

CORECURSIVE ALGEBRAS, CORECURSIVE MONADS AND BLOOM MONADS

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ABSTRACT. An algebra is called corecursive if from every coalgebra a unique coalgebra-to-algebra homomorphism exists into it. We prove that free corecursive algebras are obtained as coproducts of the terminal coalgebra (considered as an algebra) and free algebras. The monad of free corecursive algebras is proved to be the free corecursive monad, where the concept of corecursive monad is a generalization of Elgot’s iterative monads, analogous to corecursive algebras generalizing completely iterative algebras. We also characterize the Eilenberg-Moore algebras for the free corecursive monad and call them Bloom algebras.

1. INTRODUCTION

The study of structured recursive definitions is fundamental in many areas of computer science. This study can use algebraic methods extended by suitable recursion concepts. One such example are completely iterative algebras: these are algebras in which recursive equations with parameters have unique solutions, see [28]. In the present paper we study corecursive algebras. These are H -algebras for a given endofunctor H in which recursive equations without parameters have unique solutions. Equivalently, for every coalgebra there exists a unique coalgebra-to-algebra morphism. The dual concept, recursive coalgebra, was introduced by G. Osius in [29], and for endofunctors weakly preserving pullbacks P. Taylor proved that this is equivalent to being parametrically recursive, see [32]. In the dual situation, since weak preservation of pushouts is rare, the concepts of corecursive algebra and completely iterative algebra usually do not coincide. The former was studied by V. Capretta, T. Uustalu and V. Vene [21], and various counter-examples demonstrating e.g. the difference of the two concepts for algebras can be found there. In the present paper we contribute to the development of the mathematical theory of corecursive algebras. The goal

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is to eventually arrive at a useful body of results and constructions for these algebras. A major ingredient of any theory of algebraic structures is the study of how to freely endow an object with the structure of interest. So the main focus of the present paper are corecursive H -algebras freely generated by an object Y . Let FY denote the free H -algebra on Y and let T be the terminal H -coalgebra (which, due to Lambek's Lemma, can be regarded as an algebra). We prove that the coproduct of these two algebras

$$MY = T \oplus FY$$

is the free corecursive algebra on Y . Here \oplus is the coproduct in the category of H -algebras. For example for the endofunctor $HX = X \times X$ the algebra MY consists of all (finite and infinite) binary trees with finitely many leaves labelled in Y .

We also introduce the concept of a corecursive monad. This is a weakening of completely iterative monads of C. Elgot, S. Bloom and R. Tindell [23] analogous to corecursive algebras as a weakening of completely iterative ones. The monad $Y \mapsto MY$ of free corecursive algebras is proved to be corecursive, indeed, this is the free corecursive monad generated by H . For endofunctors of **Set** we also prove the converse: whenever H generates a free corecursive monad, then it has free corecursive algebras (and the free monad is then given by the corresponding adjunction).

We characterize the Eilenberg-Moore algebras for the free corecursive monad: these are H -algebras in which every recursive equation without parameters has a solution (not necessarily unique), and which allow a functorial choice of solutions. We call these algebras *Bloom algebras*; they are analogous to the complete Elgot algebras of [9] where the corresponding monad was the free completely iterative monad on H .

We also study the finitary versions of our concepts. An algebra A is called finitary corecursive if all coalgebras on finitely presentable objects have a unique coalgebra-to-algebra morphism in to A . And finitary corecursive monads are defined analogously. Every finitary endofunctor H is proved to generate a free finitary corecursive monad \mathbb{M}_H . We form the free strict functor $H_\perp = H + 1$ on H and obtain a monad \mathcal{M}^* on the category of finitary functors given by

$$\mathcal{M}^*(H) = \mathbb{M}_{H_\perp}.$$

The Eilenberg-Moore algebras for \mathcal{M}^* are called Bloom monads. They correspond to iteration theories of Bloom and Ésik: recall from [11] that the latter are precisely the Eilenberg-Moore algebras for the free-iteration-theory monad. Bloom monads are monads \mathbb{S} equipped with an operation \dagger assigning to every finitary non-parametric equation morphism a solution in free algebras for \mathbb{S} . This operation satisfies precisely the equational properties that non-parametric iteration in Domain Theory satisfies. We list some of those equational properties. The question whether our list is complete is open.

This paper is a revised and extended version of the conference paper [4]. Here we added all technical details and proofs, and our discussion of the equational properties of Bloom monads and their properties in new.

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2. CORECURSIVE ALGEBRAS

The following definition is the dual of the concept introduced by G. Osius in [29] and studied by P. Taylor [32, 31]. We assume throughout the paper that a category \mathcal{A} and an endofunctor $H : \mathcal{A} \rightarrow \mathcal{A}$ are given. We denote by $\mathbf{Alg} H$ the category of algebras $a : HA \rightarrow A$ and homomorphisms, and by $\mathbf{Coalg} H$ the category of coalgebras $e : X \rightarrow HX$ and homomorphisms. A coalgebra-to-algebra morphism from the latter to the former is a morphism $f : X \rightarrow A$ such that $f = a \cdot Hf \cdot e$.

Definition 2.1. An algebra $a : HA \rightarrow A$ is called *corecursive* if for every coalgebra $e : X \rightarrow HX$ there exists a unique coalgebra-to-algebra homomorphism $e^\dagger : X \rightarrow A$. That is, the square

$$\begin{array}{ccc} X & \xrightarrow{e^\dagger} & A \\ e \downarrow & & \uparrow a \\ HX & \xrightarrow{He^\dagger} & HA \end{array} \quad (2.1)$$

commutes. We call e an *equation morphism* and e^\dagger its *solution*.

Remark 2.2. For an endofunctor on \mathbf{Set} , we can view $e : X \rightarrow HX$ as a system of recursive equations using variables from the set X , and $e^\dagger : X \rightarrow A$ is the solution of the system. We illustrate this on classical Σ -algebras. These are the algebras for the polynomial set functor

$$H_\Sigma X = \coprod_{\sigma \in \Sigma} X^n$$

where n is the arity of σ . For every set X (of recursion variables) and every system of mutually recursive equations

$$x = \sigma(x_1, \dots, x_n),$$

one for every $x \in X$, where $\sigma \in \Sigma$ has arity n and $x_i \in X$, we get the corresponding coalgebra

$$e : X \longrightarrow H_\Sigma X; \quad x \mapsto (x_1, \dots, x_n) \quad \text{in the } \sigma\text{-summand } X^n.$$

The square (2.1) tells us that the substitution of $e^\dagger(x)$ for $x \in X$ makes the formal equations $x = \sigma(x_1, \dots, x_n)$ identities in A :

$$e^\dagger(x) = \sigma^A(e^\dagger(x_1), \dots, e^\dagger(x_n)).$$

Example 2.3.

- (1) In [21] this concept of corecursive algebras is studied and compared with a number of related concepts. A concrete example of a corecursive algebra from that paper, for the endofunctor $HX = E \times X \times X$ on \mathbf{Set} , is the set E^∞ of all streams. The operation $a : E \times E^\infty \times E^\infty \rightarrow E^\infty$ is given by $a(e, u, v)$ having head e and continuing by the merge of u and v .
- (2) If H has a terminal coalgebra $\tau : T \rightarrow HT$, then by Lambek's Lemma τ is invertible and the resulting algebra $\tau^{-1} : HT \rightarrow T$ is corecursive. In fact, this is the initial corecursive algebra, that is, for every corecursive algebra (A, a) a unique algebra homomorphism from (T, τ^{-1}) exists, see the dual of [21, Proposition 2]. There also the converse is proved (dual of Proposition 7), that is, if the initial corecursive algebra exists, then it is a terminal coalgebra (via the inverse of the algebra structure).

- (3) The trivial terminal algebra $H1 \rightarrow 1$, where 1 is the terminal object in \mathcal{A} , is clearly corecursive.
- (4) If $a : HA \rightarrow A$ is a corecursive algebra, then so is $Ha : HHA \rightarrow HA$, see [21, Proposition 6]. We generalize this in Lemma 2.5 below.
- (5) Combining (3) and (4) we conclude that the terminal ω^{op} -chain

$$1 \xleftarrow{a} H1 \xleftarrow{Ha} H H1 \xleftarrow{HHa} \dots$$

consists of corecursive algebras. Indeed, the continuation to $H^i 1$ for all ordinals (with $H^i 1 = \lim_{k \leq i} H^k 1$ for all limit ordinals) also yields corecursive algebras. This follows from the following.

Proposition 2.4. *Let \mathcal{A} be a complete category. Then corecursive algebras are closed under limits in $\mathbf{Alg} H$. Thus, limits of corecursive algebras are formed on the level of \mathcal{A} .*

Proof. It is easy to verify that limits in $\mathbf{Alg} H$ are formed on the level of \mathcal{A} . Let us prove that the product of corecursive algebras is corecursive. The proof for general limits is analogous.

Let (A, a) be the product of corecursive algebras (A_i, a_i) , with projections $p_i : A \rightarrow A_i$. For every coalgebra $e : X \rightarrow HX$ we have the unique coalgebra-to-algebra morphism $e_i^\dagger : X \rightarrow A_i$, for all $i \in I$, and the morphism $e^\dagger = \langle e_i^\dagger \rangle : X \rightarrow A = \prod_{i \in I} A_i$ is a coalgebra-to-algebra morphism. Indeed, for every $i \in I$, the diagram

$$\begin{array}{ccccc}
 & & e_i^\dagger & & \\
 & \frown & & \searrow & \\
 X & \xrightarrow{e^\dagger} & A & \xrightarrow{p_i} & A_i \\
 e \downarrow & & \uparrow a & & \uparrow a_i \\
 HX & \xrightarrow{He^\dagger} & HA & \xrightarrow{Hp_i} & HA_i \\
 & \smile & & \lrcorner & \\
 & & He_i^\dagger & &
 \end{array}$$

commutes, except perhaps for the left hand inner square; but this suffices to establish the desired commutativity of the left hand square. Since all A_i are corecursive, the uniqueness of e^\dagger follows from the observation that there is a one-one correspondence between solutions $s : X \rightarrow A$ of e in A and families of solutions $s_i : X \rightarrow A_i$ of e in A_i , for all $i \in I$. \square

In the following we write $\text{inl} : X \rightarrow X + Y$ and $\text{inr} : Y \rightarrow X + Y$ for the injections of a coproduct.

Lemma 2.5. *Let (A, a) be an algebra and $f : B \rightarrow A$ a morphism. Then (A, a) is a corecursive algebra if and only if the algebra*

$$\bar{a} \equiv H(HA + B) \xrightarrow{H[a, f]} HA \xrightarrow{\text{inl}} HA + B$$

is corecursive.

Proof. Let (A, a) be a corecursive algebra and $e : X \rightarrow HX$ be an equation morphism. Then there is a unique solution:

$$\begin{array}{ccc}
 X & \xrightarrow{e^\dagger} & A \\
 e \downarrow & & \uparrow a \\
 HX & \xrightarrow{He^\dagger} & HA
 \end{array} \tag{2.2}$$

Now inspection of the following commutative diagram shows that $\text{inl} \cdot He^\dagger \cdot e : X \rightarrow HA + B$ is a solution of e in $HA + B$.

$$\begin{array}{ccccccc}
X & \xrightarrow{e} & HX & \xrightarrow{He^\dagger} & HA & \xrightarrow{\text{inl}} & HA + B \\
\downarrow e & \nearrow & \downarrow He & & \uparrow Ha & \nwarrow H[a,f] & \uparrow \bar{a} \\
HX & \xrightarrow{He} & HHA & \xrightarrow{HHe^\dagger} & HHA & \xrightarrow{H\text{inl}} & H(HA + B)
\end{array} \tag{2.3}$$

Indeed, commutativity of the middle rectangle follows from Diagram (2.2), the lower triangle on the right is trivial and the upper triangle is the definition of \bar{a} . To show the uniqueness of the solution, suppose that $s : X \rightarrow HA + B$ is a solution for e , so we have the following commutative diagram:

$$\begin{array}{ccccc}
X & \xrightarrow{s} & HA + B & \xrightarrow{[a,f]} & A \\
\downarrow e & & \uparrow \text{inl} \cdot H[a,f] & & \uparrow a \\
HX & \xrightarrow{Hs} & H(HA + B) & \xrightarrow{H[a,f]} & HA
\end{array}$$

Since the solution e^\dagger in A is unique, $e^\dagger = [a, f] \cdot s$ and hence

$$\text{inl} \cdot He^\dagger \cdot e = \text{inl} \cdot H[a, f] \cdot Hs \cdot e = s.$$

Conversely, let $(HA + B, \text{inl} \cdot H[a, f])$ be a corecursive algebra and $e : X \rightarrow HX$ be an equation morphism. So there exists a unique solution e^\dagger of e in $HA + B$, and hence we have the above commutative diagram with e^\dagger in lieu of s . That is $[a, f] \cdot e^\dagger$ is a solution of e in the algebra (A, a) . To show uniqueness suppose that $s : X \rightarrow A$ is a solution of e , that is $s = a \cdot Hs \cdot e$. Then we have Diagrams (2.2) and (2.3) with the morphism s in lieu of e^\dagger . So, by uniqueness of solution in the corecursive algebra $(HA + B, \text{inl} \cdot H[a, f])$, we have $\text{inl} \cdot Hs \cdot e = e^\dagger$, and hence $s = a \cdot Hs \cdot e = [a, f] \cdot \text{inl} \cdot Hs \cdot e = [a, f] \cdot e^\dagger$. \square

Example 2.6. Binary algebras: For $HX = X \times X$, every algebra (given by the binary operation “ $*$ ” on a set A) which is corecursive has a unique *idempotent* $i = i * i$. This is the solution of the recursive equation

$$x = x * x$$

expressed by the isomorphism $e : 1 \xrightarrow{\sim} 1 \times 1$. Moreover the idempotent is *completely factorizable*, where the set of all completely factorizable elements is defined to be the largest subset of A such that every element a in it can be factorized as $a = b * c$, with b, c completely factorizable. The corecursiveness of A implies that no other element but i is completely factorizable: consider the system of recursive equations

$$x_\epsilon = x_0 * x_1, \quad x_0 = x_{00} * x_{01}, \quad \dots \quad x_w = x_{w0} * x_{w1}, \quad \dots$$

for all finite binary words w . Every completely factorizable element a provides a solution e^\dagger with $e^\dagger(x_\epsilon) = a$. Since solutions are unique, $a = i$.

Conversely, every binary algebra A with an idempotent i which is the only completely factorizable element is corecursive. Indeed, given a morphism $e : X \rightarrow X \times X$, the constant map $e^\dagger : X \rightarrow A$ with value i is a coalgebra-to-algebra morphism. Conversely, if e^\dagger is a coalgebra-to-algebra morphism, then for every $x \in X$ the element $e^\dagger(x)$ is clearly completely factorizable. Therefore, $e^\dagger(x) = i$.

Remark 2.7. Recall the concept of *completely iterative algebra* (*cia* for short) from [28]: it is an algebra (A, a) such that for every “flat equation” morphism $e : X \rightarrow HX + A$ there exists a unique solution, i.e. a unique morphism e^\dagger such that the square

$$\begin{array}{ccc} X & \xrightarrow{e^\dagger} & A \\ e \downarrow & & \uparrow [a, A] \\ HX + A & \xrightarrow{He^\dagger + A} & HA + A \end{array}$$

commutes. This is obviously stronger than corecursiveness because every coalgebra $e : X \rightarrow HX$ yields a flat equation morphism $\text{inl} \cdot e : X \rightarrow HX + A$. Then solutions are determined uniquely. Thus, for example, in the category of complete metric spaces with distance less than one and nonexpanding functions, all algebras for contracting endofunctors (in the sense of P. America and J. Rutten [17]) are corecursive, because, as proved in [28], they are *cia*’s. Here is a concrete example: $HX = X \times X$ equipped with the metric taking 1/2 of the maximum of the two distances is contracting. Thus every binary algebra whose operation is contracting is corecursive.

Example 2.8. The endofunctor $HX = X \times X$ has many corecursive algebras that are not *cia*’s. For example the algebra A of all binary trees with finitely many leaves. The operation is tree-tupling and the only completely factorizable tree is the complete binary tree t . Thus, A is corecursive. However, if $a \in A$ denotes the root-only tree, then the system of recursive equations

$$\begin{aligned} x &= x * y \\ y &= a \end{aligned}$$

does not have a solution in A (because the tree corresponding to x has infinitely many leaves). Thus A is not a *cia*.

Lemma 2.9. *Every homomorphism $h : (A, a) \rightarrow (B, b)$ in $\text{Alg } H$ with (A, a) and (B, b) corecursive preserves solutions. That is, given a coalgebra $e : X \rightarrow HX$ with a solution $e^\dagger : X \rightarrow A$ in the domain algebra, then $h \cdot e^\dagger : X \rightarrow B$ is the solution in the codomain one.*

Proof. This follows from the diagram

$$\begin{array}{ccccc} X & \xrightarrow{e^\dagger} & A & \xrightarrow{h} & B \\ e \downarrow & & \uparrow a & & \uparrow b \\ HX & \xrightarrow{He^\dagger} & HA & \xrightarrow{Hh} & B \end{array} \quad \square$$

We thus consider corecursive algebras as a full subcategory $\text{Alg}_C H$ of $\text{Alg } H$. We obtain a forgetful functor

$$\text{Alg}_C H \rightarrow \mathcal{A} \quad (A, a) \mapsto A$$

In Section 4 we prove that this forgetful functor has a left adjoint, that is, free corecursive algebras exist, if and only if a terminal coalgebra T exists and every object Y generates a free algebra FY (i.e., the forgetful functor $\text{Alg } H \rightarrow \mathcal{A}$ has a left adjoint). Our result holds for example for all set functors, and for them the formula for the free corecursive algebra is $T \oplus FY$, where \oplus is the coproduct in $\text{Alg } H$.

Recall from [24] that given an infinite cardinal number λ , a functor is called λ -*accessible* if it preserves λ -filtered colimits. An object X whose hom-functor $\mathcal{A}(X, -)$ is λ -accessible is called λ -*presentable*. A category \mathcal{A} is *locally λ -presentable* if it has

- (a) colimits, and
- (b) a set of λ -presentable objects whose closure under λ -filtered colimits is all of \mathcal{A} .

A category \mathcal{A} is called *locally presentable* (and a functor F *accessible*, resp.) if there exists some λ such that \mathcal{A} is locally λ -presentable (and F λ -accessible, resp.).

For a λ -accessible endofunctor H , the category $\mathbf{Alg} H$ is also locally λ -presentable, see [14]. For corecursive algebras we have:

Proposition 2.10. *Let \mathcal{A} be a locally presentable category. Then for every accessible endofunctor H , the category $\mathbf{Alg}_C H$ of corecursive algebras is locally presentable.*

Proof. Choose an uncountable cardinal number λ such that H preserves λ -filtered colimits and \mathcal{A} is locally λ -presentable. Then λ -filtered colimits in $\mathbf{Alg} H$ are clearly formed on the level of \mathcal{A} . And as proved in [13], every coalgebra is a λ -filtered colimit of λ -presentable coalgebras, and these are precisely the coalgebras carried by λ -presentable objects in \mathcal{A} .

By the Reflection Theorem (see [14, Corollary to Theorem 2.48]), in order to establish that the full subcategory $\mathbf{Alg}_C H$ of $\mathbf{Alg} H$ is locally λ -presentable, it suffices to see that it is closed in $\mathbf{Alg} H$ under limits and λ -filtered colimits. For limits see Proposition 2.4, and now we prove that λ -filtered colimits of corecursive algebras in $\mathbf{Alg} H$ are corecursive. Indeed, let $(A_t, a_t)_{t \in T}$ be a λ -filtered diagram with colimit $k_t : (A_t, a_t) \rightarrow (C, c)$. For every coalgebra $e : X \rightarrow HX$, a solution $e^\dagger : X \rightarrow A_t$ exists in (A_t, a_t) and since k_t is a homomorphism, $k_t \cdot e^\dagger$ is a solution in C , see Lemma 2.9.

To prove that solutions are unique, assume first that X is λ -presentable in \mathcal{A} . For every solution $e^\dagger : X \rightarrow C$ there exists a $t \in T$ such that e^\dagger factorizes through k_t as follows

$$\begin{array}{ccccc}
 & & s & & \\
 & & \curvearrowright & & \\
 X & \xrightarrow{e^\dagger} & C & \xleftarrow{k_t} & A_t \\
 e \downarrow & & \uparrow c & & \uparrow a_t \\
 HX & \xrightarrow{He^\dagger} & HC & \xleftarrow{Hk_t} & HA_t \\
 & & \curvearrowleft & & \\
 & & Hs & &
 \end{array}$$

The morphism k_t merges s and $a_t \cdot Hs \cdot e$:

$$\begin{aligned}
 k_t \cdot (a_t \cdot Hs \cdot e) &= c \cdot Hk_t \cdot Hs \cdot e \\
 &= c \cdot He^\dagger \cdot e \\
 &= e^\dagger \\
 &= k_t \cdot s.
 \end{aligned}$$

Consequently, since k_t is a colimit morphism of a λ -filtered colimit, there exists an object $t' \in T$ and a connecting morphism $u : A_t \rightarrow A_{t'}$ which also merges s and $a_t \cdot Hs \cdot e$, that is

$$u \cdot s = u \cdot a_t \cdot Hs \cdot e = a_{t'} \cdot Hu \cdot Hs \cdot e = a_{t'} \cdot H(u \cdot s) \cdot e.$$

This last equation proves that $u \cdot s$ is a solution of e in $A_{t'}$, thus $u \cdot s$ is uniquely determined. Hence, e^\dagger is uniquely determined from $e^\dagger = k_t \cdot s = k_{t'} \cdot u \cdot s$.

Next let X be arbitrary. Express (X, e) in the category of coalgebras as a λ -filtered colimit of coalgebras (X_i, e_i) with X_i λ -presentable in \mathcal{A} . Let $x_i : X_i \rightarrow X$ be the corresponding colimit cocone. For every solution $e^\dagger : X \rightarrow C$ each $e^\dagger \cdot x_i$ is a solution of e_i since we have the following commutative diagram

$$\begin{array}{ccccc} X_i & \xrightarrow{x_i} & X & \xrightarrow{e^\dagger} & C \\ e_i \downarrow & & \downarrow e & & \uparrow c \\ HX_i & \xrightarrow{Hx_i} & HX & \xrightarrow{He^\dagger} & HC \end{array}$$

Thus, $e^\dagger \cdot x_i$ is uniquely determined by the previous case. Since the cocone of all x_i 's is collectively epic, this proves that e^\dagger is uniquely determined. \square

Remark 2.11. We obtain from Proposition 2.10 that for an uncountable cardinal number λ , if H is λ -accessible and \mathcal{A} locally λ -presentable, then so is $\text{Alg}_C H$. And for $\lambda = \omega$, i.e. H is finitary on the locally finitely presentable category \mathcal{A} , we have that $\text{Alg}_C H$ is locally \aleph_1 -presentable.

3. BLOOM ALGEBRAS

In the case of iterative algebras, it was proved in [10] that every finitary functor H of \mathcal{A} has a free iterative algebra on every object of \mathcal{A} , and the resulting monad \mathbb{R} on \mathcal{A} is a free iterative monad. The next step was a characterization of the Eilenberg-Moore algebras for \mathbb{R} that were called Elgot algebras [9]. An Elgot algebra has for every finitary flat equation e a solution e^\dagger , but not necessarily unique. Instead, Elgot algebras are equipped with a solution operation $e \mapsto e^\dagger$ satisfying some “natural” axioms.

In the present section we take the corresponding step for corecursive algebras. We introduce Bloom algebras as algebras equipped with an operation assigning to every coalgebra e a solution e^\dagger which forms a functor. Later we prove that Bloom algebras are (analogously to Elgot algebras) precisely the Eilenberg-Moore algebras for the free corecursive monad, see Theorems 4.12 and 6.4.

Definition 3.1. A *Bloom algebra* is a triple (A, a, \dagger) where $a : HA \rightarrow A$ is an H -algebra and \dagger is an operation assigning to every coalgebra $e : X \rightarrow HX$ a coalgebra-to-algebra homomorphism $e^\dagger : X \rightarrow A$ so that \dagger is functorial. This means that we obtain a functor

$$\dagger : \text{Coalg } H \rightarrow \mathcal{A}/A.$$

More explicitly, given a coalgebra homomorphism

$$\begin{array}{ccc} X & \xrightarrow{e} & HX \\ h \downarrow & & \downarrow Hh \\ X' & \xrightarrow{f} & HX' \end{array}$$

the following triangle commutes

$$\begin{array}{ccc} X & \xrightarrow{h} & X' \\ e^\dagger \searrow & & \swarrow f^\dagger \\ & A & \end{array}$$

Example 3.2.

- (a) Every corecursive algebra is a Bloom algebra. Indeed, functoriality easily follows from the uniqueness of solutions due to the diagram

$$\begin{array}{ccccc}
 X & \xrightarrow{h} & HX & \xrightarrow{f^\dagger} & A \\
 e \downarrow & & \downarrow Hf & & \uparrow a \\
 HX & \xrightarrow{Hh} & HX' & \xrightarrow{Hf^\dagger} & HA
 \end{array}$$

- (b) A unary algebra $a : A \rightarrow A$ ($H = Id$) is a Bloom algebra iff a has a fixpoint, i. e., a morphism $t : 1 \rightarrow A$ with $a \cdot t = t$. More precisely:
- (1) Given a fixpoint, then (A, a, \dagger) is a Bloom algebra where $e^\dagger = t \cdot !$ for the unique morphism $! : X \rightarrow 1$.
 - (2) Given a Bloom algebra (A, a, \dagger) , then $id_1^\dagger : 1 \rightarrow A$ is a fixpoint of a .
- (c) Let \mathcal{A} have finite products. An algebra $a : A \times A \rightarrow A$ for $HX = X \times X$ is a Bloom algebra if and only if it has an idempotent global element, that is $i : 1 \rightarrow A$ satisfying $a \cdot (i \times i) = i$ (recall that $1 \times 1 = 1$). More precisely:
- (1) Given an idempotent i , we have a Bloom algebra (A, a, \dagger) , where \dagger is the constant function with value $e^\dagger = i \cdot !$.
 - (2) Given a Bloom algebra (A, a, \dagger) , there exists an idempotent i such that \dagger is the constant function with value $e^\dagger = i \cdot !$.

Compare this with Example 2.6. In particular every group, considered as a binary algebra in **Set**, is thus a Bloom algebra in a unique sense. But no nontrivial group is corecursive.

- (d) Every continuous algebra is a Bloom algebra if we define e^\dagger to be the least solution of e . More detailed, let H be a locally continuous endofunctor of the category **CPO** of complete ordered sets (i. e., partially ordered sets with a least element \perp and with joins of ω -chains). For every H -algebra (A, a) and every equation morphism $e : X \rightarrow HX$, we can define in $\mathbf{CPO}(X, A)$ a function $e^\dagger : X \rightarrow A$ as a join of the sequence e_n^\dagger defined by $e_0^\dagger = \text{const}_\perp$ and $e_{n+1}^\dagger = a \cdot He_n^\dagger \cdot e$. Then the least solution of e is $e^\dagger = \bigvee_{n < \omega} e_n^\dagger$ and (A, a, \dagger) is a Bloom algebra. Example (b) demonstrates that this need not be corecursive.
- (e) Every product of Bloom algebras is a Bloom algebra. We define $e^\dagger = \langle e_i^\dagger \rangle$ as in the proof of Proposition 2.4. More generally: every limit of Bloom algebras is a Bloom algebra.
- (f) Every complete Elgot algebra in the sense of [8] is a Bloom algebra.

Definition 3.3. By a *homomorphism* of Bloom algebras from (A, a, \dagger) to (B, b, \ddagger) is meant an algebra homomorphism $h : (A, a) \rightarrow (B, b)$ preserving solutions, that is, for every coalgebra $e : X \rightarrow HX$ the triangle

$$\begin{array}{ccc}
 A & \xrightarrow{h} & B \\
 e^\dagger \swarrow & & \searrow e^\ddagger \\
 & X &
 \end{array}$$

commutes. We denote by $\mathbf{Alg}_B H$ the corresponding category of Bloom algebras.

Proposition 3.4. *Let (T, τ) be a terminal coalgebra for H . The category of Bloom algebras for H is isomorphic to the slice category $(T, \tau^{-1})/\text{Alg } H$.*

Proof. Let us, for a coalgebra (X, e) , denote the unique coalgebra homomorphism from X to T by $e^\# : X \rightarrow T$. We shall define two functors between $\text{Alg}_B H$ and $\mathcal{S} = (T, \tau^{-1})/\text{Alg } H$ and show that they are mutually inverse.

(a) From Bloom algebras to the slice category \mathcal{S} : given a Bloom algebra (A, a, \dagger) we form the solution $\tau^\dagger : T \rightarrow A$ which clearly is an object (A, τ^\dagger) of \mathcal{S} . For a homomorphism $h : (A, a, \dagger) \rightarrow (B, b, \ddagger)$ of Bloom algebras, we clearly have a morphism $h : (A, \tau^\dagger) \rightarrow (B, \tau^\ddagger)$ of \mathcal{S} , since h is solution preserving. This defines a functor from $\text{Alg}_B H$ to \mathcal{S} .

(b) From \mathcal{S} to Bloom algebras: Suppose we are given an object (A, h) in \mathcal{S} , that is, an algebra homomorphism:

$$\begin{array}{ccc} T & \xleftarrow{\tau^{-1}} & HT \\ h \downarrow & & \downarrow Hh \\ A & \xleftarrow{a} & HA \end{array}$$

We define for every $e : X \rightarrow HX$ its dagger as $e^\dagger = h \cdot e^\#$. This is functorial; indeed, for every coalgebra homomorphism $k : (X, e) \rightarrow (Y, f)$ we have $f^\# \cdot k = e^\#$ by unicity of the universal property of the terminal coalgebra (T, τ) , thus

$$\begin{aligned} f^\dagger \cdot h &= h \cdot f^\# \cdot k \\ &= h \cdot e^\# \\ &= e^\dagger \end{aligned}$$

In addition, every morphism $m : (A, h) \rightarrow (B, h')$ of \mathcal{S} is a homomorphism of Bloom algebras $(A, a, \dagger) \rightarrow (B, b, \ddagger)$:

$$\begin{aligned} m \cdot e^\dagger &= m \cdot h \cdot e^\# && \text{by definition of } \dagger \\ &= h' \cdot e^\# && m \text{ is a morphism in } \mathcal{S} \\ &= e^\ddagger && \text{by definition of } \ddagger \end{aligned}$$

That this gives a functor from \mathcal{S} to $\text{Alg}_B H$ is immediate.

(c) The two functors above are mutually inverse. Indeed, it suffices to show that we have a bijection on the level of objects, since both functors are the identity maps on morphisms. So for (A, h) in \mathcal{S} we form first (A, a, \dagger) as in (b) and then (A, τ^\dagger) as in (a) and we have $\tau^\dagger = h \cdot \tau^\# = h$, as $\tau^\#$ is the identity (being the unique coalgebra homomorphism from (T, τ) to itself). Finally, given a Bloom algebra (A, a, \dagger) we first form (A, τ^\dagger) as in (a) and then (A, a, \dagger) as in (b). Then we have

$$\begin{aligned} e^\dagger &= \tau^\dagger \cdot e^\# && \text{by definition of } \dagger \text{ in (b)} \\ &= e^\ddagger && \text{by functoriality of } \ddagger \end{aligned}$$

This completes the proof. □

Remark 3.5. Being an algebra homomorphism and preserving solutions are independent concepts: neither of them implies the other one. To see this, consider for $HX = X \times X$ an algebra $A = \{a, b\}$ with a binary operation $*$ such that a and b are idempotent. We turn A into a Bloom algebra by taking $e^\dagger = \text{const}_a$ for every $e : X \rightarrow X \times X$. Then there are two homomorphisms from the one-point binary algebra (which clearly is corecursive) to A , yet only one of them is solution preserving. Thus, there exist homomorphisms which are not solution preserving.

Conversely, there exist solution preserving morphisms which are not homomorphisms. To see this, let us assume that we have $x * y = a$ for all $x \neq y$ in A . There are two different structures of Bloom algebras on A , $(A, \alpha, e \mapsto \text{const}_a)$ and $(A, \alpha, e \mapsto \text{const}_b)$. The map on A which swaps a and b is a solution preserving map between the two Bloom algebras, but not a homomorphism.

Proposition 3.6. *An initial Bloom algebra is precisely a terminal coalgebra.*

More precisely, the statement in Example 2.6 generalizes from corecursive algebras to Bloom algebras. Indeed, the proof in [21] can be used again.

Lemma 3.7. *If (A, a, \dagger) is a Bloom algebra and $h : (A, a) \rightarrow (B, b)$ is a homomorphism of algebras, then there is a unique structure of a Bloom algebra on (B, b) such that h is a solution preserving morphism. We call it, the Bloom algebra induced by h .*

Proof. For every coalgebra $e : X \rightarrow HX$ we define

$$e^* \equiv X \xrightarrow{e^\dagger} A \xrightarrow{h} B$$

and verify that e^* is a solution of e by the following commutative diagram

$$\begin{array}{ccccc}
 & & e^* & & \\
 & \frown & & \smile & \\
 X & \xrightarrow{e^\dagger} & A & \xrightarrow{h} & B \\
 \downarrow e & & \uparrow a & & \downarrow b \\
 HX & \xrightarrow{He^\dagger} & HA & \xrightarrow{Hh} & HB \\
 & \smile & & \frown & \\
 & & He^* & &
 \end{array}$$

Functoriality is easily checked too: let $g : (X, e) \rightarrow (Y, f)$ be a coalgebra homomorphism. Then the following equations hold:

$$\begin{aligned}
 f^* \cdot g &= h \cdot f^\dagger \cdot g && \text{by the definition of } (-)^* \\
 &= h \cdot e^\dagger && \text{by functoriality of } \dagger \\
 &= e^* && \text{by the definition of } (-)^*
 \end{aligned}$$

Finally, h is clearly solution preserving.

The unicity of the Bloom algebra structure given by $(-)^*$ is clear. \square

Remark 3.8. We are going to characterize the left adjoint of the forgetful functor

$$U : \text{Alg}_B H \rightarrow \mathcal{A}, \quad (A, a, \dagger) \mapsto A.$$

In other words, we characterize the free Bloom algebras: they are coproducts $T \oplus FY$ of the terminal coalgebra and free algebras. For that we first attend to the existence of those ingredients.

Lemma 3.9. *Let \mathcal{A} be a complete category. If H has a free Bloom algebra on an object Y with $\mathcal{A}(Y, HY) \neq \emptyset$, then H has a terminal coalgebra.*

Proof. The free Bloom algebra (A, a, \dagger) on Y , is weakly initial in $\mathbf{Alg}_B H$. To see this, choose a morphism $e : Y \rightarrow HY$. For every Bloom algebra (B, b, \ddagger) the solution $e^\ddagger : Y \rightarrow B$ extends to a homomorphism $h : (A, a, \dagger) \rightarrow (B, b, \ddagger)$ of Bloom algebras.

Since $\mathbf{Alg}_B H$ is complete by Example 3.2(e), we can use Freyd's Adjoint Functor Theorem. The existence of a weakly initial object implies that $\mathbf{Alg}_B H$ has an initial object. Now apply Proposition 3.6. \square

Construction 3.1. Free-Algebra Chain. Recall from [2] that if \mathcal{A} is cocomplete, we can define a chain constructing the free H -algebra on Y as follows:

$$Y \xrightarrow{\text{inr}} HY + Y \xrightarrow{H\text{inr}+Y} H(HY + Y) + Y \longrightarrow \dots$$

We mean the essentially unique chain $V : \mathbf{Ord} \rightarrow \mathcal{A}$ with

$$\begin{aligned} V_0 &= Y \\ V_{i+1} &= HV_i + Y \end{aligned}$$

and for limit ordinals i

$$V_j = \text{colim}_{k < j} V_k$$

whose connecting morphisms $v_{i,j} : V_i \rightarrow V_j$ are defined by

$$v_{0,1} \equiv \text{inr} : Y \rightarrow HY + Y \quad \text{and} \quad v_{i+1,j+1} \equiv Hv_{i,j} + Y$$

and for limit ordinals j

$$(v_{k,j})_{k < j} \text{ is the colimit cocone.}$$

This chain is called the *free-algebra chain*. If it *converges* at some ordinal λ , that is, if $v_{\lambda,\lambda+1}$ is an isomorphism, then V_λ is a free algebra on Y . More detailed: this isomorphism turns V_λ into a coproduct

$$V_\lambda = HV_\lambda + Y$$

and thus V_λ is an algebra via the left-hand coproduct injection, and the right-hand one $Y \rightarrow V_\lambda$ yields the universal arrow.

Corollary 3.10. *Every accessible endofunctor of a cocomplete category has free algebras.*

Indeed, if the given functor is λ -accessible, the free-algebra chain converges at λ .

Definition 3.11. (See [15]) We say that in a given category monomorphisms are *constructive* provided that

- (a) if $m_i : A_i \rightarrow B_i$ are monomorphisms for $i = 1, 2$ then $m_1 + m_2 : A_1 + A_2 \rightarrow B_1 + B_2$ is a monomorphism,
- (b) coproduct injections are monomorphisms, and
- (c) if $a_i : A_i \rightarrow A$, ($i < \lambda$), is a colimit of an λ -chain and $m : A \rightarrow B$ has all composites $m \cdot a_i$ monic, then m is monic.

Example 3.12. Sets, posets, graphs, abelian groups have constructive monomorphisms. If \mathcal{A} has constructive monomorphisms, then all functor categories \mathcal{A}^C do. In all locally finitely presentable categories condition (c) holds (see [14]), but (a) and (b) can fail.

Proposition 3.13. *Let \mathcal{A} be a cocomplete, wellpowered category with constructive monomorphisms. If H has a free Bloom algebra on Y and preserves monomorphisms, then it also has a free algebra on Y .*

Proof. Let (A, a, \dagger) be a free Bloom algebra and $\eta : Y \rightarrow A$ be the universal arrow.

(1) We prove that $A = HA + Y$ with coproduct injection a and η . The algebra

$$b \equiv H(HA + Y) \xrightarrow{H[a, \eta]} HA \xrightarrow{\text{inl}} HA + Y$$

is a Bloom algebra when we put, as in Lemma 2.5,

$$e^\dagger = \text{inl} \cdot He^\dagger \cdot e, \quad \text{for all } e : X \rightarrow HX.$$

Indeed, functoriality is easy to verify. Consequently, there exists a unique solution preserving homomorphism

$$h : A \rightarrow HA + Y \quad \text{with} \quad h \cdot \eta = \text{inr}.$$

We prove that h is inverse to $[a, \eta]$. For that, observe that $[a, \eta]$ is also a homomorphism

$$\begin{array}{ccccc} H(HA + Y) & \xrightarrow{H[a, \eta]} & HA & \xrightarrow{\text{inl}} & HA + Y \\ H[a, \eta] \downarrow & & & \searrow a & \downarrow [a, \eta] \\ HA & \xrightarrow{a} & & & A \end{array}$$

and also preserves solutions:

$$[a, \eta] \cdot e^\dagger = a \cdot He^\dagger \cdot e = e^\dagger.$$

The composite with h yields an endomorphism of (A, a, \dagger) such that

$$([a, \eta] \cdot h) \cdot \eta = [a, \eta] \cdot \text{inr} = \eta.$$

The universal property thus implies

$$[a, \eta] \cdot h = \text{id}_A.$$

Since h is a homomorphism,

$$h \cdot a = b \cdot Hh = \text{inl} \cdot H([a, \eta] \cdot h) = \text{inl}.$$

We conclude that h is inverse to $[a, \eta]$:

$$h \cdot [a, \eta] = [h \cdot a, h \cdot \eta] = [\text{inl}, \text{inr}] = \text{id}.$$

Therefore, $A = HA + Y$ with coproduct injections a and η .

(2) We define a cone $m_i : V_i \rightarrow A$ of the free-algebra chain by

$$m_0 \equiv \eta : Y \rightarrow A$$

and

$$m_{i+1} \equiv HV_i + Y \xrightarrow{Hm_i + \text{id}} HA + Y \xrightarrow{[a, \eta]} A.$$

More precisely: there is a unique cone with the above properties. The verification of $m_{i+1} \cdot v_{i, i+1} = m_i$ is an easy transfinite induction on i , and the limit steps then follow automatically from $V_i = \text{colim}_{k < i} V_k$. Moreover, since monomorphisms are constructive, all m_i 's are monomorphisms: m_0 is a coproduct injection of $A = HA + Y$, if m_i is a monomorphism, then so is $Hm_i + \text{id}$, hence, so is m_{i+1} , and the limit steps are clear. Since A has only a set of subobjects, there exist $j > i$ such that m_i and m_j represent the

same subobject, i. e., $v_{i,j}$ is an isomorphism. Then also $v_{i+1,j+1}$ is an isomorphism with an inverse v' , say. It follows that $v_{j,j+1}$ is an isomorphism, too: it is monomorphic since $m_{j+1} \cdot v_{j,j+1} = m_j$, and it is a split epimorphism since $v_{j,j+1} \cdot v_{i+1,j} \cdot v' = v_{i+1,j+1} \cdot v' = id$. Therefore, V_j is a free algebra on Y . \square

Corollary 3.14. *If a set functor has a free Bloom algebra on Y , it has both a terminal coalgebra T and a free algebra FY on Y .*

Proof. For the existence of T use Lemma 3.9: for $\mathcal{A} = \mathbf{Set}$ we have $\mathcal{A}(Y, HY) \neq \emptyset$ whenever $HY \neq \emptyset$ or $Y = \emptyset$, and in the remaining case where $Y \neq \emptyset$ and $HY = \emptyset$, T trivially exists since one deduces that H is the constant functor on \emptyset ; indeed, for an any set X pick some $f : X \rightarrow Y$, then $Hf : HX \rightarrow HY = \emptyset$ implies $HX = \emptyset$.

For the existence of FY apply Proposition 3.13 in the case where H preserves monomorphisms. If it does not, redefine it in \emptyset and obtain a set functor H' with $H'X = HX$, for all $X \neq \emptyset$ and $H'\emptyset = \emptyset$. Then H' preserves monomorphisms and, whenever $Y \neq \emptyset$, it has a free Bloom algebra on Y if and only if H does. The case $Y = \emptyset$ is obtained from Proposition 3.6. \square

We are going to characterize free Bloom algebras. From Lemma 3.9 and Proposition 3.13 we know that the assumption that T and FY exist is “natural”. Since the terminal coalgebra $\tau : T \rightarrow HT$ has by Lambek’s Lemma an invertible structure map τ , we can view it as an algebra. In the next proposition we assume that the coproduct of T and FY exists in $\mathbf{Alg} H$. In Section 4 we will see that this actually follows from the existence of a free Bloom algebra.

Notation 3.15. Coproducts in $\mathbf{Alg} H$ are denoted by $(A, a) \oplus (B, b)$.

Theorem 3.16. *Suppose that H has a terminal coalgebra T , a free algebra FY on Y and their coproduct $T \oplus FY$. Then the last algebra is the free Bloom algebra on Y .*

Remark 3.17. More precisely, let $\eta : Y \rightarrow FY$ be the universal arrow of the free algebra FY . Take the unique Bloom algebra structure \dagger on $T \oplus FY$ induced by $\text{inl} : T \rightarrow T \oplus FY$ following Lemma 3.7. Then this forms the free Bloom algebra $(T \oplus FY, a, \dagger)$ on Y with the universal arrow $\text{inr} \cdot \eta$. The latter means that for every Bloom algebra (B, b, \ddagger) and every morphism $g : Y \rightarrow B$ in \mathcal{A} there exists a unique homomorphism of Bloom algebras $h : T \oplus FY \rightarrow B$ such that $h \cdot \text{inr} \cdot \eta = g$.

Proof. Given a Bloom algebra (B, b, \ddagger) and morphism $g : Y \rightarrow B$, we obtain a unique homomorphism $\bar{g} : (FY, \varphi_Y) \rightarrow (B, b)$ with $g = \bar{g} \cdot \eta$. We also have a unique solution-preserving homomorphism $f : (T, \tau^{-1}, \dagger) \rightarrow (B, b, \ddagger)$, see Proposition 3.6. This yields a homomorphism $[f, \bar{g}] : T \oplus FY \rightarrow B$ which is clearly solution-preserving: recall from Lemma 3.7 that solutions in $T \oplus FY$ have the form $\text{inl} \cdot e^\dagger$. Thus, $[f, \bar{g}] \cdot (\text{inl} \cdot e^\dagger) = f \cdot e^\dagger = e^\ddagger$. And this is the desired morphism since $[f, \bar{g}] \cdot \text{inr} \cdot \eta = \bar{g} \cdot \eta = g$.

Conversely, given a solution-preserving homomorphism $h : T \oplus FY \rightarrow B$ with $h \cdot \text{inr} \cdot \eta = g$, then $h = [f, \bar{g}]$, because $h \cdot \text{inl} : T \rightarrow B$ is clearly solution-preserving, hence $h \cdot \text{inl} = f$. Also $h \cdot \text{inr}$ is a homomorphism from FY with $h \cdot \text{inr} \cdot \eta = g$, thus $h \cdot \text{inr} = \bar{g}$. \square

Corollary 3.18. *Every accessible endofunctor of a locally presentable category has free Bloom algebras. They have the form $T \oplus FY$.*

Indeed, recall that the assumptions mean that there exists an infinite ordinal λ such that

- (a) \mathcal{A} is cocomplete and has a set \mathcal{A}_λ of λ -presentable objects (that is, objects whose hom-functors preserve λ -filtered colimits) such that every object is a λ -filtered colimit of objects in \mathcal{A}_λ , and
- (b) H preserves λ -filtered colimits.

From this it follows that $\mathbf{Coalg} H$ is locally $\bar{\lambda}$ -presentable, where $\bar{\lambda} = \max\{\lambda, \aleph_1\}$ (see [13]). Thus, this category has a terminal object, T . We know from Corollary 3.10 that the free algebra FY exists. And $T \oplus FY$ exists since the category of algebras for an accessible functor on a locally presentable category is itself locally presentable and therefore comcomplete.

Remark 3.19. For concrete examples of $T \oplus FY$ see Example 4.7 below.

Proposition 3.20. *Let \mathcal{A} be a complete category. Then so is $\mathbf{Alg}_B H$ and limits of Bloom algebras are formed on the level of \mathcal{A} .*

Proof. This is completely analogous to the proof of Proposition 2.4. The verification that the function $e \mapsto \langle e_i^\dagger \rangle$ is functorial is trivial. \square

Corollary 3.21. *For a complete category \mathcal{A} , the monomorphisms in $\mathbf{Alg}_B H$ are precisely those homomorphisms carried by monomorphisms in \mathcal{A} .*

Proof. To see this use that in any category with pullbacks a morphism m is a monomorphism iff its kernel pair consists of two identity morphisms. \square

4. FREE CORECURSIVE ALGEBRAS

For accessible functors H we prove that free corecursive algebras MY exist and, in the case where H preserves monomorphisms, they coincide with the free Bloom algebras $MY = T \oplus FY$. Moreover an iterative construction of these free algebras (closely related to the free algebra chain in 3.1) is presented.

We first prove that the category of corecursive algebras is strongly epireflective in the category of Bloom algebras. That is, the full embedding is a right adjoint, and the components of the unit of the adjunction are strong epimorphisms.

Proposition 4.1. *For every accessible endofunctor of a locally presentable category, corecursive algebras form a strongly epireflective subcategory of the category of Bloom algebras. In particular, every Bloom subalgebra of a corecursive algebra in $\mathbf{Alg}_B H$ is corecursive.*

Proof. Let λ be an infinite cardinal such that \mathcal{A} is a locally λ -presentable category and H preserves λ -filtered colimits. Since λ can be chosen arbitrarily large, we can assume that λ is uncountable. The category $\mathbf{Alg}_B H$ is locally λ -presentable. The proof is analogous to that of Proposition 2.10 (the only difference is that in the proof of the uniqueness of e^\dagger we simply observe that since k_t 's are supposed to be solution-preserving, we have $e^\dagger = k_t \cdot s$, where s is the dagger of e in A_t). Consequently, $\mathbf{Alg}_B H$ is a complete, well-powered, and cowellpowered category, and it has (strong epi-mono) factorization of morphisms, see [14]. The full subcategory of corecursive algebras is closed under products by Proposition 2.4. Thus, the proof will be completed when we prove that the subcategory of corecursive algebras is closed in $\mathbf{Alg}_B H$ under subalgebras, then it is strongly epireflective (see [5, Theorem 16.8]).

Let $m : (A, a, \dagger) \rightarrow (B, b, \ddagger)$ be a monomorphism in $\text{Alg}_B H$ with (B, b, \ddagger) corecursive. From Corollary 3.21 we have that m is a monomorphism in \mathcal{A} . It is our task to prove that for every coalgebra $e : X \rightarrow HX$ the morphism e^\dagger is the unique solution in A . This follows from the fact that $m \cdot e^\dagger = e^\ddagger$. Since e^\ddagger is unique in B and m is a monomorphism in \mathcal{A} , the proof is concluded. \square

Corollary 4.2. *Every accessible endofunctor of a locally presentable category has free corecursive algebras.*

Indeed, since the functors $\text{Alg}_C H \hookrightarrow \text{Alg}_B H$ and $\text{Alg}_B H \rightarrow \mathcal{A}$ have left adjoints by Corollary 3.18 and Proposition 4.1, their composite has a left adjoint.

Remark 4.3. We believe that in the generality of the above corollary, the free corecursive algebras are $T \oplus FY$ (as in Corollary 3.18). But we can only prove this in the case where H preserves monomorphisms and monomorphisms are constructive. We are going to apply the following transfinite construction of free corecursive algebras closely related to the free algebra construction of 3.1

Construction 4.1. Free-Corecursive-Algebra Chain.

Let \mathcal{A} be cocomplete and H have a terminal coalgebra (T, τ) . We define an essentially unique chain $U : \text{Ord} \rightarrow \mathcal{A}$ by

$$\begin{aligned} U_0 &= T \\ U_{i+1} &= HU_i + Y \end{aligned}$$

and for limit ordinals j

$$U_j = \text{colim}_{k < j} U_k.$$

The connecting morphisms $u_{i,j} : U_i \rightarrow U_j$ are defined by

$$\begin{aligned} u_{0,1} &\equiv T \xrightarrow{\tau} HT \xrightarrow{\text{inl}} HT + Y \\ u_{i+1,j+1} &= Hu_{i,j} + \text{id}_Y \end{aligned}$$

and for limit ordinals j

$$(u_{k,j})_{k < j} \text{ is the colimit cocone.}$$

We say that the chain *converges at λ* if the connecting morphism $u_{\lambda,\lambda+1}$ is an isomorphism, thus $U_\lambda = HU_\lambda + Y$. Then U_λ is an algebra (via inl) connected to Y via inr .

Proposition 4.4. *Let \mathcal{A} be a cocomplete and wellpowered category with constructive monomorphisms, and let H preserve monomorphisms and have a terminal coalgebra T . If the corecursive chain for Y converges in λ steps, then $U_\lambda = T \oplus FY$.*

Remark 4.5. (a) More detailed, we prove that a free algebra FY exists and the algebra $\text{inl} : HU_\lambda \rightarrow U_\lambda$ (obtained from $U_\lambda = HU_\lambda + Y$) is a coproduct of T and FY in $\text{Alg } H$.

(b) The proposition is valid for every fixpoint of H , not necessarily a terminal coalgebra: given any isomorphism $\tau : T \rightarrow HT$ and forming the corresponding chain with $U_0 = T$ and $U_{i+1} = HU_i + Y$, then whenever it converges, it yields a coproduct of (T, τ^{-1}) and the free algebra on Y in $\text{Alg } H$.

Proof. (1) A free algebra FY exists. To prove this, we define a natural transformation $(m_i : V_i \rightarrow U_{1+i})_{i \in \text{Ord}}$ from the free-algebra chain to the corecursive chain (delayed on finite ordinals by one step, recall that $1 + i = i$ for all infinite ordinals). Put

$$m_0 = \text{inr} : Y \rightarrow HT + Y$$

and

$$m_{i+1} = Hm_i + id_Y : HV_i + Y \rightarrow HU_{1+i} + Y.$$

The first naturality square

$$\begin{array}{ccc} Y & \xrightarrow{\text{inr}} & HY + Y \\ \text{inr} \downarrow & & \downarrow H\text{inr} + id \\ HT + Y & \xrightarrow{H(\text{inl} \cdot \tau) + Y} & H(HY + Y) + Y \end{array}$$

clearly commutes, and the i -th square implies the next one easily. Therefore, the limit ordinals i define m_i automatically. Since H preserves monomorphisms, we see by transfinite induction that all m_i 's are monic. Consequently, we obtain a transfinite chain of subobjects of U_λ

$$V_i \xrightarrow{m_i} U_{1+i} \xrightarrow{u_{\lambda,1+i}^{-1}} U_\lambda, \quad \text{for } i \geq \lambda.$$

Since U_λ has only a set of subobjects, there exist ordinals σ and ρ with $\rho > \sigma \geq \lambda$ such that above monomorphisms with $i = \rho$ and $i = \sigma$ represent the same subobject. Similar reasoning as in point (2) of the proof of Proposition 3.13 then shows that $v_{\rho,\rho+1}$ is an isomorphism using the commutative triangle

$$\begin{array}{ccc} V_\rho & \xrightarrow{v_{\rho,\rho+1}} & V_{\rho+1} = HV_\rho + Y \\ m_\rho \searrow & & \swarrow m_{\rho+1} \\ U_{1+\rho} & & U_{1+\rho+1} \\ u_{\lambda,1+\rho}^{-1} \searrow & & \swarrow u_{\lambda,1+\rho+1}^{-1} \\ & U_\lambda & \end{array}$$

We thus proved that the algebra (V_ρ, inl) is free on Y with respect to $\text{inr} : Y \rightarrow V_\rho$. Shortly $FY = V_\rho$.

(2) Analogously, U_λ is an algebra with respect to $\text{inl} : HU_\lambda \rightarrow U_\lambda$ (since $u_{\lambda,\lambda+1}$ is invertible). We will prove that this algebra is the coproduct of T and FY with injections

$$\overline{\text{inl}} \equiv u_{0,\lambda} : T \rightarrow U_\lambda$$

and

$$\overline{\text{inr}} \equiv FY = V_\rho \xrightarrow{m_\rho} U_{1+\rho} \xrightarrow{u_{\lambda,1+\rho}^{-1}} U_\lambda$$

in $\text{Alg } H$. Indeed, $\overline{\text{inl}}$ is an algebra homomorphism: $\overline{\text{inl}} = u_{1,\lambda} \cdot u_{0,1}$ and we have $u_{1,\lambda+1} = Hu_{0,\lambda} + Y$, therefore the following square commutes:

$$\begin{array}{ccc}
 HT & \xrightarrow{\tau^{-1}} & T \\
 \downarrow Hu_{0,\lambda} & \swarrow \text{inl} & \swarrow u_{0,1} \\
 & HT + Y & \\
 & \searrow u_{1,\lambda} & \\
 HU_\lambda & \xrightarrow{\text{inl}} & U_\lambda \simeq HU_\lambda + Y \\
 & & \downarrow u_{0,\lambda}
 \end{array}$$

Also $\overline{\text{inr}}$ is a homomorphism: the following diagram

$$\begin{array}{ccc}
 HV_\rho & \xrightarrow{\text{inr}} & HV_\rho + Y \simeq V_\rho \\
 Hm_\rho \downarrow & & \downarrow Hm_\rho + Y \simeq m_\rho \\
 HU_{1+\rho} & \xrightarrow{\text{inr}} & HU_{1+\rho} + Y \simeq U_{1+\rho} \\
 Hu_{\lambda,1+\rho} \uparrow & & \uparrow Hu_{\lambda,1+\rho} + Y \simeq u_{\lambda,1+\rho} \\
 HU_\lambda & \xrightarrow{\text{inr}} & HU_\lambda + Y \simeq U_\lambda
 \end{array}$$

commutes and yields (by inverting $u_{\lambda,1+\rho}$) the square

$$\begin{array}{ccc}
 HFY & \xrightarrow{\text{inr}} & FY \\
 H\overline{\text{inr}} \downarrow & & \downarrow \overline{\text{inr}} \\
 HU_\lambda & \xrightarrow{\text{inr}} & U_\lambda
 \end{array}$$

To verify the universal property, let $b : HB \rightarrow B$ be an algebra and $f : T \rightarrow B$ and $g : FY \rightarrow B$ be homomorphisms. We prove that there exists a unique homomorphism $h : U_\lambda \rightarrow B$ with $f = h \cdot \overline{\text{inl}}$ and $g = h \cdot \overline{\text{inr}}$. Put $\hat{g} = g \cdot \text{inr} : Y \rightarrow B$.

Existence: Define a cocone $h_i : U_i \rightarrow B$ of the chain from Construction 4.1 by

$$h_0 = f, \quad \text{and} \quad h_{i+1} \equiv HU_i + Y \xrightarrow{Hh_i + Y} HB + Y \xrightarrow{[b, \hat{g}]} B$$

The first naturality triangle commutes because f is a homomorphism, thus, $f = b \cdot Hf \cdot \tau$:

$$\begin{array}{ccccc}
 T & \xrightarrow{\tau} & HT & \xrightarrow{\text{inl}} & HT + Y \\
 & \searrow f & \downarrow Hf & & \swarrow Hf + Y \\
 & & HB & \xrightarrow{\text{inl}} & HB + Y \\
 & & \downarrow b & & \swarrow [b, \hat{g}] \\
 & & B & &
 \end{array}$$

The further verification of the naturality, $h_i = h_{i+1} \cdot u_{i+1}$, is now an easy transfinite induction. We obtain the desired homomorphism $h_\lambda : U_\lambda \rightarrow B$. Indeed, since $h_{\lambda+1} \cdot \text{inl} = b \cdot Hh_\lambda$ (by

the above rule for h_{i+1}), the square

$$\begin{array}{ccc} HU_\lambda & \xrightarrow{\text{inl}} & HU_\lambda + Y \simeq U_\lambda \\ Hh_\lambda \downarrow & & \downarrow h_{\lambda+1} \\ HB & \xrightarrow{h} & B \end{array}$$

commutes.

The first equation $f = h_\lambda \cdot \overline{\text{inl}}$ follows from $\overline{\text{inl}} = u_{0,\lambda}$ and $h_\lambda \cdot u_{0,\lambda} = h_0 = f$. For the second one $g = h_\lambda \cdot \overline{\text{inr}} : FY \rightarrow B$ observe that both sides are algebra homomorphisms. Thus, it is sufficient to prove that the universal arrow $\text{inr} : Y \rightarrow V_\rho (\simeq HV_\rho + Y)$ merges them. Recall that $\overline{\text{inr}} = u_{\lambda,1+\rho}^{-1} \cdot m_\rho$, thus we need to prove

$$\hat{g} = h_\lambda \cdot \overline{\text{inr}} \cdot \text{inr} = h_{1+\rho} \cdot m_\rho \cdot \text{inr}.$$

This follows from $h_{1+\rho+1} = [b, \hat{g}] \cdot (Hh_{1+\rho} + Y) = [b \cdot Hh_{1+\rho}, \hat{g}]$, thus the outside of the diagram below commutes as desired:

$$\begin{array}{ccccc} Y & \xrightarrow{\text{inr}} & HV_\rho + Y & \xrightarrow{\sim} & V_\rho \\ & \searrow \text{inr} & \downarrow Hm_\rho + Y & & \downarrow m_\rho \\ & & HU_{1+\rho} + Y & \xrightarrow{\sim} & U_{1+\rho} \\ & & & \searrow [b \cdot Hh_{1+\rho}, \hat{g}] & \downarrow h_{1+\rho} \\ & & & & B \\ & \swarrow & & & \uparrow \\ & & & & \hat{g} \end{array}$$

Uniqueness: Consider an algebra homomorphism $h : U_\lambda \rightarrow B$ with $h \cdot \overline{\text{inl}} = f$ and $h \cdot \overline{\text{inr}} = g$, we prove $h \cdot u_{i,\lambda} = h_i$ by transfinite induction on $i \leq \lambda$. The case $i = \lambda$ yields $h = h_\lambda$. The initial step is clear:

$$h \cdot u_{0,\rho} = h \cdot \overline{\text{inl}} = f = h_0.$$

Assuming $h \cdot u_{i,\lambda} = h_i$, we are going to prove that the triangle

$$\begin{array}{ccc} HU_i + Y = U_{i+1} & & \\ \downarrow Hu_{i,\lambda} + Y \simeq u_{i+1,\lambda} & \searrow h_{i+1} = [b \cdot Hh_i, \hat{g}] & \\ HU_\lambda + Y \simeq U_\lambda & \xrightarrow{h} & B \end{array}$$

commutes. The left-hand component with domain HU_i commutes because h is a homomorphism, that is, $h \cdot \text{inl} = b \cdot Hh$:

$$\begin{array}{ccc} HU_i & & \\ Hh_i \searrow & & \\ Hu_{i,\lambda} \downarrow & & HB \\ HU_\lambda & \xrightarrow{Hh} & \\ \text{inl} \downarrow & & \downarrow b \\ U_\lambda & \xrightarrow{h} & B \end{array}$$

The right-hand one with domain Y follows from $g = h \cdot \overline{\text{inr}} = h \cdot u_{\lambda,1+\rho}^{-1} \cdot m_\rho$: we have

$$\hat{g} = g \cdot \text{inr} = h \cdot u_{\lambda,1+\rho}^{-1} \cdot m_\rho \cdot \text{inr} = h \cdot u_{i+1,\lambda} \cdot \text{inr},$$

where the last equation follows from the commutative diagram below expressing $u_{i+1,\lambda+1} = Hu_{i,\lambda} + Y$:

$$\begin{array}{ccccc}
& & HV_\rho + Y & \xrightarrow{\sim} & V_\rho \\
& \text{inr} \nearrow & \downarrow Hm_\rho + Y & & \downarrow m_\rho \\
Y & \xrightarrow{\text{inr}} & HU_{1+\rho} + Y & \xrightarrow{\sim} & U_{1+\rho} \\
& \text{inr} \searrow & \downarrow Hu_{\lambda,1+\rho}^{-1} + Y & & \downarrow u_{\lambda,1+\rho}^{-1} \\
& & HU_\lambda + Y & \xrightarrow{\sim} & U_\lambda \\
& \text{inr} \searrow & \downarrow Hu_{i,\lambda} + Y & & \downarrow u_{i+1,\lambda} \\
& & HU_i + Y & \xlongequal{\quad} & U_{i+1}
\end{array}$$

Finally, for a limit ordinal α we easily derive $h \cdot u_{\alpha,\lambda} = h_\alpha$ by extending with the colimit injections $u_{i,\alpha}$, $i \leq \alpha$, and using that they are jointly epic:

$$h \cdot u_{\alpha,\lambda} \cdot u_{i,\alpha} = h \cdot u_{i,\lambda} = h_i = h_\alpha \cdot u_{i,\alpha}.$$

□

Theorem 4.6. *Let \mathcal{A} be a locally presentable category with constructive monomorphisms. Every accessible endofunctor preserving monomorphisms has free corecursive algebras $MY = T \oplus FY$.*

Proof. From Corollary 3.18 we know that $T \oplus FY$ is a free Bloom algebra, thus, it is sufficient to prove that this algebra is corecursive. For that, we use Proposition 4.1 and find a corecursive algebra such that $T \oplus FY$ is its subalgebra; this will finish the proof.

The endofunctor $H(-) + Y$ is also accessible. Thus, it also has a terminal coalgebra (see the proof of Corollary 3.18). We denote it by T_Y . The components of the inverse of its coalgebra structure $T_Y \xrightarrow{\sim} H(T_Y) + Y$ are denoted by $\tau_Y : H(T_Y) \rightarrow T_Y$ and $\eta_Y : Y \rightarrow T_Y$, respectively. As proved in [28] the algebra T_Y is a cia for H , cf. Remark 2.7. We are going to prove that $T \oplus FY$ is a subalgebra of this H -algebra T_Y .

Since H is accessible and preserves monomorphisms, the terminal chain of H converges (and yields a terminal coalgebra T), as proved in [16]. That is, if we define an chain $W : \text{Ord}^{op} \rightarrow \mathcal{A}$ on objects by

$$W_0 = 1 \quad \text{and} \quad W_{i+1} = HW_i$$

with $W_i = \lim_{k < i} W_k$ for limit ordinals (and on morphisms by $w_{1,0} : H1 \rightarrow 1$ unique, $w_{i+1,j+1} = Hw_{i,j}$ and $(w_{i,k})_{k < i}$ forming a limit cone), then there exists an ordinal λ such that $w_{\lambda+1,\lambda} : HW_\lambda \rightarrow W_\lambda$ is invertible. We then get

$$T = W_\lambda \quad \text{and} \quad \tau = w_{\lambda+1,\lambda}^{-1}.$$

We can choose λ arbitrarily large, thus we can assume that λ is a cardinal such that H is λ -accessible.

Since monomorphisms are constructive, $H(-) + Y$ also preserves monomorphisms and is also λ -accessible. Denote by $\overline{W} : \text{Ord}^{op} \rightarrow \mathcal{A}$ its terminal chain. Then this chain converges and yields a terminal coalgebra T_Y . Since we again can choose an arbitrary large ordinal

for the convergence of \overline{W} , we can assume that this is the above cardinal λ , thus, $TY = \overline{W}_\lambda$ and $[\tau_Y, \eta_Y] = \overline{w}_{\lambda+1, \lambda} : H(TY) + Y \rightarrow TY$. We conclude that T is a (canonical) subalgebra of TY : define a natural transformation $m_i : W_i \rightarrow \overline{W}_i$ by $m_0 = id_1$ and $m_{i+1} \equiv HW_i \xrightarrow{Hm_i} H\overline{W}_i \xrightarrow{\text{inl}} H\overline{W}_i + Y$. The constructivity of monomorphisms implies that inl is monic, thus, we see by easy transfinite induction that m_i 's are monomorphisms for all $i \in \text{Ord}$. And $m_\lambda : T \rightarrow TY$ is a coalgebra homomorphism because the λ -th naturality square of (m_i) yields $m_\lambda \cdot \tau = \tau_Y \cdot Hm_\lambda$:

$$\begin{array}{ccc} HT & \xleftarrow{\tau} & T \\ \text{inl} \cdot Hm_\lambda \downarrow & & \downarrow m_\lambda \\ H(TY) + Y & \xleftarrow{[\tau_Y, \eta_Y]^{-1}} & TY \end{array}$$

We now define a cocone $p_i : U_i \rightarrow TY$, for $i \leq \lambda$, of the chain from Construction 4.1 by

$$p_0 = m_\lambda : T \rightarrow TY$$

and

$$p_{i+1} \equiv HU_i + Y \xrightarrow{Hp_{i+Y}} H(TY) + Y \xrightarrow{[\tau_Y, \eta_Y]} TY .$$

We need to verify compatibility, $p_i = p_{i+1} \cdot u_{i, i+1}$, from which the limit steps follow automatically. For $i = 0$, use that m_λ is a coalgebra homomorphism:

$$\begin{array}{ccccc} & & & & u_{0,1} \\ & & & & \curvearrowright \\ T & \xrightarrow{\tau} & HT & \xrightarrow{\text{inl}} & HT + Y \\ & & \downarrow Hm_\lambda & & \swarrow Hm_\lambda + Y \\ & & HTY & \xrightarrow{\text{inl}} & H(TY) + Y \\ & \searrow m_\lambda & \downarrow \tau_Y & \swarrow [\tau_Y, \eta_Y] & \\ & & TY & & \end{array}$$

The isolated step is easy because if the compatibility holds for i , then the diagram

$$\begin{array}{ccc} HU_i + Y & \xrightarrow{Hu_{i, i+1+Y}} & HU_{i+1} + Y \\ & \searrow Hp_{i+Y} \quad \swarrow Hp_{i+1+Y} & \\ & H(TY) + Y & \\ & \downarrow [\tau_Y, \eta_Y] & \\ & TY & \end{array}$$

commutes. It is obvious (by transfinite induction) that all p_i 's are monomorphisms. It remains to verify that $p_\lambda : U_\lambda \rightarrow TY$ is an algebra homomorphism. Indeed, since H preserves λ -filtered colimits, the corecursive chain U_i converges after λ steps, so that $U_\lambda =$

$T \oplus FY$, by Proposition 4.4. We have following commutative square:

$$\begin{array}{ccc} HU_\lambda & \xrightarrow{\text{inr}} & HU_\lambda + Y \simeq U_\lambda \\ Hp_\lambda \downarrow & & \downarrow p_{\lambda+1} \simeq p_\lambda \\ H(TY) & \xrightarrow{\tau_Y (= \text{inr})} & H(TY) + Y \simeq TY \end{array} \quad \square$$

Example 4.7. Free corecursive algebras MY that are obtained as U_ω .

(1) For $H = Id$ we have

$$MY = U_\omega = 1 + Y + Y + Y + \dots$$

Indeed, the terminal object 1 is T , and

$$U_1 = T + Y \qquad U_2 = T + Y + Y \qquad \dots$$

with colimit $U_\omega = 1 + Y + Y + Y + \dots$.

(2) More generally, let $H : \mathcal{A} \rightarrow \mathcal{A}$ preserve countable coproducts and have a terminal coalgebra T . Then

$$MY = U_\omega = T + Y + HY + H^2Y + \dots$$

(3) For the endofunctor $HX = X \times X$ of **Set** (of binary algebras) we have the free corecursive algebra

$$MY = \text{all binary trees with finitely many leaves, all of which are labelled in } Y.$$

Indeed, the corecursive chain U yields

$$U_0 = 1$$

which we represent by the complete binary tree t . Then

$$U_1 = 1 \times 1 + Y \simeq 1 + Y$$

is represented by t and all singleton trees labelled in Y , and

$$U_2 = (1 + Y) \times (1 + Y) + Y$$

is represented by all binary trees with leaves of depth at most 1 labelled in Y . We conclude that

$$U_n = \text{all binary trees with no leaves of depth } \geq n \text{ and leaves labelled in } Y.$$

Consequently, the corecursive chain yields

$$T \oplus FY = \text{colim}_{n < \omega} U_n = \bigcup_{n < \omega} U_n$$

which is the above set of trees. Observe that this description of MY corresponds well with the fact that MY is a free binary algebra with an additional idempotent (see Example 2.6): the unique idempotent of MY is the complete binary tree. And MY is generated by this tree and all finite trees in FY .

(4) More generally, let $\Sigma = (\Sigma_k)_{k < \omega}$ be a signature. Then Σ -algebras are precisely the algebras for the polynomial endofunctor

$$H_\Sigma X = \Sigma_0 + \Sigma_1 \times X + \Sigma_2 \times X^2 + \dots$$

Recall that the terminal coalgebra is the coalgebra of all Σ -trees, that is, trees labelled in Σ so that every node with a label of arity n has precisely n children. And FY is the

algebra of all finite $(\Sigma + Y)$ -trees, where members of Y are considered to have arity 0. Then U_n is the set of all $(\Sigma + Y)$ -trees with no leaf of depth greater than n having a label from Y . (That is, all leaves on level n or more are labelled by a nullary symbol in Σ_0 .) Consequently the free corecursive algebra $\text{colim}_{n < \omega} U_n = \bigcup_{n < \omega} U_n$ is

$$MY = T \oplus FY = \text{all } (\Sigma + Y)\text{-trees in which finitely many leaves are labelled in } Y \text{ and other leaves labelled in } \Sigma_0.$$

- (5) For the finite power set functor \mathcal{P}_f , J. Worrell [34] described the terminal coalgebra T as the coalgebra of all finitely branching, non-ordered, strongly extensional trees. Recall that a (non-ordered) tree is called *strongly extensional* if for every node, the subtrees rooted at distinct children of the given node are not tree bisimilar. The free corecursive algebra is

$$T \oplus FY = \text{all finitely branching, strongly extensional trees with finitely many leaves labelled in } Y \text{ (and other leaves unlabelled).}$$

Indeed, this is analogous to (3) since

$$\begin{aligned} U_1 &\simeq T + Y &= \text{all the trees in } T \text{ and all singleton trees labelled in } Y, \\ U_2 &= \mathcal{P}_f U_1 + Y &= \text{all finitely branching, strongly extensional trees with} \\ &&\text{some leaves of depth less than 2 labelled in } Y, \end{aligned}$$

etc.

Thus $T \oplus FY = \bigcup_{n < \omega} U_n$ is the above set of trees. The corecursive chain converges in ω steps because \mathcal{P}_f preserves ω -colimits.

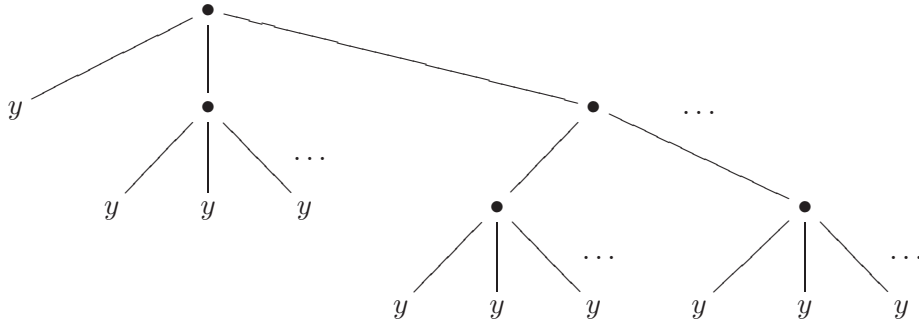
Example 4.8. Now we illustrate the need of U_i for infinite ordinals i . The functor $H : \mathbf{Set} \rightarrow \mathbf{Set}$ with $HX = X^{\mathbb{N}}$ has the following free corecursive algebras, where t_0 is the complete countably branching tree:

$$MY = \text{all countably branching trees with leaves labelled in } Y \text{ such that every infinite path contains a node whose subtree is } t_0.$$

To see this, represent $U_0 = 1$ by the single tree t_0 and obtain

$$\begin{aligned} U_1 &\simeq T + Y &= \text{all singleton trees labelled in } Y, \text{ plus } t_0 \\ U_2 &\simeq (T + Y)^{\mathbb{N}} + Y &= \text{all countably branching trees with leaves at levels at most 1,} \\ &&\text{all of which are labelled in } Y \\ &&\vdots \\ U_\omega &\simeq T + Y &= \text{all countably branching trees which, for some } n < \omega, \text{ have} \\ &&\text{all leaves at level at most } n, \text{ and they are labelled in } Y. \end{aligned}$$

But U_ω is not an algebra because it does not contain the following tree



This tree is in $U_{\omega+1} = U_{\omega}^{\mathbb{N}} + Y$.

We can identify, for every ordinal i , the set U_i with the set of all countably branching trees of type i where type 0 means the tree is t_0 , type $i+1$ means that all maximal subtrees have type i , and for limit ordinals j , type j means type i for some $i < j$. Then

$$U_{\omega_1} = \text{all countably branching trees of countable type.}$$

It is easy to verify that a tree has countable type if and only if on every infinite path there exists a node whose subtree is t_0 . Thus, $U_{\omega_1} = T \oplus FY$ is the above free corecursive algebra.

For the proof of Theorem 4.11, the main result of this section, we are going to use the following technical lemma concerning fixpoints and coproducts of algebras.

Lemma 4.9. *Let $a : HA \xrightarrow{\sim} A$ be a fixpoint and $B = HB + Y$. If the algebras (A, a) and (B, inl) have a coproduct $(C, c) = (A, a) \oplus (B, \text{inl})$ with injections i and j in $\text{Alg } H$, then we have a coproduct $C = HC + Y$ in \mathcal{A} with injections*

$$c : HC \rightarrow C \quad \text{and} \quad d \equiv Y \xrightarrow{\text{inr}} B \xrightarrow{j} C$$

Proof. For the algebra $\overline{C} = HC + Y$ with structure

$$\overline{c} \equiv H(HC + Y) \xrightarrow{H[c,d]} HC \xrightarrow{\text{inl}} HC + Y$$

we observe that $a^{-1} \cdot Hi \cdot \text{inl} : A \rightarrow \overline{C}$ and $Hj + id : B \rightarrow \overline{C}$ are homomorphisms. Indeed, for the former we have the commutative diagram

$$\begin{array}{ccccc} HA & \xrightarrow{a} & A & & \\ \text{\scriptsize } Ha^{-1} \downarrow & \searrow & \downarrow a^{-1} & & \\ HHA & \xrightarrow{Ha} & HA & & HA \\ \text{\scriptsize } HHi \downarrow & & \downarrow Hi & & \downarrow Hi \\ HHC & & HC & & HC \\ \text{\scriptsize } Hinl \downarrow & \searrow Hc & \downarrow \text{inl} & & \downarrow \text{inl} \\ H(HC + Y) & \xrightarrow{H[c,d]} & HC & \xrightarrow{\text{inl}} & \overline{C} \end{array}$$

whose middle left-hand part commutes because i is a homomorphism. For the latter one consider the diagram

$$\begin{array}{ccccc} HB & \xrightarrow{\text{inl}} & B = HB + Y & & \\ \text{\scriptsize } H(Hj+id) \downarrow & \searrow Hj & \downarrow Hj+Y & & \\ H(HC + Y) & \xrightarrow{H[c,d]} & HC & \xrightarrow{\text{inl}} & HC + Y \end{array}$$

The left-hand triangle commutes because $d = j \cdot \text{inr}$ and because j is a homomorphism, that is, $c \cdot Hj = j \cdot \text{inl}$. Therefore, we have a unique homomorphism $h : C \rightarrow \overline{C}$ with

$$h \cdot i = \text{inl} \cdot Hi \cdot a^{-1} \tag{4.1}$$

and

$$h \cdot j = Hj + id. \tag{4.2}$$

This is the inverse of $[c, d] : \overline{C} \rightarrow C$. Indeed, we prove $[c, d] \cdot h = id$ by using the universal property of coproducts. Firstly, $[c, d]$ is a homomorphism:

$$\begin{array}{ccccc} H(HC + Y) & \xrightarrow{H[c,d]} & HC & \xrightarrow{\text{inl}} & HC + Y \\ H[c,d] \downarrow & & & \searrow c & \downarrow [c,d] \\ HC & \xrightarrow{c} & & & C \end{array}$$

Therefore, $[c, d] \cdot h$ is an endomorphism of (C, c) . And we have both

$$\begin{aligned} [c, d] \cdot h \cdot i &= [c, d] \cdot \text{inl} \cdot Hi \cdot a^{-1} && \text{by (4.1)} \\ &= c \cdot Hi \cdot a^{-1} \\ &= i \cdot a \cdot a^{-1} && i \text{ is a homomorphism} \\ &= i \end{aligned}$$

and

$$\begin{aligned} [c, d] \cdot h \cdot j &= [c, d] \cdot (Hj + id) && \text{by (4.2)} \\ &= [c \cdot Hj, d] \\ &= [j \cdot \text{inl}, j \cdot \text{inr}] && \text{by (4.1)} \\ &= j. \end{aligned}$$

The remaining identity $h \cdot [c, d] = id$ translates into $h \cdot c = \text{inl}$ and $h \cdot d = \text{inr}$. The former equality follows from $[c, d] \cdot h = id$ and the fact that h is a homomorphism

$$\begin{array}{ccc} HC & \xrightarrow{c} & C \\ Hh \downarrow & \searrow & \downarrow h \\ H\overline{C} & \xrightarrow{H[c,d]} HC & \xrightarrow{\text{inl}} \overline{C} \end{array}$$

the latter one follows from (4.1) and (4.2) since $h \cdot d = h \cdot j \cdot \text{inr} = (Hj + id) \cdot \text{inr} = \text{inr}$. \square

Remark 4.10.

- (a) A *pre-fixpoint* of a functor H is an object A such that HA is a subobject of A .
- (b) A *fixpoint*, i. e. an object $A \simeq HA$, can be considered as an algebra or a coalgebra for H . When we speak about *corecursive fixpoints*, we mean fixpoints $\alpha : HA \xrightarrow{\sim} A$ that are corecursive algebras.

Theorem 4.11. *For every set functor, the following statements are equivalent:*

- (i) *free corecursive algebras exist,*
- (ii) *free algebras and a terminal coalgebra exist, and*
- (iii) *arbitrarily large pre-fixpoints and a corecursive fixpoint exist.*

They imply that the free corecursive algebra on Y is $T \oplus FY$.

Proof. We can assume, without loss of generality, that H preserves monomorphisms: see the argument for Corollary 3.14. The equivalence of (ii) and (iii) follows from the fact that (a) arbitrarily large pre-fixpoints are necessary and sufficient for the existence of free algebras, see Theorem II.4 in [33] and (b) every corecursive fixpoint is an initial corecursive algebra, thus, a terminal coalgebra, see Proposition 7 in [21].

Let us prove (i) \Leftrightarrow (ii).

(i) \Rightarrow (ii): This follows from Lemma 3.9 and Proposition 3.13 which were formulated for Bloom algebras, but hold for corecursive algebras as well. The condition $\mathbf{Set}(Y, HY) \neq \emptyset$ is automatic for all set functors except the constant functor on \emptyset , and (ii) is trivial in that case.

(ii) \Rightarrow (i): Precisely as in the proof of Theorem 4.6 we need to prove that $T \oplus FY$ exists and is a subalgebra of a corecursive algebra. The existence is clear, since $\mathbf{Alg} H$ is cocomplete. Indeed, the existence of free algebras implies that H generates a free monad \mathbb{F} , and then $\mathbf{Alg} H$ is equivalent to the category $\mathbf{Set}^{\mathbb{F}}$ of Eilenberg-Moore algebras for \mathbb{F} , see [18]. And monadic algebras over \mathbf{Set} always form cocomplete categories. We prove that the chain from Construction 4.1 converges, thus, from Proposition 4.4 we derive $T \oplus FY = U_\gamma$ for some ordinal γ . We apply Lemma 4.9 to the coproduct $(C, c) = (T, \tau^{-1}) \oplus (FY, \phi_Y)$ with injections $\overline{\text{inl}}, \overline{\text{inr}}$: put $d = \text{inr} \cdot \eta_Y : Y \rightarrow C$, then $C = HC + Y$ with injections c, d . Let us define a cocone $m_i : U_i \rightarrow C$ of the chain U_i by $m_0 = \overline{\text{inl}} : T \rightarrow C$ and

$$m_{i+1} \equiv HU_i + Y \xrightarrow{Hm_i + id} HC + Y \xrightarrow{[c, d]} C.$$

We verify that m_i form a cocone: the first triangle commutes since $\overline{\text{inl}}$ is a homomorphism:

$$\begin{array}{ccccc} T & \xrightarrow{\tau} & HT & \xrightarrow{\text{inl}} & HT + Y \\ & \searrow^{m_0 = \overline{\text{inl}}} & \downarrow^{H\overline{\text{inl}}} & & \downarrow^{H\overline{\text{inl}} + Y} \\ & & HC & \xrightarrow{\text{inl}} & HC + Y \\ & & \downarrow^c & \swarrow^{[c, d]} & \downarrow^{[c, d]} \\ & & C & \xleftarrow{m_1} & C \end{array}$$

If the i -th triangle commutes, it is immediately seen that the next one commutes, too. Furthermore, each m_i is a monomorphism. Indeed, $m_0 = \overline{\text{inl}}$ is a split monomorphism: choose any morphism $f_0 : Y \rightarrow T$ in \mathbf{Set} and extend it to a homomorphism $f : (FY, \phi_Y) \rightarrow (T, \tau^{-1})$, then $[id_T, f]$ splits $\overline{\text{inl}}$. If m_i is a monomorphism, then so is m_{i+1} because $[c, d]$ is an isomorphism. And limit steps are clear.

Since C has only a set of subobjects, we know that there exist $\delta > \gamma$ such that m_γ and m_δ represent the same subobject, i. e., $u_{\gamma, \delta} : U_\gamma \rightarrow U_\delta$ is an isomorphism. Then also $u_{\gamma+1, \delta+1} : U_{\gamma+1} \rightarrow U_{\delta+1}$ is clearly an isomorphism with an inverse $u' : U_{\delta+1} \rightarrow U_{\gamma+1}$, say. It follows that $u_{\delta, \delta+1}$ is an isomorphism, too: it is a monomorphism since $m_{\delta+1} \cdot u_{\delta, \delta+1} = m_\delta$ and a split epimorphism since $u_{\delta, \delta+1} \cdot u_{\gamma+1, \delta} \cdot u' = id$. Thus we conclude that $T \oplus FY = U_\delta$.

The rest of the proof is analogous to the proof of Theorem 4.6, except that we are not using the terminal coalgebra TY of $H(-) + Y$ (which need not exist), but one of its ‘‘approximations’’ given by the members \overline{W}_i of the terminal chain. In more detail, for every ordinal i the algebra $\overline{w}_{i+1, i} : H\overline{W}_i + Y \rightarrow \overline{W}_i$ for $H(-) + Y$ is corecursive (apply Example 2.3(5) for $H(-) + Y$). It is easy to derive that \overline{W}_i is corecursive also when considered as an algebra for H :

$$H\overline{W}_i \xrightarrow{\text{inr}} H\overline{W}_i + Y \xrightarrow{\overline{w}_{i+1, i}} \overline{W}_i$$

Now the existence of the terminal coalgebra T for H implies, as proved in [6], that the terminal chain W for H converges. Let λ be an ordinal with $T = W_\lambda$ and $\tau = w_{\lambda+1, \lambda}^{-1}$. As

in the proof of Theorem 4.6 we define a natural transformation $m_i : W_i \rightarrow \overline{W}_i$ by

$$m_0 = id_1 \quad \text{and} \quad m_{i+1} = \text{inl} \cdot Hm_i$$

and after λ steps obtain a homomorphism $m_\lambda : T \rightarrow \overline{W}_\lambda$ of algebras for H which is monic. And then we obtain a cocone of the corecursive chain $p_i : U_i \rightarrow \overline{W}_\lambda$ by

$$p_0 = m_\lambda \quad \text{and} \quad p_{i+1} \equiv HU_i + Y \xrightarrow{Hp_i + id} H\overline{W}_\lambda + Y \xrightarrow{\overline{w}_{\lambda+1, \lambda}} \overline{W}_\lambda.$$

Analogously to the proof of Theorem 4.6, if the corecursive chain converges after γ steps, then

$$p_\gamma : T \oplus FY = U_\gamma \rightarrow \overline{W}_\lambda$$

is the desired monomorphism in $\text{Alg } H$. □

Notation 4.12. Let H have free corecursive algebras MY . Denote by

$$\delta_Y : HMY \rightarrow MY$$

the algebra structure and by

$$\eta_Y : Y \rightarrow MY$$

the universal map. Then we obtain a unique homomorphism

$$\mu_Y : (MMY, \delta_{MY}) \rightarrow (MY, \delta_Y)$$

with $\mu_Y \cdot \eta_{MY} = id$:

$$\begin{array}{ccc} HMMY & \xrightarrow{\delta_{MY}} & MMY \xleftarrow{\eta_Y} MY \\ H\mu_Y \downarrow & & \downarrow \mu_Y \swarrow id_{MY} \\ HMY & \xrightarrow{\delta_Y} & MY \end{array} \quad (4.3)$$

The triple

$$\mathbb{M} = (M, \mu, \eta)$$

is the monad generated by the adjoint situation

$$\text{Alg}_C H \xleftarrow{\perp} \mathcal{A}.$$

Example 4.13. We have

$$MY = \mathbb{N} \times Y + 1$$

for the identity functor on \mathbf{Set} , see Example 4.7(1). And for $HX = X \times X$ we have

MY = binary trees with finitely many leaves, all of which are labelled in Y , see Example 4.7(3). The functor $HX = \coprod_{n < \omega} X^n$ generates the monad

MY = finitely branching trees with finitely many leaves labelled in Y

cf. Example 4.7(4) for Σ with one n -ary operation for every $n < \omega$.

Remark 4.14. Since μ_Y is, by definition, a homomorphism:

$$\mu_Y \cdot \delta_{MY} = \delta_Y \cdot H\mu_Y$$

the unit law $\mu_Y \cdot \eta_{MY} = id$ yields

$$\delta_Y = \mu_Y \cdot \delta_{MY} \cdot H\eta_{MY}.$$

It is easy to prove that the δ_Y are the components of a natural transformation $\delta : HM \rightarrow M$.

Theorem 4.15. *Let \mathcal{A} be a locally presentable category with constructive monomorphisms, and let $H : \mathcal{A} \rightarrow \mathcal{A}$ be an accessible endofunctor preserving monomorphisms. Then Bloom algebras are precisely the Eilenberg-Moore algebras for \mathbb{M} , i.e., the category $\mathbf{Alg}_B H$ is isomorphic to $\mathcal{A}^{\mathbb{M}}$.*

Proof. The forgetful functors $U_B : \mathbf{Alg}_B \rightarrow \mathcal{A}$ and $U_C : \mathbf{Alg}_C H \rightarrow \mathcal{A}$ are both right adjoints, and their left adjoints are, in both cases, defined by $Y \mapsto T \oplus FY$, see Theorem 3.16 and Proposition 4.4. Therefore the monads generated by U_B and U_C are the same, namely the monad \mathbb{M} . We are going to prove that the comparison functor from $\mathbf{Alg}_B H$ to $\mathcal{A}^{\mathbb{M}}$ is an isomorphism. By Beck's Theorem, see [20, 4.4.4], all we need to prove is that U_B creates coequalizers of U_B -split pairs. That is, we need to prove that given a parallel pair of solution preserving homomorphisms

$$f, g : (A, a, \dagger) \rightarrow (B, b, \ddagger)$$

and given morphisms k, s, t in \mathcal{A} as follows:

$$\begin{aligned} k : B &\rightarrow C & \text{with } k \cdot f &= k \cdot g \\ s : C &\rightarrow B & \text{with } k \cdot s &= id_A \end{aligned}$$

and

$$t : B \rightarrow A \quad \text{with } s \cdot k = f \cdot t \quad \text{and } id_B = g \cdot t$$

there exists a unique structure $(C, c, *)$ of a Bloom algebra such that k is a solution preserving homomorphism; moreover, k is then a coequalizer in $\mathbf{Alg}_B H$. Since H is accessible, free H -algebras exist, see Corollary 3.10. The corresponding monad is the free monad on H and its Eilenberg-Moore algebras are precisely the H -algebras, see [18]. Consequently, by Beck's Theorem there exists a unique structure $c : HC \rightarrow C$ of an algebra for which $k : (B, b) \rightarrow (C, c)$ is a coequalizer in $\mathbf{Alg} H$. And by Lemma 3.7 there exists a unique structure $(C, c, *)$ of a Bloom algebra for which k is a solution preserving algebra homomorphism. It remains to verify that k is a coequalizer in $\mathbf{Alg}_B H$. To this end, let

$$h : (B, b, \ddagger) \rightarrow (D, d, +)$$

be a solution preserving algebra homomorphism with $h \cdot f = h \cdot g$. There exists a unique homomorphism $h' : (C, c) \rightarrow (D, d)$ with $h = h' \cdot k$. And h' preserves solutions (that is for every $e : X \rightarrow HX$ we have $h' \cdot e^* = e^+$) because both k and h do:

$$h' \cdot e^* = h' \cdot k \cdot e^\ddagger = h \cdot e^\ddagger = e^+. \quad \square$$

Remark 4.16. In the case $\mathcal{A} = \mathbf{Set}$ we do not need to assume accessibility: whenever an endofunctor H has free corecursive algebras, then Bloom algebras are precisely the algebras for the monad \mathbb{M} . The proof just uses Theorem 4.11 in lieu of 4.4.

Example 4.17. A non-accessible set functor having free corecursive algebras. Let \mathbb{A} and \mathbb{B} be proper classes of infinite cardinals such that for every cardinal $\alpha \in \mathbb{A}$ the interval $[\alpha, 2^\alpha]$ is disjoint with \mathbb{B} . The functor H assigns to every set X the Set

$$HX = \{M \subseteq X \mid \text{card } M \in \mathbb{B} \text{ or } M = \emptyset\}$$

and to every function $f : X \rightarrow Y$ the following function

$$Hf(M) = \begin{cases} f[M] & \text{if } f \text{ is monic when restricted to } M \\ \emptyset & \text{otherwise} \end{cases}$$

Then H has the terminal coalgebra 1 (since $H1 = 1$). It also has free algebras. Indeed, this follows from the fact that H has arbitrary large pre-fixpoints (see the proof of Theorem 4.11): for every cardinal $\alpha \in \mathbb{A}$ the cardinal 2^α is a pre-fixpoint due to

$$\text{card } H(2^\alpha) \leq 1 + \sum_{\beta \in \mathbb{B}, \beta \leq 2^\alpha} (2^\alpha)^\beta \leq \sum_{\beta < \alpha} (2^\alpha)^\beta \leq \alpha \cdot (2^\alpha)^\omega = 2^\alpha.$$

However, H is not λ -accessible for any λ : choose $\beta \in \mathbb{B}$ with $\beta \geq \lambda$, then since $\beta \in H\beta$ we see that H does not preserve the lambda-directed colimit of β as the union of all subsets of β of cardinality less than λ .

5. CORECURSIVE MONADS

The iterative theories (or iterative monads) of C. Elgot [22] were introduced as a formalization of iteration in an algebraic setting, and in [23] completely iterative theories were studied. We first recall the concept of a completely iterative monad, and then introduce the weaker concept of a corecursive monad. The relationship between these two concepts is analogous to the relationship between cia's, see Remark 2.7, and corecursive algebras. The following definition is, for the base category \mathbf{Set} , equivalent to completely iterative theories, as shown in [1].

Definition 5.1. (See [1])

(1) An *ideal monad* is a sextuple

$$\mathbb{S} = (S, \eta, \mu, S', \sigma, \mu')$$

consisting of a monad (S, η, μ) , a subfunctor $\sigma : S' \rightarrow S$ (called the *ideal of \mathbb{S}*) such that $S = S' + Id$ with injections σ and η , and a natural transformation $\mu' : S'S \rightarrow S'$ restricting μ , i.e., with $\sigma \cdot \mu' = \mu \cdot \sigma S$.

(2) An *equation morphism with parameters* for \mathbb{S} is a morphism $e : X \rightarrow S(X + Y)$, we call X the variables and Y the parameters of e . It is called *ideal* if it factorizes through σ_{X+Y} . A solution of e is a morphism $e^\dagger : X \rightarrow SY$ such that the following square commutes:

$$\begin{array}{ccc} X & \xrightarrow{e^\dagger} & SY \\ e \downarrow & & \uparrow \mu_Y \\ S(X + Y) & \xrightarrow{S[e^\dagger, \eta_Y]} & SSY \end{array} \quad (5.1)$$

(3) An ideal monad is called *completely iterative* provided that every ideal equation morphism has a unique solution.

Example 5.2. (See [1]) Let H be an endofunctor of \mathcal{A} such that for every object Y a terminal coalgebra TY of $H(-) + Y$ exists. Then the assignment $Y \mapsto TY$ yields a monad (T, η, μ) , which is the monad of free cia's for H . This is an ideal monad w.r.t. $T' = HT$ and $\mu' = H\mu$. Moreover, this monad is completely iterative, indeed, the free completely iterative monad on H .

For example the set functor $HX = X \times X$ generates the free completely iterative monad \mathbb{T} with

$$TY = \text{all binary trees with leaves labelled in } Y.$$

Definition 5.3. Let \mathbb{S} be an ideal monad. An *equation morphism* (without parameters) is a morphism $e : X \rightarrow SX$, and e is called *ideal* if it factorizes through σ_X , i. e., there exist $e_0 : X \rightarrow S'X$ such that the diagram below commutes:

$$\begin{array}{ccc} X & \xrightarrow{e} & SX \\ & \searrow e_0 & \uparrow \sigma_X \\ & & S'X \end{array}$$

The monad \mathbb{S} is called *corecursive* if every ideal equation morphism e has a unique solution $e^\dagger : X \rightarrow SY$, i. e., the square below commutes:

$$\begin{array}{ccc} X & \xrightarrow{e^\dagger} & SY \\ e \downarrow & & \uparrow \mu_Y \\ SX & \xrightarrow{Se^\dagger} & SSY \end{array}$$

Example 5.4. Examples of corecursive monads on **Set**.

- (1) All the monads of Example 4.7 are corecursive, as we will see in Theorem 6.4 below.
- (2) All completely iterative monads are corecursive, e.g,

$$SY = \text{all finitely branching trees with leaves labelled in } Y,$$

This is the free completely iterative monad on the functor $HX = \coprod_{n < \omega} X^n$.

- (3) Consider the monad

$$RY = \text{all rational, finitely branching trees with leaves labelled in } Y,$$

where *rational* means that the tree has up to isomorphism only finitely many subtrees. This is a corecursive monad that is neither free on any endofunctor, nor completely iterative.

- (4) More generally, every submonad of S in item (2) containing the complete binary tree is corecursive.

Proposition 5.5. *The monad $\mathbb{M} = (M, \eta, \mu)$ of free corecursive algebras (of Notation 4.12) is ideal w.r.t. the ideal $M' = HM$ where $\sigma = \delta : HM \rightarrow M$ and $\mu' = H\mu : HMM \rightarrow HM$.*

Proof. (1) We prove

$$MY = HMY + Y$$

with injections δ_Y and η_Y . First apply Lemma 2.5 to the corecursive algebra MY and $\eta_Y : Y \rightarrow MY$ to see that $HMY + Y$ is a corecursive algebra, too. Using the freeness of MY we see that for the right-hand injection morphism $\text{inr} : Y \rightarrow HMY + Y$ there exists a unique homomorphism $\overline{\text{inr}} : MY \rightarrow HMY + Y$ such that $\overline{\text{inr}} \cdot \eta_Y = \text{inr}$. In order to prove that $\overline{\text{inr}}$ is an inverse for $[\delta_Y, \eta_Y]$ we consider the following diagram

$$\begin{array}{ccccc} Y & \xrightarrow{\eta_Y} & MY & \xleftarrow{\delta_Y} & HMY \\ & \searrow \text{inr} & \downarrow \overline{\text{inr}} & & \downarrow H\overline{\text{inr}} \\ & & HMY + Y & \xleftarrow{\text{inl} \cdot H[\delta_Y, \eta_Y]} & H(HMY + Y) \\ & \searrow \eta_Y & \downarrow [\delta_Y, \eta_Y] & & \downarrow H[\delta_Y, \eta_Y] \\ & & MY & \xleftarrow{\delta_Y} & HMY \end{array} \quad (5.2)$$

The upper square and triangle commute by the definition of $\overline{\text{inr}}$, and the lower triangle and square obviously do. Thus $[\delta_Y, \eta_Y]$ is homomorphism, and the composition of the two squares clearly commutes. So $[\delta_Y, \eta_Y] \cdot \overline{\text{inr}} = id_{MY}$ follows from the universal property of η_Y . Now the upper square of (5.2) yields $\overline{\text{inr}} \cdot \delta_Y = \text{inl}$, consequently $(\overline{\text{inr}} \cdot [\delta_Y, \eta_Y]) \cdot \text{inl} = \overline{\text{inr}} \cdot \delta_Y = \text{inl}$. Since also $(\overline{\text{inr}} \cdot [\delta_Y, \eta_Y]) \cdot \text{inr} = \overline{\text{inr}} \cdot \eta_Y = \text{inr}$, we conclude $\overline{\text{inr}} \cdot [\delta_Y, \eta_Y] = id$.

(2) It remains to prove $\sigma \cdot \mu' = \mu \cdot \sigma S$, that is $\delta \cdot H\mu = \mu \cdot \delta M$, which one obtains from Diagram (4.3). \square

Theorem 5.6. *The monad \mathbb{M} of free corecursive algebras is corecursive.*

Proof. We need to prove that every ideal equation morphism $e = \delta_X \cdot e_0$ where $e_0 : X \rightarrow HMX$ has a unique solution in all algebras (MY, μ_Y) . We prove a stronger statement: if $a : HA \rightarrow A$ is a corecursive algebra, then e has a unique solution in A . That means that there exists a unique morphism $e^\dagger : X \rightarrow A$ such that the square

$$\begin{array}{ccc} X & \xrightarrow{e^\dagger} & A \\ e \downarrow & & \uparrow \bar{a} \\ MX & \xrightarrow{Me^\dagger} & MA \end{array}$$

commutes, where \bar{a} is the corresponding Eilenberg-Moore algebra structure (the unique homomorphism of H -algebras from MA to A such that $\bar{a} \cdot \eta_A = id_A$). Form the equation morphism

$$\bar{e} \equiv MX \xrightarrow{[\delta_X, \eta_X]^{-1}} HMX + X \xrightarrow{[HMX, e_0]} HMX.$$

There is a unique solution s of \bar{e} :

$$\begin{array}{ccc} MX & \xrightarrow{s} & A \\ [\delta_X, \eta_X]^{-1} \left(\begin{array}{c} \uparrow \\ \downarrow \end{array} \right) [\delta_X, \eta_X] & & \uparrow a \\ HMX + X & & \\ [HMX, e_0] \downarrow & & \\ HMX & \xrightarrow{Hs} & HA \end{array} \quad (5.3)$$

Inspecting the two coproduct components of $HMX + X$ separately, we see that the commutativity of (5.3) is equivalent to the following two equations:

$$s \cdot \delta_X = a \cdot Hs \quad (5.4)$$

$$s \cdot \eta_X = a \cdot Hs \cdot e_0 \quad (5.5)$$

Equation (5.4) states that s is a homomorphism. Also since $\eta : Id \rightarrow M$ is a natural transformation we have the commutative diagram

$$\begin{array}{ccc}
X & \xrightarrow{\eta_X} & MX \\
\downarrow \eta_X & & \downarrow M\eta_X \\
MX & \xrightarrow{\eta_{MX}} & MMX \\
\downarrow s & & \downarrow Ms \\
A & \xrightarrow{\eta_A} & MA
\end{array}$$

which gives us $\bar{a} \cdot Ms \cdot M\eta_X \cdot \eta_X = \bar{a} \cdot \eta_A \cdot s \cdot \eta_X = s \cdot \eta_X$. From this the universal property of η_X implies the equation

$$\bar{a} \cdot Ms \cdot M\eta_X = s. \quad (5.6)$$

We prove that $e^\dagger = s \cdot \eta_X$ is a solution of e by inspecting the diagram

$$\begin{array}{ccccc}
& & & & e^\dagger \\
& & & & \curvearrowright \\
X & \xrightarrow{\eta_X} & MX & \xrightarrow{s} & A \\
& \searrow \text{inr} & \downarrow [\delta_X, \eta_X]^{-1} & & \nearrow a \\
& & HMX + X & & HA \\
& \searrow e_0 & \downarrow [HMX, e_0] & & \nearrow Hs \\
& & HMX & \xrightarrow{HMe^\dagger} & HMA \\
& & \nearrow \delta_{MX} & & \searrow \delta_{MA} \\
MX & \xrightarrow{Me^\dagger} & & & MA \\
& & & & \nearrow \bar{a} \\
& & & & A
\end{array} \quad (5.7)$$

The left-hand triangle commutes because e is an ideal equation. The upper right-hand triangle is Diagram (5.3), the middle triangle commutes because of Equation (5.6) and $e^\dagger = s \cdot \eta_X$, and the other parts commute obviously.

To show uniqueness, suppose that e^\dagger is an arbitrary solution of e . We take $s = \bar{a} \cdot Me^\dagger : MX \rightarrow A$, and prove that s is a solution of \bar{e} . First we note that s is an algebra homomorphism for H , since both \bar{a} and Me^\dagger are, thus Equation (5.4) holds for s . We also have $s \cdot \eta_X = e^\dagger$ using the naturality of η :

$$s \cdot \eta_X = \bar{a} \cdot Me^\dagger \cdot \eta_X = \bar{a} \cdot \eta_A \cdot e^\dagger = e^\dagger.$$

Now all the parts in Diagram (5.7) commute except, possibly, the upper right-hand triangle. So this triangle commutes when precomposed with $\eta_X : X \rightarrow MX$ from which we conclude Equation (5.5). Combining (5.4) and (5.5) we have, equivalently, that Diagram (5.3) commutes, i. e., s is a solution of \bar{e} . \square

6. FREE CORECURSIVE MONAD

In this section we prove that the corecursive monad \mathbb{M} given by the free corecursive algebras for H is a free corecursive monad on H . For that we need the appropriate concept of morphism:

Definition 6.1.

- (a) An *ideal monad morphism* from an ideal monad $(S, \eta^S, \mu^S, S', \sigma, \mu'^S)$ to an ideal monad $(U, \eta^U, \mu^U, U', \omega, \mu'^U)$ is a pair consisting of a monad morphism $\lambda : (S, \eta^S, \mu^S) \rightarrow (U, \eta^U, \mu^U)$ and a natural transformation $\lambda' : S' \rightarrow U'$ with $\lambda \cdot \sigma = \omega \cdot \lambda'$.
- (b) Given a functor H , a natural transformation $\lambda : H \rightarrow S$ is called *ideal* if it factors through $\sigma : S' \rightarrow S$.
- (c) By a *free corecursive monad* on an endofunctor H is meant a corecursive monad $\mathbb{S} = (S, \mu, \eta, S', \sigma, \mu'^S)$ together with an ideal natural transformation $\kappa : H \rightarrow S$ with the following universal property: For every ideal natural transformation $\lambda : H \rightarrow \overline{S}$, where \overline{S} is a corecursive monad, there exists a unique ideal monad morphism $(\hat{\lambda}, \hat{\lambda}') : \mathbb{S} \rightarrow \overline{S}$ such that the triangle below commutes:

$$\begin{array}{ccc} H & \xrightarrow{\kappa} & S \\ & \searrow \lambda & \downarrow \hat{\lambda} \\ & & \overline{S} \end{array}$$

Remark 6.2. Let $\text{CMon}(\mathcal{A})$ denote the category of corecursive monads and ideal monad morphisms. We have a forgetful functor to $\text{Fun}(\mathcal{A}, \mathcal{A})$, the category of all endofunctors of \mathcal{A} , assigning to every corecursive monad \mathbb{S} its ideal S' . A free corecursive monad on $H \in \text{Fun}(\mathcal{A}, \mathcal{A})$ is precisely a universal arrow from H to the above forgetful functor.

Example 6.3. If H has free corecursive algebras, then we have the corecursive monad \mathbb{M} of Proposition 5.5. And the natural transformation

$$\kappa \equiv H \xrightarrow{H\eta} HM \xrightarrow{\delta} M$$

(see Notation 4.12) is obviously ideal. We prove that κ has the universal property:

Theorem 6.4. *If an endofunctor H has free corecursive algebras, then the corresponding monad \mathbb{M} is the free corecursive monad on H .*

Remark 6.5. The proof is analogous to the corresponding theorem for free completely iterative monads, see [28, Theorem 4.3].

Proof. For every corecursive monad

$$\mathbb{S} = (S, \mu^S, \eta^S, S', \sigma, \mu'^S)$$

and every ideal natural transformation

$$\begin{array}{ccc} H & \xrightarrow{\lambda} & S \\ & \searrow \lambda' & \uparrow \sigma \\ & & S' \end{array}$$

we are going to find an ideal monad morphism $(\hat{\lambda}, \hat{\lambda}') : \mathbb{M} \rightarrow \mathbb{S}$ with $\lambda = \hat{\lambda} \cdot \kappa$, and prove that it is unique.

(a) Every object SA , $A \in \mathcal{A}$, is considered as an algebra for H via

$$\rho_A \equiv HSA \xrightarrow{\lambda_{SA}} SSA \xrightarrow{\mu_A^S} SA. \quad (6.1)$$

We prove that SA is corecursive. Every equation morphism $e : X \rightarrow HX$ yields the following equation morphism $\bar{e} \equiv X \xrightarrow{e} HX \xrightarrow{\lambda_X} SX$ w.r.t. the monad \mathbb{S} and, and since λ is an ideal natural transformation we see that e an ideal equation morphism. We now prove that e^\dagger is a solution of e in SA if and only if it is the solution of \bar{e} with respect to the corecursive monad \mathbb{S} . Let e^\dagger be a solution of e in SA , that is, the upper part of the diagram below commutes:

$$\begin{array}{ccc} X & \xrightarrow{e^\dagger} & SA \\ \downarrow e & \nearrow \rho_A & \uparrow \mu_A^S \\ HX & \xrightarrow{He^\dagger} & HSA \\ \downarrow \lambda_X & \searrow \lambda_{SA} & \downarrow \\ SX & \xrightarrow{Se^\dagger} & SSA \end{array}$$

Since the lower part commutes because of naturality of λ and the right-hand triangle by definition of ρ , we see that e^\dagger is a solution of \bar{e} .

Conversely, if e^\dagger is a solution of \bar{e} , then the outside of the above diagram commutes. Consequently, the upper part is commutative, showing e^\dagger to be a solution of e as desired. Thus, SA is a corecursive algebra.

(b) There exists a unique homomorphism $\hat{\lambda}_A : MA \rightarrow SA$ of H -algebras such that $\hat{\lambda}_A \cdot \eta_A = \eta_A^S$. Now we show that $\hat{\lambda}$ is a natural transformation. Consider $f : A \rightarrow B$, then $Sf : SA \rightarrow SB$ is a homomorphism:

$$\begin{array}{ccccc} HSA & \xrightarrow{\lambda_{SA}} & SSA & \xrightarrow{\mu_A} & SA \\ HSf \downarrow & & SSf \downarrow & & \downarrow Sf \\ HSB & \xrightarrow{\lambda_{SB}} & SSB & \xrightarrow{\mu_B} & SB \end{array} \quad (6.2)$$

The outside of the following diagram

$$\begin{array}{ccc} MA & \xrightarrow{\hat{\lambda}_A} & SA \\ \downarrow Mf & \nearrow \eta_A & \nearrow \eta_A^S \\ & A & \\ \downarrow Sf & \searrow \eta_B & \searrow \eta_B^S \\ MB & \xrightarrow{\hat{\lambda}_B} & SB \end{array}$$

commutes by the universal property of η_A .

(c) We prove next that $\hat{\lambda}$ is a monad morphism. That is, the following diagrams are commutative:

$$\begin{array}{ccc}
 M & \xrightarrow{\hat{\lambda}} & S \\
 & \eta \swarrow & \nearrow \eta^S \\
 & Id &
 \end{array}
 \qquad
 \begin{array}{ccccc}
 MM & \xrightarrow{\hat{\lambda}M} & SM & \xrightarrow{S\hat{\lambda}} & SS \\
 \mu \downarrow & & & & \downarrow \mu^S \\
 M & \xrightarrow{\hat{\lambda}} & S & &
 \end{array}$$

The left-hand triangle commutes because of the definition of $\hat{\lambda}$. For the right hand square we note that $S\hat{\lambda}_A$ is a homomorphism (cf. (6.2) with $f = \hat{\lambda}_A$), all (components of) the other natural transformations are also clearly homomorphisms, and we have the following diagram

$$\begin{array}{ccccc}
 MMA & \xrightarrow{\hat{\lambda}_{MA}} & SMA & \xrightarrow{S\hat{\lambda}_A} & SSA \\
 \eta_{MA} \swarrow & & \nearrow \eta_{MA}^S & & \nearrow \eta_{SA}^S \\
 MA & \xrightarrow{\hat{\lambda}_A} & SA & & \\
 \mu_A^M \downarrow & & & & \downarrow \mu_A^S \\
 MA & \xrightarrow{\hat{\lambda}_A} & SA & &
 \end{array}$$

The right-hand square commutes by naturality of η^S , hence the outside square is commutative.

(d) Now we have to show that $\lambda = \hat{\lambda} \cdot \kappa = \hat{\lambda} \cdot \delta \cdot H\eta$ which follows from the commutativity of the diagram below:

$$\begin{array}{ccccc}
 HA & \xrightarrow{H\eta_A} & HMA & \xrightarrow{\delta_A} & MA \\
 \lambda_A \downarrow & & \swarrow \lambda_{MA} & & \downarrow \hat{\lambda}_A \\
 SA & \xrightarrow{S\eta_A} & SMA & \xrightarrow{S\hat{\lambda}_A} & SSA \\
 & \searrow S\eta_A^S & & \swarrow \lambda_{SA} & \nearrow \rho_A \\
 & & & SSA & \xrightarrow{\mu_A^S} & SA \\
 & & & \text{---} id \text{---} &
 \end{array}$$

The left-hand upper part and the central one commute because λ is an ideal natural transformation. The right-hand upper part commutes by the definition of $\hat{\lambda}$. The lower right-hand triangle commutes by the definition of ρ , see (6.1) and the lowest part commutes by the monad laws of \mathbb{S} .

(e) Next we show that $(\hat{\lambda}, \hat{\lambda}')$, where $\hat{\lambda}' = \mu'^S \cdot \lambda' S \cdot H\hat{\lambda}$, is an ideal monad morphism. This follows from

$$\begin{array}{ccccc}
 MA & \xrightarrow{\hat{\lambda}} & SA & & \\
 \delta \uparrow & & \mu'_A \nearrow & & \uparrow \sigma \\
 & & SSA & & \\
 & \lambda_{SA} \nearrow & \uparrow \sigma_{SA} & & \\
 HMA & \xrightarrow{H\hat{\lambda}_A} & HSA & \xrightarrow{\lambda'_{SA}} & S'SA & \xrightarrow{\mu'^S} & S'A
 \end{array}$$

(f) It remains to prove that $(\hat{\lambda}, \hat{\lambda}')$ is unique. Let $(\varphi, \varphi') : \mathbb{M} \rightarrow \mathbb{S}$ be an ideal monad morphism with $\varphi \cdot \kappa = \lambda$. It is sufficient to prove that $\varphi_A : MA \rightarrow SA$ is a homomorphism of H -algebras w.r.t. the structure ρ above and $\varphi_A \cdot \eta_A = \eta'_A$, then $\varphi_A = \hat{\lambda}_A$ of (b) above. From that we derive $\varphi' = \hat{\lambda}'$ since σ is a monomorphism:

$$\sigma \cdot \varphi' = \varphi \cdot \delta = \hat{\lambda} \cdot \delta = \sigma \cdot \hat{\lambda}'.$$

The equation $\varphi_A \cdot \eta_A = \eta'_A$ follows from φ preserving the units of the monad. And the fact that φ_A is a homomorphism follows from the following diagram:

$$\begin{array}{ccccc}
 HMA & \xrightarrow{\delta_A} & MA & & \\
 \downarrow H\varphi_A & \searrow \kappa_{MA} & \downarrow \varphi_A & & \\
 & MMA & & & \\
 & \downarrow (\varphi * \varphi)_A & & & \\
 HSA & \xrightarrow{\lambda_{SA}} & SSA & \xrightarrow{\mu'_A} & SA
 \end{array}$$

For the upper triangle see Remark 4.14 (and recall that $\kappa = \delta \cdot H\eta$), the right-hand square is the preservation of the monad multiplication, and for the left-hand one we use $(\varphi \cdot \kappa)M = \lambda M$ and the naturality of λ :

$$\begin{array}{ccc}
 HMA & \xrightarrow{\kappa_{MA}} & MMA \\
 \downarrow H\varphi_A & \searrow \lambda_{MA} & \downarrow \varphi_{MA} \\
 & SMA & \\
 & \downarrow S\varphi_A & \\
 HSA & \xrightarrow{\lambda_{SA}} & SSA
 \end{array}$$

□

Example 6.6.

(1) The functor Id generates the free corecursive monad

$$MY = \mathbb{N} \times Y + 1,$$

see Example 4.7(1). This is also the free completely iterative monad, since the functor $Id + Y$ has the terminal coalgebra $\mathbb{N} \times Y + 1$.

(2) The polynomial functor H_Σ of a signature $\Sigma = (\Sigma_n)_{n < \omega}$ generates the free corecursive monad

$$MY = \text{all } (\Sigma + Y)\text{-trees in which only finitely many leaves are labelled in } Y \\ \text{(and other leaves labelled in } \Sigma_0\text{).}$$

See Example 4.7(4).

Are there any other free corecursive monads than the monads \mathbb{M} of free corecursive algebras? Not for endofunctors of **Set**:

Proposition 6.7. *If a set functor generates a free corecursive monad, then it has free corecursive algebras.*

Proof. Let $H : \mathbf{Set} \rightarrow \mathbf{Set}$ generate a free corecursive monad $\mathbb{S} = (S, \mu^S, \eta^S, S', \sigma, \mu')$, and let $\kappa : H \rightarrow S$ be the universal arrow. Following Theorem 4.11 we need to prove the existence of (a) arbitrary large pre-fixpoints and (b) a corecursive fixpoint.

The main technical statement is that the ideal S' is naturally isomorphic to HS . This proof is analogous to the same proof concerning free completely iterative monads, see Sections 5 and 6 in [1]. We therefore omit it.

Ad (a). Since $SY = S'Y + Y = HSY + Y$ for every set Y , we see that SY is a pre-fixpoint of cardinality at least $\text{card } Y$.

Ad (b). The isomorphism $\sigma_\emptyset : HS\emptyset \rightarrow S\emptyset$ defines a corecursive algebra for H . To prove this, consider an arbitrary equation morphism $e : X \rightarrow HX$ and form the equation morphism $\bar{e} = \kappa_X \cdot e : X \rightarrow SX$. Then solutions of \bar{e} w.r.t \mathbb{S} (in $S\emptyset$) are in bijective correspondence with solutions e in the algebra $S\emptyset$. This is easy to prove, the details are as in the of proof of Theorem 6.1 of [28]. \square

7. HYPER-EXTENSIVE CATEGORIES

Recall that for the more general case of recursion with parameters the equational properties of \dagger are captured by iteration theories of S. Bloom and Z. Ésik [19]. Recently *functoriality* was “added” to these properties; functoriality states that for two equation morphisms with parameters $e : X \rightarrow S(X + Y)$ and $f : Z \rightarrow S(Z + Y)$ (cf. Definition 5.1) we have

$$\begin{array}{ccc} X & \xrightarrow{e} & S(X + Y) \\ \downarrow h & & \downarrow S(h+Y) \\ Z & \xrightarrow{f} & S(Z + Y) \end{array} \quad \Longrightarrow \quad \begin{array}{ccc} X & & \\ \downarrow h & \searrow e^\dagger & \\ & & SY \\ \downarrow & \nearrow f^\dagger & \\ Z & & \end{array}$$

Being an implication, this is not equational if one takes, as in [19], the category of signatures as the base category. Instead, in [12] the presheaf category

$$\mathbf{Set}^{\mathbb{F}} \text{ (sets in context)}$$

where \mathbb{F} is the category of finite sets and functions, was suggested as a base category. Equivalently, this is the category of all finitary endofunctors on **Set**. Then functoriality is an equational property in the sense of Kelly and Power [26], and the functorial iteration theories are called *Elgot Theories* in [12]. It follows from the results in [19] that all equational properties of \dagger in Domain Theory are precisely captured by the concept of iteration theory.

More precisely, every equation that holds for a parametrized fixpoint operator \dagger given by

least fixpoints in a category of domains follows from the axioms of iteration theories (see e.g. Simpson and Plotkin [30]).

We have proved in [12] that Elgot theories are monadic over sets in context:

Theorem 7.1 ([12]). *Form the monad \mathcal{M} on $\mathbf{Set}^{\mathbb{F}}$ by assigning to every set in context H the free iterative theory on $H_{\perp} = H(-) + 1$ of C. Elgot [22]. Then the Eilenberg-Moore algebras for \mathcal{M} are precisely the Elgot theories.*

This result implies, using the results of Kelly and Power [26], that Elgot theories are equational over sets in context, and we gave one axiomatization (that includes functoriality) in [12].

Example 7.2. The polynomial set functor H_{Σ} of Example 4.7(4) defines a set in context (that we also denote by H_{Σ}) by a domain restriction to \mathbb{F} . The corresponding iterative theory $\mathcal{M}(H_{\Sigma})$ is given by

$$X \mapsto \text{all rational trees labelled in } \Sigma + X + \{\perp\}.$$

This is a subtheory of the theory \mathcal{T}_{Σ} of all trees labelled in $\Sigma + X + \{\perp\}$, which is the free continuous theory (see Example 3.2(c)).

It was proved by Bloom and Ésik in [19] that the equational properties of the operation \dagger (of solving recursive equations) of the above theory \mathcal{T}_{Σ} are precisely the equational properties that \dagger has in an impressive number of applications of iteration. Thus, the axiomatization of these properties in [19] can be understood as the summary of equational properties that \dagger is expected to have in applications.

For every finitary set functor H there exists a signature Σ such that H is a quotient of H_{Σ} (see [15]). Therefore, $\mathcal{M}(H)$ is a quotient theory of the theory $\mathcal{M}(H_{\Sigma})$ of rational trees. Thus, the equational properties of \dagger in all free Elgot theories $\mathcal{M}(H)$ for finitary set functors H are determined by those of \dagger in rational trees.

In the present section we provide the first steps to an analogous result for iteration without parameters. We introduce *finitary corecursive monads* as the analogy of Elgot's iterative theories, and we prove that every finitary endofunctor on \mathbf{Set} generates a free finitary corecursive monad. Let \mathcal{M}^* be the monad on $\mathbf{Set}^{\mathbb{F}}$ given by

$$\mathcal{M}^*(H) = \text{free finitary corecursive monad on } H_{\perp}.$$

Then we prove that the Eilenberg-Moore algebras for \mathcal{M}^* are precisely the *Bloom theories*, i. e., theories with an operation \dagger satisfying the equational properties that hold in non-parametric iteration. We list some of these properties. It is an open problem whether our list is complete.

In lieu of \mathbf{Set} we work, more generally, in a locally finitely presentable category. Thus in lieu of theories we work with finitary monads (in analogy to iterative monads of [22]). For most of the results we need to assume the category we work with is hyper-extensive. We now start by recalling this concept from [3].

Definition 7.3. (See [3]) A locally finitely presentable category \mathcal{A} is called *hyper-extensive* if every object is a coproduct of connected objects, i. e., objects A such that $\mathcal{A}(A, -)$ preserves coproducts.

Remark 7.4. In [3] the definition is different, but Theorem 2.7 of [3] states that the present formulation is equivalent. Every hyper-extensive category is extensive, i. e., coproducts are (a) disjoint (coproduct injections are monic and pairwise intersections always yield 0) and

(b) universal (preserved by pullback along any morphism).

Moreover, in hyper-extensive categories we have

(c) given pairwise disjoint monics $a_i : A_i \rightarrow B$, $i \in \mathbb{N}$, if each a_i is coproduct injection then so is $[a_i] : \coprod_{i \in \mathbb{N}} A_i \rightarrow B$.

For locally finitely presentable categories (a)–(c) are equivalent to hyper-extensivity.

Example 7.5. Sets, posets, graphs, and every presheaf category are hyper-extensive. Given a signature Σ the category of Σ -algebras is hyper-extensive iff all arities are 1.

Remark 7.6. In a hyper-extensive category a monad $\mathbb{S} = (S, \mu, \eta)$ is *ideal* (see Definition 5.1) iff that S is a coproduct $S = S' + Id$ with injections $\sigma : S' \rightarrow S$ and $\eta : Id \rightarrow S$ and the multiplication has a restriction

$$\mu' : S'S \rightarrow S'.$$

Thus, in this setting “ideal” is a property not an additional structure of a monad.

Notation 7.7. $\text{Mon}_i(\mathcal{A})$ denotes the category of ideal monads and ideal monad morphisms, i.e., morphisms $\alpha : \mathbb{S} \rightarrow \mathbb{T}$ for which a restriction to the ideals exist: we have

$$\alpha = (S = S' + Id \xrightarrow{\alpha' + Id} T' + Id = T)$$

for a natural transformation $\alpha' : S' \rightarrow T'$.

Remark 7.8. We shall prove below that every corecursive monad \mathbb{S} on a hyper-extensive category has solutions for all, not only ideal, equation morphisms. For that we need to specify an element of $S0$ which then serves for defining solutions of non-ideal equations such as $x = x$. In the following definition 1 denotes the terminal object of \mathcal{A} and 0 the initial one. The unique morphism from 0 to X is denoted by $! : 0 \rightarrow X$. Analogously $! : X \rightarrow 1$.

Definition 7.9. A *strict endofunctor* is an endofunctor H together with a morphism $\perp : 1 \rightarrow H0$. A monad is *strict* if its underlying endofunctor is. A natural transformation $\alpha : H \rightarrow K$ between strict functors is called *strict* if α_0 preserves \perp .

Every strict endofunctor has a special global element in every HX : it is the composite of $\perp : 1 \rightarrow H0$ and $H! : H0 \rightarrow HX$. We denote it again by $\perp : 1 \rightarrow HX$.

Remark 7.10. We now recall from [3] the concept of a strict solution and the fact that every equation morphism has a unique strict solution. In [3] equation morphisms with parameters Y (see Definition 5.1) were considered, here we restrict ourselves to $Y = 0$.

Definition 7.11. (See [3]) (a) For every equation morphism $e : X \rightarrow SX$ we denote by

$$i_\infty : X_\infty \rightarrow X$$

the intersection of the *derived subobjects* $i_1, i_1 \cdot i_2, i_1 \cdot i_2 \cdot i_3, \dots$ obtained by forming recursively pullbacks as follows:

$$\begin{array}{ccccccc} \cdots & X_3 & \xrightarrow{i_3} & X_2 & \xrightarrow{i_2} & X_1 & \xrightarrow{i_1} & X = X_0 \\ & e_3 \downarrow & & e_2 \downarrow & & e_1 \downarrow & & e \downarrow \\ \cdots & \xrightarrow{i_3} & X_2 & \xrightarrow{i_2} & X_1 & \xrightarrow{i_1} & X & \xrightarrow{\eta_X} & SX \end{array} \quad (7.1)$$

Remark 7.12. X_∞ represents those variables for which solutions of e have “difficulties” assigning a value. For example, if e represents the iterative equation $x = x$ or the system

$$x = y, \quad y = x$$

then $X = X_\infty$. We resolve the difficulties by assigning the value \perp to such variables:

Definition 7.13. (See [3]) Let $e : X \rightarrow SX$ be an equation morphism. A solution $e^\dagger : X \rightarrow SY$ is called *strict* if its restriction to X_∞ factorizes through \perp :

$$\begin{array}{ccc} X_\infty & \xrightarrow{\perp} & 1 \\ i_\infty \downarrow & & \downarrow \perp \\ X & \xrightarrow{e^\dagger} & SY \end{array}$$

Theorem 7.14. (See [3]) Let \mathbb{S} be a strict, corecursive monad on a hyper-extensive category. Then every equation morphism $e : X \rightarrow SX$ has a unique strict solution $e^\dagger : X \rightarrow SY$, for every object Y .

In [3] we proved this for completely iterative monads, the proof for the corecursive monads is the same.

8. FINITARY BLOOM ALGEBRAS AND MONADS

In this section we investigate the variant of iteration in which only equation morphisms $e : X \rightarrow SX$ with finitely presentable objects X (of variables) are considered.

Throughout this section we assume that the base category is locally finitely presentable and hyper-extensive. And a finitary endofunctor H is given.

Definition 8.1.

- (a) An algebra $a : HA \rightarrow A$ is said to be *finitary corecursive* if for every coalgebra $e : X \rightarrow HX$ with X finitely presentable there exists a unique solution, i. e., a unique coalgebra-to-algebra morphism $e^\dagger : X \rightarrow A$.
- (b) A *finitary Bloom algebra* is a triple (A, a, \dagger) where $a : HA \rightarrow A$ is an algebra and \dagger an operation which to every $e : X \rightarrow HX$, X finitely presentable, assigns a solution $e^\dagger : X \rightarrow A$ subject to functoriality: for every coalgebra homomorphism $h : (X, e) \rightarrow (X', e')$ with X and X' finitely presentable the following triangle commutes:

$$\begin{array}{ccc} X & \xrightarrow{h} & X' \\ & \searrow e^\dagger & \swarrow (e')^\dagger \\ & & A \end{array}$$

- (c) Homomorphisms are defined analogously to Definition 3.3.

Remark 8.2.

- (a) Every finitary corecursive algebra is a finitary Bloom algebra: the functoriality follows from the uniqueness of solutions.
- (b) Lemmas 2.9 and 3.7 hold also for finitary Bloom algebras.

Example 8.3. Consider unary algebras in \mathbf{Set} , that is, $H = Id$.

- (a) An algebra $a : A \rightarrow A$ is finitary corecursive iff a has a unique fixpoint $t = a(t)$. Indeed, for every equation morphism $e : X \rightarrow X$ with X finite the unique solution is $e^\dagger = \text{const}_t$. Thus the algebra \mathbb{Z}^+ of integers with ∞ where the unary operation is successor (and ∞ is its fixpoint) is finitary corecursive. But not corecursive: consider the system of equations given by $x_i = a(x_{i+1})$ for $i \in \mathbb{N}$. It has more than one solution in \mathbb{Z}^+ , e.g., $x_i \mapsto -i$ and $x_i \mapsto \infty$ are solutions.
- (b) An algebra $a : A \rightarrow A$ is a finitary Bloom algebra iff a has a fixpoint – this is the same as in Example 3.2(b).

Remark 8.4.

- (a) Recall from [10] the concept of an iterative algebra: it is an algebra $a : HA \rightarrow A$ such that every equation morphism $e : X \rightarrow HX + A$ with X finitely presentable has a unique solution. That is, the algebra $[a, A] : HA + A \rightarrow A$ for $H(-) + A$ is finitary corecursive. Every iterative algebra is obviously finitary corecursive (for H). An example of a finitary corecursive algebra that is not iterative is the algebra of all binary trees with finitely many leaves, all of which are labelled in Y , see Example 4.7(3).
- (b) Recall further from [10] that the category

$$\text{Coalg}_f H$$

of all coalgebras on finitely presentable objects of \mathcal{A} is filtered, and the filtered colimit of the forgetful functor to \mathcal{A} ,

$$R = \text{colim}\{X; (X, e) \in \text{Coalg}_f H\}$$

carries the structure of a coalgebra $i : R \rightarrow HR$. This structure is an isomorphism, and its inverse $\rho : HR \rightarrow R$ is the initial iterative algebra. Consequently, R is a finitary Bloom algebra as well. Indeed:

Proposition 8.5. *R is the initial finitary corecursive algebra.*

Proof. We know that R is finitary corecursive because it is even iterative. Let (A, a) be a finitary Bloom algebra. Given a (solution-preserving) homomorphism $h : R \rightarrow A$, for every $e : X \rightarrow HX$ in $\text{Coalg}_f H$ the triangle

$$\begin{array}{ccc} & X & \\ e^\sharp \swarrow & & \searrow e^\dagger \\ R & \xrightarrow{h} & A \end{array}$$

commutes, where e^\sharp denotes the solution in R . As proved in [10], these morphisms e^\sharp form the colimit cocone of R (as a colimit of the forgetful functor of $\text{Coalg}_f H$). Thus, the above triangles determine, since \dagger is functorial, a unique morphism h which is solution-preserving. It remains to prove that h is a homomorphism. For that recall from [10] that the algebra structure $\rho : HR \rightarrow R$ is defined as the inverse of the unique isomorphism $i : R \rightarrow HR$ with

$$i \cdot e^\sharp = He^\dagger \cdot e \text{ for all } e : X \rightarrow HX \in \text{Coalg}_f H.$$

Thus in order to prove $h \cdot \rho = a \cdot Hh$ we use that $h = a \cdot Hh \cdot i : R \rightarrow A$ which follows from the fact that e^\sharp are collectively epic: $h \cdot e^\sharp = e^\dagger = a \cdot He^\dagger \cdot e = a \cdot Hh \cdot He^\sharp \cdot e = a \cdot Hh \cdot i \cdot e^\sharp$. \square

Example 8.6.

- (a) For $HX = X \times X + 1$ the algebra R consists of all binary trees that are rational, i. e., have finitely many subtrees up to isomorphism, see [25].
- (b) More generally, given a finitary signature Σ the polynomial functor $H_\Sigma : \mathbf{Set} \rightarrow \mathbf{Set}$ with

$$H_\Sigma X = \Sigma_0 + \Sigma_1 \times X + \Sigma_2 \times X^2 + \dots$$

has the initial iterative algebra

$$R_\Sigma = \text{all rational } \Sigma\text{-trees.}$$

- (c) For the finite power-set functor R is the algebra of all rational, finitely branching, strongly extensional trees in the sense of J. Worrell [34].

Theorem 8.7. *Every object Y generates a free finitary Bloom algebra. This is the coproduct $R \oplus FY$ of R and the free algebra FY in $\mathbf{Alg} H$.*

Proof. The category $\mathbf{Alg} H$ is locally finitely presentable, see [14], thus \oplus exists in $\mathbf{Alg} H$. The coproduct injection $\text{inl} : R \rightarrow R \oplus FY$ induces a finitary Bloom algebra structure on $R \oplus FY$, see Lemma 3.7. And we have a canonical morphism $\text{inr} \cdot \eta_Y : Y \rightarrow R \oplus FY$ in \mathcal{A} .

The universal property of $\text{inr} \cdot \eta_Y$ is clear: let (B, b, \ddagger) be a finitary Bloom algebra and $g : Y \rightarrow B$ a morphism in \mathcal{A} .

$$\begin{array}{ccccc}
 R & \xrightarrow{\text{inl}} & R \oplus FY & \xleftarrow{\text{inr}} & FY & \xleftarrow{\eta_Y} & Y \\
 & \searrow f & \downarrow [f, \bar{g}] & \swarrow \bar{g} & & \swarrow g & \\
 & & B & & & &
 \end{array}$$

We have a unique homomorphism $\bar{g} : FY \rightarrow B$ and, due to Proposition 8.5, a unique solution-preserving homomorphism $f : R \rightarrow B$. The unique homomorphism $[f, \bar{g}] : R \oplus FY \rightarrow B$ is solution-preserving since f is and the Bloom algebra structure of $R \oplus FY$ is induced by inl . \square

Corollary 8.8. *$R \oplus FY$ is a free finitary corecursive algebra on Y .*

Indeed, the proof that solutions in $R \oplus FY$ are unique is completely analogous to that of Theorem 4.6.

Example 8.9. Let Σ be a finitary signature. The free finitary corecursive algebra $M^*Y = R_\Sigma \oplus F_\Sigma Y$ is the algebra of all rational trees with finitely many leaves labelled in Y and all other nodes (with n successors) labelled by an n -ary operation in Σ for $n = 0, 1, 2, \dots$

Notation 8.10. The monad of free finitary Bloom algebras is denoted by \mathbb{M}_H^* . It is defined on objects by assigning to Y the underlying object of $R \oplus FY$. In the case of $H = H_\Sigma$ we write \mathbb{M}_Σ^* .

Theorem 8.11. *The category of finitary Bloom algebras is isomorphic to the Eilenberg-Moore category of \mathbb{M}_H^* .*

The proof is analogous to that of Theorem 4.15.

Remark 8.12. Analogous to the concept of a corecursive monad, see Definition 5.3, we call an ideal monad \mathbb{S} *finitary corecursive* if every ideal equation morphism $e : X \rightarrow SX$ with X finitely presentable has a unique solution $e^\dagger : X \rightarrow SY$. The following result is completely analogous to Proposition 5.5 and Theorem 6.4.

Theorem 8.13. *For every finitary endofunctor H the monad \mathbb{M}_H^* of free finitary corecursive algebras is a free finitary corecursive monad on H . This monad is ideal with the ideal HM and $\mu' = H\mu$.*

Notation 8.14.

- (a) Let \mathbb{F} be a full subcategory of \mathcal{A} representing all finitely presentable objects. The functor category $\mathcal{A}^{\mathbb{F}}$ is equivalent to the category of all finitary endofunctors of \mathcal{A} .
- (b) $\mathcal{A}_{\perp}^{\mathbb{F}}$ denotes the non-full subcategory of all strict finitary endofunctors (and strict natural transformations). The embedding $\mathcal{A}_{\perp}^{\mathbb{F}} \hookrightarrow \mathcal{A}^{\mathbb{F}}$ has a left adjoint $H \mapsto H_{\perp}$ where $H_{\perp}X = HX + 1$.
- (c) We denote by

$$\mathcal{M}^*$$

the monad on $\mathcal{A}^{\mathbb{F}}$ given by free finitary corecursive monads:

$$\mathcal{M}^* : H \mapsto M_{H_{\perp}}^*.$$

More precisely \mathcal{M}^* is the monad obtained by the composite adjoint situation

$$\mathcal{A}^{\mathbb{F}} \begin{array}{c} \xrightarrow{(-)_{\perp}} \\ \xleftarrow{\perp} \end{array} \mathcal{A}_{\perp}^{\mathbb{F}} \begin{array}{c} \xrightarrow{M_{(-)}^*} \\ \xleftarrow{\perp} \\ \xleftarrow{U} \end{array} \text{FC}_{\perp}(\mathcal{A})$$

where U is the forgetful functor of the category $\text{FC}_{\perp}(\mathcal{A})$ of all strict finitary corecursive monads.

Definition 8.15. A *Bloom monad* on \mathcal{A} is an Eilenberg-Moore algebra for the monad \mathcal{M}^* .

Remark 8.16. Following Theorem 7.1, this concept is, for $\mathcal{A} = \mathbf{Set}$, completely analogous to iteration theories of Bloom and Ésik: whereas iteration theories formalizes equational properties of parametrized iteration, Bloom monads on \mathbf{Set} formalize equational properties of non-parametrized iteration. But what are Bloom monads?

- (1) Every Bloom monad is a finitary monad on \mathcal{A} . Indeed, let \mathcal{F} be the free-monad on $\mathcal{A}^{\mathbb{F}}$: to every finitary endofunctor H it assigns the free monad F_H on H . It is well-known that the Eilenberg-Moore algebras for \mathcal{F} are precisely the finitary monads on \mathcal{A} , see [27].

For every H in $\mathcal{A}^{\mathbb{F}}$ we have the unique monad morphism

$$\phi_H : \mathcal{F}(H) \rightarrow \mathcal{M}^*(H) = M_{H_{\perp}}^*$$

given by the universal property of $\mathcal{F}(H)$. These morphisms form components of a monad morphism $\phi : \mathcal{F} \rightarrow \mathcal{M}^*$ (over $\mathcal{A}^{\mathbb{F}}$). Thus, every Eilenberg-Moore algebra for \mathcal{M}^* is automatically one for \mathcal{F} , too.

- (2) Every Bloom monad $\mathbb{S} = (S, \eta, \mu)$ comes equipped with an operation \dagger assigning to every equation morphism $e : X \rightarrow SX$ with X finitely presentable and every object Y a morphism $e^{\dagger} : X \rightarrow SY$ which is a solution:

$$e^{\dagger} = \mu_Y \cdot S e^{\dagger} \cdot e.$$

Indeed, the Eilenberg-Moore structure

$$\sigma : M_{S_{\perp}}^* \rightarrow S$$

is a monad morphism, and we have also the universal arrow (see Example 6.3)

$$\bar{\kappa} \equiv S \xrightarrow{\text{inl}} S_{\perp} \xrightarrow{\kappa} M_{S_{\perp}}^*$$

which, due to the unit law of σ , fulfils

$$\sigma \cdot \bar{\kappa} = id_S. \quad (8.1)$$

For every equation morphism $e : X \rightarrow SX$ the unique strict solution (see Theorem 7.14) of $\bar{\kappa}_X \cdot e : X \rightarrow M_{S_\perp}^* X$ w.r.t. $\mathbb{M}_{S_\perp}^*$ is denoted by $e^\dagger : X \rightarrow M_{S_\perp}^* Y$. Then

$$e^\dagger = \sigma_Y \cdot e^\ddagger : X \rightarrow SY$$

is a (canonical) solution of e w.r.t. \mathbb{S} . Indeed, the following diagram, where $\bar{\mu}$ denotes the multiplication of $\mathbb{M}_{S_\perp}^*$, commutes:

$$\begin{array}{ccccc}
 X & \xrightarrow{e^\dagger} & M_{S_\perp}^* Y & \xrightarrow{\sigma_Y} & SY \\
 \downarrow e & & \uparrow \bar{\mu}_Y & & \uparrow \mu_Y \\
 & & M_{S_\perp}^* X & \xrightarrow{M_{S_\perp}^* e^\dagger} & M_{S_\perp}^* M_{S_\perp}^* Y \\
 & \nearrow \bar{\kappa}_X & & & \searrow (\sigma * \sigma)_Y \\
 SX & \xrightarrow{S(\sigma_Y \cdot e^\dagger)} & SSY & &
 \end{array}$$

The left-hand part commutes since e^\dagger is a solution of $\bar{\kappa}_X \cdot e$. The right-hand part (with $\sigma * \sigma = S\sigma \cdot \sigma M_{S_\perp}^*$) commutes because $\sigma : \mathbb{M}_{S_\perp}^* \rightarrow \mathbb{S}$ is a monad morphism. And to prove the lower part, we just need to verify

$$S e^\dagger = \sigma_{M_{S_\perp}^* Y} \cdot M_{S_\perp}^* e^\dagger \cdot \bar{\kappa}_X$$

which follows easily from $\sigma_X \cdot \bar{\kappa}_X = id$ and the naturality of σ .

Remark 8.17. (a) The operation \dagger above satisfies all the equational laws that the formation of strict solutions in all finitary corecursive monads satisfies. This follows from the fact that the Bloom monad (\mathbb{S}, \dagger) is by definition a quotient algebra (for \mathcal{M}^*) of the finitary corecursive monad $(\mathbb{M}_{S_\perp}^*, \dagger)$.

(b) For the base category **Set** we can say more. Since S is finitary, there exists a finitary signature Σ such that S is a quotient of H_Σ (see [15]). Let $\Sigma_\perp = \Sigma \cup \{\perp\}$ be the extension by a nullary operation \perp . Then S_\perp is a quotient of H_{Σ_\perp} . Indeed, there exists a signature Γ such that for suitable natural transformations $\alpha_1, \alpha_2 : H_\Gamma \rightarrow H_\Sigma$ we have a coequalizer

$$H_\Gamma \begin{array}{c} \xrightarrow{\alpha_1} \\ \xrightarrow{\alpha_2} \end{array} H_{\Sigma_\perp} \xrightarrow{e} S_\perp,$$

see [7]. The functor $\mathbb{M}_{(-)}^*$ of free finitary corecursive monads is a left adjoint, thus, it preserves coequalizers. Therefore $\mathcal{M}S_\perp$ is a quotient of $\mathcal{M}H_{\Sigma_\perp}$.

Consequently, a Bloom monad in **Set** is a finitary monad with a solution operation \dagger satisfying all the equational laws that the operation of unique strict solutions for the rational-tree monads $\mathbb{M}_{\Sigma_\perp}^*$ satisfies.

In the following example we list some equational properties of \dagger in Bloom monads. It is an open problem whether this list is complete in the sense that every equational property of \dagger holding in all Bloom monads can be derived from the properties stated.

Example 8.18. Equational properties of \dagger in Bloom monads. We use the terminology of the monograph [19].

(a) Fixpoint identity. This is the equation

$$e^\dagger = \mu_Y \cdot S e^\dagger \cdot e$$

of Definition 5.3.

(b) Functoriality (called functorial dagger implication in [19]). For every coalgebra homomorphism

$$\begin{array}{ccc} X & \xrightarrow{e} & SX \\ h \downarrow & & \downarrow Sh \\ \overline{X} & \xrightarrow{\overline{e}} & S\overline{X} \end{array}$$

we have

$$e^\dagger = \overline{e}^\dagger \cdot h.$$

To prove this recall that we can restrict ourselves to finitary corecursive monads where strict solutions are unique (Remark 8.17(a)). Thus, it is sufficient to observe that $\overline{e}^\dagger \cdot h$ solves e :

$$\mu_Y \cdot S(\overline{e}^\dagger \cdot h) \cdot e = \mu_Y \cdot S\overline{e}^\dagger \cdot \overline{e} \cdot h = \overline{e}^\dagger \cdot h$$

and that it is strict. The latter follows from the fact that \overline{e}^\dagger is strict for \overline{e} and the subobjects $i_n : X_n \rightarrow X_{n-1}$ of Definition 7.11 for e are related to the corresponding subobject $\overline{i}_n : \overline{X}_n \rightarrow \overline{X}_{n-1}$ for \overline{e} by morphisms $h_n : X \rightarrow \overline{X}_n$ such that the squares below commute:

$$\begin{array}{ccccccc} \dots & \xrightarrow{i_3} & X_2 & \xrightarrow{i_2} & X_1 & \xrightarrow{i_1} & X \\ & & \downarrow h_2 & & \downarrow h_1 & & \downarrow h=h_0 \\ \dots & \xrightarrow{\overline{i}_3} & \overline{X}_2 & \xrightarrow{\overline{i}_2} & \overline{X}_1 & \xrightarrow{\overline{i}_1} & \overline{X} \end{array}$$

For example, h_1 is the unique morphism for which the following diagram commutes:

$$\begin{array}{ccccc} X_1 & \xrightarrow{i_1} & X & & \\ \downarrow h_1 & \searrow & \downarrow h & & \\ \overline{X}_1 & \xrightarrow{\overline{i}_1} & \overline{X} & & \\ \downarrow \overline{e}_1 & \searrow & \downarrow \overline{e} & & \\ \overline{X} & \xrightarrow{\eta_X} & S\overline{X} & & \\ \downarrow h & \searrow & \downarrow Sh & & \\ X & \xrightarrow{\eta_X} & SX & & \end{array}$$

satisfying $h_{n-1} \cdot i_n = \overline{i}_n \cdot h_n$ (where $h_0 = h$). Therefore, we obtain $h_\infty : X_\infty \rightarrow \overline{X}_\infty$ such that $h \cdot i_\infty = \overline{i}_\infty \cdot h_\infty$. This gives us the derived factorization

$$(\overline{e}^\dagger \cdot h) \cdot i_\infty = \overline{e}^\dagger \cdot \overline{i}_\infty \cdot h_\infty = \perp \cdot ! \cdot i_\infty = \perp \cdot !$$

(c) Parameter identity. Given $e : X \rightarrow SX$, with X finitely presentable, and a morphism $h : Y \rightarrow SZ$, then the corresponding morphism of free Eilenberg-Moore algebras

$$\hat{h} = \mu_Z \cdot Sh : SY \rightarrow SZ$$

makes the triangle

$$\begin{array}{ccc} X & \xrightarrow{e_Z^\dagger} & SZ \\ e_Y^\dagger \downarrow & \nearrow \hat{h} & \\ & & SY \end{array}$$

commutative. Indeed, $\hat{h} \cdot e_Y^\dagger$ is strict because e_Y^\dagger is strict: from the strictness of S we get $\hat{h} \cdot \perp = \perp$, thus

$$\hat{h} \cdot e_Y^\dagger \cdot i_\infty = \hat{h} \cdot \perp \cdot ! = \perp \cdot !$$

And $\hat{h} \cdot e_Y^\dagger$ is a solution of e in SZ :

$$\begin{array}{ccccc} X & \xrightarrow{e_Y^\dagger} & SY & \xrightarrow{\hat{h}} & SZ \\ \downarrow e & & \uparrow \mu_Y & & \uparrow \mu_Z \\ & & SSY & & \\ & \nearrow Se_Y^\dagger & & \searrow S\hat{h} & \\ SX & \xrightarrow{S(\hat{h} \cdot e_Y^\dagger)} & SSZ & & \end{array}$$

(d) Double iteration identity. For every $e : X \rightarrow SX$, form $\hat{e} : SX \rightarrow SX$, then we have

$$e^\dagger = (\hat{e} \cdot e)^\dagger$$

Indeed, e^\dagger is a solution of $\hat{e}e = \mu_X \cdot Se \cdot e$:

$$\begin{array}{ccc} X & \xrightarrow{e^\dagger} & SY \\ \downarrow e & & \nearrow \mu_Y \\ SX & \xrightarrow{Se^\dagger} & SSY \\ \downarrow Se & & \uparrow S\mu_Y \\ SSX & \xrightarrow{SSe^\dagger} & SSY \\ \downarrow \mu_X & & \searrow \mu_{SY} \\ SX & \xrightarrow{Se^\dagger} & SSY \end{array}$$

The upper part states e^\dagger is a solution, and the middle part follows. The right-hand part is a monad axiom of \mathbb{S} and the lower part is the naturality of μ . The strictness of e^\dagger w.r.t. $\hat{e} \cdot e$ is trivial: since \mathbb{S} is an ideal monad on a hyper-extensive category, we

have the following diagram of pullback squares:

$$\begin{array}{ccccccc}
 X_2 & \xrightarrow{e_2} & X_1 & \xrightarrow{e_1} & X & \xlongequal{\quad} & X = X_0 \\
 \downarrow i_2 & & \downarrow i_1 & & \downarrow \eta_X & & \downarrow \eta_X \\
 X_1 & \xrightarrow{e_1} & X & \xrightarrow{e} & SX & & \\
 \downarrow i_1 & & \downarrow \eta_X & & \downarrow \eta_{SX} & & \\
 X & \xrightarrow{e} & SX & \xrightarrow{S_e} & SSX & \xrightarrow{\mu_X} & SX
 \end{array}$$

So the first derived subobject of $\hat{e} \cdot e$ is X_2 , and similarly the n -th one is X_{2n} , where the X_i are the derived subobjects of e . It follows that X_∞ is the same for e and $\hat{e} \cdot e$, thus, e^\dagger is a strict solution of $\hat{e} \cdot e$.

Observe that the above “double iteration” extends to “ n times iteration”, e.g. for $n = 3$ we get $e^\dagger = (\hat{e}\hat{e}e)^\dagger$.

(e) Dinaturality. Given morphisms $f : X \rightarrow SZ$ and $g : Z \rightarrow SX$ with X and Z finitely presentable, form equation morphisms

$$\hat{g} \cdot f : X \rightarrow SX \quad \text{and} \quad \hat{f} \cdot g : Z \rightarrow SZ.$$

Their solutions are related by the dinaturality equation

$$(\hat{g} \cdot f)^\dagger = \widehat{(\hat{f} \cdot g)^\dagger} \cdot f : X \rightarrow SY$$

for every object Y .

Indeed, the right-hand side morphism is a solution of $\hat{g} \cdot f$ since it is the composite $\mu_Y \cdot S(\hat{f} \cdot g)^\dagger \cdot f$, and the following diagram

$$\begin{array}{ccccccc}
 X & \xrightarrow{f} & SZ & \xrightarrow{S(\hat{f} \cdot g)^\dagger} & SSY & \xrightarrow{\mu} & SY \\
 \downarrow f & \searrow & \downarrow Sg & & \uparrow S\mu & & \uparrow \mu \\
 SZ & & SSX & & & & \\
 \downarrow Sg & \searrow & \downarrow SSf & & & & \\
 SSX & \xrightarrow{SSf} & SSSZ & & & & \\
 \downarrow \mu & & \downarrow S\mu & & & & \\
 SX & \xrightarrow{Sf} & SSZ & \xrightarrow{SS(\hat{f} \cdot g)^\dagger} & SSSY & \xrightarrow{S\mu} & SSY
 \end{array}$$

commutes: the middle part follows from $(\hat{f} \cdot g)^\dagger$ solving the equation morphism $\hat{f} \cdot g = \mu_Z \cdot S\hat{f} \cdot g$.

To see that $\widehat{(\hat{f} \cdot g)^\dagger} \cdot f$ is a strict solution of $\hat{g} \cdot f$, we first need to relate the derived subobjects of $\hat{g} \cdot f$ and $f \cdot \hat{g}$. For this consider the two following chains of pullbacks:

$$\begin{array}{ccccccc}
 & & X_3 & \xrightarrow{i_3} & X_2 & \xrightarrow{i_2} & X_1 & \xrightarrow{i_1} & X = X_0 \\
 & & \downarrow f_3 & & \downarrow f_2 & & \downarrow f_1 & & \downarrow f \\
 \cdots & & Z_3 & \xrightarrow{j_3} & Z_2 & \xrightarrow{j_2} & Z_1 & \xrightarrow{j_1} & Z & \xrightarrow{\eta_Z} & SZ
 \end{array}$$

$$\begin{array}{ccccccc}
& & Z_3 & \xrightarrow{j_3} & Z_2 & \xrightarrow{j_2} & Z_1 & \xrightarrow{j_1} & Z = Z_0 \\
\cdots & & \downarrow g_3 & & \downarrow g_2 & & \downarrow g_1 & & \downarrow g \\
& X_3 & \xrightarrow{i_3} & X_2 & \xrightarrow{i_2} & X_1 & \xrightarrow{i_1} & X & \xrightarrow{\eta_X} & SX
\end{array}$$

Using them and the fact that \mathbb{S} is an ideal monad in connection with (hyper-)extensivity we can compute the derived subobjects of $\hat{g} \cdot f$ as follows (all squares in following the diagram are pullbacks):

$$\begin{array}{ccccccccccc}
& & X_4 & \xrightarrow{i_4} & X_3 & \xrightarrow{i_3} & X_2 & \xrightarrow{i_2} & X_1 & \xrightarrow{i_1} & X & & \\
& & \downarrow f_4 & & \downarrow f_3 & & \downarrow f_2 & & \downarrow f_1 & & \downarrow f & & \\
\cdots & & Z_3 & \xrightarrow{j_3} & Z_2 & \xrightarrow{j_2} & Z_1 & \xrightarrow{j_1} & Z & \xrightarrow{\eta_Z} & SZ & & \\
& & \downarrow g_3 & & \downarrow g_2 & & \downarrow g_1 & & \downarrow g & & \downarrow Sg & & \\
X_4 & \xrightarrow{i_4} & X_3 & \xrightarrow{i_3} & X_2 & \xrightarrow{i_2} & X_1 & \xrightarrow{i_1} & X & \xrightarrow{\eta_X} & SX & \xrightarrow{S\eta_X} & SSX \\
\parallel & & \parallel & & \parallel & & \parallel & & \parallel & & \parallel & & \downarrow \mu_X \\
X_4 & \xrightarrow{i_4} & X_3 & \xrightarrow{i_3} & X_2 & \xrightarrow{i_2} & X_1 & \xrightarrow{i_1} & X & \xrightarrow{\eta_X} & SX & & \\
& & & & & & & & & & & & \leftarrow
\end{array}$$

$\hat{g} \cdot f$

So we see that the first derived subobject of $\hat{g} \cdot f$ is $i_1 \cdot i_2 : X_2 \rightarrow X$, and the second one is $i_1 \cdot i_2 \cdot i_3 \cdot i_4 : X_4 \rightarrow X$ etc. similarly the n -th derived subobject of $\hat{f} \cdot g$ is

$$j_1 \cdot j_2 \cdots j_{2n} : Z_{2n} \rightarrow Z.$$

Now recall from [3, Lemma 6.4] that the intersections X_∞ and Z_∞ are obtained after finitely many steps in the computation of the derived subobjects, i. e. there exists an n such that $X_\infty = X_k$ and $Z_\infty = Z_k$ for all $k \geq n$. Thus, we establish that $(\widehat{\hat{f} \cdot g})^\dagger \cdot f$ is a strict solution of $\hat{g} \cdot f$ by the commutative diagram below:

$$\begin{array}{ccccc}
& & \overset{!}{\curvearrowright} & & \\
X_\infty = X_{n+1} & \xrightarrow{f_n} & Z_\infty = Z_n & \xrightarrow{!} & 1 \\
\downarrow i_\infty & & \downarrow j_\infty & & \downarrow \perp \\
X & \xrightarrow{f} & Z & \xrightarrow{(\hat{f} \cdot g)^\dagger} & SZ \\
& & \downarrow \eta_Z & & \downarrow \eta_{SZ} \\
& & SZ & \xrightarrow{S(\hat{f} \cdot g)^\dagger} & SSZ \\
& & & & \downarrow \mu_Z \\
& & & & SZ
\end{array}$$

This completes the proof.

9. CONCLUSIONS AND OPEN PROBLEMS

For coalgebras, recursivity can be defined by the existence of unique algebra-to-coalgebra homomorphisms (no parameters are used). Or, equivalently, assuming the given endofunctor preserves weak pullbacks, by the unique solutions of all recursive systems with parameters. In contrast, in the dual situation we need to study non-equivalent variations. The present paper is dedicated to corecursive algebras A , where corecursivity means that every recursive

system of equations represented by a coalgebra has a unique solution in A . The formulation above is strictly weaker than the concept of a completely iterative algebra, where every parametrized recursive system of equations has a unique solution. For example, if we consider the endofunctor $X \mapsto X \times X$ of one binary operation in **Set**, the algebra of all binary trees with finitely many leaves is corecursive, but not completely iterative.

The main result of our paper is the description of the free corecursive algebra on Y as the coproduct $MY = T \oplus FY$ of the terminal coalgebra T and the free algebra FY in the category of all algebras. The above example of binary trees is the free corecursive algebra $M1$ on one generator. Our description is true for all accessible (= bounded) endofunctors on **Set** and, more generally, for all endofunctors on **Set** having free corecursive algebras. For accessible monos-preserving endofunctors on more general base categories (posets, groups, monoids etc.) the above description of the free corecursive algebras also holds.

We introduce the concept of a corecursive monad, a weakening of completely iterative monad. We prove that the assignment $Y \mapsto MY = T \oplus FY$ is the free corecursive monad on the given accessible endofunctor. And we characterize the Eilenberg-Moore algebras for this monad. We call them Bloom algebras in honor of Stephen Bloom. They play the analogous role that Elgot algebras, studied in [9], play for iterative monads: solutions of recursive equations are not required to be unique, but have to satisfy some “basic” properties. In the case of Bloom algebras, the only property needed is functoriality.

We further treat finitary equations: If we consider systems of recursive equations as coalgebras $e : X \rightarrow HX$, then finite systems of recursive equation are represented by coalgebras in which X is a finite set (or more generally, a finitely presentable object). We can speak about finitary corecursive algebras as those in which these finite systems have unique solutions. We prove that if R is the initial iterative algebra, then $R \oplus FY$ is a free finitary corecursive algebra.

Another question is: what is the analogy of the notion of an iteration monad of S. Bloom and Z. Ésik [19] in the realm of corecursive algebras? We do not know the answer. But at least we can formulate the question precisely. The idea of iteration monads is to collect all “equational” properties that the operation $e \mapsto e^\dagger$ of solving recursive systems e has in trees for a signature. This can be understood as forming the monad of free iterative theories (or monads) on the category $\mathbf{Set}^{\mathbb{F}}$ of sets in context, and characterizing monadic algebras: these are, as proved in [12], precisely the iteration theories of S. Bloom and Z. Ésik that are functorial. So the open problem we state is this: form the monad of free finitary corecursive theories on $\mathbf{Set}^{\mathbb{F}}$, what are its monadic algebras? We called them Bloom monads, and listed some of their properties.

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