second, the relation $\underline{\leftrightarrow}_b^\Delta$ satisfies the condition (T), with the following generalisation as a straightforward consequence.

Lemma 2.2. *Let P and Q be closed process expressions. If $P \mathrel{\underline{\leftrightarrow}}_b^\Delta Q$ and $P \longrightarrow\!\!\!\!\!\rightarrow P'' \xrightarrow{\alpha} P'$ for some closed process expressions P' and P'', then there exist closed process expressions Q' and Q'' such that $Q \longrightarrow\!\!\!\!\!\rightarrow Q'' \xrightarrow{\alpha} Q'$, $P'' \mathrel{\underline{\leftrightarrow}}_b^\Delta Q''$ and $P' \mathrel{\underline{\leftrightarrow}}_b^\Delta Q'$.*

Third, $\underline{\leftrightarrow}_b^\Delta$ also satisfies (D). In [GLT09a] several alternative definitions of divergence preservation are studied, which, in the end, all give rise to the same notion of divergence-preserving branching bisimilarity. In particular, the following alternative relational characterisations will be useful tools in the remainder.

Proposition 2.3. *Let P and Q be closed process expressions. Then*

- *$P \mathrel{\underline{\leftrightarrow}}_b^\Delta Q$ if, and only if, P and Q are related by some branching bisimulation $\mathcal{R}$ satisfying*
  (D′) *if $P \mathrel{\mathcal{R}} Q$ and there is an infinite sequence of closed process expressions $(P_k)_{k\in\omega}$ such that $P = P_0$ and $P_k \xrightarrow{\tau} P_{k+1}$, then there is an infinite sequence of closed process expressions $(Q_\ell)_{\ell\in\omega}$ and a mapping $\sigma : \omega \to \omega$ such that $Q = Q_0$, $Q_\ell \xrightarrow{\tau} Q_{\ell+1}$ and $P_{\sigma(\ell)} \mathrel{\mathcal{R}} Q_\ell$ for all $\ell \in \omega$; and*
- *$P \mathrel{\underline{\leftrightarrow}}_b^\Delta Q$ if, and only if, P and Q are related by some branching bisimulation $\mathcal{R}$ satisfying*
  (D″) *if $P \mathrel{\mathcal{R}} Q$ and there is an infinite sequence of closed process expressions $(P_k)_{k\in\omega}$ such that $P = P_0$ and $P_k \xrightarrow{\tau} P_{k+1}$, then there exists a closed process expression Q' such that $Q \xrightarrow{\tau} Q'$ and $P_k \mathrel{\mathcal{R}} Q'$ for some $k \in \omega$.*

Moreover, $\underline{\leftrightarrow}_b^\Delta$ itself satisfies (D′) and (D″).

Proof. See [GLT09a]; condition (D′) is (D3) and condition (D″) is (D2). □

And finally, it was proved in [GLT09a] that $\underline{\leftrightarrow}_b^\Delta$ satisfies the following so-called stuttering property.

Proposition 2.4. *Let P be a closed process expression and let $Q_0, \ldots, Q_k$ be closed process expressions such that $Q_0 \xrightarrow{\tau} \cdots \xrightarrow{\tau} Q_k$. If $P \mathrel{\underline{\leftrightarrow}}_b^\Delta Q_0$ and $P \mathrel{\underline{\leftrightarrow}}_b^\Delta Q_k$, then $P \mathrel{\underline{\leftrightarrow}}_b^\Delta Q_i$ for all $0 \leq i \leq k$.*

As for all variants of bisimilarity that take some form of abstraction from internal activity into account, the relation $\underline{\leftrightarrow}_b^\Delta$ is not compatible with $+$ ($\mathbf{0} \mathrel{\underline{\leftrightarrow}}_b^\Delta \tau.\mathbf{0}$ but $\mathbf{0}+a.\mathbf{0} \not\mathrel{\underline{\leftrightarrow}}_b^\Delta \tau.\mathbf{0}+a.\mathbf{0}$), and hence not a congruence for the language we are considering. In contrast to its divergence-insensitive variant, divergence-preserving branching bisimilarity is not compatible with the recursion construct either, as we will argue below. Similarly as for the divergence-insensitive variant of branching bisimilarity, it suffices to add a *root condition* to obtain the coarsest congruence for our language that is contained in $\leftrightarrow_b^\Delta$, as we shall prove in the remainder of this paper.

Definition 2.5. Let P and Q be closed process expressions. We say that P and Q are *rooted divergence-preserving branching bisimilar* (notation: $P \leftrightarrow_{rb}^\Delta Q$) if for all $\alpha \in \mathcal{A}_\tau$ the following holds:

(R1) if $P \xrightarrow{\alpha} P'$, then there exists a Q' such that $Q \xrightarrow{\alpha} Q'$ and $P' \leftrightarrow_b^\Delta Q'$; and
(R2) if $Q \xrightarrow{\alpha} Q'$, then there exists a P' such that $P \xrightarrow{\alpha} P'$ and $P' \leftrightarrow_b^\Delta Q'$.

The following proposition is a straightforward consequence of the fact that $\leftrightarrow_b^\Delta$ is an equivalence.

Proposition 2.6. *The relation $\leftrightarrow_{rb}^\Delta$ is an equivalence relation on $\mathcal{P}$.* □

Moreover, it is immediate that $\leftrightarrow_{rb}^\Delta \subseteq \leftrightarrow_b^\Delta$. It is well known, and follows immediately from the definition, that $P \leftrightarrow_{rb}^\Delta Q$ iff $P + f.0 \leftrightarrow_b^\Delta Q + f.0$ for a fresh action f, not occurring in P or Q. Using this, the problem of checking rooted divergence-preserving branching bisimilarity reduces trivially to that of checking divergence-preserving branching bisimilarity.

We have defined the notions of $\leftrightarrow_b^\Delta$ and $\leftrightarrow_{rb}^\Delta$ on closed process expressions because those are thought of as directly representing behaviour. Due to the presence of the binding construct $\mu X.\_$ it is, however, necessary to lift these notions to expressions with free variables even if the goal is simply to establish behavioural equivalence of closed process expressions.

Definition 2.7. Let E and F be process expressions, and let the sequence $\vec{X}$ of variables at least include all the variables with a free occurrence in E or F. We write $E \leftrightarrow_{rb}^\Delta F$ ($E \leftrightarrow_b^\Delta F$) if $E[\vec{P}/\vec{X}] \leftrightarrow_{rb}^\Delta F[\vec{P}/\vec{X}]$ ($E[\vec{P}/\vec{X}] \leftrightarrow_b^\Delta F[\vec{P}/\vec{X}]$) for every sequence of closed process expressions $\vec{P}$ of the same length as $\vec{X}$.

It is clear from the definition above that, since $\leftrightarrow_b^\Delta$ and $\leftrightarrow_{rb}^\Delta$ are equivalence relations on $\mathcal{P}$, their lifted versions are equivalence relations on $\mathcal{E}$. Note that $\leftrightarrow_b^\Delta$ is not compatible with the recursion construct: we have that $X \leftrightarrow_b^\Delta \tau.X$, whereas $\mu X.X \not\leftrightarrow_b^\Delta \mu X.\tau.X$. We shall prove that its rooted variant $\leftrightarrow_{rb}^\Delta$ is, however, compatible with all the constructs of the syntax, i.e., if $E \leftrightarrow_{rb}^\Delta F$, then $\alpha.E \leftrightarrow_{rb}^\Delta \alpha.F$ for all $\alpha \in \mathcal{A}_\tau$, $\mu X.E \leftrightarrow_{rb}^\Delta \mu X.F$ for all $X \in \mathcal{V}$, $E + H \leftrightarrow_{rb}^\Delta F + H$ and $H + E \leftrightarrow_{rb}^\Delta H + F$ for all process expressions H. To prove that $\leftrightarrow_{rb}^\Delta$ is compatible with $\alpha.$ and $+$ is straightforward, but for $\mu X.\_$ this is considerably more work.

3. The congruence proof

Our proof that $\leftrightarrow_{rb}^\Delta$ is compatible with $\mu X.\_$ relies on the following observation: If $\vec{Y}$ is some sequence of variables and $\vec{P}$ is a sequence of closed terms of the same length, then, on the one hand, $E \leftrightarrow_{rb}^\Delta F$ implies $E[\vec{P}/\vec{Y}] \leftrightarrow_{rb}^\Delta F[\vec{P}/\vec{Y}]$ by the definition of $\leftrightarrow_{rb}^\Delta$ on $\mathcal{E}$, and, on the other hand, if X does not occur in $\vec{Y}$, then from $\mu X.E[\vec{P}/\vec{Y}] \leftrightarrow_{rb}^\Delta \mu X.F[\vec{P}/\vec{Y}]$ it follows that $(\mu X.E)[\vec{P}/\vec{Y}] \leftrightarrow_{rb}^\Delta (\mu X.F)[\vec{P}/\vec{Y}]$ by the definition of substitution. Therefore, as formalised in the proof of Proposition 3.19, it is enough to establish that $E \leftrightarrow_{rb}^\Delta F$ implies $\mu X.E \leftrightarrow_{rb}^\Delta \mu X.F$ in the special case that E and F are process expressions with no other free variables than X; such process expressions will be called X-closed.

The rest of this section is organised as follows.

We shall first characterise, in Section 3.1, the relation $\leftrightarrow_{rb}^{\Delta}$ on X-closed process expressions in terms of the transition relation on X-closed process expressions.

Then, in Section 3.2, we shall present a suitable notion of rooted divergence-preserving branching bisimulation up to $\leftrightarrow_{rb}^{\Delta}$, and we shall prove that every pair of rooted divergence-preserving branching bisimilar X-closed process expressions (E, F) gives rise to a relation $\mathcal{R}^u$ of which we can show that it is a rooted divergence-preserving branching bisimulation up to $\leftrightarrow_{rb}^{\Delta}$. The relation $\mathcal{R}^u$ will be defined in such a way that it relates $\mu X.E$ and $\mu X.F$ and thus allows us to conclude that these process expressions are rooted divergence-preserving bisimilar.

In Section 3.3, we shall then put the pieces together and prove $\leftrightarrow_{rb}^{\Delta}$ is the coarsest congruence contained in $\leftrightarrow_{b}^{\Delta}$ for basic CCS with recursion.

3.1. $\leftrightarrow_{rb}^{\Delta}$ on X-closed process expressions.

We say that a process expression E is X-*closed* if $FV(E) \subseteq \{X\}$; the set of all X-closed process expressions is denoted by $\mathcal{P}_X$. Note that if E is X-closed and $E \xrightarrow{\alpha} E'$, then E' is X-closed too, and so the $\mathcal{A}_\tau$-labelled transition relation restricts in a natural way to X-closed process expressions.

Definition 3.1. We define when X is *exposed* in a (not necessarily X-closed) process expression E by induction on the structure of E:

(1) if $E = X$, then X is exposed in E;
(2) if $E = \mu Y.E'$, Y is a recursion variable distinct from X and X is exposed in E', then X is exposed in E;
(3) if $E = E_1 + E_2$ and X is exposed in E_1 or E_2, then X is exposed in E.

Note that the variable X is exposed in E if, and only if, E has an unguarded occurrence of X in the sense of [Mil89].

We establish a relationship between the transitions of a closed process expression $E[P/X]$ that is obtained by substituting a closed process expression P for the variable X in an X-closed process expression E, and the transitions of E and P.

Lemma 3.2. *Let E be an X-closed process expression, and let P be a closed process expression.*

(1) *If $E \xrightarrow{\alpha} E'$, then $E[P/X] \xrightarrow{\alpha} E'[P/X]$, and if X is exposed in E and $P \xrightarrow{\alpha} P'$, then $E[P/X] \xrightarrow{\alpha} P'$.*
(2) *If $E[P/X] \xrightarrow{\alpha} P'$ for some (closed) process expression P', then either there exists an X-closed process expression E' such that $E \xrightarrow{\alpha} E'$ and $P' = E'[P/X]$, or X is exposed in E, $P \xrightarrow{\alpha} P'$ and every derivation of $E[P/X] \xrightarrow{\alpha} P'$ has a derivation of $P \xrightarrow{\alpha} P'$ as a subderivation.*

Proof. Statement 1 of the lemma is established with straightforward inductions on a derivation of $E \xrightarrow{\alpha} E'$ and on the structure of E.

We proceed to establish by induction on a derivation of $E[P/X] \xrightarrow{\alpha} P'$ that there exists an X-closed process expression E' such that $E \xrightarrow{\alpha} E'$ and $P' = E'[P/X]$, or X is exposed in E, $P \xrightarrow{\alpha} P'$ and a derivation of $P \xrightarrow{\alpha} P'$ appears as a subderivation of the considered derivation of $E[P/X] \xrightarrow{\alpha} P'$. This implies statement 2.

We distinguish cases according to the structure of E:

- Clearly, E cannot be $\mathbf{0}$, for if $E = \mathbf{0}$, then $E[P/X] = \mathbf{0}$, and $\mathbf{0}$ does not admit any transitions.
- If $E = X$, then X is exposed in E and $P = E[P/X] \xrightarrow{\alpha} P'$. It is then also immediate that the considered derivation of $E[P/X] \xrightarrow{\alpha} P'$ has a derivation of $P \xrightarrow{\alpha} P'$ as a subderivation.
- If $E = \beta.E'$ for some $\beta \in \mathcal{A}_\tau$ and some X-closed process expression E', then $\beta = \alpha$ and $E \xrightarrow{\beta} E'$. Since $E[P/X] = \beta.(E'[P/X])$, rule 1 is the last rule applied in the derivation of the transition $E[P/X] \xrightarrow{\alpha} P'$, so $P' = E'[P/X]$.
- If $E = \mu Y.F$ for some process expression F with $FV(F) \subseteq \{X, Y\}$, then there are two subcases:

  On the one hand, if $Y = X$, then, since X has no free occurrence in E, we have $E = E[P/X] \xrightarrow{\alpha} P'$. We take $E' = P'$, and since E' is closed we have $E'[P/X] = E' = P'$.

  On the other hand, if $Y \neq X$, then $E[P/X] = \mu Y.(F[P/X])$, and therefore the last rule applied in the considered derivation of the transition $E[P/X] \xrightarrow{\alpha} P'$ is rule 2. Consequently, the considered derivation has a proper subderivation of the transition $F[P/X][\mu Y.(F[P/X])/Y] \xrightarrow{\alpha} P'$. Note that $F[P/X][\mu Y.(F[P/X])/Y] = (F[\mu Y.F/Y])[P/X]$. Hence, by the induction hypothesis, either there exists an E' such that $F[\mu Y.F/Y] \xrightarrow{\alpha} E'$ and $P' = E'[P/X]$, or X is exposed in $F[\mu Y.F/Y]$, $P \xrightarrow{\alpha} P'$, and the derivation of $F[\mu Y.F/Y][P/X] \xrightarrow{\alpha} P'$ has a derivation of $P \xrightarrow{\alpha} P'$ as a subderivation. In the first case, it follows from $F[\mu Y.F/Y] \xrightarrow{\alpha} E'$, by rule 2, that $E = \mu Y.F \xrightarrow{\alpha} E'$ and $P' = E'[P/X]$. In the second case, it suffices to note that X is exposed in F, hence also in E, and that a derivation of $P \xrightarrow{\alpha} P'$ appears as a subderivation of the considered derivation of $E[P/X] \xrightarrow{\alpha} P'$.
- If $E = E_1 + E_2$, then $E[P/X] = E_1[P/X] + E_2[P/X]$.

  The last rule applied in the considered derivation of the transition $E[P/X] \xrightarrow{\alpha} P'$ is either rule 3 or rule 4.

  If it is rule 3, then $E_1[P/X] \xrightarrow{\alpha} P'$, and since this transition has a derivation that is a proper subderivation of the considered derivation of $E[P/X] \xrightarrow{\alpha} P'$, by the induction hypothesis it follows that either $E_1 \xrightarrow{\alpha} E'$ and $P' = E'[P/X]$, or X is exposed in E_1, $P \xrightarrow{\alpha} P'$, and a derivation of $P \xrightarrow{\alpha} P'$ appears as a subderivation the derivation of $E_1[P/X] \xrightarrow{\alpha} P'$.

  In the first case, it remains to note that then also $E \xrightarrow{\alpha} E'$, and in the second case, it remains to note that X is also exposed in E.

  If the last rule applied in the considered derivation is rule 4, then the proof is analogous.

□

Corollary 3.3. *Let E be an X-closed process expression. If $E[\mu X.E/X] \xrightarrow{\alpha} P'$ for some (closed) process expression P', then there exists an X-closed process expression E' such that $E \xrightarrow{\alpha} E'$ and $P' = E'[\mu X.E/X]$.*

Proof. Consider a derivation of $E[\mu X.E/X] \xrightarrow{\alpha} P'$ that is minimal in the sense that it does not have a derivation of $E[\mu X.E/X] \xrightarrow{\alpha} P'$ as proper subderivation. Let $P = \mu X.E$. Since every derivation of $P \xrightarrow{\alpha} P'$ has a derivation of $E[P/X] \xrightarrow{a} P'$ as a proper subderivation (see the operational rules, and rule 2 in particular), it follows that the considered derivation of $E[\mu X.E/X] \xrightarrow{\alpha} P'$ does not have a subderivation of $P \xrightarrow{\alpha} P'$. Hence, by Lemma 3.2.2 there exists an X-closed process expression E' such that $E \xrightarrow{\alpha} E'$ and $P' = E'[\mu X.E/X]$. □

Corollary 3.4. *Let G_0 and E be X-closed process expressions. If there is an infinite sequence of closed process expressions $(P_k)_{k\in\omega}$ such that $G_0[\mu X.E/X] = P_0$ and $P_k \xrightarrow{\tau} P_{k+1}$ for all $k \in \omega$, then there is an infinite sequence of X-closed process expressions $(G_k)_{k\in\omega}$ such that $P_k = G_k[\mu X.E/X]$ and, for all $k \in \omega$, either $G_k \xrightarrow{\tau} G_{k+1}$ or X is exposed in G_k and $E \xrightarrow{\tau} G_{k+1}$.*

Proof. We construct $(G_k)_{k\in\omega}$ by induction on k. Suppose that G_k with $G_k[\mu X.E/X] = P_k$ has already been constructed. Since $P_k \xrightarrow{\tau} P_{k+1}$, by Lemma 3.2.2 there are two cases: either there is a G_{k+1} with $G_k \xrightarrow{\tau} G_{k+1}$ and $P_{k+1} = G_{k+1}[\mu X.E/X]$, in which case we are done, or X is exposed in G_k and $\mu X.E \xrightarrow{\tau} P_{k+1}$. In the latter case $E[\mu X.E/X] \xrightarrow{\tau} P_{k+1}$ (see the operational rules, and rule 2 in particular). By Corollary 3.3 there exists an X-closed process expression G_{k+1} such that $E \xrightarrow{\tau} G_{k+1}$ and $P_{k+1} = G_{k+1}[\mu X.E/X]$. □

Let E and E' be process expressions. We write $E \longrightarrow E'$ if there exists an $\alpha \in \mathcal{A}_\tau$ such that $E \xrightarrow{\alpha} E'$, and denote by $\longrightarrow^*$ the reflexive-transitive closure of $\longrightarrow$. If $E \longrightarrow^* E'$, then we say that E' is reachable from E.

Proposition 3.5 ([Gla93a, Proposition 1])**.** *If E is a process expression, then the set of all expressions reachable from E is finite.*

We now characterise the relation $\leftrightarrow^{\Delta}_{rb}$ on $\mathcal{E}$ from Definition 2.7 in the same style as Definition 2.1, but on an enriched transition system. To this end, we first define on $\mathcal{E}$ a $\mathcal{V} \uplus \mathcal{A}_\tau$-labelled transition relation $\longrightarrow \subseteq \mathcal{E} \times (\mathcal{V} \uplus \mathcal{A}_\tau) \times \mathcal{E}$ as the least ternary relation satisfying, besides the four rules of Section 2, also the rule

$$5\ \frac{}{X \xrightarrow{X} \mathbf{0}}$$

for each $X \in \mathcal{V}$. Intuitively, the $\mathcal{V} \uplus \mathcal{A}_\tau$-labelled transition relation treats a process expression E as the closed term obtained from E by replacing all free occurrences of the variable X by the closed process expression $X.\mathbf{0}$ in which X is interpreted as an action instead of as a recursion variable. Note that a variable X is exposed in an expression E according to Definition 3.1 iff $\exists F.\ E \xrightarrow{X} F$, which is the case iff $E \xrightarrow{X} \mathbf{0}$. Now let $\leftrightarrow^{\Delta}_{bX}$ and $\leftrightarrow^{\Delta}_{rbX}$ be defined exactly like $\leftrightarrow^{\Delta}_{b}$ and $\leftrightarrow^{\Delta}_{rb}$, but using the $\mathcal{V} \uplus \mathcal{A}_\tau$-labelled transition relation instead of the $\mathcal{A}_\tau$-labelled one, and applying all definitions directly to expressions with free variables, instead of applying the lifting of Definition 2.7. We proceed to show that on X-closed process expressions $\leftrightarrow^{\Delta}_{bX}$ coincides with $\leftrightarrow^{\Delta}_{b}$, and $\leftrightarrow^{\Delta}_{rbX}$ with $\leftrightarrow^{\Delta}_{rb}$. This characterisation, for weak and branching bisimilarity without preservation of divergence, stems from [Mil89] and [Gla93a]. Here we use it solely to obtain Corollaries 3.8 and 3.9.

Lemma 3.6. *The relation*

$$\mathcal{B} = \{(E[P/X], F[P/X]) \mid E, F \textit{ are } X\textit{-closed},\ E \leftrightarrow^{\Delta}_{bX} F,\ P \textit{ is closed}\}$$

is a branching bisimulation satisfying (D″) of Proposition 2.3.

Proof. It is immediate from its definition that $\mathcal{B}$ is symmetric.

We show it satisfies (T). Suppose E, F are X-closed, $E \leftrightarrow^{\Delta}_{bX} F$ and P closed. Let $E[P/X] \xrightarrow{\alpha} P'$ for some $\alpha \in \mathcal{A}_\tau$. By Lemma 3.2.2 either there exists an X-closed process expression E' such that $E \xrightarrow{\alpha} E'$ and $P' = E'[P/X]$, or X is exposed in E and $P \xrightarrow{\alpha} P'$. In the first case, since $E \leftrightarrow^{\Delta}_{bX} F$, there exist process expressions F' and F'' such that $F \twoheadrightarrow F'' \xrightarrow{(\alpha)} F'$, $E \leftrightarrow^{\Delta}_{bX} F''$ and $E' \leftrightarrow^{\Delta}_{bX} F'$. By Lemma 3.2.1 $F[P/X] \twoheadrightarrow F''[P/X] \xrightarrow{(\alpha)} F'[P/X]$. Furthermore, $E[P/X]\ \mathcal{B}\ F''[P/X]$ and $P' = E'[P/X]\ \mathcal{B}\ F'[P/X]$. In the second case, since X

is exposed in E, we have that $E \xrightarrow{X} \mathbf{0}$ and hence, since $E \leftrightarrow^{\Delta}_{bX} F$, there exist process expressions F' and F'' such that $F \twoheadrightarrow F'' \xrightarrow{X} F'$, $E \leftrightarrow^{\Delta}_{bX} F''$ and $\mathbf{0} \leftrightarrow^{\Delta}_{bX} F'$. Moreover, since $F'' \xrightarrow{X} F'$, X is exposed in F'', so by Lemma 3.2.1 $F[P/X] \twoheadrightarrow F''[P/X] \xrightarrow{\alpha} P'$. Furthermore, $E[P/X] \mathrel{\mathcal{B}} F''[P/X]$ and $P' \mathrel{\mathcal{B}} P'$.

It remains to show that $\mathcal{B}$ satisfies (D$''$). Suppose E, F are X-closed, $E \leftrightarrow^{\Delta}_{bX} F$ and P is closed, and there is an infinite sequence of closed process expressions $(P_k)_{k\in\omega}$ such that $E[P/X] = P_0$ and $P_k \xrightarrow{\tau} P_{k+1}$. By Lemma 3.2.2 either there exists an infinite sequence of X-closed process expressions $(E_k)_{k\in\omega}$ such that $E_0 = E$, $E_k \xrightarrow{\tau} E_{k+1}$ and $P_{k+1} = E_{k+1}[P/X]$ for all $k \in \omega$, or there exists a finite sequence of X-closed process expressions $(E_i)_{i\leq k}$ for some $k \in \omega$ such that $E_0 = E$, $E_i \xrightarrow{\tau} E_{i+1}$ and $P_{i+1} = E_{i+1}[P/X]$ for all $i < k$, $E_k \xrightarrow{X} \mathbf{0}$ and $P \xrightarrow{\tau} P_{k+1}$. In the first case, since $E \leftrightarrow^{\Delta}_{bX} F$, using (D$''$), there exist a process expression F' such that $F \xrightarrow{\tau} F'$ and $E_k \leftrightarrow^{\Delta}_{bX} F'$ for some $k \in N$. By Lemma 3.2.1 $F[P/X] \xrightarrow{\tau} F'[P/X]$. Furthermore, $E_k[P/X] \mathrel{\mathcal{B}} F'[P/X]$. In the second case, since $E \leftrightarrow^{\Delta}_{bX} F$, by induction on i there exists a sequence $F_0, \ldots, F_m, F_{m+1}$ and a mapping $\rho : \{0, \ldots, m\} \to \{0, \ldots, k\}$ with $\rho(m) = k$ such that $F = F_0 \xrightarrow{\tau} \cdots \xrightarrow{\tau} F_m \xrightarrow{X} F_{m+1}$ and $E_{\rho(j)} \leftrightarrow^{\Delta}_{b} F_j$ for all $j = 0, \ldots, m$. If $m = 0$, then X is exposed in F, so by Lemma 3.2.1 $F[P/X] \xrightarrow{\tau} P_{k+1}$. Furthermore, $P_{k+1} \mathrel{\mathcal{B}} P_{k+1}$. If $m > 0$, then let $F' = F_1$. By Lemma 3.2.1 $F[P/X] \xrightarrow{\tau} F'[P/X]$. Furthermore, $E_{\rho(1)}[P/X] \mathrel{\mathcal{B}} F'[P/X]$. □

For every $\alpha \in \mathcal{A}_\tau$ and $n \in \omega$, we define the closed process expression α^n inductively by $\alpha^0 = \mathbf{0}$ and $\alpha^{n+1} = \alpha.\alpha^n$. Note that, if $\alpha \neq \tau$, then $\alpha^i \leftrightarrow^{\Delta}_{b} \alpha^j$ implies $i = j$. Recall that we have assumed that $\mathcal{A}$ is non-empty; we now fix, for the remainder of this section, a particular action $a \in \mathcal{A}$.

Proposition 3.7. *Let E and F be X-closed process expressions. Then $E \leftrightarrow^{\Delta}_{bX} F$ iff $E \leftrightarrow^{\Delta}_{b} F$, and $E \leftrightarrow^{\Delta}_{rbX} F$ iff $E \leftrightarrow^{\Delta}_{rb} F$.*

Proof. We need to show that $E \leftrightarrow^{\Delta}_{bX} F$ iff $E[P/X] \leftrightarrow^{\Delta}_{b} F[P/X]$ for each closed process expression P, and likewise $E \leftrightarrow^{\Delta}_{rbX} F$ iff $E[P/X] \leftrightarrow^{\Delta}_{rb} F[P/X]$ for each closed process expression P.

"Only if": Lemma 3.6 immediately yields that $E \leftrightarrow^{\Delta}_{bX} F$ implies $E[P/X] \leftrightarrow^{\Delta}_{b} F[P/X]$ for each closed process expression P. Now let $E \leftrightarrow^{\Delta}_{rbX} F$ and $E[P/X] \xrightarrow{\alpha} P'$. By Lemma 3.2.2 either there exists an X-closed process expression E' such that $E \xrightarrow{\alpha} E'$ and $P' = E'[P/X]$, or X is exposed in E and $P \xrightarrow{\alpha} P'$. In the first case, since $E \leftrightarrow^{\Delta}_{rbX} F$, there exists a process expression F' such that $F \xrightarrow{\alpha} F'$ and $E' \leftrightarrow^{\Delta}_{bX} F'$. By Lemma 3.2.1 $F[P/X] \xrightarrow{\alpha} F'[P/X]$. Furthermore, by Lemma 3.6 $P' = E'[P/X] \leftrightarrow^{\Delta}_{b} F'[P/X]$. In the second case, since X is exposed in E we have that $E \xrightarrow{X} \mathbf{0}$, and hence, since $E \leftrightarrow^{\Delta}_{rbX} F$, there exists a process expression F' such that $F \xrightarrow{X} F'$. By Lemma 3.2.1 $F[P/X] \xrightarrow{\alpha} P'$. Furthermore, $P' \leftrightarrow^{\Delta}_{b} P'$. The other clause follows by symmetry, thus yielding $E[P/X] \leftrightarrow^{\Delta}_{rb} F[P/X]$.

"If": Let E and F be X-closed process expressions. Since by Proposition 3.5 the set of all process expressions reachable from E and F is finite, there exists a natural number $n \in \omega$ such that for all G reachable from E or F it holds that $G \not\leftrightarrow^{\Delta}_{b} a^n$, and thus $G[a^{n+1}/X] \not\leftrightarrow^{\Delta}_{b} a^n$. Let

$$\mathcal{R} = \{(E', F') \mid E \longrightarrow^{*} E',\ F \longrightarrow^{*} F',\ E'[a^{n+1}/X] \leftrightarrow^{\Delta}_{b} F'[a^{n+1}/X]\}.$$

Claim: The symmetric closure of $\mathcal{R}$ is a branching bisimulation satisfying (D$''$) w.r.t. the $\mathcal{V} \uplus \mathcal{A}_\tau$-labelled transition relation.

Proof of the claim: To prove that $\mathcal{R}$ satisfies condition (T) of Definition 2.1, let E' and F' be such that $E' \mathrel{\mathcal{R}} F'$, and suppose that $E' \xrightarrow{\alpha} E''$. Then $E'[a^{n+1}/X] \leftrightarrow_b^\Delta F'[a^{n+1}/X]$ and, using Lemma 3.2.1, $E'[a^{n+1}/X] \xrightarrow{\alpha} E''[a^{n+1}/X]$. Since $E'[a^{n+1}/X] \leftrightarrow_b^\Delta F'[a^{n+1}/X]$ there exist closed process expressions Q''' and Q'' such that $F'[a^{n+1}/X] \twoheadrightarrow Q'' \xrightarrow{(\alpha)} Q'''$, $E'[a^{n+1}/X] \leftrightarrow_b^\Delta Q''$ and $E''[a^{n+1}/X] \leftrightarrow_b^\Delta Q'''$. By Lemma 3.2.2, using that $a \neq \tau$, there exists a X-closed process expression F'' such that $F' \twoheadrightarrow F''$ and $Q'' = F''[a^{n+1}/X]$; moreover, either there exists an X-closed process expression F''' such that $F'' \xrightarrow{(\alpha)} F'''$ and $Q''' = F'''[a^{n+1}/X]$, or X is exposed in F'' and $a^{n+1} \xrightarrow{\alpha} Q'''$. In the latter case we would have $E''[a^{n+1}/X] \leftrightarrow_b^\Delta Q''' = a^n$, which is impossible by the choice of n. So the former case applies: we have $F' \twoheadrightarrow F'' \xrightarrow{(\alpha)} F'''$, $E' \mathrel{\mathcal{R}} F''$ and $E'' \mathrel{\mathcal{R}} F'''$. The case that $F' \xrightarrow{\alpha} F''$ proceeds by symmetry, so the symmetric closure of $\mathcal{R}$ satisfies condition (T).

To show that $\mathcal{R}$ (and its symmetric closure) satisfies (D$''$), let $(E_k)_{k\in\omega}$ be an infinite sequence of X-closed process expressions such that $E_k \xrightarrow{\tau} E_{k+1}$ for all $k \in \omega$, and let F_0 be such that $E_0 \mathrel{\mathcal{R}} F_0$. Then $E_0[a^{n+1}/X] \leftrightarrow_b^\Delta F_0[a^{n+1}/X]$ and by Lemma 3.2.1 $E_k[a^{n+1}/X] \xrightarrow{\tau} E_{k+1}[a^{n+1}/X]$ for all $k \in \omega$. Using (D$''$), there exist a process expression Q' such that $F_0 \xrightarrow{\tau} Q'$ and $E_k[a^{n+1}/X] \leftrightarrow_b^\Delta Q'$ for some $k \in N$. By Lemma 3.2.2, using that $a \neq \tau$, there exists a X-closed process expression F' such that $F_0 \xrightarrow{\tau} F'$ and $Q' = F'[a^{n+1}/X]$. Furthermore, $E_k \mathrel{\mathcal{R}} F'$.

Application of the claim: Let $E[P/X] \leftrightarrow_b^\Delta F[P/X]$ for each closed process expression P. Then $E[a^{n+1}/X] \leftrightarrow_b^\Delta F[a^{n+1}/X]$. The claim yields $E \leftrightarrow_{bX}^\Delta F$.

Now let $E[P/X] \leftrightarrow_{rb}^\Delta F[P/X]$ for each closed P. Then $E[a^{n+1}/X] \leftrightarrow_{rb}^\Delta F[a^{n+1}/X]$. Suppose that $E \xrightarrow{\alpha} E'$ with $\alpha \in \mathcal{A}_\tau$. Then $E[a^{n+1}/X] \xrightarrow{\alpha} E'[a^{n+1}/X]$ by Lemma 3.2.1. So there exists a Q' with $F[a^{n+1}/X] \xrightarrow{\alpha} Q'$ and $E'[a^{n+1}/X] \leftrightarrow_b^\Delta Q'$. By Lemma 3.2.2 either there exists an X-closed process expression F' such that $F \xrightarrow{\alpha} F'$ and $Q' = F'[a^{n+1}/X]$, or X is exposed in F and $a^{n+1} \xrightarrow{\alpha} Q'$. In the latter case we would have $E'[a^{n+1}/X] \leftrightarrow_b^\Delta Q' = a^n$, which is impossible by the choice of n. So the former case applies, and $E' \mathrel{\mathcal{R}} F'$. The claim yields $E' \leftrightarrow_{bX}^\Delta F'$. The other clause follows by symmetry, so $E \leftrightarrow_{rbX}^\Delta F$. □

The following is an immediate corollary of Propositions 3.7, 2.3 and 2.4.

Corollary 3.8. *Let E and F be X-closed process expressions such that $E \leftrightarrow_b^\Delta F$.*

(1) *If $E \xrightarrow{\alpha} E'$, then there exist X-closed process expressions $F_0, \ldots, F_n$ and F' such that $F = F_0 \xrightarrow{\tau} \cdots \xrightarrow{\tau} F_n \xrightarrow{(\alpha)} F'$ such that $E \leftrightarrow_b^\Delta F_i$ $(0 \le i \le n)$ and $E' \leftrightarrow_b^\Delta F'$.*
(2) *If X is exposed in E, then there exist $k \ge 0$ and X-closed process expressions $F_0, \ldots, F_k$ such that $F = F_0 \xrightarrow{\tau} \cdots \xrightarrow{\tau} F_k$, $E \leftrightarrow_b^\Delta F_i$ $(0 \le i \le k)$, and X is exposed in F_k.*
(3) *If there is an infinite sequence of X-closed process expressions $(E_k)_{k\in\omega}$ such that $E = E_0$ and $E_k \xrightarrow{\tau} E_{k+1}$, then there exists an X-closed process expression F' such that $F \xrightarrow{\tau} F'$ and $E_k \leftrightarrow_b^\Delta F'$ for some $k \in \omega$.*

Similarly, by combining Propositions 3.7 and Definition 2.5 we get the following corollary.

Corollary 3.9. *Let E and F be X-closed process expressions such that $E \leftrightarrow_{rb}^\Delta F$. If $E \xrightarrow{\alpha} E'$, then there exists an X-closed process expression F' such that $F \xrightarrow{\alpha} F'$ and $E' \leftrightarrow_b^\Delta F'$.*

3.2. **Rooted divergence-preserving branching bisimulation up to $\leftrightarrow_b^\Delta$.** As was already illustrated by Milner [Mil90], a suitable up-to relation is a crucial tool in the proof that a behavioural equivalence is compatible with the recursion construct. In [Gla93a], Milner's notion of weak bisimulation up to weak bisimilarity is adapted to branching bisimulation up to branching bisimilarity. Here we make two further modifications. Not only do we add a divergence condition; we also incorporate rootedness into the relation.

Definition 3.10. Let $\mathcal{R}$ be a symmetric binary relation on $\mathcal{P}$, and denote by $\mathcal{R}^u$ the relation $\leftrightarrow_b^\Delta ; \mathcal{R} ; \leftrightarrow_b^\Delta$. We say that $\mathcal{R}$ is a *rooted divergence-preserving branching bisimulation up to* $\leftrightarrow_b^\Delta$ if for all $P, Q \in \mathcal{P}$ such that $P \mathrel{\mathcal{R}} Q$ the following three conditions are satisfied:

(U1) if $P \xrightarrow{\alpha} P'$, then there exists Q' such that $Q \xrightarrow{\alpha} Q'$ and $P' \mathrel{\mathcal{R}^u} Q'$.

(U2) if $P \twoheadrightarrow P'' \xrightarrow{(\alpha)} P'$, then there exist Q' and Q'' such that $Q \twoheadrightarrow Q'' \xrightarrow{(\alpha)} Q'$, $P'' \mathrel{\mathcal{R}^u} Q''$ and $P' \mathrel{\mathcal{R}^u} Q'$.

(U3) if there exists an infinite sequence of closed process expressions $(P_k)_{k\in\omega}$ such that $P = P_0$, and $P_k \xrightarrow{\tau} P_{k+1}$ for all $k \in \omega$, then there also exists an infinite sequence of closed process expressions $(Q_\ell)_{\ell\in\omega}$ and a mapping $\sigma : \omega \to \omega$ such that $Q = Q_0$, and $Q_\ell \xrightarrow{\tau} Q_{\ell+1}$ and $P_{\sigma(\ell)} \mathrel{\mathcal{R}^u} Q_\ell$ for all $\ell \in \omega$.

Proposition 3.11. *Let P and Q be closed process expressions and let $\mathcal{R}$ be a rooted divergence-preserving branching bisimulation up to $\leftrightarrow_b^\Delta$. If $P \mathrel{\mathcal{R}} Q$, then $P \leftrightarrow_{rb}^\Delta Q$.*

Proof. If $P \mathrel{\mathcal{R}} Q$ and $P \xrightarrow{\alpha} P'$, then since $\mathcal{R}$ satisfies condition (U1) of Definition 3.10, there exists a Q' such that $Q \xrightarrow{\alpha} Q'$ and $P' \mathrel{\mathcal{R}^u} Q'$. Furthermore, since $\mathcal{R}$ is symmetric, whenever $P \mathrel{\mathcal{R}} Q$ also $Q \mathrel{\mathcal{R}} P$, so if $Q \xrightarrow{\alpha} Q'$, then by condition (U1) of Definition 3.10 there exists a P' such that $P \xrightarrow{\alpha} P'$ and $Q' \mathrel{\mathcal{R}^u} P'$. It remains to establish that $P' \leftrightarrow_b^\Delta Q'$, and for this, it suffices by Proposition 2.3 to prove that $\mathcal{R}^u$ is a branching bisimulation satisfying (D$'$).

Note that, since $\leftrightarrow_b^\Delta$ and $\mathcal{R}$ are both symmetric, also $\mathcal{R}^u$ is symmetric.

To prove that $\mathcal{R}^u$ satisfies (T), let P_0, P_1, Q_0 and Q_1 be closed process expressions such that $P_1 \leftrightarrow_b^\Delta P_0 \mathrel{\mathcal{R}} Q_0 \leftrightarrow_b^\Delta Q_1$, and suppose that $P_1 \xrightarrow{\alpha} P_1'$. Since $P_1 \leftrightarrow_b^\Delta P_0$ and $\leftrightarrow_b^\Delta$ satisfies (T), there exist P_0' and P_0'' such that $P_0 \twoheadrightarrow P_0'' \xrightarrow{(\alpha)} P_0'$, $P_1 \leftrightarrow_b^\Delta P_0''$ and $P_1' \leftrightarrow_b^\Delta P_0'$. So it follows by condition (U2) of Definition 3.10 that there exist Q_0' and Q_0'' such that $Q_0 \twoheadrightarrow Q_0'' \xrightarrow{(\alpha)} Q_0'$, $P_0'' \mathrel{\mathcal{R}^u} Q_0''$ and $P_0' \mathrel{\mathcal{R}^u} Q_0'$. Hence, since $Q_0 \leftrightarrow_b^\Delta Q_1$, by Lemma 2.2 there exist closed process expressions Q_1' and Q_1'' such that $Q_1 \twoheadrightarrow Q_1'' \xrightarrow{(\alpha)} Q_1'$, $Q_0'' \leftrightarrow_b^\Delta Q_1''$ and $Q_0' \leftrightarrow_b^\Delta Q_1'$. Note, moreover, that $P_1 \leftrightarrow_b^\Delta P_0'' \mathrel{\mathcal{R}^u} Q_0'' \leftrightarrow_b^\Delta Q_1''$ whence $P_1 \mathrel{\mathcal{R}^u} Q_1''$, and $P_1' \leftrightarrow_b^\Delta P_0' \mathrel{\mathcal{R}^u} Q_0' \leftrightarrow_b^\Delta Q_1'$ whence $P_1' \mathrel{\mathcal{R}^u} Q_1'$.

It remains to prove that $\mathcal{R}^u$ satisfies (D$'$) of Proposition 2.3. To this end, let P_0, P_1, Q_0 and Q_1 be closed process expressions such that $P_1 \leftrightarrow_b^\Delta P_0 \mathrel{\mathcal{R}} Q_0 \leftrightarrow_b^\Delta Q_1$, and suppose that there exists an infinite sequence of closed process expressions $(P_{1,k})_{k\in\omega}$ such that $P_1 = P_{1,0}$ and $P_{1,k} \xrightarrow{\tau} P_{1,k+1}$. Then, since $P_1 \leftrightarrow_b^\Delta P_0$, by Proposition 2.3, there exists an infinite sequence of closed process expressions $(P_{0,\ell})_{\ell\in\omega}$ and a mapping $\sigma_P : \omega \to \omega$ such that $P_0 = P_{0,0}$, $P_{0,\ell} \xrightarrow{\tau} P_{0,\ell+1}$ and $P_{1,\sigma_P(\ell)} \leftrightarrow_b^\Delta P_{0,\ell}$ for all $\ell \in \omega$. Hence, since $P_0 \mathrel{\mathcal{R}} Q_0$ and $\mathcal{R}$ is a divergence-preserving branching bisimulation up to $\leftrightarrow_b^\Delta$, there exists an infinite sequence of closed process expressions $(Q_{0,m})_{m\in\omega}$ and a mapping $\sigma_{P,Q} : \omega \to \omega$ such that $Q_0 = Q_{0,0}$, $Q_{0,m} \xrightarrow{\tau} Q_{0,m+1}$ and $P_{0,\sigma_{P,Q}(m)} \mathrel{\mathcal{R}^u} Q_{0,m}$ for all $m \in \omega$. Hence, since $Q_0 \leftrightarrow_b^\Delta Q_1$, by Proposition 2.3, there exists an infinite sequence of closed process expressions $(Q_{1,n})_{n\in\omega}$ and a mapping $\sigma_Q : \omega \to \omega$ such that $Q_1 = Q_{1,0}$, $Q_{1,n} \xrightarrow{\tau} Q_{0,n+1}$ and $Q_{0,\sigma_Q(n)} \leftrightarrow_b^\Delta Q_{1,n}$ for all $n \in \omega$. We define

$$\sigma = \sigma_P \circ \sigma_{P,Q} \circ \sigma_Q \ ,$$

and then we have that $P_{1,\sigma(n)} \leftrightarrow_b^\Delta \ ; \mathcal{R}^u \ ; \leftrightarrow_b^\Delta Q_{1,n}$, and hence $P_{1,\sigma(n)} \ \mathcal{R}^u \ Q_{1,n}$ for all $n \in \omega$. □

To prove that $\leftrightarrow_{rb}^\Delta$ is compatible with $\mu X.\_$ means to prove that if $E \leftrightarrow_{rb}^\Delta F$, then $\mu X.E \leftrightarrow_{rb}^\Delta \mu X.F$. We first do this in the special case that E and F are X-closed. A crucial step in this proof will be to show that if $E \leftrightarrow_{rb}^\Delta F$ for X-closed process expressions E and F, then the symmetric closure $\mathcal{R}_{E,F}$ of the relation

$$\{(G[\mu X.E/X], G[\mu X.F/X]) \mid G \in \mathcal{E} \text{ and } G \text{ is } X\text{-closed}\} \tag{3.1}$$

is a rooted branching bisimulation up to $\leftrightarrow_b^\Delta$. The result then follows by taking $G := X$. Until Corollary 3.18 we fix X-closed process expressions E and F such that $E \leftrightarrow_{rb}^\Delta F$.

For this application of the up-to technique from Definition 3.10, $\mathcal{R}^u$ could equally well have been defined as $\mathcal{R} \, ; \leftrightarrow_b^\Delta$. This less powerful technique is still valid by Proposition 3.11, yet is all we need in Lemmas 3.12–3.16.

Lemma 3.12. *For all X-closed process expressions G, if $G[\mu X.E/X] \xrightarrow{\alpha} P$, then there exists a Q such that $G[\mu X.F/X] \xrightarrow{\alpha} Q$ and $P \, \mathcal{R}_{E,F} \, ; \leftrightarrow_b^\Delta Q$.*

Proof. Let G be an X-closed process expression, and suppose that $G[\mu X.E/X] \xrightarrow{\alpha} P$; we proceed by induction on a derivation of this transition. By Lemma 3.2.2 there are two cases: either the transition under consideration stems directly from G, i.e., there exists a G' such that $G \xrightarrow{\alpha} G'$ and $P = G'[\mu X.E/X]$, or X is exposed in G, $\mu X.E \xrightarrow{\alpha} P$ and every derivation of $G[\mu X.E/X] \xrightarrow{\alpha} P$ has a derivation of $\mu X.E \xrightarrow{\alpha} P$ as a subderivation.

In the first case, we have $G[\mu X.F/X] \xrightarrow{\alpha} G'[\mu X.F/X]$ and $P = G'[\mu X.E/X] \ \mathcal{R}_{E,F} \ G'[\mu X.F/X]$ by Lemma 3.2.1, so, since $\leftrightarrow_b^\Delta$ is reflexive, also $P \, \mathcal{R}_{E,F} \, ; \leftrightarrow_b^\Delta G'[\mu X.F/X]$.

In the second case, since the considered derivation of the transition $G[\mu X.E/X] \xrightarrow{\alpha} P$ has a derivation of $\mu X.E \xrightarrow{\alpha} P$ as a subderivation, and the last rule applied in this subderivation must be rule 2, it follows that the considered derivation of $G[\mu X.E/X] \xrightarrow{\alpha} P$ has a derivation of $E[\mu X.E/X] \xrightarrow{\alpha} P$ as a proper subderivation. So by the induction hypothesis there exists a Q such that $E[\mu X.F/X] \xrightarrow{\alpha} Q$ and $P \, \mathcal{R}_{E,F} \, ; \leftrightarrow_b^\Delta Q$. Furthermore, since $E \leftrightarrow_{rb}^\Delta F$, whence $E[\mu X.F/X] \leftrightarrow_{rb}^\Delta F[\mu X.F/X]$, it follows that there exists an R such that $F[\mu X.F/X] \xrightarrow{\alpha} R$ and $Q \leftrightarrow_b^\Delta R$. It follows from $F[\mu X.F/X] \xrightarrow{\alpha} R$ that $\mu X.F \xrightarrow{\alpha} R$. Since X is exposed in G, Lemma 3.2.1 yields $G[\mu X.F/X] \xrightarrow{\alpha} R$. From $P \, \mathcal{R}_{E,F} \, ; \leftrightarrow_b^\Delta Q$ and $Q \leftrightarrow_b^\Delta R$ it follows that $P \, \mathcal{R}_{E,F} \, ; \leftrightarrow_b^\Delta R$. □

As an immediate corollary to Lemma 3.12 we get that if $E \leftrightarrow_{rb}^\Delta F$, then $\mathcal{R}_{E,F}$ satisfies the first condition of rooted divergence-preserving branching bisimulations up to $\leftrightarrow_b^\Delta$.

Corollary 3.13. *$\mathcal{R}_{E,F}$ satisfies condition (U1) of Definition 3.10.*

With a little more work, Lemma 3.12 can also be used to derive that $\mathcal{R}_{E,F}$ satisfies the second condition of rooted divergence-preserving branching bisimulations up to $\leftrightarrow_b^\Delta$. To this end, we first prove the following lemma.

Lemma 3.14. *Let P and Q be closed process expressions. If $P \, \mathcal{R}_{E,F} \, ; \leftrightarrow_b^\Delta Q$ and $P \xrightarrow{\alpha} P'$, then there exist Q' and Q'' such that $Q \twoheadrightarrow Q'' \xrightarrow{(\alpha)} Q'$, $P \, \mathcal{R}_{E,F} \, ; \leftrightarrow_b^\Delta Q''$ and $P' \ \mathcal{R}_{E,F} \, ; \leftrightarrow_b^\Delta Q'$.*

Proof. Suppose that $P\,\mathcal{R}_{E,F}\,;\,\leftrightarrow_b^\Delta\,Q$ and $P \xrightarrow{\alpha} P'$. Then there exists an R such that $P\,\mathcal{R}_{E,F}\,R\leftrightarrow_b^\Delta Q$, and according to the definition of $\mathcal{R}_{E,F}$ there exists an X-closed process expression G such that either $P = G[\mu X.E/X]$ and $R = G[\mu X.F/X]$ or $P = G[\mu X.F/X]$ and $R = G[\mu X.E/X]$. Without loss of generality we assume that $P = G[\mu X.E/X]$ and $R = G[\mu X.F/X]$. By Lemma 3.12, there exists an R' such that $R \xrightarrow{\alpha} R'$ and $P'\,\mathcal{R}_{E,F}\,;\,\leftrightarrow_b^\Delta\,R'$. Hence, since $R \leftrightarrow_b^\Delta Q$, there exist Q' and Q'' such that $Q \twoheadrightarrow Q'' \xrightarrow{(\alpha)} Q'$, $R \leftrightarrow_b^\Delta Q''$ and $R' \leftrightarrow_b^\Delta Q'$. It follows that $P\,\mathcal{R}_{E,F}\,;\,\leftrightarrow_b^\Delta\,Q''$ and $P'\,\mathcal{R}_{E,F}\,;\,\leftrightarrow_b^\Delta\,Q'$, so the proof of the lemma is complete. □

Using that $P\,\mathcal{R}_{E,F}\,Q$ implies $P\,\mathcal{R}_{E,F}\,;\,\leftrightarrow_b^\Delta\,Q$ by reflexivity of $\leftrightarrow_b^\Delta$, and applying Lemma 3.14 by induction on the length of a transition sequence that gives rise to $P \twoheadrightarrow P'' \xrightarrow{\alpha} P'$, it is straightforward to establish the following corollary.

Corollary 3.15. *$\mathcal{R}_{E,F}$ satisfies condition (U2) of Definition 3.10.*

It remains to establish that $\mathcal{R}_{E,F}$ satisfies the third condition of rooted divergence-preserving branching bisimulations up to $\leftrightarrow_b^\Delta$.

Lemma 3.16. *Let G and H be X-closed process expressions such that $G \leftrightarrow_b^\Delta H$. If there exists an infinite sequence of closed process expressions $(P_k)_{k\in\omega}$ such that $G[\mu X.E/X] = P_0$ and $P_k \xrightarrow{\tau} P_{k+1}$ for all $k \in \omega$, then there also exists an infinite sequence of closed process expressions $(Q_\ell)_{\ell\in\omega}$ and a mapping $\sigma : \omega \to \omega$ such that $H[\mu X.F/X] = Q_0$, $Q_\ell \xrightarrow{\tau} Q_{\ell+1}$, and $P_{\sigma(\ell)}\,\mathcal{R}_{E,F}\,;\,\leftrightarrow_b^\Delta\,Q_\ell$ for all $\ell \in \omega$.*

Proof. Suppose that there exists an infinite sequence of closed process expressions $(P_k)_{k\in\omega}$ such that $G[\mu X.E/X] = P_0$ and $P_k \xrightarrow{\tau} P_{k+1}$ for all $k \in \omega$. By Corollary 3.4 there is an infinite sequence of X-closed process expressions $(G_k)_{k\in\omega}$ such that $P_k = G_k[\mu X.E/X]$ and either $G_k \xrightarrow{\tau} G_{k+1}$ or $E \xrightarrow{\tau} G_{k+1}$ for all $k \in \omega$. We shall define simultaneously, by induction on ℓ, an infinite sequence of X-closed process expressions $(H_\ell)_{\ell\in\omega}$ with $H_0 = H$ and $H_\ell[\mu X.F/X] \xrightarrow{\tau} H_{\ell+1}[\mu X.F/X]$, and a mapping $\sigma : \omega \to \omega$, such that $G_{\sigma(\ell)} \leftrightarrow_b^\Delta H_\ell$. This will suffice, because, for all $\ell \in \omega$, defining Q_ℓ as $H_\ell[\mu X.F/X]$ we obtain $Q_\ell \xrightarrow{\tau} Q_{\ell+1}$ and $P_{\sigma(\ell)} = G_{\sigma(\ell)}[\mu X.E/X]\,\mathcal{R}_{E,F}\,G_{\sigma(\ell)}[\mu X.F/X] \leftrightarrow_b^\Delta H_\ell[\mu X.F/X] = Q_\ell$.

Suppose, by way of induction hypothesis, that H_ℓ and $\sigma(\ell)$ have been defined already, such that $G_{\sigma(\ell)} \leftrightarrow_b^\Delta H_\ell$. By Corollary 3.4 there are two cases:

(1) $G_{\sigma(\ell)+k} \xrightarrow{\tau} G_{\sigma(\ell)+k+1}$ for all $k \in \omega$. Then, since $G_{\sigma(\ell)} \leftrightarrow_b^\Delta H_\ell$, by Corollary 3.8.3 there exists an X-closed process expression H' such that $H_\ell \xrightarrow{\tau} H'$ and $G_{\sigma(\ell)+k} \leftrightarrow_b^\Delta H'$ for some $k \in \omega$. We define $H_{\ell+1} = H'$ and $\sigma(\ell+1) = \sigma(\ell) + k$. Now $H_\ell[\mu X.F/X] \xrightarrow{\tau} H_{\ell+1}[\mu X.F/X]$ by Lemma 3.2.1 and $G_{\sigma(\ell+1)} \leftrightarrow_b^\Delta H_{\ell+1}$.

(2) There is a $k \in \omega$ such that $G_{\sigma(\ell)+i} \xrightarrow{\tau} G_{\sigma(\ell)+i+1}$ for all $i < k$, X is exposed in $G_{\sigma(\ell)+k}$ and $E \xrightarrow{\tau} G_{\sigma(\ell)+k+1}$. Then, since $G_{\sigma(\ell)} \leftrightarrow_b^\Delta H_\ell$, by Corollary 3.8.1 and by induction on i there exists a sequence $H'_0, \ldots, H'_m$ and a mapping $\rho : \{0, \ldots, m\} \to \{0, \ldots, k\}$ with $\rho(m) = k$ such that $H_\ell = H'_0 \xrightarrow{\tau} \cdots \xrightarrow{\tau} H'_m$ and $G_{\sigma(\ell)+\rho(j)} \leftrightarrow_b^\Delta H'_j$. Using Corollary 3.8.2, we may furthermore assume that X is exposed in H'_m.

If $m > 0$, then we define $H_{\ell+1} = H'_1$ and $\sigma(\ell+1) = \sigma(\ell)+\rho(1)$. Now $H_\ell[\mu X.F/X] \xrightarrow{\tau} H_{\ell+1}[\mu X.F/X]$ by Lemma 3.2.1 and $G_{\sigma(\ell+1)} \leftrightarrow_b^\Delta H_{\ell+1}$.

So it remains to consider the case that $m = 0$. Since $E \leftrightarrow_{rb}^\Delta F$, there exists, by Corollary 3.9, an X-closed process expression F' such that $F \xrightarrow{\tau} F'$ and $G_{\sigma(\ell)+k+1} \leftrightarrow_b^\Delta$

F'. We now define $H_{\ell+1} = F'$ and $\sigma(\ell+1) = \sigma(\ell)+k+1$. We then have that $G_{\sigma(\ell+1)} = G_{\sigma(\ell)+k+1} \leftrightarrow_b^\Delta H_{\ell+1}$, and $F[\mu X.F/X] \xrightarrow{\tau} H_{\ell+1}[\mu X.F/X]$ by Lemma 3.2.1. So $\mu X.F \xrightarrow{\tau} H_{\ell+1}[\mu X.F/X]$ by rule 2, and Lemma 3.2.1 yields $H_\ell[\mu X.F/X] \xrightarrow{\tau} H_{\ell+1}[\mu X.F/X]$, using that X is exposed in H_ℓ. □

From Lemma 3.16 with $G = H$ we immediately get the following corollary.

Corollary 3.17. *$\mathcal{R}_{E,F}$ satisfies condition (U3) of Definition 3.10.*

The relation $\mathcal{R}_{E,F}$ is symmetric by definition and we have now also proved that it satisfies conditions (U1), (U2) and (U3), so we have established the following result.

Corollary 3.18. *$\mathcal{R}_{E,F}$ is a rooted divergence-preserving branching bisimulation up to $\leftrightarrow_b^\Delta$.*

3.3. The main results.

We can now establish that $\leftrightarrow_{rb}^\Delta$ is compatible with $\alpha.$, $\mu X.\_$ and $+$.

Proposition 3.19. *If $E \leftrightarrow_{rb}^\Delta F$, then $\alpha.E \leftrightarrow_{rb}^\Delta \alpha.F$ for all $\alpha \in \mathcal{A}_\tau$, $E + H \leftrightarrow_{rb}^\Delta F + H$ and $H + E \leftrightarrow_{rb}^\Delta H + F$ for all process expressions H, and $\mu X.E \leftrightarrow_{rb}^\Delta \mu X.F$.*

Proof. To prove that $\leftrightarrow_{rb}^\Delta$ is compatible with $\alpha.$ and $+$ is straightforward. (First, establish the property for closed terms, and then use that substitution distributes over $\alpha.$ and $+$.)

It remains to prove that $\leftrightarrow_{rb}^\Delta$ is compatible with $\mu X.\_$, i.e., that $E \leftrightarrow_{rb}^\Delta F$ implies $\mu X.E \leftrightarrow_{rb}^\Delta \mu X.F$. Note that in the special case that E and F are X-closed this immediately follows from Corollary 3.18 and Proposition 3.11. Now, for the general case, let E and F be process expressions and suppose that $E \leftrightarrow_{rb}^\Delta F$. Let $X, \vec{Y}$ be a sequence of variables that at least includes the variables with a free occurrence in E or F, and such that X does not occur in $\vec{Y}$. Then, according to the definition of $\leftrightarrow_{rb}^\Delta$ on process expressions with free variables (Definition 2.7), we have that, for every closed process expression P and for every sequence of closed process expressions $\vec{P}$ of the same length as $\vec{Y}$, $E[P,\vec{P}/X,\vec{Y}] \leftrightarrow_{rb}^\Delta F[P,\vec{P}/X,\vec{Y}]$. So, clearly, also $E[\vec{P}/\vec{Y}] \leftrightarrow_{rb}^\Delta F[\vec{P}/\vec{Y}]$, and since $E[\vec{P}/\vec{Y}]$ and $F[\vec{P}/\vec{Y}]$ are X-closed, it follows that $\mu X.E[\vec{P}/\vec{Y}] \leftrightarrow_{rb}^\Delta \mu X.F[\vec{P}/\vec{Y}]$. Since X is not among the $\vec{Y}$, we may conclude that $(\mu X.E)[\vec{P}/\vec{Y}] \leftrightarrow_{rb}^\Delta (\mu X.F)[\vec{P}/\vec{Y}]$ for every sequence of closed process expressions $\vec{P}$ of the same length as $\vec{Y}$, and hence $\mu X.E \leftrightarrow_{rb}^\Delta \mu X.F$. □

We have now obtained our main result that $\leftrightarrow_{rb}^\Delta$ is a congruence. In fact, it is the coarsest contained in $\leftrightarrow_b^\Delta$.

Theorem 3.20. *The relation $\leftrightarrow_{rb}^\Delta$ is the coarsest congruence contained in $\leftrightarrow_b^\Delta$.*

Proof. By Propositions 2.6 and 3.19, the relation $\leftrightarrow_{rb}^\Delta$ is a congruence. To prove that it is the coarsest, it suffices to prove that for every relation $\mathcal{R} \subseteq \leftrightarrow_b^\Delta$ that is compatible with $+$ we have that $\mathcal{R} \subseteq \leftrightarrow_{rb}^\Delta$. Let P and Q be closed process expressions, and suppose that $P \mathrel{\mathcal{R}} Q$.

Since by Proposition 3.5 the set of closed process expressions reachable from P and Q is finite and $\mathcal{A}$ is non-empty, there exists a natural number $n \in \omega$ such that for all R reachable from P or Q it holds that $R \not\leftrightarrow_b^\Delta a^n$. This implies that for all R' reachable from P or Q it holds that $R' \not\leftrightarrow_b^\Delta P + a^{n+1}$ and $R' \not\leftrightarrow_b^\Delta Q + a^{n+1}$; for suppose that, e.g., there exists R' reachable from P or Q such that $R' \leftrightarrow_b^\Delta P + a^{n+1}$, then, since $P + a^{n+1} \xrightarrow{a} a^n$, we have that a^n is reachable from P or Q.

Since $\mathcal{R}$ is compatible with $+$, we have that $P+a^{n+1} \mathrel{\mathcal{R}} Q+a^{n+1}$, and hence $P+a^{n+1} \leftrightarroweq_b^\Delta Q+a^{n+1}$. To prove (R1), suppose that $P \xrightarrow{\alpha} P'$. Then $P+a^{n+1} \xrightarrow{\alpha} P'$, so by Lemma 2.2 there exist closed process expressions Q' and Q'' such that $Q+a^{n+1} \twoheadrightarrow Q'' \xrightarrow{(\alpha)} Q'$, $P+a^{n+1} \leftrightarroweq_b^\Delta Q''$ and $P' \leftrightarroweq_b^\Delta Q'$. Since $a \neq \tau$, we have that $Q'' = Q+a^{n+1}$, for otherwise Q'' is reachable from Q and $Q'' \leftrightarroweq_b^\Delta P+a^{n+1}$. Moreover, $Q'' \xrightarrow{\alpha} Q'$, for otherwise $P' \leftrightarroweq_b^\Delta Q' = Q'' = Q+a^{n+1}$. Condition (R2) follows by symmetry. □

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