

CONSTRUCTIVE DOMAINS WITH CLASSICAL WITNESSES

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ABSTRACT. We develop a constructive theory of continuous domains from the perspective of program extraction. Our goal that programs represent (provably correct) computation without witnesses of correctness is achieved by formulating correctness assertions classically. Technically, we start from a predomain base and construct a completion. We then investigate continuity with respect to the Scott topology, and present a construction of the function space. We then discuss our main motivating example in detail, and instantiate our theory to real numbers that we conceptualise as the total elements of the completion of the predomain of rational intervals, and prove a representation theorem that precisely delineates the class of representable continuous functions.

INTRODUCTION

The ability to extract programs from proofs is one of the hallmark features of constructive mathematics [TvD88]: from a proof of a formula of the form $\forall x \exists y P(x, y)$ we can automatically obtain a (computable) *function* f such that $P(x, f(x))$ for all x . Within mathematics, the variables usually have *types*, such as natural or real numbers, or functions between types.

Computationally, while some of these types, such as the natural numbers, can be computed with directly, there is no immediate way to compute with others. The prime example here are the real numbers that are represented either as infinitely-long running Turing machines [Wei00], rational Cauchy sequences with modulus [BB85], linear fractional transformations [HE02], digit streams [ME07] or domains [ES98].

From the view of program extraction, the data structure that is used to represent mathematical objects is systematically derived from their definition. If we define real numbers to be Cauchy sequences with modulus, then programs extracted from an existence proof will produce just that – a Cauchy sequence with modulus. The vast majority of the work on constructive real analysis and program extraction has focussed on the Cauchy representation and its variants such as the signed digit representation, e.g. [TvD88, BB85, Ber11]. There is little work on other representations, with the notable exception of [BK09] which develops a theory of constructive domains that is instantiated to obtain representations of real numbers.

Key words and phrases: Domain Theory, Constructive Mathematics, Real Number Computation, Predomain Base, Function Spaces.

In domain theory, real numbers are represented as nested sequences of (rational, or dyadic) intervals, with the interpretation that every interval gives an upper and lower bound to the number being approximated. In other words, every sequence element gives a guaranteed enclosure of the actual result, and the successive computation of sequence elements can be halted if the actual precision, measured by the interval width, falls below a given threshold.

Compared with a representation as Cauchy sequences with modulus, domains offer two attractive features. First, every stage of approximation carries an actual error bound, rather than the worst case error, as given by the modulus of convergence for Cauchy reals. For example, computing the square root of 2 using Newton iteration (as carried out in e.g. [Sch16]) one obtains a rational Cauchy Sequence $(a_n)_n$ such that for example, $|a_5 - a_n| \leq \frac{1}{6 \cdot 2^4} \approx 5.2 \times 10^{-3}$ for all $n \geq 5$. Instantiating the same method to obtain a shrinking sequence $(a_n, b_n)_n$ of nested rational intervals such that $a_n^2 \leq 2 \leq b_n^2$ one obtains that $|b_n - a_n| \approx 5.7 \times 10^{-49}$ for $n \geq 5$. Both methods use the same initial approximation of $\sqrt{2}$, and indeed the computed Cauchy sequence is identical to the sequence of upper interval endpoints. The significant difference is explained as the modulus is a worst case estimate, whereas the differences between upper and lower interval endpoint are obtained from the actual computation and avoid over-estimation. We showcase this by means of an example in Section 7.

The second attractive feature of a domain theoretic approach is that most classes of domains are closed under the formation of function spaces, i.e. one systematically obtains a representation of the space of e.g. real-valued functions.

Both motivate the development of a more general theory of domains, as e.g. carried out in [BK09]. Our work is similar in spirit, focuses on extracted programs and data type as an end goal. Specifically, our aim is to extract (necessarily effective) functions that operate on the basis of the domains under consideration. For the special case of real numbers (and functions), our goal is to obtain algorithms in the style described in [ES98]. There, mathematical operations (such as computing square roots) are first extended to an appropriate domain (such as the interval domain), then restricted to the base of the domain, and in a third step, shown to be recursive by considering a computable enumeration of basis elements. Indeed, one of our goals is to short-circuit effectivity considerations that are often laborious and provide little insight. Our slogan is “proofs, not programs” as the constructive reasoning (via a realisability interpretation) immediately yields necessarily recursive algorithms.

Putting the extracted algorithm into the centre of attention gauges the formulation of the notion of domain, and this is where differences to [BK09] begin to emerge. The programs we are seeking to extract should embody *just* the computational essence, but no additional terms that evidence correctness. For example, when extracting a program to compute a real number, we only seek a nested sequence of intervals, but *not* a witness of the fact that the intervals are converging to zero in width. This is similar to the approach taken in [Ber11] where one freely adds (true) axioms without computational content to the theory that forms the basis of extraction. That is, we are interested in constructive existence, but are content with classical correctness. Conceptually, this can be understood as phrasing correctness in the classical (double negation) fragment of constructive logic. Technically, the (intended) consequence of this is that correctness proofs do not have any computational content, and are therefore invisible after program extraction, using a standard realisability interpretation [TvD88].

For example, subjecting the proof of the existence of the square root of two to a realisability interpretation, our aim is to extract only a nested sequence of intervals. To achieve this, the definition of equality needs to be free of computational content. We solve this by judiciously setting up the theory in such a way that treats existence of objects as constructive existence, whereas properties are usually formulated classically.

Another aspect where our theory puts the extracted algorithm into the centre of attention is the definition of completion of domain bases. It is one of the hallmark features of domains that ideal elements (such as infinite sequences, or real numbers) can be approximated by elements of a *base*. Constructively, we take the notion of a base as primitive, and recover ideal elements in the completion of the base. Classical domain theory, see e.g. [AJ94] usually considers completion by directed suprema. Here, take the same approach as [BK09] and consider completions by infinite sequences, as they are much more easily representable computationally.

Plan of the paper and main results. We introduce the notion of a *predomain base* that is similar to [BK09] in Section 2, but our definition of the way-below relation is classical, and we establish some basic lemmas, notably interpolation, for later use. We also introduce our main motivating, and running, example, the predomain base of formal intervals. In Section 3, we introduce the continuous completion of a predomain base, along with a (defined) notion of equality. As foreshadowed in the introduction, equality (defined in terms of way-below) is classical and devoid of computational content. The main result here is the extension property that allows us to extend any continuous function defined on a predomain base to its completion.

In Section 4 we align the order-theoretic notion of order-theoretic continuity to topological continuity. As expected, this necessitates a classical definition of the Scott topology which we also show to be generated by upsets of the way-below relation as in the classical theory. In particular, we can show that order-theoretic and topological continuity coincide. Our consideration of continuity naturally leads to the construction of function spaces that we carry out in Section 5. In the classical theory, function spaces are constructed as the set of Scott continuous functions, with pointwise ordering. Here, we investigate the construction of function spaces on the level of predomain bases. More specifically, we present a construction of a predomain base, the completion of which precisely captures the space of continuous functions between the completion of two bases. In Sections 6 we specialise our theory to our initial motivating example, and recapture real numbers as the total elements of the (continuous completion of the) domain of formal intervals. We show that the Euclidean topology arises as the restriction of the Scott topology to real numbers, and investigate the relationship between Cauchy reals and the domain-theoretic reals. As a consequence of our constructive existence – classical correctness approach, both notions are only equivalent if Markov’s principle holds (and in fact, we can prove Markov’s principle from their equivalence). We conclude by relating ϵ - δ continuous functions to the restrictions of Scott continuous total functions. This unearths a new notion of continuity which appears to be weaker than uniform continuity but at the same time stronger than pointwise continuity that we call *intensional non-discontinuity*. We leave the question of a more detailed analysis of this notion to future work.

Related work. We have already mentioned [BK09] which is closest to the work reported in this paper. The main differences are that our notions of way-below and equality are defined classically whereas *op.cit.* employs constructive definitions. We also present a construction of function spaces as completion of predomain bases in Section 5.

Our work stands in the tradition of Bishop-style constructive analysis, [BB85], and indeed we work in a purely constructive setting. What is different is our treatment of real numbers that we derive from the interval domain, similarly to the classical treatment of real analysis in [dG97, ES99] via continuous domains, except that we do not focus on the (classical) notion of computability. Again from a classical perspective, our real numbers (and functions) can be thought of as the total objects of (constructively understood) domains, studied in [Ber93b], although we don't investigate the notion of totality *per se*. The comparison between different notions of continuity on the induced set of real numbers is of course inspired by [Ish92]. Much of this paper is owed to discussions with Helmut Schwichtenberg. His notes [Sch16] develop constructive analysis with a view to program extraction, and the question that motivated the present paper was whether this is also possible using a domain representation of the reals, rather than a Cauchy sequence representation with a worst-case modulus of convergence.

1. PRELIMINARIES AND NOTATION

We work in standard Bishop-style constructive mathematics [BB85] that we envisage as being formalised in higher-type intuitionistic arithmetic \mathbf{HA}^ω [TvD88]. We write \mathbb{N} for the natural numbers, \mathbb{Z} for the integers and \mathbb{Q} for the rationals, and $\mathbb{Q}_{>0}$ for the positive rationals.

We use the term 'weak existence' to refer to the weak existential quantifier $\tilde{\exists} = \neg\forall\neg$ which is constructively equivalent to $\neg\neg\exists$. In informal reasoning, we often say that 'there must exist x such that A ' or 'there weakly exists x such that A ' for $\tilde{\exists}x.A$. We read defined operations universally, that is assuming that $A(x)$ defines x uniquely ($(A(x) \wedge A(y)) \rightarrow x = y$) and we let $\phi(x)$ denote 'the unique x such that $A(x)$ ', we read a formula $B(\phi(x))$ as $\forall x.A(x) \rightarrow B(x)$. In particular, if there must exist x such that $A(x)$, using $\phi(x)$ does not assert (strong) existence.

2. PREDOMAIN BASES AND INTERPOLATION

A *predomain base* is a countable ordered structure that collects finitely representable objects used to approximate elements of ideal structures, such as the real numbers. Examples of predomain bases are finite sequences (approximating infinite streams) and rational intervals (approximating real numbers). The order structure captures information content, such as the prefix ordering for finite sequences, and reverse inclusion for rational intervals.

Predomain bases are the constructive analogue of a base in classical domain theory [AJ94], where arbitrary elements of the domain can be displayed as directed suprema of base elements. In a constructive setting, the totality of the domain is not given and needs to be constructed, similar to the (constructive) notion of real numbers as rational Cauchy sequences with a modulus of convergence. This section discusses basic properties of predomain bases, and we then construct completions in Section 3.

Definition 2.1 (Predomain Bases). Let (B, \sqsubseteq) be a poset. A *chain* in B is a sequence $(b_n)_n$ such that $b_n \sqsubseteq b_{n+1}$ for all $n \in \mathbb{N}$. If (C, \sqsubseteq) is (another) poset, we call a monotone function $f : C \rightarrow D$ *Scott continuous* if $f(\bigsqcup_n b_n) = \bigsqcup_n f(b_n)$ for all chains $(b_n)_n$, provided that all suprema in the last equality exist. An element b is *way below* an element $c \in B$ if there must exist $n \in \mathbb{N}$ such that $b \sqsubseteq x_n$ whenever (x_n) is a chain in B with $\bigsqcup_n x_n \in B$ and $c \sqsubseteq \bigsqcup_n x_n$. We write $b \ll c$ if b is way below c , and also say that b *approximates* c . A chain (b_n) is an

approximating sequence of $b \in B$ if $b_n \ll b$ for all $n \in \mathbb{N}$ and $\bigsqcup_n b_n = b$. A *predomain base* is a countable poset $B = \{b_n \mid n \in \mathbb{N}\}$ with decidable ordering \sqsubseteq in which every element has an approximating sequence.

A non-empty, finite set $B_0 \subseteq B$ is *consistent*, written $\text{Cons}(B_0)$ if it must have an upper bound, i.e. there must exist $b' \in B$ such that $b \sqsubseteq b'$ for all $b \in B$. We say that consistency is *continuous* if $a_i = \bigsqcup_j a_{i,j}$ for all $i \in I$ and $\text{Cons}\{a_{i,j} \mid i \in I\}$ for all $j \in \mathbb{N}$ implies $\text{Cons}\{a_i \mid i \in I\}$ where I is a nonempty, finite set. The poset (B, \sqsubseteq) is *bounded complete* if every finite consistent subset $B_0 \subseteq B$ has a least upper bound $\bigsqcup B_0$, and *pointed* if it has a least element $\perp \in B$.

Note that all non-empty bounded complete posets are necessarily pointed, and that consistency is not necessarily continuous as we demonstrate in Example 2.8.

Remark 2.2. The notion of predomain base differs from that of [BK09] in that *op.cit.* requires that an approximating sequence be a \ll -chain. This immediately entails interpolation: if $x \ll z$ in a predomain base (where every element has an approximating \ll -chain), we have $z = \bigsqcup_n z_n$ for a \ll -chain $(z_n)_n$ so that $x \sqsubseteq z_n$ for some n by definition of \ll . But then $x \sqsubseteq z_n \ll z_{n+1} \ll z_{n+2} \sqsubseteq z$ so that $x \ll z_{n+1} \ll z$, i.e. $y = z_{n+1}$ interpolates between x and z .

We require that every element $x \in B$ can be displayed as $x = \bigsqcup_n x_n$ where each $x_n \ll x$ which is strictly weaker. As a consequence, we need additional hypotheses to establish interpolation in Corollary 2.14. On the other hand, our definition makes it easier to construct predomain bases as we don't need to ensure that approximating sequences are \ll -chains, as for example in the construction of function spaces given later in Lemma 5.10.

We are also adopting a different (weaker) definition of the way-below relation that is formulated using strong existence in *op.cit.* Both are equivalent if Markov's Principle is assumed. By directly phrasing the way-below relation in terms of weak existence, Markov's Principle can be avoided. A helpful pattern of proof that exploits weak existence is the following. Suppose that $\Gamma, \exists x.A \vdash B$ and $\Gamma \vdash \exists x.A$. Then $\Gamma \vdash \neg\neg B$. Similarly, the notion of bounded completeness, phrased in terms of weak existence, is stronger than that of *op.cit.* which uses strong existence. Technically, we need to use weak existence of an upper bound to establish that the continuous completion of a (bounded complete) predomain base has suprema of all increasing chains (Corollary 3.5). Conceptually, weak existence suffices as the witness of boundedness of a finite subset of a predomain base is not used in the construction of the least upper bound.

Example 2.3. Let B be a countable set with decidable equality, that is, $\forall b, b' \in B.(b = b') \vee \neg(b = b')$ is (constructively) provable. Then $(B, =)$ and $(B^*, \sqsubseteq_{\text{pref}})$ are predomain bases where B^* is the set of finite sequences of B and $\sqsubseteq_{\text{pref}}$ is the prefix ordering. Both are bounded complete and satisfy $x \ll x$ for all $x \in B$ (resp. $x \in B^*$).

If (B, \sqsubseteq) and (C, \sqsubseteq) are predomain bases, then so are $B \times C$ and $B + C$ with the pointwise and co-pointwise ordering. Moreover $(B_\perp, \sqsubseteq_\perp)$ is a predomain base where $B_\perp = B \cup \{\perp\}$ (we tacitly assume $\perp \notin B$) and $b \sqsubseteq c$ if either $b = \perp$ or $b \neq \perp \neq c$ and $b \sqsubseteq c$. The predomain bases $B \times C$, $B + C$ and B_\perp are the *product*, *coproduct* and *lifting* of B and C (resp. of B).

Example 2.4. The poset $\mathbb{I}\mathbb{Q} = \{(p, q) \in \mathbb{Q} \times \mathbb{Q} \mid p \leq q\}$ ordered by $(p, q) \sqsubseteq (p', q')$ iff $p \leq p' \leq q' \leq q$ is called the *predomain base of rational intervals*. We usually write $[p, q]$ for the pair $(p, q) \in \mathbb{I}\mathbb{Q}$ and think of $[p, q]$ as a rational interval. For $\alpha = [a, b] \in \mathbb{I}\mathbb{Q}$, we sometimes write $\alpha = [\underline{\alpha}, \bar{\alpha}]$ to denote the lower and upper endpoint of α , and $\alpha \pm \delta = [\underline{\alpha} - \delta, \bar{\alpha} + \delta]$ for the symmetric extension of α by $\delta \in \mathbb{Q}_{\geq 0}$.

It is not immediate (but easy) to see that $\mathbb{I}\mathbb{Q}$ is a predomain base. The negative formulation of \ll gives the following characterisation that has been established in [BK09, Proposition 7.2] using Markov's Principle.

Lemma 2.5. *Let $[p, q], [p', q'] \in \mathbb{I}\mathbb{Q}$. Then $[p, q] \ll [p', q']$ iff $p < p' \leq q' < q$.*

Proof. For the only-if direction, assume that $[p, q] \ll [p', q']$. As $[p', q'] = \sqcup_n [p' - 2^{-n}, q' + 2^{-n}]$ there must exist $n \in \mathbb{N}$ such that $[p, q] \sqsubseteq [p' - 2^{-n}, q' + 2^{-n}]$ from which we obtain that $p \leq p' - 2^{-n} < p' \leq q' < q' + 2^{-n} \leq q$ and hence $p < p' \leq q' < q$ using decidability of order on \mathbb{Q} .

For the converse, assume that $p < p' \leq q' < q$ and assume that $[p', q'] \sqsubseteq [a, b] = \sqcup_n [a_n, b_n]$. Then $a = \sup_n a_n$ and $b = \inf_n b_n$. We claim that there must exist n and m so that $a_n \geq p$ and $b_m \leq q$. So assume that $a_n \leq p$ for all $n \in \mathbb{N}$. Then p is an upper bound of $(a_n)_n$ and therefore $a \leq p$. Hence $a \leq p < p' \leq a$, contradiction. The proof of classical existence of m is analogous. Hence there must exist $N = \max\{n, m\}$ such that we have $p \leq a_N \leq b_N \leq q$, that is, $[p, q] \sqsubseteq [a_N, b_N]$. \square

Lemma 2.6. *$\mathbb{I}\mathbb{Q}$ is a predomain base.*

Proof. Let $[p, q] \in \mathbb{I}\mathbb{Q}$ be given. Then $([p - 2^{-n}, q + 2^{-n}])_n$ is a approximating sequence of $[p, q]$. \square

Lemma 2.7. *Consistency on $\mathbb{I}\mathbb{Q}$ is continuous.*

Proof. Let I be a finite set and let $\alpha_i = \sqcup_j \alpha_{i,j} \in \mathbb{I}\mathbb{Q}$ for all $i \in I$. Assume furthermore that $\{\alpha_{i,j} \mid i \in I\}$ is consistent for all $j \in \mathbb{N}$, we show that $\{\alpha_i \mid i \in I\}$ is consistent. The latter is the case if $\max\{\underline{\alpha}_i \mid i \in I\} \leq \min\{\bar{\alpha}_i \mid i \in I\}$. We have, for all $i \in I$, that $\max\{\underline{\alpha}_{i,j} \mid i \in I\} \leq \min\{\bar{\alpha}_{i,j} \mid i \in I\}$ which implies the claim. \square

Example 2.8. Consistency is not automatically continuous. Consider for instance the predomain base $B = \mathbb{I}\mathbb{Q} \setminus \{[0, 0]\}$ and two sequences $\alpha_n = [-1, 2^{-n}]$ and $\beta_n = [-2^{-n}, 1]$. Then α_n and β_n are consistent for all $n \in \mathbb{N}$ but $[-1, 0] = \sqcup_n \alpha_n$ and $[0, 1] = \sqcup_n \beta_n$ are not.

We collect some basic facts about posets and the way-below relation that are used in the proof of our first technical result, the interpolation property (Proposition 2.13 and Corollary 2.14). The majority of results are standard in (classical) domain theory, see e.g. [AJ94], and we include them here both to be self-contained and to demonstrate that they continue to hold in our framework.

Lemma 2.9. *Let (P, \sqsubseteq) be a poset for which \sqsubseteq is decidable. Then $b \sqsubseteq c$ whenever $b \ll c$. Moreover, $a \sqsubseteq b \ll c$ implies that $a \ll c$, and similarly $a \ll b \sqsubseteq c$ implies that $a \ll c$, for $a, b, c \in P$.*

Proof. For the first item, assume that $b \ll c$. As $c = \sqcup_n c$ there must exist n such that $b \sqsubseteq c$ whence $b \sqsubseteq c$.

Now suppose that $a \sqsubseteq b \ll c$. Let $c \sqsubseteq \sqcup_n c_n$. Then there is n , weakly, such that $b \sqsubseteq c_n$ whence $a \sqsubseteq c_n$, too. Now suppose that $a \ll b \sqsubseteq c$ and let $c \sqsubseteq \sqcup_n c_n$. Then $b \sqsubseteq \sqcup_n c_n$ whence there is n , weakly, such that $a \sqsubseteq c_n$. \square

The proof of the above lemma uses that \sqsubseteq is $\neg\neg$ -closed which follows from decidability.

Lemma 2.10. *Let (P, \sqsubseteq) be a poset, I be a finite set and $b_i, b'_i \in P$ with $b_i \ll b'_i$ for all $i \in I$. If $s = \sqcup_i b_i$ and $s' = \sqcup_i b'_i$ then $s \ll s'$.*

Proof. Let (x_n) be a chain in P where $s' = \sqcup_i b'_i \sqsubseteq \sqcup_n x_n$. Since $\sqcup_i b'_i$ is an upper bound of $\{b'_i \mid i \in I\}$, we have $b'_i \sqsubseteq \sqcup_n x_n$ for all $i \in I$. Moreover, by assumption $b_i \ll b'_i$ for all $i \in I$, there must exist n_i such that $b_i \sqsubseteq x_{n_i}$. Now by setting $n = \max\{n_i \mid i \in I\}$, then we have $b_i \sqsubseteq x_n$ for all $i \in I$. Hence $\sqcup_i b_i \sqsubseteq x_n$ as $\sqcup_i b_i$ is the least upper bound. \square

Corollary 2.11. *Let (B, \sqsubseteq) be a bounded complete poset and I a finite set. If $b_i, b'_i \in B$ for $i \in I$ such that $b_i \ll b'_i$ and $\{b'_i \mid i \in I\}$ is consistent, then both $\sqcup_i b_i$ and $\sqcup_i b'_i$ exist in B and $\sqcup_i b_i \ll \sqcup_i b'_i$.*

Proof. As $\{b'_i \mid i \in I\}$ is consistent, $\sqcup_i b'_i$ exists in B , and there must exist an upper bound $x \in B$ such that $b'_i \sqsubseteq x$ for all $i \in I$. Moreover, $b_i \sqsubseteq b'_i \sqsubseteq x$ for all $i \in I$ as $b_i \ll b'_i$ hence $\{b_i \mid i \in I\}$ is also consistent (with upper bound x), hence $\sqcup_i b_i$ exists in B , and the claim follows from Lemma 2.10. \square

Lemma 2.12. *Let (P, \sqsubseteq) be a poset and $(a_n)_n$ and $(b_n)_n$ chains in B . If $s = (\sqcup_n a_n) \sqcup (\sqcup_n b_n)$ exists in P and $s_n = a_n \sqcup b_n$ then $s = \sqcup_n s_n$.*

Proof. We first show that $s = a \sqcup b$ is an upper bound of $a_n \sqcup b_n$ for all $n \in \mathbb{N}$. We have that $a_n \sqsubseteq a \sqsubseteq a \sqcup b$ as a is an upper bound of a_n and $a \sqcup b$ is an upper bound of b . Similarly $b_n \sqsubseteq a \sqcup b$. Hence $a \sqcup b$ is an upper bound of $a_n \sqcup b_n$ and $a_n \sqcup b_n \sqsubseteq a \sqcup b$ follows as $a_n \sqcup b_n$ is the least upper bound.

Now we show that $a \sqcup b$ is indeed the least upper bound of $a_n \sqcup b_n$. So take another upper bound x , that is, $a_n \sqcup b_n \sqsubseteq x$ for all $n \in \mathbb{N}$. Then $a_n \sqsubseteq a_n \sqcup b_n \sqsubseteq x$ and $b_n \sqsubseteq a_n \sqcup b_n \sqsubseteq x$ for all $n \in \mathbb{N}$. Hence $a = \sqcup_n a_n \sqsubseteq x$ and $b = \sqcup_n b_n \sqsubseteq x$. As $a \sqcup b$ is the least upper bound of a and b , it follows that $a \sqcup b \sqsubseteq x$. \square

The above facts are used to prove our first result, the (weak) interpolation property.

Proposition 2.13. *Let (B, \sqsubseteq) be a bounded complete predomain base, and assume that \ll on B is decidable. Then B has the weak interpolation property, that is, whenever $x \ll z$ for $x, z \in B$ there must exist $y \in B$ such that $x \ll y \ll z$.*

Proof. We adapt the (classical) proof based on directed suprema ([AJ94, Lemma 2.2.15]) to our setting. Assume that $x \ll z$, and let $B = \{b_n \mid n \in \mathbb{N}\}$.

As $z \in B$ and B is a predomain base, we can find an element $b_n \in B$ with $b_n \ll z$ (e.g. the first element of an approximating sequence of z). By the same reasoning, we can find $b_m \in B$ with $b_m \ll b_n$. Let $o = \max\{n, m\}$ and consider the sequence

$$c_n = \sqcup \{b_i \mid 0 \leq i \leq o + n, \exists 0 \leq j \leq o + n. b_i \ll b_j \ll z\}.$$

Then c_n is well-defined, as suprema are taken over a non-empty, bounded (by z) and finite set.

We now claim that $\sqcup_n c_n = z$. First, it is clear that $c_n \sqsubseteq z$ for all $n \in \mathbb{N}$. To see that z is a least upper bound of the c_n , suppose that $c_n \sqsubseteq u$ for all $n \in \mathbb{N}$, and we show that $z \sqsubseteq u$. Let $(z_n)_n$ be an approximating sequence for z . As $z = \sqcup_n z_n$, it suffices to show that $z_n \sqsubseteq u$ for all $n \in \mathbb{N}$. So let $n \in \mathbb{N}$. As $z_n \in B$, there exists an approximating sequence $(z_n^k)_k$ for z_n , and in particular $z_n^k \ll z_n \ll z$ for all $k \in \mathbb{N}$. Now fix an arbitrary $k \in \mathbb{N}$, we show that $z_n^k \sqsubseteq u$. As B is countable, we can find $p, q \in \mathbb{N}$ such that $z_n = b_p$ and $z_n^k = b_q$. Let $r = \max\{p, q\}$. Then $b_q = z_n^k \ll z_n = b_p \ll z$ and therefore $b_q \sqsubseteq c_r$ as $p, q \leq r \leq r + o$. As $c_r \sqsubseteq u$ we have that $z_n^k = b_q \sqsubseteq u$. As k was arbitrary, this implies that $z_n = \sqcup_k z_n^k \sqsubseteq u$. By the same argument, as n was arbitrary and $z = \sqcup_n z_n$, we may conclude that $z \sqsubseteq u$, thus establishing the claim.

We now have that $x \ll z = \sqcup_n c_n$. Therefore, there weakly exists $n \in \mathbb{N}$ such that $x \sqsubseteq c_n$. Let $c_n = \sqcup\{b_i \mid i \in I\}$ where $I \subseteq \{0, \dots, o+n\}$ is a finite, non-empty set. For each $i \in I$ we can moreover find $b'_i \in \{b_0, \dots, b_{n+o}\}$ with $b_i \ll b'_i \ll z$.

By Lemma 2.10 we have that $x \sqsubseteq \sqcup\{b_i \mid i \in I\} \ll \sqcup\{b'_i \mid i \in I\} \ll z$. Therefore $y = \sqcup\{b'_i \mid i \in I\}$ is our desired interpolant. This (only) shows weak existence of an interpolant, due to the weak existence of the number n used in its construction. \square

Corollary 2.14. *Let (B, \sqsubseteq) be a bounded complete predomain base for which \ll is decidable. If $b_1, \dots, b_n \in B$ and $b_i \ll c$ for all $i = 1, \dots, n$ then there must exist an interpolant $b \in B$ such that $b_i \ll b \ll c$ for all $1 \leq i \leq n$.*

Proof. By the previous lemma, we can find interpolants \hat{b}_i for each $1 \leq i \leq n$ such that $b_i \ll \hat{b}_i \ll c$. By Lemma 2.10 we have that $b = \sqcup\{\hat{b}_i \mid 1 \leq i \leq n\}$ satisfies $b \ll c$ and moreover $b_i \ll \hat{b}_i \sqsubseteq b$ so that $b_i \ll b$ for all $1 \leq i \leq n$. \square

We conclude the section with a technical lemma on swapping the order of suprema that we will use later.

Lemma 2.15. *Suppose that P is a poset and $f : \mathbb{N} \times \mathbb{N} \rightarrow P$ is monotonic, i.e. $n \leq n'$ and $k \leq k'$ implies $f(n, k) \sqsubseteq f(n', k')$. Then*

- (1) *the sequence $(f(n, m))_n$ is monotonic for all $m \in \mathbb{N}$*
- (2) *if $\sqcup_n f(n, m)$ exists for all $m \in \mathbb{N}$, then $(\sqcup_n f(n, m))_m$ is monotonic*
- (3) *if both $\sqcup_n f(n, n)$ and $\sqcup_m \sqcup_n f(n, m)$ both exist, they are equal.*

Proof. The first item is immediate by monotonicity of f . For the second item, fix $m \in \mathbb{N}$. We show that $\sqcup_n f(n, m) \sqsubseteq \sqcup_n f(n, m+1)$. This is immediate as $\sqcup_n f(n, m+1)$ is an upper bound of $f(n, m)$ for all $n \in \mathbb{N}$.

For the last item, suppose that $\sqcup_n f(n, n)$ and $\sqcup_m \sqcup_n f(n, m)$ both exist, in particular this entails that $\sqcup_n f(n, m)$ exists for all $m \in \mathbb{N}$. We first show that $\sqcup_n f(n, n)$ is an upper bound of all $\sqcup_n f(n, m)$ for all $n \in \mathbb{N}$. By monotonicity, we have that $\sqcup_n f(n, m) = \sqcup_{n \geq m} f(n, m) \sqsubseteq \sqcup_{n \geq m} f(n, n) = \sqcup_n f(n, n)$. To finish the proof, we need to show that $\sqcup_n f(n, n)$ is the least upper bound of $\sqcup_n f(n, m)$ for all $m \in \mathbb{N}$. So let c be a competitor, i.e. $\sqcup_n f(n, m) \sqsubseteq c$ for all $m \in \mathbb{N}$. We show that $\sqcup_n f(n, n) \sqsubseteq c$. This follows once we establish that $f(n, n) \sqsubseteq c$ for all $n \in \mathbb{N}$ as $\sqcup_n f(n, n)$ is the least upper bound of all $f(n, n)$. So let $n \in \mathbb{N}$. But this is evident as $f(n, n) \sqsubseteq \sqcup_k f(k, n) \sqsubseteq c$ by assumption. \square

Corollary 2.16. *Let P be a poset that has suprema of all increasing chains, and let $f : \mathbb{N} \times \mathbb{N} \rightarrow P$ be monotone. Then both $\sqcup_n f(n, n)$ and $\sqcup_m \sqcup_n f(n, m)$ exist and are equal.*

3. COMPLETION OF PREDOMAIN BASES

We give a direct description of the rounded ideal completion of [BK09] with ideals being represented by chains. The rounded ideal (or continuous) completion is distinguished from the ideal completion by the definition of the order \sqsubseteq on the completion in terms of approximation \ll on the underlying predomain base, rather than in terms of its order.

Definition 3.1 (Continuous Completion). Let (B, \sqsubseteq) be a predomain base. The *continuous completion* of B is the set $\hat{B} = \{(b_n)_n \mid b_n \in B, b_n \sqsubseteq b_{n+1} \text{ for all } n \in \mathbb{N}\}$ of increasing sequences in B , with order relation defined by

$$(b_n) \sqsubseteq (b'_n) \text{ iff } \forall b \in B. \forall n \in \mathbb{N}. b \ll b_n \rightarrow \exists m \in \mathbb{N}. b \ll b'_m$$

for increasing sequences $(b_n)_n$ and $(b'_n)_n$ in B .

The function $i : B \rightarrow \hat{B}$ that maps $b \in B$ to the constant sequence $(b)_n$ is called the *canonical embedding*, and in the sequel we identify elements in B with their canonical embedding.

We show in Lemma 3.8 that the canonical embedding indeed preserves both \sqsubseteq and \ll which justifies our terminology. The above definition of the order \sqsubseteq on the completion of a predomain base showcases the first instance of our “constructive existence – classical correctness” approach in the classical definition of the order relation on the completion above. In particular, this implies that a realiser of $(b_n) \sqsubseteq (b'_n)_n$ carries no computational content. It is straightforward to see that the order relation \sqsubseteq defined above is in fact a preorder. We omit the straightforward proof of this fact.

Lemma 3.2. *The order relation \sqsubseteq on the continuous completion \hat{B} of a predomain base B is a preorder.*

The preorder \sqsubseteq on the continuous completion \hat{B} of a predomain base induces an equality relation on \hat{B} where $b = b'$ iff $b \sqsubseteq b'$ and $b' \sqsubseteq b$. In particular, this gives (\hat{B}, \sqsubseteq) the structure of a poset, where arbitrary suprema, if they exist, are unique up to equality, i.e. $s = \bigsqcup_i a_i$ and $s' = \bigsqcup_i a_i$ implies $s = s'$. Moreover, suprema are extensional: if $c_i = d_i$ for all $i \in I$ and $s = \bigsqcup_i c_i$, $t = \bigsqcup_i d_i$ then $s = t$.

It is an easy but very useful observation that every element of the continuous completion is equal to the supremum of the elements of (the canonical embeddings of) its representing sequence.

Lemma 3.3. *Let B be a predomain base and $x = (x_n)_n \in \hat{B}$. Then $x = \bigsqcup_n x_n$.*

Proof. First, x is an upper bound of all x_n . To see this, let $n \in \mathbb{N}$ and $a \in B$ with $a \ll x_n$. We have to show that there must exist some $k \in \mathbb{N}$ with $a \ll x_k$. But this clearly holds for $k = n$. To see that x is the least upper bound of all x_n , consider a competing upper bound $c = (c_n)_n \in \hat{B}$ with $x_n \sqsubseteq c$ for all $n \in \mathbb{N}$. To see that $x \sqsubseteq c$, let $n \in \mathbb{N}$, $a \in B$ with $a \ll x_n$. As $x_n \sqsubseteq c$ there must exist $k \in \mathbb{N}$ such that $a \ll c_k$ which is precisely what we need to show for $x \sqsubseteq c$. \square

We now show that the continuous completion \hat{B} of a predomain base B has suprema of all increasing chains. Below, we write $M \ll z$ if $x \ll z$ for all $x \in M$ and say that a predomain base B has *weak interpolation* if there weakly exists y such that $M \ll y \ll z$ whenever M is finite and $M \ll z$.

Lemma 3.4. *Let $B = \{b_n \mid n \in \mathbb{N}\}$ be a predomain base and $(b_n^k)_k$ an approximating sequence of b_n . If $(c_n)_n$ is an increasing sequence in \hat{B} and $c_n = (c_{n,m})_m$, then the following statements hold:*

- (1) *The set $D_k = \{c_{n,m}^k \mid 0 \leq n, m \leq k\}$ is consistent. If consistency is continuous, the same applies to the set $\hat{D}_k = \{c_{n,k} \mid 0 \leq n \leq k\}$.*
- (2) *The sequence $(d_k)_k = (\bigsqcup D_k)_k$ is increasing. If consistency is continuous, the same applies to the sequence $(\hat{d}_k)_k = (\bigsqcup \hat{D}_k)_k$.*
- (3) *If B has weak interpolation, we have $c_n \sqsubseteq d$ for all $n \in \mathbb{N}$. If consistency on B is moreover continuous, also $c_n \sqsubseteq \hat{d}$ for all $n \in \mathbb{N}$.*
- (4) *If $c_n \sqsubseteq u$ for all $n \in \mathbb{N}$, then $d \sqsubseteq u$. If consistency on B is continuous, then also $\hat{d} \sqsubseteq u$.*

Proof. For the first item, fix $k \in \mathbb{N}$ and let $0 \leq n, m \leq k$. As $(c_{n,m}^i)_i$ is an approximating sequence of $(c_{n,m})$ we have $c_{n,m}^k \ll c_{n,m}$. As $c_n \sqsubseteq c_k$, there must exist $r = r(n, m)$ such that $c_{n,m}^k \ll c_{k,r}$. Let $s = \max\{r(n, m) \mid 0 \leq n, m \leq k\}$. Then $c_{n,m}^k \ll c_{k,r} \sqsubseteq c_{k,s}$ so that $c_{k,s}$ is an upper bound of D_k .

Now assume that consistency is continuous. Then consistency of \hat{D}_k follows if the sets $\hat{D}_{k,i} = \{c_{n,k}^i \mid 0 \leq n \leq k\}$ are consistent for all $i \in \mathbb{N}$. Let $i \in \mathbb{N}$ and $r = \max\{i, k\}$. By what we have just demonstrated, there must exist an upper bound b of the set D_r . We show that b is an upper bound of $\hat{D}_{k,i}$. This follows since for $c_{k,n}^i \in \hat{D}_{k,i}$ we have that $c_{k,n}^i \sqsubseteq c_{k,n}^r \in D_r$ and the fact that b is an upper bound of D_r .

The second item, monotonicity of $(d_k)_k$ and $(\hat{d}_k)_k$ is clear since both $D_k \sqsubseteq D_{k+1}$ and $\hat{D}_k \sqsubseteq \hat{D}_{k+1}$.

For the third item, we begin by showing that $c_n \sqsubseteq d$. So fix $m \in \mathbb{N}$ and suppose that $x \ll c_{n,m}$, we show that there must exist $k \in \mathbb{N}$ such that $x \ll d_k$. As $c_{n,m} = \sqcup_k c_{n,m}^k$ there must exist $k \in \mathbb{N}$ such that $x \ll c_{n,m}^k$. The same relation holds if we replace k by $k' = \max\{n, m, k\}$ so we assume that $k \geq n, m$ without loss of generality. Then $x \ll c_{n,m}^k \sqsubseteq \sqcup\{c_{n,m}^k \mid 0 \leq n, m \leq k\} = d_k$ as required. Now assume that consistency is continuous. To see that $c_n \sqsubseteq \hat{d}$, fix $m \in \mathbb{N}$, $x \in B$ and assume that $x \ll c_{n,m}$. We show that there must exist $k \in \mathbb{N}$ such that $x \ll \hat{d}_k$. This holds, for example, if $k = \max\{n, m\}$ for then $x \ll c_{n,m} \sqsubseteq c_{n,k} \sqsubseteq \sqcup\{c_{n,k} \mid 0 \leq n \leq k\} = \hat{d}_k$.

For the last item, assume that $u = (u_i)_i \in \hat{B}$ and $c_n \sqsubseteq u$ for all $n \in \mathbb{N}$. We first show that $d \sqsubseteq u$. To see this, fix $k \in \mathbb{N}$, $x \in B$ and assume that $x \ll d_k$. We show that there must exist $s \in \mathbb{N}$ such that $x \ll u_s$. By assumption, we have $x \ll d_k = \sqcup\{c_{n,m}^k \mid 0 \leq n, m \leq k\}$. Fix $0 \leq n, m \leq k$. Since $c_{n,m}^k \ll c_{n,m}$ and $c_n \sqsubseteq u$, there must exist $r = r(n, m) \in \mathbb{N}$ such that $c_{n,m}^k \ll u_r$. If $s = \max\{r(n, m) \mid 0 \leq n, m \leq k\}$, we have $c_{n,m}^k \ll u_s$ for all $0 \leq n, m \leq k$. Hence, by Corollary 2.11 we obtain $d_k = \sqcup\{c_{n,m}^k \mid 0 \leq n, m \leq k\} \ll u_s$ as required. Now suppose that consistency is continuous. To see that $\hat{d} \sqsubseteq u$, fix $k \in \mathbb{N}$ and $x \in B$ such that $x \ll \hat{d}_k = \sqcup\{c_{n,k} \mid 0 \leq n \leq k\}$. We show that there must exist $s \in \mathbb{N}$ such that (as above) $x \ll u_s$. Fix $0 \leq n \leq k$. As $x \ll \hat{d}_k = \sqcup_{0 \leq k \leq n} \sqcup_i c_{n,k}^i = \sqcup_i \sqcup_{0 \leq n \leq k} c_{n,k}^i$ there must exist $i \in \mathbb{N}$ such that $x \sqsubseteq \sqcup_{0 \leq n \leq k} c_{n,k}^i$. For this $i \in \mathbb{N}$, we moreover have that $c_{n,k}^i \ll c_{n,k}$ so that there must exist $r = r(n, k, i)$ for which $c_{n,k}^i \ll u_r$ since $c_n \sqsubseteq u$. Hence for $s = \max\{r(n, k, i) \mid 0 \leq n \leq k\}$ we have that $c_{n,k}^i \ll u_s$ so that $\sqcup_{0 \leq k \leq n} c_{n,k}^i \ll u_s$ by Corollary 2.11, and finally $x \sqsubseteq \sqcup_{0 \leq k \leq n} c_{n,k}^i \ll u_s$ as desired. \square

The last lemma finally puts us into a position to show that the completion of a predomain base is in fact complete.

Corollary 3.5. *Let (B, \sqsubseteq) be a bounded complete predomain base. Then (\hat{B}, \sqsubseteq) has suprema of all increasing chains.*

Proof. For bounded complete predomain bases, we have established the weak interpolation property in Corollary 2.14. The claim follows from Lemma 3.4. \square

We have the following extension theorem.

Proposition 3.6. *Suppose that B and C are predomain bases for which consistency is continuous, and suppose that C has weak interpolation. Then every Scott continuous map $f : B \rightarrow \hat{C}$ has a Scott continuous extension $\hat{f} : \hat{B} \rightarrow \hat{C}$.*

Proof. Let $(x^i)_i$ be an approximating sequence for each element $x \in B$. Define $\hat{f}((b_n)_n) = (f(b_n))_n$ for a monotone sequence $(b_n)_n \in \hat{B}$. Then $\hat{f}((b_n)_n)$ is monotone as f is monotone. We show that \hat{f} is Scott continuous. For this we fix an increasing sequence $(b_n)_n \in \hat{B}$ where $b_n = (b_{n,m})_m$ and use Lemma 3.4 to establish that

$$A_k = [\hat{f}(\bigsqcup_n b_n)]_k = [\hat{f}(\bigsqcup_{n \leq k} (b_{n,k})_k)]_k = f(\bigsqcup_{n \leq k} b_{n,k})$$

for the k -th element $[\cdot]_k$ of $\hat{f}(\bigsqcup_n b_n)$ using that consistency on C is continuous. For the k -th element of $\bigsqcup_n \hat{f}(b_n)$ we similarly obtain

$$B_k = [\bigsqcup_n \hat{f}(b_n)]_k = [\bigsqcup_n (f((b_{n,k})_k))]_k = \bigsqcup_{n \leq k} f(b_{n,k})$$

also using Lemma 3.4 and continuity of consistency on B . For the claim, we need to establish $(A_k)_k = (B_k)_k$. To see that $(A_k)_k \sqsubseteq (B_k)_k$ fix $k \in \mathbb{N}$ and $x \in C$ such that $x \ll A_k = f(\bigsqcup_{n \leq k} b_{n,k})$. We show that there must exist $l \in \mathbb{N}$ such that $x \ll B_l = \bigsqcup_{n \leq l} f(b_{n,l})$. As C has weak interpolation, there must exist $y \in C$ such that $x \ll y \ll f(\bigsqcup_{n \leq k} b_{n,k})$. Therefore

$$y \ll f(\bigsqcup_{n \leq k} \bigsqcup_{i \in \mathbb{N}} b_{n,k}^i) = f(\bigsqcup_i \bigsqcup_{n \leq k} b_{n,k}^i) = \bigsqcup_i f(\bigsqcup_{n \leq k} b_{n,k}^i)$$

using continuity of f . Therefore there must exist $i \in \mathbb{N}$ such that $x \ll y \sqsubseteq f(\bigsqcup_{n \leq k} b_{n,k}^i)$. As $b_n \sqsubseteq b_k$ for $n \leq k$ and $b_{n,k}^i \ll b_{n,k}$ there must exist $j(n)$ such that $b_{n,k}^i \ll b_{k,j(n)}$. Let $j = \max\{j(n) \mid 0 \leq n \leq k\}$. Then $x \ll f(\bigsqcup_{n \leq k} b_{n,k}^i) \sqsubseteq f(b_{k,j})$. For $l = \max\{j, k\}$ we therefore obtain that $x \ll f(b_{k,j}) \sqsubseteq \bigsqcup_{n \leq l} f(b_{n,l}) = B_l$. For the reverse direction $(B_k)_k \sqsubseteq (A_k)_k$ fix $k \in \mathbb{N}$ and $x \in C$ such that $x \ll B_k = \bigsqcup_{n \leq k} f(b_{n,k})$. We show that there must exist $l \in \mathbb{N}$ such that $x \ll f(\bigsqcup_{n \leq l} b_{n,l})$. But this is evident for $l = k$ as $f(b_{n,k}) \sqsubseteq f(\bigsqcup_{n \leq k} b_{n,k})$ by monotonicity of f , for all $0 \leq k \leq n$, whence $x \ll \bigsqcup_{n \leq k} f(b_{n,k}) \sqsubseteq f(\bigsqcup_{n \leq k} b_{n,k})$. \square

We now show that bounded completeness (Definition 2.1) transfers from a predomain base to its completion.

Lemma 3.7. *Let (B, \sqsubseteq) be a bounded complete predomain base and suppose that consistency on B is continuous. Let $I \subseteq \hat{B}$ be finite and consistent.*

- (1) $I_n = \{x_n \mid (x_n)_n \in I\}$ is consistent for all $n \in \mathbb{N}$.
- (2) $\bigsqcup I = (\bigsqcup I_n)_n$

i.e. the completion of a bounded complete predomain base is bounded complete, and finite suprema of consistent sets are calculated pointwise.

Proof. For the first item, let $b = (b_n)_n$ be an upper bound of I , and let $(x^i)_i$ be an approximating sequence of $x \in B$. As consistency on B is continuous, it suffices to show that $\{x_n^i \mid (x_n)_n \in I\}$ is continuous for all $n, i \in \mathbb{N}$. So let $i, n \in \mathbb{N}$. For $x = (x_n)_n \in I$, as $x_n^i \ll x_n$ and $(x_n)_n \sqsubseteq b$, there must exist $k(x)$ such that $x_n^i \ll b_{k(x)}$. Let $k = \max\{k(x) \mid x \in I\}$. Then b_k is an upper bound of $\{x_n^i \mid (x_n)_n \in I\}$.

For the second item, note that $\bigsqcup I_n \in B$ exists since I_n is consistent and $(\bigsqcup I_n)_n$ is monotone as all $x \in I$ are monotone so that $(\bigsqcup I_n)_n \in \hat{B}$. We first show that $(\bigsqcup I_n)_n$ is an upper bound of all $x \in I$. So let $x \in I$, $n \in \mathbb{N}$ and assume that $z \in B$ with $z \ll x_n$. Then $z \ll \bigsqcup I_n$ whence $x \sqsubseteq (\bigsqcup I_n)_n$ by definition of \sqsubseteq on \hat{B} . We now show that $(\bigsqcup I_n)_n$ is the least upper bound of I . So assume that $x \sqsubseteq b$ for all $x \in I$. We show that $(\bigsqcup I_n)_n \sqsubseteq b$. So let $n \in \mathbb{N}$, $z \in B$ with $z \ll \bigsqcup I_n$. We show that there must exist $k \in \mathbb{N}$ with $z \ll b_k$.

As $\sqcup I_n = \sqcup\{\sqcup_i x_n^i \mid (x_n)_n \in I\} = \sqcup_i \sqcup\{x_n^i \mid (x_n)_n \in I\}$ (using Lemma 2.12) and $z \ll \sqcup I_n$, there must exist $l \in \mathbb{N}$ such that $z \sqsubseteq \sqcup\{x_n^k \mid (x_n)_n \in I\}$. For $x = (x_n)_n \in I$ we moreover have that $x_n^k \ll x_n$, and since $x \sqsubseteq b$ there must exist $l(x) \in \mathbb{N}$ such that $x_n^i \ll b_{l(x)}$. Let $l = \max\{l(x) \mid x \in I\}$. Then $z \sqsubseteq \sqcup\{x_n^k \mid (x_n)_n \in I\} \ll b_l$ by Lemma 2.10. \square

We conclude the section with properties of the canonical embedding that have already been reported in [BK09], albeit in a slight different setting.

Lemma 3.8. *Let B be a bounded complete predomain base. Then the embedding $B \hookrightarrow \hat{B}$ preserves and reflects \sqsubseteq and preserves \ll .*

Proof. It is clear that the embedding preserves and reflects the order \sqsubseteq . To see that it preserves \ll , assume that $b, c \in B$ with $b \ll c$ in B . We show that $b \ll c$ in \hat{B} where we identify b and c with the constant sequences $(b)_n$ and $(c)_n$.

To see that $b \ll c$ in \hat{B} , let $(x_n)_n$ be a chain in \hat{B} with $c \sqsubseteq \sqcup_n x_n$. Let $x_n = (x_{n,m})_m$ and choose an approximating sequence $(a^n)_n$ for every element $a \in B$. As $\sqcup_n x_n = (\sqcup\{x_{n,m}^k \mid 0 \leq n, m \leq k\})_k$ by Lemma 3.4 and $b \ll c$, there must exist $k \in \mathbb{N}$ such that $b \sqsubseteq \sqcup\{x_{n,m}^k \mid 0 \leq k \leq n\}$ in B and hence also in \hat{B} . For $0 \leq n, m \leq k$ we have $x_{n,m}^k \sqsubseteq x_{n,m} \sqsubseteq x_n \sqsubseteq x_k$ and so $b \sqsubseteq x_k$ as required. \square

4. THE SCOTT TOPOLOGY

We investigate the Scott topology on the completion of a predomain base. We show that every topologically continuous function is Scott continuous, and that the Scott topology is generated by the way-below relation. These results are also used later in the function space construction. In particular, we employ a classical definition of the Scott topology to align open sets with the way-below relation.

Definition 4.1 (Basic Notions). Let (P, \sqsubseteq) be a poset. A subset $\mathcal{O} \subseteq P$ is *Scott-open* if it is an upper set ($\forall x, y \in P. x \sqsubseteq y \wedge x \in P \rightarrow y \in P$) and inaccessible by suprema of increasing chains (if $\sqcup_n x_n \in \mathcal{O}$ for an increasing chain $(x_n)_n$ then there must exist $k \in \mathbb{N}$ such that $x_k \in \mathcal{O}$).

Lemma 4.2. *Let B be a bounded complete predomain base. Then $b \uparrow = \{x \in \hat{B} \mid b \ll x\}$ is Scott-open for all $b \in \hat{B}$.*

Proof. To see that $b \uparrow$ is an upper set, assume that $x, y \in \hat{B}$ with $x \sqsubseteq y$ and let $x \in b \uparrow$, i.e. $b \ll x$. Then $b \ll x \sqsubseteq y$ so that $b \ll y$ by Lemma 2.9. To see that $b \uparrow$ is inaccessible by suprema increasing chains, assume that $\sqcup_n x_n \in b \uparrow$, i.e. $b \ll \sqcup_n x_n$. By definition of \ll , there must exist $n \in \mathbb{N}$ such that $b \ll x_n$, hence $x_n \in b \uparrow$. \square

Lemma 4.3. *Let B be a bounded complete predomain base, $x \in \hat{B}$ and $\mathcal{O} \subseteq \hat{B}$ be Scott-open with $x \in \mathcal{O}$. Then there must exist $b \in B \cap \mathcal{O}$ such that $x \in b \uparrow \sqsubseteq \mathcal{O}$.*

Proof. Choose an approximating sequence $(a^n)_n$ for every $a \in B$ and let $x = (x_n)_n$. Then $x = \sqcup_n x_n$ by Lemma 3.3 and $x_n = \sqcup_m x_n^m$ by assumption so that $x = \sqcup_n x_n = \sqcup_n \sqcup_m x_n^m = \sqcup_n \sqcup_m (\sqcup_{i \leq n} x_i^m) = \sqcup_n \sqcup_{i \leq n} x_i^n$ by Corollary 2.16. As \mathcal{O} is Scott-open, there must exist $n \in \mathbb{N}$ such that $\sqcup_{i \leq n} x_i^n \in \mathcal{O}$. For $i \leq n$, we have $x_i^n \ll x_i \sqsubseteq x_n$ so that $\sqcup_{i \leq n} x_i^n \ll x_n$ and therefore $\sqcup_{i \leq n} x_i^n \ll x$ by Lemma 2.10. As B is bounded complete, we have $b = \sqcup_{i \leq n} x_i^n \in B$ so that $b \in B \cap \mathcal{O}$. As \mathcal{O} is an upper set, it follows that $b \uparrow \sqsubseteq \mathcal{O}$. \square

Remark 4.4. Taken together, the last two lemmas show that the Scott topology on \hat{B} is generated by sets of the form $b \uparrow$ for $b \in B$.

The following lemma shows that Scott continuity, as defined in Definition 2.1 as preservation of suprema of increasing chains, is also constructively equivalent to preservation of open sets under inverse image.

Lemma 4.5. *Let B and C be bounded complete predomain bases and $f : \hat{B} \rightarrow \hat{C}$ be Scott-continuous. Then $f^{-1}(\mathcal{O})$ is open whenever $\mathcal{O} \subseteq \hat{C}$ is open.*

Proof. Let $\mathcal{O} \subseteq \hat{C}$ be Scott-open. We show that $f^{-1}(\mathcal{O})$ is an upper set. So let $x \sqsubseteq y \in \hat{B}$ with $x \in f^{-1}(\mathcal{O})$, i.e. $f(x) \in \mathcal{O}$. By monotonicity of f , we have $f(x) \sqsubseteq f(y)$ and as \mathcal{O} is an upper set, we have $f(y) \in \mathcal{O}$, i.e. $y \in f^{-1}(\mathcal{O})$. We now show that f^{-1} is inaccessible by suprema of increasing chains. So let $\sqcup_n x_n \in f^{-1}(\mathcal{O})$. Then $\sqcup_n f(x_n) = f(\sqcup_n x_n) \in \mathcal{O}$ and since \mathcal{O} is inaccessible by directed suprema, there must exist $n \in \mathbb{N}$ such that $f(x_n) \in \mathcal{O}$ whence $x_n \in f^{-1}(\mathcal{O})$. \square

We need the following technical result before we can prove the converse of the above lemma.

Lemma 4.6. *Let B be a bounded complete predomain base with decidable \ll and choose an approximating sequence $(b^i)_i$ for all $b \in B$. Furthermore, let $x = (x_n)_n \in \hat{B}$ and $y = (y_n)_n = (\sqcup_{i \leq n} x_i^n)_n$. Then y is well defined (i.e. $\{x_i^n \mid i \leq n\}$ is consistent) for all $n \in \mathbb{N}$ and $x = y$.*

Proof. It is evident that $\{x_i^n \mid 0 \leq i \leq n\}$ is consistent for all $n \in \mathbb{N}$, as for $0 \leq i \leq n$ we have that $x_i^n \sqsubseteq x_i \sqsubseteq x_n$, hence x_n is the required upper bound.

To see that $x \sqsubseteq y$ let $n \in \mathbb{N}$ and $b \in B$ with $b \ll x_n$. We show that there must exist $k \in \mathbb{N}$ with $b \ll y_k$. First, by interpolation, there must exist $c \in B$ such that $b \ll c \ll x_n$. As $(x_n^i)_i$ is an approximating sequence, there must exist $i \in \mathbb{N}$ such that $b \ll c \sqsubseteq x_n^i$, hence $b \ll x_n^i$. Let $k = \max\{i, n\}$. Then $b \sqsubseteq x_n^i \sqsubseteq x_n^k \sqsubseteq \sqcup_{i \leq k} x_n^k = y_k$.

To see that $y \sqsubseteq x$, let $b \in \mathbb{N}$ and $b \ll y_n$. Put $k = n$. Then $b \ll y_n = \sqcup_{i \leq n} x_i^n \sqsubseteq \sqcup_{i \leq n} x_i \sqsubseteq x_n = x_k$. \square

Lemma 4.7. *Let $f : \hat{B} \rightarrow \hat{C}$ be a function between bounded complete predomain bases where \ll is decidable on C such that $f^{-1}(\mathcal{O})$ is open for all open $\mathcal{O} \subseteq \hat{C}$. Then f is Scott continuous.*

Proof. We first show that f is monotone. So let $x \sqsubseteq y \in \hat{B}$. To see that $f(x) \sqsubseteq f(y)$ let $n \in \mathbb{N}$, $c \in C$ and suppose that $c \ll f(x)_n$. We show that there must exist k such that $c \ll f(y)_k$. Since $c \ll f(x)_n \sqsubseteq f(x)$ we have that $f(x) \in c \uparrow$ so that $x \in f^{-1}(c \uparrow)$ which is open and therefore an upper set. As $x \sqsubseteq y$, we have $y \in f^{-1}(c \uparrow)$ so that $c \ll f(y)$. If $(f(y)_n^i)_i$ is an approximating sequence of $f(y)_n$, Lemma 4.6 gives $c \ll \sqcup_n \sqcup_{i \leq n} f(y)_n^i$. Therefore there must exist k such that $c \sqsubseteq \sqcup_{i \leq k} f(y)_n^k$. We have, for $i \leq k$, that $f(y)_n^i \ll f(y)_i \sqsubseteq f(y)_k$ so that $\sqcup_{i \leq k} f(y)_n^k \ll f(y)_k$ by Lemma 2.10 as required.

Now suppose that $x = \sqcup_n x_n \in \hat{B}$, we show that $f(x) = \sqcup_n f(x_n)$. As $x_n \sqsubseteq x$, it is clear that $\sqcup_n f(x_n) \sqsubseteq f(x)$. To see that $f(x) \sqsubseteq \sqcup_n f(x_n)$, let $(f(x_n)_m^i)_i$ be an approximating sequence of $f(x_n)_m$ for all $n, m \in \mathbb{N}$. Then $(\sqcup_n f(x_n))_k = \sqcup_{0 \leq n, m \leq k} f(x_n)_m^k$ by Lemma 3.4 so that we need to show that $f(x) \sqsubseteq (\sqcup_{0 \leq n, m \leq k} f(x_n)_m^k)_k$. So let $c \in C$, $n \in \mathbb{N}$ and suppose that $c \ll f(x)_n$. By Corollary 2.14, there must exist an interpolant $c' \in C$ such that $c \ll c' \ll f(x)_n$. We show that there must exist $k \in \mathbb{N}$ such that $c \ll \sqcup_{n, m \leq k} f(x_n)_m^k$.

As $c' \ll f(x)_n \sqsubseteq f(x)$ we have that $x \in f^{-1}(c' \uparrow)$ which is open by assumption. As $x = \sqcup_n x_n$, there must exist m such that $x_m \in f^{-1}(c' \uparrow)$, i.e. $c' \ll f(x_m)$. By Lemma 4.6, we

have that $f(x)_m = \sqcup_n \sqcup_{i \leq n} f(x_m)_i^n$ so that there must exist $n \in \mathbb{N}$ such that $c' \sqsubseteq \sqcup_{i \leq n} f(x_m)_i^n$. Let $k = \max\{n, m\}$. Then $c \ll c' \sqsubseteq \sqcup_{i \leq n} f(x_m)_i^n \sqsubseteq \sqcup_{i \leq n} f(x_m)_i^k \sqsubseteq \sqcup_{0 \leq n, m \leq k} f(x_m)_i^k$ as required. \square

5. FUNCTION SPACES

Given two predomain bases B and C , we construct a predomain base $B \rightarrow C$ so that the continuous completion of $B \rightarrow C$ is the space of continuous functions between the continuous completions of B and C .

Definition 5.1. Let (B, \sqsubseteq) and (C, \sqsubseteq) be predomain bases. A *single step function* of type $B \rightarrow C$ is a pair $(b, c) \in B \times C$, written $b \searrow c$. A *step function* is a finite set of single step functions, written $\sqcup_i b_i \searrow c_i$ such that

$$\text{Cons}(\{b_j \mid j \in J\}) \rightarrow \text{Cons}(\{c_j \mid j \in J\})$$

for all (finite, non-empty) $J \subseteq I$. Now suppose that \ll on B is decidable and C is bounded complete. Then a single step function $b \searrow c$ defines the function $b \searrow c : B \rightarrow C$ by $b \searrow c(x) = c$ if $b \ll x$, and $b \searrow c(x) = \perp$, otherwise. A step function $\sqcup_i b_i \searrow c_i$ defines the function $(\sqcup_i b_i \searrow c_i)(x) = \sqcup_i (b_i \searrow c_i(x))$. Step functions are ordered by $s \sqsubseteq t$ iff $s(x) \sqsubseteq t(x)$ for all $x \in B$. We write $B \rightarrow C$ for the set of step functions of type $B \rightarrow C$.

Our first goal is to show that the collection of step functions forms a predomain base, and we begin with collecting conditions that ensure decidability of the ordering. The proof needs the following condition that allows us to separate two subsets using the way below relation.

Definition 5.2. Let (B, \sqsubseteq) be predomain base. Two subsets $A, D \subseteq B$ are *separated* if there must exist $\omega \in B$ such that $a \ll \omega$ for all $a \in A$ and $d \not\ll \omega$ for all $d \in D$.

Crucially, for the ordering on step functions to be decidable, we need to require that separatedness is decidable.

Example 5.3. (1) Consider the predomain base $(B, =)$ from Example 2.3, i.e. equality = on B is decidable. Then A and D are separated if A is a singleton and $A \cap D = \emptyset$. As a consequence, separatedness between finite sets on $(B, =)$ is decidable.
 (2) For the predomain base $(B^*, \sqsubseteq_{\text{pref}})$ of finite sequences of B , we have that A and D are separated if there exists $a \in A$ such that every (other) $a' \in A$ is a prefix of a , and no $d \in D$ is a prefix of a . As a consequence, separatedness is decidable for finite subsets $A, D \subseteq B^*$.

The situation with the predomain base of rational intervals is slightly more complex, and warrants a separate lemma.

Lemma 5.4. *Two finite subsets $A, B \subseteq \mathbb{I}\mathbb{Q}$ are separated if and only if $\sqcup A$ exists and is not a singleton (i.e. $\uparrow \sqcup A$ is not empty) and $b \not\sqsubseteq \sqcup A$ for all $b \in B$.*

Proof. Assume that $A = \{[a_i, b_i] \mid i \in I\}$ and $B = \{[c_j, d_j] \mid j \in J\}$ for finite sets I and J . For the 'if' direction, assume moreover that $[a, b] = \sqcup A$ exists in $\mathbb{I}\mathbb{Q}$ and $[c_j, d_j] \not\sqsubseteq \sqcup A$ for all $j \in J$. We show that A and B are separated by constructing a witness ω of separation. Consider the sets C and D defined by

$$C = \{c_i \mid a < c_i < b\} \text{ and } D = \{d_j \mid a < d_j < b\}.$$

We first consider the case where both C and D are non-empty, and let $c = \min C$, $d = \max D$, $\epsilon = \min\{c, d\}$ and $\delta = \max\{c, d\}$. It is immediate that $a < \epsilon \leq \delta < b$ from the definition of C and D . We now claim that $\omega = [\frac{\epsilon+a}{2}, \frac{\delta+b}{2}]$ witnesses separatedness of A and B .

First note that $\frac{1}{2}(\epsilon + a) \leq \frac{1}{2}(\delta + b)$ as $\epsilon \leq \delta$ and $a < b$ so that $\omega \in \mathbb{IQ}$. Moreover, $a < \frac{\epsilon+a}{2} \leq \frac{\delta+b}{2} < b$ so that $[a, b] \ll \omega$ by Lemma 2.5 and as $[a_i, b_i] \sqsubseteq [a, b] \ll \omega$ we have that $[a_i, b_i] \ll \omega$ for all $i \in I$ using Lemma 2.9. To see that $[c_j, d_j] \not\ll \omega$ suppose that we have $j \in J$ with $[c_j, d_j] \ll \omega$. Applying Lemma 2.5 again, we obtain $c_j < \frac{\epsilon+a}{2} < \frac{\epsilon+\epsilon}{2} = \epsilon < b$ and similarly $a < \delta = \frac{\delta+\delta}{2} < \frac{\delta+b}{2} < d_j$. As we have assumed that $[c_j, d_j] \not\sqsubseteq [a, b]$, we have that $a < c_j$ or $d_j < b$. In the first case, we obtain $c_j \in C$, and hence $c \leq c_j < \epsilon \leq c$, a contradiction. In the second case, we similarly have $d_j \in D$ and obtain a contradiction as $d \leq \delta < d_j \leq d$.

We now consider the case where $C \neq \emptyset$ and $D = \emptyset$. Here, we define $\epsilon = \delta = \min C$ and let $\omega = [\frac{\epsilon+a}{2}, \frac{\delta+b}{2}]$ as before. Again, we have that $a < \delta = \epsilon < b$ and $a < \frac{\epsilon+a}{2} \leq \frac{\delta+b}{2} < b$ so that $\omega \in \mathbb{IQ}$ and $\omega \ll [a, b]$ whence $\omega \ll [a_i, b_i]$ for all $i \in I$. Again, if $[c_j, d_j] \ll \omega$ we obtain that $c_j < \frac{\epsilon+a}{2} < \frac{\epsilon+\epsilon}{2} < b$ and $\delta = \frac{\delta+\delta}{2} < \frac{\delta+b}{2} < d_j$. As we have assumed that $[c_j, d_j] \not\sqsubseteq [a, b]$ we again have two cases: $a < c_j$ or $d_j < b$. In the first case, we obtain $c_j \in C$ and as before we argue that then $c \leq c_j < \epsilon \leq c$ which is impossible. The second case is slightly different to the argument above, but we just need to observe that $d_j < b$ implies that $d_j \in D$, contradicting $D = \emptyset$.

The case where $C = \emptyset$ and $D \neq \emptyset$ is entirely analogous and left to the reader. Finally, we consider the case where both C and D are empty. Here, we put $\omega = [\frac{a+b}{2}, \frac{a+b}{2}]$. Then $[a, b] \ll \omega$ as $a < b$, and by the same argument as before, $[a_i, b_i] \sqsubseteq [a, b] \ll \omega$ whence $[a_i, b_i] \ll \omega$ for all $i \in I$. To see that $[c_j, d_j] \not\ll \omega$ we again assume that $[c_j, d_j] \ll \omega$ and show that this is impossible. From the assumption $[c_j, d_j] \ll \omega$ we obtain that $c_j < \frac{a+b}{2} < d_j$, using Lemma 2.5 one more time. Given that $c_j \notin C$, we furthermore obtain that $c_j \leq a$ or $c_j \geq b$. The latter case is impossible as then $\frac{a+b}{2} \leq \frac{b+b}{2} = b \leq c_j \leq d_j$ which cannot happen as we assumed that $[c_j, d_j] \ll \omega$. As a consequence, we have that $c_j \leq a$.

For the same reason, given that $d_j \notin D$ we have that $d_j \leq a$ or $d_j \geq b$. Again the case $d_j \leq a$ is impossible (for a similar reason) so that $d_j \geq b$. But then $c_j \leq a \leq b \leq d_j$ so that $[c_j, d_j] \sqsubseteq \sqcup A$ which contradicts our assumption. This finishes the proof of the 'if'-implication.

Conversely, suppose that A and B are separated and ω is a witness of separatedness of A and B . Then ω is an upper bound of A so that $[a, b] = \sqcup A$ exists as \mathbb{IQ} is bounded complete. As $[a, b] \ll \omega$ by Lemma 2.10 which implies that $a < b$ by Lemma 2.5. We now claim that every $[c_j, d_j] \in D$ satisfies $[c_j, d_j] \not\sqsubseteq \sqcup A$. This follows, for if $[c_j, d_j] \sqsubseteq [a, b]$ we obtain $[c_j, d_j] \sqsubseteq [a, b] \ll \omega$ so that $[c_j, d_j] \ll \omega$ which is impossible as ω witnesses the separation of A and B . \square

As the above characterisation deals with finite subset of \mathbb{IQ} and decidable properties only, the following Corollary is immediate.

Corollary 5.5. *Separatedness is decidable for finite subsets $A, D \subseteq \mathbb{IQ}$.*

We now use separatedness to characterise the order between step and single step functions which takes us one step closer to the goal of establishing decidability of the order on the set of step functions.

Theorem 5.6. *Let B be a predomain base for which \ll is decidable, and let C be bounded complete. Furthermore, let I be a finite set, $\{\alpha\} \cup \{\gamma_i \mid i \in I\} \subseteq B$ and $\{\beta\} \cup \{\delta_i \mid i \in I\} \subseteq C$ and $\sqcup_{i \in I} \gamma_i \succ \delta_i$ be a step function. Then the following are equivalent:*

- ▷ $\alpha \sqsupseteq \beta \sqsubseteq \bigsqcup_{i \in I} \gamma_i \sqsupseteq \delta_i$
- ▷ for all $x \in B$ with $\alpha \ll x$ we have that $\beta \sqsubseteq \bigsqcup \{\delta_i \mid i \in I \text{ and } \gamma_i \ll x\}$
- ▷ for any $I_0 \subseteq I$ such that $A = \{\alpha\} \cup \{\gamma_i \mid i \in I_0\}$ and $D = \{\gamma_i \mid i \notin I_0\}$ are separated, we have $\beta \sqsubseteq \bigsqcup \{\delta_i \mid i \in I_0\}$.

Proof. The equivalence of the first two items is immediate. So assume that $\alpha \ll x$ implies that $\beta \sqsubseteq \bigsqcup \{\delta_i \mid \gamma_i \ll x\}$ for all $x \in B$. Moreover, let $I_0 \subseteq I$ be finite and assume that $A = \{\alpha\} \cup \{\gamma_i \mid i \in I_0\}$ and $D = \{\gamma_i \mid i \notin I_0\}$ are separated. Note that in this case, the supremum $\bigsqcup \{\delta_i \mid i \in I_0\}$ exists, as $\bigsqcup_{i \in I} \gamma_i \sqsupseteq \delta_i$ is a step function, and C is bounded complete. As \sqsubseteq is decidable, we may argue classically to show that $\beta \sqsubseteq \bigsqcup \{\delta_i \mid i \in I_0\}$. As A and D are separated, we can (classically) find w that witnesses separatedness of A and D (as per Definition 5.2). In particular, $\alpha \ll w$ so that we have $\beta \sqsubseteq \bigsqcup \{\delta_i \mid \gamma_i \ll w\} = \bigsqcup \{\delta_i \mid i \in I_0\}$.

For the other direction, assume the last statement above and let $x \in B$ with $\alpha \sqsubseteq x$. Put $I_0 = \{i \in I \mid \gamma_i \ll x\}$. Then $A = \{\alpha\} \cup \{\gamma_i \mid i \in I_0\}$ and $D = \{\gamma_i \mid i \notin I_0\}$ are separated, and we obtain $\beta \sqsubseteq \bigsqcup \{\delta_i \mid i \in I_0\} = \bigsqcup \{\delta_i \mid \gamma_i \ll x\}$ as required. \square

The above lemma is the key stepping stone to see that the ordering on the predomain representing the function space is indeed decidable.

Corollary 5.7. *Let B and C be predomain bases where \ll on B is decidable, C is bounded complete, and separatedness on C is decidable. Then \sqsubseteq on $B \rightarrow C$ is decidable.*

Proof. Immediate by the previous lemma, as $\bigsqcup_{i \in I} \alpha_i \sqsupseteq \beta_i \sqsubseteq \phi$ if and only if $\alpha_i \sqsupseteq \beta_i \sqsubseteq \phi$ for all $i \in I$. \square

Single step functions, and step functions themselves, are automatically Scott continuous.

Lemma 5.8. *Let B and C be predomain bases where C is pointed, \ll on B is decidable, and B has weak interpolation. Then every single step function $b \sqsupseteq c$ is Scott continuous.*

Proof. Let $(x_n)_n$ be an increasing sequence in B and $x \in B$ with $x = \bigsqcup_n x_n$. We show that $b \sqsupseteq c(\bigsqcup_n x_n) = \bigsqcup_n b \sqsupseteq c(x_n)$. This is evident if $c = \perp$. So suppose $c \neq \perp$. By definition of single step functions (and the fact that \ll on B is decidable), we may distinguish two cases.

Case 1. $b \ll \bigsqcup_n x_n$ and $b \sqsupseteq c(\bigsqcup_n x_n) = c$. By weak interpolation on B , there must exist $y \in B$ such that $b \ll y \ll \bigsqcup_n x_n$, therefore there must exist $n \in \mathbb{N}$ such that $b \ll y \sqsubseteq x_n$. Therefore $\bigsqcup_n b \sqsupseteq c(x_n) = c$. Note that the latter expression is a Harrop-formula, and therefore $\neg\neg$ -stable.

Case 2. It is not the case that $b \ll \bigsqcup_n x_n$. We show that $\bigsqcup_n b \sqsupseteq c(x_n) = \perp$. This follows, if $b \sqsupseteq c(x_n) = \perp$ for all $n \in \mathbb{N}$. So pick $n \in \mathbb{N}$. We show that $\neg(b \ll x_n)$. Assume $b \ll x_n$. But then $b \ll \bigsqcup_n x_n$ whence $b \ll \bigsqcup_n x_n$ which entails $b \sqsupseteq c(\bigsqcup_n x_n) = \perp$, a contradiction. \square

The following lemma is standard in (classical) domain theory, e.g. [AJ94, Proposition 4.0.2], and helps to construct approximating sequences that in turn are required to show that step functions form a predomain base.

Lemma 5.9. *Let s be a step function. Then $b \sqsupseteq c \ll s$ whenever $c \ll s(b)$.*

Proof. Assume that $s \sqsubseteq \bigsqcup_n s_n$ where the s_n are step functions. Then $s(b) \sqsubseteq \bigsqcup_n s_n(b)$. Since $c \ll s(b)$ we can find $n \in \mathbb{N}$ such that $c \sqsubseteq s_n(b)$. Let $x \in B$, we show that $b \sqsupseteq c(x) \sqsubseteq s_n(x)$. In case $\neg(b \ll x)$ we have $b \sqsupseteq c(x) = \perp \sqsubseteq s_n(x)$. In case $b \ll x$ we have $b \sqsupseteq c(x) = c \sqsubseteq s_n(b) \sqsubseteq s_n(x)$ by monotonicity of s_n . \square

Lemma 5.10. *Let B be a countable, bounded complete predomain base and suppose that \ll and separatedness of finite sets on B are decidable, and let C be bounded complete. Then the set $B \rightarrow C$ of step functions is a predomain base.*

Proof. We have to show that every step function $s \in B \rightarrow C$ has an approximating sequence. Let $B = \{b_n \mid n \in \mathbb{N}\}$. As C is a predomain base, every $s(b_i) \in B$ has an approximating sequence $(e_i^j)_j$. Let $s_n = \bigsqcup_{0 \leq i \leq n} b_i \searrow e_i^n$.

We first show that s_n , for $n \in \mathbb{N}$, is indeed a step function. Let $J \subseteq \{0, \dots, n\}$ be a non-empty subset such that $\{b_j \mid j \in J\}$ is consistent. We need to show that $\{e_j^n \mid j \in J\}$ is consistent. Pick $j \in J$. Then

$$e_j^n \ll s(b_j) \sqsubseteq s(\bigsqcup_{j \in J} b_j)$$

so that $s(\bigsqcup_j b_j)$ is an upper bound of all e_j^n for $j \in J$.

It follows from Lemma 5.9 in combination with Corollary 2.11 that $s_n \ll s$ for all $n \in \mathbb{N}$.

We now show that $s \sqsubseteq \bigsqcup_n s_n$ for all $n \in \mathbb{N}$. Fix $n \in \mathbb{N}$ and let $x \in B$, we show that $s_n(x) \sqsubseteq s(x)$. This follows from

$$\begin{aligned} s_n(x) &= \bigsqcup \{e_i^n \mid 0 \leq i \leq n, b_i \ll x\} \\ &\sqsubseteq \bigsqcup \{s(b_i) \mid 0 \leq i \leq n, b_i \ll x\} \\ &\sqsubseteq \bigsqcup \{s(x) \mid 0 \leq i \leq n, b_i \ll x\} \\ &= s(x) \end{aligned}$$

so that s is an upper bound of the s_n in $B \rightarrow C$.

We next establish that s is in fact the least upper bound of the s_n . So let t be a step function and suppose that $s_n \sqsubseteq t$ for all $n \in \mathbb{N}$. Let $x \in B$, we show that $s(x) \sqsubseteq t(x)$. As s is a step function, we may assume that $s = \bigsqcup_{i \in I} b_i \searrow c_i$ for a finite set $I \subseteq \mathbb{N}$ and elements $c_i \in C$. Let $J = \{i \in I \mid b_i \ll x\}$ so that $s(x) = \bigsqcup \{c_j \mid j \in J\}$. By the interpolation Lemma (Corollary 2.14) we may find an interpolant \hat{x} such that $b_j \ll \hat{x} \ll x$ for all $j \in J$. Then

$$b_i \ll x \text{ iff } b_i \ll \hat{x} \text{ for all } i \in I$$

so that $s(x) = s(\hat{x})$. As B is countable, we may assume that $\hat{x} = b_i$ for some $i \in \mathbb{N}$. As $(e_i^j)_j$ is an approximating sequence for $s(x) = s(\hat{x})$, it suffices to show that $e_i^n \sqsubseteq t(x)$. Now, for $n \geq i$, we have

$$\begin{aligned} e_i^n &\sqsubseteq \bigsqcup_{0 \leq i \leq n} b_i \searrow e_i^n(x) && \text{(as } b_i = \hat{x} \ll x \text{ and } n \geq i) \\ &= s_n(x) \sqsubseteq t(x) \end{aligned}$$

so that $e_i^n \sqsubseteq t(x)$ for all $n \in \mathbb{N}$ whence $s(x) = \bigsqcup_n e_i^n \sqsubseteq t(x)$ and finally $s \sqsubseteq t$ as x was arbitrary. We note that this goal formula is stable whence applying the approximation lemma (that only guarantees classical existence of an interpolant) in fact proves the claim.

We finally need to establish that $s_n \ll s$ for all $n \in \mathbb{N}$. But this is immediate from Lemma 2.10. The last requirement for a predomain base, decidability of ordering, has already been established in Corollary 5.7. □

Remark 5.11. It is in general not true that $c \ll c'$ implies that $b \searrow c \ll b \searrow c'$. If $b' \ll b$ in addition to $c \ll c'$ we have $b \searrow c \ll b' \searrow c'$. This fact cannot be exploited in the proof of the above Lemma as we have no way of approximating base elements from above.

We now show that every Scott continuous function arises as a supremum of step functions.

Lemma 5.12. *Let B, C be predomain bases for which consistency is continuous, let B be bounded complete with decidable \ll , and let C be pointed. Then, for every Scott continuous $f : \hat{B} \rightarrow \hat{C}$ there exists $s = (s_n)_n \in \overline{B} \rightarrow \overline{C}$ such that $s(x) = \bigsqcup_n s_n(x) = f(x)$ for all $x \in \hat{B}$.*

Proof. Let $B = \{b_0, b_1, \dots\}$ and assume that $(b^i)_i$ is an approximating sequence for all $b \in B$. Suppose that $f : \hat{B} \rightarrow \hat{C}$ is Scott continuous. Let

$$s_n = \bigsqcup_{0 \leq i \leq n} b_i \searrow f(b_i)_n$$

where we write $f(b_i)_n$ for the n -th element of the sequence $f(b_i) \in \hat{C}$. Let $x = (x_n)_n \in \hat{B}$ be given.

We first show that $f(x) \sqsubseteq \bigsqcup_n s_n(x)$. Let $y_n = \bigsqcup_{0 \leq i \leq n} x_i^n$ and $y = (y_n)_n$. Then $x = y$ by Lemma 4.6 and it therefore suffices to show that

$$\bigsqcup_n f(y_n) \sqsubseteq \bigsqcup_n s_n(x)$$

as f is extensional, i.e. $f(x) = f(y)$ and Scott continuous, i.e. $f(\bigsqcup_n y_n) = \bigsqcup_n f(y_n)$. We have the following calculations for the k -th elements of the respective sequences:

$$A_k = (\bigsqcup_n f(y_n))_k = \bigsqcup_{n \leq k} f(y_n)_k = \bigsqcup_{n \leq k} f(\bigsqcup_{i \leq n} x_i^n)$$

and

$$B_k = (\bigsqcup_n s_n(x))_k = \bigsqcup_{n \leq k} s_n(x_k) = s_k(x_k) = \bigsqcup \{f(b_i)_k \mid 0 \leq i \leq k, b_i \ll x_k\}$$

by Definition of s_n and Lemma 3.4. We therefore need to show that $(A_k)_k \sqsubseteq (B_k)_k$. So let $k \in \mathbb{N}$ and $x \in C$ such that $x \ll A_k$. Choose l large enough so that $l \geq k$, $\{x_0, \dots, x_k\} \sqsubseteq \{b_0, \dots, b_l\}$ and $\bigsqcup_{i \leq n} x_i^n \in \{b_0, \dots, b_l\}$ for all $n \leq k$.

Now fix $n \leq k$. By our assumption on l , we have $j \leq l$ such that $\bigsqcup_{i \leq n} x_i^n = b_j$. Moreover, for all $i \leq n$ we have $x_i^n \ll x_i \sqsubseteq x_l$ so that $b_j = \bigsqcup_{i \leq n} x_i^n \ll x_l$. Hence

$$f(\bigsqcup_{i \leq n} x_i^n)_k = f(b_j)_k \sqsubseteq \bigsqcup \{f(b_i)_k \mid 0 \leq i \leq l, b_i \ll x_l\} = B_l.$$

As n was arbitrary, we therefore obtain $\bigsqcup_{n \leq k} f(\bigsqcup_{i \leq n} x_i^n) \sqsubseteq B_l$ and finally $x \ll B_l$ as required.

We now show that $\bigsqcup_n s_n(x) \sqsubseteq f(x)$. By Scott-continuity of f , we have $f(x) = \bigsqcup_n f(x_n)$ and the claim follows if $\bigsqcup_n s_n(x) \sqsubseteq \bigsqcup_n f(x_n)$. As above, we calculate for the k -th element of the respective sequences that

$$A_k = (\bigsqcup_n f(x_n))_k = \bigsqcup_{n \leq k} f(x_n)_k$$

and

$$B_k = (\bigsqcup_n s_n(x))_k = \bigsqcup \{f(b_i)_k \mid 0 \leq i \leq k, b_i \ll x\}$$

where we have used Lemma 3.4 for the calculation of A_k , and the calculation of B_k is as above. To see that $(B_k)_k \sqsubseteq (A_k)_k$, fix $x \in C$, $k \in \mathbb{N}$ and assume that $x \ll B_k$. Let $N = \{0 \leq i \leq k \mid b_i \ll x_k\}$ so that $B_k = \bigsqcup \{f(b_i)_k \mid i \in N\}$.

We claim that $f(b_i) \sqsubseteq f(x_k)$ for every $i \in N$. This is immediate, since $i \in N$ implies that $b_i \ll x_k$, therefore $b_i \sqsubseteq x_k$ and $f(b_i) \ll f(x_k)$ by monotonicity of f . As a consequence, $f(b_i)_k \sqsubseteq \bigsqcup_n f(b_i)_n = f(b_i) \sqsubseteq f(x_k)$ whence $\bigsqcup \{f(b_i)_k \mid i \in N\} \sqsubseteq f(x_k)$ where we view the left-hand side of the last equation as a constant sequence. As $x \ll B_k = \bigsqcup \{f(b_i)_k \mid i \in N\}$ there must exist $l \in \mathbb{N}$ such that $x \ll f(x_k)_l$. By monotonicity of $(f(x_k)_n)_n$, the same holds

for l replaced by $\max\{l, k\}$ so that we assume without loss of generality that $l \geq k$. In summary, we have obtained $x \ll f(x_k)_l \sqsubseteq \bigsqcup_{n \leq l} f(x_n)_l = A_l$ as required. \square

6. REAL NUMBERS AS TOTAL ELEMENTS OF THE INTERVAL DOMAIN

We now consider an important example of predomain bases in more detail, the predomain base of rational intervals that we have already introduced in Example 2.4. Specifically, we introduce a constructive representation of the set of real numbers as the total elements of the interval domain, and give a characterisation of continuous functions on real numbers in terms of Scott continuous functions on the interval domain. Specifically, we compare total elements of the interval domain with the (standard) constructive notion of Cauchy reals, and characterise the total elements of the interval domain as *Markov reals*, i.e. Cauchy reals where the modulus of convergence has been replaced with a modulus of non-divergence.

In summary, we establish that if Markov's principle holds, Cauchy reals and Markov reals are equivalent, and in turn equivalent to total reals in the sense of domain theory.

Definition 6.1 (Basic Notions). We write \mathbb{IR} for the continuous completion of \mathbb{IQ} , and write $\underline{\alpha}, \bar{\alpha}$ for the upper and lower endpoint of $\alpha \in \mathbb{IQ}$ as in Example 2.4, and identify $x \in \mathbb{IR}$ with the sequence $(x_n)_n$ so that $x = (x_n)_n = ([\underline{x}_n, \bar{x}_n])_n$ for $x \in \mathbb{IR}$. The *length* of $a \in \mathbb{IQ}$ is given by $\ell(a) = \bar{a} - \underline{a}$. An element $x \in \mathbb{IR}$ is a *total real* or a simply a *real*, if for all $k \in \mathbb{N}$ there must exist $n \in \mathbb{N}$ such that $\ell(x_n) \leq 2^{-k}$. We write \mathbb{R} for the set of total reals, and call a function $f : \mathbb{IR} \rightarrow \mathbb{IR}$ *total* if $f(x) \in \mathbb{R}$ whenever $x \in \mathbb{R}$. For $a, b \in \mathbb{IQ}$ we put

$$a + b = [\underline{a} + \underline{b}, \bar{a} + \bar{b}] \quad -a = [-\bar{a}, -\underline{a}]$$

and $|a| = a$ if $0 \leq \underline{a}$, $|a| = [0, \max\{-\underline{a}, \bar{a}\}]$ if $\underline{a} \leq 0 \leq \bar{a}$ and $|a| = -a$ if $\bar{a} \leq 0$. These operations are extended pointwise to elements $x, y \in \mathbb{IR}$, i.e. $x + y = (x_n + y_n)_n$, $-x = (-x_n)_n$, $|x| = (|x_n|)_n$ for all $n \in \mathbb{N}$. For $x = (x_n)_n \in \mathbb{IR}$ we put $0 \leq x$ if given $k \in \mathbb{N}$ there must exist $n \in \mathbb{N}$ such that $-2^{-k} \leq \underline{x}_n$ and $x \leq y$ if $0 \leq y - x$.

Given that the arithmetic operation above are defined on equivalence classes of (elements of) the continuous completion of \mathbb{IQ} , we need to show that they are well defined with respect to the (defined) equality in the continuous completion (Definition 3.1). This is a standard verification.

Lemma 6.2. *Let x, y, x' and $y' \in \mathbb{IR}$ and suppose that $x = x'$ and $y = y'$. Then $x + y = x' + y'$, $-x = -x'$, $|x| = |x'|$ and $0 \leq x$ if and only if $0 \leq x'$.*

Proof. We write $x = (x_n)_n$, $x_n = [\underline{x}_n, \bar{x}_n]$ and similarly for x' , y and y' . For an element $z = [\underline{z}, \bar{z}] \in \mathbb{IQ}$ and $\epsilon \in \mathbb{Q}$ with $\epsilon \geq 0$ we write $z \pm \epsilon = [\underline{z} - \epsilon, \bar{z} + \epsilon]$ as in Example 2.4. We begin with addition where it suffices to show that $x + y \sqsubseteq x' + y'$. So assume that $b \in \mathbb{IQ}$, $n \in \mathbb{N}$ and $b \ll (x + y)_n$. We show that there must exist m such that $b \ll (x' + y')_m$. Because $b \ll (x + y)_n$, there exists $\epsilon > 0$ such that $b \ll (x + y)_n \pm \epsilon$. Because $x \sqsubseteq x'$ there must exist m_x such that $x_n \pm \frac{\epsilon}{2} \ll x'_{m_x}$. For the same reason, there must exist m_y such that $y_n \pm \frac{\epsilon}{2} \ll y'_{m_y}$. For $m = \max\{m_x, m_y\}$ we therefore obtain that $b \ll (x + y)_n \pm \epsilon = x_n \pm \frac{\epsilon}{2} + y_n \pm \frac{\epsilon}{2} \sqsubseteq x'_{m_x} + y'_{m_y} \sqsubseteq (x' + y')_m$ as required.

For unary minus, it similarly suffices to show that $-x \sqsubseteq -x'$. If $b \ll -x_n$, Lemma 2.5 shows that $-b \ll x_n$ whence there must exist m such that $-b \ll x'_m$ so that $b \ll -x'_m$ by applying Lemma 2.5 again.

For the last claim, assume that $0 \leq x$, we show that $0 \leq x'$. Let $k \geq 0$ be given. We show that there must exist $m \in \mathbb{N}$ such that $-2^{-k} \leq \underline{x}'_m$. As $0 \leq x$, there must exist n such that $-2^{-(k+1)} \leq \underline{x}_n$. As $x_n \pm 2^{-(k+1)} \ll x_n$, there must exist m such that $x_n \pm 2^{-(k+1)} \ll x'_m$ as $x = x'$ by assumption. In summary, we then obtain $-2^{-k} = -2^{-(k+1)} - 2^{-(k+1)} \leq \underline{x}_n - 2^{-(k+1)} \leq x_m$ as required.

To show that $|x| = |x'|$ assume that $b \ll |x|_n$. As ordering on \mathbb{Q} is decidable, we may distinguish the following cases.

Case 1: $0 < \underline{x}_n$. It is easy to see that in this case, there must exist m_0 such that $0 < \underline{x}'_{m_0}$. Also, as $b \ll |x_n| = x_n$, there must exist m_1 such that $b \ll x'_{m_1}$. For $m = \max\{m_0, m_1\}$ we therefore have that $b \ll x'_{m_1} \sqsubseteq x'_m$ and as $0 \leq \underline{x}'_{m_0} \leq \underline{x}'_m$ we have that $|x'_m| = x'_m$ so that $b \ll |x'_m|$ as required.

Case 2: $\underline{x}_n \leq 0 \leq \bar{x}_n$. In this case we have $b \ll [0, \max\{-\underline{x}_n, \bar{x}_n\}]$ so that $\underline{b} < 0$ and $\bar{b} > \max\{-\underline{x}_n, \bar{x}_n\}$. We then have that $[-\bar{b}, \bar{b}] \ll x_n$ and because $x = x'$ there must exist m such that $[-\bar{b}, \bar{b}] \ll x'_m$. Again the decidability of order on \mathbb{Q} allows us to distinguish three subcases to relate this to $|x'_m|$.

Subcase 2a: $\underline{x}'_m > 0$. Then $\underline{b} < 0 < \underline{x}'_m \leq \bar{x}'_m < \bar{b}$ as $-b < 0$ and $[-\bar{b}, \bar{b}] \ll x'_m$. Hence $b = [\underline{b}, \bar{b}] \ll x'_m = |x'_m|$.

Subcase 2b: $\underline{x}'_m \leq 0 \leq \bar{x}'_m$. As $[-\bar{b}, \bar{b}] \ll x_m$ we have that $\max\{-\underline{x}'_m, \bar{x}'_m\} < \bar{b}$. As $\underline{b} < 0$ as noted above, this gives $\underline{b} < 0 \leq \max\{-\underline{x}'_m, \bar{x}'_m\} < \bar{b}$ so that $b \ll [0, \max\{-\underline{x}'_m, \bar{x}'_m\}] = |x_m|$.

Subcase 2c: $\bar{x}'_m < 0$. Similarly to Subcase 2a we obtain that $-\bar{b} < \underline{x}'_m \leq \bar{x}'_m < 0 < -\underline{b}$ so that $\underline{b} < 0 < -\bar{x}'_m \leq -\underline{x}'_m < \bar{b}$ and $[\underline{b}, \bar{b}] \ll [-\bar{x}'_m, -\underline{x}'_m] = |x_m|$.

Our last case is analogous to the first:

Case 3: $\bar{x}_n < 0$. As in the first case, it is easy to see that there must exist m_0 such that $x'_{m_0} < 0$. Moreover, $b \ll |x_n| = -x_n$ so that $-b \ll x_n$. As consequence, there must exist m_1 such that $-b \ll x'_{m_1}$. Let $m = \max\{m_0, m_1\}$. Then $-b \ll x'_{m_1}$ whence $b \ll -x'_{m_1}$ and $\bar{x}'_m < 0$ so that $|x'_m| = -x'_m$. Taken together this gives $b \ll |x'_m|$ as required.

This argument shows that $|x| \sqsubseteq |x'|$ which is sufficient to establish the penultimate claim. \square

The fact that the total reals coincide with the maximal elements of \mathbb{IR} is essentially a consequence that totality is formulated negatively (and, like maximality, has no computational content).

Lemma 6.3. *The total reals are precisely the maximal elements of \mathbb{IR} .*

Proof. Let $x \in \mathbb{R}$ be total and suppose that $x \sqsubseteq y$ where $y \in \mathbb{IR}$. We show $y \sqsubseteq x$. So let $a \in \mathbb{IR}$, $n \in \mathbb{N}$ and suppose that $a \ll y_n$. We show that there must exist $k \in \mathbb{N}$ such that $a \ll x_k$. By Lemma 2.5 there exists $\epsilon \in \mathbb{I}\mathbb{Q}$, $\epsilon > 0$ so that $a \ll y_n + \epsilon$. As x is an interval real, there must exist $k \in \mathbb{N}$ such that $\ell(x_k) \leq \frac{\epsilon}{2}$. We show that $a \ll x_k$. Since $x_k \pm \frac{\epsilon}{2} \ll x_k$ and $x \sqsubseteq y$ there must exist $l \in \mathbb{N}$ such that $x_k \pm \frac{\epsilon}{2} \ll y_l$. We may assume that $l \geq n$. Using Lemma 2.5 this entails that $\underline{x}_k - \frac{\epsilon}{2} < \underline{y}_l < \bar{y}_l < \bar{x}_k + \frac{\epsilon}{2}$ so that $\underline{y}_l - \bar{x}_k - \frac{\epsilon}{2} < 0$. Using this estimate, we obtain

$$\underline{a} < \underline{y}_n - \epsilon \leq \underline{y}_n - (\bar{x}_k - \underline{x}_k + \frac{\epsilon}{2}) \leq \underline{x}_k + \underline{y}_l - \bar{x}_k - \frac{\epsilon}{2} \leq \underline{x}_k.$$

One analogously establishes that $\bar{x}_k < \bar{a}$ so that $a \ll x_k$ as desired.

Now let x be maximal. We show that $\forall k \in \mathbb{N}. \exists n \in \mathbb{N}. \ell(x_n) \leq 2^{-k}$. So assume that $k \in \mathbb{N}$ and $\forall n \in \mathbb{N}. \neg \ell(x_n) \leq 2^{-k}$. We establish a contradiction.

Define $y_n = [\underline{x}_n + 2^{-n-1}, \bar{x}_n - 2^{-n-1}]$. Then $y = (y_n)_n \in \mathbb{IR}$ and $x \sqsubseteq y$. As x is maximal, we have $y \sqsubseteq x$. Since $y_k \pm 2^{-k-1} \ll y_k$ and $y \sqsubseteq x$, there must exist $m \in \mathbb{N}$ such that $y_k \pm 2^{-k-1} \ll x_m$. We may assume that $m \geq k$. Then $\ell(y_k) + 2^{-k} = \ell(y_k \pm 2^{-k-1}) < \ell(x_m)$. But then $\ell(x_k) = \ell(y_k) + 2^{-k} < \ell(x_m) \leq \ell(x_n)$, the desired contradiction. \square

We characterise total reals as Cauchy sequences where we reformulate the notion of Cauchy-ness to account for the lack of information on convergence speed.

Definition 6.4. A *classical null sequence* is a decreasing sequence $(q_n)_n$ in $\mathbb{Q}_{\geq 0} \cup \{\infty\}$ such that $\forall \epsilon > 0. \exists m \in \mathbb{N}. \forall n \geq m. q_n \leq \epsilon$. We write $(q_n)_n \searrow 0$ if $(q_n)_n$ is a classical null-sequence.

A *modulus of non-divergence* of a rational sequence $(q_n)_n$ is a classical null-sequence $(c_n)_n$ such that $\forall N \in \mathbb{N}. \forall n, m \geq N. |q_n - q_m| \leq c_N$.

Finally, *modulus of convergence* of a rational sequence $(q_n)_n$ is a non-decreasing function $M : \mathbb{N} \rightarrow \mathbb{N}$ such that $\forall n, m \geq M(k). |q_n - q_m| \leq 2^{-k}$.

A rational sequence $(q_n)_n$ is *Cauchy* if it has a modulus of convergence, and *Markov* if it has a modulus of non-divergence. A *Cauchy real* (*Markov real*) is a rational sequence together with a modulus of convergence (modulus of non-divergence).

We allow classical null sequences to begin with ∞ to account for partially defined approximations of functions later. The following is simple observation, but requires a rather technical proof to allow us to convert between moduli of (dis-) continuity.

Lemma 6.5. *Every Cauchy sequence is Markov.*

The proof of this lemma needs the following auxiliary statement that helps to convert the modulus into a (classical) null sequence.

Lemma 6.6. *Let $M : \mathbb{N} \rightarrow \mathbb{N}$. Then there exists a non-decreasing function $W : \mathbb{N} \rightarrow \mathbb{N}$ such that $W(M(0)) = 0$, $(2^{-W(n)})_n$ is a classical null sequence and $M \circ W(n) \leq n$ for all $n \geq M(0)$.*

Proof. We put $W(n) = 0$ for $n \leq M(0)$ and use induction for $n > M(0)$, i.e.

$$W(n) = \begin{cases} W(n-1) & \text{if } M(W(n-1)+1) > n \\ W(n-1)+1 & \text{if } M(W(n-1)+1) \leq n \end{cases}$$

for all $n > M(0)$. It is clear that $W(M(0)) = 0$. We first claim that W is non-decreasing, but this is evident from the definition of W . Next, we show by induction on n that $M \circ W(n) \leq n$ for all $n \geq M(0)$. For $n = M(0)$ this follows directly from the definition, as then $M \circ W(n) = M(0) \leq M(0) = n$. Now let $n > M(0)$. We distinguish two cases. If $M(W(n-1)+1) > n$, then $M \circ W(n) = M \circ W(n-1) \leq n-1 \leq n$. If on the other hand, $M(W(n-1)+1) \leq n$, we obtain $M \circ W(n) = M(W(n-1)+1) \leq n$ by assumption.

We now show that W is progressive, that is, $\forall n \geq M(0). \exists m \geq n+1. W(m) > W(n)$. Fix $n \geq M(0)$ and put $m = n+k$ where $k = \max\{1, M(W(n)+1) - M(W(n))\}$. As W is monotone, it suffices to show that $W(n) = W(n+1) = \dots = W(n+k)$ is contradictory as the latter chain of equivalences is decidable.

So assume that $W(n) = W(n+1) = \dots = W(n+k)$. In particular, as $W(n+k) = W(n+k-1)$, we obtain $M(W(n+k-1)+1) > n+k$. Since $W(n+k-1) = W(n)$ this gives $M(W(n)+1) > n+k$. Substituting the definition of k , this gives $M(W(n)+1) > n + M(W(n)+1) - M(W(n))$, i.e. $M \circ W(n) > n$, contradicting our earlier statement.

We now show that $W(n)$ is not bounded, i.e. $\forall k \in \mathbb{N}. \exists n \in \mathbb{N}. W(n) \geq k$. This follows by induction on k , and for $k = 0$ we put $n = 0$. For the inductive step, assume that there

exists $n' \in \mathbb{N}$ such that $W(n') \geq k - 1$. By the previous claim, there exists $n \geq n' + 1$ such that $W(n) > W(n') \geq k - 1$ so that $W(n) \geq k$.

This now shows that $(2^{-W(n)})_n$ is a classical null-sequence, for any $\epsilon \geq 0$ there exists $k \geq \frac{1}{\epsilon} + 1$ and in turn $n \in \mathbb{N}$ such that $W(n) \geq k$ whence $2^{-W(n)} \leq 2^{-k} \leq \frac{1}{k} \leq \epsilon$ as $k \geq 1$. \square

We now give the proof of Lemma 6.5.

Proof. Let $(q_n)_n$ be a Cauchy sequence with modulus M , i.e. $\forall k \in \mathbb{N}. \forall n, m \geq M(k), |q_n - q_m| \leq 2^{-k}$. By Lemma 6.6 there exists a non-decreasing function $W : \mathbb{N} \rightarrow \mathbb{N}$ such that $M \circ W(n) \leq n$ for all $n \geq M(0)$ and $(2^{-W(n)})_n$ is a classical null-sequence.

We define a sequence $(c_n)_n$ of non-negative rationals by $c_n = 1 + \max\{|q_k - q_l| \mid n \leq k, l \leq M(0)\}$ if $n < M(0)$ and $c_n = 2^{-W(n)}$ if $n \geq M(0)$.

We now show that if $m, n \geq N$, then $|q_n - q_m| \leq c_N$. To see this, we distinguish the cases $N < M(0)$ and $N \geq M(0)$. First suppose that $N < M(0)$. If both $n, m \leq M(0)$ this follows by definition of c_N . If both $n, m \geq M(0)$ then $|q_n - q_m| \leq 2^0 = 1 \leq c_N$. Now consider without loss of generality that $n < M(0)$ and $m \geq M(0)$. Then $|q_n - q_m| \leq |q_n - q_{M(0)}| + |q_{M(0)} - q_m| \leq \max\{|q_k - q_l| : N \leq k, l \leq M(0)\} + 1 = c_N$.

Now suppose that $N \geq M(0)$ and let $m, n \geq N$. Then $M \circ W(N) \leq N$ so that $m, n \geq M(W(N))$ whence $|q_n - q_m| \leq 2^{-W(N)} = c_N$. It remains to show that $(c_n)_n$ is a classical null-sequence which is however immediate from the fact that $2^{-W(n)}$ is a classical null sequence. \square

In fact, it turns out that the equivalence of Cauchy and Markov reals characterises Markov's principle itself.

Lemma 6.7. *If Markov's Principle holds, every Markov sequence is Cauchy.*

Proof. Let $(q_n)_n$ be a Markov sequence with (Markov-) modulus $(c_n)_n$. As c_n is a classical null-sequence, we obtain that $\forall k \in \mathbb{N}. \exists n \in \mathbb{N}. c_n \leq 2^{-k}$. By Markov's principle, we may replace the weak existential quantifier by a strong one, and number choice yields the Cauchy modulus. \square

Lemma 6.8. *If every Markov Sequence is Cauchy, then Markov's principle holds.*

Proof. Suppose that every Markov sequence is Cauchy, and let $P(n)$ be a decidable predicate on natural numbers. Suppose that $\exists n. P(n)$. We show that $\exists n. P(n)$. Define a sequence $(q_n)_n$ by $q_n = 0$ if $\forall k \leq n. \neg P(k)$, and $q_n = 1$, otherwise. We claim that $(q_n)_n$ is a Markov sequence. Put $c_n = 1 - q_n$. Then clearly $|q_n - q_m| \leq c_N$ for $n, m \geq N$ and $N \in \mathbb{N}$. If $c_N = 1$ then clearly $|q_n - q_m| \leq 1$. If, on the other hand, $c_N = 0$ we have $q_N = 1$ and therefore $q_n = q_m = 1$ as $n, m \geq N$. It remains to see that $(c_n)_n \searrow 0$, i.e. $(c_n)_n$ is a classical null-sequence. It is clear that c_n is decreasing. Now let $\epsilon > 0$. We show that $\exists n. c_n \leq \epsilon$ by showing that $\exists n. c_n = 0$. But we have $\exists n. P(n)$ hence $\exists n. q_n = 1$ whence $\exists n. c_n = 0$. \square

We are now in a position to relate Markov reals to the total reals, i.e. the maximal elements in the interval domain.

Lemma 6.9. *Every Markov real is a total real, and vice versa. More precisely, the following hold:*

- (1) *Every total real $x = (x_n)_n$ defines a Markov sequence $m(x) = (\frac{1}{2}(\bar{x}_n + \underline{x}_n))$ with Markov modulus $c_n = \bar{x}_n - \underline{x}_n$.*

(2) Every Markov-sequence $q = (q_n)_n$ with modulus $(c_n)_n$ defines a total real $t(q) = (x_n)_n = (\bigsqcup_{0 \leq i \leq n} q_i \pm c_i)_n$.

(3) If x is a total real, then $x = t(m(x))$, that is, the above constructions preserve equality.

Proof. First suppose that $x = (x_n)_n = ([\underline{x}_n, \bar{x}_n])_n$ is a total real and let $q_n = \frac{1}{2}(\bar{x}_n + \underline{x}_n)$. By definition, $c_n = \bar{x}_n - \underline{x}_n$ is a classical null-sequence, so that we just need to establish that $|q_n - q_m| \leq c_N$ whenever $n, m \geq N$. Let $k = \min\{n, m\}$ and $l = \max\{n, m\}$. Then $k, l \geq N$ and we have that $q_n - q_m = \frac{1}{2}(\bar{x}_n + \underline{x}_n - \bar{x}_m - \underline{x}_m) \leq \frac{1}{2}(\bar{x}_k + \underline{x}_l - \bar{x}_l - \underline{x}_k) = \frac{1}{2}(c_k + c_l) \leq c_N$ by monotonicity of x . One analogously establishes that $-c_N \leq q_n - q_m$ whence $|q_n - q_m| \leq c_N$.

For the converse, we just need to establish that $\{q_i \pm c_i \mid 0 \leq i \leq n\}$ is consistent, given a Markov sequence $(q_n)_n$ with modulus $(c_n)_n$ which is evident as for example $c_n \sqsubseteq q_i \pm c_i$ for all $0 \leq i \leq n$. The last claim is evident by construction. \square

If we were to define the natural equality relation on Markov reals, we would also be able to establish that the above constructions also preserve equality of Markov reals, but this is not needed for what follows.

7. EXAMPLE: COMPUTATION OF SQUARE ROOTS

Having established the interval domain as a way of representing real numbers, we now exemplify the claim that we made in the introduction by means of an example: computation with constructive domain theoretic reals gives actual-case error bounds that are much tighter than the worst case error bounds of Cauchy reals.

Throughout the section, we fix a positive rational number $q > 0$ and demonstrate how to compute a total real $s = ([\underline{s}_n, \bar{s}_n])_n$ such that $\underline{s}_n^2 \leq q \leq \bar{s}_n^2$ for all $n \in \mathbb{N}$, i.e. s represents the square root of q .

Our definition is based on Newton iteration, specifically

$$\bar{s}_{-1} = 1 \quad \bar{s}_n = \frac{1}{2} \left(\bar{s}_{n-1} + \frac{q}{\bar{s}_{n-1}} \right) \quad \underline{s}_n = q / \bar{s}_n$$

for all $n \in \mathbb{N}$. We show that $s = ([\underline{s}_n, \bar{s}_n])_n$ is indeed a total real, and represents the square root of q . This is a sequence of lemmas involving standard estimates, the proofs of which we elide. In particular:

Lemma 7.1. *Let s be given as above.*

- (1) Both $\underline{s}_n > 0$ and $\bar{s}_n > 0$ for all $n \in \mathbb{N}$.
- (2) $(\bar{s}_n)_n$ is decreasing with $\bar{s}_n^2 \geq q$ for all $n \in \mathbb{N}$.
- (3) $(\underline{s}_n)_n$ is increasing with $\underline{s}_n^2 \leq q$ for all $n \in \mathbb{N}$.
- (4) $\underline{s}_n \leq \bar{s}_n$ for all $n \in \mathbb{N}$.

We now need to demonstrate that s is indeed total, i.e. the distance $\bar{s}_n - \underline{s}_n$ can be made arbitrarily small. This is an immediate consequence of the fact that (\underline{s}_n) is increasing that we use in the following lemma.

Lemma 7.2. *We have that $\bar{s}_n - \underline{s}_n \leq \frac{1}{2}(\bar{s}_{n-1} - \underline{s}_{n-1})$ for all $n > 0$.*

Proof. Immediate, as

$$\bar{s}_n - \underline{s}_n \leq \frac{1}{2} \left(\bar{s}_{n-1} + \frac{q}{\bar{s}_{n-1}} \right) - \underline{s}_{n-1} \leq \frac{1}{2} (\bar{s}_{n-1} - \underline{s}_{n-1})$$

using that \underline{s}_n is increasing. \square

The following is now immediate:

Corollary 7.3. *With \underline{s}_n and \bar{s}_n defined as above, $s = (\underline{s}_n, \bar{s}_n)_n$ is a total real with $s^2 = q$.*

We have the following comparison to Cauchy reals.

Comparison 7.4. If $q = 2$, in particular we have that $|\bar{s}_n - \bar{s}_m| \leq \frac{1}{6 \cdot 2^n}$ for all $m \geq n$. In other words, computing the n -th iterate of the square root of two, the attained precision is $\frac{1}{6 \cdot 2^n}$. This is precisely the same modulus of convergence that was obtained for the very same example in [Sch16]. The following table compares this to the interval width obtained from a domain theoretic approach where we have used a simple (hand extracted) Haskell program to obtain the data reported.

Iterations	Interval Width	Modulus Precision
1	4.9×10^{-3}	8.3×10^{-2}
2	4.2×10^{-6}	4.2×10^{-2}
3	3.2×10^{-12}	2.1×10^{-2}
4	1.8×10^{-24}	1.0×10^{-2}
5	5.7×10^{-49}	5.2×10^{-3}

While we cannot draw any valid conclusions from this very small experiment, we nonetheless note that the difference in precision is staggering, and warrant further investigation.

8. SCOTT TOPOLOGY VS EUCLIDEAN TOPOLOGY

In Section 6, we have investigated total reals *individually* by relating them to Markov and Cauchy reals. We continue our investigation of the real line induced by the total elements of the interval domain by also considering the topology on the real line, and linking that with the Scott topology of the interval domain.

As with convergence speed, we adopt a classical definition of properties, in particular that of openness.

Definition 8.1 (Basic Notions). A subset $O \subseteq \mathbb{R}$ is *open*, if for all $x \in O$ there must exist $k \in \mathbb{N}$ such that $\forall y \in \mathbb{R}. |x - y| \leq 2^{-k} \rightarrow y \in O$.

Our first result shows that the topology defined above is the subspace topology on the set of total reals, induced by the Scott topology. That is, every open set arises as the intersection of a Scott open set with the total reals, and every such intersection is itself open.

Lemma 8.2. *Let $O \subseteq \mathbb{IR}$ be Scott-open. Then $O \cap \mathbb{R}$ is open.*

Proof. Let $x \in O \cap \mathbb{R}$. As $x \in O$, there must exist $\alpha \in \mathbb{IQ}$ so that $O \ni \alpha \ll x$ by Lemma 4.2. As $x = \bigsqcup_n x_n$ by Lemma 3.3, there must exist $n \in \mathbb{N}$ such that $\alpha \sqsubseteq x_n$ and hence $\alpha \ll x_n$. By Lemma 2.5 there exists $\epsilon \in \mathbb{Q}_{>0}$ such that $\alpha \ll x_n \pm \epsilon$. We now claim that all $y \in O$ whenever $y \in \mathbb{R}$ with $|x - y| \leq \epsilon$. So pick $y \in \mathbb{R}$ and assume that $|y - x| \leq \epsilon$. Then $\alpha \ll x_n \pm \epsilon \sqsubseteq x \pm \epsilon \sqsubseteq y$ by Lemma 8.9. Hence $\alpha \ll y$ so that $y \in O$. \square

Lemma 8.3. *Let $O \subseteq \mathbb{R}$ be open. Then*

$$O = \{\alpha \in \mathbb{IR} \mid \exists [p, q] \in \mathbb{IQ}, [p, q] \sqsubseteq \alpha \text{ and } [p, q] \subseteq O\}$$

is Scott-open and satisfies $O \cap \mathbb{R} = O$.

Proof. We show that \mathcal{O} is Scott-open. First, it is immediate that \mathcal{O} is an upper set. To see that \mathcal{O} is inaccessible by directed suprema, let $(\alpha_n)_n$ be an increasing sequence in \mathbb{IR} with $\bigsqcup_n \alpha_n \in \mathcal{O}$. This gives $[p, q] \in \mathbb{IQ}$ with $[p, q] \sqsubseteq \alpha$ and $[p, q] \subseteq O$. As O is open, there must exist $\epsilon > 0$ and $\epsilon \in \mathbb{Q}$ with $[p, q] \pm \epsilon \subseteq O$. To see that there must exist $m \in \mathbb{N}$ with $\alpha_m \in \mathcal{O}$, we show that there must exist $m \in \mathbb{N}$ with $[p, q] \pm \epsilon \subseteq \alpha_m$. Because $[p, q] \pm \epsilon \ll [p, q] \sqsubseteq \bigsqcup_n \alpha_n$ this m must exist, i.e. m must exist with $[p, q] \pm \epsilon \sqsubseteq \alpha_m$.

It remains to be seen that $\mathcal{O} \cap \mathbb{R} = O$. Let $x \in \mathcal{O} \cap \mathbb{R}$. Because $x \in \mathcal{O}$ we have $[p, q] \in \mathbb{IQ}$ with $[p, q] \sqsubseteq x$ and $[p, q] \subseteq O$. As O is open, this entails that there must exist $\epsilon > 0$ and $\epsilon \in \mathbb{Q}$ such that $[p, q] \pm \epsilon \subseteq O$. We show that $p - \epsilon \leq x \leq q + \epsilon$ which implies $x \in O$. This follows from $[p, q] \pm \epsilon \ll [p, q] \sqsubseteq x$. It is clear that $O \subseteq \mathcal{O} \cap \mathbb{R}$. \square

We state and prove the following technical lemma to help deal with the Euclidean topology.

Lemma 8.4. *Let $a, b \in \mathbb{IQ}$. Then $a \leq b$ iff $-b \leq -a$ and $|a| \leq b$ iff $-b \leq a \leq b$ and $0 \leq b$.*

Proof. For the first statement, assume that $a \leq b$. Then $\underline{a} \leq \bar{b}$ whence $\underline{-b} = -\bar{b} \leq -\underline{a} = \bar{-a}$. If $-b \leq -a$ then $a = - - a \leq - - b = b$.

For the second statement, first suppose that $|a| \leq b$. We show that $-\bar{b} = \underline{-b} \leq \bar{a}$ and $\underline{a} \leq \bar{b}$. We distinguish three cases. If $0 \leq \underline{a}$, then $a = |a| \leq b$ whence $\underline{a} \leq \bar{b}$ and $-\bar{b} \leq -\underline{a} \leq 0 \leq \bar{a}$. Now suppose that $\underline{a} \leq 0 \leq \bar{a}$. Then $[0, \max\{-\underline{a}, \bar{a}\}] = |a| \leq b$ so that $0 \leq \bar{b}$. We obtain $-\bar{b} \leq 0 \leq \bar{a}$ and $\underline{a} \leq 0 \leq \bar{b}$. Finally assume that $\bar{a} \leq 0$. Then $[-\bar{a}, -\underline{a}] = |a| \leq b$ whence $-\bar{a} \leq \bar{b}$. Then $-\bar{b} \leq \bar{a}$ and $\underline{a} \leq \bar{a} \leq 0 \leq -\underline{a} \leq \bar{b}$. In all three cases we have that $0 \leq |a| \leq \bar{b}$ so that $0 \leq b$.

Now assume that $-b \leq a$, $a \leq b$ and $0 \leq b$. We show that $|a| \leq b$, again by distinguishing three cases. If $0 \leq \underline{a}$, we have $|a| = a \leq b$. Similarly, if $\bar{a} \leq 0$, we have $-b \leq a$ and therefore $|a| = -a \leq - - b = b$ using the first statement. Now assume that $\underline{a} \leq 0 \leq \bar{a}$. Then $|a| = [0, \max\{-\underline{a}, \bar{a}\}] \leq b$ if $0 \leq \bar{b}$ which is precisely our assumption that $0 \leq b$. \square

Remark 8.5. It is in general false that $|a| \leq b$ whenever $-b \leq a \leq b$. To see this, let $a = [-2, 1]$ and $b = -1 = [-1, -1]$. Then $-b = [1, 1] \leq [-2, 1] = a$ and $a = [-2, -1] \leq [-1, -1] = b$. On the other hand, $|a| = [0, 2] \not\leq [-1, -1] = b$.

Lemma 8.6. *Let $x, y \in \mathbb{IR}$. If $x \sqsubseteq y$ then $\underline{y}_n \leq \bar{x}_m$ and $\underline{x}_m \leq \bar{y}_n$ for all $n, m \in \mathbb{N}$.*

Proof. Suppose that $x \sqsubseteq y$, let $m, n \in \mathbb{N}$ and fix a rational $\epsilon > 0$. Then $x_m \pm \epsilon \ll x_m$, hence there must exist $k \in \mathbb{N}$ such that $x_m \pm \epsilon \ll y_k$. By monotonicity of y , we may assume that $k \geq n, m$. But then $\underline{x}_m - \epsilon = \underline{x}_m - \epsilon < \underline{y}_k \leq \bar{y}_k \leq \bar{y}_n$ as $k \geq m$. Similarly $\underline{y}_n \leq \underline{y}_k \leq \bar{y}_k < \overline{x_m \pm \epsilon} = x_m + \epsilon$. The claim follows as ϵ was arbitrary. \square

In the following, recall that $\ell([x, y]) = y - x$ denotes interval length, and that we identify rationals $q \in \mathbb{Q}$ with singleton intervals.

Lemma 8.7. *Let $w, x, y \in \mathbb{IR}$ with $w \sqsubseteq x$ and $w \sqsubseteq y$. Then $|x - y| \leq \ell(w_n)$ for all $n \in \mathbb{N}$.*

Proof. Let $n, k \in \mathbb{N}$. By Lemma 8.4 it suffices to show that $-\ell(w_n) \leq \overline{(x - y)_k} = \bar{x}_k - \underline{y}_k$ and $\underline{x}_k - \bar{y}_k = \underline{(x - y)_k} \leq \ell(w_n)$. This is a consequence of Lemma 8.6, since

$$-\ell(w_n) = \underline{w}_n - \bar{w}_n \leq \bar{x}_k - \bar{w}_n \leq \bar{x}_k - \underline{y}_k$$

and similarly,

$$\underline{x}_k - \bar{y}_k \leq \bar{w}_n - \bar{y}_k \leq \bar{w}_n - \underline{w}_k$$

as required. \square

Lemma 8.8. *Let $x = (x_n)_n \in \mathbb{IR}$. Then $x \pm \ell(x_n) \sqsubseteq x$ for all $n \in \mathbb{N}$.*

Proof. Let $\alpha \in \mathbb{I}\mathbb{Q}$ and $k \in \mathbb{N}$ with $\alpha \ll x_k \pm \ell(x_n)$. We show that $\alpha \ll x_n$. By Lemma 8.6 we have $\underline{x}_k \leq \bar{x}_n$ as $x \sqsubseteq x$. Therefore $\underline{\alpha} < \underline{x}_k - (\bar{x}_n - \underline{x}_n) \leq \bar{x}_n - \bar{x}_n + \underline{x}_n = \underline{x}_n$. Analogously one shows that $\bar{x}_n < \bar{\alpha}$ so that $\alpha \ll x_n$. \square

Lemma 8.9. *Let $x \in \mathbb{I}\mathbb{R}$, $y \in \mathbb{R}$ be an interval real and $\epsilon \in \mathbb{Q}_{>0}$. Then $x \pm \epsilon \sqsubseteq y$ whenever $|x - y| \leq \epsilon$.*

Proof. Let $n \in \mathbb{N}$, $a \in \mathbb{I}\mathbb{Q}$ such that $a \ll (x \pm \epsilon)_n$. We show that there must exist $k \in \mathbb{N}$ such that $a \ll (x \pm \epsilon)_k \pm \delta$. As y is an interval real, there must exist $k \in \mathbb{N}$ such that $\ell(y_k) \leq \delta$. We may assume that $k \geq n$. As $|x - y| \leq \epsilon$, we have that

$$-\epsilon \leq \overline{(x - y)}_k = \bar{x}_k - \underline{y}_k \quad \text{and} \quad \underline{x}_k - \bar{y}_k = \underline{(x - y)}_k \leq \epsilon.$$

We obtain that

$$\underline{a} < \underline{x}_n - \epsilon - \delta \leq \underline{x}_k - \epsilon - \delta \leq \bar{y}_k - \delta \leq \bar{y}_k - w(y_k) = \underline{y}_k$$

and one analogously establishes that $\bar{y}_k < \bar{a}$ so that $a \ll y_k$. \square

We characterise Scott continuous total functions in terms of their action on total reals.

Definition 8.10. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ and $x \in \mathbb{R}$. A *modulus of non-discontinuity* of f at x is a sequence $(\delta_n, \epsilon_n)_n$ in $\mathbb{Q}_{>0} \times (\mathbb{Q}_{\geq 0} \cup \{\infty\})$ such that

- (1) $\forall y \in \mathbb{R}. |x - y| \leq \delta_n \rightarrow |f(x) - f(y)| \leq \epsilon_n$ whenever $n \in \mathbb{N}$
- (2) $(\epsilon_n)_n$ is a classical null-sequence.

We say that f is *not discontinuous at x* if there exists a modulus of non-discontinuity of f at x . A *modulus of intensional non-discontinuity* for f is a function $\omega : \mathbb{I}\mathbb{Q} \rightarrow \mathbb{Q}_{>0}$ such that $(\ell(x_n), \omega(x_n))_n$ is a modulus of non-discontinuity at x whenever $x \in \mathbb{R}$ is total. A function f is *intensionally non-discontinuous* if there exists a modulus of intensional non-discontinuity for f . The function f is *continuous at x* if, for all $\epsilon \in \mathbb{Q}_{>0}$ there exists $\delta \in \mathbb{Q}_{>0}$ such that for all y with $|x - y| \leq \delta$ we have $|f(x) - f(y)| \leq \epsilon$.

Remark 8.11. It is evident that every function that is intensionally non-discontinuous is non-discontinuous at every $x \in \mathbb{R}$. The converse is not necessarily due to the uniformity requirement.

Lemma 8.12. *If $f : \mathbb{R} \rightarrow \mathbb{R}$ is continuous at x , then f is non-discontinuous at x .*

Proof. As f is continuous at x , there exists a function $\delta : \mathbb{Q}_{>0} \rightarrow \mathbb{Q}_{\geq 0}$ such that $\forall n \in \mathbb{N}. \forall y \in \mathbb{R}. |y - x| \leq \delta(2^{-n}) \rightarrow |f(x) - f(y)| \leq 2^{-n}$. Hence the sequence $(\delta(2^{-n}), 2^{-n})_n$ is a modulus of non-discontinuity for f at x . \square

Lemma 8.13. *If every (total) function $f : \mathbb{R} \rightarrow \mathbb{R}$ that is non-discontinuous at 0 is in fact continuous at 0, then Markov's principle holds.*

Proof. Let $P(n)$ be a decidable predicate on natural numbers and assume that $\neg \forall n \in \mathbb{N}. P(n)$. We show that there exists $n \in \mathbb{N}$ such that $P(n)$ under the assumption that every function that is non-discontinuous at 0 is in fact continuous at 0. Consider, for $n \in \mathbb{N}$, the function $f_n : \mathbb{I}\mathbb{R} \rightarrow \mathbb{I}\mathbb{R}_\perp$ defined by

$$f_n(x) = \begin{cases} \perp & \forall n \leq k. \neg P(k) \\ x \cdot \min\{k \leq n \mid P(k) = 1\} & \exists k \leq n. P(k) \end{cases}$$

and let $f = \bigsqcup_n f_n$. Clearly $(f_n)_n$ is monotone so that $\bigsqcup_n f_n$ exists, and $\bigsqcup_n f_n$ defines a total function with $f(0) = 0$. We claim that the sequence (ϵ_n, δ_n) defined by $\delta_n = \frac{1}{n^2}$ and

$$\epsilon_n = \begin{cases} \infty & \forall k \leq n. \neg P(k) \\ \frac{1}{n} & \exists k \leq n. P(k) \end{cases}$$

is a modulus of non-discontinuity for f at 0. To see this, let $n \in \mathbb{N}$. If $\forall k \leq n. \neg P(n)$, then $\epsilon_n = \infty$ and there is nothing to show. So assume that $\exists k \leq n. P(n)$ and $|y - 0| \leq \delta_n = \frac{1}{n^2}$. Then $|f(y) - f(0)| = |f(y)| = |y| \cdot \min\{k \leq n \mid P(n)\} \leq |y| \cdot n \leq \delta_n \cdot n = \frac{1}{n^2} \cdot n = \frac{1}{n} = \epsilon_n$.

By assumption, f is also continuous at 0, hence for $\epsilon = 1$ there exists δ such that for all y with $|y| = |y - 0| \leq \delta$ we have $|f(y)| = |f(y) - f(0)| \leq 1$. As $\delta \in \mathbb{Q}_{>0}$ there exists n such that $n \geq \frac{1}{\delta}$. We claim that there exists $k \leq n$ such that $P(k)$. As this property is decidable, we may assume that $\neg P(k)$ for all $k \leq n$, and establish a contradiction to prove the claim. By assumption, we have $\neg \forall k. \neg P(k)$ so that it suffices to show that $\forall k. \neg P(k)$ to establish the desired contradiction. So let $k \in \mathbb{N}$, we need to show $\neg P(k)$ to prove the claim. If $k \leq n$ then $\neg P(k)$ is given. Now let $k > n$. As P is decidable, we may assume $P(k)$ and establish a contradiction to show that $\neg P(k)$. So assume that $P(k)$. Then $f(y) = \min\{k' \leq k \mid P(k')\} \cdot x$.

We now use that for $y = \delta$, we have that $|fy| = \min\{k' \leq k \mid P(k')\} \cdot \delta \leq 1$ so that $\min\{k' \leq k \mid P(k')\} \leq \frac{1}{\delta} \leq k$ so that there exists $k' \leq k$ with $P(k')$, contradiction. \square

The following lemma shows that equality is stable under adding classical null-sequences. Again, this is a consequence of the classical formulation of equality on reals.

Lemma 8.14. *Let $x = (x_n) \in \mathbb{R}$ and let (q_n) be a classical null sequence. If $y = (x_n \pm q_n)$, then $x = y$.*

Proof. We establish that $(x_n \pm q_n)_n \sqsubseteq (x_n)_n$ and $(x_n)_n \sqsubseteq (x_n \pm q_n)_n$. The first relation is immediate using the Definition 3.1, for if $\alpha \ll x_n \pm q_n$, we have $\alpha \ll x_n \pm q_n \sqsubseteq x_n$ whence $\alpha \ll x_n \pm q_n$ by Lemma 2.9. For the second (converse) relation, assume that $\alpha \ll x_n$. Using Lemm 2.5, there exists $\epsilon \in \mathbb{Q}$ with $\epsilon > 0$ and $\alpha \ll x_n \pm \epsilon$. As $(q_n)_n$ is a classical null sequence, there must exist $m \in \mathbb{N}$ such that $q_m < \epsilon$. For $k = \max\{n, m\}$ we therefore obtain that $\alpha \ll x_n \pm \epsilon \sqsubseteq x_k \pm q_k$ as required. \square

Lemma 8.15. *Suppose that $f : \mathbb{R} \rightarrow \mathbb{R}$ is Scott continuous and total. Then $f \upharpoonright \mathbb{R}$ is intensionally non-discontinuous.*

Proof. Fix an antitone function $b : \mathbb{Q}_{>0} \rightarrow \mathbb{N}$ such that $2^{-b(q)} \leq q$ and let $s(\alpha) = b(\ell(\alpha))$ for $\alpha \in \mathbb{I}\mathbb{Q}$ with $\ell(\alpha) > 0$. Then s is antitone with respect to interval length, i.e. $0 < \ell(\alpha) \leq \ell(\beta)$ implies $s(\alpha) \geq s(\beta)$ for all $\alpha, \beta \in \mathbb{I}\mathbb{Q}$.

By the approximation lemma (Lemma 5.12) we have that $f = \bigsqcup_n f_n$ is a supremum of step functions. Define $\omega : \mathbb{I}\mathbb{Q} \rightarrow \mathbb{Q}_{\geq 0} \cup \{\infty\}$ by $\omega(\alpha) = \ell(f_{s(\alpha)}(\alpha \pm \ell(\alpha)))$. We show that ω is a modulus of intensional non-discontinuity for f .

Let $x \in \mathbb{R}$ be given. We first show that $\omega(x_n)_n$ is a classical null-sequence. Let $y_n = x_n \pm \ell(x_n)$. Then $x = y$ by Lemma 8.14 and $y = \bigsqcup_n y_n$ by Lemma 3.3. Then $f(x) = f(\bigsqcup_n y_n) = \bigsqcup_n f(y_n) = \bigsqcup_n \bigsqcup_m f_n(y_m) = \bigsqcup_n f_n(y_n)$ by Corollary 2.16, and applying Lemma 3.3 this gives that $f(x) = (f_n(y_n))_n$ as a sequence. Now let $\epsilon > 0$. As $f(x)$ is total and $f(x) = (f_n(y_n))_n$, there must exist $k \in \mathbb{N}$ such that $\ell(f_k(y_k)) \leq \epsilon$. By definition of s , there moreover must exist $i \in \mathbb{N}$ such that $s(x_i) \geq k$. As s is antitone with respect to interval length and $(y_n)_n$ is increasing, we may assume that $i \geq k$. Then $\omega(x_i) = \ell(f_{s(x_i)}(y_i)) \leq \ell(f_k(y_i)) \leq \ell(f_k(y_k)) \leq \epsilon$ as required.

We now show that $(\ell(x_n), \omega(x_n))_n$ is a modulus of non-discontinuity of f at x . So let $n \in \mathbb{N}$ and fix $y \in \mathbb{R}$ with $|x - y| \leq \ell(x_n)$. Then $x_n \pm \ell(x_n) \sqsubseteq x \pm \ell(x_n) \sqsubseteq y$ by Lemma 8.9, and therefore $f(x_n \pm \ell(x_n)) \sqsubseteq f(y)$. As also $x_n \pm \ell(x_n) \sqsubseteq x \pm \ell(x_n) \sqsubseteq x$ we have, again by monotonicity of f , that $f(x_n \pm \ell(x_n)) \sqsubseteq f(y)$. As $f_{s(x_n)} \sqsubseteq f$, this gives $f_{s(x_n)}(x_n \pm \ell(x_n)) \sqsubseteq f(x)$ and $f_{s(x_n)}(x \pm \ell(x_n)) \sqsubseteq f(y)$. By Lemma 8.7 we may conclude that $|f(x) - f(y)| \leq \ell(f_{s(x)}(x \pm \ell(x_n))) = \omega(x_n)$ which finishes the proof. \square

Theorem 8.16. *Suppose that $f : \mathbb{R} \rightarrow \mathbb{R}$ is intensionally non-discontinuous. Then there exists a Scott continuous function $g : \mathbb{R} \rightarrow \mathbb{R}_\perp$ such that $g \upharpoonright \mathbb{R} = f$.*

Proof. Let ω be a modulus of intensional non-discontinuity for f and suppose that $\mathbb{I}\mathbb{Q} = \{\alpha_i \mid i \in \mathbb{N}\}$ is an enumeration of $\mathbb{I}\mathbb{Q}$. For $\alpha \in \mathbb{I}\mathbb{Q}$ let $m(\alpha) = \frac{1}{2}(\underline{\alpha} + \bar{\alpha})$ denote the midpoint of α . Put

$$g_n = \bigsqcup_{i \leq n} \alpha_i \searrow f(m(\alpha_i))_n \pm \omega(\alpha_i)$$

and let $g = \bigsqcup_n g_n$.

We first show that g_n is well-defined for all $n \in \mathbb{N}$, i.e. satisfies the consistency requirement for step functions. To see this, let $I \subseteq \mathbb{N}$ be finite and suppose that $\{\alpha_i \mid i \in I\}$ is consistent. We show that $f(m(\alpha_i))_k \pm \omega(\alpha_i)$ are consistent for arbitrary $k \in \mathbb{N}$. The claim follows for $k = \max I$. By consistency of $\{\alpha_i \mid i \in I\}$ we obtain $q \in \mathbb{Q}$ such that $\alpha_i \sqsubseteq q$ for all $i \in I$, e.g. $q = \{\max \underline{\alpha}_i \mid i \in I\}$. We now obtain that $\alpha_i = m(\alpha_i) \pm \frac{1}{2}\ell(\alpha_i) \sqsubseteq q$ and $\alpha_i \sqsubseteq m(\alpha_i)$, hence by Lemma 8.7 we obtain $|q - m(\alpha_i)| \leq \frac{1}{2}\ell(\alpha_i)$. As ω is a modulus of intensional non-discontinuity for f , this gives $|f(q) - f(m(\alpha_i))| \leq \omega(\alpha_i)$. By Lemma 8.9 this gives $f(m(\alpha_i)) \pm \omega(\alpha_i) \sqsubseteq f(q)$. As $f(m(\alpha_i))_k \pm \omega(\alpha_i) \pm 2^{-n} \ll f(m(\alpha_i))_k \pm \omega(\alpha_i)$ and $f(m(\alpha_i)) \pm \omega(\alpha_i) \sqsubseteq f(q)$, there must exist $r(i)$ such that $f(m(\alpha_i))_k \pm \omega(\alpha_i) \pm 2^{-n} \ll f(q)_{r(i)}$. For $r = \max\{r(i) \mid i \in I\}$ we therefore have $f(m(\alpha_i)) \pm \omega(\alpha_i) \pm 2^{-n} \ll f(q)_r$, that is, the set $\{f(m(\alpha_i))_k \pm \omega(\alpha_i) \pm 2^{-n} \mid i \in I\}$ is consistent for all $n \in \mathbb{N}$. As consistency on $\mathbb{I}\mathbb{Q}$ is continuous by Lemma 2.7 this shows that $\{f(m(\alpha_i))_k \pm \omega(\alpha_i) \mid i \in I\}$ is consistent.

We now demonstrate that g preserves total reals, that is, $g(x)$ is total whenever x is. Let $x \in \mathbb{R}$ be total and $y_n = x_n \pm 2^{-n}$. Then, for every $n \in \mathbb{N}$ there must exist

- $\triangleright i \in \mathbb{N}$ such that $\omega(y_i) \leq \frac{\epsilon}{4}$
- $\triangleright j \in \mathbb{N}$ such that $y_i \in \{\alpha_i \mid i \leq j\}$
- $\triangleright k \in \mathbb{N}$ such that $\ell(f(m(y_i)))_k \leq \frac{\epsilon}{2}$

so that for $l = \max\{i, j, k\}$ we have:

$$\begin{aligned} g(x) &\sqsupseteq g_l(x_i) && \text{(as } g_l = \bigsqcup_n g_n = g \text{ and } x_i \sqsubseteq \bigsqcup_n x_n = x) \\ &\sqsupseteq (y_i \searrow f(m(y_i))_l \pm \omega(y_i))(x_i) && \text{(as } y_i \in \{\alpha_i \mid i \leq j\} \sqsubseteq \{\alpha_i \mid i \leq k\}) \\ &= f(m(y_i))_l \pm \omega(y_i) && \text{(definition of step functions)} \\ &\sqsupseteq f(m(y_i))_k \pm \omega(y_i) && \text{(monotonicity of } f(m(y_i)) \text{ and } k \leq l) \end{aligned}$$

In particular, this gives that

$$\ell(g(x)) \leq \ell(f(m(y_i))_l \pm \omega(y_i)) \leq \ell(f(m(y_i))_l) + 2\omega(y_i) \leq \frac{\epsilon}{2} + 2\frac{\epsilon}{4} = \epsilon$$

as required.

We finally demonstrate that $g(x) \sqsubseteq f(x)$ for $x \in \mathbb{R}$. As $g(x) = \bigsqcup_n g_n(x_n)$, it suffices to show that $g_n(x_n) \sqsubseteq f(x)$. So let $n \in \mathbb{N}$. Then $g_n(x) = \bigsqcup\{f(m(\alpha_i))_n \pm \omega(\alpha_i) \mid 0 \leq i \leq n, \alpha_i \ll x_n\}$ so that the claim follows once we show that $f(m(\alpha)_n \pm \omega(\alpha)) \sqsubseteq f(x)$ whenever $\alpha \in \mathbb{I}\mathbb{Q}$ with $\alpha \ll x_n$. So assume that $\alpha \ll x_n$ so that in particular $\alpha \sqsubseteq x$. As also $\alpha \sqsubseteq m(\alpha)$ we have

that $|x - \alpha| \leq \ell(\alpha)$ by Lemma 8.7. As ω is a modulus of intensional non-discontinuity, this gives $|f(x) - f(m(\alpha))| \leq \omega(\alpha)$ and in turn, using Lemma 8.9 that $f(m(\alpha)) \pm \omega(\alpha) \sqsubseteq f(x)$. Using monotonicity of $f(m(\alpha))$ we obtain $f(m(\alpha))_n \pm \omega(\alpha) \sqsubseteq f(x)$ as required.

We now finish the proof by showing that $f(x) = g(x)$ whenever $x \in \mathbb{R}$ is total. But this follows from $g(x)$ being maximal by Lemma 6.3 and the fact that $g(x) \sqsubseteq f(x)$. \square

9. CONCLUSION AND DISCUSSION

The main guiding principle of our development here was “constructive existence with classical correctness”. The main goal was to constructively rationalise standard practice in constructive analysis: constructions are carried out in the universe of classical mathematics, and then a secondary argument is used to show that they are in fact effective. This is reflected in our approach that emphasises constructive existence, but contends itself with classical correctness arguments. One consequence of this is that correctness assertions have no computational content under a realisability interpretation. While this can also be achieved by achieved using different methods (e.g. non-computational quantifiers [Ber93a] or Prop-valued assertions in the calculus of constructions [CH88]), we consciously took a pragmatic approach that aligns with computable analysis. As next step, our approach should be benchmarked both mathematically (e.g. by establishing standard results of computable analysis as carried out e.g. in [Sch08]) and experimentally, by implementing our theory in a theorem prover such as Coq [BCHPM04] or Minlog [SW12].

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