

## TICKING CLOCKS AS DEPENDENT RIGHT ADJOINTS DENOTATIONAL SEMANTICS FOR CLOCKED TYPE THEORY

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**ABSTRACT.** Clocked Type Theory (CloTT) is a type theory for guarded recursion useful for programming with coinductive types, allowing productivity to be encoded in types, and for reasoning about advanced programming language features using an abstract form of step-indexing. CloTT has previously been shown to enjoy a number of syntactic properties including strong normalisation, canonicity and decidability of the equational theory. In this paper we present a denotational semantics for CloTT useful, e.g., for studying future extensions of CloTT with constructions such as path types.

The main challenge for constructing this model is to model the notion of ticks on a clock used in CloTT for coinductive reasoning about coinductive types. We build on a category previously used to model guarded recursion with multiple clocks. In this category there is an object of clocks but no object of ticks, and so tick-assumptions in a context can not be modelled using standard tools. Instead we model ticks using dependent right adjoint functors, a generalisation of the category theoretic notion of adjunction to the setting of categories with families. Dependent right adjoints are known to model Fitch-style modal types, but in the case of CloTT, the modal operators constitute a family indexed internally in the type theory by clocks. We model this family using a dependent right adjoint on the slice category over the object of clocks. Finally, we show how to model the tick constant of CloTT using a semantic substitution.

This work improves on a previous model by two of the authors which not only had a flaw but was also considerably more complicated.

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## INTRODUCTION

In recent years a number of extensions of Martin-Löf type theory (MLTT) [ML84] have been proposed to enhance the expressiveness or usability of the type theory. The most famous of these is Homotopy Type Theory [Uni13], but other directions include the related Cubical Type Theory [CCHM18], FreshMLTT [PMD15], a type theory with name abstraction based on nominal sets, and Type Theory in Color [BCM15] for internalising relational parametricity in type theory. Many of these extensions use denotational semantics to argue for consistency and to inspire constructions in the language.

This paper is part of a project to extend type theory with guarded recursion [Nak00], a variant of recursion that uses a modal type operator  $\triangleright$  (pronounced ‘later’) to preserve consistency of the logical reading of type theory. The type  $\triangleright A$  should be read as classifying data of type  $A$  available one time step from now, and comes with a map  $\text{next} : A \rightarrow \triangleright A$  and a fixed point operator mapping a function  $f : \triangleright A \rightarrow A$  to a fixed point for  $f \circ \text{next}$ . This, in combination with *guarded recursive types*, i.e., types where the recursion variable is guarded by a  $\triangleright$ , e.g.,  $\text{Str}^{\sharp} \equiv \mathbb{N} \times \triangleright \text{Str}^{\sharp}$  gives a powerful type theory in which operational models of combinations of advanced programming language features such as higher-order store [BMSS12] and nondeterminism [BBM14] can be modelled using an abstract form of step-indexing [AM01]. Combining guarded recursion with a notion of clocks, indexing the  $\triangleright$  operator with clock names, and universal quantification over clocks, one can encode coinduction using guarded recursion, allowing productivity [Coq93] of coinductive definitions to be encoded in types [AM13]. For example, if  $\text{Str}^{\kappa}$  is a type of streams guarded on the clock  $\kappa$ , i.e., satisfying the equation  $\text{Str}^{\kappa} \equiv \mathbb{N} \times \triangleright^{\kappa} \text{Str}^{\kappa}$ , then the type  $\text{Str} \stackrel{\text{def}}{=} \forall \kappa. \text{Str}^{\kappa}$  obtained by universally quantifying the clock  $\kappa$  is a coinductive type of streams satisfying the more standard type isomorphism  $\text{Str} \cong \mathbb{N} \times \text{Str}$ .

The most advanced type theory with all the above mentioned features is Clocked Type Theory (CloTT) [BGM17], which introduces the notion of ticks on a clock. Ticks are evidence that time has passed and can be used to unpack elements of type  $\triangleright^{\kappa} A$  to elements of  $A$ . In fact, in CloTT  $\triangleright^{\kappa} A$  is generalised to a special form of dependent function type  $\triangleright(\alpha : \kappa).A$  from ticks to  $A$ . The introduction rule abstracts assumptions of the form  $\alpha : \kappa$  from the context, and the elimination applies a term  $t : \triangleright(\alpha : \kappa).A$  to a tick  $\beta : \kappa$  to give an element of  $A[\beta/\alpha]$ . Special typing rules ensure that a term is never applied twice to the same tick. The combination of ticks and clocks in CloTT can be used for coinductive reasoning about coinductive types, by encoding the *delayed substitutions* of [BGC<sup>+</sup>16].

Bahr et al [BGM17] have shown that CloTT can be given a reduction semantics satisfying strong normalisation, confluence and canonicity. This establishes that productivity can indeed be encoded in types: For a closed term  $t$  of stream type, the  $n$ 'th element can be computed in finite time. These syntactic results also imply soundness of the type theory. However, these results have only been established for a core type theory without, e.g., identity types, and the arguments can be difficult to extend to larger calculi. In particular, we are interested in extending CloTT with path types as in Guarded Cubical Type Theory [BBC<sup>+</sup>19] in future work. Therefore a denotational model of CloTT can be useful, and this paper presents such a model.

The work presented here builds on a number of existing models for guarded recursion. The most basic such, modelling the single clock case, is the topos of trees model [BMSS12], in which a closed type is modelled as a family of sets  $X_n$  indexed by natural numbers  $n$ , together with restriction maps of the form  $X_{n+1} \rightarrow X_n$  for every  $n$ . In other words, a type is

a presheaf over the ordered natural numbers. In this model  $\triangleright$  is modelled as  $(\triangleright X)_0 = 1$  and  $(\triangleright X)_{n+1} = X_n$  and guarded recursion reduces to natural number recursion. The guarded recursive type  $\mathbf{Str}^{\mathfrak{E}}$  mentioned above can be modelled in the topos of trees as  $\mathbf{Str}_n^{\mathfrak{E}} = \mathbb{N}^{n+1} \times 1$ .

Bizjak and Møgelberg [BM20] recently extended this model to the case of many clocks, using a category  $\mathbf{Set}^{\mathbb{T}}$  of covariant presheaves over a category  $\mathbb{T}$  of time objects. An object of  $\mathbb{T}$  is a pair of a finite set  $\mathcal{E}$  and a map  $\delta : \mathcal{E} \rightarrow \mathbb{N}$ , and a morphism from  $(\mathcal{E}, \delta)$  to  $(\mathcal{E}', \delta')$  is a map  $\tau : \mathcal{E} \rightarrow \mathcal{E}'$  such that  $\delta'\tau \leq \delta$  in the pointwise order. Intuitively,  $\mathcal{E}$  indicates the set of clocks in play at any time in a computation, and  $\delta$  indicates the number of ticks left on each clock. The use of the inequality in the maps allows for time to pass, similarly to the passing from a larger number to a smaller number in the topos of trees model.

The main challenge when adapting the model of [BM20] to CloTT is to model ticks, which were not present in the language modelled in [BM20]. In particular, how does one model tick assumptions of the form  $\alpha : \kappa$  in a context, when there appears to be no object of ticks in the model to be used as the denotation of the clock  $\kappa$ ? In this paper we observe that these assumptions can be modelled using a left adjoint  $\blacktriangleleft^{\kappa}$  to the functor  $\blacktriangleright^{\kappa}$  used in [BM20] to model the delay modality  $\triangleright^{\kappa}$  associated to the clock  $\kappa$ . Precisely we model context extension as  $\llbracket \Gamma, \alpha : \kappa \rrbracket = \blacktriangleleft^{\kappa} \llbracket \Gamma \rrbracket$ . The modality  $\triangleright(\alpha : \kappa).A$  is then modelled as a *dependent right adjoint* to  $\blacktriangleleft^{\kappa}$ , a notion studied in detail in [CMM<sup>+</sup>20]: If  $\mathcal{C}$  is a category with family (CwF) [Dyb95] (a standard notion of model for dependent type theory) and  $L$  an endofunctor on (the underlying category of)  $\mathcal{C}$ , a dependent right adjoint to  $L$  is an operation mapping a family  $A$  over  $L\Gamma$  to a family  $RA$  over  $\Gamma$  with a bijective correspondence between elements of  $A$  and elements of  $RA$  natural in  $\Gamma$ . Dependent right adjoints model Fitch-style modal operators in type theory, a general pattern seen also in the model of fresh name abstraction of FreshMLTT [PMD15] and dependent path types in cubical type theory [CCHM18]. In CloTT the type operator  $\triangleright$  is indexed by clocks, and since the model has an object of clocks this can be understood as an internally indexed family of Fitch-style modal operators. We show how to model this as a dependent right adjoint on the slice category over the object of clocks.

Finally we show how to model the special tick constant  $\diamond$  used in CloTT to eliminate  $\triangleright^{\kappa}$  in special situations. Again, since there is no object of ticks in which  $\diamond$  can be an element, standard tools can not be used to model this. Still, we shall see that there exists a semantic substitution of  $\diamond$  for a tick variable that can be used to model application of terms to  $\diamond$ .

**Overview.** Before introducing Clocked Type Theory in full we focus on a fragment called the *tick calculus* capturing just the interaction of ticks with dependent types. Section 1 introduces this and shows how ticks can be used to program with and reason about modal types. Then we introduce the notion of dependent right adjoint and show how to use this to model the tick calculus. Section 2 introduces CloTT as an extension of the tick calculus to multiple clocks and with guarded recursion. In the original presentation of CloTT [BGM17] judgements had a separate context of clock variables. Here we use a single context, and this simplifies not only the syntax but also the semantics considerably. Section 2.1 extends basic CloTT with universes following the approach of Guarded Dependent Type Theory [BM20]. For universes to be consistent with the *clock irrelevance axiom* of CloTT these are indexed by sets of clocks that may appear freely in the elements of the universe. Inclusions between sets of clocks induce inclusions between universes and all type constructors commute on the nose with these.

Section 3 introduces the presheaf category  $\mathbf{GR}$  forming the model of CloTT and defines the object of clocks in this. This is the same category as used by Bizjak and Møgelberg [BM20] to model the related Guarded Dependent Type Theory, and it was also discovered independently by Harper and Sterling [SH18] as a model of Guarded Computational Type Theory. Section 4 constructs a dependent right adjoint on the slice category over the object of clocks, and Section 5 lifts these results to an internally indexed family of dependent right adjoints on  $\mathbf{GR}$ . Sections 6 and 7 describe the semantic structure required to model the guarded fixed point operator and the tick constant  $\diamond$ , respectively. Section 8 recalls the semantic universes of [BM20] and shows how to model the modal types of CloTT in these.

Section 9 defines the interpretation of syntax into the model and proves soundness. For this we follow the approach of Hofmann [Hof97] for modelling dependent type theories: First the interpretation of syntax is defined as a partial function, then it is proved that the interpretation is defined for all judgements that have a derivation. The latter proof is done by a simultaneous induction with proofs of soundness and a substitution lemma. As is standard, the syntax interpreted into the model is an annotated variant of the syntax presented in Section 2. Apart from the standard annotations e.g. of application terms with the  $\Pi$ -type of the function, in CloTT the term for application to the tick constant  $\diamond$  must be changed by replacing a substitution by an explicit substitution. Moreover, special lemmas for weakening substitutions must be proved to accommodate tick-weakening in CloTT. The paper ends with conclusions and future work in Section 10.

**Related work.** The two first named authors have previously published a conference publication [MM18] describing a model of CloTT. That paper contained an error in the description of the left adjoint  $\blacktriangleleft^\kappa$ , which had consequences for a number of other results in the paper. Apart from correcting this mistake the present paper also presents a greatly simplified model construction. The previous model used the original syntax of CloTT in which judgements had a separate context of clock variables  $\Delta$ , and modelled this using a diagram of categories  $\mathbf{GR}[\Delta]$  indexed by clock contexts. These categories were equivalent to slice categories of the category  $\mathbf{GR}$  used in this paper, and are also used in Section 8 to construct the semantic universes. The clock contexts  $\Delta$  allowed the modal operators to be externally indexed. In particular, each  $\kappa \in \Delta$  induced a dependent right adjoint on  $\mathbf{GR}[\Delta]$ . Unfortunately, the morphisms of the diagram induced by clock substitutions did not commute with the left adjoints of these dependent adjunctions causing great complications of the model construction. The present paper avoids these problems by using an internal indexing of the dependent adjunctions.

As described above, one of the motivations for CloTT is the encoding of coinductive types capturing the notion of productivity in types. There exist other solutions to this problem, in particular the combination of single clock guarded recursion with an ‘always’ modality  $\square$  [BGBC17, GKNB20] and sized types [HPS96, AP13, AVW17, Sac13]. We refer to [BM20] for a discussion of the relationship between these approaches.

## 1. A TICK CALCULUS

Before introducing CloTT we focus on a fragment to explain the notion of ticks and how to model these. To motivate ticks, consider the notion of applicative functor from functional programming [MP08]: a type former  $\triangleright$  with maps  $A \rightarrow \triangleright A$  and  $\triangleright(A \rightarrow B) \rightarrow \triangleright A \rightarrow \triangleright B$  satisfying a number of equations that we shall not recall. These maps can be used for

programming with the constructor  $\triangleright$ , but for reasoning in a dependent type theory, one needs an extension of these to dependent function types. For example, in guarded recursion one can prove a theorem  $X$  by constructing a map  $\triangleright X \rightarrow X$  and taking its fixed point in  $X$ . If the theorem is that a property holds for all elements in a type of guarded streams satisfying  $\mathbf{Str} \equiv \mathbb{N} \times \triangleright \mathbf{Str}$ , then  $X$  will be of the form  $\prod (xs : \mathbf{Str}) . P$ . To apply the (essentially coinductive) assumption of type  $\triangleright \prod (xs : \mathbf{Str}) . P$  to the tail of a stream, which has type  $\triangleright \mathbf{Str}$  we need an extension of the applicative functor action.

What should the type of such an extension be? Given  $a : \triangleright A$  and  $f : \triangleright (\prod (x : A) . B)$  the application of  $f$  to  $a$  should be something of the form  $\triangleright B[??/x]$ . If we think of  $\triangleright$  as a delay, intuitively  $a$  is a value of type  $A$  delayed by one time, and the  $??$  should be the value delivered by  $a$  one time step from now. One way of referring to that value is by changing the target type of the dependent applicative functor action to a `let`-expression. Here we describe a more direct approach based on ticks. Ticks should be thought of as evidence that time has passed which can be used to unpack elements of modal type.

The *tick calculus* is the extension of dependent type theory with the following four rules

$$\frac{\Gamma \vdash \quad \alpha \notin \Gamma}{\Gamma, \alpha : \text{tick} \vdash} \qquad \frac{\Gamma, \alpha : \text{tick} \vdash A \text{ type}}{\Gamma \vdash \triangleright (\alpha : \text{tick}). A \text{ type}}$$

$$\frac{\Gamma, \alpha : \text{tick} \vdash t : A}{\Gamma \vdash \lambda(\alpha : \text{tick}). t : \triangleright (\alpha : \text{tick}). A} \qquad \frac{\Gamma \vdash t : \triangleright (\alpha : \text{tick}). A \qquad \Gamma, \beta : \text{tick}, \Gamma' \vdash}{\Gamma, \beta : \text{tick}, \Gamma' \vdash t[\beta] : A[\beta/\alpha]}$$

An assumption of the form  $\alpha : \text{tick}$  in a context is an assumption that one time step has passed, and  $\alpha$  is the evidence of this. Variables on the right-hand side of such an assumption should be thought of as arriving one time step later than those on the left. Ticks can be abstracted in terms and types, so that the type constructor  $\triangleright$  now comes with evidence that time has passed that can be used in its scope. The type  $\triangleright (\alpha : \text{tick}). A$  can be thought of as a form of dependent function type over ticks, which we abbreviate to  $\triangleright A$  if  $\alpha$  does not occur free in  $A$ . The elimination rule states that if a term  $t$  can be typed as  $\triangleright (\alpha : \text{tick}). A$  before the arrival of tick  $\beta$ ,  $t$  can be opened using  $\beta$  to give an element of type  $A[\beta/\alpha]$ . Note that the causality restriction in the typing rule prevents a term like  $\lambda x. \lambda(\alpha : \text{tick}). x[\alpha][\alpha] : \triangleright \triangleright A \rightarrow \triangleright A$  being well typed; a tick can only be used to unpack the same term once. The context  $\Gamma'$  in the elimination rule ensures that typing rules are closed under weakening, also for ticks. Note that the clock object `tick` is not a type. The variable introduction rule is unchanged:  $\Gamma, x : A, \Gamma' \vdash x : A$  even if there are ticks in  $\Gamma'$ . Intuitively, this means that data is kept past time steps.

The equality theory is likewise extended with the usual  $\beta$  and  $\eta$  rules:

$$(\lambda(\alpha : \text{tick}). t)[\beta] = t[\beta/\alpha] \qquad \lambda(\alpha : \text{tick}). (t[\alpha]) = t$$

As stated, the tick calculus should be understood as an extension of standard dependent type theory. In particular one can add dependent sums and function types with standard rules.

We can now type the dependent applicative structure as

$$\lambda(x:A). \lambda(\alpha : \text{tick}). x : A \rightarrow \triangleright A$$

$$\lambda f. \lambda y. \lambda(\alpha : \text{tick}). f[\alpha](y[\alpha]) : \triangleright (\prod (x : A) . B) \rightarrow \prod (y : \triangleright A) . \triangleright (\alpha : \text{tick}). B[y[\alpha]/x]$$

**Example 1.1.** For a small example on how ticks in combination with the fixed point operator  $\text{dfix} : (\triangleright X \rightarrow X) \rightarrow \triangleright X$  can be used to reason about guarded recursive data, let  $\mathbf{Str} \equiv \mathbb{N} \times \triangleright \mathbf{Str}$  be the type of guarded recursive streams mentioned above, and suppose  $x : \mathbb{N} \vdash P(x)$  is a family

to be thought of as a predicate on  $\mathbb{N}$ . A lifting of  $P$  to streams would be another guarded recursive type  $y:\text{Str} \vdash \text{Str}_P(y)$  satisfying  $\text{Str}_P(x :: xs) \equiv P(x) \times \triangleright(\alpha : \text{tick}).\text{Str}_P(xs[\alpha])$  (where  $x :: xs$  is the pairing of  $x$  and  $xs$ ). If  $p : \prod (x : \mathbb{N}).P(x)$  is a proof of  $P$  we would expect that also  $\prod (y : \text{Str}).\text{Str}_P(y)$  can be proved, and indeed this can be done as follows. Consider first

$$f : \triangleright(\prod (y : \text{Str}).\text{Str}_P(y)) \rightarrow \prod (y : \text{Str}).\text{Str}_P(y)$$

$$f q (x :: xs) \stackrel{\text{def}}{=} (p(x), \lambda(\alpha : \text{tick}).q[\alpha](xs[\alpha]))$$

Then  $f(\text{dfix}(f))$  has the desired type.

More generally, ticks can be used to encode [BGM17] the *delayed substitutions* of [BGC<sup>+</sup>16], which have been used to reason coinductively about coinductive data. For more examples of reasoning using these see [BGC<sup>+</sup>16]. The tick calculus is an example of a Fitch-style modal calculus [Clo18, Fit52]. Most of these use a presentation where ticks are simply markers in the context, rather than carry names as here. However, names of ticks play a crucial role in the normalisation proof for CloTT [BGM17], and we therefore also use names here.

**1.1. Modelling ticks using adjunctions.** We now describe a notion of model for the tick calculus. It is based on the notion of category with family (CwF) [Dyb95], which is a standard notion of model of dependent type theory.

**Definition 1.2.** A CwF comprises

- A category  $\mathcal{C}$  with a distinguished terminal object
- For each object  $\Gamma$  of  $\mathcal{C}$  a set  $\text{Fam}(\Gamma)$  of *families* over  $\Gamma$ . We write  $\Gamma \vdash A$  to mean  $A \in \text{Fam}(\Gamma)$ .
- For each  $\Gamma$  in  $\mathcal{C}$  and each family  $A$  in  $\text{Fam}(\Gamma)$  a set  $\text{El}(A)$  of *elements* of  $A$ . We write  $\Gamma \vdash t : A$  to mean  $t \in \text{El}(A)$ .
- For each morphism  $\gamma : \Delta \rightarrow \Gamma$  in  $\mathcal{C}$  *reindexing* operations mapping  $\Gamma \vdash A$  to  $\Delta \vdash A[\gamma]$  and  $\Gamma \vdash t : A$  to  $\Delta \vdash t[\gamma] : A[\gamma]$ . These must satisfy the equations  $A[\text{id}] = A$ ,  $t[\text{id}] = t$ ,  $A[\gamma \circ \delta] = A[\gamma][\delta]$  and  $t[\gamma \circ \delta] = t[\gamma][\delta]$  for all morphisms  $\delta$  with codomain  $\Delta$ .
- A *comprehension* operation associating to each family  $\Gamma \vdash A$  the following: An object  $\Gamma.A$  in  $\mathcal{C}$ , a morphism  $\mathbf{p}_A : \Gamma.A \rightarrow \Gamma$  and an element  $\Gamma.A \vdash \mathbf{q}_A : A[\mathbf{p}_A]$ , such that for every  $\gamma : \Delta \rightarrow \Gamma$ , and  $\Delta \vdash t : A[\gamma]$  there exists a unique morphism  $\langle \gamma, t \rangle : \Delta \rightarrow \Gamma.A$  such that  $\mathbf{p}_A \circ \langle \gamma, t \rangle = \gamma$  and  $\mathbf{q}_A[\langle \gamma, t \rangle] = t$ .

The requirements on reindexing of families and elements mean that they can be described more concisely as a functor from  $\mathcal{C}^{\text{op}}$  to the category of families on sets. This is in fact Dybjer's original definition. Awodey's *natural models of type theory* [Awo18] are an elegant abstract formulation of the notion of CwF.

To model the tick calculus we need an operation  $\mathbf{L}$  modelling the extension of a context with a tick, plus an operation  $\mathbf{R}$  modelling  $\triangleright$ . In the simply typed setting,  $\mathbf{R}$  would be a right adjoint to context extension modelling the bijective correspondence between terms  $\Gamma, \alpha : \text{tick} \vdash t : A$  and terms  $\Gamma \vdash s : \triangleright(\alpha : \text{tick}).A$ . For dependent types this is not quite so, since these operations work on different objects (contexts and types respectively). Instead, we need a dependent adjunction as in the following definition, which generalises that of [CMM<sup>+</sup>20] by allowing also dependent adjunctions between different categories (not just endoadjunctions).

**Definition 1.3.** Let  $\mathcal{C}$  and  $\mathcal{D}$  be CwFs and let  $L : \mathcal{C} \rightarrow \mathcal{D}$  be a functor between the underlying categories. A *dependent right adjoint* to  $L$  consists of an operation associating to each family  $L\Gamma \vdash A$  in  $\mathcal{D}$  a family  $\Gamma \vdash RA$  in  $\mathcal{C}$  and a bijective map of elements mapping  $L\Gamma \vdash t : A$  to  $\Gamma \vdash \bar{t} : RA$  such that  $(RA)[\gamma] = R(A[L\gamma])$  and  $\bar{t}[\gamma] = t[L\gamma]$ .

We write  $\overline{(-)}$  also for the inverse direction of the bijection on terms so that  $\overline{\bar{t}} = t$ . It easily follows [CMM<sup>+</sup>20] that also the inverse direction commutes with substitution, i.e., that for  $\Gamma \vdash s : RA$  also  $\overline{s[\gamma]} = \bar{s}[L\gamma]$ .

The dependent adjunctions in this paper arise from adjunctions on the underlying categories with liftings of the right adjoint to families and elements as in the following definition.

**Definition 1.4.** Let  $\mathcal{C}$  and  $\mathcal{D}$  be CwFs and let  $R : \mathcal{C} \rightarrow \mathcal{D}$  be a functor. An *extension of  $R$  to families and elements* is a pair of operations presented here in the form of rules

$$\frac{\Gamma \vdash A}{R\Gamma \vdash R_{\text{Fam}}A} \quad \frac{\Gamma \vdash t : A}{R\Gamma \vdash R_{\text{El}}t : R_{\text{Fam}}A}$$

commuting with reindexing in the sense that  $(R_{\text{Fam}}A)[R\gamma] = R_{\text{Fam}}(A[\gamma])$  and  $(R_{\text{El}}t)[R\gamma] = R_{\text{El}}(t[\gamma])$  hold for all substitutions  $\gamma$ , and commuting with comprehension in the sense that  $\langle R\mathfrak{p}_A, R_{\text{El}}\mathfrak{q}_A \rangle : R(\Gamma.A) \rightarrow R\Gamma.R_{\text{Fam}}A$  is an isomorphism.

**Lemma 1.5.** Let  $\mathcal{C}$  and  $\mathcal{D}$  be CwFs and let  $L : \mathcal{C} \rightarrow \mathcal{D} : R$  be an adjunction of the underlying categories, such that  $R$  extends to families and elements. Let  $\eta$  be the unit and let  $\epsilon$  be the counit of the adjunction. The operation mapping  $L\Gamma \vdash A$  to  $\Gamma \vdash RA$  defined as  $RA \stackrel{\text{def}}{=} (R_{\text{Fam}}A)[\eta]$  defines a dependent right adjoint to  $L$ . The required bijection on elements maps  $L\Gamma \vdash a : A$  to  $(R_{\text{El}}a)[\eta]$  and  $\Gamma \vdash b : RA$  to  $\mathfrak{q}_A[\epsilon \circ L(\langle R\mathfrak{p}_A, R_{\text{El}}\mathfrak{q}_A \rangle^{-1} \circ \langle \eta, b \rangle)]$ .

Lemma 1.5 is a straight-forward generalisation of [CMM<sup>+</sup>20, Lemma 17]. Note the notational convention: In the setting of the lemma we overload  $R$  for both the functor on the underlying category and the dependent right adjoint, and use the more verbose  $R_{\text{Fam}}$  and  $R_{\text{El}}$  for the extension of  $R$  to families and elements. This differs from the notation used in [CMM<sup>+</sup>20], but is chosen here for notational convenience.

**1.2. Interpretation.** The tick calculus can be modelled in a CwF equipped with an endofunctor  $L$  with a dependent right adjoint and a natural transformation  $\mathfrak{p}_L : L \rightarrow \text{id}_{\mathcal{C}}$ . The latter is needed to interpret tick weakening. Defining

$$\llbracket \Gamma, \alpha : \text{tick} \vdash \rrbracket = L\llbracket \Gamma \vdash \rrbracket$$

$\mathfrak{p}_L$  allows us to define a context projection  $\mathfrak{p}_{\Gamma'} : \llbracket \Gamma, \Gamma' \vdash \rrbracket \rightarrow \llbracket \Gamma \vdash \rrbracket$  by induction on  $\Gamma'$  using  $\mathfrak{p}_L$  in the case of tick variables. We can then define the rest of the interpretation as

$$\begin{aligned} \llbracket \Gamma, x : A, \Gamma' \vdash x \rrbracket &= \mathfrak{q}_A[\mathfrak{p}_{\Gamma'}] & \llbracket \Gamma \vdash \triangleright (\alpha : \text{tick}).A \text{ type} \rrbracket &= R\llbracket A \rrbracket \\ \llbracket \Gamma \vdash \lambda(\alpha : \text{tick}).t \rrbracket &= \overline{\llbracket t \rrbracket} & \llbracket \Gamma, \beta : \text{tick}, \Gamma' \vdash t[\beta] \rrbracket &= \overline{\llbracket t \rrbracket}[\mathfrak{p}_{\Gamma'}] \end{aligned}$$

**Proposition 1.6.** *The above interpretation of the tick calculus into a CwF with adjunction and tick weakening  $\mathfrak{p}_L$  is sound.*

Proposition 1.6 can be proved using the tools of [CMM<sup>+</sup>20].

**1.3. Adding basic type constructors.** The model of the tick calculus can be extended with basic type constructors like natural numbers,  $\Pi$ - and  $\Sigma$ -types as well as identity types. Here we just recall what it means for a CwF to have extensional identity types, referring the reader to Hofmann [Hof97] for details on other constructors.

**Definition 1.7.** A CwF  $\mathcal{C}$  has *extensional identity types* if for each pair of elements  $\Gamma \vdash t : A$  and  $\Gamma \vdash u : A$  of the same family  $\Gamma \vdash A$  there is a family  $\Gamma \vdash \text{Id}_A(t, u)$  with at most one element such that  $t$  and  $u$  are equal if and only if there is an element of  $\Gamma \vdash \text{Id}_A(t, u)$ , and such that  $(\text{Id}_A(t, u))[\gamma] = \text{Id}_{A[\gamma]}(t[\gamma], u[\gamma])$ .

## 2. CLOCKED TYPE THEORY

Clocked Type Theory (CloTT) is an extension of the tick calculus with guarded recursion and multiple clocks. Rather than having a global notion of time as in the tick calculus, ticks are associated with clocks and clocks can be assumed and universally quantified. In the original presentation of CloTT [BGM17] judgements had a separate context for clock variables, i.e., assumptions of the form  $\kappa : \text{clock}$ . In this paper, clock variables are simply assumed in the context as if they were ordinary variables. This simplifies both the syntax and semantics of the language. There are no operations for forming clocks, only clock variables. It is often convenient to have a single clock constant  $\kappa_0$  and this can be achieved by a precompilation adding  $\kappa_0$  as a fresh variable to the contexts.

The rules for typing judgements and judgemental equality are given in Figure 1. These should be seen as an extension of a dependent type theory with  $\Pi$ - and  $\Sigma$ -types, as well as extensional identity types. The rules for these are completely standard, and thus are omitted from the figure. We write  $\equiv$  for judgemental equality and  $t =_A u$  for identity types. The model will also model the *identity reflection* rule

$$\frac{\Gamma \vdash p : t =_A u}{\Gamma \vdash t \equiv u : A}$$

of extensional type theory.

The type of the guarded fixed point operator  $\text{dfix}$  uses the abbreviation  $\triangleright^\kappa A$  for  $\triangleright(\alpha : \kappa).A$  where  $\alpha$  does not occur free in  $A$ . This operator is useful in combination with guarded recursive types such as a type of guarded streams  $\text{Str}^\kappa$  satisfying  $\text{Str}^\kappa \equiv \mathbb{N} \times \triangleright^\kappa \text{Str}^\kappa$ . This type is similar to the one from Example 1.1 except that the delay now is associated with a clock variable  $\kappa$ . We will see how to define such guarded recursive types in the next section. Given  $\text{Str}^\kappa$  we can use  $\text{dfix}$  for recursive programming with guarded streams, e.g., when defining a constant stream of zeros as  $\text{dfix}^\kappa(\lambda x. 0 :: x)$ . The type of  $\text{dfix}$  ensures that only productive recursive definitions are typeable, e.g.,  $\text{dfix}^\kappa(\lambda x.x)$  is not.

The tick constant  $\diamond$  gives a way to execute a delayed computation  $t$  of type  $\triangleright^\kappa A$  to compute a value of type  $A$ . In particular, if  $t$  is a fixed point, application to the tick constant unfolds the fixed point once. This explains the need to name ticks in CloTT: Substitution of  $\diamond$  for a tick variable  $\alpha$  in a term allows for all fixed points applied to  $\alpha$  in the term to be unfolded. In particular, the names of ticks are crucial for the strong normalisation result for CloTT in [BGM17].

Intuitively  $\diamond$  is a constant of type  $\kappa$  for any clock  $\kappa$ . However, since clocks are not types,  $\diamond$  can only be introduced by applying it to a term of type  $\triangleright(\alpha : \kappa).A$ , and such applications must moreover be restricted to ensure productivity. In particular a term such

$$\begin{array}{c}
\textbf{Context formation rules} \\
\frac{\Gamma \vdash A \text{ type} \quad x \notin \Gamma}{\Gamma, x : A \vdash} \quad \frac{\Gamma \vdash \quad \kappa \notin \Gamma}{\Gamma, \kappa : \text{clock} \vdash} \quad \frac{\Gamma \vdash \kappa : \text{clock} \quad \alpha \notin \Gamma}{\Gamma, \alpha : \kappa \vdash} \\
\textbf{Type formation rules} \\
\frac{\Gamma, \alpha : \kappa \vdash A \text{ type}}{\Gamma \vdash \triangleright(\alpha : \kappa).A \text{ type}} \quad \frac{\Gamma, \kappa : \text{clock} \vdash A \text{ type}}{\Gamma \vdash \forall \kappa. A \text{ type}} \quad \frac{\Gamma \vdash}{\Gamma \vdash \mathbb{N} \text{ type}} \\
\textbf{Typing rules} \\
\frac{\Gamma, \kappa : \text{clock} \vdash t : A}{\Gamma \vdash \Lambda \kappa. t : \forall \kappa. A} \quad \frac{\Gamma \vdash t : \forall \kappa. A \quad \Gamma \vdash \kappa' : \text{clock}}{\Gamma \vdash t[\kappa'] : A[\kappa'/\kappa]} \quad \frac{\Gamma, \alpha : \kappa \vdash t : A}{\Gamma \vdash \lambda(\alpha : \kappa). t : \triangleright(\alpha : \kappa). A} \\
\frac{\Gamma \vdash t : \triangleright(\alpha : \kappa). A \quad \Gamma, \alpha' : \kappa, \Gamma' \vdash}{\Gamma, \alpha' : \kappa, \Gamma' \vdash t[\alpha'] : A[\alpha'/\alpha]} \quad \frac{\Gamma, \kappa : \text{clock} \vdash t : \triangleright(\alpha : \kappa). A \quad \Gamma \vdash \kappa' : \text{clock}}{\Gamma \vdash (t[\kappa'/\kappa])[\diamond] : A[\kappa'/\kappa][\diamond/\alpha]} \\
\frac{\Gamma \vdash t : \triangleright^\kappa A \rightarrow A}{\Gamma \vdash \text{dfix}^\kappa t : \triangleright^\kappa A} \quad \frac{\Gamma \vdash t : A \quad A \equiv B}{\Gamma \vdash t : B} \quad \frac{\kappa : \text{clock} \in \Gamma}{\Gamma \vdash \kappa : \text{clock}} \quad \frac{x : A \in \Gamma}{\Gamma \vdash x : A} \\
\textbf{Judgemental equality} \\
(\Lambda \kappa. t)[\kappa'] \equiv t[\kappa'/\kappa] \quad \Lambda \kappa. (t[\kappa]) \equiv t \quad (\lambda(\alpha : \kappa). t)[\beta] \equiv t[\beta/\alpha] \\
\lambda(\alpha : \kappa). (t[\alpha]) \equiv t \quad (\text{dfix}^\kappa t)[\diamond] \equiv t(\text{dfix}^\kappa t) \quad (\lambda(\alpha : \kappa). t)[\diamond] \equiv t[\diamond/\alpha]
\end{array}$$

Figure 1: Selected typing and judgemental equality rules of Clocked Type Theory. The two  $\eta$  rules are subject to the standard conditions of  $\kappa$  and  $\alpha$ , respectively, not appearing in the term  $t$ .

as  $\lambda x : \triangleright^\kappa A. x[\diamond]$  should not be well typed, as this would give a way of inhabiting all types using  $\text{dfix}$ . The typing rule for application to the tick constant ensures this by assuming that the clock  $\kappa$  associated to the delay does not occur freely in the type of any other variable in the context of  $t$ . For example, the rule

$$\frac{\Gamma, \kappa : \text{clock} \vdash t : \triangleright(\alpha : \kappa). A}{\Gamma, \kappa : \text{clock} \vdash t[\diamond] : A[\diamond/\alpha]}$$

is admissible, which can be proved using a weakening lemma. This rule, however, is not closed under variable substitution, which is the motivation for the more general rule of Figure 1. The typing rule is a bit unusual, in that it involves substitution in the term in the conclusion. In the elaborated syntax for CloTT to be interpreted in the model in Section 9, this substitution is replaced by an explicit substitution binding  $\kappa$  in  $t$  rather than substituting it away.

Universal quantification over clocks allows for coinductive types to be encoded using guarded recursive types [AM13]. For example  $\text{Str} \stackrel{\text{def}}{=} \forall \kappa. \text{Str}^\kappa$  is a coinductive type of streams. The head and tail maps  $\text{hd} : \text{Str} \rightarrow \mathbb{N}$  and  $\text{tl} : \text{Str} \rightarrow \text{Str}$  can be defined as

$$\text{hd}(xs) \stackrel{\text{def}}{=} \pi_1(xs[\kappa_0]) \quad \text{tl}(xs) \stackrel{\text{def}}{=} \Lambda \kappa. ((\pi_2(xs[\kappa]))[\diamond])$$

using the clock constant  $\kappa_0$ . It is easily seen that  $\mathbf{Str} \equiv \forall \kappa. (\mathbb{N} \times \mathbf{Str}^\kappa) \cong \forall \kappa. \mathbb{N} \times \forall \kappa. \triangleright^\kappa \mathbf{Str}^\kappa$ . To prove that  $\forall \kappa. \mathbb{N} \cong \mathbb{N}$  and  $\forall \kappa. \triangleright^\kappa \mathbf{Str}^\kappa \cong \mathbf{Str}$  ensuring the isomorphism expected by a stream type, one needs two irrelevance axioms.

The first of these is the *clock irrelevance axiom*

$$\frac{\Gamma \vdash t : \forall \kappa. A \quad \kappa \notin \text{fc}(A)}{\Gamma \vdash \text{cirr}^\kappa t : \forall \kappa'. \forall \kappa''. t[\kappa'] =_A t[\kappa'']} \quad (2.1)$$

In the second hypothesis for the rule  $\text{fc}(A)$  stands for the free clocks of  $A$  defined in the standard way. This rule can be used to prove that  $\forall \kappa. A$  is isomorphic to  $A$  if  $\kappa$  is not free in  $A$ , in particular  $\forall \kappa. \mathbb{N} \cong \mathbb{N}$ . The second type isomorphism above requires the *tick irrelevance axiom*

$$\frac{\Gamma \vdash t : \triangleright^\kappa A}{\text{tirr}^\kappa t : \triangleright(\alpha : \kappa). \triangleright(\alpha' : \kappa). t[\alpha] =_A t[\alpha']} \quad (2.2)$$

which states that the identity of ticks is irrelevant for the equality theory, despite being crucial for the reduction semantics.

Finally we mention the fixed point unfolding axiom [BBC<sup>+</sup>19]

$$\frac{\Gamma \vdash t : \triangleright^\kappa A \rightarrow A}{\Gamma \vdash \text{pfix}^\kappa t : \triangleright(\alpha : \kappa). (\text{dfix}^\kappa t)[\alpha] =_A t(\text{dfix}^\kappa t)} \quad (2.3)$$

In an extensional type theory this implies the judgemental fixed point unfolding equality  $(\text{dfix}^\kappa t)[\diamond] \equiv t(\text{dfix}^\kappa t)$ , and so, since the model presented in this paper is extensional, it will suffice to model  $\text{pfix}^\kappa$ . We write  $\text{fix}^\kappa t \stackrel{\text{def}}{=} t(\text{dfix}^\kappa t) : A$ . Note that by extensionality, then

$$t(\lambda(\alpha : \kappa). \text{fix}^\kappa t) \equiv t(\lambda(\alpha : \kappa). (\text{dfix}^\kappa t)[\alpha]) \equiv t(\text{dfix}^\kappa t) \equiv \text{fix}^\kappa t \quad (2.4)$$

Apart from the clock irrelevance axiom, the rules for universal quantification over clocks are exactly those for a  $\Pi$ -type indexed over **clock**, except that **clock** is not a type. The latter means that **clock** can not appear positively in types, e.g.,  $\mathbf{clock} \rightarrow \mathbf{clock}$  is not wellformed. To see why **clock** should not be a type, note that clock irrelevance states that for a closed type  $A$ , all elements of  $\forall \kappa. A$  are constant functions from clocks to  $A$ . Allowing  $A = \mathbf{clock}$  would force all clocks to be equal. In the model there will be an object  $\text{Clk}$  modelling **clock** and universal quantification over clocks will be modelled as a  $\Pi$ -type.

**2.1. Universes.** In order to maintain consistency with the clock irrelevance axiom, universes in CloTT are indexed by clock contexts. To see why this is necessary, note that naively adding a closed universe  $U$ , with a map  $\triangleright : U \rightarrow \forall \kappa. U$ , the clock irrelevance principle would state that the type operation  $\triangleright^\kappa(-)$  would be independent of  $\kappa$  on small types. With the subscripting, the type operation  $\triangleright$  can be restricted on  $U_\Delta$  to the  $\kappa \in \Delta$  avoiding this problem. Note that the formulation of CloTT used here differs from that presented in [BGM17], which for simplicity used a single universe but retained consistency since the clock irrelevance axiom was not modelled (although mentioned in the paper). The presentation of universes used here follows that of GDTT [BGC<sup>+</sup>16, BM20], but extends it with ticks.

The typing rules and equalities for universes are presented in Figure 2. The subscript  $\Delta$  of a universe is a *set* of clock variables, meaning in particular, that if  $\Delta = \Delta'$  are equal as sets (contain the same elements), then  $U_\Delta \equiv U_{\Delta'}$ . The universes are Tarski style, and we restrict to a single universe level. The universes enjoy a form of polymorphism in the clock context: Inclusions of clock contexts induce inclusions of universes, and these commute with the operations on the universe. For simplicity, we just include the rules for universal

### Formation and typing rules

$$\begin{array}{c}
\frac{\Delta = \{\kappa_1, \dots, \kappa_n\} \quad \Gamma \vdash \kappa_i : \text{clock for } i = 1, \dots, n}{\Gamma \vdash U_\Delta \text{ type}} \qquad \frac{\Gamma \vdash t : U_\Delta}{\Gamma \vdash \text{El}_\Delta(t) \text{ type}} \\
\\
\frac{\Gamma \vdash t : U_\Delta \quad \Delta \subseteq \Delta'}{\Gamma \vdash \text{in}_{\Delta, \Delta'}(t) : U_{\Delta'}} \qquad \frac{\Gamma, \kappa : \text{clock} \vdash A : U_{\Delta, \kappa}}{\Gamma \vdash \bar{\forall} \kappa. A : U_\Delta} \qquad \frac{\Gamma, \alpha : \kappa \vdash A : U_\Delta \quad \kappa \in \Delta}{\Gamma \vdash \bar{\exists}(\alpha : \kappa). A : U_\Delta}
\end{array}$$

### Equations

$$\begin{array}{l}
\text{El}_\Delta(\bar{\forall} \kappa. A) \equiv \forall \kappa. \text{El}_{\Delta, \kappa}(A) \qquad \text{El}_\Delta(\bar{\exists}(\alpha : \kappa). A) \equiv \triangleright(\alpha : \kappa). \text{El}_\Delta(A) \\
\text{in}_{\Delta, \Delta'}(\bar{\forall} \kappa. A) \equiv \bar{\forall} \kappa. \text{in}_{(\Delta, \kappa), (\Delta', \kappa)}(A) \qquad \text{in}_{\Delta, \Delta'}(\bar{\exists}(\alpha : \kappa). A) \equiv \bar{\exists}(\alpha : \kappa). \text{in}_{\Delta, \Delta'}(A) \\
\text{El}_{\Delta'}(\text{in}_{\Delta, \Delta'}(t)) \equiv \text{El}_\Delta(t) \qquad \text{in}_{\Delta', \Delta''}(\text{in}_{\Delta, \Delta'}(t)) \equiv \text{in}_{\Delta, \Delta''}(t) \\
\text{in}_{\Delta, \Delta}(t) \equiv t
\end{array}$$

Figure 2: Universes in CloTT.

quantification over clocks and  $\triangleright$ . The rules for  $\Pi$ -, and  $\Sigma$ -types are the standard ones, indexed by a clock context, plus a rule stating that these commute with the universe inclusions, see [BM20] for details. We also assume a code  $\bar{\mathbb{N}}$  for natural numbers in each universe.

As mentioned above, guarded recursive types can be encoded as fixed points on the universe. For example,  $\text{Str}^\kappa \stackrel{\text{def}}{=} \text{El}_\kappa(\overline{\text{Str}}^\kappa)$  where  $\overline{\text{Str}}^\kappa \stackrel{\text{def}}{=} \text{fix}^\kappa(\lambda X. \bar{\mathbb{N}} \bar{\times} \bar{\triangleright}(\alpha : \kappa). X[\alpha])$ , and  $\bar{\times}$  is the code for binary products encoded using  $\Sigma$ -types in the standard way. By (2.4) this gives

$$\begin{aligned}
\text{Str}^\kappa &\equiv \text{El}_\kappa(\bar{\mathbb{N}} \bar{\times} \bar{\triangleright}(\alpha : \kappa). \overline{\text{Str}}^\kappa) \\
&\equiv \mathbb{N} \times \text{El}_\kappa(\bar{\triangleright}(\alpha : \kappa). \overline{\text{Str}}^\kappa) \\
&\equiv \mathbb{N} \times \triangleright(\alpha : \kappa). \text{Str}^\kappa
\end{aligned}$$

Similarly, if  $\bar{P} : \mathbb{N} \rightarrow U_\kappa$  and  $P(x) \equiv \text{El}_\kappa(\bar{P}(x))$  we can construct a lifting  $\text{Str}_P^\kappa$  of  $P$  to a predicate on guarded streams as in Example 1.1. For this, define  $\text{Str}_P^\kappa(xs) \stackrel{\text{def}}{=} \text{El}_\kappa(\overline{\text{Str}}_P^\kappa(xs))$  where

$$\overline{\text{Str}}_P^\kappa \stackrel{\text{def}}{=} \text{fix}^\kappa(\lambda X. \lambda(x :: xs). \bar{P}(x) \bar{\times} \bar{\triangleright}(\alpha : \kappa). X[\alpha](xs[\alpha])) : \text{Str}^\kappa \rightarrow U_\kappa$$

Here the type of the variable  $X$  is  $\triangleright^\kappa(\text{Str}^\kappa \rightarrow U_\kappa)$ . As above, one can then verify that  $\text{Str}_P^\kappa(x :: xs) \equiv P(x) \times \triangleright(\alpha : \kappa). \text{Str}_P^\kappa(xs[\alpha])$ .

The presentation of CloTT in [BGM17] had guarded recursive types as a primitive type formation rule. This was because the version of CloTT used there did not have identity types, and so did not have the fixed point unfolding axiom (2.3). Fixed points only unfolded when applied to  $\diamond$ . As a consequence (2.4) did not hold, so the encoding of recursive types as fixed points on the universe was not possible. Note that in an intensional version of CloTT, equality (2.4) holds only propositionally, making the guarded recursive types unfold only up to equivalence of types as in Guarded Cubical Type Theory [BBC<sup>+</sup>19].

The model constructed in this paper models extensional CloTT including the axioms (2.1), (2.2) and (2.3).

### 3. A PRESHEAF CATEGORY

The setting for the denotational semantics of CloTT is a category of covariant presheaves over a category  $\mathbb{T}$  of time objects, which we now define. This category has previously been used to give a model of GDTT [BM20] and a slight variant has been used to model Guarded Computational Type Theory [SH18].

We will assume given a countably infinite set  $\text{CV}$  of (semantic) clock variables, for which we use  $\lambda, \lambda', \dots$  to range over. A *time object* is a pair  $(\mathcal{E}; \delta)$  where  $\mathcal{E}$  is a finite subset of  $\text{CV}$  and  $\delta : \mathcal{E} \rightarrow \mathbb{N}$  is a map giving the number of ticks left on each clock in  $\mathcal{E}$ . We will write the finite sets  $\mathcal{E}$  as lists writing e.g.,  $\mathcal{E}, \lambda$  for  $\mathcal{E} \cup \{\lambda\}$  and  $\delta[\lambda \mapsto n]$  for the extension of  $\delta$  to  $\mathcal{E}, \lambda$ , or indeed for the update of  $\delta$ , if  $\delta$  is already defined on  $\lambda$ . The time objects form a category  $\mathbb{T}$  whose morphisms  $(\mathcal{E}; \delta) \rightarrow (\mathcal{E}'; \delta')$  are functions  $\tau : \mathcal{E} \rightarrow \mathcal{E}'$  such that  $\delta' \tau \leq \delta$  in the pointwise order. The inequality allows for time to pass in a morphism, but morphisms can also synchronise clocks in  $\mathcal{E}$  by mapping them to the same clock in  $\mathcal{E}'$ , or introduce new clocks if  $\tau$  is not surjective. Define  $\text{GR}$  to be the category  $\text{Set}^{\mathbb{T}}$  of *covariant* presheaves on  $\mathbb{T}$ . The topos of trees [BMSS12] can be seen as a restriction of this where time objects always have a single clock.

If  $\Gamma$  is a presheaf,  $\gamma \in \Gamma(\mathcal{E}; \delta)$  and  $\sigma : (\mathcal{E}; \delta) \rightarrow (\mathcal{E}'; \delta')$ , we will write  $\sigma \cdot \gamma$  for  $\Gamma(\sigma)(\gamma)$ , the functorial action of  $\Gamma$  applied to  $\gamma$ . With this notation, a presheaf is simply an indexed family of sets with actions satisfying

$$\sigma \cdot (\tau \cdot \gamma) = (\sigma \circ \tau) \cdot \gamma \qquad \text{id} \cdot \gamma = \gamma \qquad (3.1)$$

As for any presheaf category,  $\text{GR}$  carries a natural CwF structure in which a family over a presheaf  $\Gamma$  is a presheaf over the category  $\int \Gamma$  of elements of  $\Gamma$ . Recall that this has as objects pairs  $((\mathcal{E}; \delta), \gamma)$  such that  $\gamma \in \Gamma(\mathcal{E}; \delta)$ , and morphisms from  $((\mathcal{E}; \delta), \gamma)$  to  $((\mathcal{E}'; \delta'), \gamma')$  morphisms  $\sigma : (\mathcal{E}; \delta) \rightarrow (\mathcal{E}'; \delta')$  of  $\mathbb{T}$  such that  $\sigma \cdot \gamma = \gamma'$ . Unfolding this definition, a family is a collection of sets  $A_{(\mathcal{E}; \delta)}(\gamma)$  and maps  $\sigma : A_{(\mathcal{E}; \delta)}(\gamma) \rightarrow A_{(\mathcal{E}'; \delta')}(\sigma \cdot \gamma)$  satisfying (3.1). An element of  $A$  is a family of elements  $t_{(\mathcal{E}; \delta)}(\gamma) \in A_{(\mathcal{E}; \delta)}(\gamma)$  such that  $\sigma \cdot (t_{(\mathcal{E}; \delta)}(\gamma)) = t_{(\mathcal{E}'; \delta')}(\sigma \cdot \gamma)$ . We will often omit the subscript and simply write  $A(\gamma)$  and  $t(\gamma)$ . Abstractly, an element of  $A$  is simply a global element of  $A$  considered as a covariant presheaf over  $\int \Gamma$ . If  $f : A \rightarrow B$  is a morphism of presheaves, and  $t$  is an element of  $A$ , we can thus compose  $f$  and  $t$  to get an element  $f \circ t$  of  $B$ .

Recall the following standard lemma [Hof97].

**Lemma 3.1.** *The CwF  $\text{GR}$  models  $\Pi$ ,  $\Sigma$  and extensional identity types.*

Modelling clock as the object  $\text{Clk}$  in  $\text{GR}$  defined as

$$\text{Clk}(\mathcal{E}; \delta) = \mathcal{E} \qquad \tau \cdot \lambda = \tau(\lambda)$$

universal quantification can be modelled as a  $\Pi$ -type over  $\text{Clk}$ .

### 4. A DEPENDENT RIGHT ADJOINT

This section defines the dependent right adjoint to be used for modelling ticks in CloTT. To talk about ticks we need a clock in hand, and the smallest setting this happens in is the syntactic context  $\kappa : \text{clock}$ , modelled as  $\text{Clk}$ . In the CwF contexts extending this small context can be considered presheaves over the category  $\int \text{Clk}$  of elements of  $\text{Clk}$ , and so in the following we will construct a dependent right adjoint on this. In Section 5 we will see how to lift this to model ticks in CloTT.

We write  $\mathbb{T}_\star$  for  $\int \text{Clk}$  and  $\text{GR}_\star \stackrel{\text{def}}{=} \text{Set}^{\mathbb{T}_\star}$ . Spelling out the definition, an object of  $\mathbb{T}_\star$  is a triple  $(\mathcal{E}; \delta; \lambda)$  where  $\lambda \in \mathcal{E}$  and a morphism  $(\mathcal{E}; \delta; \lambda)$  to  $(\mathcal{E}'; \delta'; \lambda')$  is a morphism  $\sigma : (\mathcal{E}; \delta) \rightarrow (\mathcal{E}'; \delta')$  such that  $\sigma(\lambda) = \lambda'$ .

**4.1. The right adjoint  $\blacktriangleright$ .** Recall first that in the topos of trees the functor  $\blacktriangleright$  is defined as  $(\blacktriangleright F)(n+1) = Fn$  and  $(\blacktriangleright F)(0) = \{*\}$ . This generalises in a straightforward way to  $\text{GR}_\star$  by

$$(\blacktriangleright F)(\mathcal{E}; \delta; \lambda) = \begin{cases} F(\mathcal{E}; \delta[\lambda-]; \lambda) & \delta(\lambda) > 0 \\ \{*\} & \text{otherwise} \end{cases}$$

where  $\delta[\lambda-](\lambda) = \delta(\lambda) - 1$  and  $\delta[\lambda-](\lambda') = \delta(\lambda')$  for  $\lambda' \neq \lambda$ . The presheaf action of  $\blacktriangleright F$  is simply inherited from  $F$  by noticing that a map  $\sigma : (\mathcal{E}; \delta; \lambda) \rightarrow (\mathcal{E}'; \delta'; \lambda')$ , induces a map

$$\sigma^{[\lambda-]} : (\mathcal{E}; \delta[\lambda-]; \lambda) \rightarrow (\mathcal{E}'; \delta'[\lambda'-]; \lambda').$$

**Lemma 4.1.** *The functor  $\blacktriangleright : \text{GR}_\star \rightarrow \text{GR}_\star$  extends to families and elements.*

*Proof.* If  $A$  is a type over  $\Gamma$  and  $\gamma \in (\blacktriangleright \Gamma)(\mathcal{E}; \delta; \lambda)$  define

$$(\blacktriangleright_{\text{Fam}} A)_{(\mathcal{E}; \delta; \lambda)}(\gamma) = \begin{cases} \{*\} & \delta(\lambda) = 0 \\ A_{(\mathcal{E}; \delta[\lambda-]; \lambda)}(\gamma) & \text{otherwise} \end{cases}$$

To see that this commutes with comprehension, note that  $(\blacktriangleright \Gamma. \blacktriangleright_{\text{Fam}} A)_{(\mathcal{E}; \delta; \lambda)}$  equals  $\blacktriangleright(\Gamma.A)_{(\mathcal{E}; \delta; \lambda)}$  when  $\delta(\lambda) > 0$  and  $\{*\} \times \{*\}$  when  $\delta(\lambda) = 0$ . The definition for elements is similar.  $\square$

**Example 4.2.** As an example of a model of a type, recall the type of guarded streams satisfying  $\text{Str}^\kappa \equiv \mathbb{N} \times_{\triangleright^\kappa} \text{Str}^\kappa$  from Section 2. This type is definable in the clock context  $\kappa : \text{clock}$ , and so will be interpreted as a presheaf in  $\text{GR}_\star$  defined as  $\llbracket \text{Str}^\kappa \rrbracket(\mathcal{E}; \delta; \lambda) = \mathbb{N}^{\delta(\lambda)+1} \times \{*\}$ . We will assume that the products in this associate to the right, so that this is the type of tuples of the form  $(n_{\delta(\lambda)}, (\dots, (n_0, *))) \dots$ . This is needed to model the equality  $\text{Str}^\kappa \equiv \mathbb{N} \times_{\triangleright^\kappa} \text{Str}^\kappa$ , rather than just an isomorphism of types. Given a predicate  $x : \mathbb{N} \vdash P$  the lifting of  $P$  to streams  $\text{Str}_P^\kappa$ , described in Section 2.1 can be modelled as

$$\llbracket \text{Str}_P^\kappa \rrbracket_{(\mathcal{E}; \delta; \lambda)}(n_{\delta(\lambda)}, (\dots, (n_0, *))) \dots = \{(x_{\delta(\lambda)}, (\dots, (x_0, *))) \dots \mid \forall i. x_i \in \llbracket P \rrbracket_{(\mathcal{E}; \delta; \lambda)}(n_i)\}$$

It is a simple calculation (using the definitions below) that these interpretations model the type equalities mentioned above.

**4.2. The left adjoint  $\blacktriangleleft$ .** In the topos of trees, the functor  $\blacktriangleright$  defined above has a left adjoint  $\blacktriangleleft$  defined as  $(\blacktriangleleft F)n = F(n+1)$ . At first sight it would seem that one can similarly define a left adjoint  $\blacktriangleleft$  to  $\blacktriangleright$  on  $\text{GR}_\star$  by  $(\blacktriangleleft F)(\mathcal{E}; \delta; \lambda) = F(\mathcal{E}; \delta[\lambda+]; \lambda)$ , where  $\delta[\lambda+]$  is defined similarly to  $\delta[\lambda-]$ . Unfortunately,  $\blacktriangleleft F$  so described is not a presheaf because it has no well-defined action on maps since a map  $\tau : (\mathcal{E}; \delta; \lambda) \rightarrow (\mathcal{E}'; \delta'; \lambda')$  does not necessarily induce a map  $(\mathcal{E}; \delta[\lambda+]; \lambda) \rightarrow (\mathcal{E}'; \delta'[\lambda'+]; \lambda')$ : If  $\lambda'' \neq \lambda$  satisfies  $\tau(\lambda'') = \lambda'$  there is no guarantee that  $\delta'[\lambda'+](\tau(\lambda'')) \leq \delta[\lambda+](\lambda'')$ .

To define the left adjoint, we instead first give an abstract description of  $\blacktriangleright$ . Let  $\mathbb{T}_\star^\mathbb{Z}$  be the category defined as  $\mathbb{T}_\star$ , except that  $\delta$  in an object  $(\mathcal{E}; \delta; \lambda)$  is a map of type  $\mathcal{E} \rightarrow \mathbb{Z}$ , i.e., the values can be negative. There is an inclusion  $\phi : \mathbb{T}_\star \rightarrow \mathbb{T}_\star^\mathbb{Z}$  and we say that an object in  $\mathbb{T}_\star^\mathbb{Z}$  is *negative* if it is not in the image of this inclusion. Note that if  $\sigma : (\mathcal{E}; \delta; \lambda) \rightarrow (\mathcal{E}'; \delta'; \lambda')$

and  $(\mathcal{E}; \delta; \lambda)$  is negative, so is  $(\mathcal{E}'; \delta'; \lambda')$ . Recall that  $\phi$  induces a functor on presheaves  $\phi^* : \mathbf{Set}^{\mathbb{T}_*^{\mathbb{Z}}} \rightarrow \mathbf{Set}^{\mathbb{T}_*}$  by  $(\phi^* F)(\mathcal{E}; \delta; \lambda) = F(\phi(\mathcal{E}; \delta; \lambda))$ . The right adjoint

$$\phi_* : \mathbf{Set}^{\mathbb{T}_*} \rightarrow \mathbf{Set}^{\mathbb{T}_*^{\mathbb{Z}}}$$

to  $\phi^*$  can be defined as  $\phi_*(F)(\mathcal{E}; \delta; \lambda) = 1$  if  $(\mathcal{E}; \delta; \lambda)$  is negative and  $F(\mathcal{E}; \delta; \lambda)$  if not. There is a functor  $[\star-] : \mathbb{T}_*^{\mathbb{Z}} \rightarrow \mathbb{T}_*^{\mathbb{Z}}$  mapping  $(\mathcal{E}; \delta; \lambda)$  to  $(\mathcal{E}; \delta[\lambda-]; f)$ , where  $\delta[\lambda-]$  is defined as above. The functor  $\blacktriangleright$  can now be described as the composition

$$\mathbf{Set}^{\mathbb{T}_*} \xrightarrow{\phi_*} \mathbf{Set}^{\mathbb{T}_*^{\mathbb{Z}}} \xrightarrow{[\star-]^*} \mathbf{Set}^{\mathbb{T}_*^{\mathbb{Z}}} \xrightarrow{\phi^*} \mathbf{Set}^{\mathbb{T}_*}$$

Since each of the functors in this diagram has a left adjoint, so does  $\blacktriangleright$ . The left adjoint  $\blacktriangleleft$  is the composite (in the opposite order) of the left adjoints of the functors above, i.e.

$$\mathbf{Set}^{\mathbb{T}_*} \xrightarrow{\phi_!} \mathbf{Set}^{\mathbb{T}_*^{\mathbb{Z}}} \xrightarrow{[\star-]_!} \mathbf{Set}^{\mathbb{T}_*^{\mathbb{Z}}} \xrightarrow{\phi^*} \mathbf{Set}^{\mathbb{T}_*}$$

Unfolding the left Kan extensions used in the definitions of  $\phi_!$  and  $[\star-]_!$ , the left adjoint can be described concretely as follows. An element of  $(\blacktriangleleft F)(\mathcal{E}; \delta; \lambda)$  is an equivalence class of a pair of a map  $\sigma : (\mathcal{E}'; \delta'; \lambda') \rightarrow (\mathcal{E}; \delta; \lambda)$  such that  $\delta'(\lambda') > \delta(\lambda)$  and an element  $x \in F(\mathcal{E}'; \delta'; \lambda')$ , up to the equivalence relation generated by

$$(\sigma \circ \tau, x) \sim (\sigma, \tau \cdot x)$$

The presheaf action is defined by  $\tau \cdot [(\sigma, x)] = [(\tau\sigma, x)]$  and the functorial action of  $\blacktriangleleft$  is defined as  $\blacktriangleleft(f)[(\sigma, x)] = [(\sigma, f(x))]$ . We note the following.

**Lemma 4.3.** *The functor  $\blacktriangleleft : \mathbf{GR}_* \rightarrow \mathbf{GR}_*$  is left adjoint to  $\blacktriangleright$ .*

Since  $\blacktriangleright$  extends to families and elements (Lemma 4.1) by Lemma 1.5,  $\blacktriangleleft$  has a dependent right adjoint  $\blacktriangleright$  defined as  $\blacktriangleright A = (\blacktriangleright_{\mathbf{Fam}} A)[\eta]$ . There is, moreover, a projection  $\mathbf{p}_{\blacktriangleleft} : \blacktriangleleft \rightarrow \mathbf{id}$  mapping an element  $[(\sigma, x)]$  in  $\blacktriangleleft F(\mathcal{E}; \delta; \lambda)$  to  $\sigma \cdot x$ . Thus, by Proposition 1.6,  $\blacktriangleleft$  and  $\blacktriangleright$  provide a model for the tick calculus on  $\mathbf{GR}_*$ .

We note the following, which is needed for the soundness of the tick irrelevance axiom (2.2).

**Lemma 4.4.** *The projections  $\mathbf{p}_{\blacktriangleleft}, \blacktriangleleft(\mathbf{p}_{\blacktriangleleft}) : \blacktriangleleft \blacktriangleleft \rightarrow \blacktriangleleft$  are equal.*

*Proof.* This follows from a simple calculation:

$$\mathbf{p}_{\blacktriangleleft}([( \sigma, [(\tau, x)] )]) = \sigma \cdot [(\tau, x)] = [(\sigma\tau, x)]$$

and  $\blacktriangleleft(\mathbf{p}_{\blacktriangleleft})([( \sigma, [(\tau, x)] )]) = [(\sigma, \tau \cdot x)] = [(\sigma\tau, x)]$ .  $\square$

The natural transformation  $\mathbf{p}_{\blacktriangleleft} : \blacktriangleleft F \rightarrow F$  corresponds to a natural transformation  $\text{next} = \blacktriangleright(\mathbf{p}_{\blacktriangleleft}) \circ \eta : F \rightarrow \blacktriangleright F$ , which can be described as follows

$$\text{next}_{(\mathcal{E}; \delta; \lambda)}(\gamma) = \begin{cases} \text{tick} \cdot \gamma & \delta(\lambda) > 0 \\ * & \text{otherwise} \end{cases} \quad (4.1)$$

where

$$\text{tick} : (\mathcal{E}; \delta; \lambda) \rightarrow (\mathcal{E}; \delta[\lambda-]; \lambda) \quad (4.2)$$

is the map induced by the identity on  $\mathcal{E}$ .

## 5. MODELLING TICKS

We now show how to use the dependent right adjoint structure constructed in the previous section to model ticks in CloTT. First note that Clk induces a family  $\text{Clk}_\Gamma$  in any context  $\Gamma$  defined as  $(\text{Clk}_\Gamma)_{(\mathcal{E};\delta)}(\gamma) = \mathcal{E}$ , such that  $\text{Clk}_{\Gamma'}[\sigma] = \text{Clk}_\Gamma$  for any  $\sigma : \Gamma \rightarrow \Gamma'$ . The rule for extending contexts with ticks assumes a clock  $\Gamma \vdash \kappa : \text{clock}$ , which semantically corresponds to an element of  $\text{Clk}_\Gamma$ . The collection of pairs  $(\Gamma, \chi)$  where  $\Gamma$  is an object of  $\mathbf{GR}$  and  $\chi$  is an element of  $\text{Clk}_\Gamma$ , extends to a category where a morphism from  $(\Gamma, \chi)$  to  $(\Gamma', \chi')$  is a morphism  $\sigma : \Gamma \rightarrow \Gamma'$  in  $\mathbf{GR}$ , such that  $\chi = \chi'[\sigma]$ . This category is isomorphic to the slice category of  $\mathbf{GR}$  over Clk, which in turn is wellknown to be equivalent to the category  $\mathbf{GR}_*$ . The equivalence maps a pair  $(\Gamma, \chi)$  to the presheaf defined as

$$\Phi(\Gamma, \chi)(\mathcal{E}; \delta; \lambda) = \{\gamma \in \Gamma(\mathcal{E}; \delta) \mid \chi(\gamma) = \lambda\}$$

The opposite direction maps  $F$  to

$$\Psi(F)(\mathcal{E}; \delta) = \coprod_{\lambda \in \mathcal{E}} F(\mathcal{E}; \delta; \lambda)$$

and element given by the first projection.

Note, moreover, that there is an isomorphism of categories of elements

$$\int \Phi(\Gamma, \chi) \cong \int \Gamma \tag{5.1}$$

so families and elements over  $\Phi(\Gamma, \chi)$  in  $\mathbf{GR}_*$  correspond bijectively to families and elements over  $\Gamma$  in  $\mathbf{GR}$ . Since the isomorphism (5.1) is natural in  $(\Gamma, \chi)$  this bijective correspondence commutes with substitution. The dependent right adjoint constructed in the previous section therefore gives the following.

**Theorem 5.1.** *The following structure exists on  $\mathbf{GR}$ .*

- (1) *An operation mapping an object  $\Gamma$  and an element  $\chi$  of Clk over  $\Gamma$  to an object  $\blacktriangleleft^x \Gamma$ .*
- (2) *An operation mapping  $\Gamma, \chi$  as above and  $\sigma : \Gamma' \rightarrow \Gamma$  to*

$$\blacktriangleleft^x(\sigma) : \blacktriangleleft^{x[\sigma]} \Gamma' \rightarrow \blacktriangleleft^x \Gamma$$

*which is functorial, in the sense that  $\blacktriangleleft^x(\text{id}) = \text{id}$  and  $\blacktriangleleft^x(\sigma\tau) = \blacktriangleleft^x(\sigma) \circ \blacktriangleleft^{x[\sigma]}(\tau)$ .*

- (3) *A transformation, i.e., a family of maps  $\mathbf{p}_{\blacktriangleleft^x} : \blacktriangleleft^x \Gamma \rightarrow \Gamma$ , natural in the sense that  $\mathbf{p}_{\blacktriangleleft^x} \circ \blacktriangleleft^x \sigma = \sigma \circ \mathbf{p}_{\blacktriangleleft^{x[\sigma]}}$ .*
- (4) *An operation mapping families  $A$  over  $\blacktriangleleft^x \Gamma$  to families  $\blacktriangleright^x A$  over  $\Gamma$  satisfying  $(\blacktriangleright^x A)[\sigma] = \blacktriangleright^{x[\sigma]}(A[\blacktriangleleft^x(\sigma)])$ .*
- (5) *A bijection between elements of  $A$  and elements of  $\blacktriangleright^x A$ , mapping  $t$  to  $\bar{t}$  natural in the sense that  $\bar{t}[\blacktriangleleft^x \sigma] = \bar{t}[\sigma]$ .*

In explicit terms,  $\blacktriangleleft^x \Gamma$  is defined as the first component of  $\Psi(\blacktriangleleft \Phi(\Gamma, \chi))$ . By (5.1), then

$$\int \blacktriangleleft^x \Gamma \cong \int \Phi(\Psi(\blacktriangleleft(\Phi(\Gamma, \chi)))) \cong \int \blacktriangleleft(\Phi(\Gamma, \chi))$$

The operation on families in item (4) maps a family over  $\blacktriangleleft^x \Gamma$  to its corresponding family over  $\blacktriangleleft(\Phi(\Gamma, \chi))$ , then applies the dependent right adjoint operation on  $\mathbf{GR}_*$  to get a family over  $\Phi(\Gamma, \chi)$ , and finally applies (5.1) to get a family over  $\Gamma$ .

The structure of Theorem 5.1 is precisely what is required to model ticks and tick application in CloTT. Note that the operation  $\blacktriangleright$  extends to families and elements in such a way that

$$\blacktriangleright^x A = (\blacktriangleright_{\text{Fam}}^x A)[\eta] \quad (5.2)$$

where  $\eta : \Gamma \rightarrow \blacktriangleright^{x[\mathbf{p}_{\blacktriangleleft x}]} \blacktriangleleft^x \Gamma$  is the unit of the adjunction.

## 6. MODELLING GUARDED RECURSION

Guarded fixed points can be modelled essentially as in [BM20]. The aim of this section is to prove the following.

**Lemma 6.1.** *For each  $\Gamma \vdash t : \blacktriangleright^x(A[\mathbf{p}_{\blacktriangleleft x}]) \rightarrow A$  there is a unique  $\Gamma \vdash \text{dfix}_{\Gamma, A}^x(t) : \blacktriangleright^x(A[\mathbf{p}_{\blacktriangleleft x}])$  satisfying*

$$\text{dfix}_{\Gamma, A}^x(t) = \overline{\text{ev}(t, \text{dfix}_{\Gamma, A}^x(t))}[\mathbf{p}_{\blacktriangleleft x}]$$

Moreover,  $\text{dfix}_{\Gamma, A}^x(t)[\gamma] = \text{dfix}_{\Gamma', A[\gamma]}^{x[\gamma]}(t[\gamma])$  for any  $\gamma : \Gamma' \rightarrow \Gamma$ .

The lemma will be proved by proving the corresponding lemma in  $\text{GR}_*$ . First we unfold the definitions of the families involved. The next lemma refers to the map tick of (4.2).

**Lemma 6.2.** (1) *If  $A$  is a family over  $\Gamma$  in  $\text{GR}_*$ , then the family  $\blacktriangleright(A[\mathbf{p}_{\blacktriangleleft}])$  also over  $\Gamma$  can be described as*

$$\blacktriangleright(A[\mathbf{p}_{\blacktriangleleft}])_{(\mathcal{E}; \delta; \lambda)}(\gamma) = \begin{cases} \{*\} & \text{if } \delta(\lambda) = 0 \\ A_{(\mathcal{E}; \delta[\lambda-]; \lambda)}(\text{tick} \cdot \gamma) & \text{if } \delta(\lambda) = n + 1 \end{cases}$$

(2) *If  $A$  is a family over  $\Gamma$  and  $t$  is an element of  $A$ , then*

$$\overline{t[\mathbf{p}_{\blacktriangleleft x}]}_{(\mathcal{E}; \delta; \lambda)}(\gamma) = \begin{cases} * & \text{if } \delta(\lambda) = 0 \\ t_{(\mathcal{E}; \delta[\lambda-]; \lambda)}(\text{tick} \cdot \gamma) & \text{if } \delta(\lambda) = n + 1 \end{cases}$$

*Proof of Lemma 6.1.* We prove the corresponding statement in  $\text{GR}_*$ . The proof is essentially the same as in [BM20]. Unfolding the equation of the lemma gives

$$\begin{aligned} \text{dfix}_{\Gamma, A}^x(t)_{(\mathcal{E}; \delta; \lambda)}(\gamma) &= \text{ev}(t, \text{dfix}_{\Gamma, A}^x(t))_{(\mathcal{E}; \delta[\lambda-]; \lambda)}(\text{tick} \cdot \gamma) \\ &= t_{(\mathcal{E}; \delta[\lambda-]; \lambda)}(\text{tick} \cdot \gamma)(\text{id})(\text{dfix}_{\Gamma, A}^x(t)_{(\mathcal{E}; \delta[\lambda-]; \lambda)}(\text{tick} \cdot \gamma)) \end{aligned}$$

in the case where  $\delta(\lambda) > 0$  and  $*$  else. Thus  $\text{dfix}_{\Gamma, A}^x(t)_{(\mathcal{E}; \delta; f)}(\gamma)$  can be defined by induction on  $\delta(\lambda)$ . The last statement follows from uniqueness.  $\square$

## 7. MODELLING $\diamond$

To model  $\diamond$  we will construct a substitution  $d$  from the interpretation of any syntactic context of the form  $\Gamma, \kappa : \text{clock}$  to the interpretation of  $\Gamma, \kappa : \text{clock}, \alpha : \kappa$ , semantically substituting  $\diamond$  for  $\alpha$ . Omitting denotation brackets for  $\Gamma$ , the type of  $d$  is precisely

$$d : \Gamma.\text{Clk}_{\Gamma} \rightarrow \blacktriangleleft^q(\Gamma.\text{Clk}_{\Gamma})$$

Note that  $\mathbf{p}_{\blacktriangleleft^q}$  is a map in the opposite direction. We will show the following.

**Proposition 7.1.** *For any  $\Gamma$  in  $\text{GR}$ , the map  $\mathbf{p}_{\blacktriangleleft^q} : \blacktriangleleft^q(\Gamma.\text{Clk}_{\Gamma}) \rightarrow \Gamma.\text{Clk}_{\Gamma}$  is an isomorphism.*

*Proof.* Under the isomorphism of Section 5 the pair  $(\Gamma.\text{Clk}_\Gamma, \mathbf{q})$  corresponds to the object  $\Gamma_\star$  of  $\text{GR}_\star$  defined as  $\Gamma_\star(\mathcal{E}; \delta; \lambda) = \Gamma(\mathcal{E}; \delta)$ . Recall that an element of  $\blacktriangleleft(\Gamma_\star)(\mathcal{E}; \delta; \lambda)$  is an equivalence class represented by a pair  $(\sigma, x)$  where  $\sigma : (\mathcal{E}'; \delta'; \lambda') \rightarrow (\mathcal{E}; \delta; \lambda)$  such that  $\delta'(\lambda') > \delta(\lambda)$  and  $x \in \Gamma_\star(\mathcal{E}'; \delta'; \lambda')$ . The equivalence is the smallest equivalence relation relating  $(\sigma, \tau \cdot x)$  to  $(\sigma \circ \tau, x)$ . Recall also that  $\mathbf{p}_\blacktriangleleft([\!(\sigma, x)\!]) = \sigma \cdot x$ .

To construct an inverse  $d$  to  $\mathbf{p}_\blacktriangleleft$ , suppose now that  $x \in \Gamma_\star(\mathcal{E}; \delta; \lambda)$ , and let  $n = \delta(\lambda)$ . Let  $\lambda'$  be fresh, and consider the mapping  $\iota : (\mathcal{E}; \delta) \rightarrow ((\mathcal{E}, \lambda'); \delta[\lambda' \mapsto m])$  for some  $m > n$  given by the inclusion of  $\mathcal{E}$  into  $(\mathcal{E}, \lambda')$ . Define  $d(x) = [[(\lambda' \mapsto \lambda), \iota \cdot x)]$  where

$$[\lambda' \mapsto \lambda] : ((\mathcal{E}, \lambda'); \delta[\lambda' \mapsto m]; \lambda') \rightarrow (\mathcal{E}; \delta; \lambda)$$

maps  $\lambda'$  to  $\lambda$  and is the identity on all other input, and  $\iota \cdot x$  refers to the action of  $\iota$  on the object  $\Gamma$ . Clearly,  $d$  is independent of the choices of  $\lambda'$  and  $m$ . To show that  $d$  is an inverse to  $\mathbf{p}_\blacktriangleleft$ , note first that

$$\mathbf{p}_\blacktriangleleft(d(x)) = \mathbf{p}_\blacktriangleleft([\!(\lambda' \mapsto \lambda), \iota \cdot x)\!]) = [\lambda' \mapsto \lambda] \cdot \iota \cdot x = x$$

For the other direction, suppose  $[(\sigma, x)] \in \blacktriangleleft(\Gamma_\star)(\mathcal{E}; \delta; \lambda)$ , where  $\sigma : (\mathcal{E}'; \delta'; \lambda') \rightarrow (\mathcal{E}; \delta; \lambda)$ . Suppose  $\lambda''$  fresh for both  $\mathcal{E}$  and  $\mathcal{E}'$ , and let  $m$  be a number strictly greater than both  $\delta(\lambda)$  and  $\delta'(\lambda')$ . Consider the following commutative diagram in  $\mathbb{T}_\star$

$$\begin{array}{ccc} ((\mathcal{E}', \lambda''); \delta'[\lambda'' \mapsto m]; \lambda'') & \xrightarrow{[\lambda'' \mapsto \lambda']} & (\mathcal{E}'; \delta'; \lambda') \\ \sigma[\lambda'' \mapsto \lambda''] \downarrow & & \downarrow \sigma \\ ((\mathcal{E}, \lambda''); \delta[\lambda'' \mapsto m]; \lambda'') & \xrightarrow{[\lambda'' \mapsto \lambda]} & (\mathcal{E}; \delta; \lambda) \end{array}$$

and let  $\iota' : (\mathcal{E}'; \delta') \rightarrow ((\mathcal{E}', \lambda''); \delta'[\lambda'' \mapsto m])$  be given by the inclusion. Then

$$\begin{aligned} [(\sigma, x)] &= [(\sigma, [\lambda'' \mapsto \lambda'] \cdot \iota' \cdot x)] \\ &= [(\sigma \circ [\lambda'' \mapsto \lambda'], \iota' \cdot x)] \\ &= [([\lambda'' \mapsto \lambda] \circ \sigma[\lambda'' \mapsto \lambda''], \iota' \cdot x)] \\ &= [([\lambda'' \mapsto \lambda], \sigma[\lambda'' \mapsto \lambda''] \cdot \iota' \cdot x)] \\ &= [([\lambda'' \mapsto \lambda], \iota \cdot \sigma \cdot x)] \\ &= d(\mathbf{p}_\blacktriangleleft([\!(\sigma, x)\!])) \end{aligned} \quad \square$$

## 8. UNIVERSES

The universes  $U_\Delta$  of  $\text{CloTT}$  can be modelled by the semantic universes  $\mathcal{U}^\Delta$  constructed by Bizjak and Møgelberg [BM20]. This section recalls these and shows how to model the code  $\bar{\triangleright}(\alpha : \kappa).A$  which was not present in the language modelled by Bizjak and Møgelberg.

To model the universe  $U_\Delta$  we must have a  $\Delta$ -indexed set of semantic clocks in hand. Semantically, this assumption can be represented by the object  $\text{Clk}^\Delta$  of  $\text{GR}$  defined as  $\text{Clk}^\Delta(\mathcal{E}; \delta) = \mathcal{E}^\Delta$ , where the right hand side is to be understood as a set-theoretic exponent. We will therefore define  $\mathcal{U}^\Delta$  as a covariant presheaf over the category  $\int \text{Clk}^\Delta$  of elements of  $\text{Clk}^\Delta$ . Recall that the latter has as objects triples,  $(\mathcal{E}; \delta; f)$  where  $f : \Delta \rightarrow \mathcal{E}$ , and as morphisms  $(\mathcal{E}; \delta; f)$  to  $(\mathcal{E}'; \delta'; f')$  morphisms  $\sigma : (\mathcal{E}; \delta) \rightarrow (\mathcal{E}'; \delta')$  such that  $f' = \sigma \circ f$ . We

will write  $\text{GR}[\Delta]$  for the category of covariant presheaves on  $\int \text{Clk}^\Delta$ . Note that  $\text{GR} \cong \text{GR}[\emptyset]$  and  $\text{GR}_* \cong \text{GR}[\{\kappa\}]$ .

As is standard, the construction of the universe  $\mathcal{U}^\Delta$  assumes a set-theoretic universe, the inhabitants of which will be referred to as *small* sets. This notion lifts to notions of small families in  $\text{GR}$  and  $\text{GR}[\Delta]$  by requiring that all components be small. The definition of the universe  $\mathcal{U}^\Delta$  refers to the notion of invariance under clock introduction, whose definition we now recall.

**Definition 8.1.** A presheaf  $F$  in  $\text{GR}[\Delta]$  is *invariant under clock introduction* if, whenever  $\lambda \notin \mathcal{E}$ , the mapping  $\iota \cdot (-) : F(\mathcal{E}; \delta; f) \rightarrow F(\mathcal{E}, \lambda; \delta[\lambda \mapsto n]; \iota f)$ , induced by the inclusion of  $\mathcal{E}$  into  $\mathcal{E}, \lambda$ , is an isomorphism. A family  $A$  over a presheaf  $\Gamma$  is invariant under clock introduction if the mapping  $\iota \cdot (-) : A(\gamma) \rightarrow A(\iota \cdot \gamma)$  is an isomorphism for each  $\iota$  as above, and all  $\gamma$ . Note that  $A$  can be invariant under clock introduction also if  $\Gamma$  is not.

For a presheaf  $F$  in  $\text{GR}$  (considered as an object in  $\text{GR}[\emptyset]$ ) to be invariant under clock introduction is essentially equivalent to the mapping  $F \rightarrow F^{\text{Clk}}$  being an isomorphism [BM20]. So the condition captures the clock irrelevance axiom semantically. To model this axiom, it is therefore necessary that all types are interpreted as families invariant under clock introduction. It is not necessary that contexts are invariant under clock introduction, however, and they will not be, since the object  $\text{Clk}$  is not. The standard Hofmann-Streicher universe [HS99] in  $\text{GR}$  is not invariant under clock introduction, which is the semantic motivation for indexing universes by clock contexts, see [BM20] for details.

The universe  $\mathcal{U}^\Delta$  in  $\text{GR}[\Delta]$  is defined as follows. The component  $\mathcal{U}^\Delta(\mathcal{E}; \delta; f)$  is the set of small families over  $y(f[\Delta]; \delta|_{f[\Delta]})$  in  $\text{GR}$  invariant under clock introduction, where  $y$  is the yoneda embedding. In other words, an element of  $\mathcal{U}^\Delta(\mathcal{E}; \delta; f)$  is a family of small sets  $X_\tau$  indexed by morphisms  $\tau$  of time objects with domain  $(f[\Delta]; \delta|_{f[\Delta]})$  where  $f[\Delta]$  is the image of  $f$ , together with maps  $\sigma \cdot (-) : X_\tau \rightarrow X_{\sigma\tau}$  such that  $\text{id} \cdot x = x$  and  $\sigma' \cdot (\sigma \cdot x) = (\sigma' \circ \sigma) \cdot x$  for all  $x$ , and such that  $\iota \cdot (-)$  is an isomorphism, when  $\iota : (\mathcal{E}'; \delta) \rightarrow (\mathcal{E}', \lambda; \delta[\lambda \mapsto n])$  is given by the inclusion. The family of elements over  $\mathcal{U}^\Delta$  is defined as  $\mathcal{E}l^\Delta(X) = X_i$  where  $i : (f[\Delta]; \delta|_{f[\Delta]}) \rightarrow (\mathcal{E}; \delta)$  is the inclusion.

We now give a partial answer to the question of what the universes  $\mathcal{U}^\Delta$  classify.

**Lemma 8.2** ([BM20]). *Let  $A$  be a small family over an object  $\Gamma$  in  $\text{GR}[\Delta]$ . If both  $\Gamma$  and  $A$  are invariant under clock introduction, there is a unique  $\ulcorner A \urcorner : \Gamma \rightarrow \mathcal{U}^\Delta$  such that  $A = \mathcal{E}l^\Delta[\ulcorner A \urcorner]$ .*

The definition of the type operations on the universes rely on the property that all type operations preserve invariance under clock introduction.

**Lemma 8.3** ([BM20]). *The collection of families in  $\text{GR}[\Delta]$  invariant under clock introduction is closed under reindexing and under forming  $\Pi$ - and  $\Sigma$ -types as well as identity types. The objects  $\mathcal{U}^\Delta$  and families  $\mathcal{E}l^\Delta$  are invariant under clock introduction.*

To model the universes in  $\text{CloTT}$ , note that assumptions of the type formation rule for universes semantically corresponds to a presheaf  $\Gamma$  and a set  $\chi$  of elements of  $\text{Clk}_\Gamma$ . Suppose now that we are given some surjection  $\Delta \rightarrow \chi$  for some set  $\Delta$ . The reader could think of  $\Delta$  as the syntactic set of clocks in the syntactic universe  $\text{U}_\Delta$ , and the surjection as the function mapping a clock to its interpretation, but in fact the choice of  $\Delta$  does not matter. The surjection defines a map  $\langle \chi \rangle : \Gamma \rightarrow \text{Clk}^\Delta$ , and we define  $\mathcal{U}^\chi \stackrel{\text{def}}{=} \mathcal{U}^\Delta[\langle \chi \rangle]$  as a family over  $\Gamma$  and  $\mathcal{E}l^\chi \stackrel{\text{def}}{=} \mathcal{E}l^\Delta[\langle \langle \chi \rangle \circ \mathbf{p}, \mathbf{q} \rangle]$  as a family over  $\Gamma \cdot \mathcal{U}^\chi$ .

**Lemma 8.4** ([BM20]). *The object  $\mathcal{U}^\chi$  and the family  $\mathcal{E}l^\chi$  are welldefined in the sense that they are independent of the choice of  $\Delta$  and surjection  $\Delta \rightarrow \chi$ . Moreover, if  $\rho : \Gamma' \rightarrow \Gamma$  then  $\mathcal{U}^{\chi[\rho]} = \mathcal{U}^\chi[\rho]$  where  $\{\kappa_1, \dots, \kappa_n\}[\rho] = \{\kappa_1[\rho], \dots, \kappa_n[\rho]\}$ , for  $\rho : \Gamma' \rightarrow \Gamma$  and likewise for  $\mathcal{E}l$ .*

The second of these statements follows from the first, since given a surjection  $\Delta \rightarrow \chi$  the composite  $\Delta \rightarrow \chi \rightarrow \chi[\rho]$ , where the second map maps  $\kappa_i$  to  $\kappa_i[\rho]$ , is also a surjection inducing the map  $\langle \chi \rangle \circ \rho : \Gamma' \rightarrow \text{Clk}^\Delta$ . Therefore  $\mathcal{U}^{\chi[\rho]} = \mathcal{U}^\Delta[\langle \chi \rangle \circ \rho] = \mathcal{U}^\chi[\rho]$ .

In the next section we show how to model also  $\bar{\triangleright}(\alpha : \kappa).(-)$ , but first we recall the construction for universe inclusion.

Suppose  $\chi \subseteq \chi'$  and suppose that we are given a surjection  $\Delta' \rightarrow \chi'$ . Let  $\Delta \subseteq \Delta'$  be the preimage of  $\chi$ . There is a projection  $p_{\Delta, \Delta'} : \text{Clk}^{\Delta'} \rightarrow \text{Clk}^\Delta$  and by Lemma 8.3 the object  $\mathcal{U}^\Delta[p_{\Delta, \Delta'}]$  and the family  $\mathcal{E}l^\Delta[\langle p_{\Delta, \Delta'} \circ \mathfrak{p}, \mathfrak{q} \rangle]$  are invariant under clock introduction. Therefore, by Lemma 8.2, there is a unique map  $\text{in}_{\Delta, \Delta'} : \mathcal{U}^\Delta[p_{\Delta, \Delta'}] \rightarrow \mathcal{U}^{\Delta'}$  in  $\text{GR}[\Delta']$  such that  $\mathcal{E}l^\Delta[\langle p_{\Delta, \Delta'} \circ \mathfrak{p}, \mathfrak{q} \rangle] = \mathcal{E}l^{\Delta'}[\text{in}_{\Delta, \Delta'}]$ . Now let  $\langle \chi' \rangle : \Gamma \rightarrow \text{Clk}^{\Delta'}$  be the map induced by the surjection from  $\Delta'$  to  $\chi'$ , and  $\langle \chi \rangle$  the one induced by the surjection  $\Delta \rightarrow \chi$ . Then  $\langle \chi \rangle \stackrel{\text{def}}{=} p_{\Delta, \Delta'} \circ \langle \chi' \rangle : \Gamma \rightarrow \text{Clk}^\Delta$ . Reindexing  $\text{in}_{\Delta, \Delta'}$  along  $\langle \chi' \rangle$  gives a map  $\text{in}_{\chi, \chi'} : \mathcal{U}^\chi \rightarrow \mathcal{U}^{\chi'}$  in the category of covariant presheaves over  $\int \Gamma$  such that  $\mathcal{E}l^{\chi'}[\text{in}_{\chi, \chi'}] = \mathcal{E}l^\chi$ . Moreover, this map can be proved [BM20] independent of the choice of  $\Delta'$  and the surjection used to define  $\langle \chi' \rangle$ . If  $A$  is a family over  $\Gamma$  and  $\ulcorner A \urcorner$  is an element of  $\mathcal{U}^\chi$  such that  $A = \mathcal{E}l^\chi[\langle \text{id}_\Gamma, \ulcorner A \urcorner \rangle]$ , then  $\text{in}_{\chi, \chi'} \circ \ulcorner A \urcorner$  is an element of  $\mathcal{U}^{\chi'}$  and  $\mathcal{E}l^{\chi'}[\langle \text{id}_\Gamma, \text{in}_{\chi, \chi'} \circ \ulcorner A \urcorner \rangle] = \mathcal{E}l^\chi[\langle \text{id}_\Gamma, \ulcorner A \urcorner \rangle] = A$ . Universe inclusion can thus be modelled by postcomposition by  $\text{in}_{\chi, \chi'}$ .

**8.1. Modelling  $\bar{\triangleright}(\alpha : \kappa).(-)$ .** For this section let  $\Delta = \{\kappa_1, \dots, \kappa_n\}$ . Note first that the definition of the operation  $\blacktriangleright$  on  $\text{GR}_* \cong \text{GR}[\{\kappa\}]$  can be extended to define an operation on  $\text{GR}[\Delta]$  for every  $\kappa_i$ :

$$(\blacktriangleright^{\kappa_i} A)(\mathcal{E}; \delta; f) = \begin{cases} A(\mathcal{E}; \delta[f(\kappa_i)-]; f) & \delta(f(\kappa_i)) > 0 \\ \{*\} & \text{otherwise} \end{cases} \quad (8.1)$$

and that this extends to families and elements by similar constructions as for the case of  $\text{GR}_*$ . The functor  $\blacktriangleright^{\kappa_i}$  actually has a left adjoint defined similarly to  $\blacktriangleleft$ , but we shall not need that here.

One way of viewing definition (8.1) is that  $\kappa_i$  defines an element  $\hat{\kappa}_i$  of the family  $\text{Clk}$  over  $\text{Clk}^\Delta$ , and

$$(\blacktriangleright^{\kappa_i} A) = (\blacktriangleright_{\text{Fam}}^{\hat{\kappa}_i} A)[\text{next}^{\hat{\kappa}_i}] \quad (8.2)$$

where  $\text{next}^{\hat{\kappa}_i} : \text{Clk}^\Delta \rightarrow \blacktriangleright^{\hat{\kappa}_i}(\text{Clk}^\Delta)$  is the map of (4.1). Suppose now that  $\langle \chi \rangle : \Gamma \rightarrow \text{Clk}^\Delta$  is induced by some surjection  $\Delta \rightarrow \chi$  and let  $\chi_i$  be the image of  $\kappa_i$  under this surjection. If  $A$  is an object in  $\text{GR}[\Delta]$ , then by (8.2)

$$(\blacktriangleright^{\kappa_i} A)[\langle \chi \rangle] = \blacktriangleright_{\text{Fam}}^{\chi_i}(A[\langle \chi \rangle])[\text{next}^{\chi_i}] \quad (8.3)$$

Suppose now that  $B$  is a family over  $A$  in  $\text{GR}[\Delta]$ . Then  $\blacktriangleright_{\text{Fam}}^{\kappa_i} B$  defines a family over  $\text{Clk}^\Delta$ .  $\blacktriangleright^{\kappa_i} A$  in  $\text{GR}$ , and direct calculation verifies that

$$\blacktriangleright_{\text{Fam}}^{\kappa_i} B = (\blacktriangleright_{\text{Fam}}^{\hat{\kappa}_i[\mathfrak{p}]} B)[\langle \blacktriangleright_{\text{El}}^{\hat{\kappa}_i}(\mathfrak{p}), \blacktriangleright_{\text{El}}^{\hat{\kappa}_i}(\mathfrak{q}) \rangle^{-1} \circ \langle \text{next} \circ \mathfrak{p}, \mathfrak{q} \rangle] \quad (8.4)$$

Here the reindexing on the right is along the composite map

$$\text{Clk}^\Delta \cdot \blacktriangleright^{\kappa_i} A \xrightarrow{\langle \text{next}, \mathbf{p}, \mathbf{q} \rangle} \blacktriangleright^{\hat{\kappa}_i} \text{Clk}^\Delta \cdot \blacktriangleright^{\hat{\kappa}_i}_{\text{Fam}} A \xrightarrow{\langle \blacktriangleright^{\hat{\kappa}_i}(\mathbf{p}), \blacktriangleright^{\hat{\kappa}_i}_{\text{El}}(\mathbf{q}) \rangle^{-1}} \blacktriangleright^{\hat{\kappa}_i}[\mathbf{p}] (\text{Clk}^\Delta \cdot A)$$

the first of which is well-typed by (8.2). If  $\chi$  and  $\chi_i$  are as above then substituting both sides of (8.4) along  $\langle \langle \chi \rangle \circ \mathbf{p}, \mathbf{q} \rangle$  gives

$$(\blacktriangleright^{\kappa_i}_{\text{Fam}} B)[\langle \langle \chi \rangle \circ \mathbf{p}, \mathbf{q} \rangle] = (\blacktriangleright^{\chi_i[\mathbf{p}]}_{\text{Fam}} B[\langle \langle \chi \rangle \circ \mathbf{p}, \mathbf{q} \rangle])[(\blacktriangleright^{\chi_i}(\mathbf{p}), \blacktriangleright^{\chi_i}_{\text{El}}(\mathbf{q}))^{-1} \circ \langle \text{next} \circ \mathbf{p}, \mathbf{q} \rangle] \quad (8.5)$$

The next theorem gives the semantic structure needed to model  $\bar{\triangleright}(\alpha : \kappa).(-)$ .

**Theorem 8.5.** *Let  $\Gamma$  be an object of  $\text{GR}$ , let  $\chi$  be a set of elements of  $\text{Clk}_\Gamma$ , and let  $\chi_i \in \chi$ . There is a morphism  $\ulcorner \blacktriangleright_\chi^{\chi_i} \urcorner : \blacktriangleright^{\chi_i}(\mathcal{U}^{X[\mathbf{p}_{\blacktriangleleft \chi_i}]}) \rightarrow \mathcal{U}^X$  in the category of presheaves over the elements of  $\Gamma$  such that the following hold*

(1) *If  $A$  is a family over  $\blacktriangleleft^{\chi_i} \Gamma$  and  $\ulcorner A \urcorner$  is an element of  $\mathcal{U}^{X[\mathbf{p}_{\blacktriangleleft \chi_i}]}$  such that*

$$\mathcal{E}l^{X[\mathbf{p}_{\blacktriangleleft \chi_i}]}[(\text{id}_{\blacktriangleleft^{\chi_i} \Gamma}, \ulcorner A \urcorner)] = A,$$

*then  $\mathcal{E}l^X[(\text{id}_\Gamma, \ulcorner \blacktriangleright_\chi^{\chi_i} \urcorner \circ \overline{\ulcorner A \urcorner})] = \blacktriangleright^{\chi_i} A$ .*

(2) *If  $\sigma : \Gamma' \rightarrow \Gamma$  then  $\ulcorner \blacktriangleright_\chi^{\chi_i} \urcorner[\sigma] = \ulcorner \blacktriangleright_{\chi[\sigma]}^{\chi_i[\sigma]} \urcorner$ .*

(3) *If  $\chi \subseteq \chi'$  then*

$$\text{in}_{\chi, \chi'} \circ \ulcorner \blacktriangleright_\chi^{\chi_i} \urcorner \circ \overline{\ulcorner A \urcorner} = \ulcorner \blacktriangleright_{\chi'}^{\chi_i} \urcorner \circ \overline{(\text{in}_{(\chi[\mathbf{p}_{\blacktriangleleft \chi_i}]}, (\chi'[\mathbf{p}_{\blacktriangleleft \chi_i}])} \circ \ulcorner A \urcorner)}$$

The first of these items states that if  $\ulcorner A \urcorner$  is a code for  $A$ , then  $\ulcorner \blacktriangleright_\chi^{\chi_i} \urcorner \circ \overline{\ulcorner A \urcorner}$  is a code for  $\blacktriangleright^{\chi_i} A$ . The second one states that  $\ulcorner \blacktriangleright_\chi^{\chi_i} \urcorner$  is preserved by substitutions and the third that it commutes with universe inclusions. The second of these will be used in the proof of the substitution lemma (Lemma 9.1) and the other two in the proof of soundness (Theorem 9.5).

*Proof.* First note that if  $\Gamma$  is an object of  $\text{GR}[\Delta]$  and  $A$  is a small family over  $\Gamma$  such that both  $\Gamma$  and  $A$  are invariant under clock introduction, then also  $\blacktriangleright^{\kappa_i} \Gamma$  and the small family  $\blacktriangleright^{\kappa_i}_{\text{Fam}} A$  over  $\blacktriangleright^{\kappa_i} \Gamma$  are invariant under clock introduction. This is simply because the action of an inclusion  $\iota : (\mathcal{E}; \delta; f) \rightarrow (\mathcal{E}; \lambda; \delta[\lambda \mapsto n]; \iota f)$  on  $\blacktriangleright^{\kappa_i} \Gamma$  and  $\blacktriangleright^{\kappa_i}_{\text{Fam}} A$  is given by the action of  $\iota : (\mathcal{E}; \delta[f(\kappa_i)-]; f) \rightarrow (\mathcal{E}; \lambda; \delta[f(\kappa_i)-][\lambda \mapsto n]; \iota f)$  on  $\Gamma$  and  $A$ , in the case of  $f(\kappa_i) > 0$  and by the mapping  $\{*\} \rightarrow \{*\}$  when  $f(\kappa_i) = 0$ . In particular  $\blacktriangleright^{\kappa_i}(\mathcal{U}^\Delta)$  and  $\blacktriangleright^{\kappa_i}_{\text{Fam}}(\mathcal{E}l^\Delta)$  are both invariant under clock introduction, and therefore there is a unique morphism  $\ulcorner \blacktriangleright_\Delta^{\kappa_i} \urcorner : \blacktriangleright^{\kappa_i}(\mathcal{U}^\Delta) \rightarrow \mathcal{U}^\Delta$  such that  $\blacktriangleright^{\kappa_i}_{\text{Fam}}(\mathcal{E}l^\Delta) = \mathcal{E}l^\Delta[\ulcorner \blacktriangleright_\Delta^{\kappa_i} \urcorner]$ . Define  $\ulcorner \blacktriangleright_\chi^{\chi_i} \urcorner \stackrel{\text{def}}{=} \ulcorner \blacktriangleright_\Delta^{\kappa_i} \urcorner[\langle \chi \rangle]$ , which can be proved independent of the choice of  $\Delta$  and surjection inducing  $\langle \chi \rangle$  using the tools of [BM20]. To see that this has the right domain, note that by (8.3)

$$\begin{aligned} (\blacktriangleright^{\kappa_i}(\mathcal{U}^\Delta))[\langle \chi \rangle] &= (\blacktriangleright^{\chi_i}_{\text{Fam}} \mathcal{U}^X)[\text{next}^{\chi_i}] \\ &= (\blacktriangleright^{\chi_i}_{\text{Fam}} \mathcal{U}^X)[\blacktriangleright^{\chi_i}(\mathbf{p}_{\chi_i}) \circ \eta] \\ &= (\blacktriangleright^{\chi_i}_{\text{Fam}} \mathcal{U}^{X[\mathbf{p}_{\chi_i}]})[\eta] \\ &= \blacktriangleright^{\chi_i} \mathcal{U}^{X[\mathbf{p}_{\chi_i}]} \end{aligned}$$

using (5.2) in the last line. Item (1) can then be proved as follows. First note that

$$\begin{aligned}
\mathcal{E}l^X[\langle \text{id}_\Gamma, \ulcorner \triangleright_{\chi}^{\chi_i \neg} \circ \overline{\Gamma A^\neg} \rangle] &= \mathcal{E}l^\Delta[\langle \langle \chi \rangle \circ \mathbf{p}, \mathbf{q} \rangle \circ \langle \text{id}_\Gamma, \ulcorner \triangleright_{\chi}^{\chi_i \neg} \circ \mathbf{q} \rangle \circ \langle \text{id}_\Gamma, \overline{\Gamma A^\neg} \rangle] \\
&= \mathcal{E}l^\Delta[\langle \langle \chi \rangle \circ \mathbf{p}, \mathbf{q} \rangle \circ \langle \text{id}_\Gamma, \ulcorner \triangleright_{\Delta}^{\kappa_i \neg} [\langle \chi \rangle] \circ \mathbf{q} \rangle \circ \langle \text{id}_\Gamma, \overline{\Gamma A^\neg} \rangle] \\
&= \mathcal{E}l^\Delta[\langle \text{id}_\Gamma, \ulcorner \triangleright_{\Delta}^{\kappa_i \neg} \circ \mathbf{q} \rangle \circ \langle \langle \chi \rangle \circ \mathbf{p}, \mathbf{q} \rangle \circ \langle \text{id}_\Gamma, \overline{\Gamma A^\neg} \rangle] \\
&= \triangleright_{\text{Fam}}^{\kappa_i} (\mathcal{E}l^\Delta)[\langle \langle \chi \rangle \circ \mathbf{p}, \mathbf{q} \rangle \circ \langle \text{id}_\Gamma, \overline{\Gamma A^\neg} \rangle]
\end{aligned}$$

By (8.5)

$$\begin{aligned}
\triangleright_{\text{Fam}}^{\kappa_i} (\mathcal{E}l^\Delta)[\langle \langle \chi \rangle \circ \mathbf{p}, \mathbf{q} \rangle] &= (\triangleright_{\text{Fam}}^{\chi_i[\mathbf{p}]} \mathcal{E}l^\Delta[\langle \langle \chi \rangle \circ \mathbf{p}, \mathbf{q} \rangle])[(\triangleright^{\chi_i}(\mathbf{p}), \triangleright_{\text{El}}^{\chi_i}(\mathbf{q}))^{-1} \circ \langle \text{next} \circ \mathbf{p}, \mathbf{q} \rangle] \\
&= (\triangleright_{\text{Fam}}^{\chi_i[\mathbf{p}]} \mathcal{E}l^X)[\langle \triangleright^{\chi_i}(\mathbf{p}), \triangleright_{\text{El}}^{\chi_i}(\mathbf{q}))^{-1} \circ \langle \text{next} \circ \mathbf{p}, \mathbf{q} \rangle]
\end{aligned}$$

and so

$$\begin{aligned}
\mathcal{E}l^X[\langle \text{id}_\Gamma, \ulcorner \triangleright_{\chi}^{\chi_i \neg} \circ \overline{\Gamma A^\neg} \rangle] &= (\triangleright_{\text{Fam}}^{\chi_i[\mathbf{p}]} \mathcal{E}l^X)[\langle \triangleright^{\chi_i}(\mathbf{p}), \triangleright_{\text{El}}^{\chi_i}(\mathbf{q}))^{-1} \circ \langle \text{next} \circ \mathbf{p}, \mathbf{q} \rangle \circ \langle \text{id}_\Gamma, \overline{\Gamma A^\neg} \rangle] \\
&= (\triangleright_{\text{Fam}}^{\chi_i[\mathbf{p}]} \mathcal{E}l^X)[\langle \triangleright^{\chi_i}(\mathbf{p}), \triangleright_{\text{El}}^{\chi_i}(\mathbf{q}))^{-1} \circ \langle \text{next}, \overline{\Gamma A^\neg} \rangle]
\end{aligned}$$

Since  $\text{next} = \triangleright^{\chi_i}(\mathbf{p}_{\triangleleft \chi_i}) \circ \eta$  and  $\overline{\Gamma A^\neg} = \triangleright_{\text{El}}^{\chi_i}(\ulcorner \Gamma A^\neg \urcorner)[\eta]$  the above reduces to

$$\begin{aligned}
\mathcal{E}l^X[\langle \text{id}_\Gamma, \ulcorner \triangleright_{\chi}^{\chi_i \neg} \circ \overline{\Gamma A^\neg} \rangle] &= (\triangleright_{\text{Fam}}^{\chi_i[\mathbf{p}]} \mathcal{E}l^X)[\langle \triangleright^{\chi_i}(\mathbf{p}), \triangleright_{\text{El}}^{\chi_i}(\mathbf{q}))^{-1} \circ \langle \triangleright^{\chi_i}(\mathbf{p}_{\triangleleft \chi_i}), \triangleright_{\text{El}}^{\chi_i}(\ulcorner \Gamma A^\neg \urcorner) \rangle \circ \eta] \\
&= (\triangleright_{\text{Fam}}^{\chi_i[\mathbf{p}]} \mathcal{E}l^X)[\triangleright^{\chi_i}(\langle \mathbf{p}_{\triangleleft \chi_i}, \ulcorner \Gamma A^\neg \urcorner \rangle) \circ \eta] \\
&= (\triangleright_{\text{Fam}}^{\chi_i} (\mathcal{E}l^X[\langle \mathbf{p}_{\triangleleft \chi_i}, \ulcorner \Gamma A^\neg \urcorner \rangle]))[\eta] \\
&= (\triangleright_{\text{Fam}}^{\chi_i} (\mathcal{E}l^X[\mathbf{p}_{\triangleleft \chi_i}][\langle \text{id}, \ulcorner \Gamma A^\neg \urcorner \rangle]))[\eta] \\
&= (\triangleright_{\text{Fam}}^{\chi_i} (A))[\eta] \\
&= \triangleright^{\chi_i}(A)
\end{aligned}$$

For item (2) note that given a surjection  $\Delta \rightarrow \chi$ , the composite map  $\Delta \rightarrow \chi \rightarrow \chi[\sigma]$  where the last map maps  $\kappa$  to  $\kappa[\sigma]$  is also a surjection inducing the map  $\langle \chi \rangle \circ \sigma : \Gamma' \rightarrow \text{Clk}^\Delta$ . So  $\ulcorner \triangleright_{\chi[\sigma]}^{\chi_i[\sigma] \neg}$  can be defined as  $\ulcorner \triangleright_{\Delta}^{\kappa_i \neg} [\langle \chi \rangle \circ \sigma]$  which equals  $\ulcorner \triangleright_{\chi}^{\chi_i \neg} [\sigma]$ .

For item (3), first note that by (8.2)

$$(\triangleright^{\kappa_i} \mathcal{U}^\Delta)[p_{\Delta, \Delta'}] = (\triangleright_{\text{Fam}}^{\hat{\kappa}_i} \mathcal{U}^{p_{\Delta, \Delta'}})[\text{next}^{\hat{\kappa}_i}] = \triangleright^{\kappa_i}(\mathcal{U}^{p_{\Delta, \Delta'}})$$

The diagram

$$\begin{array}{ccc}
\triangleright^{\kappa_i}(\mathcal{U}^{p_{\Delta, \Delta'}}) & \xrightarrow{\ulcorner \triangleright_{\Delta}^{\kappa_i \neg} [p_{\Delta, \Delta'}] \urcorner} & \mathcal{U}^{p_{\Delta, \Delta'}} \\
\downarrow \triangleright^{\kappa_i}(\text{in}_{\Delta, \Delta'}) & & \downarrow \text{in}_{\Delta, \Delta'} \\
\triangleright^{\kappa_i}(\mathcal{U}^{\Delta'}) & \xrightarrow{\ulcorner \triangleright_{\Delta'}^{\kappa_i \neg} \urcorner} & \mathcal{U}^{\Delta'}
\end{array}$$

commutes by the uniqueness statement of Lemma 8.2 because both directions classify the family  $\triangleright_{\text{Fam}}^{\kappa_i}(\mathcal{E}l^{p_{\Delta, \Delta'}})$ . Using this, item (3) follows.  $\square$

### Typing rules

$$\begin{array}{c}
\frac{\Gamma, \kappa : \text{clock} \vdash t : A \quad \Gamma \vdash}{\Gamma \vdash \text{Lam}_{[\kappa]A}^{\forall}[\kappa]t : \forall \kappa. A} \qquad \frac{\Gamma \vdash t : \forall \kappa. A \quad \Gamma \vdash \kappa' : \text{clock}}{\Gamma \vdash \text{App}_{[\kappa]A}^{\forall}(t, \kappa') : A[\kappa'/\kappa]} \\
\\
\frac{\Gamma, \alpha : \kappa \vdash t : A}{\Gamma \vdash \text{Lam}_{[\alpha:\kappa]A}^{\triangleright}[\alpha]t : \triangleright(\alpha : \kappa).A} \qquad \frac{\Gamma \vdash t : \triangleright(\alpha : \kappa).A \quad \Gamma, \beta : \kappa, \Gamma' \vdash}{\Gamma, \beta : \kappa, \Gamma' \vdash \text{App}_{[\alpha:\kappa]A}^{\triangleright}(t, \beta) : A[\beta/\alpha]} \\
\\
\frac{\Gamma, \kappa : \text{clock} \vdash t : \triangleright(\alpha : \kappa).A \quad \Gamma \vdash \kappa' : \text{clock}}{\Gamma \vdash \text{App}_{[\kappa][\alpha]A}^{\diamond}([\kappa]t, \kappa') : A[(\diamond:\kappa')/(\alpha:\kappa)]} \qquad \frac{\Gamma \vdash t : \triangleright^{\kappa} A \rightarrow A}{\Gamma \vdash \text{dfix}_A^{\kappa} t : \triangleright^{\kappa} A}
\end{array}$$

### Tick constant substitution

$$\begin{aligned}
& (\text{Lam}_{[\kappa'']A}^{\forall}[\kappa'']t)[(\diamond:\kappa')/(\alpha:\kappa)] = \text{Lam}_{[\kappa'']A[(\diamond:\kappa')/(\alpha:\kappa)]}^{\forall}[\kappa'']t[(\diamond:\kappa')/(\alpha:\kappa)] \\
& \text{App}_{[\kappa'']A}^{\forall}(t, \kappa)[(\diamond:\kappa')/(\alpha:\kappa)] = \text{App}_{[\kappa'']A[(\diamond:\kappa')/(\alpha:\kappa)]}^{\forall}(t[(\diamond:\kappa')/(\alpha:\kappa)], \kappa') \\
& \text{App}_{[\kappa'']A}^{\forall}(t, \kappa''')[(\diamond:\kappa')/(\alpha:\kappa)] = \text{App}_{[\kappa'']A[(\diamond:\kappa')/(\alpha:\kappa)]}^{\forall}(t[(\diamond:\kappa')/(\alpha:\kappa)], \kappa''') \\
& (\text{Lam}_{[\beta:\kappa]A}^{\triangleright}[\beta]t)[(\diamond:\kappa')/(\alpha:\kappa)] = \text{Lam}_{[\beta:\kappa']A[(\diamond:\kappa')/(\alpha:\kappa)]}^{\triangleright}[\beta](t[(\diamond:\kappa')/(\alpha:\kappa)]) \\
& (\text{Lam}_{[\beta:\kappa'']A}^{\triangleright}[\beta]t)[(\diamond:\kappa')/(\alpha:\kappa)] = \text{Lam}_{[\beta:\kappa'']A[(\diamond:\kappa')/(\alpha:\kappa)]}^{\triangleright}[\beta](t[(\diamond:\kappa')/(\alpha:\kappa)]) \\
& \text{App}_{[\beta:\kappa]A}^{\triangleright}(t, \alpha)[(\diamond:\kappa')/(\alpha:\kappa)] = \text{App}_{[\beta][\beta]A}^{\diamond}([\kappa]t, \kappa') \\
& \text{App}_{[\beta:\kappa]A}^{\triangleright}(t, \alpha'')[(\diamond:\kappa')/(\alpha:\kappa)] = \text{App}_{[\beta:\kappa']A[(\diamond:\kappa')/(\alpha:\kappa)]}^{\triangleright}(t[(\diamond:\kappa')/(\alpha:\kappa)], \alpha'') \\
& \text{App}_{[\beta:\kappa'']A}^{\triangleright}(t, \alpha'')[(\diamond:\kappa')/(\alpha:\kappa)] = \text{App}_{[\beta:\kappa'']A[(\diamond:\kappa')/(\alpha:\kappa)]}^{\triangleright}(t[(\diamond:\kappa')/(\alpha:\kappa)], \alpha'') \\
& \text{App}_{[\kappa''][\beta]A}^{\diamond}([\kappa'']t, \kappa)[(\diamond:\kappa')/(\alpha:\kappa)] = \text{App}_{[\kappa''][\beta]A[(\diamond:\kappa')/(\alpha:\kappa)]}^{\diamond}([\kappa'']t[(\diamond:\kappa')/(\alpha:\kappa)], \kappa') \\
& \text{App}_{[\kappa''][\beta]A}^{\diamond}([\kappa'']t, \kappa''')[(\diamond:\kappa')/(\alpha:\kappa)] = \text{App}_{[\kappa''][\beta]A[(\diamond:\kappa')/(\alpha:\kappa)]}^{\diamond}([\kappa'']t[(\diamond:\kappa')/(\alpha:\kappa)], \kappa''')
\end{aligned}$$

Figure 3: Annotated typing rules for Clocked Type Theory.

## 9. INTERPRETATION OF SYNTAX

This section defines the interpretation of the syntax into the model. It is well known that to well define an interpretation of judgements (rather than derivations) of dependent type theory with the conversion rule, the syntax of the type theory must be annotated with typing information. For example, Hofmann [Hof97] annotates the application term for  $\Pi$  types with the type of the function being applied, writing  $\text{App}_{[x:A]B}(t, u)$  rather than the more common application term  $tu$ , in order to define an interpretation of syntax into a general CwF. Likewise lambda expressions are annotated not just with the type of the variable being abstracted, but also with the target type of the abstraction.

Following Hofmann [Hof97] we define an annotated syntax of pre-terms, pre-types and pre-contexts (allowing also judgements with no derivations), define an interpretation of this into the model as a partial function by structural induction, and finally prove that the interpretation of judgements with derivations are well-defined.

The typing rules for annotated terms are presented in Figure 3. We omit the standard cases such as  $\Pi$ - and  $\Sigma$ -types, as well as universes. Operations on universes should be annotated with the context index of the universe at which they are applied, for example  $\triangleright_{\Delta}(\alpha : \kappa).A$  contains the clocks context  $\Delta$ . The annotated syntax is a straight-forward

adaptation of the annotations used by Hofmann, except in the case of application to  $\diamond$ . The first thing to note is that the substitution of  $\kappa'$  for  $\kappa$  in the conclusion of the typing rule for  $\diamond$  has been replaced by an explicit substitution:  $\mathbf{App}_{[\kappa][\alpha]A}^{\diamond}([\kappa]t, \kappa')$  binds  $\kappa$  in  $t$  and applies it to  $\kappa'$ . This is not a completely faithful representation of the original syntax of Clocked Type Theory: In the original syntax, one might have different terms  $s$  and  $t$  such that  $t[\kappa'/\kappa] = s[\kappa'/\kappa]$  in which case the terms  $t[\kappa'/\kappa][\diamond]$  and  $s[\kappa'/\kappa][\diamond]$  are syntactically equal, but  $\mathbf{App}_{[\kappa][\alpha]A}^{\diamond}([\kappa]t, \kappa')$  and  $\mathbf{App}_{[\kappa][\alpha]A}^{\diamond}([\kappa]s, \kappa')$  are not. This is unfortunate, but seems unavoidable. See Section 10 for a discussion.

Note also that  $\mathbf{App}_{[\kappa][\alpha]A}^{\diamond}([\kappa]t, \kappa')$  binds both  $\alpha$  and  $\kappa$  in  $A$  (as indicated by the two sets of square brackets), unlike, e.g.,  $\mathbf{Lam}_{[\alpha:\kappa]A}^{\triangleright}[\alpha]t$  which only binds  $\alpha$ . The typing in the conclusion uses a special simultaneous substitution of  $\diamond$  for  $\alpha$  and  $\kappa'$  for  $\kappa$ . This is defined for terms in the figure; on types this simply distributes over the structure of types, except in universes, where it substitutes  $\kappa'$  for  $\kappa$ . The rules defining the substitution are subject to the usual side conditions avoiding capture of bound variables, but we omit these from the figure. In case multiple rules match, the top-most one should be applied. For example, the third rule only triggers when  $\kappa \neq \kappa'''$ .

The rules for judgemental equality can be formulated in the annotated syntax essentially as they are in the unannotated syntax. For example, the  $\beta$  rule for ticks in the case of application to  $\diamond$  is

$$\mathbf{App}_{[\kappa][\alpha]A}^{\diamond}([\kappa](\mathbf{Lam}_{[\alpha:\kappa]A}^{\triangleright}[\alpha]t), \kappa') = t[(\diamond:\kappa')/(\alpha:\kappa)] \quad (9.1)$$

The most important cases of the partial function defining the interpretation of syntax are given in Figure 4. Context, type, and term judgements are interpreted in the CwF structure of GR. The figure excludes the standard cases of  $\Pi$ - and  $\Sigma$ -types, extensional identity types, as well as universal quantification over clocks, which is modelled as a  $\Pi$ -type. The overlines in the interpretation of terms refer to the bijective correspondence of Theorem 5.1 and the projection  $\mathbf{p}_{\Gamma'} : \llbracket \Gamma, \alpha' : \kappa, \Gamma' \vdash \rrbracket \rightarrow \llbracket \Gamma, \alpha' : \kappa \vdash \rrbracket$  in the interpretation of tick application is defined by induction on  $\Gamma'$  in the obvious way.

**9.1. Substitution lemmas.** As is standard in models of dependent type theories [Hof97], welldefinedness of the interpretation function must be proved by induction on the structure of derivations, simultaneously with soundness and a substitution lemma, that we now describe.

Figure 5 lists the rules for wellformedness of syntactic substitutions as well as the definition of the partial function interpreting (pre-)substitutions. We define the substitution of types and terms along substitutions  $\sigma$  in the standard way using the clauses of Figure 3 in the cases involving  $\diamond$ .

**Lemma 9.1** (Substitution). *Let  $\sigma : \Gamma \rightarrow \Gamma'$  be a wellformed substitution then  $\llbracket \sigma \rrbracket$  is a welldefined morphism from  $\llbracket \Gamma \rrbracket$  to  $\llbracket \Gamma' \rrbracket$ . If  $\Gamma' \vdash A$  type then also  $\Gamma \vdash A\sigma$  type and  $\llbracket \Gamma \vdash A\sigma \text{ type} \rrbracket = \llbracket \Gamma' \vdash A \text{ type} \rrbracket[\llbracket \sigma \rrbracket]$ . If  $\Gamma' \vdash t : A$  then  $\Gamma \vdash t\sigma : A\sigma$  and  $\llbracket \Gamma \vdash t\sigma \rrbracket = \llbracket \Gamma' \vdash t \rrbracket[\llbracket \sigma \rrbracket]$ .*

Before proving Lemma 9.1 we need to establish a few facts about projections. In particular, we need Lemma 9.4 below for the case of variable introduction. Lemma 9.4 is trivial in the setting of standard type theory, but requires a little more work in a setting with ticks and tick weakening as in CloTT.

**Contexts**

$$\begin{aligned} \llbracket - \vdash \rrbracket &= 1 & \llbracket \Gamma, x : A \vdash \rrbracket &= \llbracket \Gamma \vdash \rrbracket . \llbracket \Gamma \vdash A \text{ type} \rrbracket \\ \llbracket \Gamma, \kappa : \text{clock} \vdash \rrbracket &= \llbracket \Gamma \vdash \rrbracket . \text{Clk} & \llbracket \Gamma, \alpha : \kappa \vdash \rrbracket &= \blacktriangleleft^{\llbracket \Gamma \vdash \kappa \rrbracket} \llbracket \Gamma \vdash \rrbracket \end{aligned}$$

**Types**

$$\begin{aligned} \llbracket \Gamma \vdash \triangleright (\alpha : \kappa) . A \text{ type} \rrbracket &= \blacktriangleright^{\llbracket \Gamma \vdash \kappa \rrbracket} \llbracket \Gamma, \alpha : \kappa \vdash A \text{ type} \rrbracket & \llbracket \Gamma \vdash U_{\Delta} \text{ type} \rrbracket &= \mathcal{U}^{\llbracket \Gamma \vdash \Delta \rrbracket} \\ \llbracket \Gamma \vdash \text{El}_{\Delta}(t) \text{ type} \rrbracket &= \mathcal{E}l^{\llbracket \Gamma \vdash \Delta \rrbracket}[\langle \text{id}, \llbracket \Gamma \vdash t \rrbracket \rangle] \end{aligned}$$

**Terms**

$$\begin{aligned} \llbracket \Gamma, x : A, \Gamma' \vdash x \rrbracket &= \mathbf{q}[\mathbf{p}_{\Gamma'}] \\ \llbracket \Gamma, \kappa : \text{clock}, \Gamma' \vdash \kappa \rrbracket &= \mathbf{q}[\mathbf{p}_{\Gamma'}] \\ \llbracket \Gamma \vdash \text{Lam}_{[\alpha:\kappa]A}^{\triangleright}[\alpha]t \rrbracket &= \overline{\llbracket \Gamma, \alpha : \kappa \vdash t \rrbracket} \\ \llbracket \Gamma, \alpha' : \kappa, \Gamma' \vdash \text{App}_{[\alpha:\kappa]A}^{\triangleright}(t, \alpha') \rrbracket &= \overline{\llbracket \Gamma \vdash t \rrbracket}[\mathbf{p}_{\Gamma'}] \\ \llbracket \Gamma \vdash \text{App}_{[\kappa]A}^{\diamond}([\kappa]t, \kappa') \rrbracket &= \overline{\llbracket \Gamma \vdash t \rrbracket}[d \circ \langle \text{id}_{\llbracket \Gamma \rrbracket}, \llbracket \Gamma \vdash \kappa' \rrbracket \rangle] \\ \llbracket \Gamma \vdash \text{dfix}_A^{\kappa} t \rrbracket &= \widehat{\text{dfix}}_{\llbracket \Gamma \rrbracket, [A]}^{\llbracket \Gamma \vdash \kappa \rrbracket}(\llbracket \Gamma \vdash t \rrbracket) \\ \llbracket \Gamma \vdash \overline{\triangleright}_{\Delta}(\alpha : \kappa) . A \rrbracket &= \ulcorner \blacktriangleright_{\llbracket \Gamma \vdash \Delta \rrbracket}^{\llbracket \Gamma \vdash \kappa \rrbracket} \urcorner \circ \overline{\llbracket \Gamma, \alpha : \kappa \vdash A \rrbracket} \\ \llbracket \Gamma \vdash \text{in}_{\Delta, \Delta'}(A) \rrbracket &= \text{in}_{\llbracket \Gamma \vdash \Delta \rrbracket, \llbracket \Gamma \vdash \Delta' \rrbracket} \circ \llbracket \Gamma \vdash A \rrbracket \end{aligned}$$

Figure 4: Interpretation function. The notation  $\llbracket \Gamma \vdash \Delta \rrbracket$  means  $\{\llbracket \Gamma \vdash \kappa_1 \rrbracket, \dots, \llbracket \Gamma \vdash \kappa_n \rrbracket\}$  where  $\Delta = \{\kappa_1, \dots, \kappa_n\}$ . The morphism  $d$  is the inverse of  $\mathbf{p}_{\blacktriangleleft}$  of Proposition 7.1.

**Formation rules**

$$\begin{array}{c} \frac{}{\llbracket \rrbracket : \Gamma \rightarrow \cdot} \quad \frac{\sigma : \Gamma \rightarrow \Gamma' \quad \Gamma \vdash t : A\sigma}{\sigma[x \mapsto t] : \Gamma \rightarrow \Gamma', x : A} \quad \frac{\sigma : \Gamma \rightarrow \Gamma' \quad \Gamma \vdash \kappa' : \text{clock}}{\sigma[\kappa \mapsto \kappa'] : \Gamma \rightarrow \Gamma', \kappa : \text{clock}} \\ \\ \frac{\sigma : \Gamma \rightarrow \Gamma'}{\sigma[\alpha \mapsto \beta] : \Gamma, \beta : \kappa\sigma, \Gamma'' \rightarrow \Gamma', \alpha : \kappa} \quad \frac{\sigma : \Gamma \rightarrow \Gamma' \quad \Gamma \vdash \kappa' : \text{clock}}{\sigma[(\alpha : \kappa) \mapsto (\diamond : \kappa')] : \Gamma \rightarrow \Gamma', \kappa : \text{clock}, \alpha : \kappa} \end{array}$$

**Interpretation**

$$\begin{aligned} \llbracket \llbracket \rrbracket \rrbracket &= ! & \llbracket \llbracket \sigma[x \mapsto t] \rrbracket \rrbracket &= \langle \llbracket \llbracket \sigma \rrbracket \rrbracket, \llbracket \llbracket t \rrbracket \rrbracket \rangle \\ \llbracket \llbracket \sigma[\kappa \mapsto \kappa'] \rrbracket \rrbracket &= \langle \llbracket \llbracket \sigma \rrbracket \rrbracket, \llbracket \llbracket \kappa' \rrbracket \rrbracket \rangle & \llbracket \llbracket \sigma[\alpha \mapsto \beta] \rrbracket \rrbracket &= \blacktriangleleft^{\llbracket \llbracket \kappa \rrbracket \rrbracket}(\llbracket \llbracket \sigma \rrbracket \rrbracket) \circ \mathbf{p}_{\Gamma''} \\ \llbracket \llbracket \sigma[(\alpha : \kappa) \mapsto (\diamond : \kappa')] \rrbracket \rrbracket &= d \circ \langle \llbracket \llbracket \sigma \rrbracket \rrbracket, \llbracket \llbracket \kappa' \rrbracket \rrbracket \rangle \end{aligned}$$

Figure 5: Rules for wellformedness of syntactic substitutions as well as their interpretation.

**Lemma 9.2.** *If  $\Gamma, \Gamma'$  is a wellformed context, the obvious context projection  $\text{wk}_{\Gamma, \Gamma'} : \Gamma, \Gamma' \rightarrow \Gamma$  is also wellformed, and  $\llbracket \text{wk}_{\Gamma, \Gamma'} \rrbracket = \mathbf{p}_{\Gamma'}$ .*

*Proof.* By induction on the height of  $\Gamma$ . If  $\Gamma$  is of length 0 then the statement is trivial. If  $\Gamma = \Gamma_0, x : A$ , then by induction the projection  $\Gamma_0, x : A, \Gamma' \rightarrow \Gamma_0$  is wellformed and interpreted as  $\mathbf{p}_{x:A, \Gamma'} = \mathbf{p} \circ \mathbf{p}_{\Gamma'}$ . Now, the projection  $\Gamma_0, x : A, \Gamma' \rightarrow \Gamma_0, x : A$  is interpreted

as  $\langle \mathbf{p} \circ \mathbf{p}_{\Gamma'}, \mathbf{q}[\mathbf{p}_{\Gamma'}] \rangle = \langle \mathbf{p}, \mathbf{q} \rangle \circ \mathbf{p}_{\Gamma'} = \mathbf{p}_{\Gamma'}$ . The case of extension of  $\Gamma$  with  $\kappa : \text{clock}$  is similar. In the case of  $\Gamma = \Gamma_0, \alpha : \kappa$ , by induction, the syntactic projection  $\Gamma_0 \rightarrow \Gamma_0$  is well defined and interpreted as the identity. Now, the projection  $\Gamma_0, \alpha : \kappa, \Gamma' \rightarrow \Gamma_0, \alpha : \kappa$  is by definition  $\llbracket \kappa \rrbracket(\text{id}) \circ \mathbf{p}_{\Gamma'} = \mathbf{p}_{\Gamma'}$ .  $\square$

If  $\sigma : \Gamma \rightarrow \Gamma'$  and  $\Gamma, \Gamma_0$  are well-formed we define the weakening of  $\sigma$  to be the substitution  $\sigma \circ \text{wk}_{\Gamma; \Gamma_0} : \Gamma, \Gamma_0 \rightarrow \Gamma'$  as follows

$$\begin{aligned} \llbracket \sigma \rrbracket \circ \text{wk}_{\Gamma; \Gamma_0} &= \llbracket \sigma \rrbracket \\ (\sigma[x \mapsto t]) \circ \text{wk}_{\Gamma; \Gamma_0} &= (\sigma \circ \text{wk}_{\Gamma; \Gamma_0})[x \mapsto t \text{wk}_{\Gamma; \Gamma_0}] \\ (\sigma[\kappa \mapsto \kappa']) \circ \text{wk}_{\Gamma; \Gamma_0} &= (\sigma \circ \text{wk}_{\Gamma; \Gamma_0})[\kappa \mapsto \kappa' \text{wk}_{\Gamma; \Gamma_0}] \\ (\sigma[\alpha \mapsto \beta]) \circ \text{wk}_{\Gamma; \Gamma_0} &= \sigma[\alpha \mapsto \beta] \\ \sigma[(\alpha : \kappa) \mapsto (\diamond : \kappa')] \circ \text{wk}_{\Gamma; \Gamma_0} &= (\sigma \circ \text{wk}_{\Gamma; \Gamma_0})[(\alpha : \kappa) \mapsto (\diamond : \kappa' \text{wk}_{\Gamma; \Gamma_0})] \end{aligned}$$

**Lemma 9.3.** *If  $\sigma : \Gamma \rightarrow \Gamma'$  and  $\Gamma, \Gamma_0$  are wellformed, so is  $\sigma \circ \text{wk}_{\Gamma; \Gamma_0}$  and  $\llbracket \sigma \circ \text{wk}_{\Gamma; \Gamma_0} \rrbracket = \llbracket \sigma \rrbracket \circ \mathbf{p}_{\Gamma_0}$*

*Proof.* An easy induction on  $\sigma$  using Lemma 9.1.  $\square$

The next lemma refers to the notion of prefix of a substitution, which is the reflexive-transitive closure of the following rules

$$\begin{array}{ll} \sigma \leq \sigma[x \mapsto t] & \sigma \leq \sigma[\kappa \mapsto \kappa'] \\ \sigma \circ \text{wk}_{\Gamma; \beta : \kappa \sigma, \Gamma''} \leq \sigma[\alpha \mapsto \beta] & \sigma[\kappa \mapsto \kappa'] \leq \sigma[(\alpha : \kappa) \mapsto (\diamond : \kappa')] \end{array}$$

In the case of  $\sigma[\alpha \mapsto \beta]$ , the weakening  $\text{wk}_{\Gamma; \beta : \kappa \sigma, \Gamma''}$  refers to the metavariables as in the formation rule for  $\sigma[\alpha \mapsto \beta]$ .

**Lemma 9.4.** *If  $\sigma : \Gamma \rightarrow \Gamma'_0, \Gamma'_1$  is a syntactic substitution, then there is a prefix  $\tau$  of  $\sigma$  such that  $\tau : \Gamma \rightarrow \Gamma'_0$  and  $\mathbf{p}_{\Gamma'_1} \circ \llbracket \sigma \rrbracket = \llbracket \tau \rrbracket$ .*

*Proof.* By induction on the length of  $\Gamma'_1$  simultaneously with the proof of Lemma 9.1. The case of  $\Gamma'_1$  empty is trivial. If  $\Gamma'_1 = x : A, \Gamma''_1$  then by induction, there is a prefix  $\tau$  of  $\sigma$  such that  $\llbracket \tau \rrbracket = \mathbf{p}_{\Gamma''_1} \circ \llbracket \sigma \rrbracket$ . By inspection,  $\tau$  must be of the form  $\tau'[x \mapsto t]$ , and

$$\mathbf{p}_{\Gamma'_1} \circ \llbracket \sigma \rrbracket = \mathbf{p} \circ \mathbf{p}_{\Gamma''_1} \circ \llbracket \sigma \rrbracket = \mathbf{p} \circ \langle \llbracket \tau' \rrbracket, \llbracket t \rrbracket \rangle = \llbracket \tau' \rrbracket$$

In the case that  $\Gamma'_1 = \alpha : \kappa, \Gamma''_1$ , again by induction, there is a prefix  $\tau$  of  $\sigma$  such that  $\llbracket \tau \rrbracket = \mathbf{p}_{\Gamma''_1} \circ \llbracket \sigma \rrbracket$ . Now, there are two cases for  $\tau$ . One is that  $\tau = \tau'[\alpha \mapsto \beta]$ . In this case  $\Gamma$  must be of the form  $\Gamma_0, \beta : \kappa[\sigma], \Gamma_1$  and

$$\begin{aligned} \mathbf{p}_{\Gamma'_1} \circ \llbracket \sigma \rrbracket &= \mathbf{p}_{\llbracket \kappa \rrbracket} \circ \mathbf{p}_{\Gamma''_1} \circ \llbracket \sigma \rrbracket \\ &= \mathbf{p}_{\llbracket \kappa \rrbracket} \circ \llbracket \llbracket \kappa \rrbracket \rrbracket(\llbracket \tau' \rrbracket) \circ \mathbf{p}_{\Gamma_1} \\ &= \llbracket \tau' \rrbracket \circ \mathbf{p}_{\llbracket \kappa \rrbracket} \circ \mathbf{p}_{\Gamma_1} \\ &= \llbracket \tau' \circ \text{wk}_{\Gamma_0; \beta : \kappa[\sigma], \Gamma_1} \rrbracket \end{aligned}$$

The other case for  $\tau$  is  $\tau = \tau'[(\alpha : \kappa) \mapsto (\diamond : \kappa')]$ . In this case  $\tau'[\kappa \mapsto \kappa']$  is a prefix of  $\sigma$  and

$$\begin{aligned} \mathbf{p}_{\Gamma'_1} \circ \llbracket \sigma \rrbracket &= \mathbf{p}_{\blacktriangleleft^{\llbracket \kappa \rrbracket}} \circ \mathbf{p}_{\Gamma''_1} \circ \llbracket \sigma \rrbracket \\ &= \mathbf{p}_{\blacktriangleleft^{\llbracket \kappa \rrbracket}} \circ d \circ \langle \llbracket \tau' \rrbracket, \llbracket \kappa' \rrbracket \rangle \\ &= \langle \llbracket \tau' \rrbracket, \llbracket \kappa' \rrbracket \rangle \\ &= \llbracket \tau'[\kappa \mapsto \kappa'] \rrbracket \end{aligned}$$

since  $d$  is the inverse of  $\mathbf{p}_{\blacktriangleleft^{\llbracket \kappa \rrbracket}}$ .  $\square$

*Proof of Lemma 9.1.* As in [Hof97] this is proved by induction on the sizes of the terms, contexts, and types involved, simultaneously with Theorem 9.5 and the three lemmas above. Most cases are standard, including the cases for universal quantification over clocks which is modelled here as a  $\Pi$ -type. We just explain the non-standard cases.

We start with the case of variable introduction. Suppose  $\Gamma' = \Gamma'_0, x : A, \Gamma'_1$ , and  $t = x$ . By Lemma 9.4, there must be a prefix  $\tau[x \mapsto u] : \Gamma \rightarrow \Gamma'_0, x : A$  of  $\sigma$  such that  $\mathbf{p}_{\Gamma'_1} \circ \llbracket \sigma \rrbracket = \llbracket \tau[x \mapsto u] \rrbracket$ . Now,

$$\llbracket \Gamma'_0, x : A, \Gamma'_1 \vdash x : A \rrbracket \llbracket \llbracket \sigma \rrbracket \rrbracket = \mathbf{q}[\mathbf{p}_{\Gamma'_1} \circ \llbracket \sigma \rrbracket] = \mathbf{q}[\llbracket \tau[x \mapsto u] \rrbracket] = \llbracket \Gamma \vdash u \rrbracket.$$

The case of clock introduction is similar.

The case of  $\triangleright(\alpha : \kappa).A$  is proved as follows:

$$\begin{aligned} \llbracket \Gamma \vdash \triangleright(\alpha : \kappa).A \rrbracket \llbracket \llbracket \sigma \rrbracket \rrbracket &= (\blacktriangleright^{\llbracket \kappa \rrbracket} \llbracket \Gamma, \alpha : \kappa \vdash A \rrbracket) \llbracket \llbracket \sigma \rrbracket \rrbracket \\ &= \blacktriangleright^{\llbracket \kappa \rrbracket \llbracket \llbracket \sigma \rrbracket \rrbracket} (\llbracket \Gamma, \alpha : \kappa \vdash A \rrbracket \llbracket \blacktriangleleft^{\llbracket \kappa \rrbracket} \llbracket \sigma \rrbracket \rrbracket \rrbracket) \\ &= \blacktriangleright^{\llbracket \kappa \sigma \rrbracket} (\llbracket \Gamma, \alpha : \kappa \vdash A \rrbracket \llbracket \blacktriangleleft^{\llbracket \kappa \rrbracket} \llbracket \sigma \rrbracket \rrbracket \rrbracket) \\ &= \blacktriangleright^{\llbracket \kappa \sigma \rrbracket} (\llbracket \Gamma, \alpha : \kappa \vdash A \rrbracket \llbracket \llbracket \sigma[\alpha \mapsto \alpha] \rrbracket \rrbracket) \\ &= \blacktriangleright^{\llbracket \kappa \sigma \rrbracket} \llbracket \Gamma', \alpha : \kappa \sigma \vdash A \sigma[\alpha \mapsto \alpha] \rrbracket \\ &= \llbracket \Gamma' \vdash \triangleright(\alpha : \kappa \sigma).(A \sigma[\alpha \mapsto \alpha]) \rrbracket \\ &= \llbracket \Gamma' \vdash (\triangleright(\alpha : \kappa).A) \sigma \rrbracket \end{aligned}$$

using Theorem 5.1.4 in the second equality. The proof of the case of  $\mathbf{Lam}_{[\alpha:\kappa]A}^{\triangleright}[\alpha]t$  is very similar, using Theorem 5.1.5.

Case of  $\Gamma'_0, \beta : \kappa, \Gamma'_1 \vdash \mathbf{App}_{[\alpha:\kappa]A}^{\triangleright}(t, \beta) : A[\beta/\alpha]$ : In this case the typing assumption is  $\Gamma \vdash t : \triangleright(\alpha : \kappa).A$  and

$$\llbracket \mathbf{App}_{[\alpha:\kappa]A}^{\triangleright}(t, \beta) \rrbracket \llbracket \llbracket \sigma \rrbracket \rrbracket = \overline{\llbracket t \rrbracket}[\mathbf{p}_{\Gamma'_1} \circ \llbracket \sigma \rrbracket] = \overline{\llbracket t \rrbracket}[\llbracket \tau \rrbracket]$$

for some prefix  $\tau : \Gamma \rightarrow \Gamma'_0, \alpha : \kappa$  of  $\sigma$  by Lemma 9.4. Now, there are two possible cases for  $\tau$ . The first and simplest case is  $\tau = \tau'[\beta \mapsto \beta']$ , where  $\Gamma = \Gamma_0, \beta' : \kappa \tau, \Gamma_1$  and  $\tau' : \Gamma_0 \rightarrow \Gamma'_0$ . In this case

$$\begin{aligned} \overline{\llbracket t \rrbracket}[\llbracket \tau \rrbracket] &= \overline{\llbracket t \rrbracket}[\blacktriangleleft^{\llbracket \kappa \rrbracket}(\llbracket \tau' \rrbracket) \circ \mathbf{p}_{\Gamma_1}] \\ &= \overline{\llbracket t \rrbracket}[\overline{\llbracket \tau' \rrbracket}][\mathbf{p}_{\Gamma_1}] \\ &= \overline{\llbracket t \tau' \rrbracket}[\mathbf{p}_{\Gamma_1}] \\ &= \llbracket \mathbf{App}_{[\alpha:\kappa \tau']_{A \tau'}}^{\triangleright}(t \tau', \beta') \rrbracket \\ &= \llbracket (\mathbf{App}_{[\alpha:\kappa]A}^{\triangleright}(t, \beta)) \sigma \rrbracket \end{aligned}$$

using Theorem 5.1.5 for the second equality and the induction hypothesis for the third.

The second case for  $\tau$  is  $\tau = \tau'[(\beta : \kappa) \mapsto (\diamond : \kappa')]$ . This requires  $\Gamma'_0 = \Gamma''$ ,  $\kappa : \text{clock}$  for some  $\Gamma''$ . In this case

$$\begin{aligned} \overline{[[t]]}[\tau] &= \overline{[[t]]}[d \circ \langle [[\tau']], [[\kappa']] \rangle] \\ &= \overline{[[t]]}[d \circ \langle [[\tau']] \circ \mathbf{p}, \mathbf{q} \rangle \circ \langle \text{id}, [[\kappa']] \rangle] \\ &= \overline{[[t]]}[\blacktriangleleft^{\mathbf{q}} \langle [[\tau']] \circ \mathbf{p}, \mathbf{q} \rangle \circ d \circ \langle \text{id}, [[\kappa']] \rangle] \\ &= \overline{[[t]]}[\langle [[\tau']] \circ \mathbf{p}, \mathbf{q} \rangle][d \circ \langle \text{id}, [[\kappa']] \rangle] \end{aligned}$$

using naturality of  $d$  (which follows from  $\mathbf{p} \blacktriangleleft^{\mathbf{q}}$  being natural) and Theorem 5.1.5. Now, by Lemma 9.3  $[[\tau']] \circ \mathbf{p} = [[\tau'] \circ \text{wk}_{\Gamma''; \kappa : \text{clock}}]$ , and so (not writing the weakening for simplicity)  $\langle [[\tau']] \circ \mathbf{p}, \mathbf{q} \rangle = [[\tau'[\kappa \mapsto \kappa]]]$ . Thus

$$\begin{aligned} \overline{[[t]]}[\tau] &= \overline{[[t\tau'[\kappa \mapsto \kappa]]]}[d \circ \langle \text{id}, [[\kappa']] \rangle] \\ &= \overline{[[\text{App}_{[\kappa][\alpha]}^{\diamond}(A\tau'[\kappa \mapsto \kappa][\alpha \mapsto \alpha])]}([\kappa](t\tau'[\kappa \mapsto \kappa]), \kappa')] \\ &= \overline{[[\text{App}_{[\alpha:\kappa]A}^{\diamond}(t, \beta)]\tau'[(\beta : \kappa) \mapsto (\diamond : \kappa')]]} \\ &= \overline{[[\text{App}_{[\alpha:\kappa]A}^{\diamond}(t, \beta)]\sigma]} \end{aligned}$$

Case of  $\text{App}_{[\kappa][\alpha]A}^{\diamond}([\kappa]t, \kappa')$ : In this case, the assumptions on the typing rule state that  $\Gamma, \kappa : \text{clock} \vdash t : \triangleright(\alpha : \kappa).A$  and  $\Gamma \vdash \kappa' : \text{clock}$ . The case is proved as follows

$$\begin{aligned} \overline{[[\text{App}_{[\kappa][\alpha]A}^{\diamond}([\kappa]t, \kappa')]]}[[\sigma]] &= \overline{[[t]]}[d \circ \langle \text{id}, [[\kappa']] \rangle \circ [[\sigma]]] \\ &= \overline{[[t]]}[d \circ \langle [[\sigma]] \circ \mathbf{p}, \mathbf{q} \rangle \circ \langle \text{id}, [[\kappa']] [[\sigma]] \rangle] \\ &= \overline{[[t]]}[\blacktriangleleft^{[[\kappa]]} \langle [[\sigma]] \circ \mathbf{p}, \mathbf{q} \rangle \circ d \circ \langle \text{id}, [[\kappa']\sigma] \rangle] \\ &= \overline{[[t]]}[\overline{[[\sigma[\kappa \mapsto \kappa]]]}][d \circ \langle \text{id}, [[\kappa']\sigma] \rangle] \\ &= \overline{[[t(\sigma[\kappa \mapsto \kappa])]]}[d \circ \langle \text{id}, [[\kappa']\sigma] \rangle] \\ &= \overline{[[\text{App}_{[\kappa][\alpha]A}^{\diamond}(\sigma[\kappa \mapsto \kappa][\alpha \mapsto \alpha])]}([\kappa](t(\sigma[\kappa \mapsto \kappa])), \kappa'\sigma)] \\ &= \overline{[[\text{App}_{[\kappa][\alpha]A}^{\diamond}([\kappa]t, \kappa')]\sigma]} \end{aligned}$$

The case of  $\text{dfix}_A^{\kappa} t$  follows from Lemma 6.1. The case of universes follows from Lemma 8.4 and the case of  $\overline{\triangleright}_{\Delta}(\alpha : \kappa).A$  follows from Theorem 8.5.2:

$$\begin{aligned} \overline{[\overline{\triangleright}_{\Delta}(\alpha : \kappa).A]}[[\sigma]] &= (\overline{\blacktriangleright}_{[\Delta]}^{[\kappa]} \overline{\triangleright} \circ \overline{[A]})[[\sigma]] \\ &= \overline{\blacktriangleright}_{[\Delta]}^{[\kappa]} \overline{\triangleright} [[\sigma]] \circ \overline{[A]} [[\sigma]] \\ &= \overline{\blacktriangleright}_{[\Delta]}^{[\kappa]} \overline{[[\sigma]]} \overline{\triangleright} \circ \overline{[A]} \overline{[[\sigma[\alpha \mapsto \alpha]]]} \\ &= \overline{\blacktriangleright}_{[\Delta]}^{[\kappa\sigma]} \overline{\triangleright} \circ \overline{[A\sigma[\alpha \mapsto \alpha]]} \\ &= \overline{[\overline{\triangleright}_{(\Delta\sigma)}(\alpha : \kappa\sigma).(A\sigma)]} \\ &= \overline{[\overline{\triangleright}_{\Delta}(\alpha : \kappa).A]\sigma} \end{aligned}$$

The cases of other codes on the universe follow from similar theorems found in [BM20].  $\square$

9.2. **Soundness.** The main theorem of the paper is the following.

**Theorem 9.5.** *If a judgement has a derivation, then the interpretation of it is well defined. If  $\Gamma \vdash t : A$  has a derivation then  $\llbracket \Gamma \vdash t \rrbracket$  is an element of the family  $\llbracket \Gamma \vdash A \text{ type} \rrbracket$ . Moreover, the interpretation is sound with respect to the equalities of Figure 1 and Figure 2 and models the axioms (2.1), (2.2) and (2.3).*

For the case of the clock irrelevance axiom we need the following lemma.

**Lemma 9.6.** *The interpretation of any type, if well-defined, is invariant under clock introduction in the sense of Definition 8.1.*

*Proof.* By an easy induction on the structure of types using Lemma 8.3 in most cases and arguments as in the proof of Lemma 8.5 in the case of  $\triangleright(\alpha : \kappa).A$ .  $\square$

*Proof of Theorem 9.5.* As mentioned above, the proof is by induction on typing judgements, and we just show the cases listed in Figure 3. The cases of tick abstraction and application follow straightforwardly from the bijection of Theorem 5.1.5. In the case of application, Lemma 9.1 and Lemma 9.2 ensure that the element  $\llbracket \Gamma \vdash t \rrbracket[\mathbf{p}_{\Gamma'}]$ , which a priori is an element of  $\llbracket \Gamma, \beta : \kappa \vdash A[\beta/\alpha] \text{ type} \rrbracket[\mathbf{p}_{\Gamma'}]$ , is also an element of  $\llbracket \Gamma, \beta : \kappa, \Gamma' \vdash A[\beta/\alpha] \text{ type} \rrbracket$  as required. In the case of application to  $\diamond$ , the expression  $\llbracket \Gamma \vdash t \rrbracket[d \circ \langle \text{id}_{\llbracket \Gamma \rrbracket}, \llbracket \kappa' \rrbracket \rangle]$  is by induction an element of  $\llbracket \Gamma, \kappa : \text{clock}, \alpha : \kappa \vdash A \text{ type} \rrbracket[d \circ \langle \text{id}_{\llbracket \Gamma \rrbracket}, \llbracket \kappa' \rrbracket \rangle]$ , which, since  $d \circ \langle \text{id}_{\llbracket \Gamma \rrbracket}, \llbracket \kappa' \rrbracket \rangle = \llbracket \text{id}_{\Gamma}[(\alpha, \kappa) \mapsto (\diamond, \kappa')] \rrbracket$ , by Lemma 9.1 equals  $\llbracket \Gamma \vdash A[(\diamond : \kappa')/(\alpha : \kappa)] \text{ type} \rrbracket$

Most equalities are straightforward. For example, the  $\eta$ -rules for clock and tick abstraction follow from the bijective correspondence on elements in dependent right adjoints. The  $\beta$ -rules for these are similar, but involve the substitution lemma in the case of clocks and a simple check that the interpretation is invariant under renaming of ticks in the case of ticks. The case of the  $\beta$ -rule for application to  $\diamond$ , i.e., equation (9.1), follows directly from Lemma 9.1. The judgemental equality for fixed point unfolding at  $\diamond$  follows from the fixed point unfolding axiom (2.3), since the model is extensional. To prove the latter it suffices to show that the interpretations of  $\lambda(\alpha : \kappa).t(\text{dfix}^{\kappa}t)$  and  $\text{dfix}^{\kappa}t$  are equal, which follows from Lemma 6.1. For the interpretation of the tick irrelevance axiom (2.2) it suffices to show that the interpretations of  $\Gamma, \alpha : \kappa, \alpha' : \kappa \vdash t[\alpha] : A$  and  $\Gamma, \alpha : \kappa, \alpha' : \kappa \vdash t[\alpha'] : A$  are equal. By definition  $\llbracket t[\alpha] \rrbracket = \llbracket t \rrbracket[\mathbf{p}_{\blacktriangleleft[\kappa]}]$  and by the substitution lemma  $\llbracket t[\alpha'] \rrbracket = \llbracket t \rrbracket[\mathbf{p}_{\blacktriangleleft[\kappa]}]$  which equals  $\llbracket t \rrbracket[\blacktriangleleft[\kappa](\mathbf{p}_{\blacktriangleleft[\kappa]})]$ , and so the equality follows from Lemma 4.4. The soundness of the clock irrelevance axiom (2.1) follows from Lemma 9.6 as in [BM20].

The equalities in Figure 2 were proved sound in [BM20], except the two involving  $\bar{\triangleright}(\alpha : \kappa).(-)$ . For the first of these, note that by definition

$$\begin{aligned} \llbracket \Gamma, \alpha : \kappa \vdash \text{El}_{\Delta}(A) \rrbracket &= \mathcal{E}l^{\llbracket \Gamma, \alpha : \kappa \vdash \Delta \rrbracket}[\langle \text{id}, \llbracket \Gamma, \alpha : \kappa \vdash A \rrbracket \rangle] \\ &\quad \mathcal{E}l^{\llbracket \Gamma \vdash \Delta \rrbracket[\mathbf{p}_{\blacktriangleleft[\kappa]}]}[\langle \text{id}, \llbracket \Gamma, \alpha : \kappa \vdash A \rrbracket \rangle] \end{aligned}$$

and so by Theorem 8.5.1 we get

$$\mathcal{E}l^{\llbracket \Gamma \vdash \Delta \rrbracket}[\langle \text{id}, \blacktriangleright_{\llbracket \Gamma \vdash \Delta \rrbracket}^{\llbracket \Gamma \vdash \kappa \rrbracket} \blacktriangleright \overline{\llbracket \Gamma, \alpha : \kappa \vdash A \rrbracket} \rangle] = \blacktriangleright^{\llbracket \Gamma \vdash \kappa \rrbracket} \llbracket \Gamma, \alpha : \kappa \vdash \text{El}_{\Delta}(A) \rrbracket$$

From this we deduce

$$\llbracket \text{El}_{\Delta}(\bar{\triangleright}_{\Delta}(\alpha : \kappa).A) \rrbracket = \llbracket \triangleright(\alpha : \kappa). \text{El}_{\Delta}(A) \rrbracket$$

by unfolding definitions.

The second of these can be proved using Theorem 8.5.3 as we now show.

$$\begin{aligned}
\llbracket \text{in}_{\Delta, \Delta'}(\overline{\text{p}}_{\Delta}(\alpha : \kappa).A) \rrbracket &= \text{in}_{\llbracket \Delta \rrbracket, \llbracket \Delta' \rrbracket} \circ \ulcorner \blacktriangleright_{\llbracket \Delta \rrbracket}^{\llbracket \kappa \rrbracket} \urcorner \circ \llbracket A \rrbracket \\
&= \ulcorner \blacktriangleright_{\llbracket \Delta' \rrbracket}^{\llbracket \kappa \rrbracket} \urcorner \circ \overline{(\text{in}_{(\llbracket \Delta \rrbracket[\mathbf{p}_{\blacktriangleright \llbracket \kappa \rrbracket}], \llbracket \Delta' \rrbracket[\mathbf{p}_{\blacktriangleright \llbracket \kappa \rrbracket}])} \circ \llbracket A \rrbracket)} \\
&= \ulcorner \blacktriangleright_{\llbracket \Delta' \rrbracket}^{\llbracket \kappa \rrbracket} \urcorner \circ \overline{(\text{in}_{\llbracket \Delta \rrbracket, \llbracket \Delta' \rrbracket} \circ \llbracket A \rrbracket)} \\
&= \llbracket \overline{\text{p}}_{\Delta'}(\alpha : \kappa). \text{in}_{\Delta, \Delta'}(A) \rrbracket
\end{aligned}$$

where the third equality uses  $\llbracket \Gamma \vdash \Delta \rrbracket[\mathbf{p}_{\blacktriangleright \llbracket \kappa \rrbracket}] = \llbracket \Gamma, \alpha : \kappa \vdash \Delta \rrbracket$  which follows from Lemma 9.1.  $\square$

## 10. CONCLUSION AND FUTURE WORK

As mentioned in the introduction, the model presented here improves on the previous model of CloTT [MM18] not only by fixing a mistake, but also by greatly simplifying it. The simplification is made possible by using a single-context presentation of the syntax and an internal indexing of the dependent right adjoints over the clock object.

The typing rule for application to  $\diamond$  as presented in [BGM17] and in Section 2 differs from the one of the elaborated syntax interpreted in the model, as presented in Section 9 by replacing a substitution by a delayed one. This means that for terms  $t, s$  such that  $t[\kappa'/\kappa] = s[\kappa'/\kappa]$  the application of  $t[\kappa'/\kappa]$  and  $s[\kappa'/\kappa]$  to  $\diamond$  in the original syntax are literally the same, whereas they are different in the elaborate syntax, and could potentially be different in the model. This means that the interpretation of a term  $t[\kappa'/\kappa][\diamond]$  involves a choice of  $s$  or  $t$  as above. We view this choice as part of the elaboration of terms, similarly to the choice of  $\Pi$  type for application terms  $t u$ .

We suspect that in most cases the choice mentioned above will not affect the interpretation of a term, but our attempts at proving that have failed. In particular, in the situation above, one can construct a term  $u$  such that  $u[\kappa_0, \kappa_1/\kappa, \kappa'] = t$  and  $u[\kappa_1, \kappa_0/\kappa, \kappa'] = s$  and use this to prove that the interpretations are equal. However, it is unclear if  $u$  is welltyped, and so the interpretation could fail to be welldefined.

The dependent right adjoints considered in this paper are all endo-adjunctions, i.e., the domain and codomain of the left adjoint are the same. Multimodal Dependent Type Theory [GKNB20] is a recent extension of the idea of Fitch-style modal types allowing also dependent adjunctions between different categories, and even 2-dimensional diagrams of these, referred to as *mode theories*. This is a very general framework capturing many dependent type theories with modal operators, but since the parametrisation by mode theory is given externally, we suspect that the single context presentation of CloTT falls outside this framework. It would be interesting to see if there is a generalisation of Multimodal Dependent Type Theory that captures also single context CloTT.

Our motivation for constructing this model is to study extensions of CloTT. In particular, we would like to extend CloTT with path types as in [BBC<sup>+</sup>19]. This requires an adaptation of the model to the cubical setting, using  $\mathbb{T}$ -indexed families of cubical sets [CCHM18] rather than just sets. The resulting variant of CloTT should be closer to the intensional type theory presented in [BGM17] than the extensional type theory modelled here. For example, fixed points should unfold only up to path equality. We also hope that formulating the clock irrelevance axiom using path equality the universes can be modelled differently, simplifying the perhaps most technical part of the present model construction.

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